COMP9024: Data Structures and Algorithms

Analysis of Algorithms

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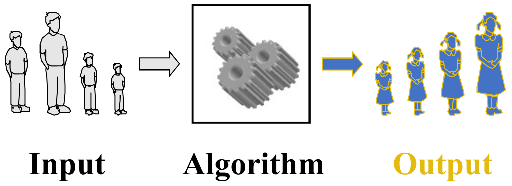
- Big-oh notation
- Big-theta notation
- Big-omega notation
- Asymptotic algorithm analysis

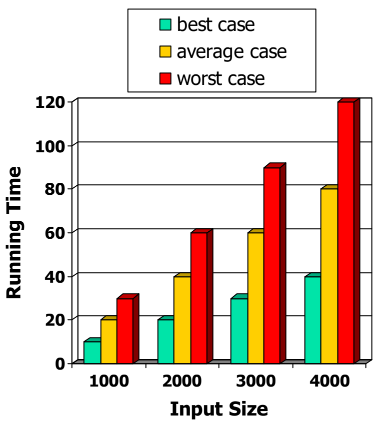
Analysis of Algorithms



An **algorithm** is a step-by-step procedure for solving a problem in a finite amount of time.

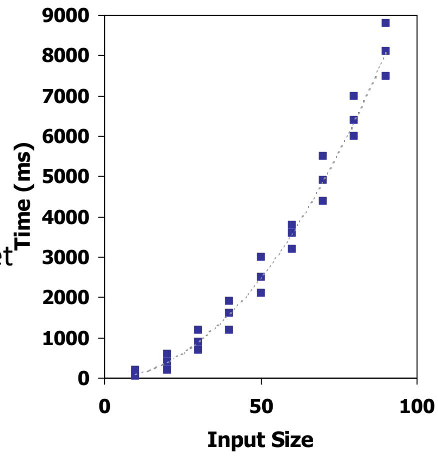
Running Time

- Most algorithms transform input objects into output objects.
- The running time of an algorithm typically grows with the input size.
- Average case time is often difficult to determine.
- We focus on the worst case running time.
  - Easier to analyze
  - Crucial to applications such as games, finance and robotics



Experimental Studies

- Write a program implementing the algorithm
- Run the program with inputs of varying size and composition
- Use a method like System.currentTimeMillis() to get an accurate measure of the actual running time
- Plot the results



Limitations of Experiments

- It is necessary to implement the algorithm, which may be difficult
- Results may not be indicative of the running time on other inputs not included in the experiment.
- In order to compare two algorithms, the same hardware and software environments must be used

Theoretical Analysis

- Uses a high-level description of the algorithm instead of an implementation
- Characterizes running time as a function of the input size, *n*.
- Takes into account all possible inputs
- Allows us to evaluate the speed of an algorithm independent of the hardware/software environment

Pseudocode

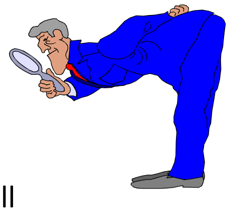
- High-level description of an algorithm
- More structured than English prose
- Less detailed than a program
- Preferred notation for describing algorithms
- Hides program design issues

Example: find max element of an array

```
Algorithm arrayMax(A, n)
 {
   Input array A of n integers
   Output maximum element of A

   currentMax = A[0]
   for ( i=1; i<n; i++)
      if  A[i] > currentMax
          currentMax =A[i]
    return currentMax
 }
```

C-Like Pseudocode Details



- Control flow
  - **if** … [**else** …]
  - **while** …
  - **do** … **while** …
  - **for** …
- Method declaration

  Algorithm *method* (*arg* [, *arg*…])

  Input …

  Output …
- Method call

  var.method (*arg* [, *arg*…])
- Return value

  return *expression*
- Expressions

  = Assignment

  = Equality testing

  n^2 Superscripts and other mathematical formatting allowed

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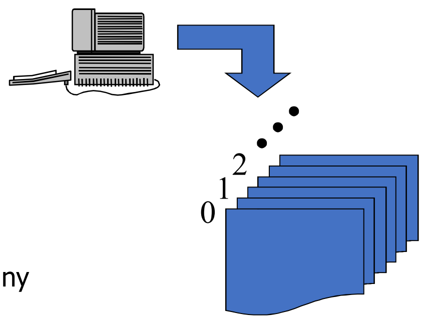
The Random Access Machine (RAM) Model

- A **CPU**
- An potentially unbounded bank of **memory** cells, each of which can hold an arbitrary number or character
  - Memory cells are numbered and accessing any cell in memory takes unit time.



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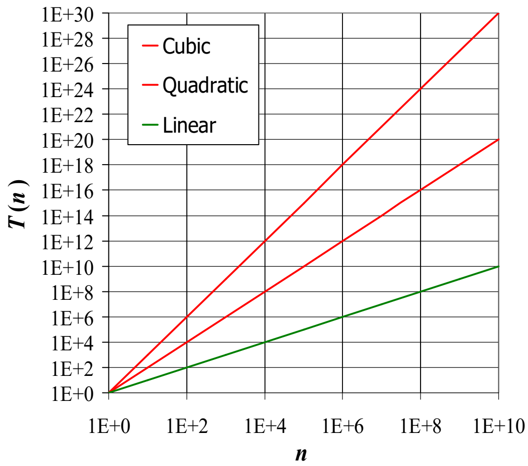
Seven Important Functions

- Seven functions that often appear in algorithm analysis:
  - Constant ≈ *1*
  - Logarithmic ≈ log n
  - Linear ≈ n
  - N-Log-N ≈ n log n
  - Quadratic ≈ n^2
  - Cubic ≈ n^3
  - Exponential ≈ 2^n
- In a log-log chart, the slope of the line corresponds to the growth rate of the function



Primitive Operations

- Basic computations performed by an algorithm
- Identifiable in pseudocode
- Largely independent from the programming language
- Exact definition not important (we will see why later)
- Assumed to take a constant amount of time in the RAM model

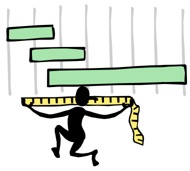
- Examples:
  - Evaluating an expression
  - Assigning a value to a variable
  - Indexing into an array
  - Calling a method
  - Returning from a method

Counting Primitive Operations

- By inspecting the pseudocode, we can determine the maximum number of primitive operations executed by an algorithm, as a function of the input size

Algorithm ***arrayMax(A, n)***	
{	# operations
currentMax = *A*[0]	1
for (*i =1; i<n-1; i++*)	*n*
if *A*[*i*] > *currentMax*	$n - 1$
currentMax = *A*[*i*]	$n - 1$
// increment counter *i*	$n - 1$
return *currentMax*	1
}	Total $4n - 1$

Estimating Running Time



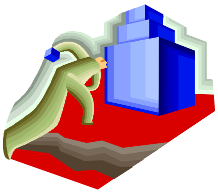
- Algorithm ***arrayMax*** executes $4n - 1$ primitive operations in the worst case. Define:
  - a = Time taken by the fastest primitive operation
  - b = Time taken by the slowest primitive operation
- Let $T(n)$ be worst-case time of ***arrayMax.*** Then

$$a\,(4n - 1) \leq T(n) \leq b(4n - 1)$$

- Hence, the running time $T(n)$ is bounded by two linear functions

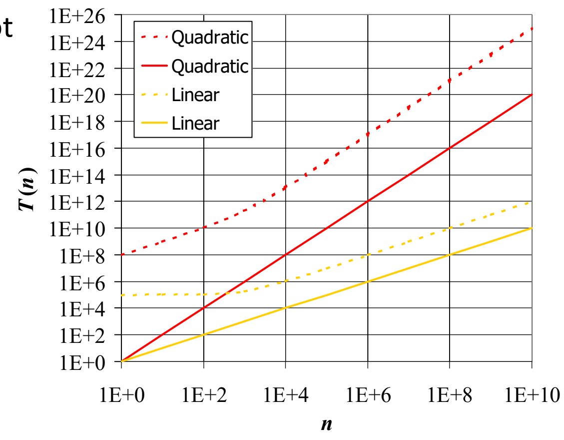
Growth Rate of Running Time

- Changing the hardware/ software environment
  - Affects $T(n)$ by a constant factor, but
  - Does not alter the growth rate of $T(n)$
- The linear growth rate of the running time $T(n)$ is an intrinsic property of algorithm ***arrayMax***



Constant Factors

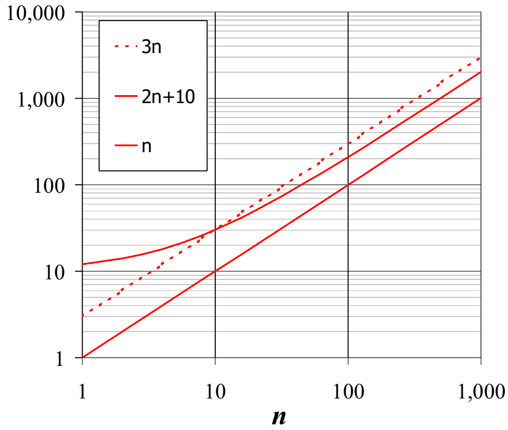
- The growth rate is not affected by
  - constant factors or
  - lower-order terms
- Examples
  - $10^2n + 10^5$ is a linear function
  - $10^5n^2 + 10^8n$ is a quadratic function



Big-Oh Notation

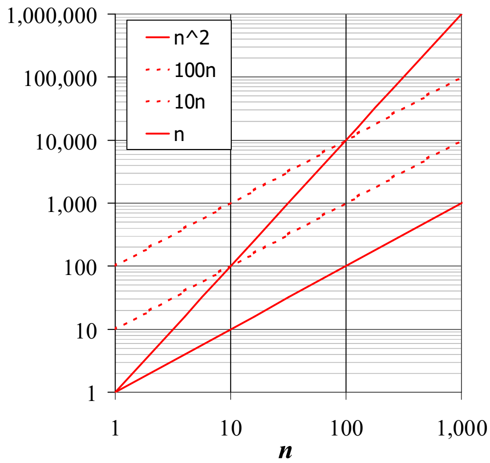
- Given functions $f(n)$ and $g(n)$, we say that $f(n)$ is $O(g(n))$ if there are positive constants c and n_0 such that

  $f(n) \leq cg(n)$ for $n \geq n_0$
- Example: $2n + 10$ is $O(n)$
  - $2n + 10 \leq cn$
  - $(c - 2)\, n \geq 10$
  - $n \geq 10/(c - 2)$
  - Pick $c = 3$ and $n_0 = 10$

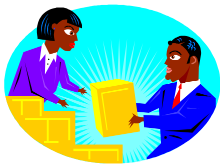


Big-Oh Example

- Example: the function n^2 is not $O(n)$
  - $n^2 \leq cn$
  - $n \leq c$
  - The above inequality cannot be satisfied since c must be a constant



More Big-Oh Examples



- 7n-2

  7n-2 is O(n)
  need $c > 0$ and $n_0 \geq 1$ such that $7n-2 \leq c \cdot n$ for $n \geq n_0$
  this is true for $c = 7$ and $n_0 = 1$

- $3n^3 + 20n^2 + 5$

  $3n^3 + 20n^2 + 5$ is $O(n^3)$
  need $c > 0$ and $n_0 \geq 1$ such that $3n^3 + 20n^2 + 5 \leq c \cdot n^3$ for $n \geq n_0$
  this is true for $c = 4$ and $n_0 = 21$

- 3 log n + 5

  3 log n + 5 is O(log n)
  need $c > 0$ and $n_0 \geq 1$ such that $3 \log n + 5 \leq c \cdot \log n$ for $n \geq n_0$
  this is true for $c = 8$ and $n_0 = 2$

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Big-Oh and Growth Rate

- The big-Oh notation gives an upper bound on the growth rate of a function
- The statement "$f(n)$ is $O(g(n))$" means that the growth rate of $f(n)$ is no more than the growth rate of $g(n)$
- We can use the big-Oh notation to rank functions according to their growth rate

	$f(n)$ is $O(g(n))$	$g(n)$ is $O(f(n))$
$g(n)$ grows more	Yes	No
$f(n)$ grows more	No	Yes
Same growth	Yes	Yes

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Big-Oh Rules (1/3)

- If is $f(n)$ a polynomial of degree d, then $f(n)$ is $O(n^d)$, i.e.,
  1. Drop lower-order terms
  2. Drop constant factors
- Use the smallest possible class of functions
  - Say "$2n$ is $O(n)$" instead of "$2n$ is $O(n^2)$"
- Use the simplest expression of the class
  - Say "$3n + 5$ is $O(n)$" instead of "$3n + 5$ is $O(3n)$"

Big-Oh Rules (2/3)

- $5n^5+20n^4-3n^3\log^2 n+10^7 n$ is $O(n^5)$

  Step 1: Drop lower-order terms: $5n^5$

  Step 2: Drop constant factors: n^5

  Therefore, $5n^5+20n^4-3n^3\log^2 n+10^7 n$ is $O(n^5)$.

- $10n^5+200n^4\log n-3n^3\log^2 n+5000n$

  Step 1: Drop lower-order terms: $10n^5$

  Step 2: Drop constant factors: n^5

  Therefore, $10n^5+200n^4\log n-3n^3\log^2 n+5000n$ is $O(n^5)$.

- $10*2^n+200n^{400}-3n^3\log^2 n+500n$

  Step 1: Drop lower-order terms: $10*2^n$

  Step 2: Drop constant factors: 2^n

  $10*2^n+200n^{400}-3n^3\log^2 n+500n$ is $O(2^n)$.

Big-Oh Rules (3/3)

- $1+2^3+3^3+\ldots+n^3$

  Step 1: Drop lower-order terms: n^3

  Step 2: Drop constant factors: n^3

  ~~Therefore, $1+2^3+3^3+\ldots+n^3$ is $O(n^3)$~~ ×

- The drop-constant-factor rule is only applicable to an arithmetic expression with a constant number of terms.

- $1+2^3+3^3+\ldots+n^3 < n*n^3=O(n^4)$

Asymptotic Algorithm Analysis

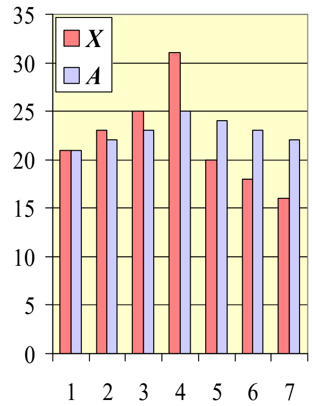
- The asymptotic analysis of an algorithm determines the running time in big-Oh notation.
- To perform the asymptotic analysis
  - We find the worst-case number of primitive operations executed as a function of the input size
  - We express this function with big-Oh notation
- Example:
  - We determine that algorithm ***arrayMax*** executes at most $4n - 1$ primitive operations
  - We say that algorithm ***arrayMax*** "runs in $O(n)$ time"
- Since constant factors and lower-order terms are eventually dropped anyhow, we can disregard them when counting primitive operations

Computing Prefix Averages

- We further illustrate asymptotic analysis with two algorithms for prefix averages
- The i-th prefix average of an array X is average of the first $(i + 1)$ elements of X:

$$A[i] = (X[0] + X[1] + \ldots + X[i])/(i+1)$$

- Computing the array A of prefix averages of another array X has applications to financial analysis



Prefix Averages (Quadratic)

- The following algorithm computes prefix averages in quadratic time by applying the definition

Algorithm *prefixAverages1*(X, n)
{
  Input array X of n integers
  Output array A of prefix averages of X — #operations
  A = new array of n integers; — n
  for (i = 0; $i < n$; i++) — n+1
   { $s = X[0]$; — n
    for (j =1; $j \le i$; j++) — $1 + 2 + \ldots + (n - 1)$
     $s = s + X[j]$; — $1 + 2 + \ldots + (n - 1)$
    $A[i] = s / (i + 1)$; — n
   }
  return A; — 1
}

Arithmetic Progression

- The total number of primitive operations of **prefixAverages1** is
  $n+n+1+n+1 + 2 + \ldots + (n - 1)+1 + 2 + \ldots + (n - 1)+n+1$
  $=n^2+3n+2=O(n^2)$.

- Thus, algorithm ***prefixAverages1*** runs in $O(n^2)$ time, or we say the *time complexity* of ***prefixAverages1*** is $O(n^2)$.

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Prefix Averages (Linear)

- The following algorithm computes prefix averages in linear time by keeping a running sum

Algorithm	#operations
Algorithm *prefixAverages2(X, n)*	
{ **Input** array ***X*** of ***n*** integers	
Output array ***A*** of prefix averages of ***X***	
A = new array of ***n*** integers	***n***
s = 0	1
for (***i*** = 0; ***i*** < ***n***; ***i***++)	***n***
{ ***s*** = ***s*** + ***X***[***i***]	***n***
A[***i***] = ***s*** / (***i*** + 1)	***n***
}	
return ***A***	1
}	

- Algorithm ***prefixAverages2*** runs in $O(n)$ time

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Binary Search (1/4)

The following recursive algorithm searches for a value in a sorted array:

```
BinarySearch(v, a, lo, hi)
 Input  value v
        array a[lo..hi] of values
 Output true if v in a[lo..hi]
        false otherwise

  mid=(lo+hi)/2
  if lo>hi  return false
  if a[mid]=v  return true
  else if a[mid]<v
        return BiarySearch(v,a,mid+1,hi)
      else
        return BinarySearch(v,a,lo,mid-1)
```

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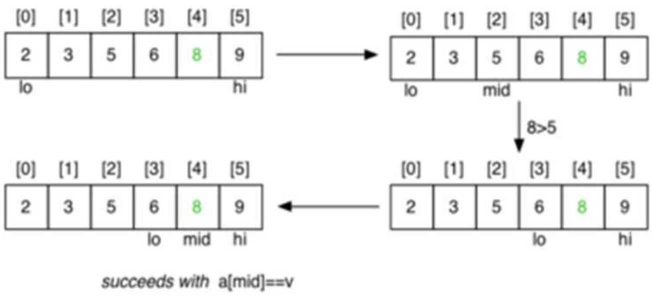
Binary Search (2/4)

Successful search for a value of 8:



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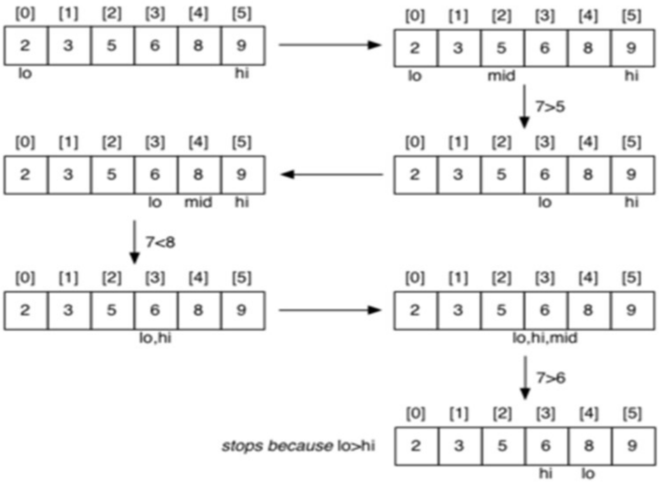
Binary Search (3/4)

Unsuccessful search for a value of 7:

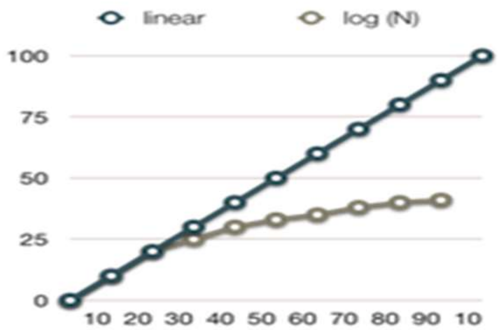


Binary Search (4/4)

Time complexity analysis:

- A single call of BinarySearch() takes O(1) time
- The number of calls of BinarySearch() is O(log n) in the worst case
- Therefore, the time complexity of the binary search is O(log n)

Linear Time vs Logarithmic Time



A logarithmic time algorithm is much faster than a linear time one

Computing Powers (1/3)

- The power function, $p(x,n)=x^n$, can be defined recursively:

$$p(x,n)=\begin{cases} 1 & \text{if } n=0 \\ x \cdot p(x,n-1) & \text{else} \end{cases}$$

- This leads to an power function that runs in O(n) time (for we make n recursive calls).
- We can do better than this, however.

Computing Powers (2/3)

- We can derive a more efficient linearly recursive algorithm by using repeated squaring:

$$p(x,n) = \begin{cases} 1 & \text{if } x = 0 \\ x \cdot p(x,(n-1)/2)^2 & \text{if } x > 0 \text{ is odd} \\ p(x,n/2)^2 & \text{if } x > 0 \text{ is even} \end{cases}$$

- For example,

$2^4 = 2^{(4/2)2} = (2^{4/2})^2 = (2^2)^2 = 4^2 = 16$
$2^5 = 2^{1+(4/2)2} = 2(2^{4/2})^2 = 2(2^2)^2 = 2(4^2) = 32$
$2^6 = 2^{(6/2)2} = (2^{6/2})^2 = (2^3)^2 = 8^2 = 64$
$2^7 = 2^{1+(6/2)2} = 2(2^{6/2})^2 = 2(2^3)^2 = 2(8^2) = 128.$

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Computing Powers (3/3)

Time complexity analysis:
- Each call of Power() takes O(1) time
- There are O(log n) calls
- Time complexity: O(log n)

Algorithm Power(*x, n*)
{
 Input : A number *x* and integer *n* = 0
 Output : The value x^n
 if n = 0 **return** 1
 if n is odd
  { y = Power(x, (n - 1)/ 2)
   return x*y*y }
 else
  { y = Power(x, n/ 2)
   return y*y }
}

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Computing Fibanacci Numbers (1/3)

- Fibonacci numbers are defined recursively:

  $F_0 = 0$
  $F_1 = 1$
  $F_i = F_{i-1} + F_{i-2}$ for $i > 1$.
- As a recursive algorithm (first attempt):

```
Algorithm BinaryFib(k)
 { Input : Nonnegative integer k
   Output : The kth Fibonacci number F_k
    if ( k =0 or 1) return k;
    else
       return BinaryFib(k - 1) + BinaryFib(k - 2);
 }
```


Computing Fibanacci Numbers (2/3)

- Let n_k denote number of recursive calls made by BinaryFib(k). Then
  - $n_0 = 1$
  - $n_1 = 1$
  - $n_2 = n_1 + n_0 + 1 = 1 + 1 + 1 = 3$
  - $n_3 = n_2 + n_1 + 1 = 3 + 1 + 1 = 5$
  - $n_4 = n_3 + n_2 + 1 = 5 + 3 + 1 = 9$
  - $n_5 = n_4 + n_3 + 1 = 9 + 5 + 1 = 15$
  - $n_6 = n_5 + n_4 + 1 = 15 + 9 + 1 = 25$
  - $n_7 = n_6 + n_5 + 1 = 25 + 15 + 1 = 41$
  - $n_8 = n_7 + n_6 + 1 = 41 + 25 + 1 = 67$.
- Note that the value at least doubles for every other value of n_k. That is, $n_k > 2^{k/2}$. It is exponential!
- Time complexity: $O(2^k)$

Computing Fibanacci Numbers (3/3)

To compute F_k's only once, we remember the value of each F_k:

```
Algorithm LinearFibonacci(k):
 {  Input : A nonnegative integer k
    Output : Pair of Fibonacci numbers (F_k, F_k-1)
     if  ( k = 1) return (k, 0);
     else
       {
         (i,  j)  =  LinearFibonacci(k - 1);
         return (i +j, i);
       }
  }
```

Time complexity: O(k)

* Space Complexity Analysis for Recursive Algorithms (1/9)

- In general, space complexity analysis is easier than time complexity analysis.
- The hard part in analyzing the space complexity of a recursive algorithm is in the stack space complexity.
- We need to understand how a recursive method is executed on computers.
- Key concept: stack frame or activation record.

* Space Complexity Analysis for Recursive Algorithms (2/9)

- A stack frame for a function stores the local variables, some parameters, the return address, and some other stuff such as values of some registers.
- A stack frame is created in the stack space whenever a function is called.
- A stack frame is freed when the function returns.
- The fame size of each frame can be determined at compile time.

* Space Complexity Analysis for Recursive Algorithms (3/9)

- A call graph is a weighted directed graph G = (V, E, W) where
  - $V=\{v_1, v_2, \ldots, v_n\}$ is a set of nodes each of which denotes an execution of a function;
  - $E=\{v_i \rightarrow v_j: v_i \text{ calls } v_j\}$ is a set of directed edges each of which denotes the caller-callee relationship, and
  - $W=\{w_i\ (i=1, 2, \ldots, n): w_i \text{ is the frame size of } v_i\}$ is a set of stack frame sizes.
- The maximum size of stack space needed for method calls can be derived from the call graph.

* Space Complexity Analysis for Recursive Algorithms (4/9)

- How to compute the maximum size of stack space needed for a method call?

  Step 1: Draw the call graph.

  Step 2: Find the longest weighted path in the call graph.

  The total weight of the longest weighted path is the maximum stack size needed for the function calls.

* Space Complexity Analysis for Recursive Algorithms (5/9)

Assumptions:

- func3() is called 20 times
- Frame sizes (bytes):
  - main(): 10
  - func1(): 20
  - func2(): 60
  - func3(): 80
  - func4(): 10
  - func5(): 30

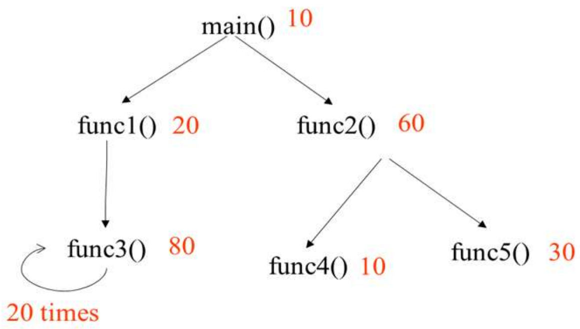
```
int main(void)
{ …
   func1();
   …
   func2();
}

void func1()
 { …
   func3();
   …
 }
```

```
void func2()
{ …
  func4();
  …
  func5();
 …
}

int func3()
{ …
   x=func3();
   …
}
```

* Space Complexity Analysis for Recursive Algorithms (6/9)



The longest path is main() → func1() → func3() … → func3() with a length (total weight) of 10+20+80*20=1630. So the maximum stack space needed for main() is 1630 bytes.

* Space Complexity Analysis for Recursive Algorithms (7/9)

The above approach can be generalized to recursive algorithms.

- The frame size of each algorithm is represented by big O.
- Compute the longest path length in terms of big O.

Consider the previous example.

- Assume that the frame sizes of all the methods except func4() are O(1), and the frame size of func4() is O(n). The space complexity of main() is O(n).
- Assume that the frame sizes of all the methods are O(1). The space complexity of main() is O(1).

* Space Complexity Analysis for Recursive Algorithms (8/9)

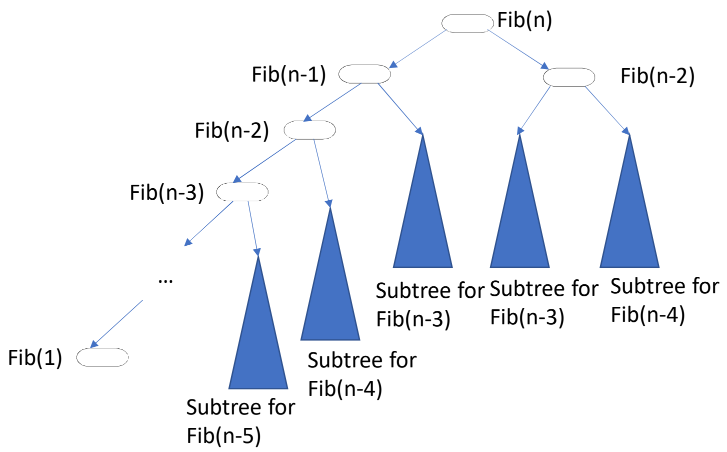
Recursive algorithm for Fibonacci numbers:

Algorithm Fib(k)
{ **Input :** Nonnegative integer k
Output : The kth Fibonacci number F_k
if (k =0 or 1) **return** k;
else
return Fib(k - 1) + Fib(k - 2);
}

What is the space complexity of Fib(n) in terms of big-O?

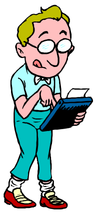
* Space Complexity Analysis for Recursive Algorithms (9/9)

What is the space complexity of BinaryFib(n) in terms of big-O?



The space complexity: the number of node on the longest path * frame size=n*c=O(n)

Math you need to Review



- Summations
- Logarithms and Exponents
  - **properties of logarithms:**
    $\log_b(xy) = \log_b x + \log_b y$
    $\log_b (x/y) = \log_b x - \log_b y$
    $\log_b x^a = a\log_b x$
    $\log_b a = \log_x a / \log_x b$
  - **properties of exponentials:**
    $a^{(b+c)} = a^b a^c$
    $a^{bc} = (a^b)^c$
    $a^b/a^c = a^{(b-c)}$
    $b = a^{\log_a b}$
    $b^c = a^{c*\log_a b}$
- Proof techniques
- Basic probability

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Relatives of Big-Oh

- **Big-Omega**
  - f(n) is $\Omega(g(n))$ if there is a constant $c > 0$ and an integer constant $n_0 \geq 1$ such that $f(n) \geq c \bullet g(n)$ for $n \geq n_0$
- **Big-Theta**
  - f(n) is $\Theta(g(n))$ if there are constants $c' > 0$ and $c'' > 0$ and an integer constant $n_0 \geq 1$ such that $c' \bullet g(n) \leq f(n) \leq c'' \bullet g(n)$ for $n \geq n_0$

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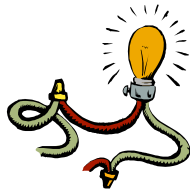
Intuition for Asymptotic Notation



Big-Oh

- f(n) is O(g(n)) if f(n) is asymptotically **less than or equal** to g(n)

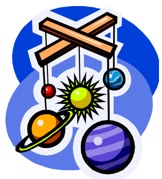
Big-Omega

- f(n) is Ω(g(n)) if f(n) is asymptotically **greater than or equal** to g(n)

Big-Theta

- f(n) is Θ(g(n)) if f(n) is asymptotically **equal** to g(n)

Example Uses of the Relatives of Big-Oh



- **$5n^2$ is $\Omega(n^2)$**

  $f(n)$ is $\Omega(g(n))$ if there is a constant $c > 0$ and an integer constant $n_0 \geq 1$ such that $f(n) \geq c \cdot g(n)$ for $n \geq n_0$

  let $c = 5$ and $n_0 = 1$

- **$5n^2$ is $\Omega(n)$**

  $f(n)$ is $\Omega(g(n))$ if there is a constant $c > 0$ and an integer constant $n_0 \geq 1$ such that $f(n) \geq c \cdot g(n)$ for $n \geq n_0$

  let $c = 1$ and $n_0 = 1$

- **$5n^2$ is $\Theta(n^2)$**

  $f(n)$ is $\Theta(g(n))$ if it is $\Omega(n^2)$ and $O(n^2)$. We have already seen the former, for the latter recall that $f(n)$ is $O(g(n))$ if there is a constant $c > 0$ and an integer constant $n_0 \geq 1$ such that $f(n) \leq c \cdot g(n)$ for $n \geq n_0$

  Let $c = 5$ and $n_0 = 1$

Summary

- Big-Oh, big-theta and big-omega notations
- Asymptotic analysis of algorithms
- Examples of algorithms with logarithmic, linear, polynomial, exponential time complexity
- Suggested reading:
  - Sedgewick, Ch.2.1-2.4,2.6