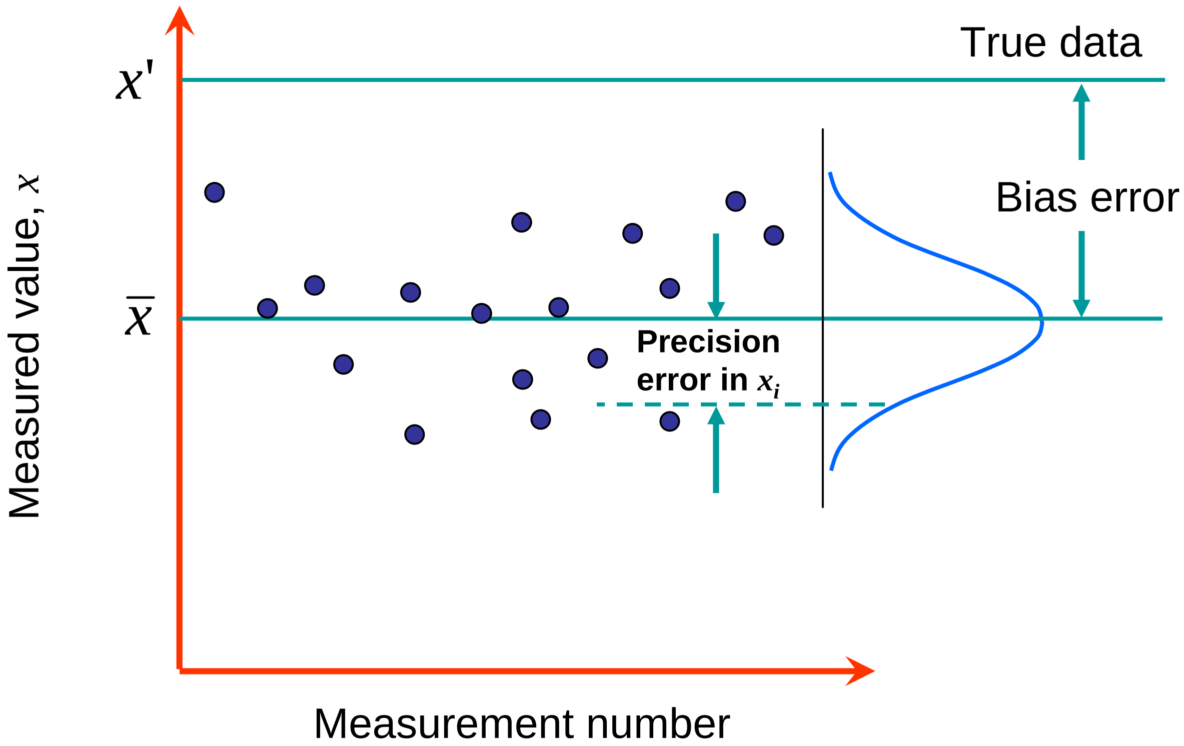
2141-375
Measurement and Instrumentation

Uncertainty Analysis

Measurement Error



Uncertainty defines an interval about the measured value within which we suspect the true value must fall
We call the process of identifying and quantifying errors as uncertainty analysis.

Design-Stage Uncertainty Analysis

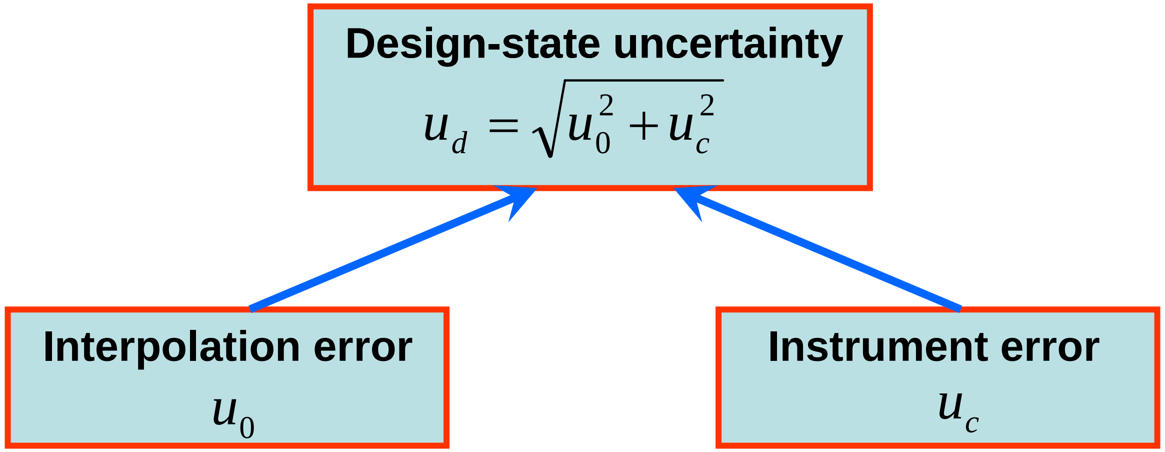
Design-stage uncertainty analysis refers to an initial analysis performed prior to the measurement

Useful for selecting instruments, measurement techniques and to estimate the minimum uncertainty that would result from the measurement .

Design-Stage Uncertainty Analysis

$$u_d = \sqrt{u_0^2 + u_c^2} \quad (P\%)$$

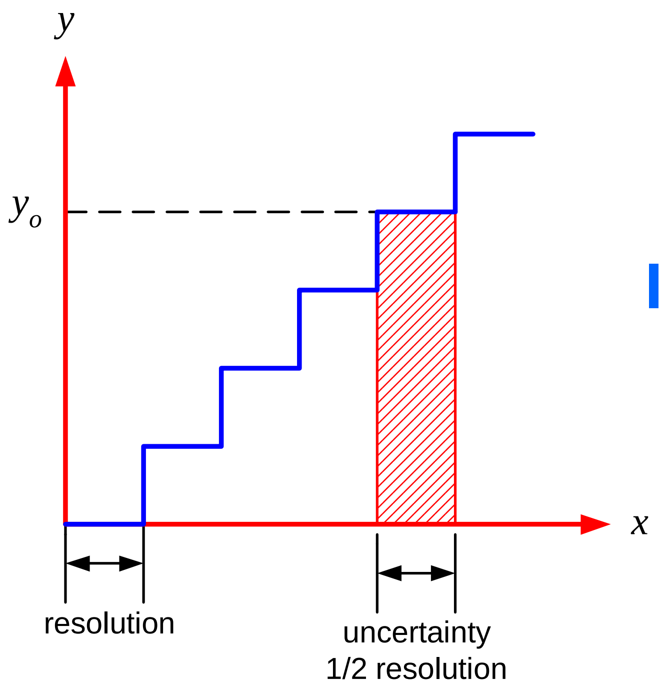
RSS method for combining error



Design-Stage Uncertainty Analysis

Zero-Order Uncertainty (Interpolation Error)

Even when all error are zero, the value of the measurand must be affected by the ability to resolve the information provided by the instrument. This is called zero-order uncertainty. At zero-order, we assume that the variation expected in the measurand will be less than that caused by the instrument resolution. And that all other aspects of the measurement are perfectly controlled (ideal conditions)



$$u_0 = \pm 1/2 \text{ resolution} \quad (95\%)$$

Instrument Uncertainty, u_c

This information is available from the manufacturer's catalog

Design-Stage Uncertainty Analysis

Specifications: Typical Pressure Transducer

Operation	
Input range	0-1000 cm H_2O
Excitation	±15 V dc
Output range	0-5 V
Temperature range	0-50°C nominal at 25°C
Performance	
Linearity error e_L	±0.5%FSO
Hysteresis error e_h	Less than ±0.15%FSO
Sensitivity error e_S	±0.25%of reading
Thermal sensitivity error e_{ST}	0.02%/°C of reading from 25°C
Thermal zero drift e_{ZT}	0.02%/°C FSO from 25°C

The root of sum square approach:

$$e_{rss} = \sqrt{e_1^2 + e_2^2 + e_3^2 + \cdots e_n^2} \qquad (95\%)$$

Design-Stage Uncertainty Analysis

Example: Consider the force measuring instrument described by the catalog data that follows. Provide an estimate of the uncertainty attributable to this instrument and the instrument design state uncertainty.

Force measuring instrument

Resolution:	0.25 N
Range:	0 - 100 N
Linearity:	within 0.20 N over range
Repeatability:	within 0.30 N over range

Known: Instrument specifications

Assume: Values representation of instrument 95% probability

Solution:

Design-state uncertainty

$$u_d = \sqrt{u_0^2 + u_c^2}$$

$$u_d = \pm\sqrt{0.125^2 + 0.36^2} = \pm 0.38 \text{ N}$$

u_0

½ Resolution = 0.125 N

u_c

$$\sqrt{e_l^2 + e_r^2} = \pm\sqrt{0.2^2 + 0.3^2} = \pm 0.36 \text{ N}$$

Design-Stage Uncertainty Analysis

Example: A voltmeter is to be used to measure the output from a pressure transducer that outputs an electrical signal. The nominal pressure expected will be ~3 psi (3 lb/in2). Estimate the design-state uncertainty in this combination. The following information is available:

Voltmeter	
Resolution:	10 μV
Accuracy:	within 0.001% of reading
Transducer	
Range:	$\pm$5 psi
Sensitivity:	1 V/psi
Input power:	10 Vdc $\pm$ 1%
Output:	$\pm$5 V
Linearity:	within 2.5 mV/psi over range
Repeatability:	within 2 mV/psi over range
Resolution:	negligible

Known: Instrument specifications

Assume: Values representation of instrument 95% probability

Solution:

Design-Stage Uncertainty Analysis

Design-state uncertainty

$$u_d = \sqrt{(u_d)_E^2 + (u_d)_P^2}$$

Design-state uncertainty

$$(u_d)_E = \sqrt{(u_0)_E^2 + (u_c)_E^2}$$

Design-state uncertainty

$$(u_d)_P = \sqrt{(u_0)_P^2 + (u_c)_P^2}$$

Error Propagation

Computation of the overall uncertainty for a measurement system consisting of a chain of components or several instruments

Let R is a known function of the n independent variables x_{i1}, x_{i2} , x_{i3}, …, x_{iL}

$$R = f(x_1, x_2, \ldots, x_L)$$

L is the number of independent variables. Each variable contains some uncertainty (u_{x1}, u_{x2}, u_{x3},…, u_{xL}) that will affect the result R.

Application of Taylor's expansion gives, (neglect the higher order term)

$$\bar{R} \pm \Delta R = f(\bar{x}_1 \pm u_{x1}, \bar{x}_2 \pm u_{x2}, \ldots, \bar{x}_L \pm u_{xL}) \approx f(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_L) +$$

$$\frac{\partial f}{\partial x_1} u_{x1} + \frac{\partial f}{\partial x_2} u_{x2} + \ldots + \frac{\partial f}{\partial x_L} u_{xL}$$

The best estimate value, R'

$$R' = \bar{R} \pm u_R \quad (P\%)$$

Where $\bar{R} = f(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_L)$

Error Propagation

The combination of uncertainty of all variables (probable estimate of u_R)

$$u_R = \pm\sqrt{\left(\frac{\partial f}{\partial x_1}u_{x1}\right)^2 + \left(\frac{\partial f}{\partial x_2}u_{x2}\right)^2 + \ldots + \left(\frac{\partial f}{\partial x_L}u_{xL}\right)^2}$$

$$= \pm\sqrt{\sum_{i=1}^{L}\left(\theta_i u_{xi}\right)^2} \quad (P\%)$$

Where θ_i is the sensitivity index relate to the uncertainty of x_i

$$\theta_i = \frac{\partial f}{\partial x_i}$$

Error Propagation

Example: For a displacement transducer having a calibration curve $y = KE$, estimate the uncertainty in displacement y for E = 5.00 V, if K = 10.10 mm/V with u_k = ±0.10 mm/V and u_E = ±0.01 V at 95% confidence

Known: $y = KE$

E = 5.00 V u_E = 0.01 V

K = 10.10 mm/V u_k = 0.10 mm/V

Solution: Find u_y

$$y' = \bar{y} \pm u_y = KE \pm u_y$$

$$\boxed{u_y = \pm\sqrt{(\theta_E u_E)^2 + (\theta_K u_K)^2}}$$

$$\theta_E = \frac{\partial y}{\partial E} = K$$

u_E = 0.01 V

$$\theta_K = \frac{\partial y}{\partial K} = E$$

u_K = 0.10 mm/V

$$u_y = \pm\sqrt{(Ku_E)^2 + (Eu_K)^2}$$

$$= \pm\sqrt{(10.10\ \mathrm{mm/V} \times 0.01\ \mathrm{V})^2 + (5\ \mathrm{V} \times 0.10\ \mathrm{mm/V})^2} = \pm 0.51\ \mathrm{mm}$$

Sequential Perturbation

A numerical approach can also be used to estimated the propagation of uncertainty. This refers to as sequential perturbation. This method is straightforward and uses the finite difference to approximate the derivatives (sensitivity index)

1) Calculate the average result from the independent variables

$$\bar{R} = f(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_L)$$

2) Increase the independent variables by their respect uncertainties and recalculate the result based on each of these new values. Call these values R_i^+

$$R_1^+ = f(\bar{x}_1 + u_1, \bar{x}_2, \ldots, \bar{x}_L),$$

$$R_2^+ = f(\bar{x}_1, \bar{x}_2 + u_2, \ldots, \bar{x}_L)$$

$$R_L^+ = f(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_L + u_L)$$

3) Decrease the independent variables by their respect uncertainties and recalculate the result based on each of these new values. Call these values R_i^-

Sequential Perturbation

$$R_1^- = f(\bar{x}_1 - u_1, \bar{x}_2, ..., \bar{x}_L),$$
$$R_2^- = f(\bar{x}_1, \bar{x}_2 - u_2, ..., \bar{x}_L)$$
$$R_L^- = f(\bar{x}_1, \bar{x}_2, ..., \bar{x}_L - u_L)$$

4) Calculate the difference for each element

$$\delta R_i^+ = R_i^+ - \bar{R}$$
$$\delta R_i^- = R_i^- - \bar{R}$$

5) Finally, evaluate the approximation of the uncertainty contribution from each variables

$$\delta R_i = \frac{\left|\delta R_i^+\right| + \left|\delta R_i^-\right|}{2} \approx \theta_i u_i$$

The uncertainty in the result

$$u_R = \pm \left[\sum_{i=1}^{L} \left(\delta R_i\right)^2 \right]^{1/2}$$

Error Propagation

Example: For a displacement transducer having a calibration curve $y = KE$, estimate the uncertainty in displacement y for E = 5.00 V, if K = 10.10 mm/V with u_k = ±0.10 mm/V and u_E = ±0.01 V at 95% confidence

Known: $y = KE$

E = 5.00 V u_E = 0.01 V

K = 10.10 mm/V u_k = 0.10 mm/V

Solution: Find u_y

$$y' = \bar{y} \pm u_y = KE \pm u_y$$

$$u_y = \pm\sqrt{(\delta R_E)^2 + (\delta R_K)^2}$$

$$\bar{y} = KE = (10.10)(5) = 50.50 \text{ mm}$$

i	x_i		u_i	x_i+u_i	x_i-u_i	R_i^+	R_i^-	δR_i^+	δR_i^-	δR_i
1	E	5	0.01	5.01	4.99	50.60	50.40	0.10	-0.10	0.10
2	K	10.1	0.1	10.20	10.00	51.00	50.00	0.50	-0.50	0.50

Error Sources

Steps in measurement process

1) Calibration
2) Data-acquisition
3) Data-reduction (Analysis)

Calibration error $e_{11}, e_{12}, \ldots$	Data-acquisition error $e_{21}, e_{22}, \ldots$	Data-reduction error $e_{31}, e_{32}, \ldots$

e_{ij}

i = Error source group

i = 1 for Calibration Error

i = 2 for Data-acquisition Error

i = 3 for Data-reduction Error

j = Elemental error

Calibration Error Source Group

Element (*j*)	Error Source
1	Primary to interlab standard
2	Interlab to transfer standard
3	Transfer to lab standard
4	Lab standard to measurement system
5	Calibration technique
Etc.	

Data-Acquisition Error Source Group

Element (*j*)	Error Source
1	Measurement system operating conditions
2	Sensor-transducer stage (instrument error)
3	Signal conditioning stage (instrument error)
4	Output stage (instrument error)
5	Process operating conditions
6	Process installation effects
7	Environmental effects
8	Spatial variation error
9	Temporal variation error
Etc.	

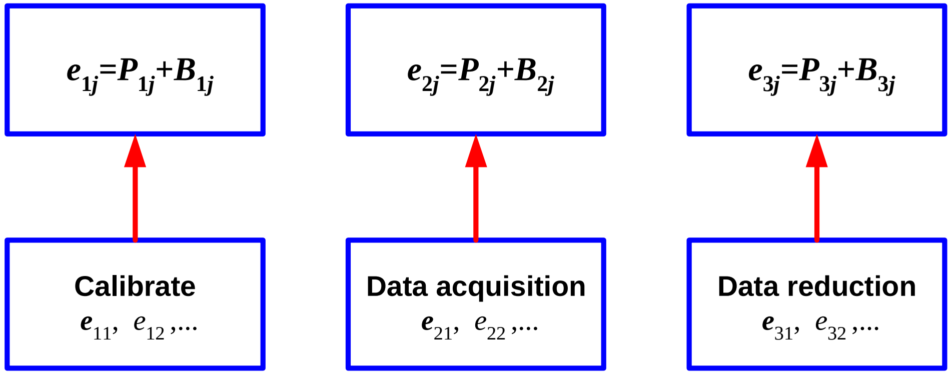
Data-Reduction Error Source Group

Element (*j*)	Error Source
1	Calibration curve fit
2	Truncation error
Etc.	

Multiple-Measurement Uncertainty Analysis

This section develops a method for the estimate of the uncertainty in the value assigned to a measured variable based on repeated measurements

The procedure for a multiple-measurement uncertainty analysis



Identify the elemental errors in each of the three source groups (calibration, data acquisition, and data reduction)

Estimate the magnitude of bias and precision error in each of the elemental errors

Estimate any propagation of uncertainty through to the result

Multiple-Measurement Uncertainty Analysis

Consider the measurement of variable, x which is subject to elemental precision errors, P_{ij} and bias, B_{ij} in each of three source groups. Let i = 1, 2, 3 refer to the error source groups (calibration error i = 1, data acquisition error i = 2, data-reduction i = 3) and j = 1,2,…,K refer to each of up to any K error elements of error e_{ij}

Source Precision index P_i

$$P_i = \left[P_{i1}^2 + P_{i2}^2 + \ldots + P_{ik}^2\right]^{1/2} \qquad i = 1, 2, 3$$

Measurement Precision index P

$$P = \left[P_1^2 + P_2^2 + P_3^2\right]^{1/2}$$

Source Bias limit B_i

$$B_i = \left[B_{i1}^2 + B_{i2}^2 + \ldots + B_{ik}^2\right]^{1/2} \qquad i = 1, 2, 3$$

Measurement Bias limit B

$$B = \left[B_1^2 + B_2^2 + B_3^2\right]^{1/2}$$

Multiple-Measurement Uncertainty Analysis

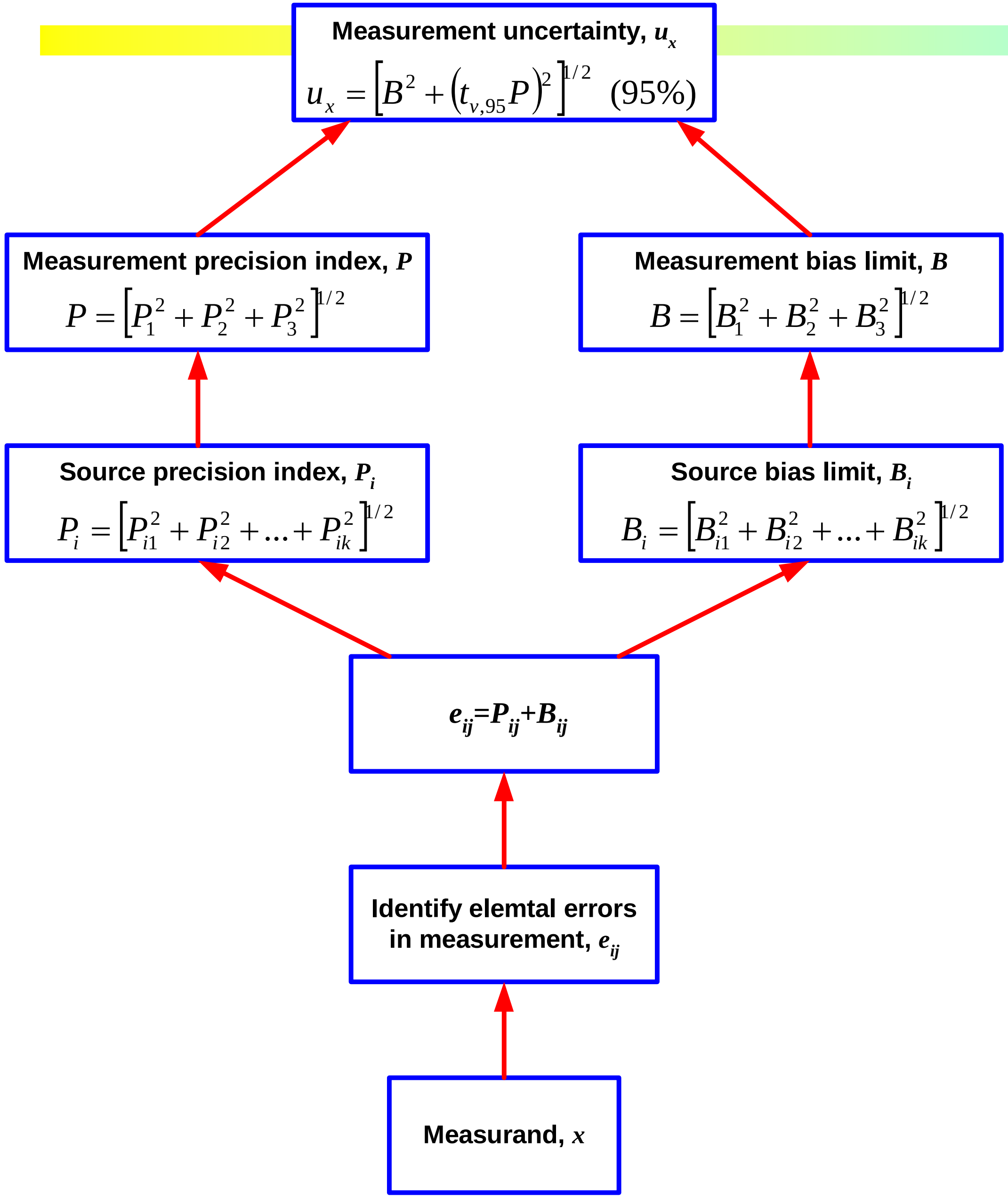
The measurement uncertainty in *x*, u_x

$$u_x = \sqrt{B^2 + \left(t_{\nu,95}\, P\right)^2} \quad (95\%)$$

The degrees of freedom, ν (Welch-Satterthwaite formula)

$$\nu = \frac{\left(\sum_{i=1}^{3}\sum_{j=1}^{K} P_{ij}^2\right)^2}{\sum_{i=1}^{3}\sum_{j=1}^{K}\left(P_{ij}^4 / \nu_{ij}\right)}$$

Multiple-Measurement Uncertainty Analysis



Multiple-Measurement Uncertainty Analysis

Example: After an experiment to measure stress in a load beam, an uncertainty analysis reveals the following source errors in stress measurement whose magnitude were computed from elemental errors

$B_1 = 1.0$ N/cm^2	$B_2 = 2.1$ N/cm^2	$B_3 = 0$ N/cm^2
$P_1 = 4.6$ N/cm^2	$P_2 = 10.3$ N/cm^2	$P_3 = 1.2$ N/cm^2
$v_1 = 14$	$v_2 = 37$	$v_3 = 8$

If the mean value of the stress in the measurement is 223.4 N/cm^2, determine the best estimate of the stress

Known: Experimental error source indices

Assume: All elemental error have been included

Solution: Find u_σ

Measurement uncertainty, u_x

$$u_x = \left[B^2 + \left(t_{v,95}P\right)^2\right]^{1/2} \quad (95\%)$$

Measurement precision index, P

$$P = \left[P_1^2 + P_2^2 + P_3^2\right]^{1/2}$$

Measurement bias limit, B

$$B = \left[B_1^2 + B_2^2 + B_3^2\right]^{1/2}$$

Propagation Uncertainty Analysis to a result

Consider the result, R which is determined from the function of the n independent variables x_{i1}, x_{i2} , x_{i3}, …, x_{iL}

$$R' = \overline{R} \pm u_R \quad (P\%)$$

The measurement uncertainty, u_R

$$u_R = \sqrt{B_R^2 + \left(t_{\nu,95}\, P_R\right)^2} \quad (95\%)$$

where

$$P_R = \pm\sqrt{\sum_{i=1}^{L} [\theta_i P_{xi}]^2} \qquad B_R = \pm\sqrt{\sum_{i=1}^{L} [\theta_i B_{xi}]^2}$$

The degrees of freedom, v

$$v = \frac{\left(\sum_{i=1}^{L} [\theta_i P_{xi}]^2\right)^2}{\sum_{i=1}^{L} \left\{[\theta_i P_{xi}]^4 / v_{xi}\right\}}$$

Propagation Uncertainty Analysis to a result

Example: The density of a gas, ρ, which is believed to follow the ideal gas equation of state, $\rho = p/RT$, is to be estimated through separate measurements of pressure, p, and temperature, T. the gas is housed with in a rigid impermeable vessel. The literature accompanying the pressure measurement system states an accuracy to within 1% of the reading an that accompanying the temperature measuring system suggest 0.6°R. Twenty measurements of pressure, $N_p = 20$, and ten measurements of temperature, $N_T = 10$, are made with the following statistical outcome:

$$\bar{p} = 2253.91\,\text{psfa} \qquad S_p = 167.21\,\text{psfa}$$

$$\bar{T} = 560.4^{\circ}\text{R} \qquad S_T = 3.0^{\circ}\text{R}$$

Where psfa refers to lb/ft^2 absolute. Determine a best estimate of the density. The gas constant is R = 54.7 ft lb/lb$_m$ °R

Known: $\bar{p}, S_p, \bar{T}, S_T$

$$\rho = P/RT \quad R = 54.7 \text{ ft lb/lb}_\text{m}\ {}^{\circ}\text{R}$$

Assume: Gas behaves as an ideal gas

Solution: Find $\rho' = \bar{\rho} + u_\rho$

Propagation Uncertainty Analysis to a result

$$u_\rho = \left[B^2 + \left(t_{\nu,95} P\right)^2\right]^{1/2} \quad (95\%)$$ **where** $$\nu = \frac{\left[\left(\theta_p P_p\right)^2 + \left(\theta_T P_T\right)^2\right]^2}{\left(\theta_p P_p\right)^4 / \nu_p + \left(\theta_T P_T\right)^4 / \nu_T}$$

$$B = \pm\sqrt{\left(\theta_p B_p\right)^2 + \left(\theta_T B_T\right)^2}$$

$$P = \pm\sqrt{\left(\theta_p P_p\right)^2 + \left(\theta_T P_T\right)^2}$$

where $\rho = P / RT \quad R = 54.7 \ \text{ft lb/lb}_\text{m} \ {}^\circ\text{R}$

$$\theta_p = \frac{\partial \rho}{\partial p} = \frac{1}{RT}$$

$$\theta_T = \frac{\partial \rho}{\partial T} = -\frac{p}{RT^2}$$