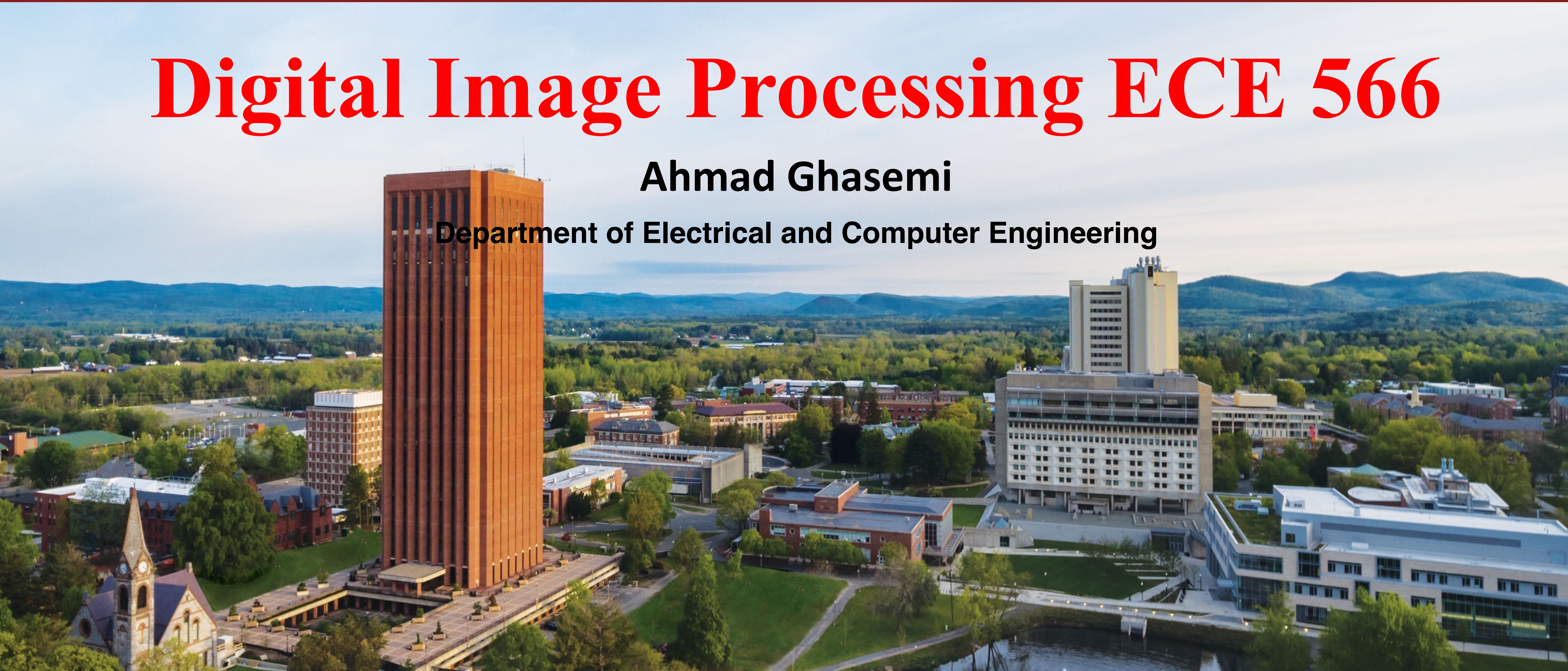
University of Massachusetts Amherst

Digital Image Processing ECE 566

Ahmad Ghasemi

Department of Electrical and Computer Engineering



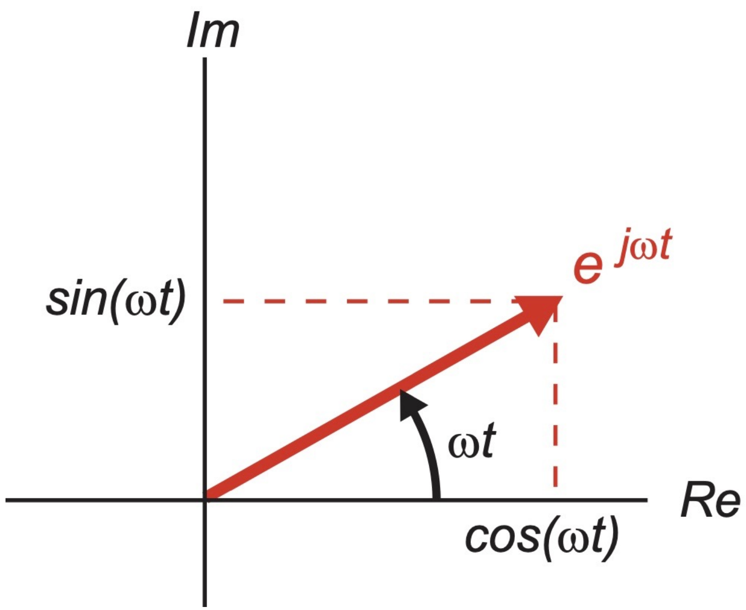
The Complex Exponential as a Vector

Euler's Identity

$$e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)$$

$$j = \sqrt{-1}$$

Real part   Imaginary part



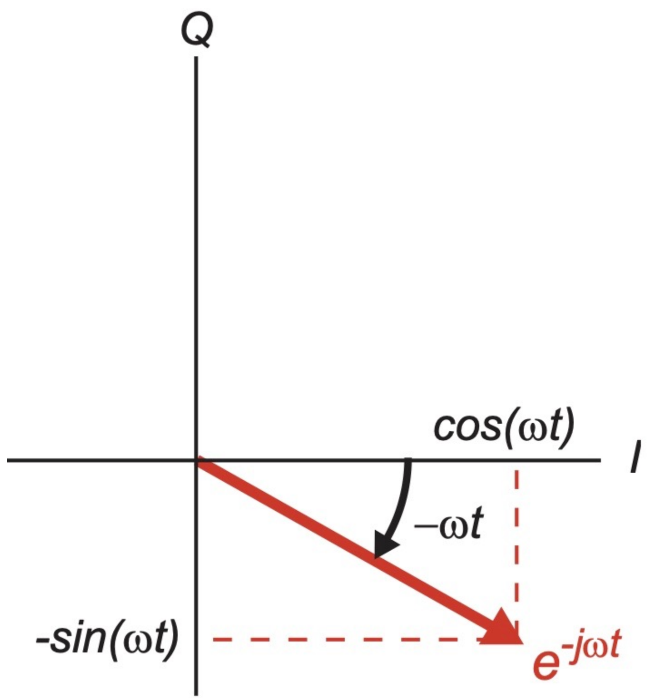
✓ As t increases, vector rotate counterclockwise

✓ We consider $e^{i\omega t}$ to have positive frequency

Negative Frequency

$$e^{-j\omega t} = \cos(\omega t) - j\sin(\omega t)$$

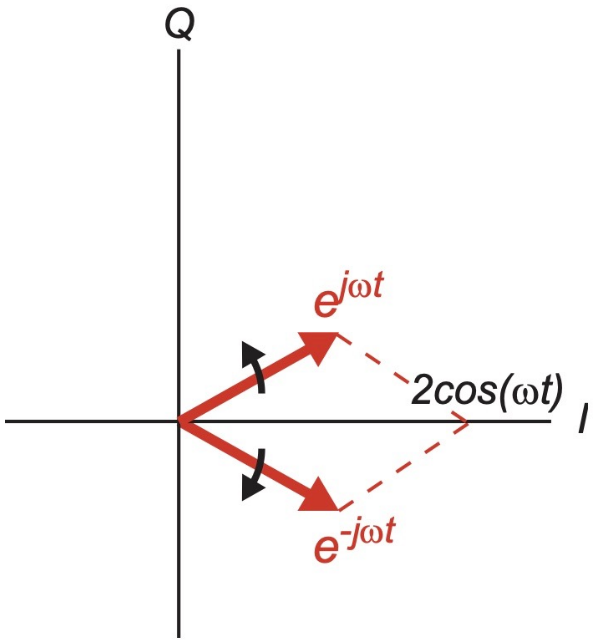
Real part   Imaginary part



✓ As t increases, vector rotate clockwise

✓ We consider $e^{-i\omega t}$ to have negative frequency

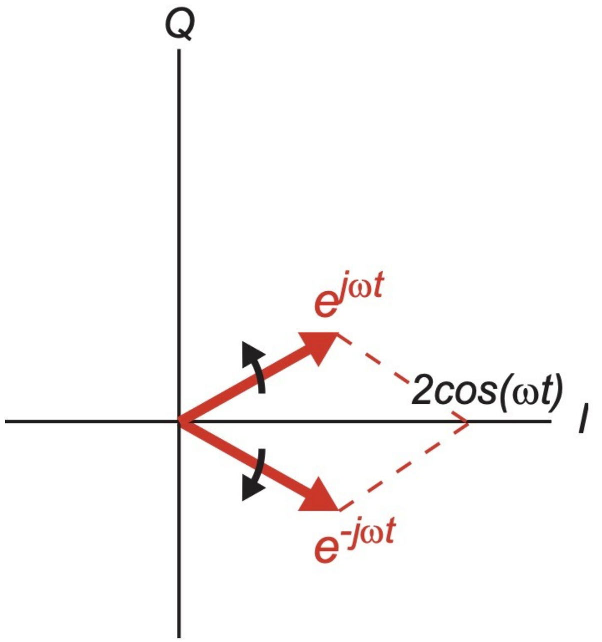
Add / Subtract Positive and Negative Frequencies



$$2\cos(\omega t) = e^{i\omega t} + e^{-i\omega t}$$

- ✓ It leads to a cosine wave
- ✓ It is purely real and considered to have a positive frequency

Add / Subtract Positive and Negative Frequencies



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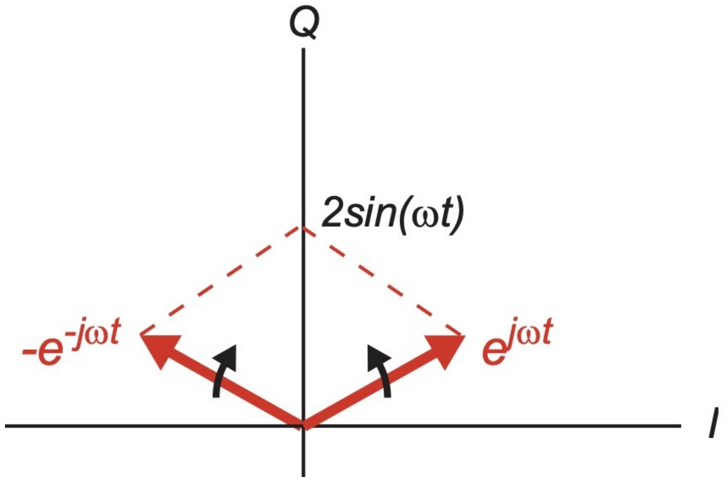


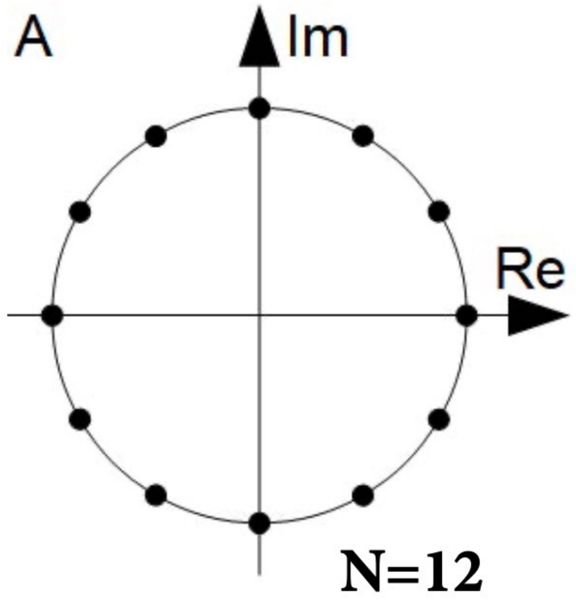
$$j2\sin(\omega t) = e^{i\omega t} - e^{-i\omega t}$$

- ✓ It leads to a sine wave
- ✓ It is purely imaginary and considered to have a positive frequency

Periodic Complex Exponentials

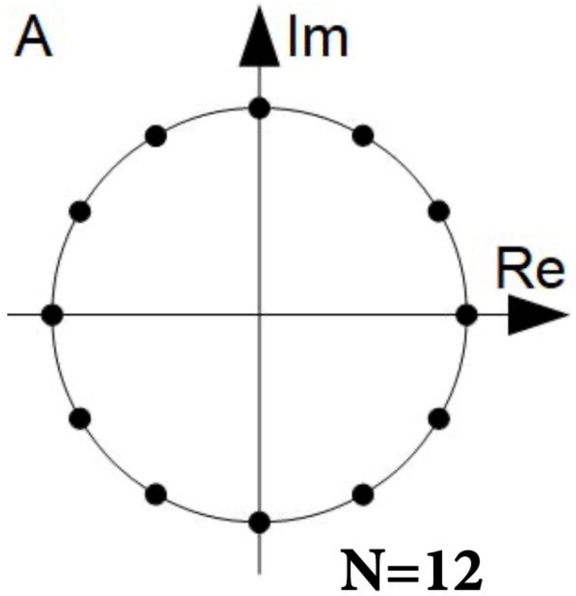
$$x[n] = Ae^{j(k\frac{2\pi}{N})n}$$

What is its period?



Periodic Complex Exponentials

$$x[n] = Ae^{j(k\frac{2\pi}{N})n}$$



✓ It is periodic with period N

$$x[n+N] = Ae^{j\left(k\frac{2\pi}{N}\right)(n+N)} = Ae^{j(k\frac{2\pi}{N})n} \times e^{jk2\pi} = Ae^{j(k\frac{2\pi}{N})n} = \mathrm{x[n]}$$

Fourier Series vs. Fourier Transform

The Fourier Series deals with periodic signals

The Fourier Transform deals with non-periodic signals

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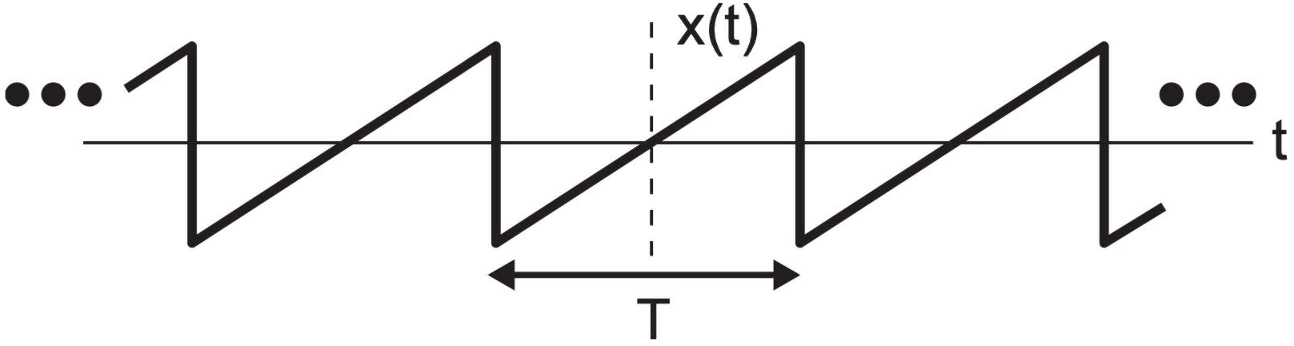
Fourier Series

It is compactly defined using complex exponentials

$$x(t) = \sum_{n=-\infty}^{\infty} \hat{X}_n e^{jn\omega_o t}$$

A periodic signal



$$\boxed{\omega_o = \frac{2\pi}{T}}$$

Fourier Series

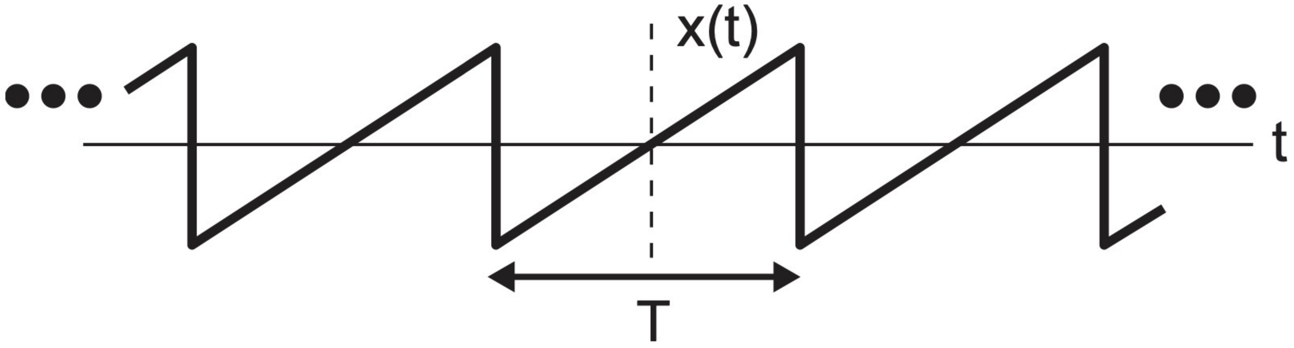
It is compactly defined using complex exponentials

$$x(t) = \sum_{n=-\infty}^{\infty} \hat{X}_n e^{jn\omega_o t}$$

$$\hat{X}_n = \frac{1}{T} \int_{t_o}^{t_o+T} x(t)\, e^{-jn\omega_o t} dt$$

$$\hat{X}_n = A_n + jB_n$$

A periodic signal



$$\boxed{\omega_o = \frac{2\pi}{T}}$$

Fourier Series

It can be defined using cosines and sines

$$x(t) = a_0 + \sum_{i=1}^{\infty} a_n \cos(n\omega_o t) + b_n \sin(n\omega_o t)$$, where $\omega_o = \frac{2\pi}{T}$

Where: $$a_0 = \frac{1}{T} \int_{t_o}^{t_o+T} x(t)\, dt$$

$$a_n = \frac{2}{T} \int_{t_o}^{t_o+T} x(t) \cos(n\omega_o t)\, dt, \qquad n > 0$$

$$b_n = \frac{2}{T} \int_{t_o}^{t_o+T} x(t) \sin(n\omega_o t)\, dt$$

Fourier Series

FS complex exponentials relationship with FS cosines and sines

$$x(t) = a_0 + \sum_{i=1}^{\infty} a_n \cos(n\omega_o t) + b_n \sin(n\omega_o t)$$

$$x(t) = \sum_{n=-\infty}^{\infty} \hat{X}_n e^{jn\omega_o t} = \sum_{n=-\infty}^{\infty} (A_n + jB_n) e^{jn\omega_o t}$$

$$A_0 = a_0$$

$$2A_n = a_n, \qquad n > 0$$

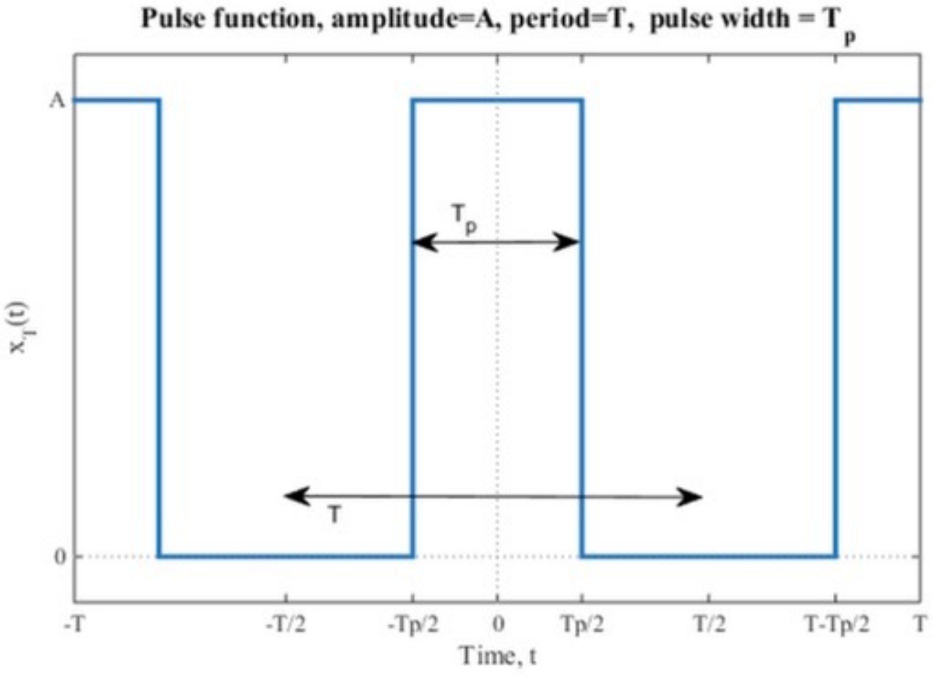
$$-2B_n = b_n,$$

Fourier Series: Examples

Even Pulse Function (Cosine Series)

Consider the periodic pulse function shown below. It is an even function with period T. The function is a pulse function with amplitude A, and pulse width T_p. The function can be defined over one period (centered around the origin) as:

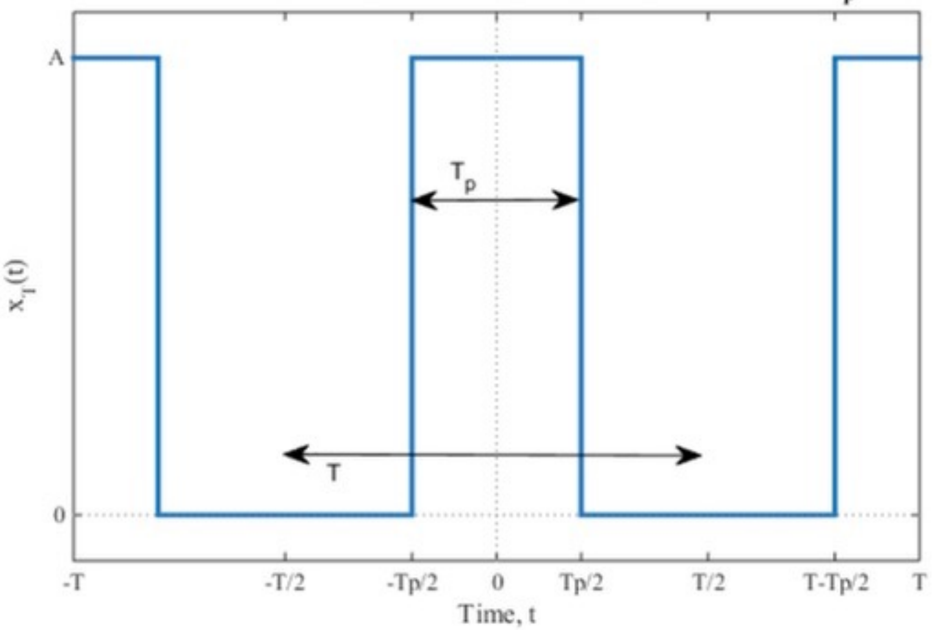
$$x_T(t) = \begin{cases} A, & |t| \leq \frac{T_p}{2} \\ 0, & |t| > \frac{T_p}{2} \end{cases}, \qquad -\frac{T}{2} < t \leq \frac{T}{2}$$



Fourier Series: Examples

$$x_T(t) = \begin{cases} A, & |t| \leq \frac{T_p}{2} \\ 0, & |t| > \frac{T_p}{2} \end{cases}, \qquad -\frac{T}{2} < t \leq \frac{T}{2}$$

Pulse function, amplitude=A, period=T, pulse width = T_p



Even function.

$$x(t) = a_0 + \sum_{i=1}^{\infty} a_n \cos(n\omega_o t) + b_n \sin(n\omega_o t)$$

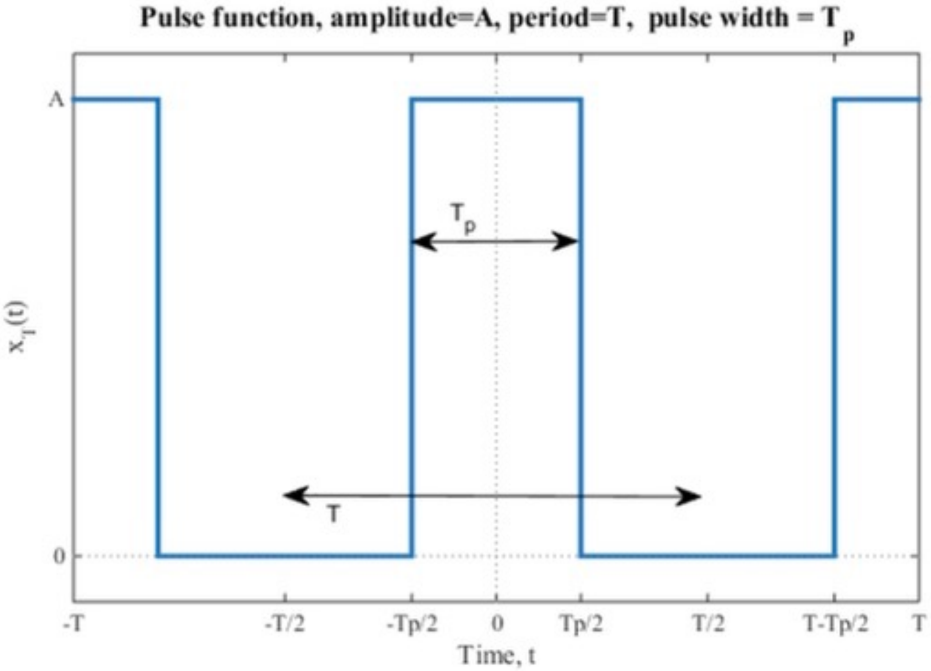
$$a_0 = \frac{1}{T} \int_{t_o}^{t_o+T} x(t)\, dt$$

$$a_n = \frac{2}{T} \int_{t_o}^{t_o+T} x(t) \cos(n\omega_o t)\, dt, \qquad n > 0$$

$$b_n = \frac{2}{T} \int_{t_o}^{t_o+T} x(t) \sin(n\omega_o t)\, dt$$

Fourier Series: Examples

$$x_T(t) = \begin{cases} A, & |t| \le \frac{T_p}{2} \\ 0, & |t| > \frac{T_p}{2} \end{cases}, \qquad -\frac{T}{2} < t \le \frac{T}{2}$$

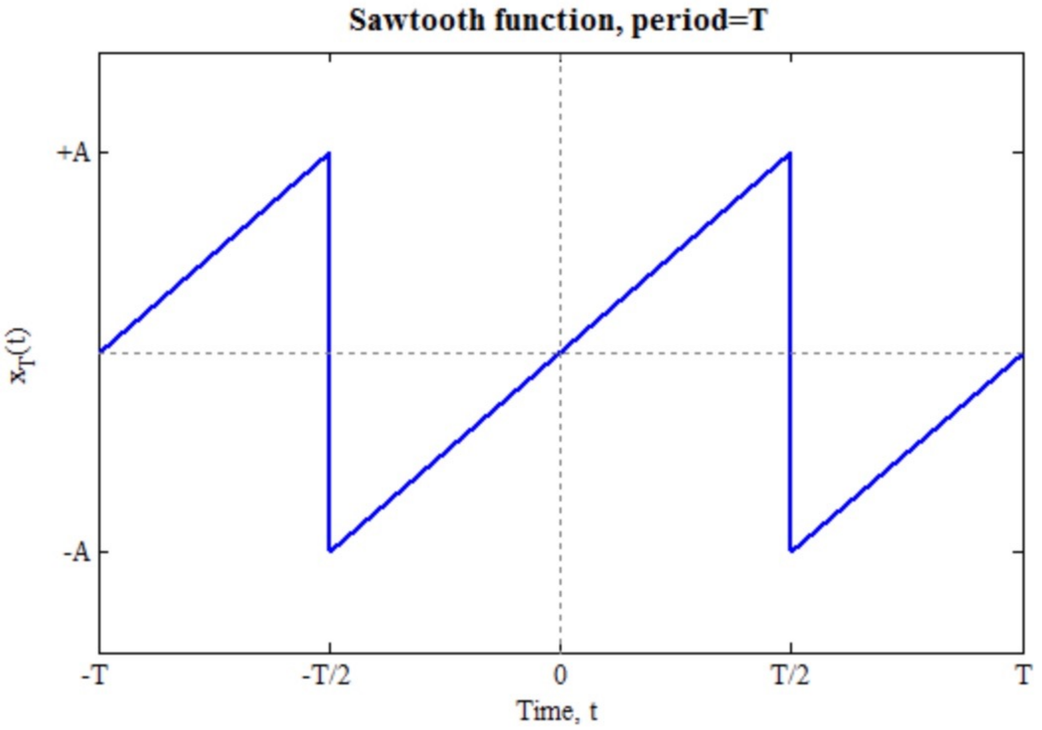


Even function.

$$x(t) = \sum_{n=-\infty}^{\infty} \hat{X}_n e^{jn\omega_o t}$$

$$\hat{X}_n = \frac{1}{T} \int_{t_o}^{t_o+T} x(t)\, e^{-jn\omega_o t} dt$$

Fourier Series: Examples



Odd function.

$$x(t) = a_0 + \sum_{i=1}^{\infty} a_n \cos(n\omega_o t) + b_n \sin(n\omega_o t)$$

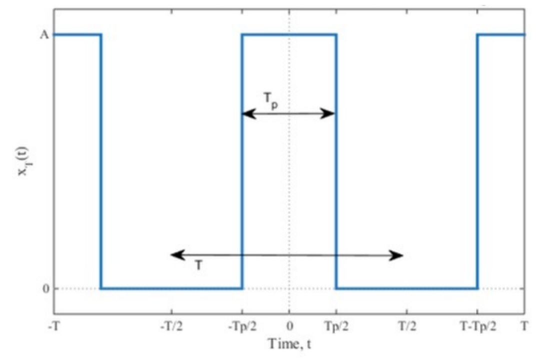
$$a_0 = \frac{1}{T} \int_{t_o}^{t_o+T} x(t)\, dt$$

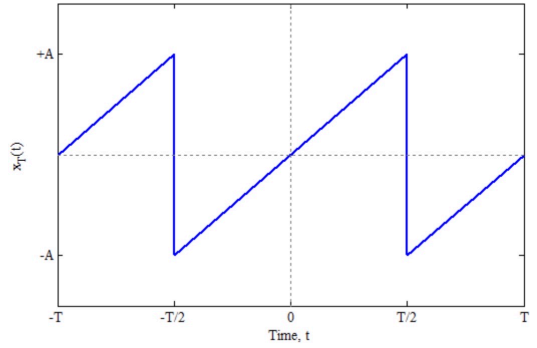
$$a_n = \frac{2}{T} \int_{t_o}^{t_o+T} x(t) \cos(n\omega_o t)\, dt, \qquad n > 0$$

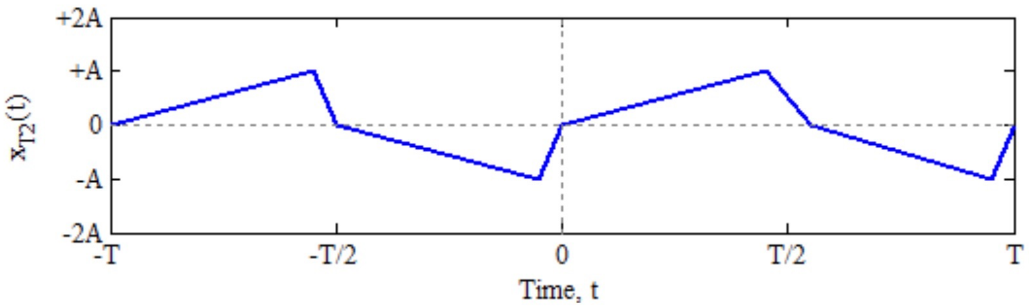
$$b_n = \frac{2}{T} \int_{t_o}^{t_o+T} x(t) \sin(n\omega_o t)\, dt$$

Fourier Series: Examples

Symmetry	Simplification
$x_T(t)$ is even	$a_0 = average$ $a_n = \frac{4}{T}\int_{0}^{+\frac{T}{2}} x_T(t)\cos(n\omega_0 t)dt, \quad n \neq 0$ $b_n = 0$
$x_T(t)$ is odd	$a_n = 0$ $b_n = \frac{4}{T}\int_{0}^{+\frac{T}{2}} x_T(t)\sin(n\omega_0 t)dt$
$x_T(t)$ has *half-wave symmetry* A function can have half-wave symmetry without being either even or odd.	$a_n = b_n = 0, \quad n\ even$ $a_n = \frac{4}{T}\int_{0}^{+\frac{T}{2}} x_T(t)\cos(n\omega_0 t)dt, \quad n\ odd$ $b_n = \frac{4}{T}\int_{0}^{+\frac{T}{2}} x_T(t)\sin(n\omega_0 t)dt, \quad n\ odd$

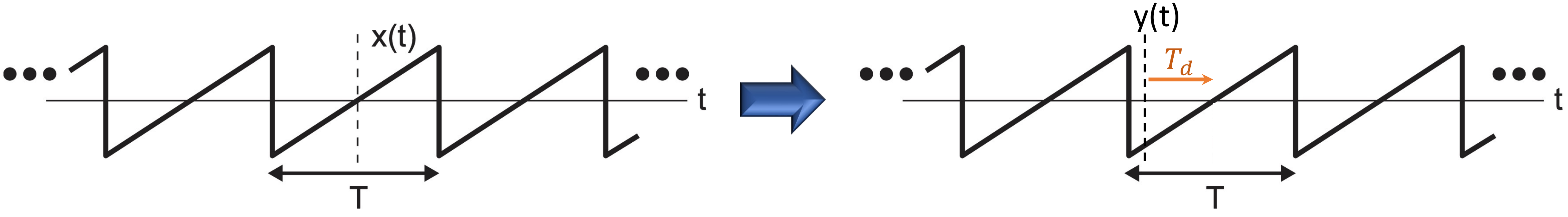






The Impact of a Time (Phase) Shift

Consider shifting a signal x(t) in time by T_d



$$\hat{Y}_n = \frac{1}{T} \int_{t_o}^{t_o+T} y(t)\, e^{-jn\omega_o t} dt$$

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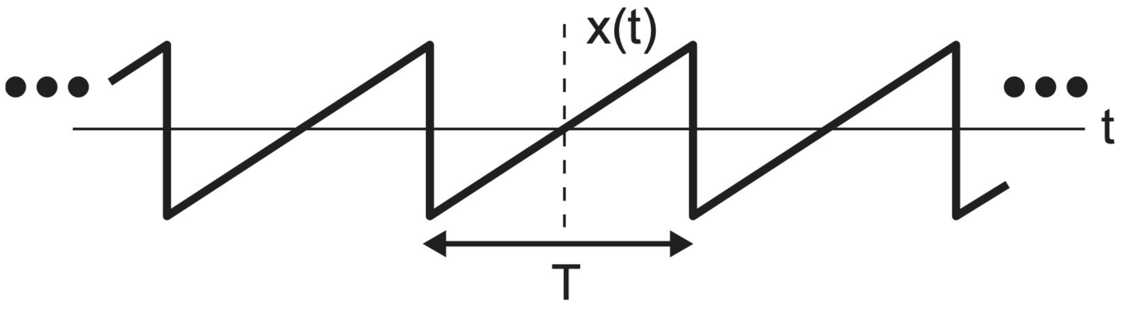

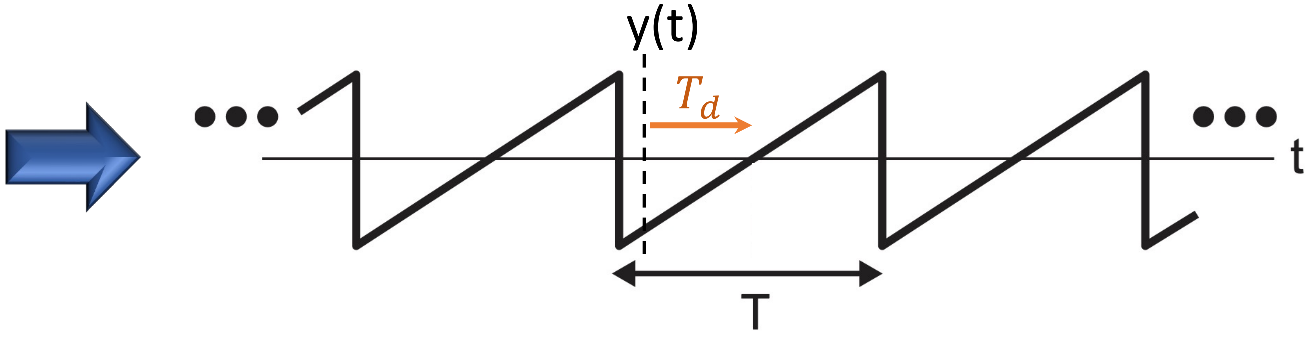

$$\hat{Y}_n = \frac{1}{T}\int_{t_o}^{t_o+T} y(t)\, e^{-jn\omega_o t} dt = \frac{1}{T}\int_{t_o}^{t_o+T} x(t - T_d)\, e^{-jn\omega_o t} dt$$

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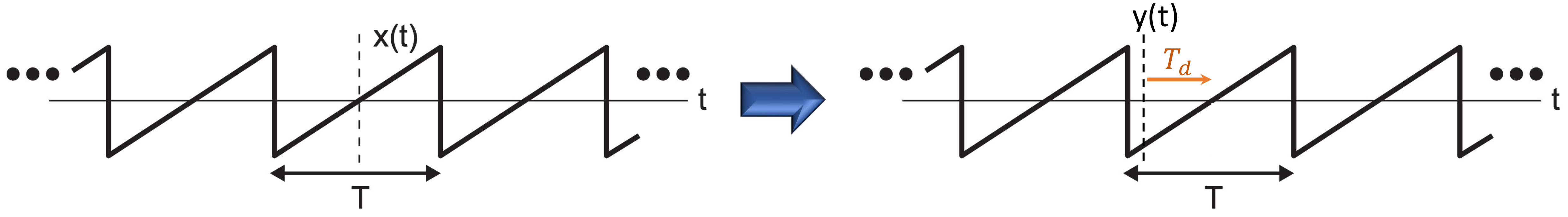

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Define $\tau = t - T_d \Rightarrow d\tau = dt$

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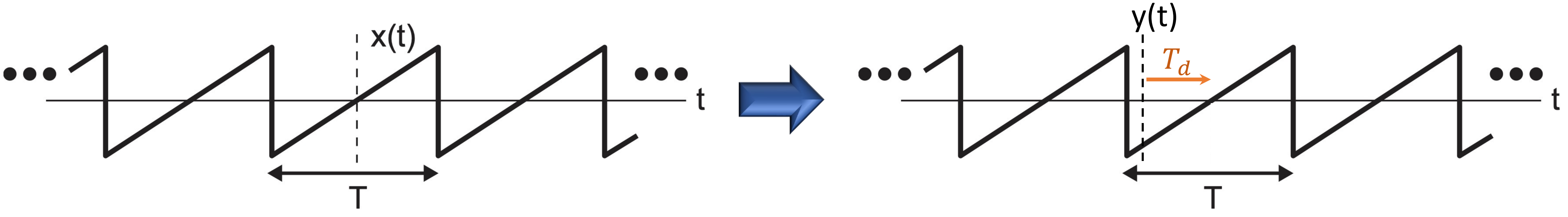

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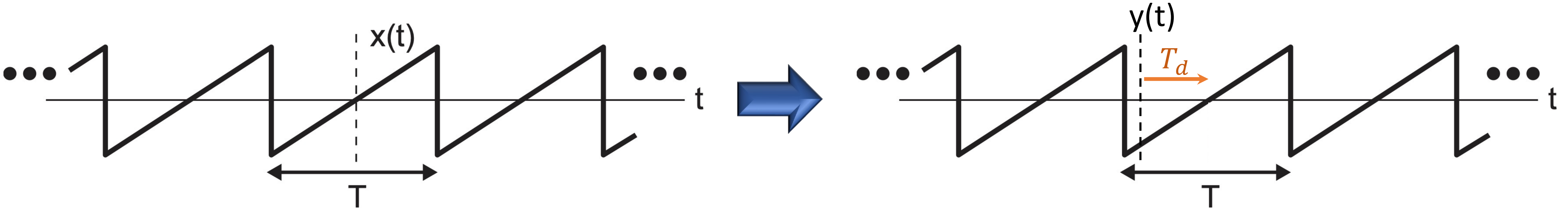
Define $\tau = t - T_d \Rightarrow d\tau = dt$

$$\hat{Y}_n = \frac{1}{T}\int_{t_o - T_d}^{t_o+T-T_d} x(\tau)\, e^{-jn\omega_o(\tau + T_d)} d\tau$$

The Impact of a Time (Phase) Shift

Consider shifting a signal x(t) in time by T_d



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Define $\tau = t - T_d \Rightarrow d\tau = dt$

$$\Rightarrow \hat{Y}_n = \frac{1}{T}\int_{t_o-T_d}^{t_o+T-T_d} x(\tau)\, e^{-jn\omega_o(\tau+T_d)} d\tau = e^{-jn\omega_o T_d}\left(\frac{1}{T}\int_{t_o-T_d}^{t_o+T-T_d} x(\tau)\, e^{-jn\omega_o \tau} d\tau\right) = e^{-jn\omega_o T_d}\hat{X}_n$$

$t' = t_o - T_d$

Magnitude and Phase

The Fourier coefficients can also be represented in term of magnitude and phase

$$\hat{X}_n = A_n + jB_n = |\hat{X}_n| e^{j\phi_n}$$

Where

$$|\hat{X}_n| = \sqrt{A_n^2 + B_n^2} \qquad \phi_n = \tan^{-1}\left(\frac{B_n}{A_n}\right)$$

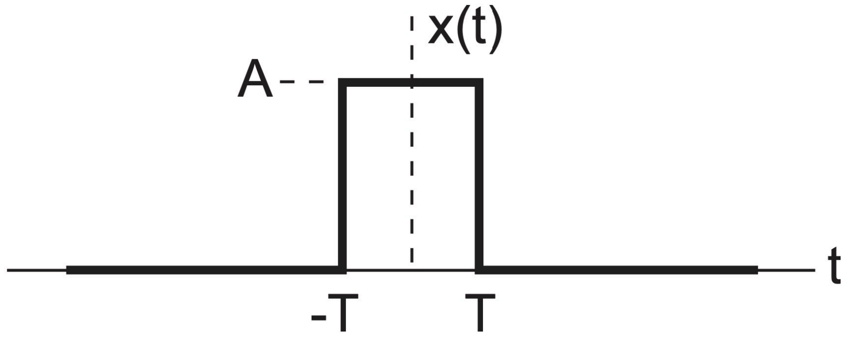
Fourier Transform

The Fourier Transform deals with non-periodic signals

$$x(t) = \int_{-\infty}^{\infty} X(j2\pi f) e^{j2\pi f t} dt \qquad X(j2\pi f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt$$

Fourier Transform: Example

$$X(j2\pi f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi f t}dt$$

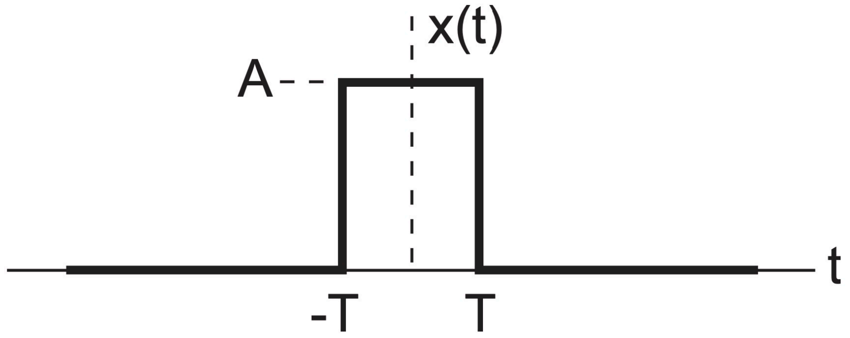


A non-periodic signal

Fourier Transform: Example

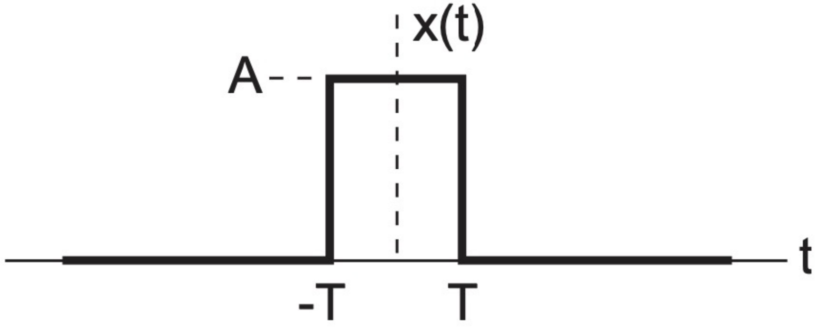
$$X(j2\pi f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi f t}dt = \int_{-T}^{T} Ae^{-j2\pi f t}dt$$

$$= \frac{A}{-j2\pi f t}e^{-j2\pi f t}\Big|_{-T}^{T} = \frac{A\sin(2\pi f T)}{\pi f}$$



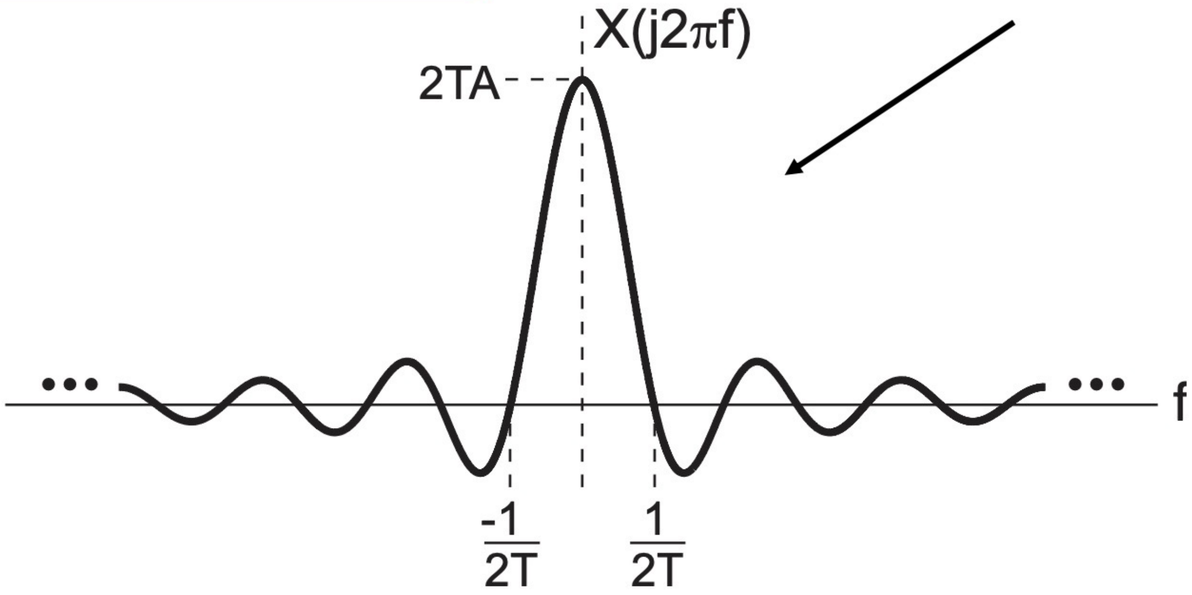
A non-periodic signal

Graphical View of Fourier Transform



$$\boxed{X(j2\pi f) = \frac{A\sin(2\pi f T)}{\pi f}}$$

This is called a sinc function



Fourier Transform: Example

$$f(x) = \begin{cases} 1 - x^2 & |x| < 1 \\ 0 & |x| > 1 \end{cases}$$

$$X(j2\pi f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt$$

Duality of Multiplication And Convolution

Multiplication in time leads to convolution in frequency:

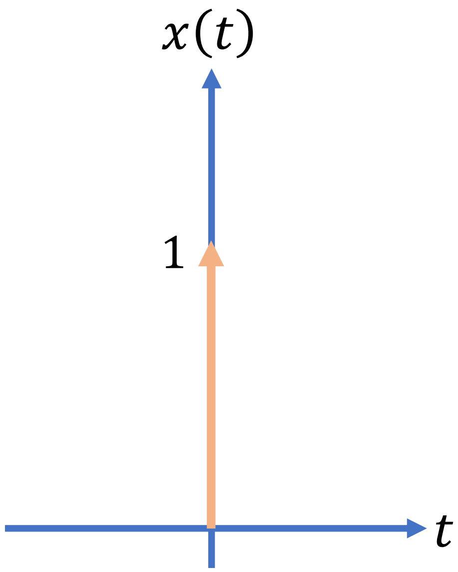
$$x(t)y(t) \iff X(j2\pi f)\star Y(j2\pi f)$$

Convolution in time leads to multiplication in frequency:

$$x(t)\star y(t) \iff X(j2\pi f)Y(j2\pi f)$$

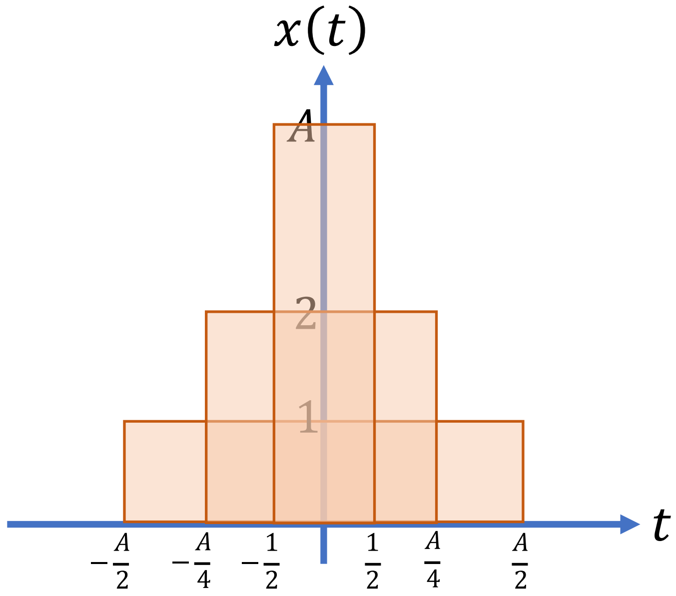
Impulse Function $\delta(t)$

- ✓ AKA a Dirac delta function.
- ✓ It is a pulse having a total area of 1 and its amplitude goes to infinity.



Impulse function at $t=0$

denoted as $\delta(t)$



Rectangular pulses with the same area A

Impulse Function $\delta(t)$

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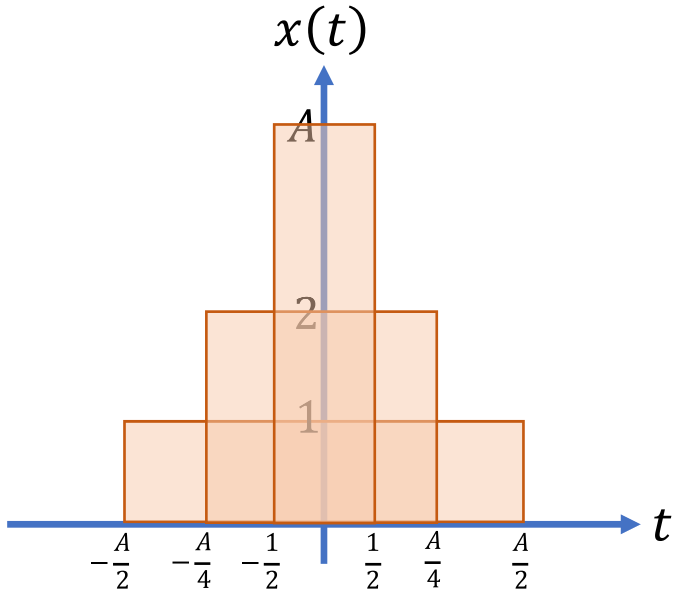
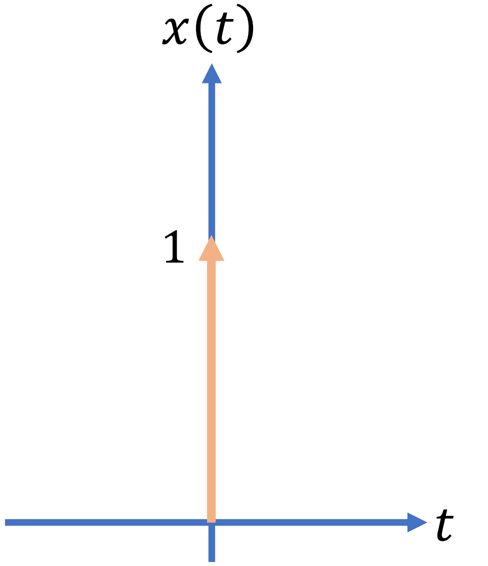


Rectangular pulses with the same area A



Impulse function at $t = 0$

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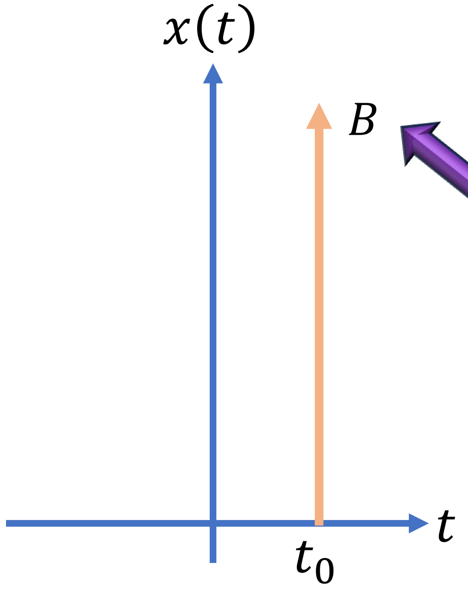


Impulse function at $t = t_0$

denoted as $\mathrm{B}\delta(t - t_0)$

Area under the impulse

Impulse Function $\delta(t)$: Properties

✓ Area

$$\int_{t=-\infty}^{\infty} \mathrm{B}\delta(t-t_0)dt = B\int_{t=-\infty}^{\infty} \delta(t-t_0)dt = B$$

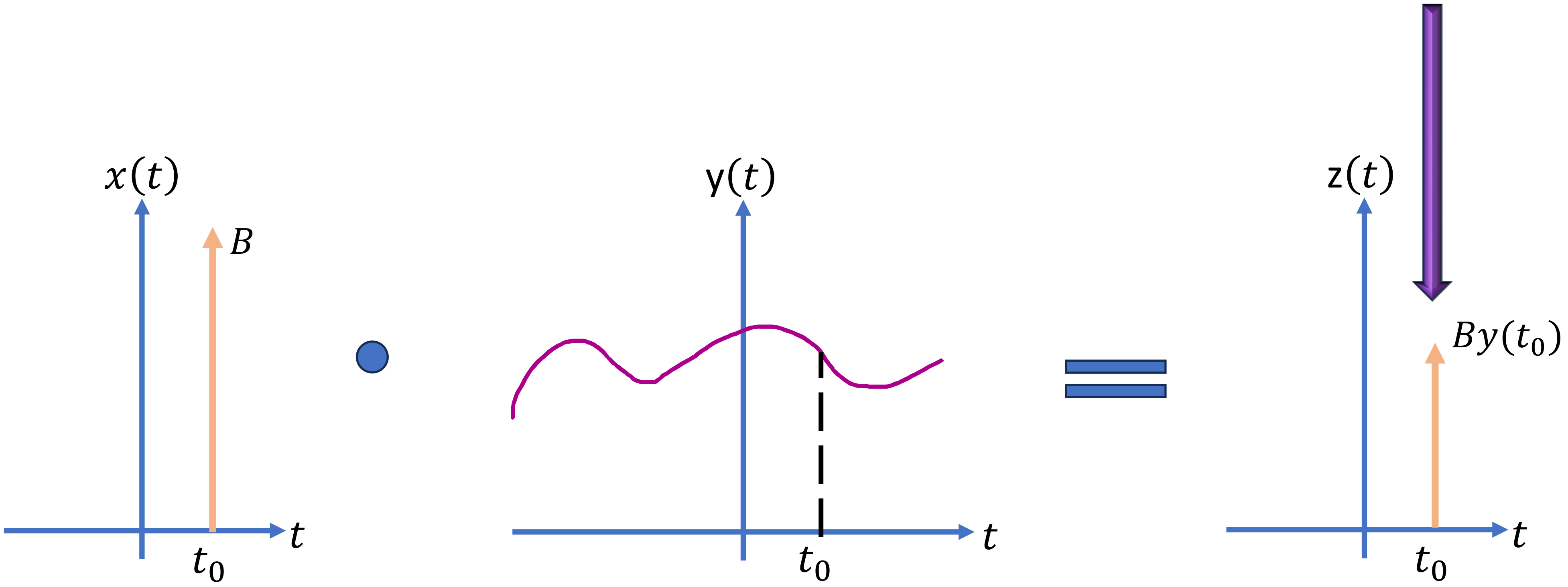
✓ Fourier Transform

$$\mathrm{B}\delta(t-t_0) \Leftrightarrow Be^{-i\omega t_0}$$ where $\boxed{\omega = 2\pi f}$

Why?

Impulse Function $\delta(t)$: Sampling

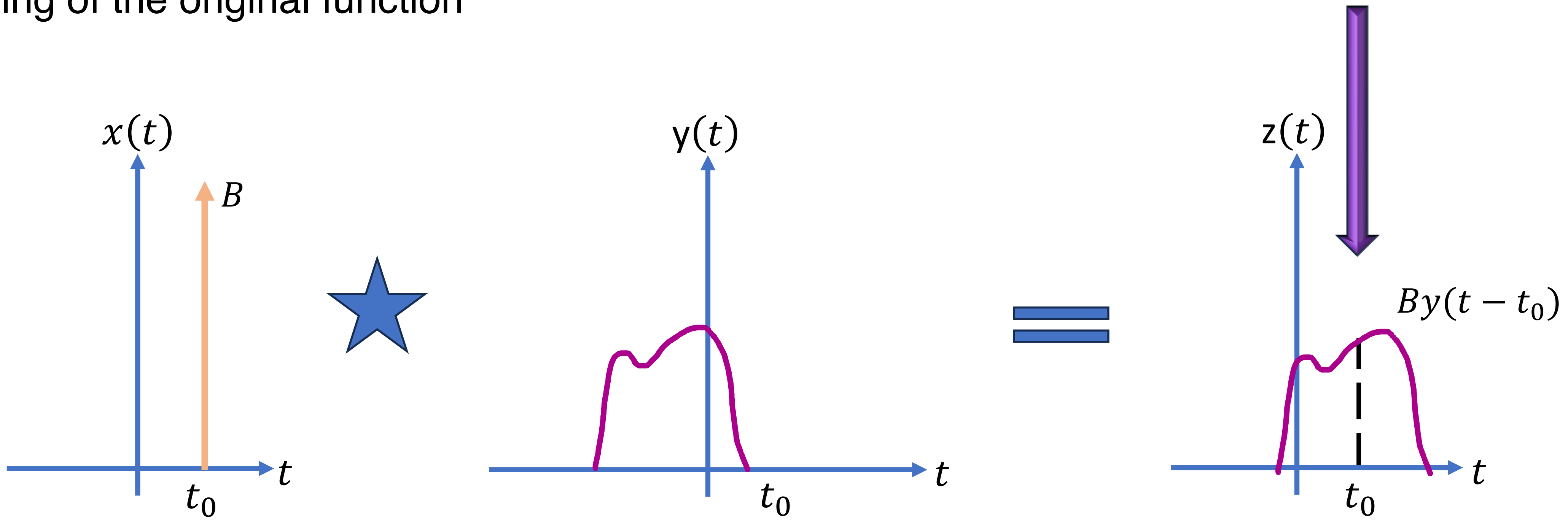
- ✓ Multiplication of an impulse and a continuous function leads to scaling of the original impulse
- ✓ The scale factor corresponds to the sample value of the continuous function at the impulse location



$$\boxed{B\delta(t-t_0)y(t) = By(t_0)\delta(t-t_0)}$$

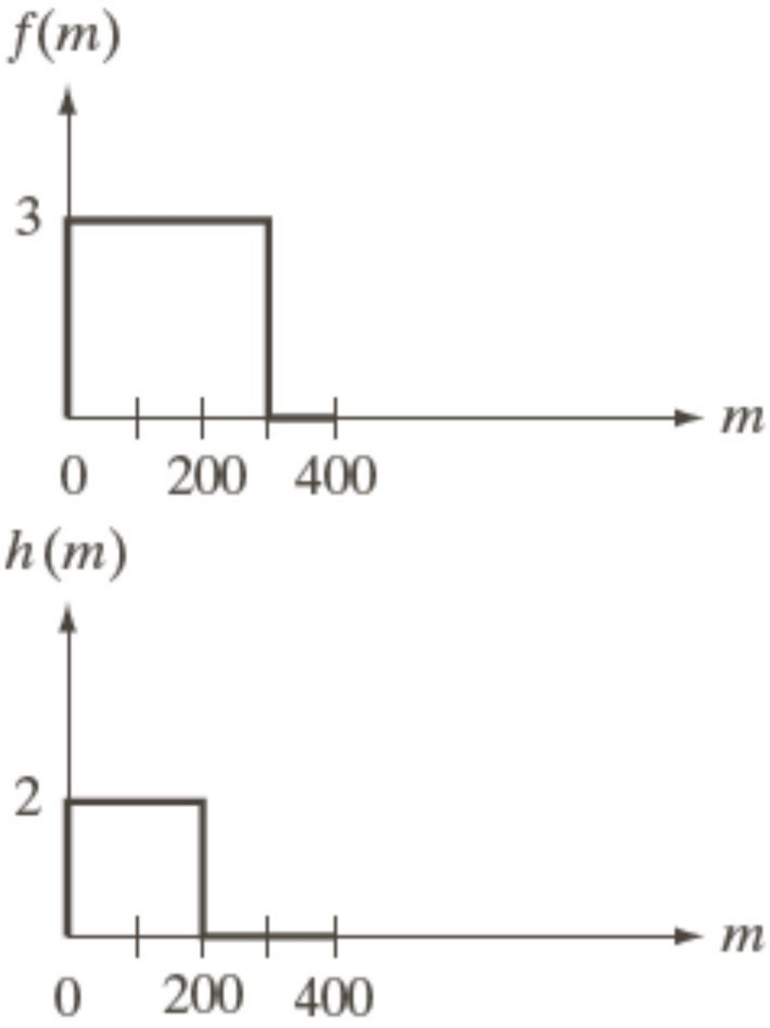
Impulse Function $\delta(t)$: Convolution

- ✓ Convolution of an impulse and a function leads to shifting and scaling of the original function
- ✓ The scale factor corresponds to the area of the impulse
- ✓ The shift value corresponds to the location of the impulse

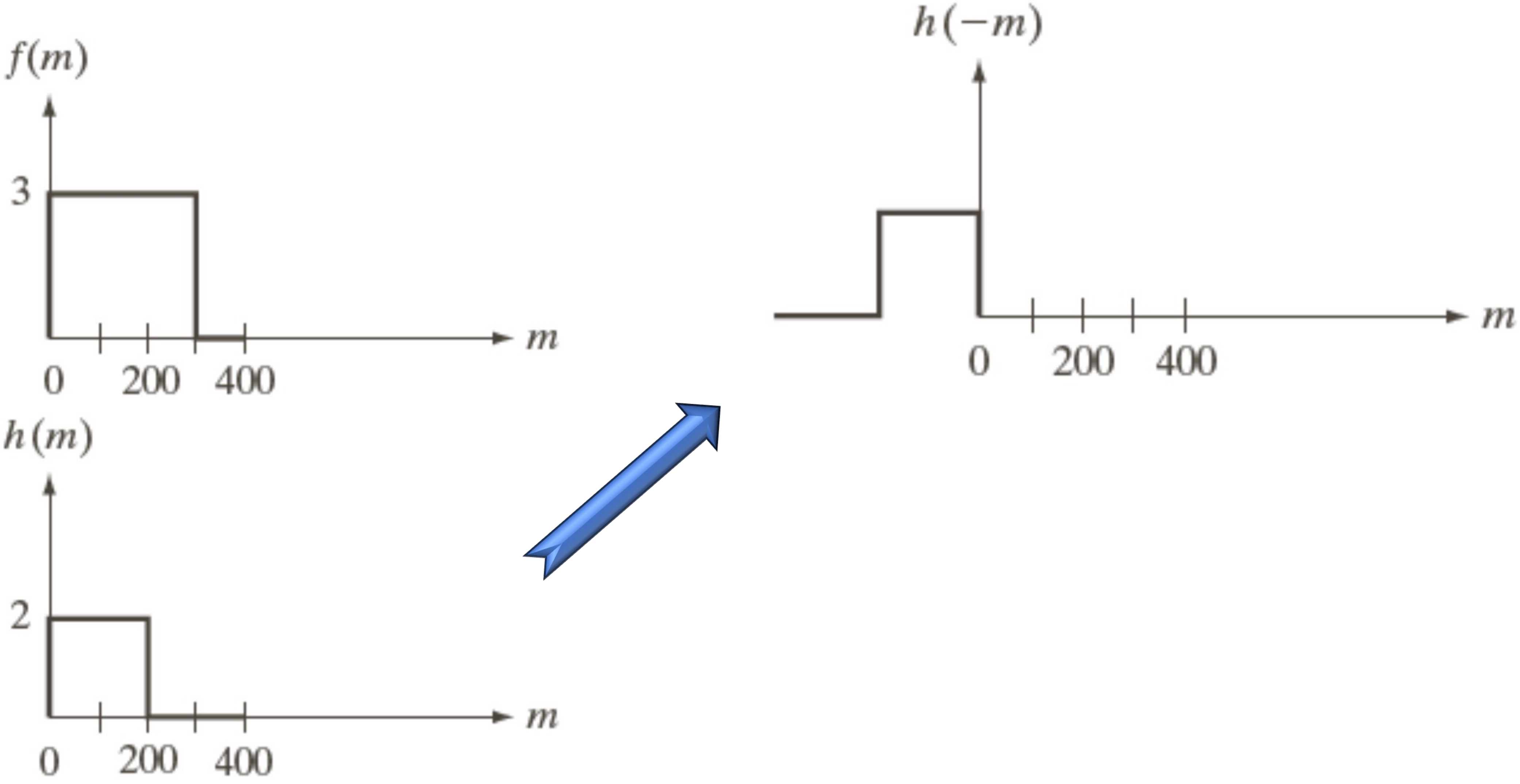


$$f(t) \star g(t) = \int_{-\infty}^{\infty} f(\tau) g(t-\tau) d\tau \Rightarrow \boxed{B\delta(t-t_0) \star y(t) = By(t-t_0)}$$

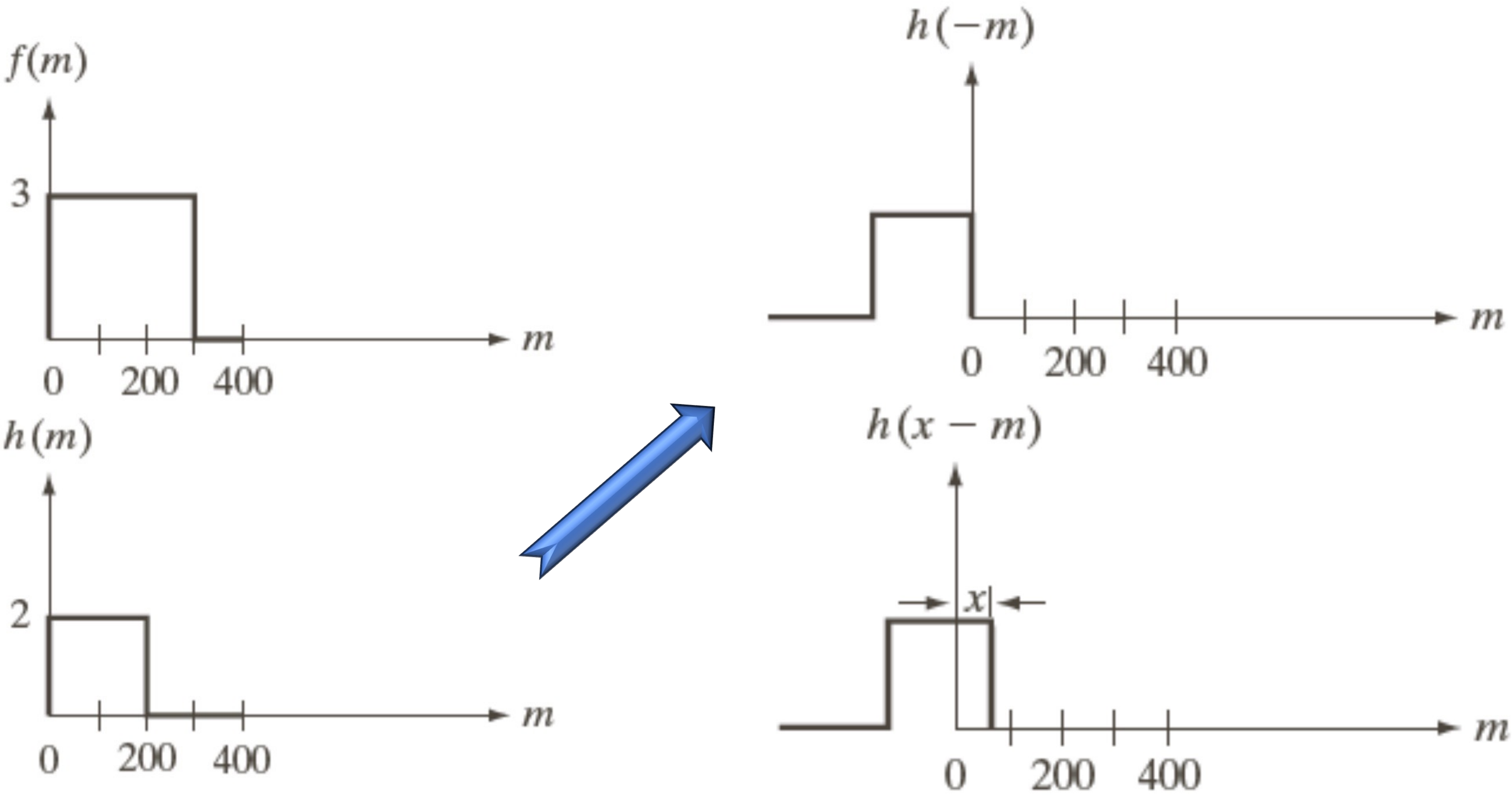
Convolution



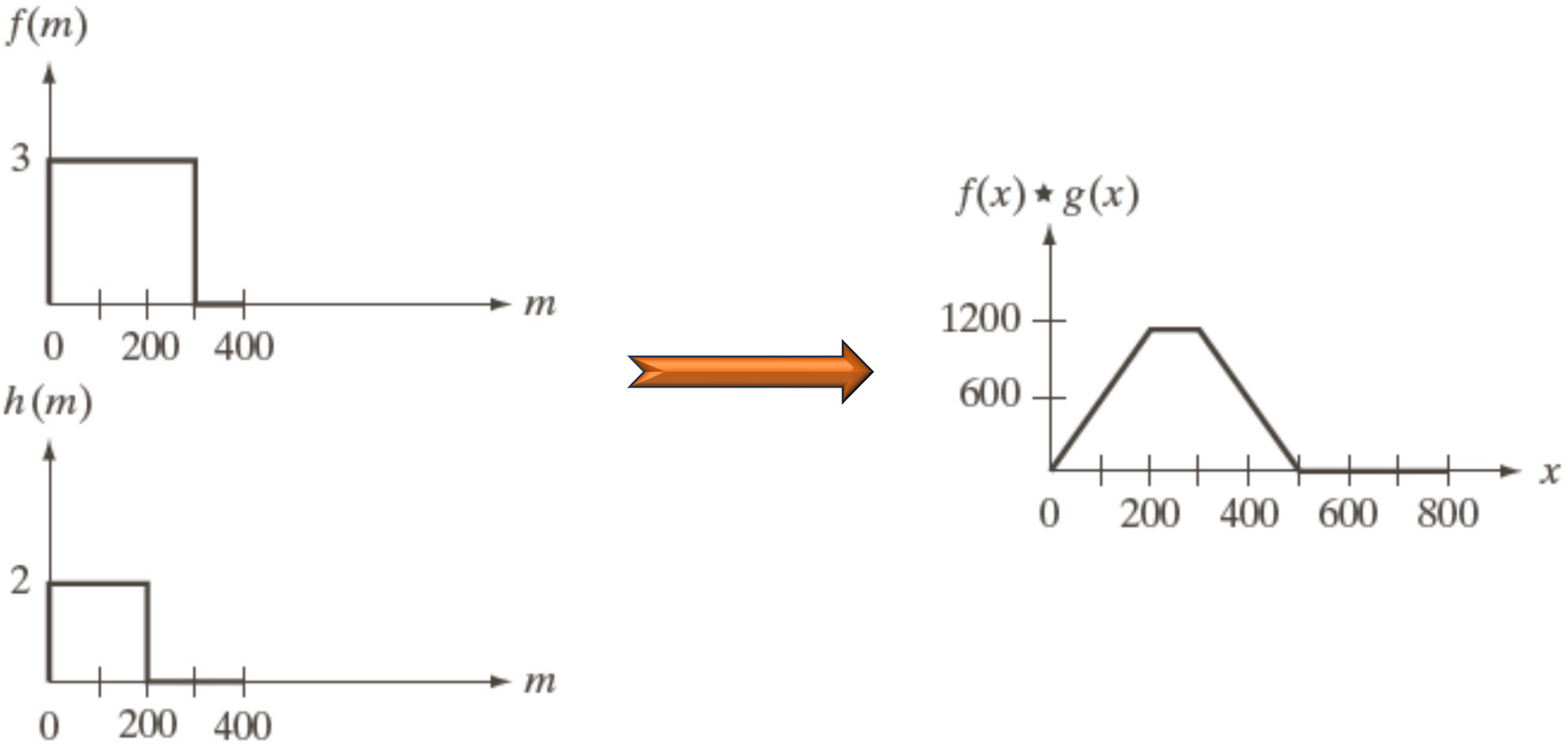
Convolution



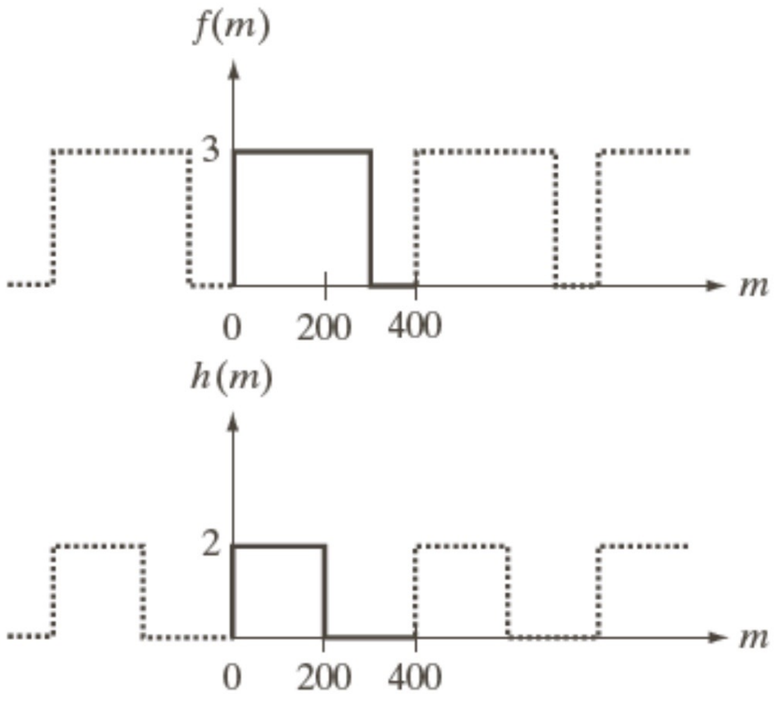
Convolution



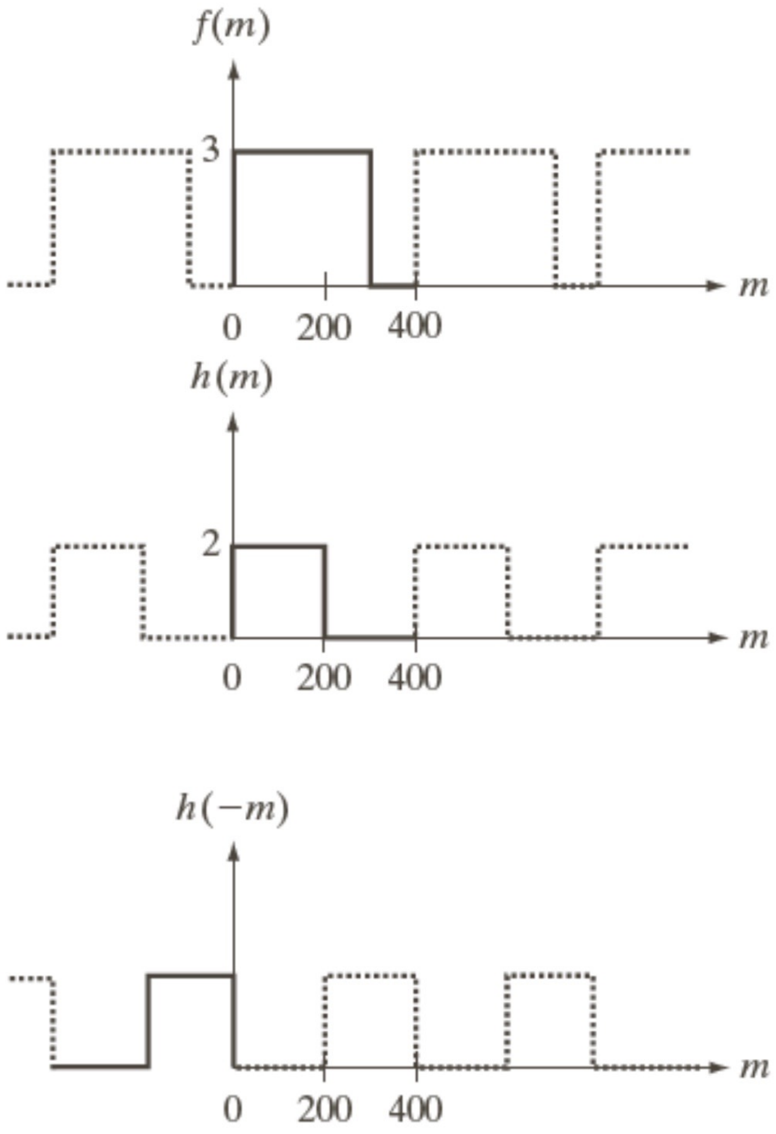
Convolution



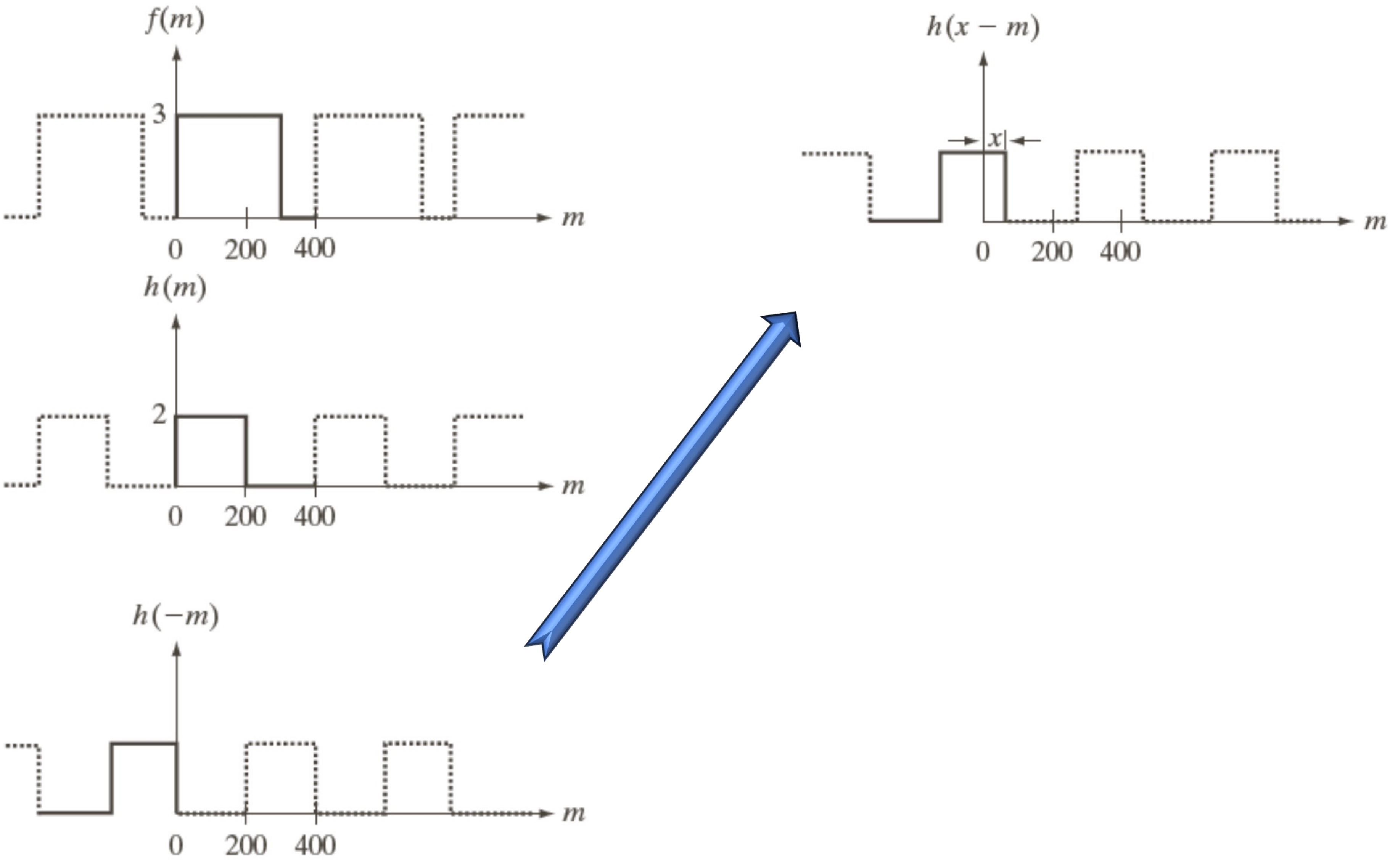
Circular Convolution



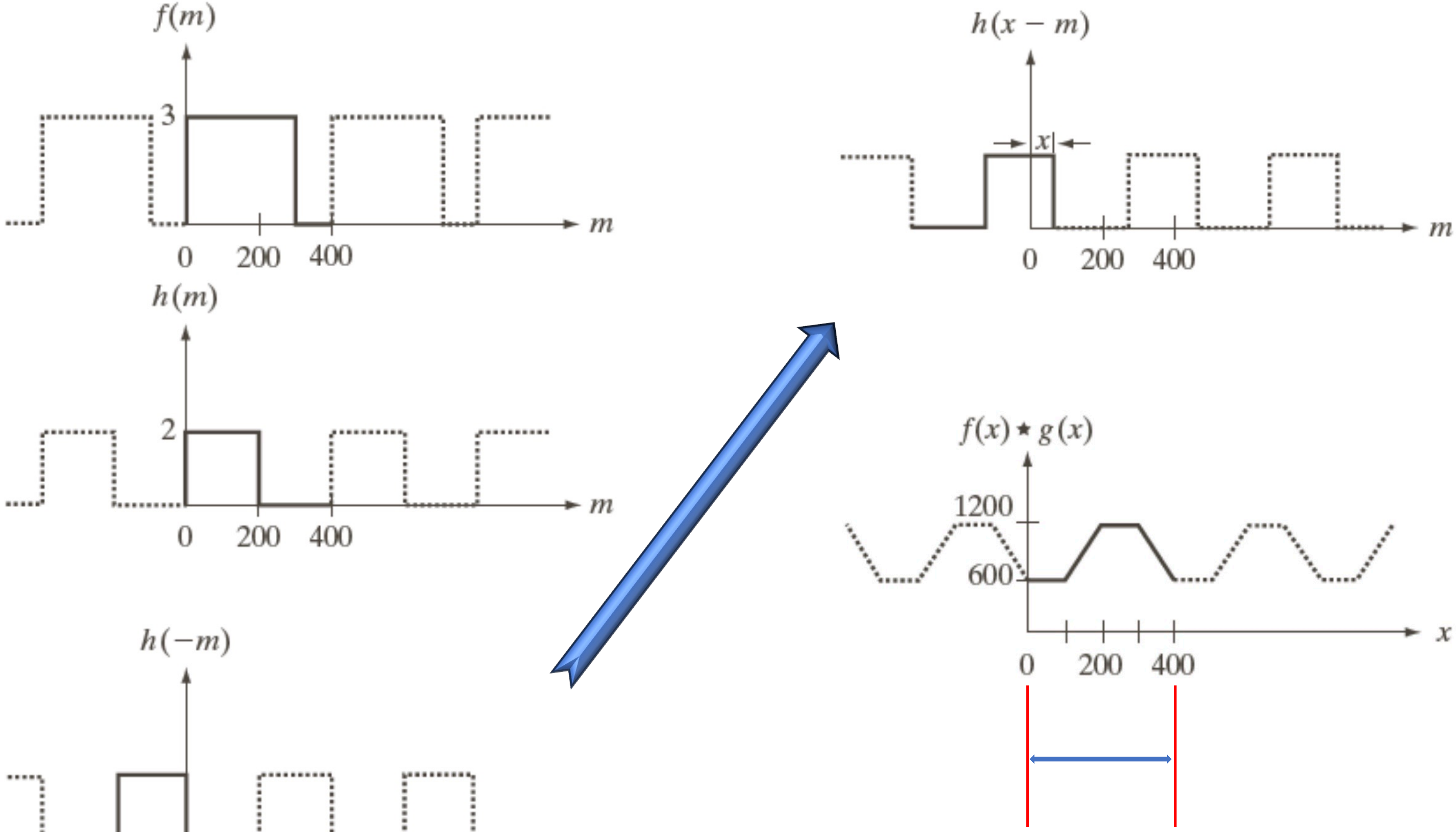
Circular Convolution



Circular Convolution



Circular Convolution



Range of FT computation

2D discrete convolution theorem

$$f(x,y) \star h(x,y) = \sum_{m=0}^{M} \sum_{n=0}^{N} f(m,n)h(x-m, y-n)$$

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Multiplication in time leads to convolution in frequency:

$$f(x,y)h(x,y) \iff \frac{1}{MN} F(u,v) \star H(u,v)$$

Convolution in time leads to multiplication in frequency:

$$f(x,y) \star h(x,y) \iff F(u,v)H(u,v)$$