Unit 3: Modeling with first-order differential equations

3.1 Analytical solutions

We'll consider first-order differential equations that can be written in the form $\frac{dy}{dx} = f(x,y)$, for some suitably nice function *f*. There are usually infinitely many solutions to an equation of this type.

Definition: The *general solution* to this equation is an expression that encompasses all possible solutions.

For first-order equations, the general solution is characterized by one arbitrary constant.

Example: Show that $y = ce^x - x - 1$ is the general solution of $\frac{dy}{dx} = x + y$.

By substitution, we have $\frac{dy}{dx} = ce^x - 1 = x + y$. So every solution of the form $y = ce^x - x - 1$ satisfies the equation. Since there is one arbitrary constant, then we are led to believe that this is the general solution, although proving that there are no others takes some work.

If we are given additional information such as a point on the solution curve, then we can solve for the arbitrary constant. In example above, if $y(0) = 3$, then $c = 4$.

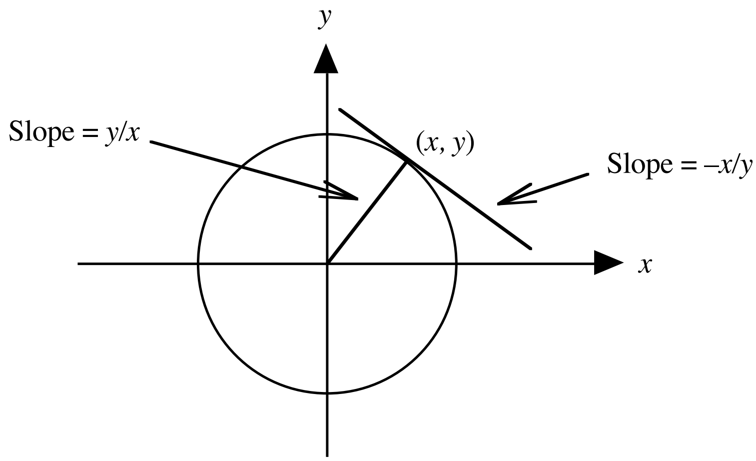
Solution methods

- **Separation of variables:** In some cases, the equation can be written in the form $g(y)dy = h(x)dx$. (This treats $\frac{dy}{dx}$ as a "fraction", which technically it isn't.) If *g* and *h* have closed-form antiderivatives, then we can integrate both sides to get *y* in terms of *x*.

Example: $\frac{dy}{dx} = \frac{-x}{y}$.

We can rewrite as $y\,dy = -x\,dx$. So $\frac{y^2}{2} = \frac{-x^2}{2} + c$. (The constant is only needed on one side of the equation.). This is equivalent to $y^2 + x^2 = c$ (a different *c*), which are circles of radius $\sqrt{c}$ centered at the origin.

This makes sense since the slope of the radius to the point (x,y) is $\frac{y}{x}$. Since the differential equation says that the slope of the tangent (i.e. $\frac{dy}{dx}$) at the point (x,y) is $\frac{-x}{y}$, then the radius and the tangent must be perpendicular, which is true for circles.



- **Integrating factors:** This method applies to equations that can be written in the form $\frac{dy}{dx} + p(x)y = q(x)$ for some continuous functions p and q. These are linear, nonhomogeneous (unless $q(x) \equiv 0$) equations.

Let $\mu(x) = e^{\int p(x)dx}$ be an integrating factor. Then $\mu'(x) = e^{\int p(x)dx} p(x) = \mu(x)p(x)$. Multiply both sides of the differential equation by $\mu(x)$.

$$\mu(x)\frac{dy}{dx} + \mu(x)p(x)y = \mu(x)q(x)$$

The left-hand side is $\mu(x)\frac{dy}{dx} + \mu'(x)y = \frac{d}{dx}[\mu(x)y]$. Integrating both sides (if possible) gives $\mu(x)y = \int \mu(x)q(x)dx + c$, which in principle, can be solved for y.

Example: $\frac{dy}{dx} + \frac{1}{x}y = e^{x^2}$

$\mu(x) = e^{\int \frac{1}{x}dx} = e^{\ln x} = x$. After multiplying both sides by x, we get $\frac{d}{dx}(xy) = xe^{x^2}$. Integrating gives $xy = \frac{1}{2}e^{x^2} + c$, so $y = \frac{1}{2x}e^{x^2} + \frac{c}{x}$.

We can check by substitution:

$$\frac{dy}{dx} + \frac{1}{x}y = \left(-\frac{1}{2x^2}e^{x^2} + e^{x^2} - \frac{c}{x^2}\right) + \frac{1}{x}\left(\frac{1}{2x}e^{x^2} + \frac{c}{x}\right) = e^{x^2}$$

Notice that the solution consists of two terms. One term $y_h = \frac{c}{x}$ is the solution to the corresponding homogeneous equation $\frac{dy}{dx} + \frac{1}{x}y = 0$. (In linear algebra terminology, it is the *kernel* or *null space* of the linear transformation $L[y] = \frac{dy}{dx} + \frac{1}{x}y$. Since this is a first-order equation, the kernel is a one-dimensional subspace of the set of differentiable functions.) The other term $y_p = \frac{1}{2x}e^{x^2}$ is a specific solution satisfying this particular right-hand side. Changing the right-hand side of the original equation changes this particular solution, but not the homogeneous solution.

Recall that, for any linear equation, or system of linear equations (algebraic or differential), the solution always consists of two terms just like this.

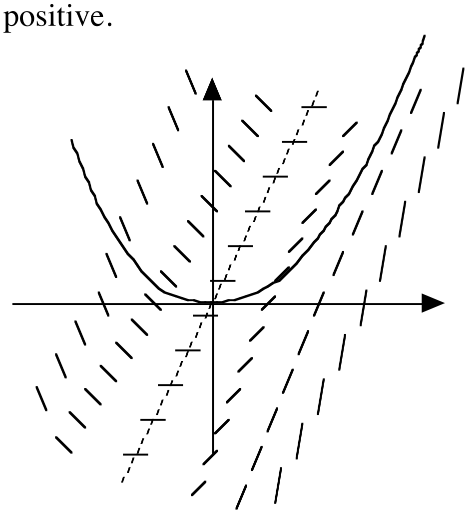
3.2 Graphical techniques

It is often possible to sketch the solutions to a first-order differential equation without solving it first. To do so, we can draw a *slope field* (or *direction field*).

Consider the equation $\frac{dy}{dx} = f(x,y)$. At each point (x,y), draw a short line segment whose slope is $f(x,y)$. Given an initial point (x_0, y_0), we can trace the solution curve.

Example: $\frac{dy}{dx} = 2x - y$

The slope is 0 along the line $y = 2x$. Above the line the slopes are negative; below the line, they are positive.



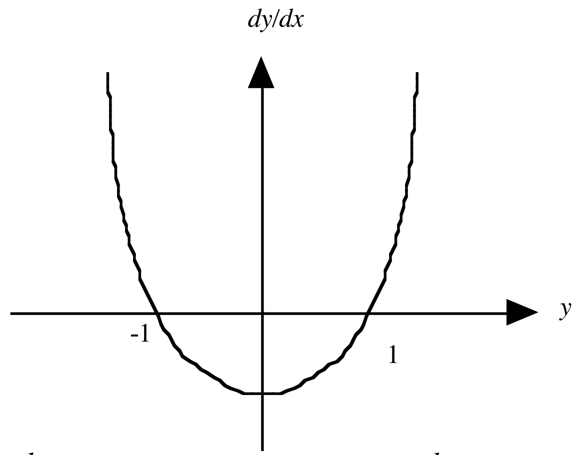
We can solve this equation using an integrating factor. Rewrite as $\frac{dy}{dx} + y = 2x$. Then $\mu(x) = e^{\int 1dx} = e^x$. Upon multiplying, we get $\frac{d}{dx}\left[e^x y\right] = 2xe^x$. Integrating by parts gives $e^x y = \int 2xe^x dx = 2(x-1)e^x + c$ from which $y = 2(x-1) + ce^{-x}$.

Or we could have solved the homogeneous equation $\frac{dy}{dx} + y = 0$, obtaining $y = ce^{-x}$. The particular solution must be of the form $y = ax + b$. Substituting in the equation gives $a + ax + b = 2x$. Since this must be true for all x, then $a = 2$ and $b = -2$. Thus, the complete solution is $y = ce^{-x} + 2x - 2$, as we got before.

Suppose $y(0) = 0$ as in the graph above. Then $c = 2$. Notice that for large x, the second term vanishes and y is approximately a straight line $y = 2x - 2$, as the graph shows. Also, if $c > 0$, then $y > 2x - 2$, while if $c < 0$, $y < 2x - 2$, for all x. Therefore, the solution curve can never cross the line $y = 2x - 2$.

Now consider the special case $\frac{dy}{dx} = f(y)$, where the right side is independent of x. We can draw a graph with $\frac{dy}{dx}$ on the vertical axis and y on the horizontal axis. This is called a *phase-plane portrait*. It can be used to determine the values of y for which the solution is increasing, decreasing, concave up, concave down, etc.

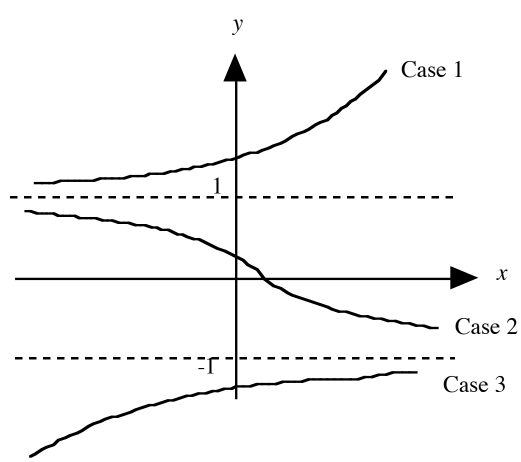
Example: $\frac{dy}{dx} = y^2 - 1$.



Observe that $\frac{dy}{dx} > 0$ for $y > 1$ or $y < -1$, and $\frac{dy}{dx} < 0$ for $-1 < y < 1$.

By the chain rule, $\frac{d^2y}{dx^2} = 2y\frac{dy}{dx} = 2y(y^2 - 1)$. So, $\frac{d^2y}{dx^2} > 0$ for $y > 1$ or $-1 < y < 0$, and $\frac{d^2y}{dx^2} < 0$ for $y < -1$ or $0 < y < 1$. There are three possible cases:

1. $y_0 > 1$
2. $-1 < y_0 < 1$
3. $y_0 < -1$



Note that the graph can never cross the lines $y = \pm 1$ since $\frac{dy}{dx} = 0$ there.

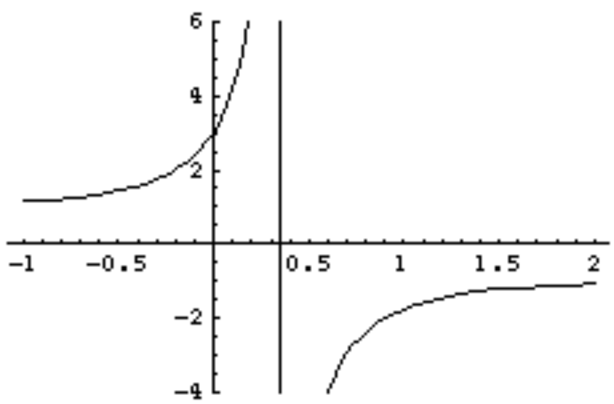
We can solve this equation by separation of variables:

$$\frac{1}{y^2-1}\,dy = dx$$

Using a table of integrals (or a technique called *partial fractions*), we get $\frac{1}{2}\ln\left|\frac{y-1}{y+1}\right| = x + c$. Then $\ln\left|\frac{y-1}{y+1}\right| = 2x + c$ from which $\left|\frac{y-1}{y+1}\right| = Ce^{2x}$. If the initial population is y_0, then $C = \left|\frac{y_0-1}{y_0+1}\right|$. Note that $C > 0$ for all y_0, implying that $\left|\frac{y-1}{y+1}\right| \to \infty$ as $x \to \infty$.

- Case 1: $y_0 > 1$. Then $y > 1$ for all x, so $\left|\frac{y-1}{y+1}\right| = \frac{y-1}{y+1}$. After cross-multiplying and solving for y, we get $y = \frac{y_0+1+(y_0-1)e^{2x}}{y_0+1-(y_0-1)e^{2x}}$. This would appear to approach –1 as $x \to \infty$. However, there is a vertical asymptote at $x = \frac{1}{2}\ln\left(\frac{y_0+1}{y_0-1}\right)$, which is a positive number. So y increases without bound for x less than this value. For x greater than this value, y approaches –1.

Here is the graph for $y_0 = 3$. The asymptote is at $x = \frac{\ln 2}{2} \approx .35$.



- Case 2: $-1 < y_0 < 1$. Then $-1 < y < 1$ for all x—that is, y is bounded, so there can be no asymptote. The only way that $\left|\frac{y-1}{y+1}\right| \to \infty$ as $x \to \infty$ is if $y \to -1$.
- Case 3: $y_0 < -1$. The expression for y in Case 1 is still valid; however, the x-value of the vertical asymptote is now negative since $\frac{y_0+1}{y_0-1} < 1$. So, y approaches –1, as the graph shows.

Under no circumstances does y approach 1. (If $y_0 = 1$, then $y = 1$ for all x.)

3.3 Equilibria and stability

Definition: The equation $\frac{dy}{dx} = f(x,y)$ has an *equilibrium solution* $y = y^*$ if $f(x, y^*) = 0$ for all x.

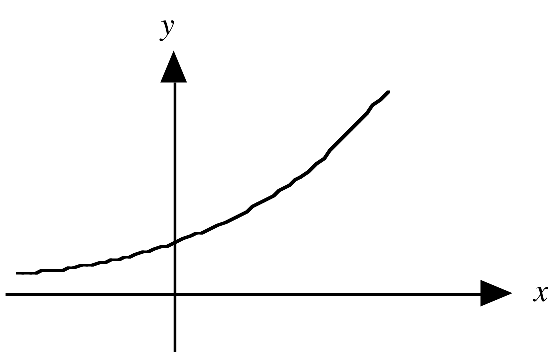
Example:
- $\frac{dy}{dx} = xy$ has an equilibrium at $y = 0$.
- $\frac{dy}{dx} = y^2 - 1$ has equilibria at $y = \pm 1$.
- $\frac{dy}{dx} = x + y$ has no equilibria since there is no y-value that makes $x + y = 0$ for all x.

Most of the equations we'll see are of the form $\frac{dy}{dx} = f(y)$, where the right side is independent of x. It is easier to determine and analyze the equilibria for these equations.

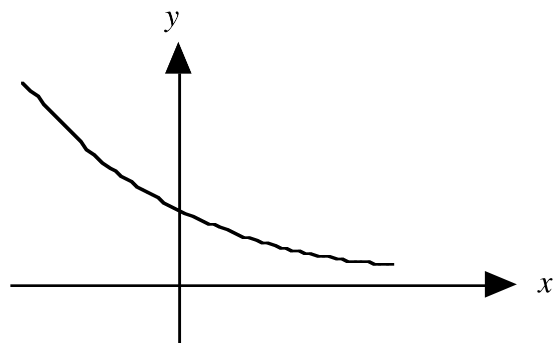
Definition: $y = y^*$ is a *stable* equilibrium if solutions close to y^* converge to y^* as x approaches ∞. Otherwise, $y = y^*$ is *unstable*.

In other words, y^* is unstable if, once the solution moves away from y^*, it doesn't return.

Example:
- $\frac{dy}{dx} = 4y$: $y = 0$ is an unstable equilibrium since $\frac{dy}{dx} > 0$ for $y > 0$ and $\frac{dy}{dx} < 0$ for $y < 0$. Hence, the solutions move away from $y = 0$.



- $\frac{dy}{dx} = -4y$: $y = 0$ is a stable equilibrium since $\frac{dy}{dx} < 0$ for $y > 0$ and $\frac{dy}{dx} > 0$ for $y < 0$. Hence, the solutions move towards $y = 0$.



- $\dfrac{dy}{dx} = y^2 - 1$: $y = 1$ is unstable and $y = -1$ is stable as the earlier graph shows.

Recall that any "nice" function $f(y)$ can be written as a Taylor series about the point $y = a$. Specifically,
$$f(y) = f(a) + f'(a)(y-a) + \frac{f''(a)}{2}(y-a)^2 + \frac{f'''(a)}{3!}(y-a)^3 + \cdots = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(y-a)^k,$$
where $f^{(k)}(a)$ is the k^{th} derivative of f evaluated at $y = a$. The 0^{th} derivative is the function itself.

Often we write the series about $y = 0$, in which case the series is sometimes called a *MacLaurin* series. Every Taylor series has an *interval of convergence*, which is the set of y-values for which the series produces a finite value. There are only 3 possibilities for the interval of convergence: (i) all y (ii) $y = a$ only or (iii) $a - R < y < a + R$, where R is called the *radius of convergence*, and one or both of the endpoints of the interval may be included. The radius of convergence can be determined by the ratio test for infinite series, or other methods. The endpoints need to be checked separately.

Example: Determine the Taylor series for e^y about $y = 0$. Since $f^{(k)}(y) = e^y$ for all k, then $f^{(k)}(0) = 1$. Hence, the Taylor series is $\sum_{k=0}^{\infty} \frac{1}{k!} y^k = 1 + y + \frac{y^2}{2} + \frac{y^3}{3!} + \cdots$. Since $\lim_{k\to\infty} \left| \frac{y^{k+1}}{(k+1)!} \div \frac{y^k}{k!} \right| = \lim_{k\to\infty} \left| \frac{y}{k+1} \right| = 0$ for all y, then the series converges everywhere.

Example: Determine the Taylor series for $\ln(1+y)$ about $y = 0$. It is easy to see that $f^{(k)}(y) = \frac{(-1)^{k-1}(k-1)!}{(1+y)^k}$ from which $f^{(k)}(0) = (-1)^{k-1}(k-1)!$. Hence, the Taylor series is $y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + \cdots \sum_{k=1}^{\infty} \frac{(-1)^{k-1} y^k}{k}$. Since, $\lim_{k\to\infty} \left| \frac{(-1)^k y^{k+1}}{k+1} \div \frac{(-1)^{k-1} y^k}{k} \right| = \lim_{k\to\infty} \left| \frac{ky}{k+1} \right| = |y|$, the ratio test tells us the series converges for $-1 < y < 1$. At $y = 1$, the series gives us the negative of the harmonic series, which diverges. (Also, ln 0 is undefined.) At $y = -1$, we get the alternating harmonic series, which converges (to ln 2). Hence, the Taylor series converges for $-1 < y \le 1$.

Now suppose y^* is an equilibrium solution. Then $f(y^*) = 0$, and the Taylor series for f about $y = y^*$ simplifies to:

$$f(y) = f'(y^*)(y - y^*) + \frac{f''(y^*)}{2}(y - y^*)^2 + \frac{f'''(y^*)}{3!}(y - y^*)^3 + \cdots$$

If y is close to y^*, then we can ignore all terms after the first. Thus, the differential equation can be written approximately as $\frac{dy}{dx} = f(y) \approx f'(y^*)(y - y^*)$. We say we have *linearized* the equation, a technique we shall use again later.

This leads to the following theorem which characterizes equilibria.

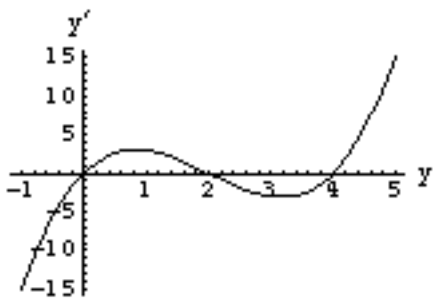
Theorem 3.1: Let y^* be a value of y such that $f(y^*) = 0$. Then y^* is a stable equilbrium of the equation $\frac{dy}{dx} = f(y)$ if and only if $f'(y^*) < 0$.

Proof: Suppose $f'(y^*) < 0$. If $y > y^*$, then $\frac{dy}{dx} < 0$. Hence, the solution decreases towards y^*. Likewise, if $y < y^*$, then $\frac{dy}{dx} > 0$ and the solution increases towards y^*.

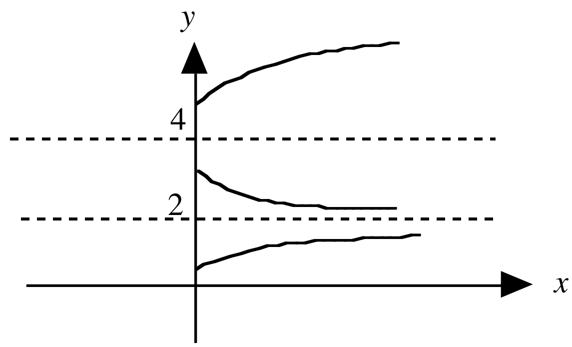
We can also assess stability from the phase-plane portrait. The equilibria are the points where the graph of $f(y)$ crosses the horizontal axis. If the graph crosses from "northwest to southeast", the equilibrium is stable. Otherwise it is unstable.

Note: There is such a thing as a semi-stable equilibrium. If $f(y^*) = 0$, but $f(y)$ is either strictly positive or strictly negative on both sides of y^*, then y^* is semi-stable. In other words the graph of $\frac{dy}{dx}$ is tangent to the y-axis at y^*. From a practical point of view, these are unstable.

Example: $\frac{dy}{dx} = y(y-2)(y-4)$



The equilibria are $y = 0$ (unstable), $y = 2$ (stable), $y = 4$ (unstable). This means that any solution with $0 < y_0 < 4$ will converge to $y = 2$. Solutions with $y_0 > 4$ or $y_0 < 0$ will diverge.



This is an example where writing the solution in closed form is not easy, even though the equation is separable. The resulting integral $\int \frac{1}{y(y-2)(y-4)}\,dy$ is not that bad if we note that $\frac{1}{y(y-2)(y-4)} = \frac{1}{8y} - \frac{1}{4(y-2)} + \frac{1}{8(y-4)}$. It is solving for y in terms of x that is the hard part.

Often it is sufficient to assess stability (long-term behavior) of the solution to a differential equation.

3.4 Applications

3.4.1 Radioactive decay

Let $y(t)$ = amount (in grams) of a radioactive substance present at time t. Assume the amount of substance decays at an instantaneous rate that is proportional to the amount present. This means $\frac{dy}{dt} = -ky$, for some $k > 0$.

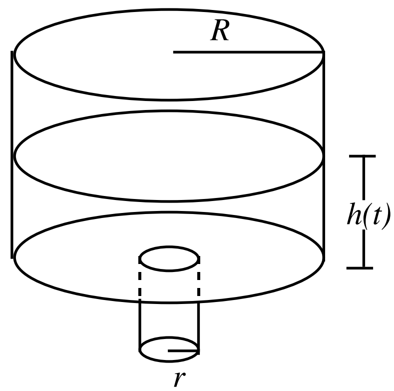
Note: For simplicity, we'll treat all constants as positive and insert negative signs when we want them to be negative.

Separating variables gives $\frac{1}{y}dy = -k\,dt$, from which $\ln y = -kt + c$ or $y = Ce^{-kt}$. Given $y(0) = y_0$, we get $C = y_0$, so $y = y_0 e^{-kt}$. This is a decreasing function such that $\lim_{t\to\infty} y(t) = 0$. That is, $y = 0$ is a stable equilibrium.

Let $t_{1/2}$ = half-life, the time it takes for one-half of the substance to decay. Because of the exponential decay, $t_{1/2}$ is a constant independent of the amount present. To find $t_{1/2}$, solve $\frac{1}{2}y_0 = y_0 e^{-kt}$ for t. We get $t_{1/2} = \frac{\ln(1/2)}{-k} = \frac{\ln 2}{k}$. As k increases (substance decays faster), $t_{1/2}$ decreases.

3.4.2 A hole-in-the-can experiment

Imagine a cylindrical can of radius R with a hole of radius r in the bottom. The can is filled with water to initial height h_0 and allowed to drain through the hole. How long does it take for the can to empty?



Let $h(t)$ be the height of water in the can at time t. In a time interval of length Δt, the height of the water decreases by Δh, so the volume of water decreases by $\pi R^2 \Delta h$. During the same time interval, the amount of water that passes through the hole is $\pi r^2 \Delta s$ where Δs is the distance that the water at the bottom of the can moves.

Now $\Delta s = v\Delta t$, where v is the velocity of the water. So the amount of water that passes through the hole is $\pi r^2 v\Delta t$. By conservation principles, $\pi r^2 v\Delta t \approx -\pi R^2 \Delta h$. (The minus sign is needed because $\Delta h < 0$, while v and Δt are both positive.)

Clearly, v depends on h. Torricelli, a student of Gallileo, proposed a law which says that $v = \sqrt{2gh}$, where g is the acceleration due to gravity. (This can be proved by the conservation of energy principle of physics.) So, upon substitution, we get:

$\pi r^2 \sqrt{2gh}\Delta t \approx -\pi R^2 \Delta h$, which implies

$\dfrac{\Delta h}{\Delta t} \approx -k\sqrt{h}$, for some constant k which is proportional to $\dfrac{r}{R}$, the ratio of the radius of the hole to the radius of the can.

Now let Δt approach 0. Then $\dfrac{\Delta h}{\Delta t} \to \dfrac{dh}{dt}$ and so $h(t)$ satisfies the differential equation:

$$\frac{dh}{dt} = -k\sqrt{h}$$

Solving by separating variables, we get:

$$2\sqrt{h} = -kt + c$$

Substituting the initial condition and rearranging gives:

$$h(t) = \left(\frac{-k}{2}t + \sqrt{h_0}\right)^2$$

Thus, h is a quadratic function of time. The time for the can to empty is $t_e = \dfrac{2\sqrt{h_0}}{k}$ which is proportional to $\dfrac{R}{r}$. So the time to empty increases if: (i) the radius of the can increases, (ii) the radius of the hole decreases, or (iii) the initial height increases.

Torricelli's Law does not take into account friction, surface tension or viscosity ("thickness") of the liquid. In other words, water, oil and molasses would give the same answer. We can modify the law to $v = \alpha\sqrt{2gh}$, where $0 < \alpha < 1$ is a constant. For water flowing through a sufficiently large hole, $\alpha \approx .6$.

Neither the hole nor the can need be circular. The results above are easily modified for any shape, as long as the can remains cylindrical (meaning it has the same cross-section at every height). Just replace πr^2 and πR^2 by the area of the hole and the cross-sectional area of the can. The analysis for noncylindrical (perhaps conical) cans is not much harder; the complication comes from the fact that R now depends on h.

3.4.3 Population models

Let $y(t)$ be the size of a population at time t. Earlier we considered the model in which the rate of growth is proportional to the current population; that is, $\frac{dy}{dt} = ry$, where r is the net growth rate (birth rate – death rate). If r is constant, then $y = y_0 e^{rt}$ which increases without bound. This is unreasonable due to environmental limitations.

Then we let r be a decreasing function of y since the death rate increases as the population grows. Let $r = a - by$, where a and b are constants. Then $\frac{dy}{dt} = (a - by)y$. This equation has two equilibria: $y = 0$ (unstable) and $y = \frac{a}{b}$ (stable). The ratio $\frac{a}{b}$ (which we earlier called L) is the carrying capacity of the environment (maximum sustainable population). Furthermore, by differentiating with respect to t (invoking the chain rule):

$$\frac{d^2y}{dt^2} = (a - 2by)\frac{dy}{dt} = y(a - by)(a - 2by),$$

so there are inflection points at $y = 0$, $y = \frac{a}{b}$ and $y = \frac{a}{2b}$. Only the latter is meaningful.

So if the initial population $y_0 < \frac{a}{2b}$ then the population increases in a concave manner until $y = \frac{a}{2b}$. Then it becomes concave down, leveling off at $y = \frac{a}{b}$.

Now let's solve the equation by separation of variables:

$$\frac{1}{y(a - by)}\,dy = dt\,.$$

From a table of integrals or from the fact that $\frac{1}{y(a - by)} = \frac{1}{ay} + \frac{b}{a(a - by)}$, we get:

$$\frac{1}{a}\ln\left(\frac{y}{a - by}\right) = t + c \text{ from which } \qquad \frac{y}{a - by} = Ce^{at}.$$

If the initial population is y_0, then $\frac{y_0}{a - by_0} = C$. Upon solving for y, we get:

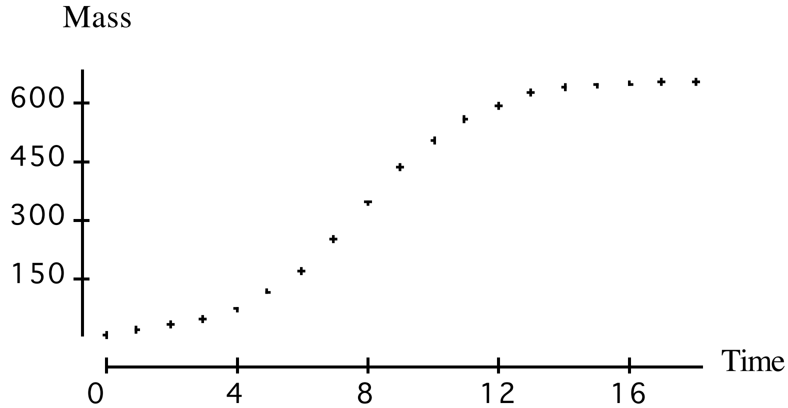
$$y = \frac{ay_0 e^{at}}{a + by_0\left(e^{at} - 1\right)}.$$

This is called a *logistic curve*. Note that $\lim_{t \to \infty} y(t) = \frac{a}{b}$, the stable equilibrium.

Let's see if the model agrees with observations. The following data was reported in "The Growth of Population", by R. Pearl in *Quart. Rev. Biol.* 2 (1927), pg. 532 – 548. He grew yeast in a culture and recorded the mass (in grams) every hour for 18 hours.

Time	0	1	2	3	4	5	6	7	8	9
Mass	9.6	18.3	29.0	47.2	71.1	119.1	174.6	257.3	350.7	441.0

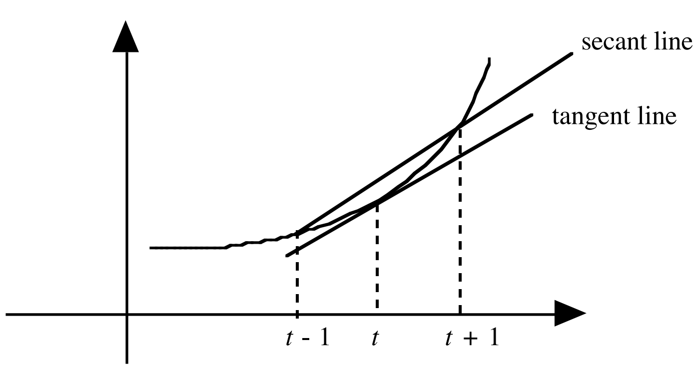
Time	10	11	12	13	14	15	16	17	18
Mass	513.3	559.7	594.8	629.4	640.8	651.1	655.9	659.6	661.8



We want to see if this data fit the logistic model $y(t) = \dfrac{ay_0e^{at}}{a + by_0\left(e^{at} - 1\right)}$, for some a and b.

It is difficult to estimate a and b directly from this model since it is not linear and can't be linearized. So we will estimate them by fitting the data to the original differential equation $\dfrac{dy}{dt} = ay - by^2$. In order to do this, we will have to estimate values of $\dfrac{dy}{dt}$ at each given t. One way to do this is to use the slope of the secant line joining the two points on either side of the given t; that is,

$$\frac{dy}{dt} \approx \frac{y(t+1) - y(t-1)}{2}, \quad t = 1, 2, 3, \ldots, 17.$$



For example, at $t = 1$, $\frac{dy}{dt} \approx \frac{y(2) - y(0)}{2} = \frac{29 - 9.6}{2} = 9.7$.

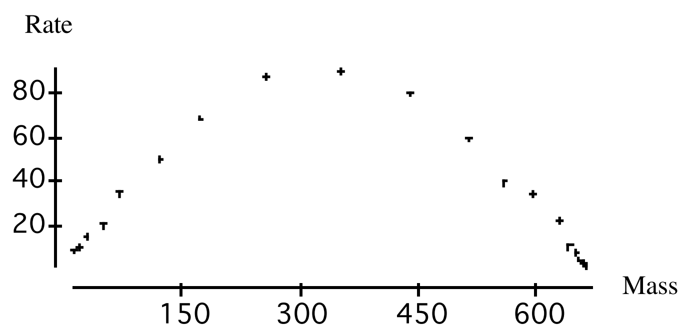
Some modification is needed at the endpoints since there is no value at $t = -1$ and $t = 19$. For $t = 0$, we can estimate $\frac{dy}{dt} \approx y(1) - y(0) = 8.7$ and for $t = 18$, we can estimate $\frac{dy}{dt} \approx y(18) - y(17) = 2.2$.

The complete set of estimated values of the derivative is:

Time	0	1	2	3	4	5	6	7	8	9
Rate	8.7	9.7	14.45	21.05	35.95	51.75	69.1	88.05	91.85	81.3

Time	10	11	12	13	14	15	16	17	18
Rate	59.35	40.75	34.85	23	10.85	7.55	4.25	2.95	2.2

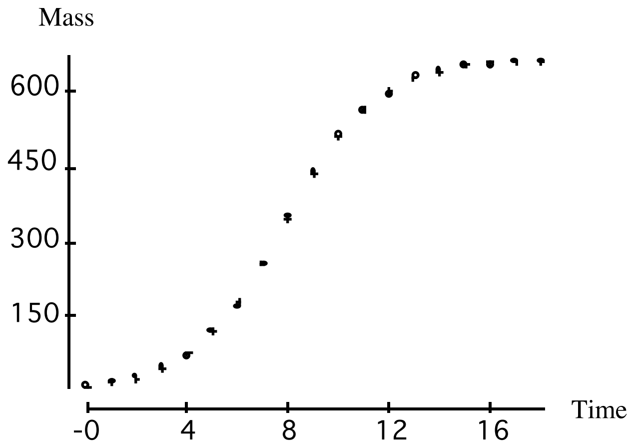
We plot these against the y-values and then fit a quadratic model with no constant term.



The fitted model is $\frac{dy}{dt} = .541y - .000814y^2$.

The maximum size of the population (i.e. carrying capacity of the environment) is $y_{\max} = \frac{.541}{.000814} = 664.6$ which is a stable equilibrium ($y = 0$ is an unstable equilibrium). This is a plausible value based on the observed values.

The graph below shows the corresponding logistic curve superimposed on the original data. The circles represent the observed values; the + signs represent the predicted values from this model. The residuals are: $\{0, -1.98, -1.45, -1.25, 4.11, 0.38, 7.18, 3.71, -0.94, -4.92, -4.05, 4.67, 7.53, -2.52, 1.32, 0.23, 0.92, 0.46, 0.15\}$. These are sufficiently randomly scattered about zero to suggest that the model is good.



Suppose the population is harvested (by hunting or fishing). As we said earlier, a reasonable assumption is that the harvest rate is proportional to the size of the population; that is, a constant fraction of the population is removed per unit time. Then

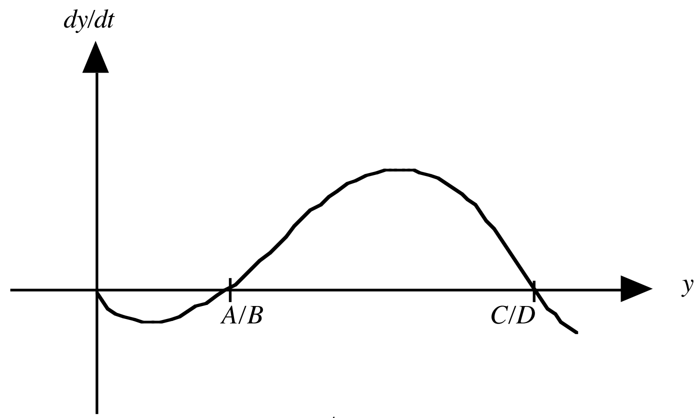
$$\frac{dy}{dt} = y(a - by) - hy = y(a - h - by).$$

Again there are two equilibria: $y = 0$ and $y = \frac{a-h}{b}$. If $h < a$, then $y = 0$ is unstable and $y = \frac{a-h}{b}$ is stable. If $h > a$, then $y = 0$ is stable and the population dies out.

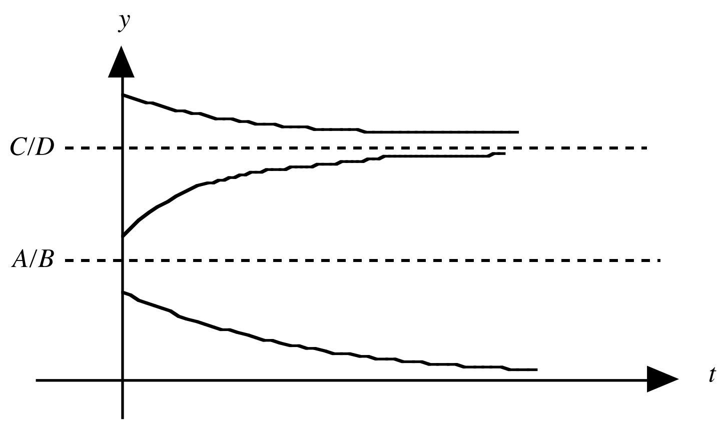
The yield (number of fish caught per unit time) is $hy_{eq} = \frac{h(a-h)}{b}$, assuming the equilibrium is reached. The objective is to maximize the yield. This occurs when $h = \frac{a}{2}$. So the regulations should be written to allow harvesting at that rate (by limiting the length of season, number of nets, etc.) Any other value of h less than a will result in a smaller yield; any value of h bigger than a will wipe out the population.

There are other possible models for population growth. For example, the birth rate might increase while the population is small and then level off. The death rate might be constant for small populations, then increase as the population gets larger. This makes the net growth rate increase for small populations, then decrease. One possibility is a quadratic model: $r = (A - By)(-C + Dy)$. [Note that this is a concave down parabola if B and D are positive.]

Thus, $\frac{dy}{dt} = y(A - By)(-C + Dy)$. There are 3 equilibria: $y_1 = 0$, $y_2 = \frac{A}{B}$ and $y_3 = \frac{C}{D}$. Assume $\frac{A}{B} < \frac{C}{D}$. Let $f(y) = y(A - By)(C + Dy) = -ACy + (AD + BC)y^2 - BDy^3$. Then $f'(y) = -AC + 2(AD + BC)y - 3BDy^2$. Since $f'(0) = -AC < 0$, then $y_1 = 0$ is a stable equilibrium. Since $f'\left(\frac{A}{B}\right) = \frac{A}{B}(BC - AD) > 0$, then $y_2 = \frac{A}{B}$ is unstable. Since $f'\left(\frac{C}{D}\right) = \frac{C}{D}(AD - BC) < 0$, then $y_3 = \frac{C}{D}$ is stable.



So, if the initial population $y_0 < \frac{A}{B}$, then the population dies out. If $y_0 > \frac{A}{B}$, then the population increases to $\frac{C}{D}$. In other words, if the population dips below some critical value (called the *threshold population*), the birth rate cannot compensate for the death rate. (The graph below is approximate; there may be inflection points within each band.)



3.4.4 Modeling motion in one dimension

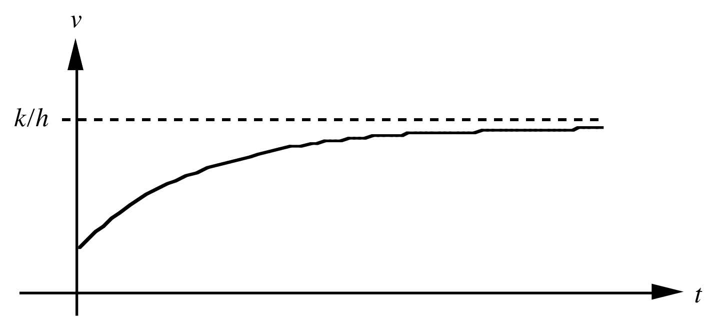
Suppose an object is moving in one dimension (horizontal, vertical or on an incline). Let $y(t)$ be its position at time t and $v(t) = y'(t)$ be its velocity. Newton's Law says that the sum of the forces acting on the object is equal to the product of its mass and acceleration. In other words, $m\frac{dv}{dt} = \sum F$.

If the motion is vertical and we ignore air resistance, then the only force is gravity. Thus, $m\frac{dv}{dt} = -mg$, where g is the accleration due to gravity. (The negative sign is needed because the positive direction is up and gravity pulls downward.) Then, upon integrating, we get $v(t) = -gt + v_0$, where v_0 is the initial velocity. Integrating again gives $y = -\frac{1}{2}gt^2 + v_0t + y_0$, where y_0 is the initial position.

Now let's take air resistance into account. At slow velocities, experimental evidence shows that air resistance is proportional to the velocity. (At higher velocities, it may be proportional to the *square* of the velocity.) Then $\frac{dv}{dt} = -g - hv$, for some constant h. Again, the negative sign in front of the hv term is because the air resistance opposes the direction of motion.

This equation has a stable equilibrium at $v_{\text{term}} = \frac{-g}{h}$, which is called the *terminal velocity*. More generally, if the object undergoes a "natural" acceleration k, then in the absence of any other forces, $\frac{dv}{dt} = k - hv$ and the terminal velocity is $v_{\text{term}} = \frac{k}{h}$.

Here's a graph of the velocity as a function of time:



Now let's solve. Upon separating variables, we have:

$$\frac{1}{k - hv}\, dv = dt.$$

Integrating gives $\frac{-1}{h}\ln(k - hv) = t + c$ from which $v(t) = \frac{k}{h} - Ce^{-ht}$, where $C = \frac{k}{h} - v_0$.

Thus, $v(t) = \frac{k}{h} - \left(\frac{k}{h} - v_0\right)e^{-ht} = v_{\text{term}} - (v_{\text{term}} - v_0)e^{-ht}$.

Now we can integrate once more to obtain the position function:

$$y(t) = v_{\text{term}}t + \frac{1}{h}(v_{\text{term}} - v_0)(e^{-ht} - 1) + y_0,$$ where y_0 is the initial position.

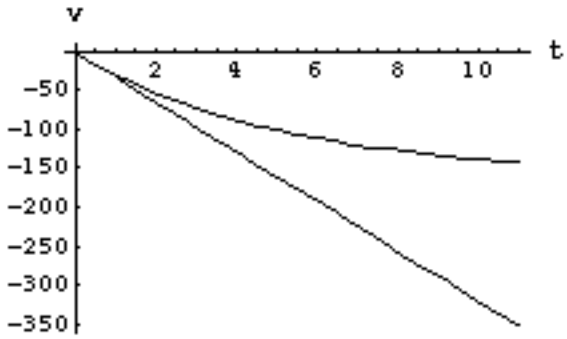
As t increases, the second term approaches a constant and the position becomes a linear function of time (with slope equal to the terminal velocity).

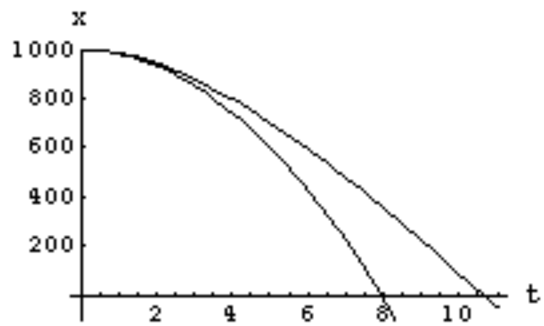
Example: A ball is dropped from a 1000 ft. skyscraper. The drag coefficient is $h = 0.2$. What is the terminal velocity? How long does it take for the ball to reach the ground and what is its velocity at that time? Compare to the results if there were no drag.

Solution: Here, $k = -g = -32$ ft/sec^2, so $v_{term} = \frac{-32}{0.2} = -160$ ft/sec.

The position of the ball is $y(t) = -160t - 800(e^{-0.2t} - 1) + 1000$. Upon setting $y(t) = 0$, we get $160t + 800e^{-0.2t} = 1800$, an equation that can't be solved analytically. Using numerical or graphical methods, we find that the ball hits the ground at approximately $t = 10.7$ sec. The velocity at that time is approximately $v = -141$ ft/sec.

If there were no drag, then the velocity would be $v(t) = -32t$ and the position would be $y(t) = -16t^2 + 1000$. The ball would hit the ground at time $t = \sqrt{\frac{1000}{16}} \approx 7.9$ sec, and its velocity would be approximately –253 ft/sec, considerably above the terminal velocity. The graphs below show the relationship between velocity and position versus time with and without drag.





3.4.5 Modeling motion in two dimensions

In two-dimensional motion, the position of the object is specified by two functions, $x(t)$ and $y(t)$, representing the horizontal and vertical coordinates, respectively, at time *t*. The corresponding velocities are $v_x(t) = \frac{dx}{dt}$ and $v_y(t) = \frac{dy}{dt}$. The *speed* is the magnitude of the velocity; that is, $s(t) = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$.

We need to model the horizontal and vertical components of the acceleration, $a_x(t)$ and $a_y(t)$. The simplest model assumes that the only force acting upon the object is due to gravity, which is in the vertical direction. In this case, $a_x(t) = 0$ and $a_y(t) = -g$, where $g = 9.8$ m/sec^2 $= 32$ ft/sec^2 is the acceleration of gravity at the earth's surface.

In order for the object to move, it must be given some initial velocity $v_x(0)$ in the horizontal direction and $v_y(0)$ in the vertical direction. Using elementary trigonometry, we can show that $v_x(0) = s_0 \cos\theta$ and $v_y(0) = s_0 \sin\theta$, where $s_0 = \sqrt{(v_x(0))^2 + (v_y(0))^2}$ is the initial speed and $\theta = \tan^{-1}\left(\frac{v_y(0)}{v_x(0)}\right)$ is the initial angle at which the object is launched.

Now introduce a coordinate system in which the origin is the point on the ground directly below the spot from which the object is launched. The initial positions are $x(0) = 0$ and $y(0) = y_0$, which is the initial height off the ground. Then:

$$v_x(t) = s_0 \cos\theta \qquad \text{and} \qquad v_y(t) = -gt + s_0 \sin\theta$$

from which the horizontal and vertical positions are:

$$x(t) = (s_0 \cos\theta)t \qquad \text{and} \qquad y(t) = -\frac{1}{2}gt^2 + (s_0 \sin\theta)t + y_0$$

These are parametric equations that give the position of the object at any time t.

Example: Suppose an object is launched from the ground with initial speed $s_0 = 100$ ft/sec and initial angle $\theta = \frac{\pi}{4} = 45^\circ$. What is the maximum height of the object? How far does it travel?

Solution: Substituting the given speed and angle in the equations above gives $x(t) = 50\sqrt{2}\ t$ and $y(t) = -16t^2 + 50\sqrt{2}\ t$.

The maximum height occurs when $v_y(t) = 0$, that is, at $t = \frac{50\sqrt{2}}{32} \approx 2.21$ sec. The corresponding height is $y \approx 78$ ft. The object hits the ground when $y = 0$, which occurs at time $t = \frac{50\sqrt{2}}{16} \approx 4.42$ sec. (This is exactly twice as long as it takes for the object to reach its maximum height since it is launched from the ground and there is a symmetry to the path it travels.) The corresponding x-value is 312.5 ft.

In general, by setting $v_y(t) = 0$, we see that the maximum height occurs at $t_{\max} = \frac{s_0 \sin\theta}{g}$. The maximum height is :

$$y_{\max} = y(t_{\max}) = \frac{-1}{2}g\left(\frac{s_0 \sin\theta}{g}\right)^2 + [s_0 \sin\theta]\left(\frac{s_0 \sin\theta}{g}\right) + y_0 = \frac{s_0^2 \sin^2\theta}{2g} + y_0.$$

Sometimes it is more convenient to have a rectangular equation for the path of the object, rather than parametric equations. While we lose the notion of time, we can more easily compute the y-coordinate at any given x. To get the rectangular equation, take the x-equation, solve for t and substitute in the y-equation, obtaining:

(*) $$y = \frac{-g\sec^2\theta}{2s_0^2}x^2 + (\tan\theta)x + y_0.$$

Thus, the object follows a parabolic path (in the absence of air resistance).

[Recall: $\sec\theta = \frac{1}{\cos\theta}$, $\frac{d}{d\theta}(\sec\theta) = \sec\theta\tan\theta$.]

Now let's find the angle θ that maximizes the horizontal range. If $y_0 = 0$ (meaning the object starts on the ground), then it takes as long for the object to hit its maximum height as it does for it to return to the ground. Hence, the object hits the ground at time $t_g = 2t_{\max} = \frac{2s_0 \sin\theta}{g}$. The corresponding range is:

$$x(t_g) = s_0 \cos\theta\left(\frac{2s_0 \sin\theta}{g}\right) = \frac{s_0^2 \sin(2\theta)}{g}.$$

Since the largest value of $\sin(2\theta)$ is 1, which occurs when $2\theta = \frac{\pi}{2}$, then the maximum range is $\frac{s_0^2}{g}$, occurring when $\theta = \frac{\pi}{4}$.

If $y_0 \neq 0$, then the problem is more complicated. Substituting $y = 0$ in (*) gives:

$$y_0 = -(\tan\theta)x + \frac{g\sec^2\theta}{2s_0^2}x^2.$$

This is an implicit expression for x in terms of θ. Differentiating the entire expression with respect to θ, we get:

$$0 = -(\sec^2\theta)x - (\tan\theta)\frac{dx}{d\theta} + \frac{g}{2s_0^2}\left[(\sec^2\theta)2x\frac{dx}{d\theta} + (2\sec^2\theta\tan\theta)x^2\right].$$

When x is maximized, $\frac{dx}{d\theta} = 0$, so this equation simplifies to $1 = \frac{g}{s_0^2}[(\tan\theta)x]$ from which:

$$x = \frac{s_0^2}{g}\cot\theta.$$

[Recall: $\cot\theta = \frac{1}{\tan\theta}$, $\frac{d}{d\theta}(\cot\theta) = -\csc^2\theta = \frac{-1}{\sin^2\theta}$. Also, $1 + \cot^2\theta = \csc^2\theta$.]

Then $y_0 = -\frac{s_0^2}{g} + \frac{s_0^2}{2g}\csc^2\theta$ from which $-2gy_0 = s_0^2 - s_0^2\cot^2\theta$. This implies that the angle that maximizes the range is given by:

$$\theta = \tan^{-1}\left(\frac{s_0}{\sqrt{s_0^2 + 2gy_0}}\right).$$

Note that for given initial speed s_0, the optimal angle is a decreasing function of the initial height y_0. Furthermore, if $y_0 < 0$ (which could happen when a golfer is trying to hit a ball out of a sand trap), then the expression under the square root may become negative. This means there is no angle at which we could hit the ball to get it back to ground level (for a given initial speed).

All the calculations we have done so far are based on a model that ignores air resistance, turbulence, spin and all other "real world" factors that can affect the flight of the ball.

Here is actual data[1] for a ball that was launched at a 35° angle at an initial speed of 110 mph = 161.3 ft/sec. Assume the ball is hit off the ground or, equivalently, that the *y*-values are measured in relation to the initial height.

t	0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0
x	0	61.5	117.9	161.5	202.6	238.5	271.8	302.6	330.8	356.4	379.5
y	0	41.8	74.4	90.7	104.7	107.0	100.0	90.7	74.4	51.2	23.3

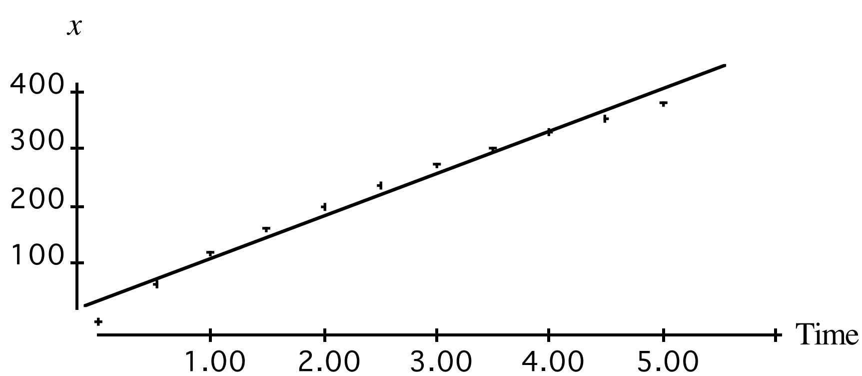
Substituting $\theta = 35^\circ$, $s_0 = 161.3$ into the simple equations gives:

$$x = 131.13t \qquad y = -16t^2 + 92.52t$$

t	0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0
x	0	65.5	131.1	196.7	262.2	327.8	393.3	459.0	524.5	590.0	655.7
y	0	42.2	76.5	102.8	121.0	131.3	133.6	127.8	114.1	92.3	62.6

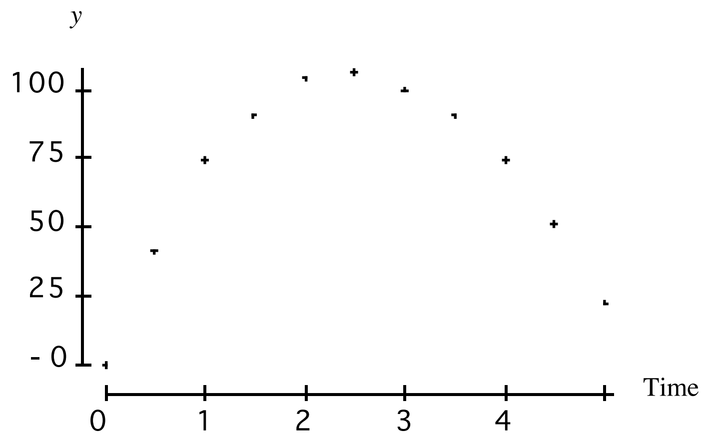
There is a considerable discrepancy between the empirical and predicted results. The simple model predicts the ball to travel $x = \dfrac{s_0^2 \sin(2\theta)}{g} \approx 764$ ft. It travelled only about 400 ft. The predicted maximum height is approximately 133 ft. compared to the actual maximum height of approximately 107 ft.

Observe that the horizontal velocity is decreasing over time. So there must be a horizontal force causing the ball to decelerate. This force is predominantly the drag, or friction between the ball and the air.

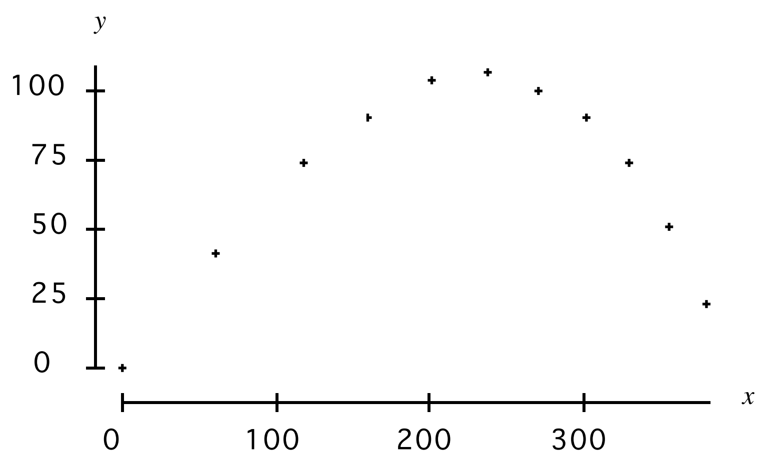


The *y*-values do appear to follow the parabolic relationship predicted by the simple model. (The coefficients must be slightly different to account for the vertical component of the drag force or other forces, such as spin, that act in the vertical direction.)

[1] Data was extracted from graphs in *The Physics of Baseball*, by Robert K. Adair, Harper Perennial, 1994.



In Eq. (*), the simple model predicts that y is a quadratic function of x. The scatterplot below shows that the plot is not symmetric about a central axis. The highest point is near $x = 250$ ft. but the ball hits the ground approximately at $x = 400$ ft. Hence, a quadratic function is not appropriate.



We can either try to fit a curve to each of x and y as functions of t, or to y as a function of x. We'll try the second approach.

Let's try to fit a cubic curve of the form $y = ax^3 + bx^2 + cx$ to the data. (The lack of a constant term in the model ensures that $y = 0$ when $x = 0$.) Using a computer to do least-squares regression with x, x^2 and x^3 as predictor variables and y as the response variable, we get $y = -4.8 \times 10^{-6} x^3 + 2.58 \times 10^{-4} x^2 + .659x$.

We can use this model to predict the y-values at each x.

x	0	61.5	117.9	161.5	202.6	238.5	271.8	302.6	330.8	356.4	379.5
y	0	41.8	74.4	90.7	104.7	107.0	100.0	90.7	74.4	51.2	23.3
pred y	0	40.4	73.4	92.9	104.2	106.8	101.8	90.1	72.5	50.4	24.9

The absolute deviations between actual and predicted y-values is never more than 2.2 feet. That's good.

Had we fit a quadratic model with no constant term, we would have gotten $y = -2.35 \times 10^{-3} x^2 + .983x$. The predicted values for this model are:

0	51.6	83.3	97.6	102.9	101.1	93.9	82.7	68.6	52.5	35.3

which are considerably worse. The absolute deviations are as much as 12 feet.

To make this analysis really useful, we should analyze the trajectories of other batted baseballs under identical conditions – i.e. same wind, same speed of pitch, same weather and altitude, etc. Just change the speed of the bat and the angle of impact. We could then try to determine how the coefficients in the cubic model depend on these factors.

Now let's see if we can provide some analytical justification for the results above. We'll begin by incorporating a drag force into the equations. Assume that drag is proportional to velocity, with the same drag coefficient *h* in both dimensions. Then the horizontal and vertical accelerations are given by:

$$\frac{dv_x}{dt} = -hv_x \qquad \text{and} \qquad \frac{dv_y}{dt} = -g - hv_y$$

The solutions to these equations are:

$$v_x = (s_0 \cos\theta)e^{-ht} \qquad \text{and} \qquad v_y = (s_0 \sin\theta)e^{-ht} + \frac{g}{h}\left(e^{-ht} - 1\right).$$

Note that as $h \to 0$ (meaning that the drag force has a lesser effect), $v_x \to s_0 \cos\theta$ and $v_y \to s_0 \sin\theta - gt$. Also, $v_x \to 0$ and $v_y \to \dfrac{-g}{h}$ as $t \to \infty$, which means that, eventually, the object stops moving horizontally and approaches a terminal vertical velocity.

Now integrate to find the position functions:

$$x(t) = \frac{1}{h}(s_0 \cos\theta)\left(1 - e^{-ht}\right) \qquad \text{and}$$
$$y(t) = \frac{1}{h}(s_0 \sin\theta)\left(1 - e^{-ht}\right) + \frac{g}{h^2}\left(1 - e^{-ht}\right) - \frac{g}{h}t + y_0$$

Once again, these agree with earlier results as $h \to 0$. We can eliminate *t* from these equations by noting that $1 - e^{-ht} = \dfrac{xh}{s_0 \cos\theta}$, from which $t = \dfrac{-1}{h}\ln\left(1 - \dfrac{xh}{s_0 \cos\theta}\right)$.

$$y(t) = (\tan\theta)x + \frac{gx}{hs_0 \cos\theta} + \frac{g}{h^2}\ln\left(1 - \frac{xh}{s_0 \cos\theta}\right) + y_0$$

Modifying an earlier example, we see that the Taylor series for $\ln(1-z)$ is $-\left(z+\frac{z^2}{2}+\frac{z^3}{3}+\dots\right)$, which can be shown to converge for $-1 \le z < 1$. Therefore, $\ln\left(1-\frac{xh}{s_0\cos\theta}\right) = -\left(\frac{xh}{s_0\cos\theta}+\frac{1}{2}\left(\frac{xh}{s_0\cos\theta}\right)^2+\frac{1}{3}\left(\frac{xh}{s_0\cos\theta}\right)^3+\dots\right)$. Thus

$$y(t) = (\tan\theta)x - \frac{gx^2}{2s_0^2\cos^2\theta} - \frac{ghx^3}{3s_0^3\cos^3\theta} - \dots + y_0$$

This provides some justification for the cubic polynomial that we found empirically. This is particularly true for large initial speeds since the coefficients of higher degree terms will become quite small. Also notice that as $h \to 0$, we get (*).

So far, we have only considered drag force. Other factors such as the spin on the ball, the wind, and the stitches (baseball or football) or dimples (golfball) on the ball affect the flight. Analysis of these aspects requires an understanding of aerodynamics, which is beyond the scope of this course.

3.4.6 Kepler's Laws of Planetary Motion

In the seventeenth century, Johannes Kepler, using observational data collected by Tycho Brahe, proposed three laws about the motion of planets around the sun.

1. Planets move in elliptical orbits, with the sun at one focus.
2. The radius drawn from the sun to the planet sweeps out areas at a constant rate.
3. The square of the period of the orbit (i.e. the time it takes to complete one revolution around the sun) is proportional to the cube of the semi-major axis of the ellipse.

Later in that century, Isaac Newton used his newly-developed calculus and physics to prove these laws. We'll look at Newton's proof (updated to modern terminology).

The first thing we need is a coordinate system. Ordinary rectangular (x and y) coordinates are insufficient for two reasons: The equation of an ellipse, $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, is awkward (it is an implicit expression since y is not a unique function of x in the usual sense), and there is no way of incorporating time into the equation. Parametric equations $x = a\cos t$, $y = b\sin t$ are time-dependent, but these put the origin at the center of the ellipse. This makes it difficult to talk about the radius from one of the foci. Adjusting the equations so that the origin is at one focus makes things worse.

We shall use polar coordinates instead. Each point is characterized by an angle θ, measured counterclockwise with respect to some horizontal axis, and a radius r. Polar coordinates and rectangular coordinates are related as follows:

$$x = r\cos\theta,\quad y = r\sin\theta,\quad x^2+y^2=r^2,\quad \tan\theta = \frac{y}{x}$$

Obviously, there are infinitely many values of θ associated with a given point (all of which differ by integer multiples of π). Also, r need not be positive.

To incorporate the time dependence, we shall think of r and θ as functions of t.

The acceleration of the planet at any given point is a vector which we will resolve into a radial component a_r in the direction out from the sun, and an angular component a_θ, which is perpendicular to a_r (The angular component is not tangential to the path of the planet, unless the path is a circle.)

Using some fairly intricate chain rule arguments which we mercifully omit, it can be shown that:

(1) $$a_r = \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2$$ and

(2) $$a_\theta = 2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}.$$

Now we need a model. Let's assume that the planets are far enough apart and small enough in mass that the only appreciable force acting on the planet is the gravitational attraction of the sun, which is in the radial direction. This, in turn, means that the angular component of the force is 0. And, since acceleration is proportional to force, this means:

(3) $$2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2} = 0.$$

Multiply this equation by r, obtaining:

(4) $$2r\frac{dr}{dt}\frac{d\theta}{dt} + r^2\frac{d^2\theta}{dt^2} = 0.$$

Observe that the left-hand side of this equation is $\frac{d}{dt}\left[r^2\frac{d\theta}{dt}\right]$. Hence, we can integrate both sides to get:

(5) $$r^2\frac{d\theta}{dt} = k,$$ where k is a constant.

It can be shown that the area of a sector bounded by the polar function $r = f(\theta)$ and the rays $\theta = \alpha$ and $\theta = \beta$ is given by $A = \frac{1}{2}\int_\alpha^\beta r^2\, d\theta$. Suppose that the θ-coordinate of the planet at time $t = a$ is $\theta = \alpha$, and at time $t = b$ is $\theta = \beta$. Integrating both sides of (5) with respect to t gives:

(6) $$2A = \int_\alpha^\beta r^2\, d\theta = \int_a^b k\, dt = k(b-a)$$

But $b - a$ is the length of time it takes for the planet to go from $\theta = \alpha$ to $\theta = \beta$. Therefore, the area swept out depends only on the length of the interval.

So, we have proved that if the only force acting on the planet is radial (in the direction of the sun), then Kepler's second law holds.

To prove Kepler's other laws, we need an assumption about the magnitude of the force. Newton's Law of Universal Gravitation says that the gravitational attraction between two bodies is directly proportional to the product of the masses and inversely proportional to the square of the distance between them; that is, $F = -\frac{GMm}{r^2}$, where M is the mass of the sun, m is the mass of the planet, and G is the universal gravitation constant ($6.67 \times 10^{-11}\ \mathrm{m}^3 / \mathrm{kg} \cdot \mathrm{sec}^2$).

Now, $a_r = \frac{F}{m}$. Hence, Eq. (1) implies:

$$(7) \qquad \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = -\frac{h}{r^2}\text{, for some constant } h.$$

From (5), we have $\frac{d\theta}{dt} = \frac{k}{r^2}$. Substituting in (7) gives:

$$(8) \qquad \frac{d^2r}{dt^2} - \frac{k^2}{r^3} = -\frac{h}{r^2} \quad \text{or, equivalently, } r^3\frac{d^2r}{dt^2} + hr = k^2.$$

This is a second-order, nonlinear, nonhomogeneous differential equation which is hard to solve. However, we want to show that the solution is an ellipse. So we will determine the polar equation of an ellipse and substitute in (7) to see if it works.

Let $r = \frac{p}{1 + q\cos\theta}$. We shall show that, depending on the values of p and q, this is the equation of one of the conic sections. Cross-multiplying gives $r + rq\cos\theta = p$. Substituting $x = r\cos\theta$ and $r = \sqrt{x^2 + y^2}$ gives $\sqrt{x^2 + y^2} = p - qx$. Squaring and rearranging gives $y^2 = p^2 - 2pqx + \left(q^2 - 1\right)x^2$. Hence, if $q^2 - 1 = 0$, we have a parabola; if $q^2 - 1 > 0$, we have a hyperbola; if $q^2 - 1 < 0$, we have an ellipse. (The special case $q = 0$ gives a circle of radius p.)

Now, consider the case in which $p = \frac{k^2}{h}$ and $q = cp$, for some arbitrary constant c. We shall show that the corresponding conic $r = \frac{p}{1 + cp\cos\theta}$ satisfies (8). Differentiating with respect to t gives:

$$(9) \qquad \frac{dr}{dt} = \frac{cp^2\sin\theta}{\left(1 + cp\cos\theta\right)^2}\frac{d\theta}{dt} = cr^2\sin\theta\frac{d\theta}{dt} = ck\sin\theta,$$

where the last equality comes from (5). Differentiating again gives:

$$(10) \qquad \frac{d^2r}{dt^2} = ck\cos\theta\frac{d\theta}{dt} = \frac{ck^2\cos\theta}{r^2}.$$

Substituting in (8), we get:

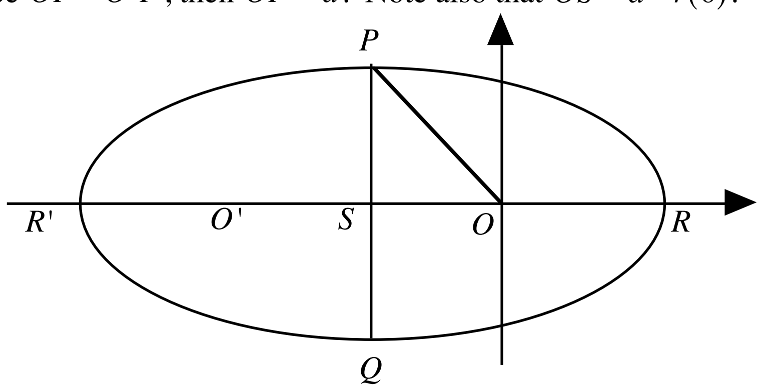
(11) $$r^3\frac{d^2r}{dt^2}+hr=rck^2\cos\theta+hr=hr(1+cp\cos\theta)=hp=k^2.$$

Hence, the conic section satisfies the differential equation. Since we know the planetary orbits are closed curves, they must be ellipses. Thus, we have shown that if Newton's Law of Gravitation holds, the Kepler's first law follows.

The quantity cp is called the *eccentricity* of the ellipse. If $cp = 0$, then the orbit is a circle. As cp approaches 1, the ellipse becomes "flatter."

Now for the third law. In the diagram below, let $a = SR$ be the length of the semi-major axis and $b = SP$ be the length of the semi-minor axis. The foci are at O and O'. Then $OR = r(0) = \dfrac{p}{1+cp}$ and $OR' = r(\pi) = \dfrac{p}{1-cp}$. Thus, $a = \dfrac{OR+OR'}{2} = \dfrac{p}{1-c^2p^2}$.

Ellipses have the property that the sum of the distances from any point on the ellipse to the two foci is equal to $2a$, the length of the major axis. In particular, $OP + O'P = 2a$. Since $OP = O'P$, then $OP = a$. Note also that $OS = a - r(0)$.



By the Pythagorean theorem, $b^2 = (SP)^2 = (OP)^2 - (OS)^2 = a^2 - (a - r(0))^2$ $= 2ar(0) - r(0)^2 = \left(\dfrac{2p}{1-c^2p^2}\right)\left(\dfrac{p}{1+cp}\right) - \left(\dfrac{p}{1+cp}\right)^2 = \dfrac{p^2}{1-c^2p^2}$. Therefore, $b = \dfrac{p}{\sqrt{1-c^2p^2}} = \sqrt{pa}$.

The area of the entire ellipse is $A = \pi ab = \pi\sqrt{p}a^{3/2}$. Kepler's second law says that the area is swept out at a constant rate k per unit time. Thus, $A = kT$, where T is the time it takes to make one complete orbit – that is, the period of the planet. So, we have $kT = \pi\sqrt{p}a^{3/2}$, which implies that T^2 is proportional to a^3. This is Kepler's third law.

Problems

1. A 100 gallon tank is filled with water and 20 lb. of salt. Fresh water is pumped in at a rate of 2 gal/min. The mixture is continuously stirred and overflows to keep the tank at the 100 gallon level.

(a) Let $S(t)$ = amount of salt in the tank at time t. Without doing any calculation, sketch a reasonable graph of $S(t)$ vs. t.

(b) Write and solve a differential equation which can be used to find $S(t)$ explicitly.

(c) Let t_p = time at which the amount of salt is p percent of its original value. For example, t_{50} = time when there is 10 lbs. of salt in the tank, t_{25} = time when there is 5 lbs., etc. Indicate t_{75}, t_{50} and t_{25} on your graph in (a). Determine t_p from your answer to (b). How are t_{50} and t_{25} related?

(d) Certainly, the answer to (b) depends on the rate at which fresh water is pumped in. Let k denote this rate. Graph $S(t)$ for several values of k on the same axes. What happens to t_{50} as k increases? Explain briefly.

(e) Rather than pumping in fresh water, suppose we pump in salt water whose concentration is 1 lb./gal. at a rate of 2 gal/min. Modify your equation in (b) and solve. Graph the solution. What happens to $S(t)$ as t increases? Why does this make sense?

2. A population grows according to the logistic model we derived in class. Let t^* be the time at which the population reaches half its maximum value.

(a) Express t^* in terms of a, b and y_0.

(b) Show that $y(t)$ can be written as $y(t) = \dfrac{a}{b\left(1+e^{-a(t-t^*)}\right)}$.

3. It has been observed that if the number of whales in a population falls below a certain level m, the population becomes extinct. In addition, the environment will not support a population bigger than $M > m$. Let $Y(t)$ be the number of whales at time t.

(a) Argue that a reasonable model is $\dfrac{dY}{dt} = k(M-Y)(Y-m)$, where $k > 0$.

(b) Discuss the equilibria and stability of the equation in (a).

(c) Solve the equation in (a) using the fact that

$$\frac{1}{(M-Y)(Y-m)} = \frac{1}{M-m}\left(\frac{1}{M-Y} + \frac{1}{Y-m}\right)$$

(d) Show that your solution in (c) approaches M as t approaches ∞.

4. A population grows according to the logistic model. In addition, there is emigration from the population at a constant rate of h individuals per unit time. (This is not the same as the fishing model, where h was the fraction of the population harvested per unit time.)

(a) Write a differential equation for $y(t)$, the population at time t.

(b) What is the maximum value of h for which the population can continue to exist? Explain.

(c) Assuming h is small enough for the population to exist, does the population level off? If so, at what value?

5. In Section 3.4.5, we showed that the angle that maximizes the horizontal range of an object moving in two dimensions is given by $\theta = \tan^{-1}\left(\frac{s_0}{\sqrt{s_0^2 + 2gy_0}}\right)$, where s_0 is the initial speed and y_0 is the initial height.

(a) For fixed s_0, how does θ behave as a function of y_0?

(b) For fixed y_0, how does θ behave as a function of s_0?

6. A skier moves down a mountain that makes a constant angle θ with the horizontal. There are three forces acting on the skier: gravity, friction and drag. The magnitude of the component of gravity in the downhill direction is $mg\sin\theta$. Friction is usually modeled as proportional to the component of the weight of the skier perpendicular to the mountain. Hence, the magnitude of the friction force is $\mu mg\cos\theta$, where μ is the coefficient of friction. At the speeds at which the skier is moving, drag is proportional to the square of the velocity and to the cross-sectional area *A* of the skier. Hence, the magnitude of the drag force is kAv^2, for some constant *k*. Therefore, the motion of the skier is governed by the equation

(**) $$m\frac{dv}{dt} = mg\sin\theta - \mu mg\cos\theta - kAv^2.$$

(a) Express the terminal velocity in terms of m, g, k, A, μ and θ. Explain how changing μ, *k*, *A* and *m* affect the terminal velocity. What are the practical implications of these changes? (In other words, what does it mean to change μ? Etc.)

(b) We can rewrite (**) as $\frac{dv}{dt} = \beta^2 - \alpha^2 v^2$, where $\beta = \sqrt{g(\sin\theta - \mu\cos\theta)}$ and $\alpha = \sqrt{\frac{kA}{m}}$. Solve this equation for *v* in terms of *t*. Use your answer to verify the terminal velocity obtained in (a). [Hint: $\frac{1}{\beta^2 - \alpha^2 v^2} = \frac{1}{2\beta}\left(\frac{1}{\beta + \alpha v} + \frac{1}{\beta - \alpha v}\right)$]

(c) Show that, if the initial position (i.e. top of the mountain) corresponds to $s_0 = 0$, then the position function is given by $s(t) = -\frac{\beta}{\alpha}t + \frac{1}{\alpha^2}\ln\left(\frac{1 + e^{2\alpha\beta t}}{2}\right)$.