Unit 4: Modeling with systems of first-order differential equations

4.1 Solution of 2 x 2 systems

Let x and y be functions of t. Consider the homogeneous system of first-order linear differential equations:

$$\frac{dx}{dt} = a_1 x + b_1 y$$
$$\frac{dy}{dt} = a_2 x + b_2 y$$

which can be written in matrix form as $\begin{bmatrix} dx/dt \\ dy/dt \end{bmatrix} = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$.

For example, $\frac{dx}{dt} = x + 2y$, $\frac{dy}{dt} = 4x + 3y$ or, equivalently,

(*) $$\begin{bmatrix} dx/dt \\ dy/dt \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

One way to solve this is to convert it to a single second-order equation. Differentiating the first equation, we get:

$\frac{d^2x}{dt^2} = \frac{dx}{dt} + 2\frac{dy}{dt} = \frac{dx}{dt} + 2(4x + 3y) = \frac{dx}{dt} + 8x + 3\left(\frac{dx}{dt} - x\right)$, which simplifies to $\frac{d^2x}{dt^2} - 4\frac{dx}{dt} - 5x = 0$.

This is a homogeneous equation with constant coefficients. If it were a first-order equation (say $\frac{dx}{dt} - 5x = 0$), the solution would be an exponential function ($x = e^{5t}$). Hence, we'll assume the solution is of the form $x = e^{rt}$ for some constant r. Then $\frac{dx}{dt} = re^{rt}$ and $\frac{d^2x}{dt^2} = r^2 e^{rt}$. Substituting gives $e^{rt}(r^2 - 4r - 5) = 0$. Since e^{rt} is never 0, then we get the *characteristic equation* $r^2 - 4r - 5$. The solutions are $r = -1$ and $r = 5$.

Therefore, $x_1 = e^{-t}$ and $x_2 = e^{5t}$ are solutions to the equation. By the theory of second-order differential equations, the general solution is $x = c_1 e^{-t} + c_2 e^{5t}$. In linear algebra terminology, the solution space is a two-dimensional subspace of the set of twice-differentiable functions. Since $x_1 = e^{-t}$ and $x_2 = e^{5t}$ are linearly independent, they form a basis for the set of solutions. This means that every solution is a linear combination of these vectors. (It is easy to see that every linear combination is a solution. It takes more work to show that these are the only solutions.)

Note that the roots of the characteristic equation are real and distinct. Had that not been the case, we would have to modify this approach. In particular, if the roots are real and equal, the general solution is $x = c_1 e^{rt} + c_2 te^{rt}$, where r is the lone root. If the roots are not real, then they must occur as a conjugate pair, $r = a \pm bi$. Then the general solution of the differential equation is $x = e^{at}[c_1 \cos bt + c_2 \sin bt]$.

In order to obtain a specific solution, we need to be given additional information. For second order equations, it can be shown that specifying values of $x(t)$ and $\frac{dx}{dt}$ at some value of t (typically $t = 0$) will guarantee a unique solution.

Now back to the system (*). Since $x = c_1 e^{-t} + c_2 e^{5t}$, then $\frac{dx}{dt} = -c_1 e^{-t} + 5c_2 e^{5t}$. Substituting in the first equation gives $y = \frac{1}{2}\left(\frac{dx}{dt} - x\right) = -c_1 e^{-t} + 2c_2 e^{5t}$.

We can write the solution in vector form:

$$\begin{bmatrix} x \\ y \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{5t}.$$

Now specifying $x(0)$ and $y(0)$ will allow us to solve for c_1 and c_2.

This vector representation suggests another approach. Let's review some ideas from linear algebra.

Definition: Let A be a square matrix. The number λ is said to be an *eigenvalue* of A if there exists a nonzero vector $\vec{v}$ such that $A\vec{v} = \lambda\vec{v}$. The corresponding vector $\vec{v}$ is called the *eigenvector* of A associated with the eigenvalue λ.

The equation $A\vec{v} = \lambda\vec{v}$ is equivalent to $(A - \lambda I)\vec{v} = \vec{0}$, where I is an identity matrix. This is a homogeneous system of linear algebraic equations. In order for there to be a nonzero (i.e. nontrivial) solution, the coefficient matrix must be singular (noninvertible). This is equivalent to saying that the determinant of the coefficient matrix must be 0.

For an n x n matrix, the expression $f(\lambda) = \det(A - \lambda I)$ is an n^{th} degree polynomial function of λ, and is called the *characteristic polynomial* of the matrix A.

In the example (*), $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$, $A - \lambda I = \begin{bmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{bmatrix}$. So, the characteristic polynomial is $f(\lambda) = (1-\lambda)(3-\lambda) - 8 = \lambda^2 - 4\lambda - 5$. Thus, the eigenvalues are $\lambda = -1$ and $\lambda = 5$. Note that the characteristic polynomial of the matrix is the same as the characteristic equation of the corresponding second-order differential equation. The eigenvalues are the roots of that equation.

To get the eigenvectors, substitute the eigenvalue in $A - \lambda I$ and reduce the matrix to row-echelon form. The reduced matrix must have at least one row of zeroes; otherwise, it would not be singular. In most (but not all) cases, the set of eigenvectors associated with a given eigenvalue – i.e., the *eigenspace* – forms a one-dimensional subspace of R^n, meaning that all the eigenvectors for that eigenvalue are scalar multiples of each other.

- $\lambda = -1$: $A - \lambda I = \begin{bmatrix} 2 & 2 \\ 4 & 4 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$. The equation $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is satisfied by $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ or any scalar multiple thereof. Thus, we can use $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ as an eigenvector for this eigenvalue.

- $\lambda = 5$: $A - \lambda I = \begin{bmatrix} -4 & 2 \\ 4 & -2 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix}$. The equation $\begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is satisfied by $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ or any scalar multiple thereof. Thus, we can use $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ as an eigenvector for this eigenvalue.

Useful facts:

- The sum of the eigenvalues of a matrix is equal to the sum of the entries on the main diagonal (often called the *trace* of the matrix).

- The product of the eigenvalues is equal to the determinant of the matrix. This implies that all the eigenvalues of a nonsingular (invertible) matrix must be nonzero.

- If A is a diagonal matrix or a triangular matrix (meaning that either all the elements below the main diagonal or all the elements above the main diagonal are 0), then the eigenvalues of A are the entries on the main diagonal.

Theorem 4.1: (a) Let $A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$. If A has *distinct, real* eigenvalues λ_1 and λ_2, then the general solution to the system of differential equations $\begin{bmatrix} dx/dt \\ dy/dt \end{bmatrix} = A\begin{bmatrix} x \\ y \end{bmatrix}$ is $\begin{bmatrix} x \\ y \end{bmatrix} = c_1\vec{v}_1 e^{\lambda_1 t} + c_2\vec{v}_2 e^{\lambda_2 t}$, where $\vec{v}_1$ and $\vec{v}_2$ are the corresponding eigenvectors.

(b) If the eigenvalues are not real, then they must be conjugate pairs $\lambda_1 = a + bi$, $\lambda_2 = a - bi$. The corresponding eigenvectors are also conjugate pairs $\vec{v}_1 = \vec{r} + \vec{s}\,i$, $\vec{v}_2 = \vec{r} - \vec{s}\,i$. Then the solution to the system of differential equation is:

$$\begin{bmatrix} x \\ y \end{bmatrix} = e^{at}\left[c_1(\vec{r}\cos bt + \vec{s}\sin bt) + c_2(\vec{s}\cos bt - \vec{r}\sin bt)\right].$$

This can be verified by substitution.

We will not consider the case of equal eigenvalues here. (See the Problems at the end of this unit.)

Graphical solution: The solution to a system of two differential equations constitutes a set of parametric equations $x = f(t), y = g(t)$ that define a curve C in the xy-plane. We can plot the points (x, y) as t varies and indicate the direction in which the curve is traversed as t increases.

In most cases, a parametric representation is the only convenient way to describe the curve since the curve is not usually the graph of a function $y = f(x)$. There are often multiple y-values for any given x-value. For example, the curve might be a circle or spiral.

Even without solving the equations, we can analyze what the curve looks like by determining the sign of $\frac{dx}{dt}$ and $\frac{dy}{dt}$ in different regions of the xy-plane.

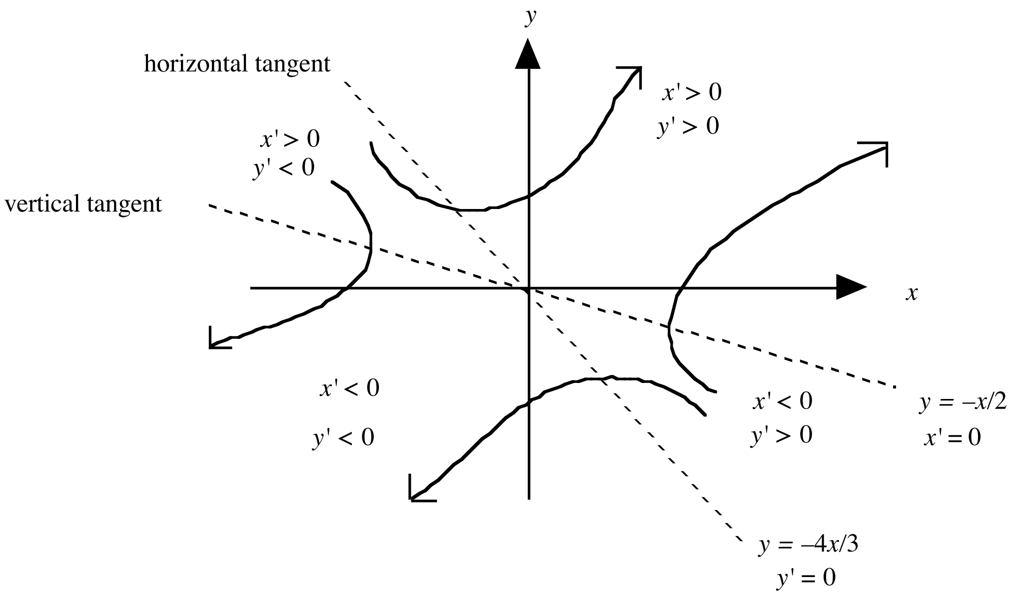
By the chain rule, the slope of the tangent to C is $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$. Therefore, C has a horizontal tangent wherever $\frac{dy}{dt} = 0$ and a vertical tangent (undefined slope) wherever $\frac{dx}{dt} = 0$ (provided both derivatives are not simultaneously 0 at that point).

Step 1: Draw the lines where $\frac{dy}{dt} = 0$ and where $\frac{dx}{dt} = 0$. For a homogeneous system of differential equations with nonsingular coefficient matrix, these should intersect at the origin and divide the xy-plane into 4 regions.

Step 2: Determine the sign of $\frac{dx}{dt}$ and $\frac{dy}{dt}$ in each region. If, for example $\frac{dx}{dt} > 0$ and $\frac{dy}{dt} > 0$, then x and y are both increasing functions of t. This means that the curve moves upward to the right as t increases. If $\frac{dx}{dt} > 0$ and $\frac{dy}{dt} < 0$, then the curve moves downward to the right. And so on.

Step 3: Draw curves satisfying the conditions of step 2. You may get different shapes depending on the initial point (which determines the values of c_1 and c_2). Remember to cross the line where $\frac{dy}{dt} = 0$ horizontally and the line where $\frac{dx}{dt} = 0$ vertically.

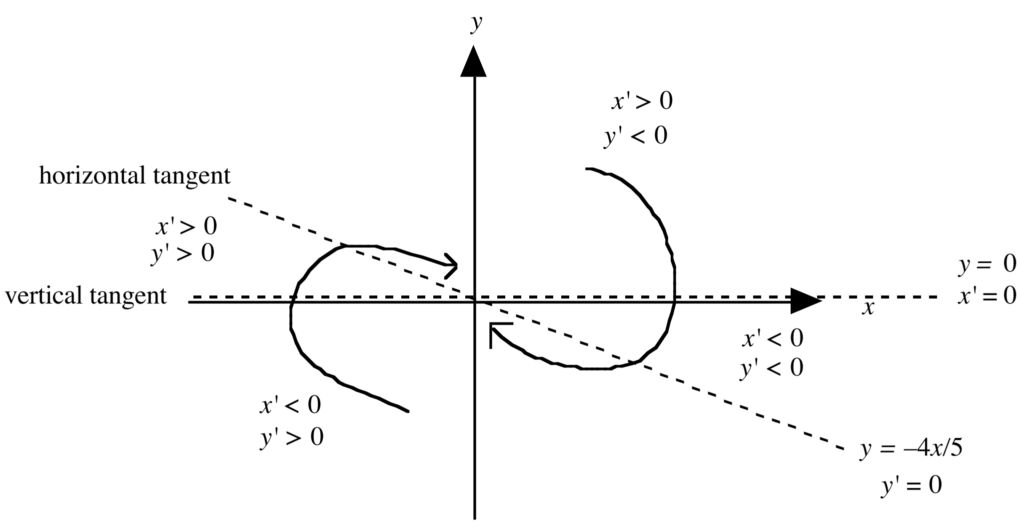
In our example (*), $\frac{dx}{dt} = x + 2y = 0$ when $y = -\frac{1}{2}x$ and $\frac{dy}{dt} = 4x + 3y = 0$ when $y = -\frac{4}{3}x$. The picture below shows the signs and some typical solutions.



Notice that $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is an equilibrium solution for every homogeneous system of this form, as it is for systems of algebraic equations. (Just take $c_1 = c_2 = 0$.) In this example, the equilibrium appears to be unstable since all solution curves move away from the origin.

Example: $\dfrac{dx}{dt} = y$, $\dfrac{dy}{dt} = -4x - 5y$

First, a graphical approach.



Since the trajectories all move toward the origin, we claim that the origin is stable. Now let's solve the system:

$$A=\begin{bmatrix}0 & 1\\ -4 & -5\end{bmatrix},\ A-\lambda I=\begin{bmatrix}-\lambda & 1\\ -4 & -5-\lambda\end{bmatrix},\ \det(A-\lambda I)=\lambda^2+5\lambda+4.$$

So the eigenvalues are $\lambda=-1$ and $\lambda=-4$.

- $\lambda=-1$: $A-\lambda I=\begin{bmatrix}1 & 1\\ -4 & -4\end{bmatrix}\Rightarrow\begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix}$. An eigenvector is $\begin{bmatrix}1\\ -1\end{bmatrix}$.

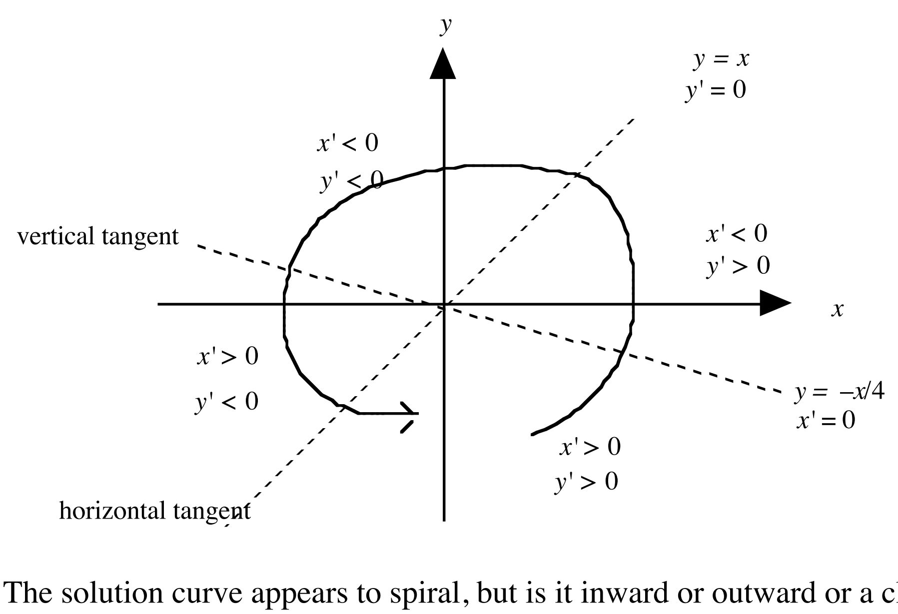
- $\lambda=-4$: $A-\lambda I=\begin{bmatrix}4 & 1\\ -4 & -1\end{bmatrix}\Rightarrow\begin{bmatrix}4 & 1\\ 0 & 0\end{bmatrix}$. An eigenvector is $\begin{bmatrix}1\\ -4\end{bmatrix}$.

Therefore, $x=c_1e^{-t}+c_2e^{-4t}$, $y=-c_1e^{-t}-4c_2e^{-4t}$. Notice that $\lim_{t\to\infty}x(t)=\lim_{t\to\infty}y(t)=0$, which confirms the stability of the origin.

Suppose we have initial conditions $x(0)=3$, $y(0)=-9$. Then $c_1+c_2=3$, $-c_1-4c_2=-9$. The solution is $c_1=1$, $c_2=2$, so $x=e^{-t}+2e^{-4t}$, $y=-e^{-t}-8e^{-4t}$. The initial point $(3,-9)$ is in the region where $\frac{dx}{dt}<0$ and $\frac{dy}{dt}>0$. Hence, the solution curve heads northwest to the origin for $t>0$.

In special cases, it is possible to write an xy-equation for the solution. For example, if $c_2=0$, then $y=-x$. Any initial condition on this line will give $c_2=0$. The trajectory will move toward the origin along the line $y=-x$, but won't reach the origin in finite time.

Example: $\frac{dx}{dt}=-x-4y$, $\frac{dy}{dt}=x-y$



The solution curve appears to spiral, but is it inward or outward or a closed curve?

Let $A=\begin{bmatrix}-1 & -4\\ 1 & -1\end{bmatrix}$, $A-\lambda I=\begin{bmatrix}-1-\lambda & -4\\ 1 & -1-\lambda\end{bmatrix}$, $\det(A-\lambda I)=\lambda^2+2\lambda+5$. The eigenvalues are complex numbers: $\lambda=-1\pm 2i$.

$\lambda=-1+2i$: $A-\lambda I=\begin{bmatrix}-2i & -4\\ 1 & -2i\end{bmatrix}\Rightarrow\begin{bmatrix}1 & -2i\\ 0 & 0\end{bmatrix}$. An eigenvector is $\begin{bmatrix}2\\ -i\end{bmatrix}$.

$\lambda=-1-2i$: $A-\lambda I=\begin{bmatrix}2i & -4\\ 1 & 2i\end{bmatrix}\Rightarrow\begin{bmatrix}1 & 2i\\ 0 & 0\end{bmatrix}$. An eigenvector is $\begin{bmatrix}2\\ i\end{bmatrix}$.

Note that the eigenvectors are complex conjugates of the form $\vec{r}\pm\vec{s}\,i$, with $\vec{r}=\begin{bmatrix}2\\ 0\end{bmatrix}$ and $\vec{s}=\begin{bmatrix}0\\ 1\end{bmatrix}$.

Therefore, $x=e^{-t}\left[2c_1\cos(2t)-2c_2\sin(2t)\right]$, $y=e^{-t}\left[c_1\sin(2t)+c_2\cos(2t)\right]$. The trigonometric factors are bounded; hence, the behavior is governed by the exponential factor which, due to the negative coefficient, goes to 0 as t increases. Therefore, $\lim_{t\to\infty}x(t)=\lim_{t\to\infty}y(t)=0$, implying that the spiral must be inward and the origin is a stable equilibrium.

Example: $\dfrac{dx}{dt}=x-y$, $\dfrac{dy}{dt}=x+y$

Graphically, we get a similar picture of a spiral. In this case: $A=\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}$, $A-\lambda I=\begin{bmatrix}1-\lambda & -1\\ 1 & 1-\lambda\end{bmatrix}$, $\det(A-\lambda I)=\lambda^2-2\lambda+2$. Hence, the eigenvectors are $\lambda=1\pm i$. The solution is of the form $x=e^{t}[\text{trigonometric part}]$, $y=e^{t}[\text{trigonometric part}]$. Since the trigonometric part is bounded, but the exponential factor has a positive exponent, then the spiral is outward.

Example: $\dfrac{dx}{dt}=-y$, $\dfrac{dy}{dt}=x$.

Now the eigenvalues are purely imaginary: $\lambda=\pm i$. So there is no exponential factor. The solution consists only of trigonometric terms which are periodic. Therefore, the solution curve is closed (a circle or an ellipse). This means that the origin is an unstable equilibrium, even though the solution curves remain bounded.

The eigenvectors are $\vec{v}_1=\begin{bmatrix}1\\ i\end{bmatrix}$ and $\vec{v}_2=\begin{bmatrix}1\\ -i\end{bmatrix}$. Hence, the general solution is $x=c_1\cos t-c_2\sin t$, $y=c_1\sin t+c_2\cos t$. We can determine that the solution is a circle by noting that $x^2+y^2=\left(c_1^2+c_2^2\right)\left(\sin^2 t+\cos^2 t\right)=c_1^2+c_2^2$.

We can also confirm that the solutions are circles by eliminating t from the differential equations, as follows:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = -\frac{x}{y},$$

which upon solving by separation of variables gives $x^2 + y^2 = c$.

Putting a coefficient in front of the *x* or *y* in either differential equation will make the solution ellipses. The eigenvalues will still be purely imaginary.

These examples should convince us of the following:

Theorem 4.2: The system of differential equations $\begin{bmatrix} dx/dt \\ dy/dt \end{bmatrix} = A \begin{bmatrix} x \\ y \end{bmatrix}$ has an equilibrium at the origin. The equilibrium is stable if and only if the real parts of *all* the eigenvalues of *A* are negative.

Note: This presumes that *A* is nonsingular. If *A* is singular, then there are infinitely many equilibria, and one of the eigenvalues of *A* must be 0. See Section 4.2.3.

4.2 Applications

4.2.1 The game of dodgeball

A dodgeball game consists of two teams, *X* and *Y*, with a large number of players each. There are several balls used in the game. Players are eliminated when they are hit by a ball thrown by the other team or when a ball they throw is caught by a member of the other team. The game ends when one team runs out of players.

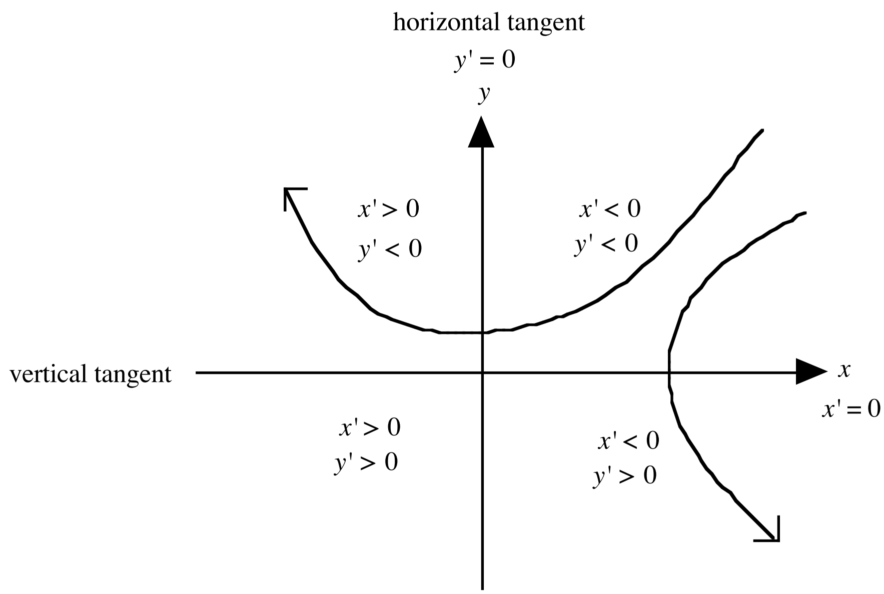
Let $x(t)$ and $y(t)$ be the number of players on teams *X* and *Y*, respectively, remaining in the game at time *t*. For now, let's assume that no balls are caught; eliminations only occur when players are hit. Then it seems logical that the rate at which either team loses players depends on how many players the other team has left. A simple model that reflects this is:

$$\frac{dx}{dt} = -ay$$
$$\frac{dy}{dt} = -bx,$$

where *a* and *b* are measures of the "skill" of the players on teams *Y* and *X*, respectively. In other words, a large value of *a* means that *Y*'s players are very adept at knocking out *X*'s players.

The coefficient matrix $A = \begin{bmatrix} 0 & -a \\ -b & 0 \end{bmatrix}$ has eigenvalues $\lambda = \pm\sqrt{ab}$, which are real. Since they are not both negative, then the origin is an unstable equilibrium. The corresponding eigenvectors are $\begin{bmatrix} 1 \\ \sqrt{b/a} \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -\sqrt{b/a} \end{bmatrix}$. Hence:

$$x(t) = c_1 e^{\sqrt{ab}\,t} + c_2 e^{-\sqrt{ab}\,t} \qquad \text{and} \qquad y(t) = \sqrt{\frac{b}{a}}\left(c_1 e^{\sqrt{ab}\,t} - c_2 e^{-\sqrt{ab}\,t}\right).$$



Depending on the initial conditions, either *X* eliminates all of *Y*'s players, or vice versa. To find out which team wins, let's solve the equations by eliminating *t*. Upon dividing, we get:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{bx}{ay},$$

which is a separable equation.

$$ay\,dy = bx\,dx \text{ implies } \frac{1}{2}ay^2 = \frac{1}{2}bx^2 + c \text{ or, equivalently, } ay^2 - bx^2 = C.$$

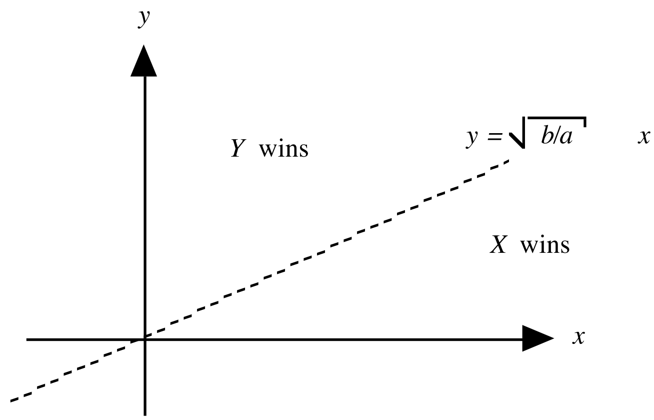
Substituting initial conditions $x(0) = x_0$, $y(0) = y_0$ implies $C = ay_0^2 - bx_0^2$, so the complete solution is $ay^2 - bx^2 = ay_0^2 - bx_0^2$.

This is the equation of a hyperbola. If $ay_0^2 - bx_0^2 > 0$, then the hyperbola intercepts the *y*-axis. This means that there is some time at which *X* has no players left while *Y* still does; that is, *Y* wins. If $ay_0^2 - bx_0^2 < 0$, then it intercepts the *x*-axis. This means *X* wins.

Conclusion: *Y* wins if and only if $ay_0^2 - bx_0^2 > 0$, which is equivalent to $\frac{y_0}{x_0} > \sqrt{\frac{b}{a}}$.

So *Y* wins if it starts with a lot of players (large y_0) or if its players are very skilled (large *a*).

The line $y = \sqrt{\frac{b}{a}}x$ divides the first quadrant into 2 regions. Initial points above the line mean *Y* wins; initial points below the line mean *X* wins.



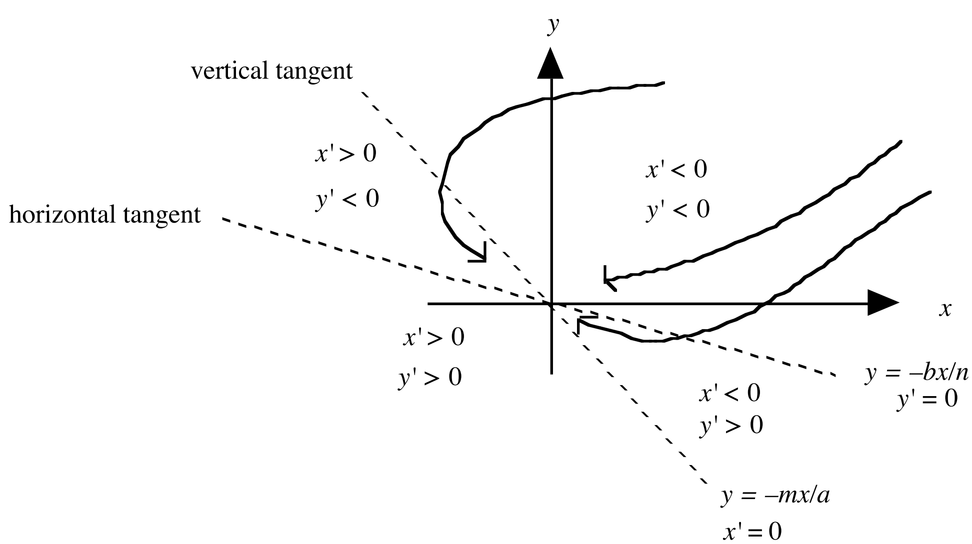
Now let's complicate the model by allowing catches. The rate at which X loses players due to balls being caught by Y depends on how many players both X and Y have. (The more players X has, the more balls it controls, the more throws it makes, the more likely it is that a Y player will catch one. The more players Y has, the more likely it is to catch one of X's throws.) One possible model is:

$$\frac{dx}{dt} = -ay - mx$$
$$\frac{dy}{dt} = -bx - ny$$

The coefficient matrix is $A = \begin{bmatrix} -m & -a \\ -b & -n \end{bmatrix}$. The characteristic polynomial is $\det(A - \lambda I) = \lambda^2 + (m+n)\lambda + (mn - ab)$. The eigenvalues are:

$$\lambda = \frac{-(m+n) \pm \sqrt{(m+n)^2 - 4(mn - ab)}}{2}$$

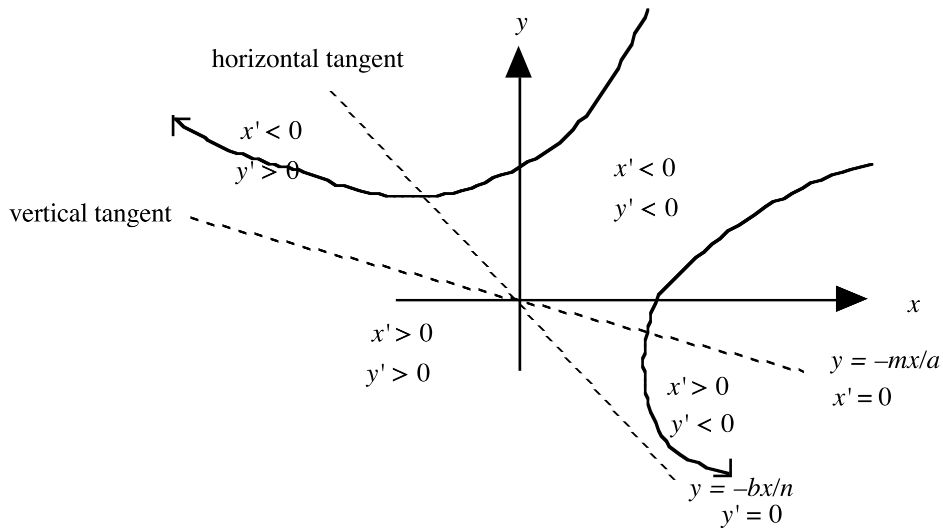
Case I: $mn - ab > 0$ Then $\sqrt{(m+n)^2 - 4(mn-ab)} < m + n$ and both eigenvalues are negative. This makes the origin a stable equilibrium.



Depending on the initial conditions, one team may wipe out the other in a finite amount of time. Or both teams may dwindle, meaning that it takes an infinite amount of time to reach the origin.

Case II: $mn - ab < 0$

Then $\sqrt{(m+n)^2 - 4(mn-ab)} > m+n$ and the eigenvalues are of opposite signs. This makes the origin unstable.



Now one team or the other is sure to win in finite time.

For example, suppose $m = 1$, $n = 3$, $a = 2$, $b = 4$, $x_0 = 80$, $y_0 = 70$. The eigenvalues are –5 and 1, with corresponding eigenvectors $\begin{bmatrix}1\\2\end{bmatrix}$ and $\begin{bmatrix}1\\-1\end{bmatrix}$. After substituting the initial conditions, we get $x = 50e^{-5t} + 30e^{t}$, $y = 100e^{-5t} - 30e^{t}$. Here, $x > 0$ for all t, but $y = 0$ when $t = \frac{1}{6}\ln\left(\frac{10}{3}\right) \approx .2$. This is when Y runs out of players and X wins.

In some versions of dodgeball, a large "medicine ball" is used. This is capable of eliminating several players on the opposing team simultaneously. Now the rate at which X loses players again depends on the number of players both X and Y have. (Since the play is confined to a finite arena, the more players X has, the greater the density, the more that are likely to be hit by one medicine ball thrown by Y.)

While the model above might still be appropriate, another possibility is:

$$\frac{dx}{dt} = -axy$$
$$\frac{dy}{dt} = -bxy.$$

This is a nonlinear model so the results about eigenvalues don't apply. But we can solve easily by dividing the two equations:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-bxy}{-axy} = \frac{b}{a}.$$

This means the slope of the solution curve is constant, so the solutions are straight lines of the form $y - y_0 = \frac{b}{a}(x - x_0)$. It is easy to show that *Y* wins iff $\frac{y_0}{x_0} > \frac{b}{a}$ (as opposed to $\frac{y_0}{x_0} > \sqrt{\frac{b}{a}}$ in the earlier model where the solution curves were hyperbolas).

4.2.2 Pollutant decomposition

An organic pollutant enters a lake at a constant rate. Bacteria decompose the pollutant at a rate proportional to its mass. In doing so, the dissolved oxygen in the water is used up at the same rate that the pollutant decomposes. However, oxygen from the air re-enters the lake at a rate proportional to the difference between the maximum level the lake can support and its current value.

Let $x(t)$ = amount of pollutant in the lake at time *t*

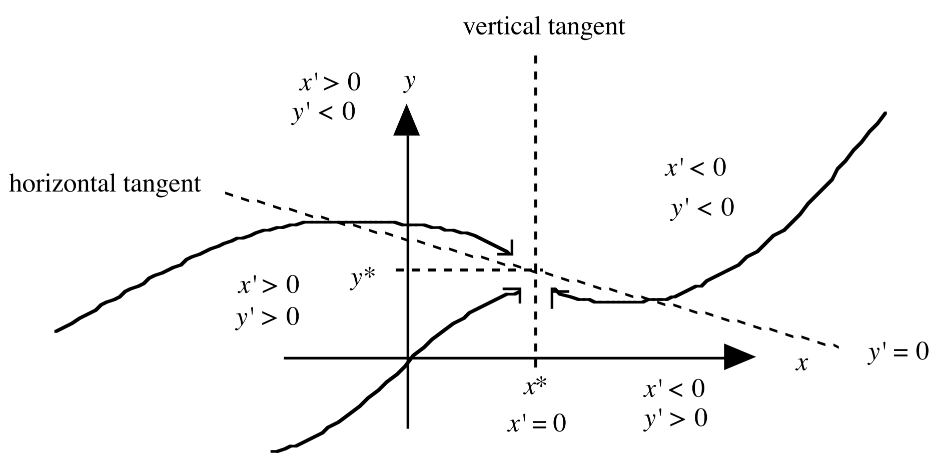
$y(t)$ = amount of oxygen in the lake at time *t*

Then:

$$\frac{dx}{dt} = \alpha - \beta x$$
$$\frac{dy}{dt} = -\beta x + \gamma(M - y),$$

where α = rate at which pollutant enters the lake, β = rate at which the bacteria decompose the pollutant, γ = rate at which the lake absorbs oxygen from the air, and *M* = maximum amount of oxygen the lake can support.

This is a linear, *nonhomogeneous* system. However, we can still analyze the equilibria. By setting both equations equal to 0, we see that there is an equilibrium at $x^* = \frac{\alpha}{\beta}$, $y^* = M - \frac{\alpha}{\gamma}$. Assume *M* is sufficiently large so that $y^* > 0$.



Note that no solution can cross the line $x = x^*$.

It appears that the equilibrium is stable. We can check by "homogenizing" the system – that is, introduce new variables so that the system becomes homogeneous and the equilibrium is at the origin in the new coordinates.

Let $u = x - x^*$, $v = y - y^*$. Then $\frac{du}{dt} = \frac{dx}{dt}$, $\frac{dv}{dt} = \frac{dy}{dt}$ and the system becomes:

$$\frac{du}{dt} = -\beta u\,,\ \ \frac{dv}{dt} = -\beta u - \gamma v\,,$$

which is homogeneous with an equilibrium at the origin.

The coefficient matrix $A = \begin{bmatrix} -\beta & 0 \\ -\beta & -\gamma \end{bmatrix}$ has eigenvalues $\lambda_1 = -\beta$, $\lambda_2 = -\gamma$, which are both negative. Hence, the equilibrium is stable.

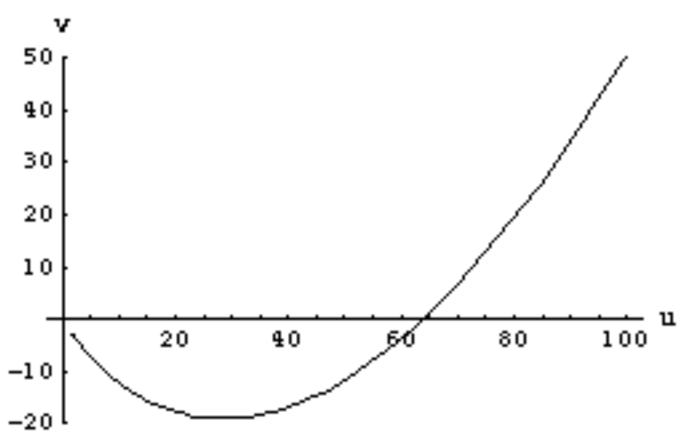
We can solve the system either by using eigenvectors or directly since the first equation is independent of v. We'll do the latter. The solution to the first equation is $u = u_0 e^{-\beta t}$. Then the second equation can be written as $\frac{dv}{dt} + \gamma v = -\beta u_0 e^{-\beta t}$. The integrating factor is $\mu = e^{\int \gamma\, dt} = e^{\gamma t}$. Hence, $\frac{d}{dt}\left(ve^{\gamma t}\right) = -\beta u_0 e^{(\gamma - \beta)t}$.

$$ve^{\gamma t} = -\frac{\beta}{\gamma - \beta} u_0 e^{(\gamma - \beta)t} + c$$

$$v = -\frac{\beta}{\gamma - \beta} u_0 e^{-\beta t} + c e^{-\gamma t}$$

If $v(0) = v_0$, then $c = v_0 + \frac{\beta}{\gamma - \beta} u_0$, from which $v = -\frac{\beta}{\gamma - \beta} u_0 e^{-\beta t} + \left(v_0 + \frac{\beta}{\gamma - \beta} u_0 \right) e^{-\gamma t}$.
Note that this solution is independent of α. To get the original variables x and y, just add the corresponding equilibrium values – which do depend on α – to u and v.)

Suppose $u_0 = 100$, $v_0 = 50$, $\gamma = 3$, $\beta = 2$. Then $u = 100e^{-2t}$, $v = -200e^{-2t} + 250e^{-3t}$. The graph is shown below; the motion is towards the origin.



4.2.3 Diffusion through a membrane

Two chambers are separated by a permeable membrane. A substance can diffuse through the membrane at a rate proportional to the difference between the concentrations of the substance in the two chambers. The diffusion is from the chamber with the higher concentration to the chamber with the lower concentration.

Let v_1, v_2 be the volumes of the chambers and let $x_1(t), x_2(t)$ be the amount of substance in each chamber. Therefore, the concentration in chamber *i* is $y_i(t) = \frac{x_i(t)}{v_i}$, i = 1, 2. The rate of change in the amount of substance in chamber 1 is:

$$\frac{dx_1}{dt} = k\big(y_2(t) - y_1(t)\big) \text{, where } k \text{ is a constant;}$$

for chamber 2, it is:

$$\frac{dx_2}{dt} = k\big(y_1(t) - y_2(t)\big).$$

Since $\frac{dx_i}{dt} = v_i \frac{dy_i}{dt}$, we can rewrite the equations as:

$$\frac{dy_1}{dt} = \frac{k}{v_1}\big(y_2(t) - y_1(t)\big) \qquad \text{and} \qquad \frac{dy_2}{dt} = \frac{k}{v_2}\big(y_1(t) - y_2(t)\big).$$

The coefficient matrix is $A = \begin{bmatrix} -\frac{k}{v_1} & \frac{k}{v_1} \\ \frac{k}{v_2} & -\frac{k}{v_2} \end{bmatrix}$, which is singular since its determinant is equal to 0. The eigenvalues are $\lambda_1 = 0$ (which is always the case for a singular matrix) and $\lambda_2 = -k\left(\frac{1}{v_1} + \frac{1}{v_2}\right) < 0$. The corresponding eigenvectors are $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -\frac{v_1}{v_2} \end{bmatrix}$. Hence, the solution is $y_1(t) = c_1 + c_2 e^{\lambda_2 t}$, $y_2(t) = c_1 - c_2 \frac{v_1}{v_2} e^{\lambda_2 t}$.

You may have already noticed that this system has infinitely many equilibria; any point on the line $y_2 = y_1$ makes both derivative vanish. In other words, after a long time, the concentration of substance in the two chambers is the same.

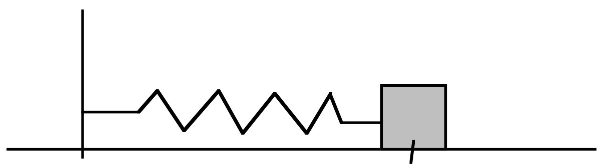
The solution above confirms this. Since $\lambda_2 < 0$, then $\lim_{t\to\infty} y_1(t) = \lim_{t\to\infty} y_2(t) = c_1$. This means that the equilibria are all stable. However, the specific equilibrium attained depends on the initial conditions. (That was not the case in earlier examples.)

Suppose we start with *a* grams of the substance in chamber 1 and 0 grams in chamber 2. Then the initial concentration in chamber 1 is $y_1(0) = c_1 + c_2 = \frac{a}{v_1}$ and the initial concentration in chamber 2 is $y_2(0) = c_1 - \frac{v_1}{v_2} c_2 = 0$. This implies $c_1 = \frac{a}{v_1 + v_2}$,

$c_2 = \dfrac{av_2 / v_1}{v_1 + v_2}$. So, after a long time, the concentration in each chamber is $\dfrac{a}{v_1 + v_2}$, which is the total amount of substance divided by the total volume of the two chambers.

4.2.4 Mass-spring model

Suppose an object of mass m rests on a horizontal frictionless table. It is attached to a spring which, in turn, is anchored to a vertical wall. Let $x(t)$ be the position of the spring at time t; $x = 0$ when the spring is neither stretched nor compressed. Assume that the object encounters air resistance which, as we said in a previous model, is proportional to its velocity.



Hooke's Law says that the force exerted by a spring is proportional to the amount by which the spring is stretched or compressed from its rest position. Then the motion of the object is governed by the second-order equation:

$$m\frac{d^2x}{dt^2} = -kx - h\frac{dx}{dt},$$

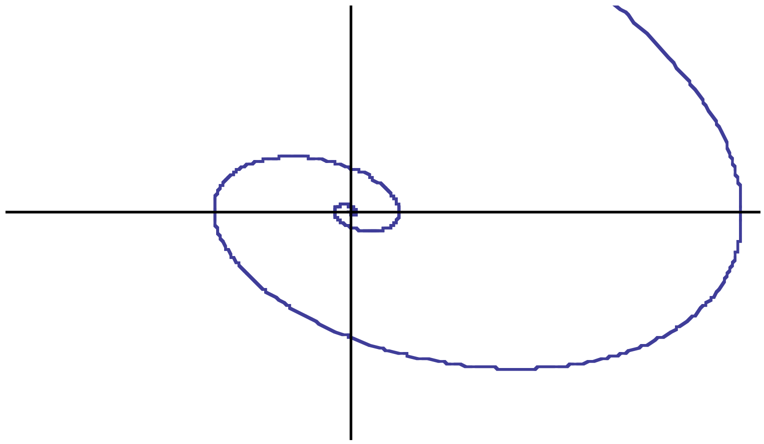
where k measures the strength of the spring (in units of kg/sec^2) and h measures the amount of air resistance (in units of kg/sec).

We can solve this equation either as a second-order equation (as described at the beginning of this unit), or by converting to a system of first-order equations, as follows. Let $y(t) = \dfrac{dx}{dt}$. Then we have the system:

$$\frac{dx}{dt} = y$$
$$\frac{dy}{dt} = -ax - by,$$

where $a = \dfrac{k}{m}$, $b = \dfrac{h}{m}$. The coefficient matrix $A = \begin{bmatrix} 0 & 1 \\ -a & -b \end{bmatrix}$ has eigenvalues $\lambda = \dfrac{-b \pm \sqrt{b^2 - 4a}}{2}$. If $b^2 - 4a > 0$, the both eigenvalues are real and negative. This makes the origin stable and the solutions do not spiral. So, depending on the initial conditions, the mass might move a bit to the right or left, then return towards the rest position without oscillating. The condition $b^2 - 4a > 0$ means that the air resistance is large compared to the strength of the spring. We call this *overdamping*.

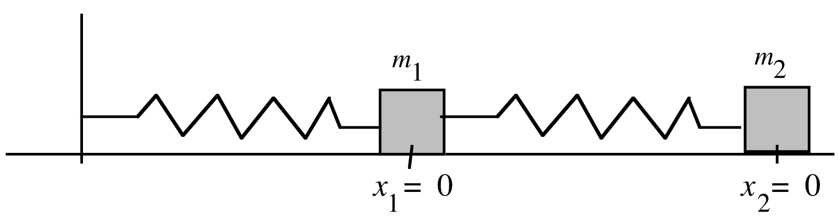
If $b^2 - 4a < 0$, the both eigenvalues are imaginary with negative real parts. This means the solution spirals towards the origin.



As the solution spirals, the x-coordinate (the position of the spring) oscillates with decreasing amplitude. The frequency of the oscillations is $\sqrt{4a-b^2}$, which is the coefficient of t in the trigonometric part of the solution; it remains constant over time. For fixed mass m, the frequency increases if the spring is made stronger (increasing a) or the resistance is lessened (decreasing b).

The y-coordinate (the velocity of the spring) also oscillates with decreasing amplitude, but it is out of phase with the position. The velocity is 0 every time the position attains a local maximum or minimum.

Now consider a system with two objects and two springs as shown below. The masses of the objects are m_1 and m_2; their positions are denoted by $x_1(t)$ and $x_2(t)$, respectively. The distance between the two objects when the springs are at rest is irrelevant. The amount by which the second spring is stretched or compressed is $x_2 - x_1$.



For the first object, we have to account for the forces from both springs; for the second object, only one spring is relevant. Hence, the differential equations for this system are:

$$m_1 \frac{d^2x_1}{dt^2} = -k_1x_1 - h_1\frac{dx_1}{dt} + k_2(x_2 - x_1) \quad \text{and}$$

$$m_2 \frac{d^2x_2}{dt^2} = -h_2\frac{dx_2}{dt} - k_2(x_2 - x_1)$$

In principle, we can solve this by converting to a system of four first-order linear equations. Let $y_1 = \frac{dx_1}{dt}$, $y_2 = \frac{dx_2}{dt}$. That's two of the equations. The other two are:

$$m_1 \frac{dy_1}{dt} = -k_1x_1 - h_1y_1 + k_2(x_2 - x_1) \qquad \text{and}$$

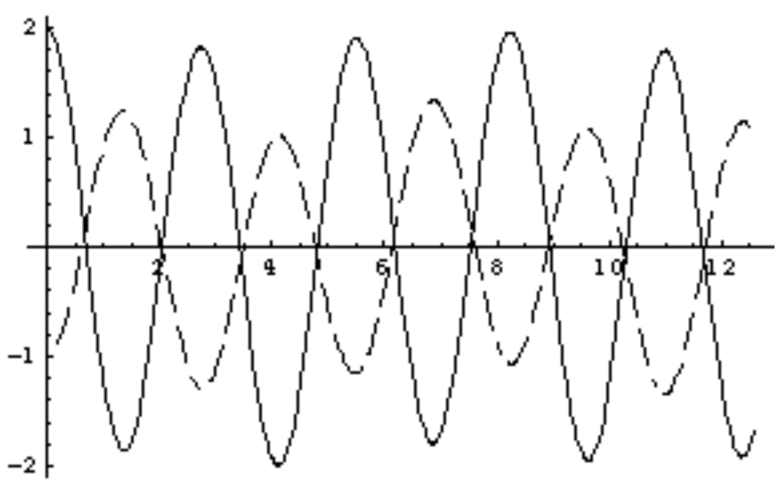
$$m_2 \frac{dy_2}{dt} = -h_2y_2 - k_2(x_2 - x_1)$$

We shall not attempt to solve this in general. However, let's consider the special case in which there is no air resistance: $h_1 = h_2 = 0$. Let $a = \frac{k_1}{m_1}$, $b = \frac{k_2}{m_1}$, and $c = \frac{k_2}{m_2}$. Then the system, in matrix form, is:

$$\begin{bmatrix} dx_1/dt \\ dy_1/dt \\ dx_2/dt \\ dy_2/dt \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -a-b & 0 & b & 0 \\ 0 & 0 & 0 & 1 \\ c & 0 & -c & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \end{bmatrix}$$

This matrix has distinct eigenvalues $\lambda = \pm\sqrt{\frac{-p \pm \sqrt{p^2 - 4ac}}{2}}$, where $p = a + b + c$. Since $\sqrt{p^2 - 4ac} < p$, then $-p \pm \sqrt{p^2 - 4ac} < 0$. Thus, all the eigenvalues are purely imaginary. There are two pairs, one with magnitude $\sqrt{\frac{p + \sqrt{p^2 - 4ac}}{2}}$, the other with magnitude $\sqrt{\frac{p - \sqrt{p^2 - 4ac}}{2}}$. These are the frequencies of the oscillations of the two masses.

Suppose $a = 2$, $b = 2$, $c = 2$. (One way for this to happen is if both masses are 1 and both spring constants are 2.) Then the eigenvalues are $\pm 2.288i$ and $\pm 0.874i$. We used Mathematica™ to solve the differential equations subject to the initial conditions $x_1(0) = 2$, $x_2(0) = -1$, $y_1(0) = y_2(0) = 0$. This corresponds to move the first object 2 units to the right, the second object 1 unit to the left and releasing them. The graph below shows $x_1(t)$ (solid) and $x_2(t)$ (dashed), for $0 \le t \le 4\pi$.



The curves intersect at $t = \{.678823, 2.05751, 3.44191, 4.80117, \ldots\}$. The common values of $x_1(t)$ and $x_2(t)$ at these points are $\{.120962, -.0329118, -.144617, -.071990\ldots\}$, so the intersections are not on the t-axis.

Problems

1. Consider the following homogeneous system:

$$\frac{dx}{dt} = 2x + y \qquad\qquad \frac{dy}{dt} = -3x + 6y$$

(a) Solve by converting to a second-order equation.

(b) Solve by the eigenvalue method.

(c) The lines $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} = 0$ divide the xy-plane into 4 regions. Sketch the trajectories corresponding to initial conditions in each of those regions.

(d) Determine the specific solution satisfying the initial conditions $x(0) = 4$, $y(0) = 6$.

2. Solve the following 3 x 3 system by the eigenvalue method.

$$\frac{dx}{dt} = 3x + 2y + 2z \qquad \frac{dy}{dt} = x + 4y + z \qquad \frac{dz}{dt} = -2x - 4y - z$$

3. Consider the system:

$$\frac{dx}{dt} = y \qquad\qquad \frac{dy}{dt} = -x + 2y$$

(a) Show that the coefficient matrix has only one eigenvalue with one corresponding eigenvector.

(b) In view of (a), we don't have a complete solution to this system. So we need to find another independent piece. Let $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} (At + B)e^t \\ (Ct + D)e^t \end{bmatrix}$. Determine values of A, B, C and D such that these functions satisfy the system of equations.

(c) Write the complete solution to this system of equations.

4. Let $x(t)$ and $y(t)$ represent the sizes of two competing populations, X and Y, at time t. Assume that they are governed by the system of equations:

$$\frac{dx}{dt} = 4x - 6y \qquad\qquad \frac{dy}{dt} = 8x - 10y$$

(a) These equations imply that Y is a parasite that lives off the X's. Explain.

(b) Show that both populations are headed for extinction.

(c) Change the second equation to $\frac{dy}{dt} = kx - 10y$, where $k > 0$. What is the smallest value of k for which both populations become extinct? What happens if k is smaller than that value?

(d) Change the first equation to $\frac{dx}{dt} = mx - 6y$, where $m > 0$. Keep the second equation as $\frac{dy}{dt} = 8x - 10y$. For which positive values of m do the populations become extinct? For which values do they exhibit spiral behavior? For which values do they grow without bound?

5. In the pollutant decomposition example of Section 4.3.2, we obtained the equations $u = 100e^{-2t}$, $v = -200e^{-2t} + 250e^{-3t}$ for a specific set of values of the constants in the system.

(a) Solve for v in terms of u.

(b) What is the minimum value of v? When does it occur?

6. In Problem 1, we considered the homogeneous system

$$\frac{dx}{dt} = 2x + y \qquad \frac{dy}{dt} = -3x + 6y.$$

Now let's consider the nonhomogeneous system

$$(*) \qquad \frac{dx}{dt} = 2x + y + f(t) \qquad \frac{dy}{dt} = -3x + 6y + g(t),$$

where f and g are continuous functions.

(a) Show that this system is equivalent to the second-order equation

$$(**) \qquad \frac{d^2y}{dt^2} - 8\frac{dy}{dt} + 15y = -2g(t) + g'(t) - 3f(t).$$

(b) Write the solution to the corresponding homogeneous equation.

(c) Consider the special case $f(t) = g(t) = e^t$. Show that the equation (**) becomes $\frac{d^2y}{dt^2} - 8\frac{dy}{dt} + 15y = -4e^t$.

(d) Assume that $y_p = Ae^t$ is a particular solution to this equation. Determine A. What is the general solution to (**) in this case?

(e) Determine x in (*) for the special case $f(t) = g(t) = e^t$.