Programming Assignment 4

Angel Hernandez

December 3, 2014

1 Part 1

Given the 4 dimension nonlinear system of equations defined by

$$x_i = \exp\left(\cos\left(i\sum_{j=1}^{4} x_j\right)\right)$$

Let $u = \sum_{j=1}^{4} x_j$, so x_i can be written as $x_i = e^{\cos(iu)}$. To find the entires in the Jacobian Matrix, first note that they are given by $J_{ij} = \dfrac{\partial f_i}{\partial x_j}$. Since we want a solution to $f(x) = 0$, write $f(x_i) = x_i - e^{\cos(iu)}$. Then for $i = j = 1$ we have:

$$\begin{aligned}
\frac{\partial f_1}{\partial x_1} &= \frac{\partial}{\partial x_1}\left(x_1 - e^{\cos u}\right) \\
&= 1 - e^{\cos u}\frac{\partial}{\partial x_1}\cos u \\
&= 1 + e^{\cos u}\sin u\frac{\partial}{\partial x_1}u \\
&= 1 + e^{\cos u}\sin u
\end{aligned}$$

Repeating this for $j = 2,\ 3,\ 4$, we have results of the form $\dfrac{\partial f_1}{\partial x_j} = e^{\cos u}\sin u$ for the rest of the 1st row. More generally, for some $i = j$, we have:

$$\begin{aligned}
\frac{\partial f_i}{\partial x_i} &= \frac{\partial}{\partial x_i}\left(x_i - e^{\cos iu}\right) \\
&= 1 - e^{\cos iu}\frac{\partial}{\partial x_i}\cos iu \\
&= 1 + e^{\cos iu}\sin iu\frac{\partial}{\partial x_i}iu \\
&= 1 + ie^{\cos iu}\sin iu
\end{aligned}$$

If $i \neq j$, then we have $\dfrac{\partial f_i}{\partial x_i} = ie^{\cos iu}\sin iu$. So the Jacobian Matrix J, is of the form:

$$J = I_4 + \begin{bmatrix} e^{\cos u}\sin u & \cdots & e^{\cos u}\sin u \\ 2e^{\cos 2u}\sin 2u & \cdots & 2e^{\cos 2u}\sin 2u \\ \vdots & \vdots & \vdots \\ 4e^{\cos 4u}\sin 4u & \cdots & 4e^{\cos 4u}\sin 4u \end{bmatrix}$$

where I_4 is the 4×4 identity matrix representing the 1s obtains from differentiating x_i when $i = j$. So for the missing portion of `JJ.m` we write `J(i, j) = J(i, j) + (i*exp(cos(i*u))*sin(i*u))`.

2 Part 2

The system of stiff ODE's given by:

$$
\begin{aligned}
y_1' &= -0.04y_1 + y_2y_3 \\
y_2' &= 400y_1 - 10000y_2y_3 - 3000y_2^2 \\
y_3' &= 0.3y_2^2
\end{aligned}
$$

We can find the associated Jacobian Matrix by differentiating y_i' with respect to y_i for $i = 1,\ 2,\ 3$. Beginning with first row of the Jacobian we have:

$$\frac{\partial y_1'}{\partial y_1} = \frac{\partial}{\partial y_1}(-0.04y_1 + y_2y_3) = -0.04,\ \frac{\partial y_1'}{\partial y_2} = \frac{\partial}{\partial y_2}(-0.04y_1 + y_2y_3) = y_3,\ \frac{\partial y_1'}{\partial y_3} = \frac{\partial}{\partial y_3}(-0.04y_1 + y_2y_3) = y_2$$

For the second row we have:

$$\frac{\partial y_2'}{\partial y_1} = \frac{\partial}{\partial y_1}(400y_1 - 10000y_2y_3 - 3000y_2^2) = 400,\ \frac{\partial y_2'}{\partial y_2} = \frac{\partial}{\partial y_2}(400y_1 - 10000y_2y_3 - 3000y_2^2) = -10000y_3 - 6000y_2,$$
$$\frac{\partial y_2'}{\partial y_3} = \frac{\partial}{\partial y_3}(400y_1 - 10000y_2y_3 - 3000y_2^2) = -10000y_2$$

For the third row we have:

$$\frac{\partial y_3'}{\partial y_1} = \frac{\partial}{\partial y_1}(0.3y_2^2) = 0,\ \frac{\partial y_3'}{\partial y_2} = \frac{\partial}{\partial y_1}(0.3y_2^2) = 0.6y_2,\ \frac{\partial y_3'}{\partial y_3} = \frac{\partial}{\partial y_1}(0.3y_2^2) = 0$$

Thus, we have the Jacobian Matrix for this system of stiff ODE's given by:

$$J = \begin{bmatrix} -0.04 & y_3 & y_2 \\ 400 & -10000y_3 - 6000y_2 & -10000y_2 \\ 0 & 0.6y_2 & 0 \end{bmatrix}$$

In MATLab, this translates into the function

```
J = @(y) [-0.04, y(3), y(2); 400, -1e4*y(3) - 6000*y(2), -1e4*y(2); 0, 0.6*y(2), 0]
```

Now for the missing parts in:

```
ah = h*A(i,i);
g = @(k) ???;  you figure out what goes here (involves f)
dg = @(k) ???;  and here.. (involves J)
```

Since $g(k)$ is defined by $g(k) = k - f(z + ha_{ii}k)$. In MATLab, this becomes:

```
g = @(k) k - f(z + ah*k)
```

For `dg` we differentiate $g(k)$ with respect to k_j for $i = j$ to find that:

$$\frac{\partial g_i}{\partial k_j} = \frac{\partial}{\partial k_i}(k_i - f(z + ha_{ii}k_i)) = 1 - \frac{\partial f_i}{\partial k_i}\left(\frac{\partial z}{\partial k_i} + ha_{ii}\right) = 1 - ha_{ii}\frac{\partial f_i}{\partial k_i}\Big|_{z+ha_{ii}k}$$

If $i \neq j$, we have

$$\frac{\partial g_i}{\partial k_j} = -\frac{\partial f_i}{\partial k_i}\left(\frac{\partial z}{\partial k_i} + ha_{ii}\right) = -ha_{ii}\frac{\partial f_i}{\partial k_i}\Big|_{z+ha_{ii}k}$$

Since z is dependent on previously computed values of k_j, we have that $\frac{\partial z}{\partial k_i} = 0$. Futhermore, here the f_i's represent the corresponding y_i' for some i. So, the previously derived Jacobian is given by $\frac{\partial f_i}{\partial k_j}$, which is evaluated at $z + ha_{ii}k$ and scaled by ha_{ii}. We also have 3×3 identity matrix as a result of the $i = j$ condition. So we have the code:

```
dg = @(k) eye(3) - ah*J(z+ah*k)
```