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These are the derivations that I have been transferring to a Latex document and I plan to add as many as possible throughout my gap year and review them along the way.

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Quantum Mechanics (Sakurai)

$|S_y;\pm\rangle = \frac{1}{\sqrt{2}}|S_z;+\rangle \pm \frac{i}{\sqrt{2}}|S_z;-\rangle$
$|S_x;\pm\rangle = \frac{1}{\sqrt{2}}|S_z;+\rangle \pm \frac{1}{\sqrt{2}}|S_z;-\rangle$
$S_z.|S_z;\pm\rangle = \pm\frac{\hbar}{2}|S_z;\pm\rangle$
nothing special about z direction
<u>Note:</u> $S_x|S_x;\pm\rangle = \pm\frac{\hbar}{2}|S_x;\pm\rangle$ for example
<u>Note:</u> $|S_z;+\rangle = |+\rangle;\ |S_z;-\rangle = |-\rangle$

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$$\underline{X = \sum_{a''}\sum_{a'}|a''\rangle\langle a''|X|a'\rangle\langle a'|} \tag{785}$$
<u>recall:</u> $1 = \sum_{a'}|a'\rangle\langle a'|(completeness)$
$X = (\sum_{a''}|a''\rangle\langle a''|)X(\sum_{a'}|a'\rangle\langle a'|)$
$= \sum_{a''}\sum_{a'}(|a''\rangle\langle a''|)X(|a'\rangle\langle a'|)$
$= \sum_{a''a'}(|a''\rangle\langle a''|)(X|a'\rangle)(\langle a'|)$
$= \sum_{a''a'}|a''\rangle\langle a''|X|a'\rangle\langle a'|$

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<u>Note:</u> $S_+ \equiv \hbar|+\rangle\langle-|;\ S_- \equiv \hbar|-\rangle\langle+|$
$S_+ turns|-\rangle into|+\rangle and|+\rangle \to 0,$ etc.

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<u>Note:</u> these objects can also be written in matrix notation, in z basis
$|+\rangle = \begin{pmatrix}1\\0\end{pmatrix},\ |-\rangle = \begin{pmatrix}0\\1\end{pmatrix}$
$S_z \doteq \frac{\hbar}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix};\ S_+ \doteq \hbar\begin{pmatrix}0&1\\0&0\end{pmatrix};\ S_- \doteq \hbar\begin{pmatrix}0&0\\1&0\end{pmatrix}$

$\doteq\sim$ means "represented by"

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(786) $\underline{\langle A\rangle = \sum_{a'} a'|\langle a'|\alpha\rangle|^2}$
$\langle A\rangle = \langle\alpha|A|\alpha\rangle = \langle\alpha|(\sum_{a''}|a''\rangle\langle a''|)A(\sum_{a'}|a'\rangle\langle a'|)|\alpha\rangle$
$= \sum_{a'a''}\langle\alpha|a''\rangle\langle a''|A|a'\rangle\langle a'|\alpha\rangle$
$= \sum_{a'a''}\langle\alpha|a''\rangle a'\eta_{a''a'}\langle a'|\alpha\rangle$
$= \sum_{a'} a'\langle\alpha|a'\rangle\langle a'|\alpha\rangle$
$= \sum_{a'} a'|\langle a'|\alpha\rangle|^2$
Note: $\langle a'|\alpha\rangle = \langle\alpha|a'\rangle^*$

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$\Lambda_{a'}|\alpha\rangle = |a'\rangle\langle a'|\alpha\rangle$ (Projection operator)

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(787) $\underline{S_x = \frac{\hbar}{2}[|+\rangle\langle-|) + (|-\rangle\langle+|)}$
z not special
$\implies S_x = \frac{\hbar}{2}[(|S_x;+\rangle\langle S_x' + | - (|S_x;-\rangle\langle S_x;-1|]$
insert $|S_x;\pm\rangle = \frac{1}{\sqrt{2}}|+\rangle \pm \frac{1}{\sqrt{2}}|-\rangle$
$\implies S_x = \frac{\hbar}{2}[(|+\rangle\langle-|) + (|-\rangle\langle+|)]$

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(788) $\underline{S_y = \frac{\hbar}{2}[-i(|+\rangle\langle-|) + i(|-\rangle\langle+|)}$
y ain't special
$S_y = \frac{\hbar}{2}[(|S_y;+\rangle\langle S_y;+|) - (|S_y;-\rangle\langle S_y;-|)]$
recall: $|S_y;\pm\rangle = \frac{1}{\sqrt{2}}|+\rangle \pm \frac{i}{\sqrt{2}}|-\rangle$
insert
$\implies S_y = \frac{\hbar}{2}[-i(|+\rangle\langle-|) + i(|-\rangle\langle|)]$

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Note: $\vec{S} = (S_x, S_y, S_z)$ can be measured along any direction with $\vec{S}\cdot\hat{n} = (S_x, S_y, S_z)\cdot(n_x, n_y, n_z)$
$|\langle+|\alpha\rangle|^3 \sim$ probability $|\alpha\rangle$ will be in $|+\rangle$ state
$\Delta S_x = S_x - \langle S_x\rangle$

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(789) $\underline{\rangle(\Delta S_x)^2\rangle = \langle S_x^2\rangle - \langle S_x\rangle^2}$
$\langle(\Delta S_x)^2\rangle = \langle(S_x - \langle S_x\rangle)^2\rangle = \langle S_x^2 - 2S_x\langle S_x\rangle + \langle S_x\rangle^2\rangle$
$= \langle S_x^2\rangle - 2\langle S_x\rangle^2 + \langle S_x\rangle^2 = \langle S_x^2\rangle - \langle S_x\rangle^2$

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(790) $\underline{[S_i, S_j] = i\hbar\epsilon_{ijk}S_k}$
$[S_x, S_y] = S_xS_y - S_yS_x$
$\frac{\hbar^2}{4}i[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}]$
$= \frac{\hbar^2}{4}i[\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}]$
$= \frac{2\hbar^2}{4}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = i\hbar\frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
$= i\hbar S_z = i\hbar\epsilon_{xyz}S_z$
now $x \to y,\ y \to z,\ z \to x$
$\implies [S_y, S_z] = i\hbar\epsilon_{yzx}S_x$
so on so forth, by cyclic permutations
$[S_i, S_j] = i\hbar\epsilon_{ijk}S_k$

$\{S_i, S_j\} = \frac{\hbar^2}{2}\delta_{ij}$
Note: degenerate eigen functions are in general not orthogonal

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$B = UAU^{-1}A(diagonal)U \sim$ unitary
U has columns that are eigenvectors of B, A has eigenvalues, i.e., $U \sim (\vec{v}_1, \vec{v}_2, \dots)$

$$A \sim \begin{pmatrix} \lambda_1 & \dots & 0 \\ \vdots & \lambda_i & \vdots \\ 0 & \dots & \lambda_n \end{pmatrix}$$

Note: $det(e^A) = e^{trA}$

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(791) $\underline{[x, F(p_x)] = \frac{\partial F}{\partial p_x}}$
recall: $[f, g] = \sum_{i=1}^{N}(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i})$
$\implies [x, F(p_x)] = \frac{\partial x}{\partial x}\frac{\partial F}{\partial p_x} - \frac{\partial F}{\partial x}\frac{\partial x}{\partial p} = \frac{\partial F}{\partial p_x}$

Note: with commutators to get to quantum just insert $i\hbar, i.e., [x, F(p_x)] = i\hbar\frac{\partial F}{\partial p_x}$

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(792) $\underline{[x_i, p_j] = i\hbar\delta_{ij}}$
$[x, p_x]f = (xp_x - p_xx)f$

$= -i\hbar(x\frac{\partial f}{\partial x} - \frac{\partial}{\partial x}(xf)) = -i\hbar(x\frac{\partial f}{\partial x} - f - x\frac{\partial f}{\partial x})$
$= i\hbar$
$\implies [x_i, p_j] = i\hbar\delta_{ij}$

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(793) $\underline{[x_i, G(\hat{\vec{p}})] = i\hbar\frac{\partial G}{\partial p_i}}$ (quantum)
assume WLOG $G(\hat{\vec{p}}) = \sum_{nm\ell} a_{nm\ell} p_i^n p_j^m p_k^\ell$
i, j, k not equal
$\implies x_i G(\hat{\vec{p}}) = \sum_{nm\ell} a_{nm\ell} x_i p_i^n p_j^m p_k^\ell$
<u>recall:</u> $[x_i, p_j] = i\hbar\delta_{ij} \implies [x_i, p_j] = i\hbar$
$x_i p_i^n = (x_i p_i) p_i^{n-1} = (i\hbar + p_i x_i) p_i^{n-1}$
$= i\hbar p_i^{n-1} + p_i x_i p_i^{n-1} = i\hbar p_i^{n-1} + p_i x_i p_i p_i^{n-2}$
$= i\hbar p_i^{n-1} + p_i(i\hbar + p_i x_i) p_i^{n-2}$
$= i\hbar p_i^{n-1} + i\hbar p_i^{n-1} + p_i^2 x_i p_i^{n-2}$
$= 2i\hbar p_i^{n-1} + p_i^2 x_i p_i^{n-2} = \dots$
$= ni\hbar p_i^{n-1} + p_i^n x_i p_i^{n-n} = i\hbar n p_i^{n-1} + p_i^n x_i$
$\implies [x_i, G(\hat{\vec{p}})] = \sum_{nm\ell} a_{nm\ell}(x_i p_i^n - p_i^n x_i) p_j^m p_k^\ell$
$= \sum_{nm\ell} a_{nm\ell}(i\hbar n p_i^{n-1} + p_i^n x_i - p_i^n x_i) p_j^m p_k^\ell$
$= \sum_{nm\ell} a_{nm\ell} i\hbar n p_i^{n-1} p_j^m p_k^\ell$
$= i\hbar \sum_{nm\ell} a_{nm\ell} \frac{\partial(p_i^n p_j^m p_k^\ell)}{\partial p_i}$
$= i\hbar \frac{\partial}{\partial p_i} G(\hat{\vec{p}})$
likewise $[p_i, F(\hat{\vec{x}}) = -i\hbar\frac{\partial F}{\partial x_i}$

<u>Note:</u> $[AB, C] = A[B, C] + [A, C]B$
$[A, BC] = [A, B]C + B[A, C]$
$T(\vec{\ell}) = \exp(-\frac{i}{\hbar}\vec{p} \cdot \vec{\ell}),\ T(\vec{\ell})|\vec{x}'\rangle = |\vec{x}' + \vec{\ell}\rangle$

<u>Note:</u> p' are eigenvalues p is an operator

(794) $\underline{\langle x'|p'\rangle = \frac{1}{\sqrt{2\pi\hbar}}\exp(\frac{i}{\hbar}p'x')}$
$\langle x'|p|p'\rangle = -i\hbar\frac{\partial}{\partial x'}\langle x'|p'\rangle = p'\langle x'|p'\rangle$
ODE $\implies \langle x'|p'\rangle = \frac{1}{\sqrt{2\pi\hbar}}\exp(\frac{i}{\hbar}p'x')$

$\Phi(p') = \langle p'|\alpha\rangle;\ \psi(x') = \langle x'|\alpha\rangle$

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(795) $\underline{\Phi(p') = \frac{1}{\sqrt{2\pi\hbar}} \int dx' \exp(-\frac{i}{\hbar}p'x')\psi(x')}$

$\langle p'|\alpha\rangle = \langle p'|(\int dx'|x'\rangle\langle x'|)|\alpha\rangle$
$= \int dx' \langle p'|x'\rangle\langle x'|\alpha\rangle$
$= \int dx' \langle p'|x'\rangle\langle x'|\alpha\rangle$
$= \int dx' \frac{1}{\sqrt{2\pi\hbar}} \exp(-\frac{i}{\hbar}p'x')\psi(x')$
$= \frac{1}{\sqrt{2\pi\hbar}} \int dx' \exp(-\frac{i}{\hbar}p'x')\psi(x')$

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(796) $\underline{\psi(x') = \frac{1}{\sqrt{2\pi\hbar}} \int dp' \exp(\frac{i}{\hbar}p'x')\Phi_p(p')}$

$\psi(x') = \langle x'|\alpha\rangle = \langle x'|(\int dp'|p'\rangle\langle p'|)|\alpha\rangle$
$= \int dp' \langle x'|p'\rangle\langle p'|\alpha\rangle = \frac{1}{\sqrt{2\pi\hbar}} \int dp'(\frac{i}{\hbar}p'x')\Phi_p(p')$

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<u>Note:</u> $H|\alpha\rangle = i\hbar\frac{\partial}{\partial t}|\alpha\rangle$ it might so happen that $|\alpha\rangle = |E\rangle i.e. H|E\rangle = E|E\rangle$ in which case $|\alpha\rangle$ is an eigen function of H

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(797) $\underline{\langle x'|p|\alpha\rangle = -i\hbar\frac{\partial}{\partial x'}\langle x'|\alpha\rangle}$

$\langle x'|p|\alpha\rangle = \langle x'|p(\int dp'|p'\rangle\langle p'|)|\alpha\rangle$
$= \int dp'p'\langle x'|p'\rangle\langle p'|\alpha\rangle$
$= \int dp''p'\langle ;' x'|p'\rangle\langle p'|\alpha\rangle$
$- \int dp'p' \frac{1}{\sqrt{2\pi\hbar}} \exp(\frac{i}{\hbar}p'x')\Phi(p')$
$- \frac{\hbar}{i}\frac{\partial}{\partial x'}(\int dp' \frac{1}{\sqrt{2\pi\hbar}} \exp(\frac{i}{\hbar}p'x')\Phi(p'))$
$= -i\hbar\frac{\partial}{\partial x'}\psi(x') = -i\hbar\frac{\partial}{\partial x'}\langle x'|\alpha\rangle$

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(798) $\underline{\langle p'|x|\alpha\rangle = i\hbar\frac{\partial}{\partial p'}\langle p'|\alpha\rangle}$

$\langle p'|x|\alpha\rangle = \langle p'|x(\int dx'|x'\rangle\langle x'|)|\alpha\rangle$
$= \int dx' \langle p'|x.|x'\rangle\langle x'|\alpha\rangle$
$= \int dx' x' \langle x'|p'\rangle^* \langle x'|\alpha\rangle$
$= \int dx' - \frac{\hbar}{i}\frac{\partial}{\partial p'}\langle x'|p'\rangle^* \langle x'|\alpha\rangle$
$= i\hbar\frac{\partial}{\partial p'}\langle p'|\alpha\rangle$

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<u>Note:</u> $U(t, t_0) \exp(-\frac{i}{\hbar}H(t0 - t_0))(if H(t) = H)$
$\implies x(t) = U(t, t_0)xU(t, t_0) = U^\dagger x U$

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$|\psi, t\rangle = U|\psi\rangle$
$\langle A\rangle(t) = \langle\psi|U^\dagger AU|\psi\rangle$

$N \equiv a^\dagger a,\ H = \hbar\omega(N + \frac{1}{2})$

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(799) $\underline{H|n\rangle = E_n|n\rangle \implies N|n\rangle = n|n\rangle}$
Proof:
Assume$H|n\rangle = E_n|n\rangle$
$H|n\rangle = \hbar\omega(N + \frac{1}{2})|n\rangle = \frac{\hbar\omega}{2}|n\rangle + \hbar\omega N|n\rangle = E_n|n\rangle$
$\implies \hbar\omega N|n\rangle = (E_n - \frac{\hbar\omega}{2})|n\rangle$
$\implies N|n\rangle = \frac{1}{\hbar\omega}(E_n - \frac{\hbar\omega}{2})|n\rangle \equiv n|n\rangle$

$a = \sqrt{\frac{m\omega}{2\hbar}}(x + \frac{ip}{m\omega}),\ a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}(x - \frac{ip}{m\omega})$

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(800) $\underline{[a, a^\dagger] = 1}$
$[a, a^\dagger] = aa^\dagger - a^\dagger a = \frac{m\omega}{2\hbar}[(x + \frac{ip}{m\omega})(x - \frac{ip}{m\omega}) - (x - i\frac{p}{m\omega})(x + \frac{ip}{m\omega})]$
$= \frac{m\omega}{2\hbar}[x^2 - \frac{i}{m\omega}xp + \frac{i}{m\omega}px + (\frac{1}{m\omega})^2p^2 - x^2 - \frac{i}{m\omega}xp + \frac{i}{m\omega}px - \frac{p^2}{(m\omega})^2]$
$= \frac{m\omega}{2\hbar}[-\frac{2i}{m\omega}xp + \frac{2i}{m\omega}px]$
$= -\frac{2im\omega}{2\hbar}\frac{1}{m\omega}[x, p] = -\frac{i}{\hbar}i\hbar = 1$

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(801) $\underline{[N, a] = -a;\ [N, a^\dagger = a^\dagger}$
$[N, a] = [a^\dagger a, a] = a^\dagger[a, a] + [a^\dagger, a]a$
$= -a$

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(802) $\underline{Na^\dagger|n\rangle = (n+1)a^\dagger|n\rangle;\ Na|n\rangle = (n-1)a|n\rangle}$
$Na^\dagger|n\rangle = a^\dagger aa^\dagger|n\rangle = a^\dagger(1 + a^\dagger a)|n\rangle$
$= a^\dagger|n\rangle + a^\dagger N|n\rangle = .a^\dagger|n\rangle + na^\dagger|n\rangle$
$= (n+1)a^\dagger|n\rangle$

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this tells us that $Na^\dagger|n\rangle$
results in $a^\dagger|n\rangle$ with an eigenvalue increased by 1 and justifies labeling the eigenstate $a^\dagger|n\rangle by|n+1\rangle$
or actually $a^\dagger|n\rangle \propto |n+\rangle$, need to normalize $|n+1\rangle$

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(803) $\underline{a|n\rangle = \sqrt{n}|n-1\rangle;\ a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle}$
$\underline{\text{Note:}}$ $Na^\dagger|n\rangle = (n+1)a^\dagger|n\rangle \implies a^\dagger|n\rangle = c|n+1\rangle$
$\implies \langle n|aa^\dagger|n\rangle = |c|^2$
but $\langle n|aa^\dagger|n\rangle = \langle n|(1+a^\dagger a)|n\rangle = 1 + \langle n|N|n\rangle = 1+n$
$\implies c = \sqrt{n.+1}$
$\therefore a^\dagger|n\rangle = c|n+1\rangle$

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(804) $\underline{\frac{dp}{dt} = -m\omega^2 x}$
$\underline{\text{recall:}}$ $\frac{d\langle p\rangle}{dt} = \frac{i}{\hbar}\langle[H,p]\rangle \implies \frac{dp}{dt} = \frac{i}{\hbar}[H,p]$
$\underline{\text{recall:}}$ $H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2x^2$
$[H,p] = \frac{1}{2m}[p^2,p] + \frac{1}{2}m\omega^2[x^2,p]$
$= \frac{1}{2}m\omega^2(x[x,p] + [x,p]x)$
$= \frac{1}{2}m\omega^2(2i\hbar x) = i\hbar m\omega^2 x$
$\implies \frac{dp}{dt} = \frac{i}{\hbar}i\hbar m\omega^2 x = -m\omega^2 x$

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(805) $\underline{\frac{dx}{dt} = \frac{p}{m}}$
$\underline{\text{recall:}}$ $\frac{dx}{dt} = \frac{i}{\hbar}[H,x];\ H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2x^2$
$[H,x] = \frac{1}{2m}[p^2,x] = \frac{1}{2m}(p[p,x] + [p,x]p)$
$= \frac{1}{2m}(-2i\hbar p) = -i\hbar\frac{p}{m}$
$\implies \frac{dx}{dt} = \frac{i}{\hbar}(-i\hbar\frac{p}{m}) = \frac{p}{m}$

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(806) $\underline{a(t) = a(0)exp(-i\omega t);\ a^\dagger(t)a^\dagger(0)exp(i\omega t)}$
$\underline{\text{recall:}}$ $\frac{da}{dt} = \frac{i}{\hbar}[H,a];\ H = \hbar\omega(a^\dagger a + \frac{1}{2})$
$[H,a] = ([a^\dagger a, a] + [\frac{1}{2}, a])\hbar\omega$
$= \hbar\omega(a^\dagger[a,a] + [a^\dagger,a]a) = \hbar\omega[a^\dagger,a]a = \hbar\omega a$
$\implies \frac{da}{dt} = -\frac{i}{\hbar}\hbar\omega a = -i\omega a$
likewise $\frac{da^\dagger}{dt} = i\omega a^\dagger$
Solve
$\implies a(t) = a(0)\exp(-i\omega t);\ a^\dagger(t) = a^\dagger(0)\exp(i\omega t)$

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(807) $\underline{x(t) = x(0)\cos\omega t + \frac{p(0)}{m\omega}\sin\omega t;\ p(t) = -m\omega x(0)\sin\omega t + p(0)\cos\omega t}$
$\underline{\text{recall:}}$ $a = \sqrt{\frac{m\omega}{2\hbar}}(x + \frac{ip}{m\omega});\ a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}(x - \frac{ip}{m\omega})$
$a(t) = a(0)\exp(-i\omega t)$
$\sqrt{\frac{m\omega}{2\hbar}}(x(t) + \frac{ip(t)}{m\omega}) = \sqrt{\frac{m\omega}{2\hbar}}(x + \frac{ip}{m\omega})\exp(-i\omega t)$

$\implies x(t) + \frac{ip(t)}{m\omega} = x\cos\omega t - ix\sin\omega t + \frac{ip}{m\omega}\cos\omega t + \frac{p}{m\omega}\sin\omega t$
$\implies x(t)+\frac{ip(t)}{m\omega} = (x\cos\omega t+\frac{p}{m\omega}\sin\omega t)+i(-x\sin\omega t+\frac{p}{m\omega}\cos\omega t)$
$\implies x(t) = x(0)\cos\omega t + \frac{p(0)}{m\omega}\sin\omega t;\ p(t) = -m\omega x(0)\sin\omega t + p(0)\cos\omega t$

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(808) $\underline{\frac{dN}{d\epsilon} = \frac{mL^2}{\pi\hbar^2}}$
$N = \sum_{n_X,n_y} = 2\int_0^{\pi/2}\int_0^n ndnd\theta$
only positive n_x, n_y, and $2 \sim$ spin
$\implies N = \frac{\pi}{2}\frac{1}{2}n^2 = \frac{\pi n^2}{2}$
$\psi(0) = \psi(L),\ \psi(x) = Ae^{ikx} + Be^{-ikx}$
$\implies \sin kL = 0 \implies k = \frac{npi}{L}$
$\epsilon = \frac{\hbar^2k^2}{2m} = \frac{\hbar^2n^2\pi^2}{2mL^2} \implies n^2 = \frac{2mL^2\epsilon}{\hbar^2\pi^2}$
$\implies \frac{dN}{d\epsilon} = \frac{\pi}{2}\frac{2mL^2}{\hbar^2\pi^2} = \frac{mL^2}{\pi\hbar^2}$

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spose $J^2|a,b\rangle = a|a,b\rangle;\ J_z|a,b\rangle = b|a,b\rangle$
i.e. simultaneous eigenkets of J^2 and J_z
J is generalized angular momenta, i.e. either $\vec{L}or\vec{S}$ or a combination of both

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$J_+|a,b\rangle$ is an eigenket whose J_z eigenvalue increased by $n\hbar$
J^2 eigenvalue unaltered
this process cannot continue forever

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(809) $\underline{a \geq b^2}$ b upper limit
<u>recall:</u> $J^2 - J_z^2 = \frac{1}{2}(J_+J_+^\dagger + J_+^\dagger J_+)$
$J_+J_+^\dagger,\ J_+^\dagger J_+$ have non negative expectation values because
$J_+^\dagger|a,b\rangle \overset{\text{DC}}{\leftrightarrow} \langle a,b|J_+,\ J_+|a,b\rangle \overset{\text{DC}}{\leftrightarrow} \langle a,b|J_+^\dagger$
so for example $J_+^\dagger|a,b\rangle = |\lambda\rangle$
then $\langle\lambda|\lambda\rangle = \langle a,b|J_+J_+^\dagger|a,b\rangle > 0$
$\implies \langle a,b|(J^2 - J_z^2)|a,b\rangle \geq 0$
$\implies \langle a,b|J^2|a,b\rangle \geq \langle a,b|J_z^2|a,b\rangle$
$\implies a \geq b^2 \implies \sqrt{a} \geq b \implies$ upper bound for b
call it $b_{max} \implies J_+|a,b_{max}\rangle = 0$

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(810) $\underline{a = b_{max}(b_{max} + \hbar)}$

$J_+|a, b_{max}\rangle = 0 \implies J_-J_+|a, b_{max}\rangle = 0$

<u>recall:</u> $J_-J_+ = J_x^2 + J_y^2 - i(J_yJ_x - J_xJ_y)$

$= J^2 - J_z^2 - \hbar J_z$

$\implies (J^2 - J_z^2 - \hbar J_z)|a, b_{max}\rangle = 0$

$\implies a - b_{max}^2 - \hbar b_{max} = 0$

$\implies a = b_{max}(b_{max} + \hbar)$

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similarly $J_-|a, b_{min}\rangle = 0$

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(811) $\underline{a = b_{min}(b_{min} - \hbar)}$; $-b_{max} \leq b \leq b_{max}$

$J_-|a, b_{min}\rangle = 0 \implies J_+J_-|a, b_{min}\rangle = 0$

<u>recall:</u> $J_+J_- = J^2 - J_z^2 + \hbar J_z$

$\implies (J^2 - J_z^2 + \hbar J_z)|a, b_{min}\rangle = 0$

$\implies a - b_{min}^2 + \hbar b_{min} = 0$

$\implies a = b_{min}(b_{min} - \hbar)$

$\implies b_{min}(b_{min} - \hbar) = b_{max}(b_{max} + \hbar)$

$\implies b_{min} = -b_{max},\ b_{min} = b_{max} + \hbar$

but b_{min} cannot be greater than b_{max}

$\implies b_{min} = -b_{max}$

$\implies -b_{max} \leq b \leq b_{max}$

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(812) $\underline{b_{max} = b_{min} + n\hbar}$; $b_{max} = \frac{n\hbar}{2}$

b increases in units of $\hbar$

$\implies b_{max} = b_{min} + n\hbar = -b_{max} + n\hbar$

$\implies 2b_{max} = n\hbar \implies b_{max} = \frac{n\hbar}{2}$

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(813) $\underline{J^2|j, m\rangle = j(j+1)\hbar^2|j, m\rangle;\ J_z|j, m\rangle = m\hbar|j, m\rangle}$

define $j = \frac{b_{max}}{\hbar} = \frac{n}{2}$

$\implies a = \hbar^2 j(j+1),\ b \equiv m\hbar$ (convenient since b increases in units of $\hbar$)

$m = -j, -j+1, \ldots, j-1, j(2j+1 state)$

$\underline{recall\ :}J^2|a, b\rangle = a|a, b\rangle;\ a = b_{max}(b_{max} + \hbar)$

$b_{max} = \frac{n\hbar}{2}$

$\implies a = \frac{n}{2}\hbar^2(\frac{n}{2} + 1);\ motivates j \equiv \frac{n}{2}$

$\implies a = j\hbar^2(j+1)$

$\underline{recall\ :}b_{min} = b_{max} - n\hbar;\ b_{max} = \frac{n\hbar}{2}$

$\Longrightarrow\ b_{min} = \frac{n\hbar}{2} - n\hbar = -\frac{n}{2}\hbar = -j\hbar$
$and b_{min} increases in units of \hbar \Longrightarrow b = m\hbar$
$(j is a half integer or integer and all m values will be either half integers or integers) \Longrightarrow$
$\begin{cases} J^2|j,m\rangle = j(j+1)\hbar^2|j,m\rangle \\ J_z|j,m\rangle = m\hbar|j,m\rangle \end{cases}$

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(814) $\underline{\mathscr{J}(d\vec{x}')|\vec{x}'\rangle \equiv |\vec{x}' + d\vec{x}'\rangle}$
Properties:
$|\alpha\rangle \to \mathscr{J}(d\vec{x}')|\alpha\rangle = \mathscr{J}(d\vec{x}')\int d^3x'|\vec{x}'\rangle\langle\vec{x}'|\alpha\rangle$
$= \int d^3x'|\vec{x}' + d\vec{x}'\rangle\langle\vec{x}'|\alpha\rangle$
$or \int d^3x'|\vec{x}' + d\vec{x}'\rangle\langle\vec{x}'|\alpha\rangle = \int d^3x'|\vec{x}'\rangle\langle\vec{x}' - d\vec{x}'|\alpha\rangle$
$\langle\alpha|\alpha r\angle = \langle\alpha\mathscr{J}^\dagger(d\vec{x}')\mathscr{J}(d\vec{x}')|\alpha\rangle$
(translated state must be normalized)
$\Longrightarrow\ \mathscr{J}^\dagger(d\vec{x}')\mathscr{J}(d\vec{x}') = 1(unitary)$
$\mathscr{J}(d\vec{x}'')\mathscr{J}(d\vec{x}') = \mathscr{J}(d\vec{x}' + d\vec{x}'')$
$\mathscr{J}(-d\vec{x}') = \mathscr{J}^{-1}(d\vec{x}')$
$\lim_{d\vec{x}'\to 0}\mathscr{J}(d\vec{x}') = 1$

- -

(815) $\underline{\mathscr{J}(d\vec{x}') = 1 - i\vec{K}\cdot d\vec{x}'}$
we show this operator satisfies defining operators for $\mathscr{J}(d\vec{x}')$
$\mathscr{J}^\dagger(d\vec{x}')\mathscr{J}(d\vec{x}') = (1 + i\vec{K}^\dagger\cdot d\vec{x}')(1 - i\vec{K}\cdot d\vec{x}')$
$= 1 - i\vec{K}\cdot d\vec{x}' + i\vec{K}^\dagger\cdot d\vec{x}' + O[(d\vec{x}')^2]$
$\approx 1 - i(\vec{K} - \vec{K}^\dagger)\cdot d\vec{x}' \approx 1$
$\mathscr{J}(d\vec{x}'')\mathscr{J}(d\vec{x}') = (1 - i\vec{K}\cdot d\vec{x}'')(1 - i\vec{K}\cdot d\vec{x}')$
$\approx 1 - i\vec{K}\cdot(d\vec{x}' + d\vec{x}'') = \mathscr{J}(d\vec{x}' + d\vec{x}'')$

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(816) $\underline{[\vec{x}, \mathscr{J}(d\vec{x}')] = d\vec{x}'I}$ see page 76 in Sakuri
$\vec{x}\mathscr{J}(d\vec{x}')|\vec{x}'\rangle = \vec{x}|\vec{x}' + d\vec{x}'\rangle = (\vec{x}' + d\vec{x}')|\vec{x}' + d\vec{x}'\rangle$
$\mathscr{J}(d\vec{x}')\vec{x}|\vec{x}'\rangle = \vec{x}'\mathscr{J}(d\vec{x}')|\vec{x}'\rangle = \vec{x}'|\vec{x}' + d\vec{x}'\rangle$
$\Longrightarrow\ [\vec{x}, \mathscr{J}(d\vec{x}')]|\vec{x}'\rangle = d\vec{x}'|\vec{x}'d\vec{x}'\rangle \approx d\vec{x}'|\vec{x}'\rangle$
$\Longrightarrow\ [\vec{x}, \mathscr{J}(d\vec{x}')] = d\vec{x}'$

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(817) $\underline{[x_i, K_j] = i\delta_{ij}I}$
racall: $[\vec{x}\mathscr{J}(d\vec{x}')] = d\vec{x}'$
Choose $d\vec{x}'$ in direction of $\hat{x}_j$, take scalar product $\hat{x}_i$
Note: $\hat{x}_i$ is a unit vector not an operator
$\Longrightarrow\ \hat{x}_i\cdot[\vec{x}, \mathscr{J}(d\vec{x}')] = \hat{x}_i\cdot[\vec{x}, 1 - i\vec{K}\cdot d\vec{x}']$
$= -i\hat{x}_i\cdot[\vec{x}, \vec{K}\cdot d\vec{x}'] = -i(x_i\vec{K}\cdot d\vec{x}' - \vec{K}\cdot d\vec{x}'x_i)$

$= -i(x_iK_j - K_jx_i)dx'_j = \hat{x}_i \cdot d\vec{x}' = dx'_j\delta_{ij}$
$\implies [x_i, K_j] = i\delta_{ij}$

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(818) $\underline{\mathscr{J}(d\vec{x}') = 1 - i\vec{p}\cdot d\vec{x}'/\hbar}$
it appears we cans et
$\vec{K} = \frac{\vec{p}}{universalconstantwithunitsofaction} = \frac{\vec{p}}{\hbar}$
$\implies \mathscr{J}(d\vec{x}') = 1 - i\frac{\vec{p}\cdot d\vec{x}'}{\hbar}$

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$\mathscr{J}(\Delta x'\hat{x}) = \lim_N \to \infty(1 - \frac{ip_x\Delta x'}{Nx})^N = \exp(-\frac{ip_x\Delta x'}{\hbar})$

- -

$|\alpha, t_0; t\rangle = \mathscr{U}(t, t_0)|\alpha, t_0\rangle$
has similar properties to translation operator $\mathscr{J}(d\vec{x}')$
$\mathscr{U}(t_0 + dt, t_0) = 1 - i\Omega dt$
$\Omega = \frac{H}{\hbar}$
$\mathscr{U}(t_0 + dt, t_0)1 - \frac{iHdt}{\hbar};\ \mathscr{U}(t, t_0)\exp(-\frac{iH}{\hbar}(t - t_0))$

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(819) $\underline{i\hbar\frac{\partial}{\partial t}\mathscr{U}(t, t_0) = H\mathscr{U}(t, t_0)};\ i\hbar\frac{\partial}{\partial t}|\alpha, t_0;\ t\rangle = H|\alpha, t_0; t\rangle$
$\mathscr{U}(t + dt, t_0) = \mathscr{U}(t + dt, t)\mathscr{U}(t, t_0) = (1 - \frac{iHdt}{\hbar})\mathscr{U}(t, t_0)$
$\mathscr{U}(t + dt, t_0) - \mathscr{U}(t, t_0) = -i\frac{H}{\hbar}dt\mathscr{U}(t, t_0)$
$\implies i\hbar\frac{\partial}{\partial t}\mathscr{U}(t, t_0) = H\mathscr{U}(t, t_0)$
$i\hbar\frac{\partial}{\partial t}\mathscr{U}(t, t_0)|\alpha, t_0\rangle = H\mathscr{U}(t, t_0)|\alpha, t_0\rangle$
$i\hbar\frac{\partial}{\partial t}|\alpha, t_0;\ t\rangle = H|\alpha, t_0; t\rangle$

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$|\alpha\rangle_R = \mathscr{D}(R)|\alpha\rangle$

- -

(820) $\underline{\mathscr{D}(\hat{n}, d\phi) = 1 - i(\frac{\vec{J}\cdot\hat{n}}{\hbar})d\phi}$
$U_\varepsilon = 1 - iG\varepsilon$ (general infititesimal operator)
$G \to \frac{J_k}{\hbar}\varepsilon \to d\phi,\ U_\varepsilon \to \mathscr{D}(\hat{x}_k, d\phi)$
$\implies \mathscr{D}(\hat{x}_k, d\phi) = 1 - i\frac{\vec{J}\cdot\hat{x}_k}{\hbar}d\phi$
or let $\hat{n} = \hat{x}_k$
$\therefore \mathscr{D}(\hat{n}, d\phi)1 - i(\frac{\vec{J}\cdot\hat{n}}{\hbar})d\phi$

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<u>Note:</u> $\vec{J} canbe \vec{L} = \vec{x} \times \vec{p} or \vec{S}$ or both

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Note: $\forall R 3\times 3$ rotation matrix acting on $\vec{v}$
$\exists \mathscr{D}(R)$ acting on a ket in ket space

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(821) $\underline{\mathscr{D}_{\hat{n}}(\phi) = \lim_{N\to\infty}[1 - i(\frac{\vec{J}\cdot\hat{n}}{\hbar})(\frac{\phi}{N})]^N}$

$\mathscr{D}_{\hat{n}}(\phi) = \lim_{N\to\infty}[1 - i(\frac{\vec{J}\cdot\hat{n}}{\hbar})(\frac{\phi}{\hbar})]^N = \exp(-i\frac{\vec{J}\cdot\hat{n}}{\hbar}\phi)$

- -

(822) $\underline{\exp(\frac{-i\vec{\sigma}\cdot\hat{n}\phi}{2}) = \mathbf{1}\cos(\frac{\phi}{2}) - i\vec{\sigma}\cdot\hat{n}\sin\frac{\phi}{2}}$

recall: $\mathscr{D}_{\hat{n}}(\phi) = \exp(-i\frac{\vec{J}\cdot\hat{n}}{\hbar}\phi),\ \vec{J} = \vec{S} = \frac{\hbar}{2}\vec{\sigma};\ (\vec{\sigma}\cdot\vec{a})^2 = |\vec{a}|^2$

$\implies \mathscr{D}_{\hat{n}}(\phi) = \exp(-i\frac{\vec{\sigma}\cdot\hat{n}}{2}\phi) = \sum_{n=0}^{\infty}\frac{1}{n!}(-i\frac{\vec{\sigma}\cdot\hat{n}}{2}\phi)^n$

$= \sum_{n=0}\frac{1}{(2n+1)!}(-i\frac{\vec{\sigma}\cdot\hat{n}}{2}\phi)^{2n+1} + \sum_{n=0}\frac{1}{(2n)!}(-i\frac{\vec{\sigma}\cdot\hat{n}}{2}\phi)^{2n}$

$= -i\sum_n \frac{1}{(2n+1)!}(-1)^n(\vec{\sigma}\cdot\hat{n})^{2n}(\frac{\phi}{2}^{2n+1}(\vec{\sigma}\cdot\hat{n})$

$+\sum_n\frac{1}{(2n)!}(-1)^n(\vec{\sigma}\cdot\hat{n})^{2n}(\frac{\phi}{2})^{2n}$

$(\vec{\sigma}\cdot\hat{n})^{2n} = ((\vec{\sigma}\cdot\hat{n})^2)^n = |\hat{n}|^n = 1$

$\therefore \mathscr{D}_{\hat{n}}(\phi) = \mathbf{1}\cos(\frac{\phi}{2}) - i\vec{\sigma}\cdot\hat{n}\sin\frac{\phi}{2}$

- -

(823) $\underline{(\vec{\sigma}\cdot\vec{a})(\vec{\sigma}\cdot\vec{b}) = \vec{a}\cdot\vec{b} + i\vec{\sigma}\cdot(\vec{a}\times\vec{b}) \implies (\vec{\sigma}\cdot\vec{a})^2 = |\vec{a}|^2}$

$(\vec{\sigma}\cdot\vec{a})(\vec{\sigma}\cdot\vec{b}) = \sum_{i,j}\sigma_i a_i \sigma_i b_j = \sum_{i,j}(\frac{1}{2}\{\sigma_i,\sigma_j\} + \frac{1}{2}[\sigma_i,\sigma_j])a_i b_j$

recall: $\{\sigma_i,\sigma_j\} = 2\delta_{ij}\mathbf{1};\ [\sigma_i,\sigma_j] = 2i\epsilon_{ijk}\sigma_k;$

$\implies (\vec{\sigma}\cdot\vec{a})(\vec{\sigma}\cdot\vec{b}) = \sum_{i,j}(\delta_{jk} + i\epsilon_{ijk}\sigma_k)a_i b_j,\ (\vec{a}\times\vec{b})_k = \epsilon_{ijk}a_i b_j$

$= \sum_{i,j}(\delta_{ij}a_i b_j + i a_i b_j \epsilon_{ijk}\sigma_k)$

$= \sum_{ij}(\delta_{ij}a_i b_j + i(\vec{a}\times\vec{b}_k\sigma_k)$

$= \vec{a}\cdot\vec{b} + i\vec{\sigma}\cdot(\vec{a}\times\vec{b})$

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(824) $\underline{|n\rangle = |n^{(0)}\rangle + \lambda\sum_{k\neq n}|k^{(0)}\rangle\frac{V_{kn}}{E_n^{(0)}-E_k^{(0)}} + \lambda^2(\sum_{k\neq n}\sum_{\ell\neq n}\frac{|k^{(0)}\rangle V_{k\ell}V_{\ell n}}{(E_n^{(0)}-E_k^{(0)})(E_n^{(0)}-E_\ell^{(0)})} - \sum_{k\neq n}\frac{|k^{(0)}\rangle V_{nn}}{(E_n^{(0)}-E}}$

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(825) $\underline{(E_n^{(0)} - H_0)|n\rangle = (\lambda V - \Delta_n)|n\rangle}$

$H_0|n^{(0)}\rangle = E_n^{(0)}|n^{(0)}\rangle$

$(H_0 + \lambda V)|n\rangle_\lambda = E_n^{(\lambda)}|n\rangle_\lambda;\ |n\rangle \equiv |n\rangle_\lambda,\ E_n^{(\lambda)} = E_n$

$\Delta_n \equiv E_n - E_n^{(0)}$

$(E_n - H_0)|n\rangle = \lambda V|n\rangle$

$\implies (\Delta_n + E_n^{(0)} - H_0)|n\rangle = \lambda V|n\rangle$

$\implies (E_n^{(0)} - H_0)|n\rangle = (\lambda V - \Delta_n)|n\rangle$

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(826) $\underline{\langle n^{(0)}|(\lambda V - \Delta_n)|n\rangle = 0}$

<u>recall:</u> $(E_n^{(0)} - H_0)|n\rangle = (\lambda V - \Delta_n)|n\rangle$

$\langle n^{(0)}|(E_n^{(0)} - H_0)|n\rangle = E_n^{(0)}\langle n^{(0)}|n\rangle - E_n^{(0)}\langle n^{(0)}|n\rangle = 0$

$= \langle n^{(0)}|(\lambda V - \Delta_n)|n\rangle = \lambda\langle n^{(0)}|V|n\rangle - \Delta_n\langle n^{(0)}|n\rangle$

$\implies \langle n^{(0)}|n\rangle = 0???$

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(827)

(828) $\underline{|n\rangle = c_n(\lambda)|n^{(0)}\rangle + \frac{1}{E_n^{(0)} - H_0}\phi_n(\lambda V - \Delta_n)|n\rangle}$

$\phi_n \equiv 1 - |n^{(0)}\rangle\langle n^{(0)}| = \sum_k |k^{(0)}\rangle\langle k^{(0)}| - |n^{(0)}\rangle\langle n^{(0)}|$

$= \sum_{k\neq n}|k^{(0)}\rangle\langle k^{(0)}|$

we would like to invert $(E_n^{(0)} - H_0)$

but we cant do this if $(E_n^{(0)} - H_0)$ acts on $|n^{(0)}\rangle$

<u>Note:</u> $\frac{1}{E_n^{(0)} - H_0}|n^{(0)}\rangle = \frac{1}{E_n^{(0)} - E_n^{(0)}}|n^{(0)}\rangle =$ undefined but we just showed $\langle n^{(0)}|(\lambda V - \Delta_n)|n\rangle = 0$ so I'm not sure why $\frac{1}{E_n^{(0)} - H_0}$ is not well defined since it wont act on $|n^{(0)}\rangle$, but whatever.

Since $\frac{1}{E_n^{(0)} - H_0}$ might act on $|n^{(0)}\rangle$ lets take out $|n^{(0)}\rangle$, recall we can define an operator $H = \sum_{k,k'}\langle k'|H|k\rangle$ but if it is diagonal in this basis then $H = \sum_k \langle |H|k\rangle = \sum_k H|k\rangle\langle k|$

so let's do

$\frac{\phi_n}{E_n^{(0)} - H_0} = \sum_{k\neq n}\frac{1}{E_n^{(0)} - E_k^{(0)}}|k^{(0)}\rangle\langle k^{(0)}|$

<u>Note:</u> $\phi(\lambda V - \Delta_n)|n\rangle = (1 - |n^{(0)}\rangle\langle n^{(0)}|)(\lambda V - \Delta_n)|n\rangle$

$= (\lambda V - \Delta_n)|n\rangle - |n^{(0)}\rangle\langle n^{(0)}|(\lambda V - \Delta_n)|n\rangle$

<u>recall:</u> $\langle n^{(0)}|(\lambda V - \Delta_n)|n\rangle = 0$

$\implies \phi_n(\lambda V - \Delta_n)|n\rangle = (\lambda V - \Delta_n)|n\rangle$

Ansatz: $(E_n^{(0)} - H_0)|n\rangle = \phi_n(\lambda V - \Delta_n)|n\rangle$

$\implies |n\rangle = \frac{1}{E_n^{(0)} - H_0}\phi_n(\lambda V - \Delta_n)|n\rangle$?

this doesn't work because we need $|n\rangle \to |n^{(0)}\rangle$

as $\lambda \to 0$ but $|n\rangle \to 0$ currently so force it to work

$|n\rangle = c_n(\lambda)|n^{(0)}\rangle + \frac{1}{E_n^{(0)} - H_0}\phi_n(\lambda V - \Delta_n)|n\rangle$

<u>Note:</u> $\lim_{\lambda\to 0} c_n(\lambda) = 1$; $\langle n^{(0)}|n\rangle = c_n(\lambda) + \langle n^{(0)}|\frac{1}{E_n^{(0)} - H_0}\phi_n(\lambda V - \Delta_n)|n\rangle$

$= c_n(\lambda) + \langle n^{(0)}|\frac{1}{E_n^{(0)} - E_k^{(0)}}\sum_{k\neq n}|k^{(0)}\rangle\langle k^{(0)}|$

(I don't think $\frac{1}{E_n^{(0)}-H_0}$ is Hermitean?)
$(\lambda V - \Delta_n)|n\rangle$
$= c_n(\lambda) + 0 = c_n(\lambda)$
set $c_n(\lambda) = 1$
$\therefore |n\rangle = |n^{(0)}\rangle + \frac{\phi_n}{E_n^{(0)}-H_0}(\lambda V - \Delta_n)|n\rangle$

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(829) $\underline{\Delta_n = \lambda\langle n^{(0)}|V|n\rangle}$
recall: $\langle n^{(0)}|(\lambda V - \Delta_n)|n\rangle = 0$
$\therefore \lambda\langle n^{(0)}|V|n\rangle = \Delta_n$

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(830) $\underline{\Delta_n^{(N)} = \langle n^{(0)}|V|n^{(N-1)}\rangle}$
$|n\rangle = |n^{(0)}\rangle + \lambda|n^{(1)}\rangle + \lambda^2|n^{(2)}\rangle + \cdots$
$\Delta_n = \lambda\Delta_n^{(1)} + \lambda^2\Delta_n^{(2)} + \cdots$
recall: $\Delta_n = \lambda\langle n^{(0)}|V|n\rangle$
$= \lambda\langle n^{(0)}|V(|n^{(0)}\rangle + \lambda|n^{(1)}\rangle + \lambda^2|n^{(2)}\rangle + \cdots)$
$= \lambda\langle n^{(0)}|V|n^{(0)}\rangle + \lambda^2\langle n^{(0)}|V|n^{(1)}\rangle + \lambda^3\langle n^{(0)}|V|n^{(2)}\rangle + \cdots$
compare with Δ_n expansion
$\therefore \Delta_n^{(N)} = \langle n^{(0)}|V|n^{(N-1)}\rangle$

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(831) $\underline{|n^{(1)}\rangle = \frac{\phi_n}{E_n^{(0)}-H_0}V|n^{(0)}\rangle}$
recall: $|n\rangle = |n^{(0)}\rangle + \frac{\phi_n}{E_n^{(0)}-H_0}(\lambda V - \Delta_n)|n\rangle$
plug in Δ_n and $|n\rangle$ expansion
$\implies |n\rangle = |n^{(0)}\rangle + \frac{\phi_n}{E_n^{(0)}-H_0}(\lambda V - \lambda\Delta_n^{(1)} - \lambda^2\Delta_n^{(2)} - \cdots)(|n^{(0)}\rangle + \lambda|n^{(1)}\rangle + \cdots)$
$= |n^{(0)}\rangle + \frac{\phi_n}{E_n^{(0)}-H_0}\lambda V|n^{(0)}\rangle - \frac{\phi_n}{E_n^{(0)}-H_0}\lambda\Delta_n^{(1)}|n^{(0)}\rangle$
$+ \frac{\phi_n}{E_n^{(0)}-H_0}\lambda^2 V|n^{(1)}\rangle - \frac{\phi_n}{E_n^{(0)}-H_0}\lambda^2\Delta_n^{(2)}|n^{(0)}\rangle$
$= |n^{(0)}\rangle + \lambda(\frac{\phi_n}{E_n^{(0)}-H_0}V|n^{(0)}\rangle - \frac{\phi_n}{E_n^{(0)}-H_0}\Delta_n^{(1)}|n^{(0)}\rangle$
but $\phi_n|n^{(0)}\rangle = (\sum_{k\neq n}|k^{(0)}\rangle\langle k^{(0)}|)|n^{(0)}\rangle = 0$
$\therefore |n^{(1)}\rangle = \frac{\phi_n}{E_n^{(0)}-H_0}V|n^{(0)}\rangle + \lambda^2\frac{\phi_n}{E_n^{(0)}-H_0}(V - \Delta_n^{(1)})n^{(1)}\rangle$
$|n\rangle = |n^{(0)}\rangle + \lambda|n^{(1)}\rangle + \lambda^2|n^{(2)}\rangle + \cdots$
$\Delta_n = \lambda_n^{(1)} + \lambda^2\Delta_n^{(2)} + \cdots$
$|n\rangle = |n^{(0)}\rangle + \sim (\lambda V - \Delta_n)|n\rangle$
$= |n^{(0)}\rangle + \sim (\lambda V - \lambda\Delta_n^{(1)} - \lambda^2\Delta_n^{(2)} - \cdots)(|n^{(0)}\rangle + \lambda|n^{(1)}\rangle + \lambda^2|n^{(2)}\rangle + \cdots)$

$= |n^{(0)}\rangle + \sim (\lambda V - \lambda\Delta_n^{(1)}) + \sim (\lambda^2 V|n^{(1)}\rangle \Delta_n^{2)} \lambda^2 |n^{(1)}\rangle)$
$= |n^{(0}\rangle + \sim (\lambda V - \lambda\Delta_n^{0)}) + \sim \lambda^2 (V - \Delta_n^{(1)})|n^{(1)}\rangle + \cdots$
$|n^{(2)}\rangle = \sim (V - \Delta_n^{(1)})|n^{(1)}\rangle$
$= \sim V|n^{(1)}\rangle - \sim \Delta_n^{(1)}|n^{(1)}\rangle$
$= \frac{\phi_n}{E_n^{(0)} - H_0} V \frac{\phi_n}{E_n^{(0} - H_0} V|n^{(0)}\rangle - \frac{\Phi_n}{E_n^{(0)} - H_0} \langle N^{(0)}|V|N^{(0)}\rangle |n^{(1)}\rangle$
$= \frac{\phi_n}{E_n^{(0)} - H_0} V \frac{\phi_n}{E_n^{(0)} - H_0} V|n^{(0)}\rangle - \frac{\phi_n}{E_n^{(0)} - H_0} \langle n^{(0)}|V|n^{(0)}\rangle \frac{\phi_n}{E_n^{(0)} - H_0} V|n^{(0)}\rangle$

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(832) $\underline{|n^{(2)}\rangle = \frac{\phi_n}{E_n^{(0)} - H_0} V \frac{\phi_n}{E_n^{(0)} - H_0} V|n^{(0)}\rangle - \frac{\phi_n}{E_n^{(0)} - H_0} \langle n^{(0)}|V|n^{0}\rangle \frac{\phi_n}{E_n^{(0)} - H_0} V|n^{(0)}\rangle}$

<u>recall:</u> $|n^{(2)}\rangle = \frac{\phi_n}{E_n^{(0)} - H_0} (V - \Delta_n^{(1)})|n^{(1)}\rangle; l,\ |n^{(1)}\rangle = \frac{\phi_n}{E_n^{(0)} - H_0} V|n^{0}\rangle$
$\Delta_n^{(1)} = \langle n^{(0)}|V|n^{(0)}\rangle$
Plug in
$\therefore |n^{(2)}\rangle = \frac{\phi_n}{E_n^{(0)} - H_0} V \frac{\phi_n}{E_n^{(0)} - H_0} V|n^{(0)}\rangle - \frac{\phi_n}{E_n^{(0)} - H_0} \langle n^{(0)}|V|n^{(0)}\rangle \frac{\phi_n}{E_n^{(0)} - H_0} V|n^{(0)}\rangle$

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(833) $\underline{\Delta_n^{(1)} = \langle n^{(0)}|V|n^{(0)}\rangle;\ \Delta_n^{(2)} = \langle n^{(0)}|V \frac{\phi_n}{E_n^{(0)} - H_0} V|n^{(0)}\rangle}$

<u>recall:</u> $\Delta_n^{(2)} = \langle n^{(0)}|V|n^{(1)}\rangle$
$= \langle n^{(0)}|V \frac{\phi_n}{E_n^{(0)} - H_0} V|n^{(0)}\rangle$

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(834) $\underline{\Delta_n \equiv E_n - E_n^{(0)} = \lambda V_{nn} + \lambda^2 \sum_{k \neq n} \frac{|V_{nk}|^2}{E_n^{(0)} - E_k^{(0)}} + \cdots}$

<u>recall:</u> $\Delta_n = \lambda \langle n^{(0)}|V|n\rangle;\ |n\rangle = |n^{(0)}\rangle + \lambda \frac{\phi_n}{E_n^{(0)} - H_0} V|n^{(0)}\rangle + \cdots$
$\implies \Delta_n = \lambda \langle n^{(0)}|V(|n^{()}\rangle + \lambda \frac{\phi_n}{E_n^{(0)} - H_0} V|n^{(0)}\rangle + \cdots)$
$= \lambda \langle n^{(0)}|V|n^{(0)}\rangle + \lambda^2 \langle n^{(0)}|V \frac{\phi_n}{E_n^{(0)} - H_0} V|n^{(0)}\rangle + \cdots$
$\langle n^{(0)}|V \frac{\phi_n}{E_n^{(0)} - H_0} V|n^{(0)}\rangle$
$= \langle n^{(0)}|V \frac{1}{E_n^{(0)} - H_0} (\sum_{k \neq n} |k^{90)}\rangle \langle k^{(0)}|) V|n^{(0)}\rangle$
$= \sum_{k \neq n} \frac{1}{E_n^{(0)} - E_k^{(0)}} \langle n^{(0)}|V|k^{(0)}\rangle \langle k^{(0)}|V|n^{(0)}\rangle$
$= \sum_{k \neq n} \frac{1}{E_n^{(0)} - E_k^{(0)}} V_{nk} V_{nk}^*$
$= \sum_{k \neq n} \frac{|V_{nk}|^2}{E_n^{(0)} - E_k^{(0)}}$
$\therefore \Delta_n = \lambda V_{nn} + \lambda^2 \sum_{k \neq n} \frac{|V_{nk}|^2}{E_n^{(0)} - E_k^{(0)}} + \cdots$
where $V_{nk} \equiv \langle n^{(0)}|V|k^{(0)}\rangle$

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(835) $\underline{|n\rangle = |n^{(0)}\rangle + \lambda \sum_{k\neq n} |k^{(0)}\rangle \frac{V_{kn}}{E_n^{(0)} - E_k^{(0)}}}$

(836) $\underline{+\lambda^2 (\sum_{k\neq n}\sum_{\ell\neq n} \frac{|k^{(0)}\rangle V_{k\ell}V_{\ell n}}{(E_n^{(0)}-E_k^{(0)})(E_n^{(0)}-E_\ell^{(0)})} - \sum_{k\neq n}\frac{|k^{(0)}\rangle V_{nn}V_{kn}}{(E_n^{(0)}-E_k^{(0)})^2}) + \cdots}$

recall: $|n\rangle = |n^{90)}\rangle + \lambda \frac{\phi_n}{E_n^{(0)}-H_0} V|n^{(0)}\rangle$
$+ \lambda^2 (\frac{\phi_n}{E_n^{(0)}-H_0} V \frac{\phi_n}{E_n^{(0)}-H_0} V|n^{(0)}\rangle - \frac{\phi_n}{E_n^{(0)}-H_0}\lambda n^{(0)}|V|n^{(0)}\rangle$
$\frac{\phi_n}{E_n^{(0)}-H_0} V|n^{(0)}\rangle) + \cdots$
$= |n^{(0)}\rangle + \lambda \frac{1}{E_n^{(0)}-H_0}(\sum_{k\neq n}|k^{(0)}\rangle\langle k^{(0)}|)V|n^{(0)}\rangle$
$+\lambda^2 (\frac{1}{E_n^{(0)}-H_0}(\sum_{k\neq n}|k^{(0)}\rangle\langle k^{(0)}|)V\frac{1}{E_n^{(0)}-H_0}(\sum_{k\neq n}|k^{(0)}\rangle\langle k^{(0)})V|n^{(0)}\rangle$
$-\frac{1}{E_n^{(0)}-H_0}(\sum_{k\neq n}|k^{(0)}\rangle\langle k^{(0)}|)\langle n^{(0)}|V|n^{(0)}\rangle \frac{1}{E_n^{(0)}-H_0}(\sum_{\ell\neq n}|\ell^{(0)}\rangle\langle \ell^{(0)}|)V|n^{(0)}\rangle$
$= |n^{(0)}\rangle + \lambda \sum_{k\neq n} \frac{|k^{(0)}\rangle V_{kn}}{E_n^{(0)}-E_k^{(0)}}$
$+ \lambda^2\{(\sum_{k\neq n}\frac{|k^{(0)}\rangle\langle k^{(0)}|}{E_n^{(0)}-E_k^{(0)}})V(\sum_{\ell\neq n}\frac{|\ell^{(0)}\rangle\langle \ell^{(0)}|}{E_n^{(0)}-E_\ell^{(0)}})V|n^{(0)}\rangle$
$- (\sum_{k\neq n}\frac{|k^{(0)}\rangle\langle k^{(0)}|}{E_n^{(0)}-E_k^{(0)}})V_{nn}(\sum_{\ell\neq n}\frac{|\ell^{(0)}\rangle\langle \ell^{(0)}|}{E_n^{(0)}-E_\ell^{(0)}})V|n^{(0)}\rangle\}$
$O(\lambda^2) : \sum_{k\neq n}\sum_{\ell\neq n}\frac{|k^{(0)}\rangle V_{k\ell}V_{\ell n}}{(E_n^{(0)}-E_k^{(0)})(E_n^{(0)}-E_\ell^{(0)})}$
$- \sum_{k\neq n}\sum_{\ell\neq n}\frac{|k^{(0)}\rangle \delta_{k\ell}V_{nn}V_{\ell n}}{(E_n^{(0)}-E_k^{(0)})(E_n^{(0)}-E_\ell^{(0)})}$
$= \sum_{k\neq n}\sum_{\ell\neq n}\frac{|k^{(0)}\rangle V_{k\ell}V_{\ell n}}{(E_n^{(0)}E_k^{(0)})(E_n^{(0)}-E_\ell^{(0)})}$
$- \sum_{k\neq n}\frac{|k^{(0)}\rangle V_{nn}V_{kn}}{(E_n^{()}-E_k^{(0)})^2}$
$\therefore |n\rangle = |n^{(0)}\rangle + \lambda\sum_{k\neq n}|k^{(0)}\rangle\frac{V_{kn}}{E_n^{(0)}-E_k^{(0)}}$
$+ \lambda^2(\sum_{k\neq n}\sum_{\ell\neq n}\frac{|k^{(0)}\rangle V_{k\ell}V_{\ell n}}{(E_n^{(0)}-E_k^{(0)})(E_n^{(0)}-E_\ell^{(0)})} - \sum_{k\neq n}\frac{|k^{(0)}\rangle V_{nn}V_{kn}}{(E_n^{(0)}-E_k^{(0)})^2})$
$+\cdots$

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Classical mechanics

(837) $\underline{\frac{d}{dt}(\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} = Q_j}$

System in equillibrium
$\implies \vec{F}_i = 0$ (net force on each particle)
$\implies \sum_i \vec{F}_i \cdot \delta\vec{r}_i = 0$
$\vec{F}_i = \vec{F}_i^{(a)} + \vec{f}_i,\ \vec{F}_i^{(a)} \sim applied,\ \vec{F}_i \sim$ constraint
$\implies \sum_i \vec{F}_i^{(a)}\cdot\delta\vec{r}_i + \sum_i \vec{f}_i\delta\vec{r}_i = 0$
only consider $\sum_i \vec{f}_i\cdot\delta\vec{r}_i = 0$
(think of a particle sliding on a table top, the normal force is perpendicular to the displacement)

$\implies \sum_i \vec{F}_i^{(a)} \cdot \delta\vec{r}_i = 0$
$\vec{F}_i = \dot{\vec{p}}_i$ (not in equillibrium)
$\implies \vec{F}_i - \dot{\vec{p}}_i = 0$ (new "effective" force)
$\implies \sum_i (\vec{F}_i - \dot{\vec{p}}_i) \cdot \delta\vec{r}_i = 0;\ \vec{F}_i = \vec{F}_i^{(a)} + \vec{f}_i$
$\implies \sum_i (\vec{F}_i^{(a)} + \vec{f}_i - \dot{\vec{p}}_i) \cdot \delta\vec{r}_i = 0$
$\implies \sum_i (\vec{F}_i^{(a)} - \dot{\vec{p}}_i) \cdot \delta\vec{r}_i = 0$
$\vec{r}_i = \vec{r}_i(q_j, t)$
$\implies \vec{v}_i \equiv \frac{d\vec{r}_i}{dt} = \sum_k \frac{\partial\vec{r}_i}{\partial q_k}\dot{q}_k + \frac{\partial\vec{r}_e i}{\partial t}$
$similarly \delta\vec{r}_i = \sum_j \frac{\partial\vec{r}_i}{\partial q_j}\delta q_j$
$\sum_i \vec{F}_i \cdot \delta\vec{r}_i = \sum_{i,j} \vec{F}_i \cdot \frac{\partial\vec{r}_i}{\partial q_j}\delta q_j = \sum_j (\sum_i \vec{F}_i \cdot \frac{\partial\vec{r}_i}{\partial q_j})\delta q_j$
$= \sum_j Q_j \delta q_j;\ Q_j = \sum_i \vec{F}_i \cdot \frac{\partial\vec{r}_i}{\partial q_j}$
$\sum_i \dot{\vec{p}}_i \cdot \delta\vec{r}_i = \sum_i m_i \vec{r}_i \cdot \delta\vec{r}_i = \sum_i m_i \ddot{\vec{r}}_i \cdot (\sum_j \frac{\partial\ddot{\vec{r}}_i}{\partial q_j}\delta q_j) = \sum_{i,j} m_i \ddot{\vec{r}}_i \cdot \frac{[}{\partial}\vec{r}q_j \delta q_j$
<u>Note:</u> $\sum_i m_i \ddot{\vec{r}}_i \cdot \frac{\partial\vec{r}_i}{\partial q_j} = \sum_i [\frac{d}{dt}(m_i \dot{\vec{r}}_i \cdot \frac{\partial\vec{r}_i}{\partial q_j}) - m_i \dot{\vec{r}}_i \cdot \frac{d}{dt}(\frac{\partial\vec{r}_i}{\partial q_j})]$
$\frac{d}{dt}\frac{\partial\vec{r}_i}{\partial q_j} = \sum_k \frac{\partial^2\vec{r}_i}{\partial q_j \partial q_k}\dot{q}_k + \frac{\partial^2\vec{r}_i}{\partial q_j \partial t}$
$= \frac{\partial}{\partial q_j}(\sum_k \frac{\partial\vec{r}_i}{\partial q_k}\dot{q}_k + \frac{\partial\vec{r}_i}{\partial t}) = \frac{\partial\vec{v}_i}{\partial q_j}$
$\frac{\partial\vec{v}_i}{\partial \dot{w} q_j} = \sum_k \frac{\partial\vec{r}_i}{\partial q_k}\delta_{kj} = \frac{\partial\vec{r}_i}{\partial q_j}$
$\implies \sum_i m_i \ddot{\vec{r}}_i \cdot \frac{\partial\vec{r}_i}{\partial q_j} = \sum_i [\frac{d}{dt}(m_i \vec{v}_i \cdot \frac{\partial\vec{v}_i}{\partial\dot{q}_j}) - m_i \vec{v}_i \cdot \frac{\partial\vec{v}_i}{\partial q_j}]$
<u>recall:</u> $\sum_i (\vec{F}_i - \dot{\vec{p}}_i) \cdot \delta\vec{r}_i = 0$
$\implies \sum_i \vec{p}_i \cdot \delta\vec{r}_i - \sum_i Q_i \delta q_i = 0$
$\implies \sum_i (m_i (\frac{d}{dt}\vec{v}_i) \cdot \delta\vec{r}_i - Q_i \delta q_i) = 0$
$\vec{v}_i \frac{\partial\vec{v}_i}{\partial\dot{q}_j} = \frac{1}{2}\frac{\partial}{\partial\dot{q}_j}v_i^2$
$(\frac{d}{dt}\vec{v}_i) \cdot \delta\vec{r}_i = (\frac{d}{dt}\vec{v}_i) \cdot \sum_j \frac{\partial\vec{r}_i}{\partial q_j}\delta q_j$
$= \sum_j (\frac{d}{dt}\vec{v}_i) \cdot \frac{\partial\vec{r}_i}{\partial q_j}\delta q_j$
$= \sum_j (\frac{d}{dt}(\vec{v}_i \cdot \frac{\partial\vec{r}_i}{\partial q_j}) - \frac{d}{dt}(\frac{\partial\vec{r}_i}{\partial q_j}) \cdot \vec{v}_i)\delta q_j$
$\sum_i (\frac{d}{dt}(\vec{v}_i \cdot \frac{\partial\vec{v}_i}{\partial\dot{q}_j}) - \frac{d}{dt}(\frac{\partial\vec{r}_i}{\partial\dot{q}_j}) \cdot \vec{v}_i)\delta q_j$
$= \sum_i (\frac{d}{dt}(\vec{v}_i \cdot \frac{\partial\vec{r}_i}{\partial\dot{q}_j}) - \frac{\partial\vec{v}_i}{\partial q_j} \cdot \vec{v}_i)\delta q_j$
$= \sum_i (\frac{d}{dt}(\frac{\partial}{\partial\dot{q}_j}\frac{1}{2}v_i^2) - \frac{\partial}{\partial q_j}\frac{1}{2}v_i^2)\delta q_j$

$\implies \sum_i (v\vec{F}_i - \dot{\vec{p}}_i) \cdot d\delta\vec{r}_i$
$= \sum_j (\frac{d}{dt}[\frac{\partial}{\partial\dot{q}_j}(\sum_i \frac{1}{2}m_i v_i^2)] - \frac{\partial}{\partial q_j}(\sum_i \frac{1}{2}m_i v_i^2) - Q_j)\delta q_j$
$= \sum_i [\frac{d}{dt}(\frac{\partial T}{\partial\dot{q}_j}) - \frac{\partial T}{|partial q_j} - Q_j]\delta q_j = 0$
$\implies \frac{d}{dt}(\frac{\partial T}{\partial\dot{q}_j}) - \frac{\partial T}{\partial q_j} = Q_j$

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(838) $\underline{L = \frac{m_1+m_2}{2}\dot{\vec{R}}^2 + \frac{1}{2}\frac{m_1 m_2}{m_1+m_2}\dot{\vec{r}}^2 - U(\vec{r}, \dot{\vec{r}}, \dots)}$

assume $L = T(\vec{r}_1, \vec{r}_2) - U(\vec{r}_2 - \vec{r}_1, \dot{\vec{r}}_2 - \dot{\vec{r}}_1)$

$\implies$ 6 deg of freedom, choose generalized coordinates to be $\vec{R} = \vec{R}_{cm} and \vec{r} = \vec{r}_2 - \vec{r}_1$

$\implies L = T(\dot{\vec{R}}, \dot{\vec{r}}) - U(\vec{r}, \dot{\vec{r}}, \dots)$

$T = \frac{1}{2}(m_1 + m_2)\dot{\vec{R}}^2 + T'$

$T' = \frac{1}{2}m_1\dot{\vec{r}}_1'^2 + \frac{1}{2}m_2\dot{\vec{r}}_2'^2,\ \vec{r}_i'(relativetocm)$

$use \vec{r}_1 = \vec{R} + \vec{r}_1\ \vec{r}_2 = \vec{R} + \vec{r}_2';\ \vec{R} = \frac{1}{m_1+m_2}(m_1\vec{r}_1 + m_2\vec{r}_2)$

then solve first 2 for $\vec{r}_1',\ \vec{r}_2'$

$\implies \begin{cases} \vec{r}_1' = \frac{m_2}{m_1+m_2}(\vec{r}_1 - \vec{r}_2) = -\frac{m_2}{m_1+m_2}\vec{r} \\ \vec{r}_2' = \frac{m_1}{m_1+m_2}(\vec{r}_2 - \vec{r}_1) \end{cases}$ (mathematica)

plug into T' (mathematica)

$\implies T' = \frac{1}{2}\frac{m_1 m_2}{m_1+m_2}\dot{\vec{r}}^2$

$\therefore L = \frac{m_1+m_2}{2}\dot{\vec{R}}^2 + \frac{1}{2}\frac{m_1 m_2}{m_1+m_2}\dot{\vec{r}}^2 - U(\vec{r}, \dot{\vec{r}}, \dots)$

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Note: $\vec{R}$ is at rest or moving uniformly due to U and will not appear in EOM for $\vec{r}$, so drop it

$\implies L = \frac{1}{2}\mu\dot{\vec{r}}^2 - U(\vec{r}, \dot{\vec{r}}, \dots)$

- -

(839) $\underline{\ell = mr^2\dot{\theta} = const.}$

$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r)$ (central force

$\frac{\partial L}{\partial \theta} = \frac{dx}{Dt}\frac{\partial L}{\partial \dot{\theta}}$

$\implies \frac{d}{dt}p_\theta = 0 \implies p_\theta = \ell = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}$

$\therefore \ell = mr^2\dot{\theta}$ Note: $\ell = |\vec{r} \times \vec{p}| = rm(v\sin\theta) = mrv_\theta = mr^2\dot{\theta}$

- -

(840) $\underline{m\ddot{r} - \frac{\ell^2}{mr^3} = f(r)}$

$\frac{\partial L}{\partial r} = \frac{d}{dt}\frac{\partial L}{\partial \dot{r}}$

$\implies mr\dot{\theta}^2 - V'(r) = m\ddot{r}$

$m\ddot{r} - mr\dot{\theta}^2 = -V'(r) = f(r),\ f(r) \sim$ force along $\hat{r}$

recall: $\ell = mr^2\dot{\theta} \implies \dot{\theta} = \frac{\ell}{mr^2},\ plugin$

$\implies m\ddot{r} - \frac{\ell^2}{mr^3} = f(r)$

- -

(841) $\underline{\frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + V(r) = const.}$

recall: $m\ddot{r} - \frac{\ell^2}{mr^3} = -\frac{\partial V}{\partial r} = f(r)$

$\implies m\ddot{r} = -\frac{d}{dr}(V + \frac{1}{2}\frac{\ell^2}{mr^2})$

$\implies m\dot{r}\ddot{r} = -\dot{r}\frac{d}{dr}(V + \frac{1}{2}\frac{\ell^2}{mr^2})$
$\implies \frac{d}{dt}(\frac{1}{2}m\dot{r}^2) = -\frac{d}{dt}(V + \frac{1}{2}\frac{\ell^2}{mr^2})$
$\implies \frac{d}{dt}(\frac{1}{2}m\dot{r}^2 + \frac{1}{2}\frac{\ell^2}{mr^2} + V) = 0$
$\implies \frac{1}{2}m\dot{r}^2 + \frac{1}{2}\frac{|ell^2}{mr^2} + V = \text{const.}$
but $\frac{\ell^2}{mr^2} = \frac{m^2r^4\dot{\theta}^2}{mr^2} = mr^2\dot{\theta}^2$
$\therefore \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + V(r) = E = \text{const.}$

- -

(842) $\dot{r} = \sqrt{\frac{2}{m}(E - V - \frac{\ell^2}{2mr^2})}$

<u>recall:</u> $\frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + V(r) = E$
$\implies \frac{1}{2}m\dot{r}^2 + \frac{1}{2}\frac{\ell^2}{mr^2} + V = E$
$\implies \frac{1}{2}m\dot{r}^2 = E - V - \frac{1}{2}\frac{\ell^2}{mr^2}$
$\therefore \dot{r} = \sqrt{\frac{2}{m}(E - V - \frac{\ell^2}{2mr^2})}$
$\implies dt = \frac{dr}{\sqrt{\frac{2}{m}(E - V - \frac{\ell^2}{2mr^2})}}$
$\therefore t = \int_{r_0}^{r} \frac{dr}{\sqrt{\frac{2}{m}(E - V - \frac{\ell^2}{2mr^2})}}$

- -

(843) $d\theta] = \frac{\ell dr}{\sqrt{mr^2\frac{2}{m}(E - V(r) - \frac{\ell^2}{2mr^2})}}$

<u>recall:</u> $\ell = mr^2\dot{\theta} \implies \ell dt = d\theta;\ dt = \frac{dr}{\sqrt{\frac{2}{m}(E - V - \frac{\ell^2}{2mr^2})}}$
$\therefore d\theta = \frac{\ell dr}{\sqrt{mr^2\frac{2}{m}(E - V(r) - \frac{\ell^2}{2mr^2})}}$

- -

(844) $\theta = \theta_0 - \int_{u_0}^{u} \frac{du}{\sqrt{\frac{2mE}{\ell^2} - \frac{2ma}{\ell^2}u^{-n-1} - u^2}}$ (potential $\sim r^{n+1}$)

$d\theta = \frac{\ell dr}{mr^2\sqrt{\frac{2}{m}(E - V(r) - \ell^2 2mr^2)}}$
$= \frac{\ell dr}{mr^2\sqrt{\frac{2}{m}\frac{\ell^2}{2m}(\frac{2mE}{\ell^2} - \frac{2mV}{\ell^2} - \frac{1}{r^2})}}$
$= \frac{dr}{r^2\sqrt{\frac{2mE}{\ell^2} - \frac{2mV}{\ell^2} - \frac{1}{r^2}}}$
$u - \frac{1}{r},\ du = -\frac{1}{r^2}dr$
$\therefore \theta = \theta_0 - \int_{u_0}^{u} \frac{du}{\sqrt{\frac{2mE}{\ell^2} - \frac{2m}{\ell^2}V - u^2}}$
most important potentials: $V = ar^{n+1}(force \sim r^n)$
$n = 0, 1, \ldots$
$\therefore \theta = \theta_0 - \int_{u_0}^{u} \frac{du}{\sqrt{\frac{2mE}{\ell^2} - \frac{2ma}{\ell^2}u^{-n-1} - u^2}}$

- -

(845) $\frac{1}{r} = \frac{mk}{\ell^2}(1 + \sqrt{1 + \frac{2E\ell^2}{mk^2}}\cos(\theta - \theta'))$(gtravitational force)

$\underline{\theta(u)\text{ equation above set } n = -2, a = k}$

$\implies \theta = \theta' - \int \frac{du}{\sqrt{\frac{2mE}{\ell^2} - \frac{2mku}{\ell^2} - u^2}}$

<u>Note:</u> $\theta' \neq \theta_0$

use $\int \frac{dx}{\sqrt{\alpha + \beta x + \gamma x^2}} = \frac{1}{\sqrt{-\gamma}} \arccos - \frac{\beta + 2\gamma x}{\sqrt{q}}$

$q = \beta^2 - 4\alpha\gamma$

$\implies \theta = \theta' - \arccos \frac{\frac{\ell^2 u}{mk} - 1}{\sqrt{1 + \frac{2E\ell^2}{mk^2}}}$ (solve for u $= \frac{1}{r}$)

$\therefore \frac{1}{r} = \frac{mk}{\ell^2}(1 + \sqrt{1 + \frac{2E\ell^2}{mk^2}}\cos(\theta - \theta'))$ (orbit equation)

- -

<u>Note:</u> general equation of conic w one focus at origin

$\frac{1}{r} = C[1 + e\cos(\theta - \theta')],\ e \sim$ eccentricity

$\implies e = \sqrt{1 + \frac{2E\ell^2}{mk^2}}$

$e > 1$ (hyperbola) $\implies E > 0$

$e = 1$ (parabola) $\implies E = 0$

$e < 1(ellipse) \implies E < 0$

$e = 0(circle) \implies E = -\frac{mk^2}{2\ell^2}$

- -

$\implies \frac{1}{r} = \frac{mk}{\ell^2}(1 + e\cos(\theta - \theta'))$

- -

(846) $\underline{V' = -\frac{k}{r} + \frac{\ell^2}{2mr^2};\ w/f' = -\frac{\partial V'}{\partial r}}$

<u>recall:</u> $m\ddot{r} - \frac{\ell^2}{mr^3} = f(r) = -\frac{\partial V}{\partial r}$

$\implies m\ddot{r} = -\frac{\partial V}{\partial r} - \frac{\partial}{\partial r}\frac{\ell^2}{2mr^2} = -\frac{\partial}{\partial r}(V + \frac{\ell^2}{2mr^2})$

$\implies f' \equiv m\ddot{r} = -\frac{\partial}{\partial r}(V + \frac{\ell^2}{2mr^2}) \equiv -\frac{\partial}{\partial r}V'$

- -

$f' = f + \frac{\ell^2}{mr^2} Let f = -\frac{k}{r^2}$

$\implies V' \equiv -\frac{k}{r} + \frac{\ell^2}{2mr^2}$

- -

$\beta^2 = 3 + \frac{r}{f}\frac{df}{dr}|_{r=r_0}$ (derive)

- -

spose we shoot a beam of particles at V repulsive, fixed particle s $\sim$ impact param

- -

(847) $\underline{\ell = mv_0 s = s\sqrt{2mE}}$
$\ell = |\vec{r} \times \vec{p} = rp\sin\theta = mv_0 s$
$E = \frac{1}{2}mv_0^2$ (at $r = \infty V(\infty) = 0$)
$\implies v_0 = \sqrt{\frac{2E}{m}}$
$\implies mv_0 s = s\sqrt{2mE}$
$\therefore \ell = mv_0 s = s\sqrt{2mE}$

- -

(848) $\underline{\sigma(\Theta) = \frac{s}{\sin\Theta}|\frac{ds}{d\Theta}|}$
Note: Θ is the scattering angle from horizontal and is $0 < \Theta < \pi\Theta$ is similar to spherical angle θ so this tells us scattering can happen in any direction

- -

$\frac{dN}{dt} \propto I$
constant of proportionality is $d\sigma$ or $\sigma d\Omega$
$\sigma(\Omega)d\Omega = \frac{\text{Number of particles scattered into } d\Omega \text{ per unit time}}{\text{incident intensity}}$
Note: the beam incident intensity is the intensity (or flux) perpendicular to beam
$d\Omega = \frac{dA}{r^2} = 2\pi\sin\Theta d\Theta$
incoming particles in a shell corresponds to an outgoing shell / time
$2\pi sI|ds|$ is a ring of incoming particles, s is the impact parameter $2\pi sI|ds| = I\sigma(\Omega)|d\Omega| = I2\pi\sin\Theta|d\Theta|\sigma(\Theta)$
$\implies \sigma(\Theta) = \frac{s}{\sin\Theta}|\frac{ds}{d\Theta}|$

- -

(849) $\underline{\Theta(s) = \pi - 2\int_0^{u_m} \frac{sdu}{\sqrt{1 - \frac{V(u)}{E} - s^2u^2}}}$
Note: draw a line to closest approach, the angle to incoming angle to outgoing angle are equal due to time symmetry
$\implies 2\Psi + \Theta = \pi \implies \Theta = \pi - 2\Psi$
recall: $\theta = \int_{r_0}^{r} \frac{dr}{r^2\sqrt{\frac{2mE}{\ell^2} - \frac{2mV}{\ell^2} - \frac{1}{r^2}}} + \theta_0$
ψ is angle between incoming direction $\theta_0 = \pi$ and r_m (distance of closest approach) thus
$\theta_0 = \pi \implies r_0 = \infty when r = r_m \implies \theta = \pi - \Psi$

$\implies \pi - \Psi = -\int_{r_m}^{\infty} \frac{dr}{r^2\sqrt{\frac{2mE}{\ell^2}-\frac{2mV}{\ell^2}-\frac{1}{r^2}}} + \pi$

$\implies \Psi = \int_{r_m}^{\infty} \frac{dr}{r^2\sqrt{\frac{2mE}{\ell^2}-\frac{2mV}{\ell^2}-\frac{1}{r^2}}}$

Note: Θ and θ are similar but Θ is fixed (angle of outgoing radius) θdescribes its path.

recall: $\Theta = \pi - 2\Psi$

$\therefore \Theta(s) = \pi - 2\int_{r_m}^{\infty}(\frac{dr}{r^2\sqrt{\frac{2mE}{\ell^2}-\frac{2mV}{\ell^2}-\frac{[}{1}r^2}})$

recall: $s = \frac{\ell}{\sqrt{2mE}}$

$\frac{dr}{r^2\sqrt{\frac{2mE}{\ell^2}-\frac{2mVE}{\ell^2 E}-\frac{1}{r^2}}} = \frac{dr}{\frac{r^2}{r}\sqrt{\frac{r^2}{s^2}-\frac{r^2}{Es^2}-1}}$

$= \frac{sdr}{r\sqrt{r^2-\frac{Vr^2}{E}-s^2}} = \frac{sdr}{rr^2(1-\frac{V(r)}{E})-s^2}$

$r \to \frac{1}{u}$

$\therefore \Theta(s) = \pi - 2\int_0^{u_m} \frac{sdu}{\sqrt{1-\frac{V(u)}{E}-s^2u^2}}$

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(850) $\epsilon = \sqrt{1+(\frac{2Es}{ZZ'e^2})^2}$

use $f = \frac{ZZ'e^2}{r^2}$ (repulsive)

$\implies k = -ZZ'e^2$

recall: $\epsilon = \sqrt{1+\frac{2E\ell^2}{mk^2}}$; $\ell = \sqrt{2mE}s$

$\implies \epsilon = \sqrt{1+\frac{2E\ell^2}{m(ZZ'e^2)^2}} = \sqrt{1+\frac{2E(2mE)r^2}{m(ZZ'e^2)^2}}$

$= \sqrt{1+(\frac{2Es}{ZZ'e^2})^2}$

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(851) $\frac{1}{r} = \frac{mZZ'e^2}{\ell^2}(\epsilon\cos\theta - 1)$

recall: $\frac{1}{r} = \frac{mk}{\ell^2}(1+\epsilon\cos(\theta-\theta'))$

$\theta' = \pi \implies \theta = 0$ is pariapsis (this is because $k < 0$ here)

$k = -ZZ'e^2$

$\implies \frac{1}{r} = \frac{mZZ'e^2}{\ell^2}(\epsilon\cos\theta - 1)$

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(852) $\cos\Phi = \frac{1}{\epsilon}$

when $r \to \infty$, $\theta = \Psi$ (see my diagram in book)

$\implies \epsilon\cos\theta - 1 = 0$

$\implies \cos\Phi = \frac{1}{\epsilon}$

Math Phys

turning differential operator into matrix example

Suppose we have the differential operator

$\hat{p} = \frac{d}{dx} + a$ acting on the space spanned by $(1, x)$ then the matrix is obtained by acting on the basis, extracting the matrix then transposing the result, we transpose it because it works (lol)
$(\frac{d}{dx} + a)1 = a \cdot 1 + 0 \cdot x;\ (\frac{d}{dx} + a)x = 1 + a \cdot x$
$\implies M = \begin{pmatrix} a & 0 \\ 1 & a \end{pmatrix}$
now $M \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \hat{p} \cdot 1 \neq a$
but $M^\dagger \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = a$
so $\hat{p} \to M^\dagger$

- -

greens functions (ODE's)
initial value
$L_t y(t) = f(t) solve L_t g[t;\ t_p] = \delta(t - t_p)$
$\implies y(t) = \int_{-\infty}^{t} g[t; t_p] f(t_p) dt_p$

- -

(853) Example
$m''(t) = -kx(t) + f(t)$
$x(0) = 0,\ x'(0) = 0,\ f(t) = 0,\ t < 0$
$mg''(t, t_p) + kg(t, t_p) = \delta(t - t_p)$
$if t \neq t_p$
$\implies mg''(t, t_p) + kg(t, t_p) = 0$
$\implies g(t, t_p) = A\cos(\sqrt{\frac{k}{m}}t) + B\sin(\sqrt{\frac{k}{m}}t) t > t_p$
Note: t_p is the time the pulse is delivered so for $t < t_p$
$g(t, t_p) = 0 t < t_p$
to balance $\delta g''$ must be delta function g' is step and g is cts (kink), at $t = t_p$
$\implies g(t, t_p) = \begin{cases} 0 t < t_p \\ A\cos\sqrt{\frac{k}{|}m t} + B\sin\sqrt{\frac{k}{m}}t \end{cases}$
integrate $L_t g = \delta(t - t_p)$ to get discontinuity condition t_p and another condition $g(t_p - \epsilon =; t_p) = g(t_p + \epsilon; t_p) as \epsilon \to 0$
which allows us to solve for A, B
$\implies y(t) = \int_{-\infty}^{t} g(t; t_p) f(t_p) dt_p$

- -

<u>greens function, boundary cond</u>
$L_x y(x) = q(x);\ y(0) = 0,\ y(L) = 0$
$L_x g(x; x_p) = \delta(x - x_p);\ g(0; x_p) = 0,\ g(L; x_p) = 0$
$y(x) = \int_0^L g(x; x_p) q(x_p) dx_p$
easy to show this satisfies boundary condition
and $L_x y(x) = \int_0^L \delta(x - x_p) q(x_p) dx_p = q(x)$

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(854) <u>Example</u>
$\frac{d^2T}{dx^2} = -\frac{1(x)}{\kappa} T(0) = 0,\ T(L) = 0$
$-\kappa \frac{d^2}{dex}^2 g(x; x_p) = \delta(x - x_p),\ g(0; x_p) = 0,\ g(L; x_p) = 0$
solve for $x < x_p$:
$\implies g''(x; x_p) = 0;\ g(0; x_p) = 0$
$\implies g(x; x_p) = cx$
solve for $x > x_p$:
$g''(x; x_p) g(L; x_p) = 0$
$ax + b = g(x; x_p)$
$g(L; x_p) = aL + b = 0 \implies b = -aL$
$\implies g(x; x_p) = ax - aL = a(x - L),\ x > x_p$
$g''(x; x_p) must be \delta function \implies g' step g cts at x_p$
$\implies -\kappa \int_{x_p - \epsilon}^{x_p + \epsilon} g''(x; x_p) dx = 1$
$\implies g'(x_p + \epsilon; x_p) - g'(x_p - \epsilon; x_p) = -\frac{1}{\kappa}$
$\implies a - c = -\frac{1}{\kappa}$
$g(x_p + \epsilon; x_p) = g(x_p - \epsilon; x_p)$
$\implies a(x_p - L) = cx_p, solve for a, c$
$\implies a = -\frac{x_p}{L\kappa},\ c = \frac{-L + x_p}{L\kappa}$
$$g(x; x_p) = \begin{cases} \frac{L - x_p}{L\kappa} x,\ x < x_p \\ -\frac{x_p}{L\kappa}(x - L),\ x > x_p \end{cases}$$

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(855) <u>greens functions using eigenfunctions</u>
L_x hermitian w/ $\langle y_2, L_x y_1 \rangle = \langle L_x y_2, y_1 \rangle$
$\implies$ L_x has orthonormal eigenfunctions $e_i(x)$ s.t.
$L_x e_i(x) = \lambda_i e_i(x) w/ e_i(0) = 0,\ e_i(L) = 0$
$\langle e_i(x), e_j(x) \rangle = \delta_{ij},\ e_i(x)$ complete
$\implies y(x) = \sum_i a_i e_i(x) = a_i e_i(x)$
<u>recall:</u> $L_x y(x) = q(x)$
$\implies a_i \lambda_i \langle e_i, e_j \rangle = \langle q(x), e_j(x) \rangle$
$\implies a_j \lambda_j = \langle q(x), e_j(x) \rangle$
$\implies a_j = \frac{\langle q(x), e_j(x) \rangle}{\lambda_j}$

$\implies y(x) = \frac{\langle q(x), e_i(x)\rangle}{\lambda_i} e_i(x)$
$q(x) = \delta(x - x_p)$
$\implies g(x; x_p) = \frac{\langle \delta(x-x_p), e_i(x)\rangle}{\lambda_i} e_i(x) = \frac{e_i(x_p)e_i(x)}{\lambda_i}$

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(856) Example
$\frac{d^2T}{dx^2} = -\frac{q(x)}{\kappa} \implies -\kappa e''(x) = \lambda e(x),\ e(0) = 0 \implies e(x) = C\sin(\sqrt{\frac{\lambda}{\kappa}}x)$
$e(L) = 0 \implies \lambda = n\pi$
$e_n(x)$ are not normalized i.e. $\langle e_n, e_n\rangle \neq \delta_{ij}$
after normalizing obtain
$e_n(x) = \sqrt{\frac{2}{L}}\sin(\frac{n\pi x}{L})$
$\implies g(x; x_p) = \frac{e_n(x_p)e_n(x)}{\lambda_n} = \sum_n \frac{2}{L}\frac{\sin\frac{n\pi x_p}{L}\sin\frac{n\pi x}{L}}{n\pi}$
this is essentially a fourier series of the previous method and approaches the greens function as $n \to \infty$

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(857) $\underline{g(z, w) = \sum_n \frac{e_n(w,z)e_n^*(w_s,z_s)}{\lambda_n}}$
$Lg(w, z) = \delta(w - w_s)\delta(z - z_s)$
find complete set of eigenfunctions
$Le_n(w, z) = \lambda_n e_n(w, z) \implies g(z, w) = \sum_n g_n e_n(w, z)$
$\implies Lg(w, z) = \sum_n g_n Le_n(w, z) = \sum_n g_n\lambda_n e_n(w, z)$
$= \delta(w - w_s)\delta(z - z_s)$
$\implies \sum_n g_n\lambda_n\langle e_n, e_m\rangle = \langle\delta(w - w_s)\delta(z - z_s), e_m\rangle$
$= e_m^*(w_s, z_s) = g_m\lambda_m$
$\implies g_m(w_s, z_s) = \frac{e_m^*(w_s,z_s)}{\lambda_m}$
$\implies g(z, w) = \sum_n \frac{e_n(w,z)e_n^*(w_s,z_s)}{\lambda_n}$

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separable case $L = L_w + L_z,\ L_w w_n(w) = \lambda_n W_n(w),\ \langle W_n, W_m\rangle = \delta_{nm}$
$L_z Z_n(z) = \mu_n Z_n(z);\ \langle Z_n, Z_m\rangle = \delta_{nm}$
$\implies L(W_n(w)Z_n(z)) = (L_w + L_z)(W_n(w)Z_n(z))$
$= \lambda_n W_n(w)Z_n(z) + \mu_n W_n(w)Z_n(z) = (\lambda_n + \mu_n)W_n(w)Z_n(z)$
choose w_n or Z_n and expand g in it
$\implies g(w, z) = \sum_n g_n(z)W_n(w)$
$\implies (L_w + L_z)\sum_n g_n(z)W_n(w) = \sum_n(\lambda_n g_n W_n + W_n L_z g_n)$
$= \delta(w - w_s)\delta(z - z_s)$
$\implies \sum_n[\lambda_n g_n\langle W_n, W_m\rangle + \langle W_n, W_m\rangle L_z g_n] = \langle\delta(w-w_s), W_m\rangle\delta(z-z_s)$

Electroweak theory (redo)

$L_i = \frac{1-\gamma_5}{2}\begin{pmatrix}\psi_{\nu_i}\\ \psi_i\end{pmatrix};\ R_i = \frac{1+\gamma_5}{2}\psi_i,\ i = e, \mu, \tau$

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(858) $\underline{\gamma^\alpha(1-\gamma_5) = 2\frac{1+\gamma_5}{2}\gamma^\alpha\frac{1-\gamma_5}{2}}$
$\gamma^\alpha(1-\gamma_5) = \frac{1}{2}\gamma_5\gamma^\alpha(1-\gamma_5) - \frac{1}{2}\gamma_5\gamma^\alpha(1-\gamma_5) + \frac{1}{2}\gamma^\alpha(1-\gamma_5) + \frac{1}{2}\gamma^\alpha(1-\gamma_5)$
$= \frac{1}{2}(1+\gamma_5)\gamma^\alpha(1-\gamma_5) + \frac{1}{2}(1-\gamma_5)\gamma^\alpha(1-\gamma_5)$
but $(1-\gamma_5)\gamma^\alpha(1-\gamma_5) = (1-\gamma_5)(\gamma^\alpha + \gamma_5\gamma^\alpha)$
$= \gamma^\alpha + \gamma_5\gamma^\alpha - \gamma_6\gamma^\alpha - \gamma_5^2\gamma^\alpha = \gamma^\alpha - \gamma^\alpha = 0$
here $\gamma_5\gamma^\alpha = -\gamma^\alpha\gamma_5;\ \gamma_5^2 = 1$ was used
$\therefore \gamma^\alpha(1-\gamma_5) = 2(\frac{1+\gamma_5}{2})\gamma^\alpha(\frac{1-\gamma_5}{2})$

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(859) $\underline{J_-^{(e)\alpha} = 2\bar{L}_e\gamma^\alpha\hat{T}_-L_e}$
$J_-^{(e)\alpha} = \bar{\psi}_e\gamma^\alpha(1-\gamma_5)\psi_{\nu_e} = 2\bar{\psi}_e\frac{1+\gamma_5}{2}\gamma^\alpha\frac{1-\gamma_5}{2}\psi_{\nu_e}$
<u>Note:</u> $(\bar{\psi}_e\ \bar{\psi}_{\nu_e} = (\bar{\psi}_{\nu_e}\bar{\psi}_e)\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\begin{pmatrix}\psi_{\nu_e}\\ \psi_e\end{pmatrix}$
$\implies J_-^{(e)\alpha} = 2(\bar{\psi}_{\nu_e}\ \bar{\psi}_e)\frac{1+\gamma_5}{2}\gamma^\alpha\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\frac{1-\gamma_5}{2}\begin{pmatrix}\psi_{\nu_e}\\ \psi_e\end{pmatrix}$
$= 2\bar{L}_e\gamma^\alpha\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}L_e = 2\bar{L}_e\gamma^\alpha\hat{T}_-L_e$

<u>Note:</u> $\underline{(\hat{P}u)} = (\hat{P}u)^\dagger\gamma_0$
$\implies \bar{L}_e = (\frac{1-\gamma_5}{2}\begin{pmatrix}\psi_{\nu_e}\\ \psi_e\end{pmatrix}) = (\frac{1-\gamma_5}{2}\begin{pmatrix}\psi_{\nu_e}\\ \psi_e\end{pmatrix})^\dagger\gamma_0$
$= (\psi_{\nu_e}\ \psi_e)(\frac{1-\gamma_5}{2})\gamma_0 = (\psi_{\nu e}ll\psi_e)\gamma_0\frac{1+\gamma_5}{2} = (\bar{\psi}_{\nu_e}\bar{\psi}_e)\frac{1+\gamma_5}{2}$
$\hat{T}_\pm = \hat{T}_1 \pm i\hat{T}_2;\ \hat{\vec{T}} = (\hat{T}_1, \hat{T}_2, \hat{T}_3)$
$\hat{T}_i = \frac{1}{2}\sigma_i;\ \sigma_1 = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\ \sigma_2 = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix},\ \sigma_3 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$

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(860) $\underline{L_{int}^{(e)} = g(\bar{L}_e\gamma^\alpha\hat{\vec{T}}L_e)\cdot\vec{A}_\alpha - g'[\frac{1}{2}(\bar{L}_e\gamma^\alpha L_e) + (\bar{R}_e\gamma^\alpha R_e)]B_\alpha}$
from $J_-^{(e)\alpha},\ J_+^{(e)\alpha},\ J_{EM}^{(e)\alpha}$ we see there are 2 types of current involved, i.e. isptriplets $\bar{L}_e\gamma^\alpha\hat{T}_iL_e$ and isosinglet $\frac{1}{2}(\bar{L}_e\gamma^\alpha L_e) + (\bar{R}_e\gamma^\alpha R_e)$Analogous to EM where we take $J_{EM}^\alpha A_\alpha = \mathscr{L}_{int}$ we can take a linear combination of couplets $gg_i(\bar{L}_e\gamma^\alpha\hat{T}_iL_e)\cdot A_{i,\alpha} - g'[\frac{1}{2}(\bar{L}_e\gamma^\alpha L_e) + (\bar{R}_e\gamma^\alpha R_e)]B_\alpha$
Let $g_iA_i \to A_i$ (can we do this?)

(dont understand why we dont just $g\vec{A}_\mu \to \vec{A}_\mu$)
$\therefore L^{(e)}_{int} = g(\bar{L}_e \gamma^\alpha \hat{\vec{T}} L_e)\cdot \vec{A}_\alpha - g'[\frac{1}{2}(\bar{L}_e\gamma^\alpha L_e) + (\bar{R}_e\gamma^\alpha R_e)]B_\alpha$

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(861) $\underline{A_\mu = \cos\theta B_\mu + \sin\theta A^3_\mu}$ (A_μ ~photon field (dont know how to derive/justify?)
A_μ is not equal to $\vec{A}_{|mu}, B_\mu$ since it couples to $J^{(e)\alpha}_{EM}$
<u>recall:</u> $J^{(e)\alpha}_{EM} A_\alpha = (g\bar{L}_e gamma^\alpha \hat{T}_3 L_e A^3_\alpha - g'[\frac{1}{2}(\bar{L}_e\gamma^\alpha L_e) + (\bar{R}_e\gamma^\alpha R_e)]B_\alpha$
(don't know how to transform one into the other, it gets into Weinberg mixing angles but I dont know how to derive it)

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photon field (A_μ) couples to $\bar{\psi}\gamma^\alpha\psi$
$\implies A_\mu = \cos B_\mu + \sin\theta A^3_\mu$,
$Z_\mu = -\sin\theta B_\mu + \cos\theta A^3_\mu (Z_\mu isorthoto A_\mu)$
i.e. $Z_\mu A^\mu = 0$ (don't understand where $A_{|mu}, Z_\mu$ come from)
Also $W^{(\pm)}_\mu = \frac{1}{\sqrt{2}}(A^1_\mu \mp iA^2_\mu)$

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(862) $\underline{\bar{L}^{(e)}_{int} \equiv \frac{g}{2\sqrt{2}}(J^{(e)\alpha}_{-} W^{(-)}_\alpha + J^{(e)\alpha}_{+} W^{(+)}_\alpha + J^{(e)\alpha}_0 Z_\alpha) - eJ^{(e)\alpha}_{EM} A_\alpha}$
<u>recall:</u> $L^{(e)}_{int} = g(\bar{L}_e\gamma^\alpha \hat{\vec{T}} L_e)\cdot\vec{A}_\alpha - g'[\frac{1}{2}(\bar{L}_e\gamma^\alpha L_e) + (\bar{R}_e\gamma^\alpha R_e)]B_\alpha$
insert $\vec{A}_\alpha, B_\alpha$ and
<u>Note:</u> $W^{(\pm)}_{mu} \implies A^1_\mu = \frac{1}{\sqrt{2}}(W^{(+)}_\mu + W^{(-)}_\mu); A^2_\mu = \frac{i}{\sqrt{2}}(W^{(+)}_\mu - W^{(-)}_\mu)$
$\implies L^{(e)}_{int} = \frac{g}{\sqrt{2}}\bar{L}_e\gamma^\alpha(\hat{T}_- W^{(-)}_\alpha + \hat{T}_+ W^{(+)}_\alpha)L_e + [g\cos\theta\bar{L}_e\gamma^\alpha\hat{T}_3 L_e + g''\sin\theta(\frac{1}{2}\bar{L}_e\gamma^\alpha L_e + \bar{R}_e\gamma^\alpha R_e)]Z_\alpha + [-g'\cos\theta(\frac{1}{2}\bar{L}_e\gamma^\alpha L_e + \bar{R}_e\gamma^\alpha R_e) + g\sin\theta\bar{L}_e\gamma^\alpha\hat{T}_3 L_e]A_\alpha$ (mathematica)
<u>recall:</u> $J^{(e)\alpha}_{EM} = \bar{L}_e\gamma^\alpha(\frac{1}{2} - \hat{T}_3)L_e + \bar{R}_e\gamma^\alpha R_e$;
$J^{(e)\alpha}_{-} = 2\bar{L}_e\gamma^\alpha\hat{T}_- L_e; J^{(e)\alpha}_{+} = 2\bar{L}_e\gamma^\alpha\hat{T}_+ L_e$
$\implies L^{(e)}_{int} = \frac{g}{2\sqrt{2}}(J^{(e)\alpha}_{-} W^{(-)}_\alpha + J^{(e)\alpha}_{+} W^{(+)}_\alpha + J^{(e)\alpha}_0 Z_\alpha - eJ^{(e)\alpha}_{EM} A_\alpha$
this defines
$J^{(e)\alpha}_0 = 2\sqrt{2}[\cos\theta\bar{L}_e\gamma^\alpha\hat{T}_3 L_e + \frac{g'}{g}\sin\theta(\frac{1}{2}\bar{L}_e\gamma^\alpha L_e + \bar{R}_e\gamma^\alpha R_e)]$

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(863) $\underline{L = i\bar{\Psi}\gamma^\mu(\partial_\mu - ig\vec{A}_\mu\cdot\hat{\vec{T}})\Psi = i\bar{\Psi}\gamma^\mu\hat{D}_\mu\Psi}$
start with ansats $i\bar{\Psi}\gamma^\mu\partial_\mu\Psi$
forcing local symmetry, i.e. $\Psi \to U\Psi = e^{i\theta}e^{ig\hat{\vec{T}}\cdot\vec{\lambda}}$
$e^{i\theta}$ is automatically satisfied
$\implies \Psi \to e^{ig\hat{\vec{T}}\cdot\vec{\lambda}}\Psi;\ e^{i\theta(x)}$ symmetry automatifcally obeyed (dont

understand)

$\partial_\mu \to \partial_\mu - ig\vec{A}_\mu \cdot \hat{\vec{T}}$ (analogous to QED)

$\therefore i\bar{\Psi}\gamma^\mu(\partial_\mu - i\vec{A}_\mu \cdot \hat{\vec{T}})\Psi = L$

<u>Note:</u> This implies $L = i\bar{\Psi}\gamma^\mu D_\mu\Psi = L' = i\bar{\Psi}'\gamma^\mu(\partial_\mu - i\vec{A}'_\mu \cdot \hat{\vec{T}})\Psi'$

Next lets figure out how $\vec{A}_\mu$ transforms to make this true

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(864) $\underline{\vec{A}'_\mu \cdot \hat{\vec{T}} = \hat{U}\vec{A}_\mu \cdot \hat{\vec{T}}U^{-1} + \frac{i}{g}\hat{U}(\partial_\mu \hat{U}^{-1})}$

$L = i\bar{\Psi}\gamma^\mu\partial_\mu\Psi + g\bar{\Psi}\gamma^\mu\vec{A}_\mu \cdot \hat{\vec{T}}\Psi$

$= i\bar{\Psi}U^{-1}U\gamma^\mu\partial_\mu U^{-1}U\Psi + g\bar{\Psi}U^{-1}U\gamma^\mu\vec{A}_\mu \cdot \hat{\vec{T}}U^{-1}U\Psi$

$= i\bar{\Psi}'U\gamma^\mu\partial_\mu(U^{-1}\Psi') + g\bar{\Psi}'U\gamma^\mu\vec{A}_\mu \cdot \hat{\vec{T}}U^{-1}\Psi'$

$= i\bar{\Psi}'U\gamma^\mu U^{-1}\partial_\mu\Psi' + i\bar{\Psi}'U\gamma^\mu(\partial_\mu U^{-2})\Psi + g\bar{\Psi}'U\gamma^\mu\vec{A}_\mu \cdot \hat{\vec{T}}U^{-1}\Psi'$

$= i\bar{\Psi}'\gamma^\mu\partial_\mu\Psi' + g\bar{\Psi}'\gamma^\mu U\vec{A}_\mu \cdot \hat{\vec{T}}U^{-1} + \frac{i}{g}U(\partial_\mu U^{-1})]\Psi'$

$= L' = i\bar{\Psi}'\gamma^\mu\partial_\mu\Psi' + g\bar{\Psi}\gamma^\mu\vec{A}'_\mu \cdot \hat{\vec{T}}\Psi'$

$\therefore \vec{A}'_\mu \cdot \hat{\vec{T}} = U\vec{A}_\mu \cdot \hat{\vec{T}}U^{-1} + \frac{i}{g}U\partial_\mu(U^{-1})$

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(865) $\underline{\vec{F}_{\mu\nu} = \partial_\mu\vec{A}_\nu - \partial_\nu\vec{A}_\mu + \frac{2q}{\hbar c}(\vec{A}_\nu \times \vec{A}_\mu)}$(QCD)

<u>recall:</u> $[D_\mu, D_\nu] = \frac{iq}{\hbar c}\vec{\tau} \cdot \vec{F}_{\mu\nu}$

$D_\mu = \partial_\mu + i\frac{q}{\hbar c}\vec{\tau} \cdot \vec{A}_\mu = \partial_\mu + iq\vec{\tau} \cdot \vec{A}_\mu$

$[D_\mu, D_\nu] = [\partial_\mu + iq\vec{\tau} \cdot \vec{A}_\mu, \partial_\nu + iq\vec{\tau} \cdot \vec{A}_\nu]$

$= [\partial_\mu, \partial_\nu] + [\partial_\mu, iq\vec{\tau} \cdot \vec{A}_\nu] + [iq\vec{\tau} \cdot \vec{A}_\mu, \partial_\nu] + [iq\vec{\tau} \cdot \vec{A}_\mu, iq\vec{\tau} \cdot \vec{A}_\nu]$

$= 0 + i\vec{\tau} \cdot [\partial_\mu, \vec{A}_\nu] + iq\vec{\tau} \cdot [\vec{A}_\mu, \partial_\nu] - q^2[\tau_i, \tau_j]A^i_\mu A^j_\nu$

$= iq\vec{\tau} \cdot (\partial_\mu\vec{A}_\nu - \partial_\nu\vec{A}_\mu) - q^2[\tau_i, \tau_j]A^i_\mu A^j_\nu$

<u>recall:</u> $[\tau^i, \tau^j] = 2i\epsilon_{ijk}\tau^k$

$[\tau_i, \tau_j]A^i_\mu A^j_\nu = 2i\epsilon_{ijk}\tau^k A^i_\mu A^j_\nu = 2i(\vec{A}_\mu \times \vec{A}_\nu)_k\tau^k$

$= 2i\vec{\tau} \cdot (\vec{A}_\mu \times \vec{A}_\nu)$

$\therefore [D_\mu, D_\nu] = \frac{iq}{\hbar c}\vec{\tau} \cdot (\partial_\mu\vec{A}_\nu - \partial_\nu\vec{A}_\mu - \frac{2q}{\hbar c}\vec{A}_\mu \times \vec{A}_\nu)$

$\therefore \vec{F}_{\mu\nu} = \partial_\mu\vec{A}_\nu - \partial_\nu\vec{A}_\mu - \frac{2q}{\hbar c}\vec{A}_\mu \times \vec{A}_{|nu}$

factor of $\frac{1}{\hbar c}$ comes from q^2 term

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QFT

(866) $\underline{\hat{H} = \sum_{k=1}^N \hbar\omega_{\vec{k}}(\hat{a}^\dagger_{\vec{k}}\hat{a}_{\vec{k}} + \frac{1}{2})}$

<u>recall:</u> $\hat{H} = \sum_j \frac{\hat{p}_j^2}{2m} + \frac{1}{2}K(\hat{x}_{j+1} - \hat{x}_j)^2$

<u>recall:</u> $\psi(x) = \int_{-\infty}^{\infty} e^{ikx}\phi(k)dk \to \sum e^{ikx}\tilde{\psi}$
on a lattice $\psi \to x_j$
$\implies \begin{cases} x_j - = \frac{1}{\sqrt{N}} \sum_k \tilde{x}_k e^{ikja} \\ p_j = \frac{1}{\sqrt{N}} \sum_k \tilde{p}_k e^{ikja} \end{cases}$
(could understand this part better and i dont understand $\frac{1}{\sqrt{N}} \implies$
$\begin{cases} \tilde{x}_k = \frac{1}{\sqrt{N}} \sum_j x_j e^{-ikja} \\ \tilde{p}_k = \frac{1}{\sqrt{N}} \sum_j p_j e^{-ikja} \end{cases}$
force periodic boundary condition $e^{ikja} = e^{ik(j+N)a} \implies k = \frac{2\pi m}{Na}$
$-\frac{N}{2} \leq m \leq \frac{N}{2}$
<u>Note:</u> $\sum_j e^{ikja} = N\delta_{k0}$
$\implies [x_j, p_{j'}] = i\hbar\delta_{jj'}$
$[\tilde{x}_k, \tilde{p}_{k'}] = i\hbar\delta_{k,-k'}$ (verify)
$\sum_j p_j T^2 = \sum_k \tilde{p}_k\tilde{p}_{-k}$ (verify)
$\sum_j (x_{j+1} - x_j)^2 = \sum_k \tilde{x}_k\tilde{x}_{-k}(4\sin\frac{ka}{2}^2)$ (verify)
and so
$\hat{H} = \sum_j \frac{p_j^2}{2m} + \frac{1}{2}K(x_{j+1} - x_j)^2$
becomes
$\hat{H} = \sum_k \frac{1}{2m}\tilde{p}_k\tilde{p}_{-k} + \frac{1}{2}k\tilde{x}_k\tilde{x}_{-k}(4\sin^2\frac{ka}{2})$
$k4\sin^2\frac{ka}{2} = m\omega^2$
$\implies \omega = \sqrt{\frac{4k\sin^2\frac{ka}{2}}{m}}$
$\implies \hat{H} = \sum_k \frac{1}{2m}(\tilde{p}_k\tilde{p}_{-k} + \frac{1}{2}m\omega_k^2\tilde{x}_k\tilde{x}_{-k})$
(we require $p_k^\dagger = p_{-k}\hat{x}_k^\dagger = \hat{x}_{-k}$ (since $x_j p_j$ are hermiatian?)
recall expressions for $a,\ a^\dagger$
invert them and plug in for x, p
$\implies \hat{H} = \sum_{k=1}^{N} \hbar\omega_k(a_k^\dagger a_k + \frac{1}{2})$
$\implies H = \int d^3p E_p a_p^\dagger a$, pretty sure $\frac{\hbar\omega}{2}$ term gets subtracted off because it diverges when we go to the oscillator picture
<u>Note:</u> $a_p = \frac{1}{2}(\sqrt{2\omega_p}\tilde{\phi}(\vec{p}) + i\sqrt{\frac{2}{\omega_p}}\tilde{\pi}(\vec{p}))$
$a_{\vec{p}}^\dagger = \frac{1}{2}(\sqrt{2\omega_{\vec{p}}}\tilde{\phi}(-\vec{p}) - i\sqrt{\frac{2}{\omega_p}}\tilde{\pi}(-\vec{p}))$
$\phi(\vec{x}) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}e^{i\vec{p}\cdot\vec{x}}(a_{\vec{p}} + a_{-\vec{p}}^\dagger)$ (dont understand how to get here, should ask for help)
$\pi(\vec{x}) = -i\int \frac{d^3p}{(2\pi)^3}\sqrt{\frac{\omega}{\vec{p}}2}e^{\vec{p}\cdot\vec{x}}(a_{\vec{p}} - a_{-\vec{p}}^\dagger)$
derive $L = \int d^3x[\frac{1}{2}\rho$ bla bla see pg 56 in amateur

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Schrodinger $\to$ Interaction
Operator $O \to O_t^I = e^{iH_0 t} O e^{-iH_0 t}$ (interaction part gets absorbed into $|\psi\rangle$
state $|\psi_t\rangle = e^{-iHt}|\psi\rangle \to |\psi_t^I\rangle = e^{iH_0 t} e^{-iHt}|\psi\rangle$
Note: $e^{iH_0 t - iHt} \implies H_0 - H = H_{ont}$
$t = 0 \to t'$
$\implies |\psi_{t'}^I\rangle = e^{iH_0 t} e^{-iHt'}|\psi\rangle$
$\implies \psi_{t'}^I\rangle = e^{iH_0 t'} e^{-iH(t'-t)} e^{-iH_0 t}(e^{iH_0 t} e^{-iHt})|\psi\rangle$
$= U(t', t)(|\psi_t^I\rangle)$
Let $\Delta = \frac{t'-t}{2}$
$U(t', t) = U(t', t' - \Delta)U(t - \Delta, t' - 2\Delta) \cdots U(t + \Delta, t)$
$\implies U(t + \Delta, t) = e^{iH_0(t+\Delta)} e^{-iH\Delta} e^{-iH_0 t} = e^{iH_0 t} e^{-H_{int}\Delta} e^{-iH_0 t}$
$= e^{-iH_{int}(t)\Delta}$ (time evolved)
Note: $e^A e^B \neq e^{A+B}$ you must use Baker - campbell hausdorff formula

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(867) $U(t', t) = T \exp(-i \int_t^{t'} d\tau H_{int}^I(\tau)),\ n \to \infty$
$U(t + \Delta, t) = e^{-iH_{int}^I(t)\Delta} \implies U(t', t' - \Delta) = e^{-iH_{int}^I(t'-\Delta)\Delta}$
$\implies U(t', t) = e^{-iH'_{int}(t-\Delta)\Delta} \cdot e^{-iH_{int}^I(t'-2\Delta)\Delta} \cdots e^{-iH_{int}^I(t)\Delta}$
$= Te^{-iH_{int}^I(t'-\Delta)\Delta} e^{-iH_{int}^I(t'-2\Delta)\Delta} \cdots e^{-iH_{int}^I\Delta}$
$= Te^{-iH_{int}^I(t'-\Delta)\Delta - iH_{int}^I(t'-2\Delta)\Delta + \cdots - iH_{int}^I(t)\Delta}$
$= T \exp(-i \int_t^{t'} d\tau H_{int}^I(\tau)) n \to \infty$

Note: $T\phi(t_1)\phi(t_2) = \begin{cases} \phi(t_1)\phi(t_2) t_1 \geq t_2 \\ \phi(t_2)\phi(t_1) t_2 > t_1 \end{cases}$

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(868) $H_{int}^I(t) = \int d^3x \frac{\lambda}{4!}(\phi^I(x))^4$
we've already discussed $V(\phi) = \frac{m^2}{2}\phi^2$
now consider next term which is symmetric under $\phi \to -\phi$; $\mathcal{H}_{int} = -\mathcal{L}_{int} = \frac{\lambda}{4!}\phi^4$
$H_{int}^I(t) = e^{iH_0 t} \int d^3x (\frac{\lambda}{4!}(\phi(\vec{x}))^4) e^{-iH_0 t}$
$= \int d^3x \frac{\lambda}{4!} e^{iH_0 t} \phi e^{-iH_0 t} e^{H_0 t} \cdots \phi e^{-iH_0 t}$
$= \int d^3x \frac{\lambda}{4!}(\phi^I(x))^4$

H^I includes terms like $a_{\vec{p}_1'}^\dagger a_{\vec{p}_2'}^\dagger a_{\vec{p}_1} a_{\vec{p}_2}$
so 2-2 scattering is expected at leading order

$S = \lim_{(t',t)\to(\infty,-\infty)} = T\exp(-i\int_{-\infty}^{\infty} dt H^I_{int}(t))$
$= T\exp(i\int d^4x\mathcal{L}^I_{int}(x))$
$S_{fi} = \langle p_1' p_2' | T\exp(i\int d^4x\mathcal{L}^I_{int}) | p_1 p_2\rangle$
(matrix elements)
(I don't understand motivation for this?)

Thermodynamics

(869) $\underline{\int_0^{\epsilon_F} a(\epsilon)d\epsilon = N}$ (Fermions)
$N = 2\sum_{\vec{k}} = 2\frac{V}{(2\pi)^3}\int d\vec{k} = \frac{2\cdot 4\pi V}{(2\pi)^3}\int_0^{k_F} k^2 dk$
$= \frac{2\cdot 4\pi V}{h^3}\int_0^{p_F} p^2 \frac{dp}{d\epsilon} d\epsilon$
$a(\epsilon) = \frac{2\cdot 4\pi V}{h^3} p^2 \frac{dp}{d\epsilon}$
in general
$a(\epsilon) = \frac{g\cdot 4\pi V}{h^3} p^2 \frac{dp}{d\epsilon}$

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(870) $\underline{E_0 = \frac{4\pi g V}{h^3}\int_0^{p_F} \frac{p^2}{2m} p^2 dp}$ (fermion gas, ground state)
$E_0 = 2\sum_{\vec{k}} \epsilon(\vec{k}) = \frac{2V}{(2\pi)^3}\int \epsilon d\vec{k}$
$= \frac{2V}{(2\pi)^3} 4\pi \int \epsilon k^2 dk = \frac{2V}{h^3} 4\pi \int \epsilon p^2 dp$
$= \frac{2V}{h^3} 4\pi \int_0^{p_F} (\frac{p^2}{2m}) p^2 dp$
$= \int_0^{p_F} \epsilon a(\epsilon) d\epsilon$

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(871) $\underline{\frac{\partial S}{\partial V}|_{T,N} = \frac{\partial P}{\partial T}|_{V,N}}$ (a maxwell's relation)
we want to find the maxwell relation involving $\frac{\partial S}{\partial V}|_{T,N}$
we look for a thermo Identity that involes S alone and dV
<u>recall:</u> $dF = -SdT - PdV + \mu dN$
$\implies \frac{\partial F}{\partial T}|_{V,N} = -S, \ \frac{\partial F}{\partial V}|_{T,N} = -P$
$\implies \frac{\partial^2 F}{\partial V \partial T}|_N = -\frac{\partial S}{\partial V}|_{T,N}; \ \frac{\partial^2 F}{\partial T \partial V}|_N = -\frac{\partial P}{\partial T}|_{V,N}$
$\therefore \frac{\partial S}{\partial V}|_{T,N} = \frac{\partial P}{\partial T}_{V,N}$

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(872) $\underline{f(\vec{p}) = \frac{n}{(2\pi m k_B T)^{3/2}}\exp(-p^2/2mk_BT)}$
<u>Note:</u> $f = PN, \ \mathcal{P} \sim$ probability
$\mathcal{P} = \frac{1}{Z}\exp(-p^2/2mk_BT)$
$Z = V\int d^3p\exp(-p^2/2mk_BT) = V4\pi\int_0^{\infty} p^2 dp\exp(-p^2/2mk_BT)$
$= 4\pi V\sqrt{\frac{\pi}{2}}(mk_BT)^{3/2} = V(2\pi m k_B T)^{3/2}$
$\implies \mathcal{P} = \frac{n}{(2\pi m k_B T)^{3/2}}\exp(-p^2/2mk_BT)$

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(873) $\underline{S(E,N) = -Nk_B[(\frac{E}{N\epsilon})\ln(\frac{E}{N\epsilon}) + (1-\frac{E}{N\epsilon})\ln(1-\frac{E}{N\epsilon})}$

(two state system w/ $E = 0, \epsilon$)
$\Omega(E,N) = \frac{N!}{N_1!(N-N_1)!}$ (number of ways to choose N_1 impurities to be excited)
$\implies \ln\Omega = \ln N! - \ln N! - \ln(N-N_1)!$
<u>recall:</u> $\ln N! = N\ln N - N$
$\implies \ln\Omega = N\ln N - N - N_1\ln N_1 + N_1 - (N-N_1)\ln(N-N_1) + N - N_1$
$= N\ln N - N_1\ln N_1 - (N-N_1)\ln(N-N_1)$
$= N\ln N - N_1\ln N_1 - (N-N_1)\ln(N-N_1) + (N-N_1)\ln N - (N-N_1)\ln N$
$= N\ln N - N_1\ln N_1 - (N-N_1)\ln\frac{N-N_1}{N} - (N-N_1)\ln N$
$= -N_1\ln N_1 + N_1\ln N - (N-N_1)\ln\frac{N-N_1}{N}$
$= -N_1\ln\frac{N_1}{N} - (N-N_1)\ln\frac{N-N_1}{N}$
$= -N(\frac{N_1}{N}\ln\frac{N_1}{N} + \frac{(N-N_1)}{N}\ln\frac{N-N_1}{N})$
$\implies S \approx -Nk_B(\frac{N_1}{N}\ln\frac{N_1}{N} + \frac{(N-N_1)}{N}\ln\frac{N-N_1}{N})$
$= -Nk_B(\frac{E}{N\epsilon}\ln(\frac{E}{N\epsilon}) + \frac{(N-\frac{E}{\epsilon})}{N}\ln\frac{N-\frac{E}{\epsilon}}{N})$

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(874) $\underline{E(T) = \frac{N\epsilon}{(\exp(\frac{\epsilon}{k_BT})+1)}}$

<u>recall:</u> $S(E,N) = -Nk_B[(\frac{E}{N\epsilon})\ln(\frac{E}{N\epsilon}) + (1-\frac{E}{N\epsilon})\ln(1-\frac{E}{N\epsilon})$
$dE = TdS - PdV + \mu dN$
$\implies \frac{\partial S}{\partial E}|_{V,N} = \frac{1}{T}$
$\implies k_BN\{\frac{1}{N\epsilon}\ln(\frac{E}{\epsilon N}) - \frac{1}{N\epsilon}\ln(1-\frac{E}{\epsilon N})\} = \frac{\partial S}{\partial E} = \frac{1}{T}$
(Mathematica)
now solve for E
$\implies E(T) = \frac{N\epsilon}{1+\exp(\epsilon\beta)}$

<u>Note:</u> $C_V = \frac{\partial E}{\partial T}|_V$ but $E(T)$ (above does not depend on volume so $\frac{\partial E}{\partial T} = \frac{dE}{dT} = C$
<u>Note:</u> $E(T) \to \frac{N\epsilon}{2}$ as $T \to \infty$ If we started with $\frac{N}{2}+1$ excited atoms this would give negative temperature
we can see this from
$\frac{1}{T} = -\frac{k_B}{\epsilon}\ln(\frac{E}{N\epsilon - E})$ is $\frac{E}{n\epsilon - E} > 1$
$\implies E > N\epsilon - E, E > \frac{N\epsilon}{2}$
$\implies E > \frac{N\epsilon}{2} \implies$ negative temperature.

$\mathcal{P}(n_1) = \frac{\Omega(E-n_1\epsilon, N-1)}{\Omega(E,N)}$ (Probability of exciting an impurity)

Electrodynamics

(875) $\underline{\frac{1}{|\vec{x}-\vec{x}'|} = \sum_{\ell=0}^{\infty} \frac{r_<^\ell}{r_>^{\ell+1}} P_\ell(\cos\gamma)}$
$r_<$ is the smaller of $|\vec{x}|$ and $|\vec{x}'|$
while $r_>$ is the larger of $|\vec{x}|$ and $|\vec{x}'|$
recall: $\frac{1}{|\vec{x}-\vec{x}'|} = \frac{1}{\sqrt{x^2+x'^2-2xx'\cos\gamma}} = \sum_{\ell=0}^{\infty} \frac{(x')^\ell}{x^{\ell+1}} P_\ell(\cos\gamma)$
if $x >> x'$ but the middle expression is x, x'
symmetric so
$\frac{1}{|\vec{x}-\vec{x}'|} = \sum_{\ell=0}^{\infty} \frac{x^\ell}{x'^{\ell+1}} P_\ell(\cos\gamma)$ if $x < x'$
$\therefore \frac{1}{|\vec{x}-\vec{x}'|} = \sum_{\ell=0}^{\infty} \frac{r_<^\ell}{r_>^{\ell+1}} P_\ell(\cos\gamma)$

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(876) $\underline{\frac{1}{|\vec{x}-\vec{x}'|} = 4\pi \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{1}{2\ell+1} \frac{r_<^\ell}{r_>^{\ell+1}} Y_{\ell m}^*(\theta',\phi') Y_{\ell m}(\theta,\phi)}$
recall: $P_\ell(\cos\gamma) = \frac{4\phi}{2\ell+1} \sum_{m=-\ell}^{\ell} Y_{\ell m}^*(\theta',\phi') Y_{\ell m}(\theta,\phi)$
(Addition theorem for spherical harmonics)
$\frac{1}{|\vec{x}-\vec{x}'|} = \sum_{\ell=0}^{\infty} \frac{r_<^\ell}{r_>^{\ell+1}} P_\ell(\cos\gamma)$
$= \sum_{\ell=0}^{\infty} \frac{r_<^\ell}{r_>^{\ell+1}} \frac{4\pi}{2\ell+1} \sum_{m=-\ell}^{\ell} Y_{\ell m}^*(\theta',\phi') Y_{\ell m}(\theta,\phi)$
$= 4\pi \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{1}{2\ell+1} Y_{\ell m}^*(\theta',\phi') Y_{\ell m}(\theta,\phi) \frac{r_<^\ell}{r_>^{\ell+1}}$

Note: γ is the angle between $\vec{x}$ and $\vec{x}'$ which does not necessarily need to be θ

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(877) $\underline{P_\ell(\cos\gamma) = \frac{4\pi}{2\ell+1} \sum_{m=-\ell}^{\ell} Y_{\ell m}^*(\theta',\phi') Y_{\ell m}(\theta,\phi)}$
recall: $Y_{\ell m}(\theta,\phi)$ are complete;
$\implies g(\theta,\phi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} A_{\ell m} Y_{\ell m}(\theta,\phi)$;
$A_{\ell m} = \int d\Omega Y_{\ell m}^*(\theta,\phi) g(\theta,\phi)$; $\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi - \phi')$
$\implies P_\ell(\cos\gamma) = \sum_{\ell'=0}^{\infty} \sum_{m=-\ell'}^{\ell'} A_{\ell m} Y_{\ell' m}(\theta,\phi)$
claim: only $\ell' = \ell$ term appears
proof:
recall: the PDE for spherical harmonics is
$\nabla^2 Y_{\ell m}(\theta,\phi) + \frac{\ell(\ell+1)}{r^2} Y_{\ell m}(\theta,\phi) = 0$
fix $\vec{x}' on the z axis \implies \gamma \to \theta$

$\implies \nabla'^2 P_\ell(\cos\gamma) + \frac{\ell(\ell+1)}{r^2} P_\ell(\cos\gamma) = 0$

$\nabla'^2 = \nabla^2$, since $\nabla \cdot \nabla f = \nabla' \cdot \nabla' f$, i.e. scalar products are invariant under rotation so we can rotate it so that $\vec{x}'$ doesnt have to be on the z- axis

$\implies \nabla^2 P_\ell(\cos\gamma + \frac{\ell(\ell+1)}{r^2} P_\ell(\cos\gamma) = 0$

$\implies P_\ell$ is a spherical harmonic of order ℓ

$\implies P_\ell(\cos\gamma) = \sum_{m=-\ell}^{\ell} A_m(\theta', \phi') Y_{\ell m}(\theta, \phi)$

$\int P_\ell(\cos\gamma) Y^*_{\ell' m'}(\theta, \phi) d\Omega = \sum_{m=-\ell}^{\ell} A_m(\theta', \phi') \int Y_{\ell m}(\theta, \phi) Y_{\ell' m'}(\theta, \phi) d\Omega$

$\implies \sum_{m=-\ell}^{\ell} \delta_{\ell\ell'} \delta_{mm'} A_m = \delta_{\ell\ell'} A_m = \int P_\ell(\cos\gamma) Y^*_{\ell m'}(\theta, \phi) d\Omega$

$\ell = \ell' \implies A_m = \int P_\ell(\cos\gamma) Y^*_{\ell' m'}(\theta, \phi) d\Omega$

this gives a vague way to also see that above claim is true.

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(878) $\underline{\nabla^2 \phi_m = \rho_m;\ \Phi_M(\vec{x}) = -\frac{1}{4\pi} \int \frac{\nabla' \cdot \vec{M}(\vec{x}')}{|\vec{x} - \vec{x}'} d^3x'}$

<u>recall:</u> $\vec{H} = -\nabla \phi_m$

<u>recall:</u> $\nabla \cdot \vec{B} = 0 = \mu_0 \nabla \cdot (\vec{H} + \vec{M}) = 0$

$\implies \nabla \cdot \vec{H} = -\nabla \cdot \vec{M} \implies \nabla^2 \phi_M = -\nabla \cdot \vec{M}$

$\rho_M = -\nabla \cdot \vec{M}$

$\implies \phi_m = -\frac{1}{4\pi} \int \frac{\nabla' \cdot \vec{M}(\vec{x}')}{|\vec{x} - \vec{x}'|} d^3x'$ (no boundary surfaces)

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(879) $\underline{\Phi_M = -\frac{1}{4\pi} \int_V \frac{\nabla' \cdot \vec{M}(\vec{x}')}{|\vec{x} - \vec{x}'|} d^3x' + \frac{1}{4\pi} \oint_S \frac{\hat{n}' \cdot \vec{M}(\vec{x}')}{|\vec{x} - \vec{x}'|} da'}$

<u>recall:</u> $\sigma_M = \hat{n} \cdot \vec{M};\ \rho_M = -\nabla \cdot \vec{M}$

$\Phi_m = \frac{1}{4\pi} \oint \frac{\rho_M}{|\vec{x} - \vec{x}'|} d^3x' + \frac{1}{4\pi} \oint \frac{\sigma_M}{|\vec{x} - \vec{x}'|} da'$

$= -\frac{1}{4\pi} \int_V \frac{\nabla' \cdot \vec{M}(\vec{x}')}{|\vec{x} - \vec{x}'|} d^3x' + \frac{1}{4\pi} \oint_S \frac{\hat{n}' \cdot \vec{M}(\vec{x}')}{|\vec{x} - \vec{x}'|} da'$

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$\vec{F} = \int \vec{J}(\vec{x} \times \vec{B}(\vec{x}) d^3;\ \vec{N} = \vec{x} \times (\vec{J} \times \vec{B}) d^3x$

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(880) $\underline{\vec{F} = \nabla(\vec{m} \cdot \vec{B}) - m(\nabla \cdot \vec{B})}$

<u>recall:</u> $(\vec{J} \times \vec{B})_i = \epsilon_{ijk} J_j B_k;\ \vec{F} = \int \vec{J}(\vec{x}) \times \vec{B}(\vec{x}) d^3x$

$B_k(\vec{x}) = B_k(0) + \vec{x} \cdot \nabla B_k(0) + \cdots$

$F_i = \int \epsilon_{ijk} J_j B_k d^3x$

$= \sum_{jk} \epsilon_{ijk} \int J_j B_k d^3x$

$= \sum_{jk} \epsilon_{ijk} [\int J_j (B_k(0) + \vec{x} \cdot \nabla B_k(0) + \cdots) d^3x']$

$= \sum_{jk} \epsilon_{ijk} [B_k(0) \int J_j(\vec{x}') d^3x' + \int J_j(\vec{x}') \vec{x}' \cdot \nabla B_k(0) d^3x'$

$\int J_j(\vec{x}') d^3x' = 0$ (steady current)

$F_i \approx \sum_{jk} \epsilon_{ijk} \int J_j(\vec{x}') \vec{x}' \cdot \nabla B_k(0) d^3x'$

$= \int \vec{x} \cdot [\vec{J} \times \nabla B_k(0)] d^3x'$

$= -\int \vec{x} \cdot [\nabla B_k(0) \times \vec{J}] d^3x'$
$= -\int \nabla B_k(0) \cdot [\vec{J} \times \vec{x}] d^3x'$
$= -2 \int \nabla B_k(0) \cdot \vec{m} d^3x'$

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(881) $\underline{\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi} \int \vec{J}(\vec{x}') \frac{e^{ik|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|} d^3x'}$
<u>recall:</u> $\vec{J}(\vec{x}, t) = \vec{J}(\vec{x}) e^{-i\omega t}$
<u>recall:</u> $\vec{A}(\vec{x}, t) = \frac{\mu_0}{4\pi} \int d^3x' \int dt' \frac{\vec{J}(\vec{x}', t')}{|\vec{x}-\vec{x}'|} \delta(t' + \frac{|\vec{x}-\vec{x}'|}{c} - t)$
$= \frac{\mu_0}{4\pi} [\int d^3x' \frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|} e^{ik|\vec{x}-\vec{x}'|}] e^{-i\omega t}$
$= \vec{A}(\vec{x} e^{-i\omega t}$
$\therefore \vec{A}(\vec{x} = \frac{\mu_0}{4\pi} \int d^3x' \frac{\vec{J}(\vec{x}') e^{ik|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|} d^3x'$

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(882) $\underline{\lim_{kr\to 0} \vec{A}(\vec{x}) = \frac{\mu_0}{4\pi} \sum_{\ell,m} \frac{4\pi}{2\ell+1} \frac{Y_{\ell m}(\theta,\phi)}{r^{\ell+1}} \int \vec{J}(\vec{x}') r'^{\ell} Y^*_{\ell m}(\theta', \phi') d^3x'}$ (near field)
<u>recall:</u> $\vec{A}(\vec{x}) = \frac{\mu_)}{4\pi} \int \vec{J}(\vec{x}') \frac{e^{ik|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|} d^3x'$;
$\frac{1}{|\vec{x}-\vec{x}'|} = 4\pi \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{1}{2\ell+1} \frac{r_<^{\ell}}{r_>{}^{"}{}^{\ell+1}} Y^*_{\ell m}(\theta', \phi') Y_{\ell m}(\theta, \phi)$
$\implies \vec{A}(\vec{x}) = \frac{\mu_0}{4\pi} \int \vec{J}(\vec{x}') e^{ik|\vec{x}-\vec{x}'|} (4\pi \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{1}{2\ell+1} \frac{r_<^{\ell}}{r_>^{\ell+1}} Y^*_{\ell m}(\theta', \phi') Y_{\ell m}(\theta, \phi)) d^3x'$
<u>recall:</u> $|\vec{x} - \vec{x}'| \approx r - \hat{n} \cdot \vec{x}'$
$\implies \lim_{kr\to 0} \vec{A}(\vec{x}) = \lim_{kr\to 0} \frac{\mu_0}{4\pi} \int \vec{J}(\vec{x}') e^{ikr} e^{-ik\hat{n}\cdot\vec{x}'} (4\pi \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{1}{2\ell+1} \frac{r_<^{\ell}}{r_>^{\ell+1}} Y^*_{\ell m}(\theta', \phi') Y_{\ell m}($
$0 \implies r < r'$
$= \frac{\mu_0}{4\pi} \sum_{\ell,m} \frac{4\pi}{2\ell+1} Y_{\ell m}(\theta, \phi)$
don't understand this step we might have to take a detour through Griffiths.