ALEC HEWITT

Thermal Physics

Chapter 1

$C \equiv \frac{Q}{\Delta T}$; $c \equiv \frac{C}{m}$

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(276) $\underline{C_P = (\frac{\partial U}{\partial T})_P + P(\frac{\partial V}{\partial T})_P}$
$C_P = \frac{Q}{\Delta T} = (\frac{\Delta U + P\Delta V}{\Delta T})_P = (\frac{\partial U}{\partial T})_P + P(\frac{\partial V}{\partial T})_P$

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(277) $\underline{C_V = (\frac{\partial U}{\partial T})_V}$
$C_V = \frac{Q}{\Delta T} = (\frac{\Delta U}{\Delta T})_V = (\frac{\partial U}{\partial T})_V$

If quadratic degrees of freedom
$\implies U = \frac{1}{2}NfkT \implies (\frac{\partial U}{\partial T})_V = \frac{1}{2}Nfk$

$C = \frac{Q}{\Delta T} = \frac{Q}{0} = \infty(during phase transformation)$
$L \equiv \frac{Q}{m}$ (Latent heat)
$H \equiv U + PV$ (enthalpy)
U is the energy required to create object and PV is the energy required to make room for it.

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(278) $\underline{\Delta H = Q + W_{other}}$
$\Delta H = \Delta U + P\Delta V (P \sim \text{const.})$
$\Delta U = Q - P\Delta V + W_{other}$
$\implies \Delta H = Q + W_{other}$

$C_P = (\frac{\partial H}{\partial T})_P$
skipped 1.7

$\frac{Q}{\Delta t} = -k_t A \frac{dT}{dx}$

consider a window separating cold and hot rooms, we expect the heat trransferring from hot to cold to be proportional to window area and inversely proportional oto Δx (thickness of window)
proportional to the time Δt and proportional to difference in temperature $\Delta T = T_2 - T_1$
$\implies Q \propto \frac{A \Delta T \delta t}{\Delta x} \implies \frac{Q}{\Delta t} \propto A \frac{dT}{d}$
$\implies \frac{Q}{\Delta t} = -k_t A \frac{dT}{dx}$

Chapter 2

$$\Omega(N, n) = \frac{N!}{n! \cdot (N-n)!} = \binom{N}{n} \tag{279}$$

pobability of n heads $= \frac{\Omega(n)}{\Omega(\text{all})}$
suppose you have 100 coins, the number of ways that there are 2 heads is
$\Omega(2) = \frac{100 \cdot 99}{2}$100 places for the first head 99 ways for the second divided by 2 since each head is indistinguishable
$\Omega(3) = \frac{100 \cdot 99 \cdot 98}{3 \cdot 2} = \frac{100 \cdot 99 \cdot (100-3+1)}{3 \cdot 2}$
$3 \cdot 2$ because that is the number of ways you can arrange 3 indistinguishable heads
$\Omega(n) = \frac{100 \cdot 99 \cdot 98 \cdots (100-n+1)}{n!} = \frac{100 \cdot 99 \cdot 98 \cdots (100-n+1)(100-n)!}{n! \cdot (100-n)!} = \frac{100!}{n! \cdot (100-n)!}$
or, in general
$\therefore \Omega(N, n) = \frac{N!}{N!(N-n)!} = \binom{N}{n}$ " N choose n"

$\Omega(N_\uparrow) = \binom{N}{N_\uparrow} = \frac{N!}{N_\uparrow!(N-N_\uparrow)!} = \frac{N!}{N_\uparrow! N_\downarrow!}$

Einstein solid: $\cdot | \cdots || \cdots$
— represents partition between two oscillators

$$\Omega(N, q) = \binom{q+N-1}{q} = \frac{(q+N-1)!}{q!(N-1)!} \tag{280}$$

$N \sim$ solids $\implies N-1 \sim$ lines, q units of energy
$\implies q + N - 1 \sim$ symbols

We choose q of these symbols to be energy so there are $\Omega(N,q) = \binom{q+N-1}{q}$ ways to arrange q energies with $q+N-1$ symbols
$\Longrightarrow \; \Omega(N,q) = \frac{(q+N-1)!}{q!(q+N-1-q)!} = \frac{(1+N-1)!}{q!(N-1)!}$

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(281) 2 Second Law of Thermodynamics
Consider an islolated system of two solids (weakly coupled) that exchange energy w/
$N_A = N_B = 3;\; q_{total} = q_A + q_B = 6$
$\Omega_A = \frac{(q_A+N_A-1)!}{q_A!(N_A-1)!}$

q_A	Ω_A	q_B	Ω_B	$\Omega_{tot} = \Omega_A\Omega_B$
0	1	6	28	28
1	3	5	21	63
2	6	4	15	90
3	10	3	10	100
4	15	2	6	90
5	21	1	3	63
6	28	0	1	28

fundamental assumption of statistical mechanics: In an isolated system in thermal equilibrium all accessible microstates are equally probable
In this example, this means that if we started with an arbitrary state, later we would most likely find the system in a macrostate where Ω_{tot} is maximized, this is the second law of thermodynamics.

Large numbers $\sim 10^{23} + 23 = 10^{23}$
very large numbers $\sim 10^{10^{23}} \times 10^{23} = 10^{10^{23}+23} = 10^{10^{23}}$

$N! \approx N^N e^{-N}\sqrt{2\pi N}$ sometimes $\sqrt{2\pi N}$ is ignored (Stirling's approximation) $N >> 1$
$\ln N! \approx \ln N^N e^{-N} = N \ln N - N$

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(282) $\underline{\Omega(N,q) \approx (\frac{eq}{N})^N}$ $q >> N$

$\Omega(N,q) = \binom{q+N-1}{q} = \frac{(q+N-1)!}{q!(N-1)!} \approx \frac{(q+N)!}{q!N!}$

$\ln\Omega = \ln(\frac{(q+N)!}{q!N!}) = \ln(q+N)! - \ln q! - \ln N!$

$\approx (q+N)\ln(q+N) - (q+N) - q\ln q + q - N\ln N + N$

use $q >> N$

$\implies \ln(q+N) = \ln[q(1+\frac{N}{q})] = \ln q + \ln(1+\frac{N}{q})$

recall: $\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^n}{n} \approx x$

$\implies \ln(q+N) \approx \ln q + \frac{N}{q}$

$\implies \ln\Omega \approx (q+N)(\ln q + \frac{N}{q}) - (q+N) - q\ln q + q - N\ln N + N$

$= q\ln q + N + N\ln q + \frac{N^2}{q} - q - N - q\ln q + q - N\ln N + N$

$= N\ln\frac{q}{N} + N + \frac{N^2}{q} \approx N\ln\frac{q}{N} + N$

$\therefore \Omega = e^{N\ln\frac{q}{N}+N} = e^N(\frac{q}{N})^N = (\frac{eq}{N})^N$

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(283) $\underline{x = \frac{q}{2\sqrt{N}}}$ (how sharp is Ω peak for interacting solids?)

recaall: $\Omega_{tot} = \Omega_A\Omega_B$; $\Omega = (\frac{eq}{N})^N$

$\implies \Omega_{tot} = (\frac{eq_A}{N})^N(\frac{eq_B}{N})^N = (\frac{e}{N}^{2N}(q_Aq_B)^N$

(assumed each oscillator has N solids sharp peak occurs at $q_A = \frac{q}{2}$

$\implies \Omega_{max} = (\frac{e}{N})^{2N}(\frac{q}{2})^{2N}$

What does it look like near peak?

$q_A = \frac{q}{2} + x,\ q_B = \frac{q}{2} - x,\ x << q$

$\implies \Omega = (\frac{e}{N})^{2N}((\frac{q}{2})^2 - x^2)^N$

$\ln[(\frac{q}{2})^2 - x^2]^N = N\ln[(\frac{q}{2})^2(1-(\frac{2x}{q})^2)]$

$N[\ln(\frac{q}{2})^2 + \ln(1-(\frac{2x}{q})^2)]$

$\approx N[\ln(\frac{q}{2})^2 - (\frac{2x}{q})^2]$

$\implies ((\frac{q}{2})^2 - x^2)^N = e^{N\ln(\frac{q}{2})^2}e^{-N(2x/q)^2}$

plug in

$\implies \Omega = (\frac{e}{N})^{2N}e^{N\ln(q/2)^2}e^{-N(2x/q)^2} = \Omega_{max}e^{-N(2x/q)^2}$

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(284) $\underline{\Omega(U,V,N) = f(N)V^NU^{3N/2}}$(multiplicity of monatomic ideal gas)

<u>1 molecule</u>

If we have a molecule in a box and we double the volume then we double the number of states

$\implies \Omega_1 \propto V$

also if we double the number of allowed momentum(or volume of momentum space)

this should also double Ω
$\implies \Omega_1 \propto VV_p$
constraint on momentum
$\implies U = \frac{1}{2m}(p_x^2 + p_y^2 + p_z^2) \implies p_x^2 + p_y^2 + p_z^2 = 2mU$
this gas is isolated, so the sum of all particle kinetic energy must equal U
this is a sphere in momentum space with radius $\sqrt{2mU}$
space cts $\implies$ invoke QM
$\implies (\Delta x)(\Delta p_x) = h$
$\Omega^{1D} = \frac{L}{\Delta x}\frac{L_{px}}{\Delta p_x} = \frac{LL_p}{h}$
$\implies \Omega_1 = \frac{VV_p}{h^3}$
<u>2 molecules</u>
$\implies p_{1x}^2 + p_{1y}^2 + p_{1z}^2 + p_{2x}^2 + p_{2y}^2 + p_{2z}^2 = 2mU$
$\implies$ 6 - D sphere
$\Omega_2 \propto V_1V_2V_p \implies \Omega_2^{1D} = \frac{L}{\Delta x}\frac{L}{\Delta x}\frac{A}{(\Delta p)^2}\frac{L^2A_p}{h^2}$
$A_p \sim$ Area of momentum hypersphere
if $1, 2$ are indistinguishable
$\implies \Omega_2^{3D} = \frac{1}{2}\frac{V^2A_p}{h^6}$
$\implies \Omega_N = \frac{1}{N!}\frac{V^N}{h^3N}A_p$
$A_p = \frac{2\pi^{d/2}}{(\frac{d}{2}-1)!}r^{d-1},\ d = 3N,\ r = \sqrt{2mU}$
$\implies \Omega_N = \frac{1}{N!}\frac{V^N}{h^{3N}}\frac{2\pi^{3N/2}}{(\frac{3N}{2}-1)!}(\sqrt{2mU})^{3N-1}$
$\approx \frac{1}{N!}\frac{V^N}{h^{3N}}\frac{\pi^{3N/2}}{(\frac{3N}{2})!}(\sqrt{2mU})^{3N}$
or
$\Omega(U, V, N) = f(N)V^NU^{3N/2}$

<u>Interacting Ideal gasses</u>
$\Omega_{tot} = [f(N)]^2(V_AV_B)^N(U_AU_B)^{3N/2}$
width of peak $= \frac{U_{total}}{\sqrt{3N/2}}$

$S \equiv k\ln\Omega$
$S_{tot} = kln\Omega_A\Omega_B = S_A + S_B$
<u>recall:</u> $\Omega_N = \frac{1}{N!}\frac{V^N}{h^{3N}}\frac{\pi^{3N/2}}{(3N/2)!}(\sqrt{2mU})^{3N}$
$\ln\Omega_n = \ln\frac{1}{N!} + \ln\frac{V^N}{h^{3N} + \ln(\frac{\pi^{3N/2}}{(3N/2}!}) + \frac{3N}{2}\ln 2mU$
$= -\ln N! + N\ln V - 3N\ln N + \frac{3N}{23}\ln\pi - \ln(\frac{3N}{2})!$
$+ \frac{3N}{2}\ln 2mU$
<u>recall:</u> $\ln N! \approx N\ln N - N$

$\implies \Omega = -N\ln N + N + N\ln V - 3N\ln h + \frac{3N}{2}\ln\pi - \frac{3N}{2}\ln\frac{3N}{2} + \frac{3N}{2}$

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$S = Nk[\ln(\frac{V}{N}(\frac{4\pi mU}{3Nh^2})^{3/2}) + \frac{5}{2}]$ (Derive)
$\Delta U = W + Q = 0$ (freely expanding gas does no work and heat does not flow into or out of the gas)

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(285) $\underline{\Delta S_{total} = \Delta S_A + \Delta S_B = 2Nk\ln 2}$ (entropy of mixing)
recall: $S = Nk[\ln(\frac{V}{N}(\frac{4\pi mU}{3NhT^2})^{3/2}) + \frac{5}{2}]$
const U, N; changing V
$\implies \Delta S = Nk\ln(\frac{V_f}{V_i})$
Lets mix two gases with equal volume initially separated by a partition
$\Delta S_A = Nk\ln(\frac{2V}{V}) = Nk\ln 2;\ \Delta S_B = Nk\ln 2$
$\therefore \Delta S_{tot} = \Delta S_A + \Delta S_B = 2Nk\ln 2$

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Chapter 3

After two objects have been in contact long enough, we say they are in thermal equilibrium

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(286) $\underline{\frac{1}{T} \equiv (\frac{\partial S}{\partial U})_{N,V} \implies T = (\frac{\partial U}{\partial S})_{N,V}}$
temperature is the thing that is the same when they are in thermal equilibrium
entropy for the total system is maximized in thermal equilibrium, which means if I added a small amount of energy to A in the combined system the entropy does not change
$\implies \frac{\partial S_{tot}}{\partial q_A} = 0 \implies \frac{\partial S_{tot}}{\partial U_A} = 0$
$\implies \frac{\partial S_A}{\partial U_A} + \frac{\partial S_B}{\partial U_A} = \frac{\partial S_A}{\partial U_A} - \frac{\partial S_B}{\partial U_B} = 0$
$\implies \frac{\partial S_A}{\partial U_A} = \frac{\partial S_B}{\partial U_B}$
analytic dimensions $\frac{J/k}{J} = \frac{J}{J}\frac{1}{k} = \frac{1}{K}$
$\implies (\frac{\partial S}{\partial U})_{V,N} = \frac{1}{T} \implies T = (\frac{\partial U}{\partial S})_{N,V}$

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$C_V \equiv (\frac{\partial U}{\partial T})_{N,V},\ U = NkT(EinsteinSolid)(from$
$T = (\frac{\partial S}{\partial U})^{-1})$

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$\Delta S = \int \frac{dU}{T} = \int \frac{1}{T}(\frac{\partial U}{\partial T})_{N,V} dT = \int_0^{T_f} \frac{C_V}{T} dT = S_f - S(0)$
$S(0) = 0$ (3rd law of thermodynamics)

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Magnetic dipole

recall: $-\vec{\mu} \cdot \vec{B} = \epsilon_{dipole}$
$E = N_\uparrow \epsilon_{dip}^\uparrow + N_\downarrow \epsilon_{dip}^\downarrow = -N_\uparrow \mu B + N_\downarrow \mu B$
$= (N_\downarrow - N_\uparrow)\mu B = \mu B(N - 2N_\uparrow)$
$M = \mu(N_\uparrow - N_\downarrow) = -\frac{U}{B}$

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(287) $M = N\mu \tanh(\frac{\mu B}{kT});\ U = -N\mu B \tanh(\frac{\mu B}{kT})$
recall: $\Omega(N_\uparrow) = \frac{N!}{N_\uparrow ! N_\downarrow !};\ \ln N! \approx N \ln N - N$
$\implies \frac{S}{k} = \ln \Omega(N_\uparrow) = \ln N! - \ln N_\uparrow ! - \ln N_\downarrow !$
$= \ln N! - \ln N_\uparrow ! - \ln(N - N_\uparrow)!$
$\approx N \ln N - N - N_\uparrow \ln N_\uparrow + N_\uparrow - (N - N_\uparrow)\ln(N - N_\uparrow) + (N - N_\uparrow)$
$= N \ln N - N_\uparrow \ln N_\uparrow - (N - N_\uparrow)\ln(N - N_\uparrow)$
$\implies \frac{1}{T} = (\frac{\partial S}{\partial U})_{N,B} = \frac{\partial N_\uparrow}{\partial U}\frac{\partial S}{\partial N_\uparrow}$
recall :$U = \mu B(N - 2N_\uparrow) \implies \frac{\partial N_\uparrow}{\partial U} = -2\mu B$
$\implies \frac{1}{T} = -\frac{1}{2\mu B}\frac{\partial S}{\partial N_\uparrow}$
$\frac{\partial S}{\partial N_\uparrow} = k(-\ln N_\uparrow - \frac{N_\uparrow}{N_\uparrow} + \ln(N - N_\uparrow) + 1)$
$= k(\ln(N - N_\uparrow) - \ln N_\uparrow) = k \ln(\frac{N - N_\uparrow}{N_\uparrow})$
recall: $U = \mu B(N - 2N_\uparrow) \implies N_\uparrow = \frac{N}{2} - \frac{U}{2\mu B}$
$\implies \frac{\partial S}{\partial N_\uparrow} = k \ln(\frac{N - \frac{N}{2} + \frac{U}{2\mu B}}{\frac{N}{2} - \frac{U}{2\mu B}}) = k \ln(\frac{N + \frac{U}{\mu B}}{N - \frac{U}{\mu B}})$
$\implies \frac{1}{T} = \frac{k}{2\mu B}\ln(\frac{N - \frac{U}{\mu B}}{N + \frac{U}{\mu B}})$
Solve for U
$\implies U = N\mu B(\frac{1 - e^{-2\mu B/kT}}{1 + e^{2\mu B/kT}}) = -N\mu B \tanh(\frac{\mu B}{kT})$
$M = -\frac{U}{B} = N\mu \tanh(\frac{\mu B}{kT});\ C_B = (\frac{\partial U}{\partial T})_{N,B}$

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(288) $P \equiv T(\frac{\partial S}{\partial V})_{U,N}$
Pressure is the ' thing' that is the same when two systems are in mechanical equilibrium (sort of)
$\implies \frac{\partial S_{tot}}{\partial U_A} = 0,\ \frac{\partial S_{tot}}{\partial V_A} = 0$
$\implies \frac{\partial S_A}{\partial V_A} + \frac{\partial S_B}{\partial V_A} = \frac{\partial S_A}{\partial V_A} - \frac{\partial S_B}{\partial V_B} = 0$
$\implies \frac{\partial S_A}{\partial V_A} = \frac{\partial S_B}{\partial V_B} \implies \frac{J/K}{m^3} = \frac{Nm}{Km^3} = \frac{N}{Km^2} = \frac{1}{T}P$
$\implies P = T(\frac{\partial S}{\partial V}_{U,N}$

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(289) $\underline{PV = NkT}$
$\underline{\text{recall:}}\ \Omega = f(N)V^N U^{3N/2}$
$S = k\ln\Omega = k\ln f(N) + Nk\ln V + \frac{3N}{2}\ln U$
$(\frac{\partial S}{\partial V})_{U,N} = \frac{Nk}{V} = \frac{P}{T} \implies PV = NkT$

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(290) $\underline{dS = \frac{1}{T}dU + \frac{P}{T}dV}$
$dS = \frac{\partial S}{\partial U}dU + \frac{\partial S}{\partial V}dV = \frac{1}{T}dU + \frac{P}{T}dV$

$dU = TdS - pdV$

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(291) $\underline{Q = TdS}$ (quasistatic)
$\underline{\text{recall:}}\ dU = TdS - PdV,\ dU = Q + W$
quasistatic changes in volume $\sim$ pressure remains constant, i.e., no energy is wasted in compression $\implies W = -PdV$
$\implies Q = TdS$

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isentropic = adiabatic ($Q = 0$) + quasistatic

If you push harder than needed $\implies W > -PdV \implies$ $dU = W + Q > -PdV + Q$
$\implies -PdV + TdS > -PdV + Q$
$\implies TdS > Q$
$\implies dS > \frac{Q}{T}$ i.e. you add extra entropy

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(292) $\underline{\mu \equiv -T(\frac{\partial S}{\partial N})_{U,V}}$
$(\frac{\partial S_{tot}}{\partial U_A})_{N_A,V_A} = 0,\ (\frac{\partial S_{tot}}{\partial N_A})_{U_A,V_a} = 0($ diffusive equillibrium)
$\implies \frac{\partial S_A}{\partial N_A} = \frac{\partial S_B}{\partial N_B}$ (at equillibrium)
need energy units so multiply by $-T$
(negative by convention)
$\implies -T\frac{\partial S_A}{\partial N_A} = -T\frac{\partial S_B}{\partial N_B}$
$\therefore \mu \equiv -T(\frac{\partial S}{\partial N})_{U,V}$

$dS = (\frac{\partial S}{\partial U})_{N,V} dU + (\frac{\partial S}{\partial V})_{U,N} dV + (\frac{\partial S}{\partial N})_{U,V} dS$
$= \frac{1}{T} dU + \frac{P}{T} dV - \frac{\mu}{T} dN$

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μ from Sakur Tetrode equation unfinished
$dU = TdS - PdV + \sum_i \mu_i dN_i$ (Thermodynamic identities) (multiple species)
$H \equiv U + PV$ (Enthalpy) $\sim$ the energy you would recover if you completely annihilated the system
$F \equiv U - TS$ (Helmholtz free energy) $\sim$ energy needed to create system minus energy you can get for free from environment.

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Chapter 5

$G \equiv U - TS + PV$ (Gibbs free energy)
Constant P and T then G is the amount of energy required for you to put in to create system from nothing

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(293) $\underline{\Delta F \leq W, \text{ Const. } T}$
$\Delta F = \Delta U - T\Delta S = Q + W - T\Delta S$
$T\Delta S \geq Q$ if "new" entropy is created
$\implies \Delta F \leq W$

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(294) $\underline{\Delta G \leq W_{other} const. T, P}$
$\Delta G = \Delta U - T\Delta S + P\Delta V = Q + W - T\Delta S + P\Delta V$
$W = W_{other} + W_{byenvironment} = -P\Delta V + W_{other}$
$\implies \Delta G \leq W_{other}$

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(295) $\underline{dS_{total} = -\frac{1}{T} dF}$
$S \sim$ system $S_R \sim$ reservoir
$dS_{total} = dS + dS_R$
$dS = \frac{1}{T} dU + \frac{P}{T} dV - \frac{\mu}{T} dN$
$V_R,\ N_R$ fixed $\implies dS_R = \frac{1}{T_R} dU_R$
$\implies dS_{total} = dS + \frac{1}{T_R} dU_R$
$dU_R = -dU,\ T_R = T$
$\implies dS_{tot} = dS - \frac{1}{T} dU = -\frac{1}{T}(dU - TdS) = -\frac{1}{T} dF$
assumed T constant

$\implies fixed T, V, N \implies$ entropy increase decreases F for system

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likewise const T, P
$\implies dS_{tot} = dS - \frac{1}{T}dU - \frac{P}{T}dV = -\frac{1}{T}(dU - TdS + PdV)$
$= -\frac{1}{T}dG$

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quantities that double if you double the amount of "Stuff" are extensive
the quantities that are unchanged are intensive
Extensive: V, N, S, U, H, F, G, mass
intensive: T, P, μ, density

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(296) $\underline{dH = TdS + VdP + \mu dN}$
recall: $H = U + PV$
$\implies dH = dU + VdP + PdV$
recall: $dU = TdS - PdV + \mu dN$
$dH = TdS - PdV + \mu dN + VdP + PdV$
$= TdS + \mu dN + VdP$

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(297) $\underline{dF = -SdT - PdV + \mu dN}$
recall: $F = U - TS$
$dF = dU - TdS - SdT$
$= TdS - PdV + \mu dN - TdS - SdT$
$= -SdT - PdV + \mu dN$

$\implies S = -(\frac{\partial F}{\partial T})_{V,N};\ P = -(\frac{\partial F}{\partial V})_{T,N};\ \mu = (\frac{\partial F}{\partial N})_{T,V}$

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(298) $\underline{dG = -SdT + VdP + \mu dN}$
recall: $G = U - TS + PV$
$\implies dG = dU - TdS - SdT + PdV + VdP$
$\implies dG = -SdT + VdP + \mu dN$

$S = -(\frac{\partial G}{\partial T})_{P,N};\ V = (\frac{\partial G}{\partial P})_{T,N};\ \mu = (\frac{\partial G}{\partial N})_{T,P}$

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(299) $\underline{G = N\mu}$
recall: $\mu = (\frac{\partial G}{\partial N})_{T,P}$

$G \sim$ extensive
$\implies \mu = \frac{G}{N} \implies G = N\mu$

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More generally $G = \sum_i N_i \mu_i$

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(300) $\underline{\mu(T,P) = \mu^\circ(T) + kT\ln(P/P^\circ)}$(ideal gas)
<u>recall:</u> $G = N\mu;\ V = \frac{\partial G}{\partial P}$
$\frac{\partial \mu}{\partial P} = \frac{1}{N}\frac{\partial G}{\partial P} = \frac{V}{N}$
<u>recall:</u> $PV = NkT$
$\implies \frac{\partial \mu}{\partial P} = \frac{kT}{P}$
$\implies \mu(T,P) - \mu(T,P^\circ) = kT\ln(\frac{P}{P^\circ})$

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(301) $\underline{\frac{dP}{dT} = \frac{L}{T\Delta V}}$ (Clausius-Clapeyron relation)
consider the boundary of PT diagram between liquid and gas.
Diffusive equilibrium $\implies$ Chemical potential are equal $\implies$ $G_\ell = G_g$ (at phase boundary change dT and dP so they remain equally stable
$\implies dG_\ell = dG_g$
$\implies -S_\ell dT + V_\ell dP = -S_g dT + V_g dP$
$\implies V_\ell \frac{dP}{dT} - S_\ell = -S_g + V_g\frac{dP}{dT}$
$\implies \frac{S_g - S_\ell}{V_g - V_\ell} = \frac{dP}{dT}$
<u>*recall* :</u>$dH = TdS + VdP$
$\implies dH = TdS$
$\implies \frac{dH}{T} = dS$
$dH \equiv L$
$\therefore \frac{L}{\Delta V T} = \frac{dP}{dT}$

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(302) $\underline{P = \frac{NkT}{V - Nb} - \frac{aN^2}{V^2} or (P + \frac{aN^2}{V^2})(V - Nb) = NkT}$
P, V, T relation is called an equation of state
<u>recall:</u> $PV = NkT$
$\implies P(V - Nb)(-Nb$ makes it uncompressible to volumes smaller than $Nb)$
Potential of a single molecule is $\propto \frac{N}{V}$ so the total potential energy
is $\propto \frac{N^2}{V} \implies U = -\frac{aN^2}{V}$
but $P = -\frac{\partial U}{\partial V} \implies P = -\frac{aN^2}{V^2}$ (pressure due to potential energy)

$\implies P = \frac{NkT}{V-Nb} - \frac{aN^2}{V^2}$

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Thermal Physics

Chapter 6

$E(s) = E(s_1, s_2)$; s is the state of a two atom system which is equivalent to saying when the system is in state s then atom 1 is in state s_1 and atom 2 is in state s_2
Note: $E(particle1, particle2)$ so indistinguishable means $E(s_1, s_2) = E(s_2, s_1)$

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(303) $\underline{Z_{tot} = Z_1 Z_2}$ (non-interacting, distinguishable)
$E_{tot}(s) = E_1(s) + E_2(s)$
$\implies Z_{total} = \sum_s e^{-\beta[E_1(s)+E_2(s)]} = \sum_s e^{-\beta E_1(s)} e^{-\beta E_2(s)}$
Note: $E(s) = E(s_1, s_2) = E_1(s_1) + E_2(s_2)$
$\implies Z_{total} = \sum_{s_1} \sum_{s_2} e^{-\beta E_1(s_1)} e^{-\beta E_2(s_2)} = Z_1 Z_2$
Note if I solved the schrodinger equation, without interactions it would be separable and so Z_{tot} would loop over every possible state, which turns into a double sum. Note that energy does not have to be the same since the system is interacting with the resevoir all of the energies are accessible.

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(304) $\underline{Z_{tot} \approx \frac{1}{N!} Z_1^N}$ (non-interacting, indistinguishable, not dense)
First Note $Z_{tot} \approx \frac{1}{2} Z_1 Z_2$ this is because if I have two particles in two different states, then it is the same state if I switch them; it is not exactly equal since they could be in the same state. More rigorously, for distinguishable particles
$Z = Z_1 Z_2 = \sum_{s_1} \sum_{s_2} e^{-\beta E(s_1, s_2)}$
$= e^{-\beta E(1,1)} + e^{-\beta E(1,2)} + e^{-\beta E(2,1)} + e^{-\beta E(2,2)}$
Not too dense means $(1,1)$ or $(2,2)$ cant happen
$\implies Z_{dist} = Z_1 Z_2 = e^{-\beta E(1,2)} + e^{-\beta E(2,1)}$
indistinguishable; $E(1,2) = E(2,1)$
$\implies Z_{dist} = Z_1 Z_2 = \left(e^{-\beta E(1,2)} + e^{-\beta E(2,1)}\right)$
$Z_{indist} = e^{-\beta E(1,2)} = \frac{1}{2}\left(e^{-\beta E(1,2)} + e^{-\beta E(2,1)}\right)$
$= \frac{1}{2} Z_{dist} = \frac{1}{2} Z_1 Z_2$
In general $Z_{tot} = \frac{1}{N!} Z_1^N$

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$Z_{tot} = Z_1 Z_2 \ldots Z_N$ (noninteracting, distinguishable)

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(305) $\underline{Z_1 = Z_{tr} Z_{int}}$
$Z_1 = \sum_s e^{-E(s)/kT} = \sum_{tr} \sum_{int} e^{-E_{tr}/kT} e^{-E_{int}} kT$
$= (\sum_{tr} e^{-E_{tr}/kT})(\sum_{int} e^{-E_{int}/kT}) = Z_{tr} Z_{int}$
Note: energy is not a constant in these partition functions, E can be basically anything since our system exchanges energy with a resevoir.

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(306) $\underline{\ell_Q \equiv \frac{\hbar}{\sqrt{2\pi mkT}}}$ (Quantum length)
Molecule in a box
recall: $E_n = \frac{h^2 n^2}{8mL^2} = E_{trans}$ (infinite square well) (particle has no potential in the box)
$Z_{1D} = \sum_n e^{-E_n/kT} = \sum_n e^{-h^2 n^2/8mL^2 kT}$
$\rightarrow \frac{1}{\Delta n} \int_0^\infty e^{-h^2 n^2/8mL^2 kT} dn = \int_0^\infty e^{-h^2 n^2/8mL^2 kT} dn$
$= \frac{\sqrt{\pi}}{2} \sqrt{\frac{8mL^2 kT}{h^2}} = \frac{L}{\frac{h}{\sqrt{2\pi mkT}}} = \frac{L}{\ell_Q}$
$\therefore \ell_Q = \frac{h}{\sqrt{2\pi mkT}}$

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(307) $\underline{V_Q = \ell_Q^3 = \frac{h}{2\pi mkT}}$
$Z_{tr} = \sum_s e^{-E_{tr}/kT} = \frac{L_x}{\ell_Q} \frac{L_y}{\ell_Q} \frac{L_z}{\ell_Q} = \frac{V}{v_Q}$
$\therefore v_Q = \ell_Q^3$

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(308) $\underline{\mu = 0}$ for photons
photons are bosons
$\implies \bar{n}_{BE} = \frac{1}{e^{(\epsilon - \mu)/kT} - 1}$
but they also follow
$\bar{n}_{Pl} = \frac{1}{e^{hf/kT} - 1} \implies \epsilon = hf, \ \mu = 0$
N is not conserved for photons and F is minimized at equilibrium
but $(\frac{\partial F}{\partial N})_{T,V} = \mu = 0$ (F is minimized)

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(309) $\underline{U = \int_0^\infty dn \int_0^{\pi/2} d\theta \int_0^{\pi/2} d\phi n^2 \sin\theta \frac{hcn}{L} \frac{1}{e^{hcn/2LkT} - 1}}$ (energy of photons in a box)
recall: $\lambda = \frac{2L}{n}; \ p = \frac{hn}{2L}$ (photons in a box)
recall: $E^2 = (pc)^2 + (m_0 c^2)^2, \ m_0 = 0$ (photons) $\implies E \equiv \epsilon = pc = \frac{hcn}{2L}$ (relativistic) (1D)

$\epsilon = c\sqrt{p_x^2+p_y^2+p_z^2} = \frac{hc}{2L}\sqrt{n_x^2+n_y^2+n_z^2} = \frac{hcn}{2L}(3D)$
$U = 2\sum_{n_x}\sum_{n_y}\sum_{n_z}\epsilon\bar{n}_{Pl}(\epsilon) = \sum_{n_x,n_yn_z}\frac{hcn}{L}\frac{1}{e^{hcn/2LkT}-1}$
2 comes from the fact that every wave can hold photons with two independent polarizations
$\implies U = \frac{1}{\Delta n^3}\int_0^\infty dn\int_0^{\pi/2}d\theta\int_0^{\pi/2}d\phi n^2\sin\theta\frac{hcn}{L}\frac{1}{e^{hcn/2LkT}-1}$
$\Delta n = 1$

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(310) $\underline{\frac{U}{V} = \frac{8\pi^5(kT)^4}{15(hc)^3}}$
recall: $U = \int_0^\infty dn\int_0^{\pi/2}d\theta\int_0^{\pi/2}d\phi n^2\sin\theta\frac{hcn}{L}\frac{1}{e^{hcn/2LkT}-1} = \frac{\pi}{2}\int_0^\infty dnn^2\frac{hcn}{L}\frac{1}{e^{hcn/2LkT}-1}$
recall: $\epsilon = \frac{hcn}{2L} \implies \frac{2L}{hc}d\epsilon = dn$
$U = \frac{\pi}{2}\frac{8L^2}{h^2c^2}\int_0^\infty d\epsilon\frac{2L}{hc}\frac{\epsilon^3}{e^{\epsilon/kT}-1}$
$= 8\pi(\frac{L}{hc})^3\int_0^\infty\frac{\epsilon^3}{e^{\epsilon/kT}-1}d\epsilon$
$\frac{U}{V} = \int_0^\infty\frac{8\pi\epsilon^3/(hc)^3}{e^{\epsilon/kT}-1}d\epsilon = \int_0^\infty u(\epsilon)d\epsilon$
$u(\epsilon) = \frac{8\pi}{(hc)^3}\frac{\epsilon^3}{e^{\epsilon/kT}-1}$(spectrum or energy density per unit photon energy)
$x = \frac{\epsilon}{kT}$
$\implies \frac{U}{V} = \frac{8\pi(kT)^4}{(hc)^3}\int_0^\infty\frac{x^3}{e^x-1}dx = \frac{8\pi^5(kT)^4}{15(hc)^3}$

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(311) $\underline{S(T) = \frac{32\pi^5}{45}V(\frac{kT}{hc})^3k}$ (entropy of a photon gas)
recall: $dU = TdS$ (constant volume); $C_V = (\frac{\partial U}{\partial T})_V \implies dU = C_VdT$
$\implies \frac{dU}{T} = \frac{C_VdT}{T}$
$\implies S = \int_0^T\frac{dU}{T} = \int_0^T\frac{C_V(T')}{T'}dT'$
this step used S(T=0)=0 recall: $U = \frac{8\pi^5}{15}\frac{(kT)^4}{(hc)^3}V$
$\implies \frac{\partial U}{\partial T} = \frac{8\pi^5}{15}k4\frac{(kT)^3}{(hc)^3}V = 4aT^3$
$\implies S = \int_0^T\frac{C_V}{T'}dT' = 4a\int_0^T(T')^2dT' = \frac{4}{3}aT^3$
$= \frac{32\pi^5}{45}V(\frac{kT}{hc})^3k$
Weinberg 1977 has a discussion of early universe dynamics. problem 7.49 is a good problem

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<u>power per unit area $= \sigma T^4$; (Stefan's law $\sigma = \frac{2\pi^5 k^4}{15h^3c^2}$ (Stefan-Boltzmann constant)</u>
given photons in a box we want to determine how many will escape through a hole, refer to diagram 7.23
Volume of chunk $= Rd\theta R\sin\theta d\phi cdt$
<u>recall:</u> $\frac{U}{V} = \frac{8\pi^5}{15}\frac{(kT)^4}{(hc)^3}$
energy in chunk $= \frac{U}{V}cdtR^2\sin\theta d\theta d\phi$
probability of escape $= \frac{Areaviewedfromchunk}{Areaofspherearoundchunk} = \frac{A'}{4\pi R^2} = \frac{A\cos\theta}{4\pi R^2}$
A' is the area of the hole that lies on the surface of the sphere
energy escaping from chunk $= \frac{A\cos\theta}{4\pi}\frac{U}{V}cdt\sin\theta d\theta d\phi$
total energy escaping $= \int_0^{2\pi} d\phi \int_0^{\pi/2} d\theta \frac{A\cos\theta}{4\pi}\frac{U}{V}cdt\sin\theta$($\pi/2$ since we integrate over half of a sphere)
$= 2\pi\frac{A}{4\pi}\frac{U}{V}cdt\int_0^{\pi/2}\cos\theta\sin\theta d\theta = \frac{A}{4}\frac{U}{V}cdt$
$\implies$ power per unit area $= \frac{c}{4}\frac{U}{V}$
$\therefore$ power per unit area $= \frac{2\pi^5}{15}\frac{(kT)^4}{h^3c^2 2} = \sigma T^4$

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<u>Theorem</u>

A black body and a hole with the same size and same temperature, emit em waves with the same intensity, so Stefan's law works for a black body

<u>Proof</u>

Assume the hole and the black body are the same size and same temperature. Suppose the black body emits less power than the hole, then more energy will flow from the hole to the black object and the black object will get hotter, this violates the second law of thermodynamics since we know heat travels from hotter objects to colder objects.
The proof in the case that the black body emits more is similar
$\therefore$ they must emit the same energy
<u>Note:</u> A black body does not reflect light , it only absorbs it.
not only are the intensities identical, their spectrum must be too.
If it is not black, but reflects some amount of light then Stefan's Law becomes Power $= \sigma eAT^4$

(Skipped 7.108 - 7.117 (Debye))

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(312) $\underline{N_0 = N - N_{excited} = [1 - (\frac{T}{T_c})^{3/2}]N}$ $(T < T_c)$

Lets consider bosons as atoms with integer spin, assume they are in a box of length L

$\implies \epsilon_0 = \frac{h^2}{8mL^2}(1^2 + 1^2 + 1^2) = \frac{3h^2}{8mL^2}$ (ground state)

recall: $\bar{n}_{BE} = \frac{1}{e^{(\epsilon-\mu)/kT} - 1}$ (Bose-Einstein distribution)

$\implies N = \int_0^\infty g(\epsilon)n(\epsilon)d\epsilon = \int_0^\infty g(\epsilon)\frac{1}{e^{(\epsilon-\mu)/kT}-1}d\epsilon$

$g(\epsilon) = \frac{g_{electrons}(\epsilon)}{2} = \frac{2}{\sqrt{\pi}}(\frac{2\pi m}{h^2})^{3/2}V\sqrt{\epsilon}$

(only one spin orientatin)

for $\mu = 0$ (low temp, N_0 large)

$x = \epsilon/kT$

$\implies \frac{2}{\sqrt{\pi}}(\frac{2\pi m}{h^2})^{3/2}V\int_0^\infty \frac{\sqrt{\epsilon}}{e^x-1}d\epsilon$

$= \frac{2}{\sqrt{\pi}}(\frac{2\pi mkT}{h^2})^{3/2}V\int_0^\infty \frac{\sqrt{x}}{e^x-1}dx$

$\int_0^\infty \frac{\sqrt{x}}{e^x-1}dx \approx 2.315$ (numerical)

$\implies N = 2.315\frac{2}{\sqrt{\pi}}(\frac{2\pi mkT}{h^2})^{3/2}V$

$\approx 2.612(\frac{2\pi mkT}{h^2})^{3/2}V$ (this is wrong number of atoms, does not depend on T)

there is one temperature where this would work $T \equiv T_c$

$\implies N = 2.612(\frac{2\pi mkT_c}{h^2})^{3/2}V$

When $T > T_c\mu$ must be less than zero $\mu < 0 \implies N$ smaller

$T < T_c$ changing sum to integral is not valid

$\implies N_{excited} = 2/612(\frac{2\pi mkT}{h^2})^{3/2}V$ $(T < T_c)$

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$N_{excited} = (\frac{T}{T_c})^{3/2}N(T < T_c)$

$N_{ground} \equiv N_0 = N - N_{excited} = [1 - (\frac{T}{T_c})^{3/2}]N(T < T_c)$

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(313) $\underline{\mathcal{P} = \frac{1}{Z}e^{-E(s)/kT}}$(Boltzmann distribution)

If the atom were isolated, then its energy would be fixed and all microstates are equally probable, however when it interacts with a resovoir, some states are more probable, unless microstates have same energy, the atom is not isolated but the resevoir and atom is isolated so all microstates are equally probable. Consider $\Omega_R(s_1)$ the multiplicity of the resevoir when the atom is in state s_1 $\frac{\mathcal{P}(s_2)}{\mathcal{P}(s_1)} = \frac{\Omega_R(s_2)}{\Omega_R(s_1)}$

recall: $S = kln\Omega \implies \Omega = e^{S/k}$

$\implies \frac{\mathcal{P}(s_2)}{\mathcal{P}(s_1)} = e^{[S_R(s_2) - S_R(s_1)]/k}$

recall: $dS_R = \frac{1}{T}(dU_R + PdV_R - \mu dN_R)$

$dV_R \approx 0,\ dN_R = 0$
$\implies dS_R = \frac{1}{T} dU_R$ (thermal equillibrium)
$\implies S_R(s_2) - S_R(s_1) = \frac{1}{T}[U_R(s_2) - U_R(s_1)]$
$= -\frac{1}{T}[E(s_2) - E(s_1)]$
$\implies \frac{\mathcal{P}(s_2)}{\mathcal{P}(s_2)} = e^{-[E(s_2)-E(s_1)]/kT} = \frac{e^{-E(s_2)/kT}}{e^{-E(s_1)/kT}}$
Boltzmann factor $\equiv e^{-E(s)/kT}$
$\implies \frac{\mathcal{P}(s_2)}{e^{-E(s_2)/kT}} = \frac{\mathcal{P}(s_1)}{e^{-E(s_1)/kT}} = \frac{1}{Z}$
$\therefore \mathcal{P}(s) = \frac{1}{Z} e^{-E(s)/kT}$

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(314) $\underline{Z = \sum_s e^{-E(s)/kT}}$
$\sum_s \mathcal{P}(s) = \frac{1}{Z}\sum_s e^{-E(s)/kT} = 1$
$\therefore Z = \sum_s e^{-E(s)/kT}$
(315) $\underline{\bar{X} = \sum_s X(s)P(s)}$
$\bar{X} = \frac{\sum_s X(s)N(s)}{N} = \sum_s X(s)P(s)$

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$U = N\bar{E}$ (identical independent particles)

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(316) $\underline{M = N\bar{\mu}_z = N\mu\tanh(\beta\mu B);\ U = -N\mu B\tanh(\beta\mu B)}$
<u>recall:</u> $U = -\vec{\mu}\cdot\vec{B};\ U_{up} = -\mu B;\ U_{down} = \mu B$
$Z = \sum_s e^{-\beta E(s)} = e^{\beta\mu B} + e^{-\beta\mu B} = 2\cosh\beta\mu B$
$P_\uparrow = \frac{e^{\beta\mu B}}{2\cosh(\beta\mu B)};\ P_\downarrow = \frac{e^{-\beta\mu B}}{2\cosh(\beta\mu B)}$
$\bar{E} = \sum_s E(s)P(s) = -\mu BP_\uparrow + \mu BP_\downarrow = -\mu B(P_\uparrow - P_\downarrow)$
$= -\mu B\frac{e^{\beta\mu B} - e^{-\beta\mu B}}{2\cosh(\beta\mu B)} = -\mu B\tanh(\beta\mu B)$
$U = N\bar{E} = -N\mu B\tanh(\beta\mu B)$
$\bar{\mu}_z = \sum_s \mu_z(s)P(s) = \mu P_\uparrow - \mu P_\downarrow = \mu\tanh(\beta\mu B)$
$\therefore M = N\bar{\mu}_z = N\mu\tanh(\beta\mu B)$

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(317) $\underline{E_n = \frac{\hbar^2}{2I}n(n+1)}$
<u>recall:</u> $E = \frac{1}{2}I\omega^2 = \frac{\ell^2}{2I};\ \ell^2 = \hbar^2 n(n+1)$
$\therefore E_n = \frac{1}{2I}\hbar^2 n(n+1)$
$\star$
(318) $\underline{Z_{rot} \approx \frac{kT}{\epsilon}(kT >> \epsilon)}$ (diatomic distinguishable)
$kT >> \epsilon$ makes the approximation of turning the sum into an integral accurate. next time check if $j = \ell$ in this case,

pretty sure it does, in that case it would make perfect sense that the degeneracy of ψ_ℓ^m is $2\ell+1$, since $-\ell \le m \le \ell$ recall: $E(j) = j(j+1)\epsilon, E_n = n(n+1)\hbar^2/2I, H\psi = E\psi$(165Griffiths)
$\mathrm{L}^2\psi = \hbar^2\ell(\ell+1)\psi,\ [\mathrm{L}^2, \mathrm{H}] = 0$
$\Longrightarrow$ simultaneous eigen functions $\Longrightarrow$ ψ_ℓ^m is eigenfunction of H but $-\ell \le m \le \ell$
$\Longrightarrow$ each value of E_n has $2n+1$ eigen functions (not completely correct but we are on the right track
$\Longrightarrow\ Z_{rot} = \sum_{j=0}^{\infty} N(j)e^{-E(j)/kT} = \sum_{j=0}^{\infty}(2j+1)e^{-j(j+1)\epsilon/kT}\frac{\Delta j}{\Delta j}$
where $N(j) = (2j+1)$ is the number of states with energy $E(j)$ (degenerate)
$\Delta j = 1$
$\Longrightarrow\ Z_{rot} \approx \int_0^\infty (2j+1)e^{-j(j+1)\epsilon/kT}dj = \frac{kT}{\epsilon}(kT >> \epsilon)$

If atoms are identical, then switching the two atoms would result in the same state so $Z_{rot} = \frac{kT}{2\epsilon}$ (indistinguishable $kT >> \epsilon$)

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(319) $\underline{\bar{E}_{rot} = kT}$ $(kT >> \epsilon)$
recall: $Z_{rot} = \frac{kT}{\epsilon} = \frac{1}{\epsilon\beta}$ $\bar{E}_{rot} = -\frac{1}{Z}\frac{\partial Z}{\partial \beta} = -\beta\frac{\epsilon}{\epsilon}\frac{\partial}{\partial\beta}\frac{1}{\beta} = \beta\frac{1}{\beta^2} = \frac{1}{\beta} = kT$

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(320) $\underline{\overset{\star}{\bar{E}} = \frac{1}{2}kT}$ (equipartition theorem)
assume energy in the form of quadratic degrees of freedom
Since this describes an object of temperature T, the spacing in energy of each molecule is spaced very close together. $\Longrightarrow$ $E(q) = cq^2,\ q \sim$ coordinate variable
$\Longrightarrow\ Z = \sum_q e^{-\beta E(q)} = \sum_q e^{-\beta cq^2} = \frac{1}{\Delta q}\sum_q e^{-\beta cq^2}\Delta q$
$= \frac{q}{\Delta q}\int_{-\infty}^{\infty} e^{-\beta cq^2}dq;\ x = \sqrt{\beta c}q \Longrightarrow dq = \frac{dx}{\sqrt{\beta c}}$
$\Longrightarrow\ Z = \frac{q}{\Delta q}\frac{1}{\sqrt{\beta c}}\int_{-\infty}^{\infty} e^{-x^2}dx = \frac{\sqrt{\pi c}}{\Delta q}\beta^{-1/2} = C\beta^{-1/2}$
$\bar{E} = -\frac{1}{Z}\frac{\partial Z}{\partial\beta} = -\frac{1}{C\beta^{-1/2}}\frac{\partial}{\partial\beta}C\beta^{-1/2}$
$= \frac{1}{2}\beta^{-1} = \frac{1}{2}kT$
Skipped 6.4

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(321) $\underline{F = -kT\ln Z}$
Z is like Ω so $\ln\Omega$ increases $\Longrightarrow$ $\ln Z$ increases

$\implies -\ln Z$ decreases and $\ln Z$ dimensionless
F decreases and $\frac{F}{kT}$ is dimensionless
$\therefore \frac{F}{kT} = -\ln Z \implies F = -kT\ln Z$

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(322) $\underline{F = -kT\ln Z}$
<u>proof</u>
<u>recall:</u> $F = U - TS$; $(\frac{\partial F}{\partial T})_{V,N} = -S$
Solve for $S \implies \frac{F-U}{T} = -S$
$\implies (\frac{\partial F}{\partial T})_{V,N} = \frac{F-U}{T}$
We show $F = -kT\ln Z$ obeys same ODE with same initial condition
Let $\tilde{F} = -kT\ln Z$
$\implies \frac{\partial \tilde{F}}{\partial T} = \frac{\partial}{\partial T}(-kT\ln Z) = -kT\ln Z - kT\frac{\partial}{\partial T}\ln Z$
$\frac{\partial}{\partial T}\ln Z = \frac{\partial \beta}{\partial T}\frac{\partial}{\partial \beta}\ln Z$
$= -\frac{1}{kT^2}\frac{1}{Z}\frac{\partial Z}{\partial \beta} = -\frac{U}{kT^2}$
$\implies \frac{\partial \tilde{F}}{\partial T} = -k\ln Z - kT\frac{U}{kT^2} = \frac{\tilde{F}}{T} - \frac{U}{T} = \frac{\tilde{F}-U}{T}$
<u>recall:</u> $F = U - TS$ so at $T = 0 F = U = U_0$
$\implies Z(T=0) = e^{-U_0/kT}$
(a horrible explanation for why the above equation is true: at $T = 0$ all boltzmann factors are infinitely suppressed and everything must occupy a minimum energy U_0)
$\implies \tilde{F}(0) = -kT\ln Z(0) = U_0 = F(0)$
$\therefore \tilde{F} = F$

Chapter 7

(323) $\underline{\bar{n}_{Boltzmann} = e^{-(\epsilon-\mu)/kT}}$ (dont understand)
<u>Note:</u> They are independent particles so their energies do not add when more particles are added to the state
$\bar{n}_{Boltzmann} = \sum_n nP(s) = NP(s) = \frac{N}{Z_1}e^{-\epsilon/kT}$
<u>*recall :*</u> $\mu = -kTln(\frac{Z}{N}) \implies Ne^{-\mu/kT} = Z_1$
$\implies \bar{n}_{Boltzmann} = e^{-(\epsilon-\mu)/kT}$

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(324) $\underline{\bar{n} = \frac{1}{e^{(\epsilon-\mu)/kT}+1}}$ (fermions)
$P(n) = \frac{1}{Z}e^{-(n\epsilon-\mu n)/kT} = \frac{1}{Z}e^{-n(\epsilon-\mu)/kT}$
$Z = \sum_n e^{-(n\epsilon-\mu n)/kT} = 1 + e^{-(\epsilon-\mu)/kT}$
$\bar{n} = \sum_n nP(n) = 0 \cdot P(0) + P(1)$

$= \frac{e^{-(\epsilon-\mu)/kT}}{1+e^{-(\epsilon-\mu)/kT}} = \frac{1}{e^{(\epsilon-\mu)/kT}+1}$

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(325) $\bar{n} = \frac{1}{e^{(\epsilon-\mu)/kT}-1}$ (bosons)
$Z = \sum_{n=0}^{\infty} e^{-n(\epsilon-\mu)/kT} = \sum_{n=0}^{\infty}(e^{-(\epsilon-\mu)/kT})^n = \frac{1}{1-e^{-(\epsilon-\mu)/kT}}$
$\bar{n} = -\frac{1}{Z}\frac{\partial Z}{\partial x}; x = \frac{\epsilon-\mu}{kT}$
$\bar{n} = -(1-e^{-x})\frac{\partial}{\partial x}(1-e^{-x})^{-1} = (1-e^{-x})e^{-x}(1-e^{-x})^{-2}$
$= \frac{e^{-x}}{1-e^{-x}} = \frac{1}{e^x-1} = \frac{1}{e^{(\epsilon-\mu)/kT}-1}$

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Note: $\epsilon_F = \mu(T=0)$ (Fermi energy) (Fermi-Dirac distribution becomes step function)

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degenerate gas when a gas of fermions is cold enough that most states are below ϵ_F

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(326) $\lambda_n = \frac{2L}{n};\ p_n = \frac{h}{\lambda_n} = \frac{hn}{2L}$ (infinite square well)
recall:$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V\psi = E\psi;\ V = 0$ if $0 < x < L$; ∞ o.w.
$\psi'' = \frac{-2mE}{\hbar^2}\psi \implies k = \pm\sqrt{\frac{2mE}{\hbar^2}}i$
$\implies \psi(x) = A\cos(kx) + B\sin(kx)$
$\psi(0) = A = 0,\ \psi(L) = B\sin(kL) = 0 \implies \sin(kL) = 0$
$\implies kL = n\pi \implies k = \frac{n\pi}{L} \implies \frac{2\pi}{\lambda} = \frac{n\pi}{L} \implies \lambda_n = \frac{2L}{n}$
$p_n = \frac{h}{\lambda_n} = \frac{hn}{2L},\ E = \frac{\hbar^2k^2}{2m} = \frac{h^2}{8mL^2}|\vec{n}|^2$

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$U = 2\sum_{n_x}\sum_{n_y}\sum_{n_z}\epsilon(\vec{n}) = 2\iiint \epsilon(\vec{n})dn_xdn_ydn_z$ at (T=0)

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$N = 2\sum_{n_x}\sum_{n_y}\sum_{n_z} = 2\frac{1}{\Delta n_x\Delta n_y\Delta n_z}\iiint dn_xdn_ydn_z = 2\frac{1}{8}(\frac{4}{3}\pi n_{max}^3)$
$= \frac{\pi n_{max}^3}{3} \implies \epsilon_F = \frac{h^2n^2}{8mL^2} = \frac{h^2}{8m}(\frac{3N}{\pi V})^{2/3}$at (T=0)

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(327) $U = \frac{3}{5}N\epsilon_F$
$U = 2\sum_{n_x}\sum_{n_y}\sum_{n_z}\epsilon(\vec{n}) = 2\iiint \epsilon(\vec{n})dn_xdn_ydn_z$
$U = 2\int_0^{n_{max}} dn\int_0^{\pi/2} d\theta\int_0^{\pi/2} d\phi n^2\sin\theta\epsilon(n)$
recall: $\epsilon = \frac{|\vec{p}|^2}{2m} = \frac{h^2}{8mL^2}(n_x^2+n_y^2+n_z^2) = \frac{h^2}{8mL^2}n^2$
$\implies U = \pi\int_0^{n_{max}}\epsilon(n)n^2dn = \frac{\pi h^2}{8mL^2}\int_0^{n_{max}} n^4dn = \frac{\pi h^2 n_{max}^5}{40mL^2} =$

$\frac{3}{5}N\epsilon_F$

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(328) $\underline{\frac{V}{N} << v_Q \implies kT << \epsilon_F}$

recall: $v_Q = (\frac{h}{\sqrt{2\pi mkT}})^3,\ \epsilon_F = \frac{h^2}{8m}(\frac{3N}{\pi V})^{2/3}$

$\implies \frac{V}{N} << \frac{h^3}{(2\pi mkT)^(3/2)} \implies (kT)^{3/2} << \frac{h^3 N}{V(2\pi m)^{3/2}}$

$\implies kT << \frac{h^2 N^{2/3}}{V^{2/3}(2\pi m)}$

$kT << \frac{h^2 N^{2/3}}{V^{2/3}(2\pi m)} = \frac{h^2}{2\pi m}(\frac{N}{V})^{2/3} \approx \frac{h^2 3^{2/3}}{8\pi^{2/3}m}(\frac{N}{V})^{2/3}$

$= \frac{h^2}{8m}(\frac{3N}{\pi V})^{2/3} = \epsilon_F$

$\star\star$ thermal

(329) $\underline{C_V = (\frac{\partial U}{\partial T})_V = \frac{\pi^2 Nk^2 T}{2\epsilon_F}}$

Suppose temperature of degenerate electron gas is almost zero, if we increase T, the number of electrons that can jump to higher energy is proportional to T

(# affected electrons) $\propto N;\ N \sim$ extensive

additional energy $\propto$ (# affected electrons)(energy acquired by each)

$\propto (NkT)\cdot(kT) \propto N(kT)^2 \sim$ units of J^2

so divide by ϵ_F to get J

$\implies$ additional energy $\propto \frac{N(kT)^2}{\epsilon_F}$

$\implies U_{add} = \frac{\pi^2}{4}N\frac{(kT)^2}{\epsilon_F}$

recall: $U = \frac{3}{5}N\epsilon_F($ at $T = 0)$

$\implies U_{tot} = \frac{3}{5}N\epsilon_F + \frac{\pi^2}{4}N\frac{(kT)^2}{\epsilon_F}$ (small T)

$\therefore C_V = (\frac{\partial U}{\partial T})_V = \frac{\pi^2 Nk^2 T}{2\epsilon_F}$

$\star\star$

(330) $\underline{g(\epsilon) = \frac{\pi(8m)^{3/2}}{2\hbar^3}V\sqrt{\epsilon} = \frac{3N}{2\epsilon_F^{2/3}}\sqrt{\epsilon}}$ I wonder if we could obtain this using $N = 2\sum_{\vec{n}}$

recall: $p_n = \frac{h}{\lambda_n};\ \epsilon = \frac{p^2}{2m} \implies \epsilon = \frac{h^2}{8mL^2}n^2$

$\implies n = \sqrt{\frac{8mL^2}{h^2}}\sqrt{\epsilon} \implies dn = \sqrt{\frac{8mL^2}{h^2}}\frac{1}{2}\frac{1}{\sqrt{\epsilon}}d\epsilon$

recall: $U = 2\iiint \epsilon(\vec{n})dn_x dn_y dn_z = 2\int_0^{n_{max}} dn \int_0^{\pi/2} d\theta \int_0^{\pi/2} d\phi n^2 \sin\theta \epsilon(n) = 2\frac{\pi}{2}\int_0^{n_{max}} dn n^2 \epsilon(n);\ \epsilon = \frac{h^2}{8mL^2}n^2$; plug dn, n in

$\implies U = 2\frac{\pi}{2}\int_0^{\epsilon_F} d\epsilon(\sqrt{\frac{8mL^2}{h^2}}\frac{1}{2}\frac{1}{\sqrt{\epsilon}})(\sqrt{\frac{8mL^2}{h^2}}\sqrt{\epsilon})^2\epsilon$
$= \frac{\pi}{2}\int_0^{\epsilon_F} d\epsilon((\frac{8mL^2}{h^2})^{3/2}\sqrt{\epsilon})\epsilon d\epsilon$
$\therefore g(\epsilon) = \frac{\pi(8m)^{3/2}}{2h^3}V\sqrt{\epsilon} = \frac{3N}{2\epsilon_F^{3/2}}\sqrt{\epsilon}$

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Note: $N = \int_0^{\epsilon_F} g(\epsilon)d\epsilon(atT = 0)$

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$N = \int_0^{\infty} g(\epsilon)$ (probability state is occupied) (I think this line is wrong since int n_{FD} doesn't equal 1) $d\epsilon = \int_0^{\infty} g(\epsilon)\bar{n}_{FD}(\epsilon)d\epsilon = \int_0^{\infty} g(\epsilon)\frac{1}{e^{(\epsilon-\mu)/kT}+1}d\epsilon(anyT)$
$U = \int_0^{\infty} \epsilon g(\epsilon)\bar{n}_{FD}(\epsilon)d\epsilon = \int_0^{\infty} \epsilon g(\epsilon)\frac{1}{e^{(\epsilon-\mu)/kT}+1}d\epsilon(anyT)$

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★★

(331) $\underline{N = \frac{2}{3}g_0\mu^{3/2} + \frac{1}{4}g_0(kT)^2\mu^{-1/2}\cdot\frac{\pi^2}{3} + \dots}$ (assume $kT << \epsilon_F$) (Sommerfeld expansion)
recall: $N = \int_0^{\infty} g(\epsilon)\bar{n}_{FD}(\epsilon)d\epsilon;\ g(\epsilon) = \frac{3N}{2\epsilon_F^{3/2}}\sqrt{\epsilon} = g_0\sqrt{\epsilon}$
$\rightarrow= g_0\int_0^{\infty}\sqrt{\epsilon}\bar{n}_{FD}(\epsilon)d\epsilon$
$U = \bar{n}_{FD}(\epsilon) \rightarrow dU = \frac{d\bar{n}_{FD}(\epsilon)}{d\epsilon}d\epsilon;\ dV = g_0\epsilon^{1/2}d\epsilon \implies V = g_0\frac{2}{3}\epsilon^{3/2}$
$(\frac{2}{3}g_0\epsilon^{3/2}\bar{n}_{FD}(\epsilon)|_0^{\infty} + \frac{2}{3}g_0\int_0^{\infty}\epsilon^{3/2}(-\frac{d\bar{n}_{FD}(\epsilon)}{d\epsilon})d\epsilon$
$= \frac{2}{3}g_0\int_0^{\infty}\epsilon^{3/2}(-\frac{d\bar{n}_{FD}(\epsilon)}{d\epsilon})d\epsilon$
$-\frac{d\bar{n}_{FD}}{d\epsilon} = -\frac{d}{d\epsilon}(e^{(\epsilon-\mu)/kT}+1)^{-1} = \frac{1}{kT}\frac{e^x}{(e^x+1)^2};\ x = \frac{\epsilon-\mu}{kT}$
$N = \frac{2}{3}g_0\int_0^{\infty}\frac{1}{kT}\frac{e^x}{(e^x+1)^2}\epsilon^{3/2}d\epsilon = \frac{2}{3}g_0\int_{-\mu/kT}^{\infty}\frac{e^x}{(e^x+1)^2}\epsilon^{3/2}dx$
$\epsilon^{3/2} = \sum_{n=0}^{\infty}\frac{f^{(n)}(\epsilon-\mu)^n}{n!} = \mu^{3/2}+(\epsilon-\mu)\frac{d}{d\epsilon}\epsilon^{3/2}|_{\mu}+\frac{1}{2}(\epsilon-\mu)^2\frac{d^2}{d\epsilon^2}\epsilon^{3/2}|_{\epsilon=\mu}+\dots = \mu^{3/2}+\frac{3}{2}(\epsilon-\mu)\mu^{1/2}+\frac{3/8}{(}\epsilon-\mu)^2\mu^{-\frac{1}{2}}+\dots$
$\implies N = \frac{2}{3}g_0\int_{-\infty}^{\infty}\frac{e^x}{(e^x+1)^2}[\mu^{3/2}+\frac{3}{2}xkT\mu^{1/2}+\frac{3}{8}(xkT)^2\mu^{-1/2}+\dots]dx$ Can extend to $-\infty$ since this part is negligible
(dont understand since $\epsilon^{3/2}$ is imaginary.)
1st term: $\int_{-\infty}^{\infty}\frac{e^x}{(e^x+1)^2}dx = -\int_{-\infty}^{\infty}\frac{d}{?}\bar{n}_{FD}d\epsilon d\epsilon = \bar{n}_{FD}(-\infty) - \bar{n}_{FD}(\infty) = 1 - 0 = 1$
$\underline{2nd term:}\int_{-\infty}^{\infty}\frac{xe^x}{(e^x+1)^2}dx = \int_{-\infty}^{\infty}\frac{x}{e^{-x}(e^x+1)^2}dx$
$= \int_{-\infty}^{\infty}\frac{x}{(e^x+1)(e^{-x}+1)}dx = \int_0^{\infty}\frac{x}{(e^x+1)(e^{-x}+1)}dx + \int_{\infty}^{0}\frac{x}{(e^x+1)(e^{-x}+1)}dx$
$= \int_0^{\infty}\frac{x}{(e^x+1)(e^{-x}+1)}dx - \int_0^{\infty}\frac{x}{(e^x+1)(e^{-x}+1)}dx = 0$
3rd term $\int_{-\infty}^{\infty}\frac{x^2e^x}{(e^x+1)^2}dx = \frac{\pi^2}{3}$ (difficult, can look up)
$\implies N = \frac{2}{3}g_0\mu^{3/2} + \frac{1}{4}g_0(kT)^2\mu^{-1/2}\frac{\pi^2}{3} + \dots$

<u>recall:</u> $\frac{3N}{2\epsilon_F^{3/2}}\sqrt{\epsilon} = g(\epsilon) = g_0\sqrt{\epsilon}$, plug in g_0

$\implies N = N(\frac{\mu}{\epsilon_F})^{3/2} + N\frac{\pi^2}{8}\frac{(kT)^2}{\epsilon_F^{3/2}}\mu^{1/2}$

Cancel N's $\implies \frac{\mu}{\epsilon_F} \approx 1;\ solve for \frac{\mu}{\epsilon_F}$

$[1 - \frac{\pi^2}{8}\frac{(kT)^2}{\epsilon_F^{3/2}\mu^{1/2}} + \dots]^{2/3} = \frac{\mu}{\epsilon_F}$

use $\mu \approx \epsilon_F$

$\implies \frac{\mu}{\epsilon_F} = [1 - \frac{\pi^2}{8}(\frac{kT}{\epsilon_F})^2 + \dots]^{2/3}$

<u>recall:</u> $(1-\epsilon)^{2/3} = \sum_{n=0}^{\infty}(-1)^n \begin{pmatrix} \frac{2}{3} \\ n \end{pmatrix} \epsilon^n = \begin{pmatrix} \frac{2}{3} \\ 0 \end{pmatrix} - \begin{pmatrix} \frac{2}{3} \\ 1 \end{pmatrix} \epsilon + \dots = 1 - \frac{2}{3}\epsilon + \dots;\ \begin{pmatrix} \alpha \\ n \end{pmatrix} = \frac{\alpha(\alpha-1)(\alpha-2)\dots(\alpha-n+1)}{n!}$

$\implies \frac{\mu}{\epsilon_F} = [1 - \frac{2}{3}(\frac{\pi^2}{8}(\frac{kT}{\epsilon_F})^2) + \dots]$

$= 1 - \frac{\pi^2}{12}(\frac{kT}{(\epsilon_F^2} + \dots$

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$U = \frac{3}{5}N\frac{\mu^{5/2}}{\epsilon_F^{3/2}} + \frac{3\pi^2}{8}N\frac{(kT)^2}{\epsilon_F} + \dots$

$U = \frac{3}{5}N\epsilon_F + \frac{\pi^2}{4}N\frac{(kT)^2}{\epsilon_F} + \dots$

(Learn these later pg. 284)

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(332) <u>$\bar{n}_{pl} = \frac{1}{e^{hf/kT}-1}$</u> (Planck distribution)

<u>recall:</u> $E_n = (n + \frac{1}{2})\hbar\omega = 2\pi f\frac{h}{2\pi}(n + \frac{1}{2}) = hf(n + \frac{1}{2})$

measure from ground state $\Delta E_n \equiv E_n \implies E_n = hfn$

<u>recall:</u> $\bar{n} = -\frac{1}{\mathcal{Z}}\frac{\partial \mathcal{Z}}{\partial x};\ x = \frac{(\epsilon-\mu)}{kT}$

$\bar{n}_{Pl} = -(1 - e^{-\beta hf})\frac{\partial}{\partial x}\frac{1}{1-e^{-x}}$

$= -(1 - e^{-\beta hf})(e^{-x})(-\frac{1}{(1-e^{-\beta hf})^2})$

$= \frac{e^{-\beta hf}}{1-e^{-\beta hf}} = \frac{1}{e^{\beta hf}-1}$

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<u>Solid State</u>

Chapter 2

(333) <u>$\langle E\rangle = \hbar\omega(n_B(\beta\hbar\omega) + \frac{1}{2})$ (Einstein)</u>

<u>recall:</u> $E_n = \hbar\omega(n + \frac{1}{2})$ (Quantum harmonic oscillators)

$Z_{1D} = \sum_n e^{-E(n)\beta} = \sum_n e^{-\beta\hbar\omega(n+\frac{1}{2})} = e^{-\beta\hbar\omega/2}\sum_n(e^{-\beta\hbar\omega})^n = \frac{e^{-\beta\hbar\omega/2}}{1-e^{-\beta\hbar\omega}} = \frac{1}{e^{\beta\hbar\omega/2}-e^{-\beta\hbar\omega/2}} = \frac{1}{2\sinh(\beta\hbar\omega/2)}$

$\langle E\rangle = -\frac{1}{Z_{1D}}\frac{\partial Z_{1D}}{\partial\beta} = -2\sinh(\beta\hbar\omega/2)\frac{\partial\frac{\beta\hbar\omega}{2}}{\partial\beta} = \frac{\beta\hbar\omega}{2}\coth\frac{\beta\hbar\omega}{2}$

<u>Note:</u> $\coth x = \frac{\cosh x}{\sinh x} = \frac{e^x+e^{-x}}{e^x-e^{-x}} = \frac{e^{-x}(e^{2x}+1)}{e^{-x}(e^{2x}-1)} = \frac{e^{2x}+1+1-1}{e^{2x}-1} = \frac{e^{2x}-1+2}{e^{2x}-1} = \frac{2}{e^{2x}-1} + 1$

$\implies \langle E\rangle = \frac{\hbar\omega}{2}\coth\frac{\beta\hbar\omega}{2} = \frac{\hbar\omega}{2}(\frac{2}{e^{\beta\hbar\omega}-1}+1)$
<u>*recall* :</u>$n_B(x) = \frac{1}{e^x-1}$ (Bosons)
$\therefore \langle E\rangle = \hbar\omega(n_B(\hbar\omega\beta)+\frac{1}{2})$

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(334) $\underline{C_{3D} = C_{1D} = 3\frac{\partial\langle E\rangle}{\partial T}}$
<u>Aside:</u> $E_{n_x,n_y,n_z} = \hbar\omega[(n_x+\frac{1}{2})+(n_y+\frac{1}{2})+(n_z+\frac{1}{2})]$
$Z_{3D} = \sum_{n_x,n_y,n_z} e^{-\beta E_{n_x,n_y,n_z}} = (Z_{1D})^3$
$\implies \langle E\rangle_{3D} = -\frac{1}{Z_{3D}}\frac{\partial Z_{3D}}{\partial\beta} = -\frac{1}{Z_{1D}^3}\frac{\partial Z_{1D}}{\partial\beta}3Z_{1D}^2$
$= -\frac{1}{Z_{1D}^3}3Z_{1D}^2\frac{\partial Z_{1D}}{\partial\beta} = -\frac{3}{Z_{1D}}\frac{\partial Z_{1D}}{\partial\beta} = 3\langle E\rangle_{1D}$

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<u>Note:</u> $e^{ikr} = e^{ik(r+L)} \implies e^{ikL} = 1 \implies k = \frac{2\pi n}{L}$

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(335) $\underline{\langle E\rangle = \int_0^\infty d\omega g(\omega)(\hbar\omega)(n_B(\beta\hbar\omega)+\frac{1}{2})}$ (expectation energy of an atom in a solid); (3D) $g(\omega) = \frac{12\pi\omega^2}{(2\pi)^3v^3}L^3$
$\langle E\rangle = 3\sum_{\vec{k}}\hbar\omega(\vec{k})(n_B(\beta\hbar\omega(\vec{k}))+\frac{1}{2})$
<u>Note:</u> $\sum_{\vec{k}} = \frac{1}{\Delta k^3}\int d\vec{k};\ k = \frac{2\pi n}{L} \implies \Delta k = \frac{2\pi}{L}$
$\implies \sum_{\vec{k}} = \frac{L^3}{(2\pi)^3}\int d\vec{k}$
$\implies \langle E\rangle = 3\frac{L^3}{(2\pi)^3}\int d\vec{k}\hbar\omega(\vec{k})(n_B(\beta\hbar\omega(\vec{k}))+\frac{1}{2})$
$\int d\vec{k} = \int_0^{2\pi}d\phi\int_0^\pi d\theta\sin\theta\int_0^\infty k^2dk = 4\pi\int k^2dk$
$\omega(\vec{k}) = v|\vec{k}| \implies k = \frac{\omega}{v}$
$\implies 4\pi\int_0^\infty k^2dk = 4\pi\int_0^\infty\frac{\omega^2}{v^2}\frac{1}{v}d\omega = 4\pi\int_0^\infty\frac{1}{v^3}\omega^2d\omega$
$\implies \langle E\rangle = 3\frac{4\pi L^3}{(2\pi)^3}\int_0^\infty\omega^2d\omega\frac{1}{v^3}\hbar\omega(n_B(\beta\hbar\omega)+\frac{1}{2})$
$= \int_0^\infty d\omega g(\omega)\hbar\omega(n_B(\beta\hbar\omega)+\frac{1}{2})$
$g(\omega) \equiv L^3(\frac{12\omega^2\pi}{(2\pi)^3v^3}) = N\frac{12\pi\omega^2}{(2\pi)^3nv^3} = N\frac{9\omega^2}{\omega_d^3}$
$\omega_d^3 = 6\pi^2nv^3$

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(336) $\underline{\overset{\star}{C} = \frac{\partial\langle E\rangle}{\partial T} = Nk_B\frac{T^3}{(T_{Debye})^3}\frac{12\pi^4}{5}}$
<u>recall:</u> $n_B(\beta\hbar\omega) = \frac{1}{e^{\beta\hbar\omega}-1}$
$\implies \langle E\rangle = \frac{9N}{\omega_d^3}\hbar\int_0^\infty d\omega\frac{\omega^3}{e^{\beta\hbar\omega}-1} + const$
$x = \beta\hbar\omega$
$\implies \langle E\rangle = \frac{9N\hbar}{\omega_d^3(\beta\hbar)^4}\int_0^\infty dx\frac{x^3}{e^x-1} + const.$
$\int_0^\infty dx\frac{x^3}{e^x-1} = \frac{\pi^4}{15}$

$\implies \langle E \rangle = \frac{9N\hbar}{\omega_d^3(\beta\hbar)^4}\frac{\pi^4}{15} = 9N\frac{(k_BT)^4}{(\hbar\omega_d)^3}\frac{\pi^4}{15} + const.$
$\implies \frac{\partial\langle E\rangle}{\partial T} = Nk_B\frac{(k_BT)^3}{(\hbar\omega_d)^3}\frac{12\pi^4}{15} = C \sim T^3$
$k_BT_{Debye} = \hbar\omega$
$\therefore C = \frac{\partial\langle E\rangle}{\partial T} = Nk_B\frac{T^3}{T_{Debye}^3}\frac{12\pi^4}{15}$

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We want $C \to 3k_BN$ for large T
$\implies 3N = \int_0^{\omega_{cutoff}} d\omega g(\omega)$ (there can only be as many modes as there are degrees of freedom in the system)

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Drude Theory
Assumptions
(1) electrons scatter in time τ. Probability of scatter is $\frac{dt}{\tau}$
(2) after scatter electron momentum $\vec{p} = 0$
(3) between scatter electrons respond to $\vec{E}_{ext},\ \vec{B}_{ext}$
$\implies \langle\vec{p}(t+dt)\rangle = \sum_i \mathcal{P}(t_i)\vec{p}(t_i)$
$= (1 - \frac{dt}{\tau})(\vec{p}(t) + \vec{F}dt) + \vec{0}\frac{dt}{\tau}$
i.e., it either scatters in $t = t_0 + dt$ or it doesn't
$\implies \langle\vec{p}(t+dt)\rangle = \vec{p}(t) + \vec{F}dt - \vec{p}(t)\frac{dt}{\tau}$
$\implies \frac{\langle\vec{p}(t+dt)\rangle - \vec{p}(t)}{dt} = \frac{d\vec{p}}{dt} = \vec{F} - \frac{\vec{p}}{\tau}$
$\vec{F} = -e(\vec{E} + \vec{v}\times\vec{B})$
$\vec{E} = 0;\ \vec{B} = 0 \implies \frac{d\vec{p}}{dt} = -\frac{\vec{p}}{\tau} \implies \vec{p}(t) = \vec{p}_0 e^{-t/\tau}$

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(337) $\underline{\vec{J} = \sigma\vec{E} = \frac{e^2\tau n}{m}\vec{E}}$
recall: $\frac{d\vec{p}}{dt} = -e(\vec{E} + \vec{v}\times\vec{B}) - \frac{\vec{p}}{\tau}$
$\vec{B} = 0 \implies \frac{d\vec{p}}{dt} = -e\vec{E} - \frac{\vec{p}}{\tau},\ \frac{d\vec{p}}{dt} = 0$ (steady state)
$\implies \vec{p} = -\tau e\vec{E} = m\vec{v} \implies \vec{v} = -\frac{e\tau}{m}\vec{E}$
$\therefore \vec{J} = \frac{\vec{I}}{A} = \frac{q\vec{i}_e}{A} = -\frac{e\vec{i}_e}{A} = -\frac{eN\vec{v}}{\ell A} = -en\vec{v} = \frac{e^2\tau n}{m}\vec{E}$

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(338) $\underline{F_n = \kappa(\delta x_{n+1} - \delta x_n) + \kappa(\delta x_{n-1} - \delta x_n)}$ (solid state)
$\delta x_n = x_n - x_n^{eq};\ x_n^{eq} = na$
Ansatz: $\delta x_n = Ae^{i\omega t - ikna}$

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(339) $\underline{\omega = 2\sqrt{\frac{\kappa}{m}}|\sin(\frac{ka}{2})|}$
recall: $m(\delta\ddot{x}_n) = \kappa(\delta x_{n+1} + \delta x_{n-1} - 2\delta x_n);\ \delta x_n = Ae^{i\omega t - ikna}$

plug in; $\delta\dot{x}_n = i\omega\delta x_n,\ \delta\ddot{x}_n = -\omega^2\delta x_n$
$\implies -m\omega^2\delta x_n = \kappa(e^{-ikna}\delta x_n + e^{ika}\delta x_n - 2\delta x_n)$
$\implies -m\omega^2 = \kappa(e^{-ika} + e^{ika} - 2)$
<u>recall:</u> $\cos(ka) = \frac{e^{-ika}+e^{ika}}{2}$
$\implies m\omega^2 = 2\kappa 2\frac{1}{2}(1-\cos(2\frac{ka}{2})) = 4\kappa\sin^2(\frac{ka}{2})$
$\therefore \omega = 2\sqrt{\frac{\kappa}{m}}|\sin(\frac{ka}{2})|$

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<u>principle:</u> a system periodic in real space (period a), is periodic in reciprocal space (k space) with periodicity $\frac{2\pi}{a}$

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<u>ω periodic in $k \to k + \frac{2\pi}{a}$</u>
$\sin(\frac{(k+b)a}{2}) = \sin(\frac{ka}{2} + \frac{ba}{2})$
periodic if $\frac{ba}{2} = 2\pi n \implies b = \frac{4\pi n}{a}$
$\implies \omega = 2\sqrt{\frac{\kappa}{m}}|\sin(\frac{ka}{2})| \implies \omega$ periodic with period $\frac{b}{2} = \frac{2\pi}{a}$

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<u>Brillouin zone:</u> unit-cell in k space.
$k = \pm\frac{\pi}{a}$Brillouin-zone boundaries

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(340) <u>$x_n = na; G_n = \frac{2\pi r}{a}$</u>
<u>recall:</u> $\delta x_n = Ae^{i\omega t - ikna},\ k \to k + \frac{2\pi}{a}$
Note that δx_n is not periodic with $k = \frac{2\pi}{na}$ since the $\omega(k)$ term in the exponential is not periodic under this transformation
$\delta x_n = Ae^{i\omega t - i(k+\frac{2\pi}{a})na} = Ae^{i\omega t - ikna}e^{-i2\pi n}$
$= Ae^{i\omega t - ikna};\ k \to k + \frac{2\pi p}{a}$ Yields same result
$k = \frac{2\pi p}{a}$ is equivalent to $k = 0$
$\therefore x_n = na;\ n \in \mathbb{Z}$ (direct lattice)
$\therefore G_n = \frac{2\pi p}{a};\ n \in \mathbb{Z}$ (reciprocal lattice)
<u>Note:</u> $e^{iG_m x_n} = e^{i(\frac{2\pi m}{a})na} = (e^{i2\pi})^{nm} = 1$

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(341) <u>$v_{sound} = a\sqrt{\frac{\kappa}{m}}$</u> (long wavelength)
<u>sound:</u> $\lambda \sim large \implies k \sim small$
$\omega = 2\sqrt{\frac{\kappa}{m}}|\sin(\frac{ka}{2})| \approx 2\sqrt{\frac{\kappa}{m}}|\frac{ka}{2}| = ka\sqrt{\frac{\kappa}{m}}$
$\implies \frac{\omega}{k} = \frac{d\omega}{dk} = v_{sound} = a\sqrt{\frac{\kappa}{m}}$

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for large k, v_{sound} splits into a group velocity and phase velocity $v_{group} = \frac{d\omega}{dk}$; $v_{phase} = \frac{\omega}{k}$

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modes$= N$

periodic boundary conditions: $x_{n+N} = x_n \implies e^{i\omega t - ikna} = e^{i\omega t - ik(n+N)a} \implies e^{-ikNa} = 1 = e^{ikNa}$

$\implies k = \frac{2\pi p}{Na} = \frac{2\pi p}{L} \implies$ # modes $= \frac{range k}{\Delta k} = \frac{\frac{2\pi}{a}}{\frac{2\pi}{Na}} = N$

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(342) $U_{total} = \frac{Na}{2\pi}\int_{-\pi/a}^{\pi/a} dk\hbar\omega(k)(n_B(\beta\hbar\omega(k)) + \frac{1}{2})$

$n_B(\beta\hbar\omega) = \frac{1}{e^{\beta\hbar\omega}-1}$ (phonons)

$E_k = \hbar\omega(k)(n_B(\beta\hbar\omega(k)) + \frac{1}{2})$ (Expectation energy)

$U_{total} = \sum_k \hbar\omega(k)(n_B(\beta\hbar\omega(k)) + \frac{1}{2})$

sum over $k = \frac{2\pi p}{Na} s.t. -\pi/a \leq k < \pi/a$

$\implies -\frac{N}{2} \leq p < \frac{N}{2} \implies -\frac{N}{2} \leq p \leq \frac{N}{2} - 1$

$\implies \sum_{k s.t. k=\frac{2\pi p}{Na} and -\frac{\pi}{a}<k<\frac{\pi}{2}} \to \sum_{p=-\frac{N}{2}}^{p=-\frac{N}{2}-1}$

$\sum_k \to \frac{Na}{2\pi}\int_{-\pi/2}^{\pi/2} dk$

$\therefore U_{total} = \frac{Na}{2\pi}\int_{-\pi/a}^{\pi/a} dk\hbar\omega(k)(n_B(\beta\hbar\omega(k)) + \frac{1}{2})$

Note: $\frac{Na}{2\pi}\int_{-\pi/a}^{\pi/a} dk = N$

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(343) $g(\omega) = 2\frac{Na}{2\pi}|\frac{dk}{d\omega}|$

$\frac{Na}{2\pi}\int_{-\pi/a}^{\pi/a} dk = 2\frac{Na}{2\pi}\int|\frac{dk}{d\omega}|d\omega = \int g(\omega)d\omega$

$\therefore g(\omega) = 2\frac{Na}{2\pi}|\frac{dk}{d\omega}|$

factor of 2 because for every value of ω there are two corresponding k values.

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unit cell: the part that repeats in a solid when all atoms are in equillibrium

---

reference point = lattice point

in each unit cell there is one reference point.

the position of the lattice point is $r_n = na$ the position of atoms in the unit cell is measured from reference point. For example light grey atom in equillibrium position located at $x_n^{eq} = an - \frac{3a}{40}$; dark grey located at $y_n^{eq} = an + \frac{7a}{20}$

- -

(344) $\omega_\pm = \sqrt{\frac{\kappa_1+\kappa_2}{m} \pm \frac{1}{m}\sqrt{(\kappa_1+\kappa_2)^2 - 4\kappa_1\kappa_2\sin^2(\frac{ka}{2})}}$

Two atoms, same mass, different spring constants
$\implies m\delta\ddot{x}_n = \kappa_2(\delta y_n - \delta x_n) + \kappa_1(\delta y_{n-1} - \delta x_n)$; $m\delta\ddot{y}_n = \kappa_1(\delta x_{n+1} - \delta y_n) + \kappa_2(\delta x_n - \delta y_n)$
$\delta x_n = A_x e^{i\omega t - ikna}$; $\delta y_n = A_y e^{i\omega t - ikna}$
$\implies -\omega^2 m A_x e^{i\omega t - ikna} = \kappa_2 A_y e^{i\omega t - ikna} + \kappa_1 A_y e^{i\omega t - ik(n-1)a} - (\kappa_1+\kappa_2)A_x e^{i\omega t - ikna}$;
$-\omega^2 m A_y e^{i\omega t - ikna} = \kappa_1 A_x e^{i\omega t - ik(n+1)a} + \kappa_2 A_x e^{i\omega t - ikna} - (\kappa_1 + \kappa_2)A_y e^{i\omega t - ikna}$
$\implies -\omega^2 m A_x = \kappa_2 A_y + \kappa_1 A_y e^{ika} - (\kappa_1+\kappa_2)A_x$
$\implies -\omega^2 m A_y = \kappa_1 A_x e^{-ika} + \kappa_2 A_x - (\kappa_1+\kappa_2)A_y$
$\implies m\omega^2 \begin{pmatrix} A_x \\ A_y \end{pmatrix} = \begin{pmatrix} (\kappa_1+\kappa_2) & -\kappa_2 - \kappa_1 e^{ika} \\ -\kappa_2 - \kappa_1 e^{-ika} & (\kappa_1+\kappa_2) \end{pmatrix} \begin{pmatrix} A_x \\ A_y \end{pmatrix}$
<u>recall:</u> We care about nontrivial (A_x, A_y) so $\lambda\vec{x} = A\vec{x}$ where $\lambda = m\omega^2$ (nontrivial $\vec{x}$) $\implies |A - \lambda I| = 0$
$\implies 0 = \begin{vmatrix} (\kappa_1+\kappa_2) - m\omega^2 & -\kappa_2 - \kappa_1 e^{ika} \\ -\kappa_2 - \kappa_1 e^{-ika} & (\kappa_1+\kappa_2) - m\omega^2 \end{vmatrix} = |(\kappa_1+\kappa_2) - m\omega^2|^2 - |\kappa_2 + \kappa_1 e^{ika}|^2$ (important)
$\implies ((\kappa_1+\kappa_2) - m\omega^2) = \pm|\kappa_1 + \kappa_2 e^{ika}|$
$\implies m\omega^2 = (\kappa_1+\kappa_2) \pm |\kappa_1 + \kappa_2 e^{ika}|$
$|\kappa_1 + \kappa_2 e^{ika}| = \sqrt{(\kappa_1 + \kappa_2 e^{-ika})(\kappa_1 + \kappa_2 e^{ika})}$
$= \sqrt{\kappa_1^2 + \kappa_2^2 + \kappa_1\kappa_2(e^{ika} + e^{-ika})}$
<u>recall:</u> $\cos(ka) = \frac{1}{2}(e^{ika} + e^{-ika})$
$\implies m\omega^2 = (\kappa_1+\kappa_2) \pm \sqrt{\kappa_1^2 + \kappa_2^2 + 2\kappa_1\kappa_2\cos(ka)}$
$\implies \omega_\pm = \sqrt{\frac{\kappa_1+\kappa_2}{m} \pm \frac{1}{m}\sqrt{\kappa_1^2 + \kappa_2^2 + 2\kappa_1\kappa_2\cos(ka)}}$
$\kappa_1^2 + \kappa_2^2 + 2\kappa_1\kappa_2\cos(ka) = \kappa_1^2 + 2\kappa_1\kappa_2 + \kappa_2^2 - 2\kappa_1\kappa_2(1 - \cos(ka))$
$4\sin^2 x = 2(1 - \cos 2x) \implies (\kappa_1+\kappa_2)^2 - 4\kappa_1\kappa_2\sin^2\frac{ka}{2}$
$\therefore \omega_\pm = \sqrt{\frac{\kappa_1+\kappa_2}{m} \pm \frac{1}{m}\sqrt{(\kappa_1+\kappa_2)^2 - 4\kappa_1\kappa_2\sin^2\frac{ka}{2}}}$

- -

(345) $\begin{pmatrix} A_x \\ A_y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \sim acoustic \begin{pmatrix} A_x \\ A_y \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \sim optical(k \to 0)$

$\omega_- \sim acoustick = 0\omega_-(k=0) = 0$

$\underline{recall\ :} m\omega^2 \begin{pmatrix} A_x \\ A_y \end{pmatrix} = \begin{pmatrix} (\kappa_1+\kappa_2) & -\kappa_2-\kappa_1 e^{ika} \\ -\kappa_2-\kappa_1 e^{-ika} (\kappa_1+\kappa_2) & \end{pmatrix} \begin{pmatrix} A_x \\ A_y \end{pmatrix}$
$k=0$
$\omega^2 \begin{pmatrix} A_x \\ A_y \end{pmatrix} = \frac{\kappa_1+\kappa_2}{m} \begin{pmatrix} 1^{-1} \\ -1^1 \end{pmatrix} \begin{pmatrix} A_x \\ A_y \end{pmatrix} = 0$
$\implies \begin{pmatrix} A_x \\ A_y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \implies$ oscillate in phase
$\omega_+ \sim optical\ \omega_+^2(k=0) = \frac{2(\kappa_1+\kappa_2)}{m}$
$\implies \frac{2(\kappa_1+\kappa_2)}{m} \begin{pmatrix} A_x \\ A_y \end{pmatrix} = \frac{\kappa_1+\kappa_2}{m} \begin{pmatrix} 1^-1 & \\ -1 & 1 \end{pmatrix} \begin{pmatrix} A_x \\ A_y \end{pmatrix}$
$\implies \begin{pmatrix} A_x \\ A_y \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \sim$ eigenvector $\implies$ oscillate out of phase

- -

Note: $N \sim k \implies 2N$ modes; $\omega_+ \sim$ optic; $\omega_- \sim$ acoustic For each atom in the unit cell there is another mode that appears, the first one is an acoustic mode, they oscillate in phase (when k=0), the rest are optical modes, they oscillate out of phase (when k=0).
The Brillouin zone $|k| \leq \pi/a$ is the reduced scheme, all of the ω curves superimposed on the Brillouin zone while after 'unfolding' these curves result in the extended scheme.

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Chapter 11

$|n\rangle \sim$ orbital of n^{th} atom

- -

(346) $\underline{\overset{\star}{\sum_m} H_{nm}\phi_m = E\phi_m}$ assume $\langle n|m\rangle = \delta_{nm}$
recall: for $|\beta\rangle = \hat{Q}|\alpha\rangle,\ Q_{nm} \equiv \langle n|\hat{Q}|m\rangle$
spose $\hat{H}|\Psi\rangle = E|\Psi\rangle,\ w/,\ |\Psi\rangle = \sum_m \phi_m|m\rangle$
$\implies \langle n|\hat{H}|\Psi\rangle = \sum_m \phi_m \langle n|\hat{H}|m\rangle = \sum_m H_{nm}\phi_m$
and $\langle n|E|\Psi\rangle = E\langle n|\Psi\rangle = E\sum_m \phi_m\langle n|m\rangle = E\phi_n$
$\therefore \sum_m H_{nm}\phi_m = E\phi_m$

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(347) $\underline{H_{n,m} = \epsilon_0\delta_{n,m} - t(\delta_{n+1,m} + \delta_{n-1,m})}$ Let $H = K + \sum_j V(\vec{r} - \vec{R}_j) = K + \sum_j V_j;\ K = \frac{\hat{p}^2}{2m}$
$\sum_j V_j$ is the total potential of nuclei in lattice acting on the electron at $\vec{r}$ (which is attached to atomic site m)

$H|m\rangle = (K + \sum_j V_j)|m\rangle = (K + \sum_{j\neq m} V_j + V_m)|m\rangle$
$= (K + V_m)|m\rangle + \sum_{j\neq m} V_j|m\rangle$
$(K + V_m)|m\rangle + \sum_{j\neq m} V_j|m\rangle$
$(K + V_m)|m\rangle = \epsilon_{atomic}|m\rangle$, ϵ_{atomic} energy of electron from its nucleus
$H_{n,m} = \langle n|H|m\rangle = \epsilon_{atomic}\langle n|m\rangle + \sum_{j\neq m}\langle n|V_j|m\rangle$
$= \epsilon_{atomic}\delta_{nm} + \sum_{j\neq m}\langle n|V_j|m\rangle$
$\sum_{j\neq m}\langle n|V_j|m\rangle$ is a hopping term
assuming electrons can only hop to nearest neighbor
$\implies \sum_{j\neq m}\langle n|V_j|m\rangle = \begin{cases} V_0\text{ifn = m} \\ -t;\ n = m \pm 1 \\ 0 o.w. \end{cases}$ $n = m \implies \sum_{j\neq m}\langle n|V_j|m\rangle =$
$V_0 = V_0\delta_{nm}$
$m = n + 1 \implies \sum_{j\neq m}\langle n|V_j|m\rangle = V_0\delta_{n,m} - t\delta_{n+1,m}$
$m = n - 1 \implies \sum_{j\neq m}\langle n|V_j|m\rangle = -t = V_0\delta_{n,m} - t(\delta_{n+1,m} + \delta_{n-1,m})$
$\implies H_{n,m} = \langle n|H|m\rangle = \epsilon_{atomic}\delta_{n,m} + \sum_{j\neq m}\langle n|V_j|m\rangle$
$= \epsilon_{atomic}\delta_{n,m} + V_0\delta_{n,m} - t(\delta_{n+1,m} + \delta_{n-1,m})$
$= \epsilon_0\delta_{n,m} - t(\delta_{n+1,m} + \delta_{n-1,m})$; $\epsilon_0 \equiv \epsilon_{atomic} + V_0$
$\therefore H_{n,m} = \epsilon_0\delta_{n,m} - t(\delta_{n+1,m} + \delta_{n-1,m})$

★

(348) $\underline{\phi_n = \frac{1}{\sqrt{N}}e^{-ikna};\ |\Psi\rangle = \sum_n \phi_n|n\rangle}$
$\phi_n = Ae^{-ikna}$ (guess); want $\langle\Psi|\Psi\rangle = 1$
$\langle\Psi|\Psi\rangle = \sum_{n,m}\phi_n^*\phi_m\langle n|m\rangle = \sum_{n,m}\phi_n^*\phi_m\delta_{nm} = \sum_n|\phi_n|^2 = |A|^2\sum_n = N|A|^2 = 1 \implies A = \frac{1}{\sqrt{N}}$

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(349) $\underline{E = \epsilon_0 - 2t\cos(ka)}$
<u>recall:</u> $H_{n,m} = \epsilon_0\delta_{nm} - t(\delta_{n+1,m} + \delta_{n-1,m})$; $\sum_m H_{nm}\phi_m = E\phi_n$
$\implies \sum_m(\epsilon_0\delta_{nm} - t(\delta_{n+1,m} + \delta_{n-1,m}))e^{-ikma} = Ee^{-ikna}$
$\implies \epsilon_0 e^{-ikna}| - te^{-ik(n+1)a} - te^{-ik(n-1)a} = Ee^{-ikna}$
$\implies \epsilon_0 - te^{-ika} - te^{ika} = E$
<u>recall:</u> $\frac{1}{2}(e^{ika} + e^{-ika}) = \cos ka$
$\therefore E = \epsilon_0 - 2t\cos ka$

----

compare with $\omega^2 = 2\frac{\kappa}{m} - 2\frac{\kappa}{m}\cos ka$
the reason ω^2 and E is because $F = ma = m\frac{\partial^2 x}{\partial t^2}$ and $\hat{H}\Psi =$

$i\hbar\frac{\partial\Psi}{\partial t}$

- -

band - energy range for which eigenstates exist, since E is a function there is only 1 band and ω^2 has 2.

- -

(350) $\underline{m^* = \frac{\hbar^2}{2ta^2}}$ (effective mass)

recall: $E = \epsilon_0 - 2t\cos(ka)$; $\cos x \approx 1 - \frac{x^2}{2}$(bottom of band)

$E = \epsilon_0 - 2t\cos ka \approx \epsilon_0 - 2t(1 - \frac{k^2a^2}{2})$

$= \epsilon_0 - 2t + tk^2a^2 = const. + tk^2a^2$

recall: $E_{free}(k) = \frac{\hbar^2k^2}{2m}$

$\implies \frac{\hbar^2k^2}{2m^*} = tk^2a^2 \implies \frac{\hbar^2}{2a^2t} = m^*$

- -

If we had one atom per unit cell and multiple orbitals we would have multiple bands pop up (understand better)

- -

Suppose we have N atoms in our tight binding model contributing 1 electron each (valence =1), each band has N k-states but each k-state can be occupied by 2 electrons so the band would only be half filled.

- -

When an electric field is applied, this shifts the filled k-states (or the fermi surface) and induces a current, needs justification, a filled band cannot have an induced current from a smal magnetic field.

- -

A divalent model would have a completely filled band

- -

Definition 12.1 A lattice is an infinite set of points defined by integer sums of a set of linearly independent primitive lattice vectors.

- -

$R_{[n_1n_2n_3]} = n_1\vec{a}_1 + n_2\vec{a}_2 + n_3\vec{a}_3,\ n_1, n_2, n_3 \in \mathbb{Z}$

- -

$\vec{a}_1, \vec{a}_2, \vec{a}_3$ primitive lattice vectors.
recall: this is analogous to $R = na$ (1D crystal)

- -

Equivalent Definition 12.1.1
A lattice is an infinite set of vectors where addition of any two vectors in the set gives a third vector in the set

- -

Equivalent Defintion 12.1.2
A lattice is a set of points where the environment of any given point is equivalent to the environment of any other given point.

- -

Defintion 12.2 A unit cell is a region of space such that when many identical units are stacked together it tiles (completlely fills) all of space and reconstructs the full structure.

- -

Equivalent Definition 12.2.1 A unit cell is the repeated motif which is the elementary building block of the periodic structure.

- -

Definition 12.3
A primitive unit cell for a periodic crystal is a unit cell containing exactly one lattice point.

- -

Definition:
a lattice point is a point ofn the lattice
Note: inside a unit cell, you can have fractional lattice points.

- -

Definition 12.4
Given a lattice point, the set of all points in space which are closer to that given lattice point than to any other lattice point constitute the Wigner-Seitz cell of the given lattice point.

- -

Definition 12.5
The description of objects in the unit cell with respect to the

reference lattice point in the unit cell is known as a basis.

- -

review 12.2, Note that [0,0] i fig 12.8 is the reference point

- -

Body centered cubic lattice (bcc)- a regular cubic lattice with an extra point right in the center of each cube, contains 2 lattice points (recall fractional lattice points)

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Face centered cubic (fcc)- a cubic lattice except each face of the cube has a point right in the middle, contains 4 lattice points

- -

Of course, there are tons of other lattice structures as well.

- -

<u>Definition 13.1</u>
Spose $\vec{R}$ is a point indirect lattice $\vec{G}$ element of reciprocal lattice $\iff$ $e^{i\vec{G}\cdot\vec{R}} = 1$

- -

(1) reciprocal lattice is a lattice in reciprocal space
(2) primitive lattice $\vec{a}_i$ and reciprocal lattice $(\vec{b}_j)$ are related by $\vec{a}_i \cdot \vec{b}_j = 2\pi\delta_{ij}$

- -

$\vec{b}_1 = \frac{2\pi\vec{a}_2\times\vec{a}_3}{\vec{a}_1\cdot(\vec{a}_2\times\vec{a}_3)}$
$\vec{b}_2 = \frac{2\pi\vec{a}_3\times\vec{a}_1}{\vec{a}_1\cdot(\vec{a}_2\times\vec{a}_3)}$
$\vec{b}_3 = \frac{2\pi\vec{a}_1\times\vec{a}_2}{\vec{a}_1\cdot(\vec{a}_2\times\vec{a}_3)}$
easy to check $\vec{a}_1 \cdot \vec{b}_1 = \frac{2\pi\vec{a}_1\cdot(\vec{a}_2\times\vec{a}_3)}{\vec{a}_1\cdot(\vec{a}_2\times\vec{a}_3)} = 2\pi$
just like $k = \frac{2\pi}{a}$ $\vec{b}_1 = \frac{2\pi\vec{Q}}{V} butV = \vec{a}_1 \cdot (\vec{a}_2 \times \vec{a}_3)$
we want $\vec{a}_1 \cdot \vec{b}_1 = 2\pi$ so a good choice of $\vec{Q} = \vec{a}_2 \times \vec{a}_3$

- -

$$\underline{\mathcal{F}[\rho(\vec{r})] = \sum_{\vec{R}} e^{i\vec{k}\cdot\vec{R}} = \frac{(2\pi)^D}{V} \sum_{\vec{G}} \delta^D(\vec{k} - \vec{G})} \tag{351}$$

$\rho(r) = \sum_n \delta(r - an)$ (Density of a 1-D lattice)
$\mathcal{F}[\rho(r)] = \int dre^{ikr}\rho(r) = \sum_n \int dre^{ikr}\delta(r - an)$
$= \sum_n e^{ikan} = \frac{2\pi}{|a|} \sum_m \delta(k - \frac{2\pi m}{a})$ (Poisson resumation formula)
In general,

$\therefore \mathcal{F}[\rho(\vec{r})] = \sum_{\vec{R}} e^{i\vec{k}\cdot\vec{R}} = \frac{(2\pi)^D}{V}\sum_{\vec{G}}\delta^D(\vec{k}-\vec{G})$

---

(352) $\underline{\mathcal{F}[\rho(\vec{r})] = (2\pi)^D\sum_{\vec{G}}\delta^D(\vec{k}-\vec{G})S(\vec{k});\ S(\vec{k}) = \int_{unit-cell} d\vec{x}e^{i\vec{k}\cdot\vec{x}}\rho(\vec{x})}$
periodic lattice
recall: $\mathcal{F}[\rho(\vec{r}) = \int dre^{i\vec{k}\cdot\vec{r}}\rho(\vec{r})$ all of space
lets instead integrate over a unit cell and add up all unit cells.
Let $\vec{r} = \vec{R} + \vec{x},\ \vec{x}$ within unit cell
$\implies \mathcal{F}[\rho(\vec{r})] = \sum_{\vec{R}}\int_{unit-cell} d\vec{x}e^{i\vec{k}\cdot(\vec{x}+\vec{R})}\rho(\vec{x}+\vec{R})$
$= \sum_{\vec{R}} e^{i\vec{k}\cdot\vec{R}}\int_{unit-cell} d\vec{x}e^{i\vec{k}\cdot\vec{x}}\rho(\vec{x})$
recall: $\sum_{\vec{R}} e^{i\vec{k}\cdot\vec{R}} = \frac{(2\pi)^D}{V}\sum_{\vec{G}}\delta^D(\vec{k}-\vec{G})$

---

(353) $\underline{E^i = \tilde{\rho}J^i;\ \tilde{\rho} = (\frac{1}{ne}\epsilon^i_{jk}B^k - \frac{m}{ne\tau}\delta^i_j)J^j}$
recall: $\frac{d\vec{p}}{dt} = -e(\vec{E}+\vec{v}\times\vec{B}) - \frac{\vec{p}}{\tau}$
$\frac{d\vec{p}}{dt} = 0$ (steady state)
recall: $\vec{J} = 0ne\vec{v} \implies \vec{v} = \frac{\vec{J}}{-ne}$
$\implies 0 = -e(\vec{E} + -\frac{\vec{J}}{ne}\times\vec{B}) - \frac{\vec{p}}{\tau}$
$\implies \vec{E} = \frac{1}{ne}\vec{J}\times\vec{B} - \frac{1}{\tau}\frac{m\vec{J}}{ne}$
$\implies (\vec{E})^i = E^i = \frac{1}{ne}\epsilon^i_{jk}J^jB^k - \frac{m}{ne\tau}J^i$
$= \frac{1}{ne}\epsilon^i_{jk}J^jB^k - \frac{m}{ne\tau}\delta^i_jJ^j$
$= (\frac{1}{ne}\epsilon^i_{jk}B^k - \frac{m}{ne\tau}\delta^i_j)J^j$

Skipped section 3.2

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(354) $\underline{N = 2\frac{V}{(2\pi)^3}\int d\vec{k}n_F(\beta(\epsilon(\vec{k})-\mu))}$
recall: $n_F(\beta(E-\mu)) = \frac{1}{e^{\beta(E-\mu)}+1}$
$\implies N = 2\sum_{\vec{k}} n_F(\beta(\epsilon(\vec{k})-\mu)) = 2\frac{V}{(2\pi)^3}\int d\vec{k}n_F(\beta(\epsilon(\vec{k})-\mu))$

Definition: Fermi energy (E_F) chemical potential at $T=0$
$E_F = \frac{\hbar^2k_F^2}{2m}$ (fermi wave vector (implicit definition))
$p_F = \hbar k_F$ (fermi momentum)
$v_F = \hbar k_F/m$ (fermi velocity)

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(355) $N = 2\frac{V}{(2\pi)^3}(\frac{4}{3}\pi k_F^3);\ k_F = (3\pi^2 n)^{1/3};\ E_F = \frac{\hbar^2(3\pi^2 n)^{2/3}}{2m}$

Consider 3D metal $N \sim$ electrons $T = 0$

$\implies n_F(\beta(\epsilon(\vec{k} - \mu)) = \Theta(E_F - \epsilon(\vec{k}))$ (step function)

$\implies N = 2\frac{V}{(2\pi)^3}\int d\vec{k}\Theta(E_f - \epsilon(\vec{k})) = 2\frac{V}{(2\pi)^3}\int^{|k|<k_F} d\vec{k}$

Note: $\Theta(x) = \begin{cases} 1, & x \geq 0, \\ 0, & x < 0 \end{cases}$

$\implies E_F - \epsilon(k) \geq 0 \implies \epsilon(k) \leq E_F \implies |k| < k_F$

$\therefore N = 2\frac{V}{(2\pi)^3}(\frac{4}{3}\pi k_F^3) \implies k_F = (3\pi^2 n)^{1/3} \implies E_F = \frac{\hbar^2(3\pi^2 n)^{2/3}}{2m}$

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(356) $N = 2\frac{V}{(2\pi)^3}\frac{4}{3}\pi k_F^3 = \frac{V}{3\pi^2}k_F^3$ (alternative method)

$N = 2\sum_{n_x,n_y,n_z} = 2\int_0^{\pi/2}\int_0^{\pi/2}\int_0^{n_{max}} \sin\theta n^2 dn d\theta d\phi$

$= 2\frac{\pi}{2}\frac{n_{max}^3}{3}$

recall: $k = \frac{\pi n}{L} \implies k_F = \frac{\pi n_{max}}{L}$

$\implies N = \frac{\pi}{3}(\frac{Lk_F}{\pi})^3 = \frac{\pi}{3}\frac{Vk_F}{\pi^3} = \frac{V}{3\pi^2}k_F^3$

----

(357) $g(\epsilon) = \frac{3n}{2E_F}(\frac{\epsilon}{E_F})^{1/2}$

recall: $E_{tot} = \frac{2V}{(2\pi)^3}\int d\vec{k}\epsilon(\vec{k})n_F(\beta(\epsilon(\vec{k}) - \mu)) = \frac{2V}{(2\pi)^3}\int_0^\infty 4\pi k^2 dk\epsilon(\vec{k})n_F(\beta(\epsilon(\vec{k}) - \mu))$

----

(358) $E_{tot} = V\int_0^\infty d\epsilon\epsilon g(\epsilon)n_F(\beta(\epsilon - \mu));\ N = V\int_0^\infty d\epsilon g(\epsilon)n_F(\beta(\epsilon - \mu))$

$E_{tot} = \frac{2V}{(2\pi)^3}\int d\vec{k}\epsilon(\vec{k}n_F(\beta(\epsilon(\vec{k}) - \mu)) = \frac{2V}{(2\pi)^3}\int_0^\infty 4\pi k^2 dk\epsilon(\vec{k})n_F(\beta(\epsilon(\vec{k}) - \mu))$

$k = \sqrt{\frac{2\epsilon m}{\hbar^2}} \implies dk = \sqrt{\frac{m}{2\epsilon\hbar^2}}d\epsilon$

$\implies E_{tot} = \frac{2V}{(2\pi)^3}\int_0^\infty 4\pi\frac{2\epsilon m}{\hbar^2}\sqrt{\frac{m}{2\epsilon\hbar^2}}d\epsilon\epsilon n_F(\beta(\epsilon - \mu))$

$= V\int_0^\infty d\epsilon\epsilon g(\epsilon)n_F(\beta(\epsilon - \mu))$

where $g(\epsilon) = \frac{2}{(2\pi)^3}4\pi\frac{2\epsilon m}{\hbar^2}\sqrt{\frac{m}{2\epsilon\hbar^2}}$

$= \frac{(2m)^{3/2}}{2\pi^2\hbar^3}\epsilon^{1/2}$

also

$N = V\int_0^\infty d\epsilon g(\epsilon)n_F(\epsilon - \mu)$

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skipped 6.7 come back to on weekend

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Skipped 4.13, 4.14

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(359) $\underline{v = \sqrt{\frac{\kappa a^2}{m}}}$

isolate an atom in a 1-D solid

Imagine the equillibrium position is at $x_e q$, if you perturb it the potential is approximately hooks law $\implies -\kappa(x - x_{eq}) = -\kappa(\delta x_{eq}) = F =$

recall: $\beta = -\frac{1}{V}\frac{\partial V}{\partial P}$ (compressibility) $\beta = -\frac{1}{L}\frac{\partial L}{\partial F}$ (1-D)

recall: $v = \sqrt{\frac{B}{\rho}} = \sqrt{\frac{1}{\rho\beta}}$, $B = \frac{1}{\beta}$ (bulk modulus)

$\beta = -\frac{1}{L}\frac{\partial L}{\partial F} = -\frac{1}{a}(\frac{\partial x}{\partial F})$

$= -\frac{1}{a}(\frac{\partial F}{\partial x})^{-1} = -\frac{1}{a}(-\frac{1}{\kappa}) = \frac{1}{\kappa a}$; $\rho = \frac{m}{a}$

$\therefore v = \sqrt{\frac{1}{\rho\beta}} = \sqrt{\frac{\kappa a^2}{m}} = \sqrt{\frac{\kappa}{m}}a$

- -

Definition 13.2

Definition 13.3

Claim 13.1

Proof:

We know direct and reciprocal lattice are defined by $e^{i\vec{G}\cdot\vec{r}} = 1 \implies \vec{G}\cdot\vec{r} = 2\pi m$, this defines an infinite set of parallel planes

Claim

The spacing between planes is $d = \frac{2\pi}{||\vec{G}||}$

Proof:

Consider $\vec{r}-1$ point on a plane and $\vec{r}_2$ a point on adjacent plane

$\implies \vec{G}\cdot(\vec{r}_1 - \vec{r}_2) = 2\pi(m+1) - 2\pi m = 2\pi$

$\implies ||\vec{G}||\hat{G}\cdot(\vec{r} - 1 - \vec{r}_2) = 2\pi$

$\implies \hat{G}\cdot(\vec{r}_1 - \vec{r}_2) = \frac{2\pi}{||\vec{G}||}$

Thus the distance between two planes in the direction of normal $\hat{G} isd = \frac{2\pi}{||\vec{G}||}$

Choose $\vec{G}$ to be the smallest such $\vec{G}$ (largest distance) such that every plane contains lattice points.

Note: $\vec{G} = \frac{2\pi\hat{G}}{d}$

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Definition 13.4 Brillouin zone is primitive unit cell of the reciprocal lattice
Definition 13.5

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recall: $R = \frac{2\pi}{\hbar}|\frac{V_{if}}{2}|^2\rho(E_f) = \frac{2\pi}{\hbar}|\frac{\langle i|V|f\rangle}{2}|^2\rho(E_f)$
analogously
$\Gamma(\vec{k}', \vec{k}) = \frac{2\pi}{\hbar}|\langle\vec{k}'|V|\vec{k}\rangle|^2\delta(E_{\vec{k}'} - E_{\vec{k}})$

---

(360) $\underline{\langle\vec{k}'|V|\vec{k}\rangle = \frac{1}{L^3}\int d\vec{r}e^{-i(\vec{k}'-\vec{k})\cdot\vec{r}}V(\vec{r})}$
trap a particle in a box of length L
actually no, we are shooting a particle at a crystal lattice, the lattice represents potential and $|\vec{k}\rangle$ represents the free particle (say electron) that we are shooting at the lattice
$\implies \psi_{\vec{k}} = \frac{1}{\sqrt{L^3}}e^{i\vec{k}\cdot\vec{r}}$
Note: dont use the regular def for $\langle x|p\rangle$ because the domain is $(-\infty, \infty)$ instead use $0, L$. $\implies \psi_{\vec{k}} = Ae^{ipx/\hbar}$ but $p = \hbar k \implies \psi_{\vec{k}} = Ae^{ikx}$. normalizing gives us $\int_0^\infty |A|^2dx = 1$ for $(k = k')$ so $A = \frac{1}{\sqrt{L}} \implies \langle\vec{k}'|V|\vec{k}\rangle = \langle\vec{k}'|V(\hat{\vec{x}})(\int d\vec{x}|\vec{x}\rangle\langle\vec{x}|)|\vec{k}\rangle$
$= \int d\vec{x}\langle\vec{k}'|V(\hat{\vec{x}})|\vec{x}\rangle\langle\vec{x}|\vec{k}\rangle$
$= \int d\vec{x}V(\vec{x})\langle\vec{k}'|\vec{x}\rangle\langle\vec{x}|\vec{k}\rangle = \frac{1}{L^3}\int d\vec{r}e^{-i\vec{k}'\cdot\vec{r}}Ve^{i\vec{k}\cdot\vec{r}}$
$= \frac{1}{L^3}\int d\vec{r}e^{-i(\vec{k}'-\vec{k})}V(\vec{r})$

---

(361) $\underline{\langle\vec{k}'|V|\vec{k}\rangle = \frac{1}{L^3}[\sum_{\vec{R}} e^{-i(\vec{k}'-\vec{k})\cdot\vec{R}}][\int_{unit-cell} d\vec{x}e^{-i(\vec{k}'-\vec{k})\cdot\vec{x}}V(\vec{x})]}$
recall: $\langle\vec{k}'|V|\vec{k}\rangle = \frac{1}{L^3}\int d\vec{r}e^{-i(\vec{k}'-\vec{k})\cdot\vec{r}}V(\vec{r})$
$= \frac{1}{L^3}\sum_{\vec{R}}\int_{unit-cell} d\vec{x}e^{-i(\vec{k}'-\vec{k})\cdot(\vec{x}+\vec{R})}V(\vec{x}+\vec{R})$
$= \frac{1}{L^3}[\sum_{\vec{R}} e^{-i(\vec{k}'-\vec{k})\cdot\vec{R}}][\int_{unit-cell} d\vec{x}e^{-i(\vec{k}'-\vec{k})\cdot\vec{x}}V(\vec{x}+\vec{R})]$
Potential periodic $\implies V(\vec{x}+\vec{R}) = V(\vec{x})$
$\therefore \langle\vec{k}'|V|\vec{k}\rangle = \frac{1}{L^3}[\sum_{\vec{R}} e^{-i(\vec{k}'-\vec{k})\cdot\vec{R}}][\int_{unit-cell} d\vec{x}e^{-i(\vec{k}'-\vec{k})\cdot\vec{x}}V(\vec{x})]$
recall: $\sum_{\vec{R}} e^{i\vec{k}\cdot\vec{R}} = \frac{(2\pi)^D}{V}\sum_{\vec{G}}\delta^D(\vec{k}-\vec{G})$
$\implies \sum_{\vec{R}} e^{-i(\vec{k}'-\vec{k})\cdot\vec{R}} = \frac{(2\pi)^D}{V}\sum_{\vec{G}}\delta^D(\vec{k}'-\vec{k}-\vec{G})$
$\implies \vec{k}'-\vec{k} = \vec{G}$, otherwise $\langle\vec{k}'|V|\vec{k}\rangle = 0$

$|\vec{k}| = |\vec{k}'|$ conservation of energy (dont understand)
$n\lambda = 2d\sin\theta$ (bragg condition for constructive interference

---

(362) $\underline{2d\sin\theta = n\lambda}$
$\hat{k}\cdot\hat{G} = \cos(90-\theta) = \sin\theta = -\hat{k}'\cdot\vec{G} = -\cos(90+\theta)$
$= \sin\theta$
$\vec{k} - \vec{k}' = \vec{G}$ (Laue condition); $|\vec{k}| = |\vec{k}'| = \frac{2\pi}{\lambda}$
$\implies \vec{k} - \vec{k}' = \frac{2\pi}{\lambda}(\hat{k} - \hat{k}' = \vec{G}$
$\implies \frac{2\pi}{\lambda}\hat{G}\cdot(\hat{k} - \hat{k}') = \hat{G}\cdot\vec{G} = |\vec{G}|$
$\implies \frac{2\pi}{\lambda}2\hat{G}\cdot\hat{k} = \frac{2\pi}{\lambda}2\sin\theta = |\vec{G}|$
$\implies \frac{2\pi}{|\vec{G}|}2\sin\theta = 2d\sin\theta = \lambda$
recall: $d = \frac{2\pi}{|\vec{G}|}$
but if $\vec{G}$ is reciprocal lattice vector then $n\vec{G}$ is too $\implies 2d\sin\theta = n\lambda$

$S(\vec{G}) = \int_{unit-cell} d\vec{x}e^{i\vec{G}\cdot\vec{x}}V(\vec{x})$ (structure factor)
$\vec{G} = \vec{k} - \vec{k}'$ (reciprocal lattice vector)
$I_{(hk\ell)} \propto |S_{(hk\ell)}|^2$ (scattering intensity)
assume $V(\vec{x}) = \sum_{atomj} V_j(\vec{x} - \vec{x}_j)$
i.e. the interactions of atoms does not affect in coming waves.

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(363) $\underline{S(\vec{G}) \sim \sum_{atomsinjunit-cell} b_j e^{i\vec{G}\cdot\vec{x}_j}}$ (scattering neutrons)
Short range nuclear forces
$\implies V(\vec{x}) = \sum_{atomsj} f_j\delta(\vec{x} - \vec{x}_j)$
$\vec{x}_j$ position of j^{th} atom in unit cell
f_j form factor (strenth of scatter)
$V(\vec{x} \propto \sum_{atomj} b_j\delta(\vec{x} - \vec{x}_i)f_j = \frac{2\pi\hbar b_j}{m}$
$\therefore S(\vec{G}) = \int_{unit-cell} d\vec{x}e^{i\vec{G}\cdot\vec{x}}V(\vec{x})$
$= \sum_{atomj}\int_{unit-cell} d\vec{x}e^{i\vec{G}\cdot\vec{x}}f_j\delta(\vec{x} - \vec{x}_j)$
$= \sum_{atomj} e^{i\vec{G}\cdot\vec{x}_j}f_j \propto \sum_{atomj} b_j e^{i\vec{G}\cdot\vec{x}_j}$

---

(364) $\underline{S(\vec{G}) = \sum_{atominjunit-cell} f_j(\vec{G})e^{i\vec{G}\cdot\vec{x}_j}}$
(x-rays scatter from electrons)
$V_j(\vec{x} - \vec{x}_j) = Z_j g_j(\vec{X} - \vec{X}_j)$
$g_j \sim$ short range; $Z_j \sim$ atomic number of jth atom
$\implies S(\vec{G}) = \int_{unit-cell} d\vec{x}e^{i\vec{G}\cdot\vec{x}}V(\vec{x})$
$= \sum_j \int d\vec{X}e^{i\vec{G}\cdot\vec{x}}V_j(\vec{X})$

$= \sum_{jthatominunit-cell} f_j(\vec{G})e^{i\vec{G}\cdot\vec{X}_j}$ (don't understand)
shifts all eigen energies by constant
$\Longrightarrow$ may assume $V_0 = 0$
$\epsilon(\vec{k}) = \epsilon_0(\vec{k}) + \sum_{\vec{k}'=\vec{k}+\vec{G}} \frac{|\langle\vec{k}'|V|\vec{k}\rangle|^2}{\epsilon_0(\vec{k})-\epsilon_0(\vec{k}')}$

- -

(365) $\underline{\epsilon(\vec{k}) = \epsilon_0(\vec{k}) + \langle\vec{k}|V|\vec{k}\rangle = \epsilon_0(\vec{k}) + V_0}$(first order; nondegenerate)
$H_0 = \frac{\hat{p}^2}{2m}$; $\epsilon_0(\vec{k}) = \frac{\hbar^2|\vec{k}|^2}{2m}$
$H = H_0 + V(\vec{r})$ (perturbation)
$V(\vec{r}) = V(\vec{r}+\vec{R})$
recall: $\langle\vec{k}'|V|\vec{k}\rangle = \frac{1}{L^3}\int d\vec{r}e^{i(\vec{k}-\vec{k}')\cdot\vec{r}}V(\vec{r}) \equiv V_{\vec{k}'-\vec{k}}$
this is zero unless $\vec{k}'-\vec{k}$ is a reciprocal Lattice vector
$\Longrightarrow$ $\epsilon(\vec{k}') = \langle\vec{k}|H|\vec{k}\rangle = \langle\vec{k}|\epsilon_0|\vec{k}\rangle + \langle\vec{k}|V|\vec{k}\rangle = \epsilon_0(\vec{k}) + V_0$
this is for non-degenerate case.
its possible that $\epsilon_0(\vec{k}) = \epsilon_0(\vec{k}')$ but $\vec{k}' = \vec{k}+\vec{G}$

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(366) $\underline{(\epsilon_0(\vec{k})-E)(\epsilon_0(\vec{k}+\vec{G})-E) - |V_{\vec{G}}|^2 = 0}$ (degenerate)
$\langle\vec{k}|H|\vec{k}\rangle = \epsilon_0(\vec{k})$, spose we have to plane wave states $|\vec{k}'\rangle = |\vec{k}+\vec{G}\rangle$ and $|\vec{k}\rangle$
$\langle\vec{k}'|H|\vec{k}'\rangle = \epsilon_0(\vec{k}') = \epsilon_0(\vec{k}+\vec{G})$
$\langle\vec{k}|H|\vec{k}'\rangle = V_{\vec{k}-\vec{k}'} = V^*_{\vec{G}}$
$\langle\vec{k}'|H|\vec{k}\rangle = V_{\vec{k}'-\vec{k}} = V_{\vec{G}}$
$V_{-\vec{G}} = V^*_{\vec{G}}$
$|\psi\rangle = \alpha|\vec{k}\rangle + \beta|\vec{k}'\rangle = \alpha|\vec{k}\rangle + \beta|\vec{k}+\vec{G}\rangle$
$H = \begin{pmatrix} \epsilon_0(\vec{k}) & V^*_{-\vec{G}} \\ V_{\vec{G}} & \epsilon_0(\vec{k}+\vec{G}) \end{pmatrix}$
$\Longrightarrow \begin{vmatrix} \epsilon_0(\vec{k})-E & V^*_{-\vec{G}} \\ V_{\vec{G}} & \epsilon_0(\vec{k}+\vec{G})-E \end{vmatrix} = 0$
$\therefore (\epsilon_0(\vec{k})-E)(\epsilon_0(\vec{k}+\vec{G})-E) - |V_{\vec{G}}|^2 = 0$

- -

(367) $\underline{E_\pm = \epsilon_0(\vec{k}) \pm |V_{\vec{G}}|}$
$\vec{k}$ on zone boundary $\Longrightarrow$ $\epsilon_0(\vec{k}) = \epsilon_0(\vec{k}+\vec{G})$
recall: $(\epsilon_0(\vec{k})-E)(\epsilon_0(\vec{k}+\vec{G})-E) - |V_{\vec{G}}|^2 = 0$
$\Longrightarrow$ $E_\pm = \epsilon_0(\vec{k}) \pm |V_{\vec{G}}|$

- -

(368) $$\begin{cases} \psi_+ \sim e^{ix\pi/a} + e^{-ix\pi/a} \propto \cos(\frac{x\pi}{a}) \\ \psi_- \sim e^{ix\pi/a} - e^{-ix\pi/a} \end{cases} \quad (1D)$$

$V(x) = \tilde{V}\cos(2\pi x/a);\ \tilde{V} > 0$ Brillouin zones at $k = \frac{\pi}{a} and k' = -k = -\frac{\pi}{a} \implies k' - k = G = -\frac{2\pi}{a}$ and $\epsilon_0(\vec{k}) = \epsilon_0(\vec{k}')$

recall: $\begin{pmatrix} \epsilon_0(\vec{k}) & V^*_{\vec{G}} \\ V_{\vec{G}} & \epsilon_0(\vec{k}+\vec{G}) \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = E \begin{pmatrix} \alpha \\ \beta \end{pmatrix};\ |\psi\rangle = \alpha|\vec{k}\rangle + \beta|\vec{k}'\rangle$

$\implies \begin{pmatrix} \epsilon_0(\vec{k} & V_{\vec{G}} \\ V_{\vec{G}} & \epsilon_0(\vec{k}) \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = E \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$

$\implies \alpha = \pm\beta \implies |\psi_\pm\rangle = \frac{1}{\sqrt{2}}(|k\rangle \pm |k'\rangle)$

$\implies \langle x|k\rangle \to e^{ikx} = e^{ix\pi/a}$

$\implies \langle x|k'\rangle \to e^{-ik'x} = e^{-ix\pi/a}$

$\implies \begin{cases} \langle x|\psi_+\rangle \sim e^{ix\pi/a} + e^{-ix\pi/a} \propto \cos(x\pi/a) \\ \langle x|\psi_-\rangle \sim e^{ix\pi/a} - e^{-ix\pi/a} \propto \sin(x\pi/a) \end{cases}$

- -

(369) $E_\pm = \frac{\hbar^2(n\pi/a)^2}{2m} \pm |V_G| + \frac{\hbar^2\delta^2}{2m}[1 \pm \frac{\hbar^2(n\pi/a)^2}{m}\frac{1}{|V_G|}]$ (k just off zone boundaries $k = \pm\frac{n\pi}{a}$)

gap at zone boundaries is$\pm|V_G|$

consider $k = n\pi/a + \delta$ can scatter to $k' = -\frac{n\pi}{a} + \delta$

$\implies \epsilon_0(k) = \epsilon_0(\frac{n\pi}{a} + \delta = \frac{\hbar^2 k^2}{2m} = \frac{\hbar^2}{2m}(\frac{n\pi}{a} + \delta)^2 = \frac{\hbar^2}{2m}[(\frac{n\pi}{a})^2 + \frac{2\pi n\delta}{a} + \delta^2]$

$\epsilon_0(k') = \epsilon_0(-\frac{n\pi}{a} + \delta) = \frac{\hbar^2}{2m}[(n\pi/a)^2 - 2n\pi\delta/a + \delta^2]$

recall :$(\epsilon_0(\vec{k}) - E)(\epsilon_0(\vec{k}+\vec{G}) - E) - |V_{\vec{G}}|^2 = 0$

$\implies (\frac{\hbar^2}{2m}[(n\pi/a)^2 + \delta^2] - E + \frac{\hbar^2}{2m}2n\pi\delta/a) \cdot (\frac{\hbar^2}{2m}[(n\pi/a)^2 + \delta^2] - E - \frac{\hbar^2}{2m}\frac{2n\pi\delta}{a}) - |V_{\vec{G}}|^2 = 0$

$\implies (\frac{\hbar^2}{2m}[(n\pi/a) + \delta^2] - E)^2 = (\frac{\hbar^2}{2m}2n\pi\delta/a)^2 + |V_G|^2$

$\implies E_\pm = \frac{\hbar^2}{2m}[(n\pi/a)^2 + \delta^2] \pm \sqrt{(\frac{\hbar^2}{2m}2n\pi\delta/a)^2 + |V_G|^2}$

small $\delta \implies \sqrt{x+1} \approx 1 + \frac{1}{2}x$

$\implies E_\pm = \frac{\hbar^2}{2m}[(n\pi/a)^2 + \delta^2] \pm (1 + (\frac{\hbar^2}{2m|V_G|^2}\frac{n\pi\delta}{a})^2)|V_G|$

$\implies E_\pm = \frac{\hbar^2}{2m}(n\pi/a)^2 \pm |V_G| + \frac{\hbar^2\delta^2}{2m}[1 \pm \frac{\hbar^2}{2m}(\frac{n\pi}{a})^2\frac{1}{|V_G|}]$

- -

this equation can be rewritten as $E_\pm(G+\delta) = C_{pm} \pm \frac{\hbar^2\delta^2}{2m^*_\pm}C_\pm$ (constants)

where $m^*_\pm = \frac{m}{|1 \pm \frac{\hbar^2(n\pi/a)^2}{m}\frac{1}{|V_G|}|}$

Do exercise 32 chapter 9 in GR
Lets restrict the gauge:
Lets use TT gauge transverse to direction of motion which has unit vector $n^j = x^j/r$ this simplifies the wave.
Choose axes so that at the point we measure the wave it travels in the z direction (assume plane waves)
$\implies \bar{h}^{TT}_{zi} = -2\Omega^2 D_{zi} e^{i\Omega(r-t)}/r$
$= -2\Omega^2 \int T^{00} x_z x_i d^3x e^{i\Omega(r-t)}/r$

Chapter 2

(370) $\underline{\Delta x^{\bar{\alpha}} = \sum_{\beta=0}^{3} \Lambda^{\bar{\alpha}}_{\beta} \Delta x^{\beta}}$
recall: $\Delta x^{\bar{0}} = \frac{\Delta x^0}{\sqrt{1-v^2}} - \frac{v\Delta x^1}{\sqrt{1-v^2}}$
$\Delta x^{\bar{0}} = \Lambda^{\bar{0}}_1 \Delta x^1 + \Lambda^{\bar{0}}_2 \Delta x^2 = \Lambda^{\bar{0}}_{\alpha} \Delta x^{\alpha}$
$\therefore \Delta x^{\bar{\alpha}} = \Lambda^{\bar{\alpha}}_{\beta} \Delta x^{\beta}$

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recall: $(\vec{e}_\alpha)^\beta = \delta^\beta_\alpha$($\beta^{th}$ component of α^{th} basis vector)

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(371) $\underline{\vec{e}_\alpha = \Lambda^{\bar{\beta}}_{\alpha} \vec{e}_{\bar{\beta}}}$
$\vec{A} = A^\alpha \vec{e}_\alpha = A^{\bar{\alpha}} \vec{e}_{\bar{\alpha}} \implies \Lambda^{\bar{\alpha}}_{\beta} A^\beta \vec{e}_{\bar{\alpha}} = A^\alpha \vec{e}_\alpha \implies A^\beta \Lambda^{\bar{\alpha}}_{\beta} \vec{e}_{\bar{\alpha}} = A^\alpha \vec{e}_\alpha$
$\beta \to \alpha, \bar{\alpha} \to \bar{\beta} \implies A^\alpha \Lambda^{\bar{\alpha}}_{\alpha} \vec{e}_{\bar{\alpha}} = A^\alpha \vec{e}_\alpha \implies A^\alpha(\Lambda^{\bar{\alpha}}_{\alpha} \vec{e}_{\bar{\alpha}} - \vec{e}_\alpha) = 0$
$\implies \vec{e}_\alpha = \Lambda^{\bar{\alpha}}_{\alpha} \vec{e}_\alpha$

---

(372) $\underline{\Lambda^{\bar{\beta}}_{\alpha}(\vec{v}) \Lambda^{\nu}_{\bar{\beta}}(-\vec{v}) = \delta^{v}_{\alpha}}$
$\Lambda^{\bar{\beta}}_{\alpha} = \Lambda^{\bar{\beta}}_{\alpha}(\vec{v}),\ \vec{e}_\alpha = \Lambda^{\bar{\beta}}_{\alpha}(\vec{v}) \vec{e}_{\bar{\beta}},\ \vec{e}_{\bar{\mu}} = \Lambda^{\nu}_{\bar{\mu}}(-\vec{v}) \vec{e}_\nu$
$\implies \vec{e}_{\bar{\beta}} = \Lambda^{\nu}_{\bar{\beta}}(-\vec{v}) \vec{e}_\nu$
$\therefore \Lambda^{\bar{\beta}}_{\alpha}(\vec{v}) \Lambda^{\nu}_{\bar{\beta}}(-\vec{v}) = \delta^{\nu}_{\alpha} \implies \vec{e}_\alpha = \delta^{\nu}_{\alpha} \vec{e}_\nu$

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(373) $\underline{A^{\bar{\beta}} = \Lambda^{\bar{\beta}}_{\alpha}(\vec{v}) A^\alpha \implies \Lambda^{\nu}_{\bar{\beta}}(-\vec{v}) A^{\bar{\beta}} = A^\nu}$
$A^{\bar{\beta}} = \Lambda^{\bar{\beta}}_{\alpha}(\vec{v}) A^\alpha \implies \Lambda^{\nu}_{\bar{\beta}}(-\vec{v}) A^{\bar{\beta}} = \Lambda^{\nu}_{\bar{\beta}}(-\vec{v}) \Lambda^{\bar{\beta}}_{\alpha} A^\alpha = \delta^{\nu}_{\alpha} A^\alpha = A^\nu$

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momentarily comoving reference frame (MCRF) - an inertial frame that momentarily has the same velocity as the accelerated particle

$\vec{U}$ (four velocity) - vector tangent to world line, length s.t. stretches 1 unit of time in particles reference frame, $\vec{U} := (\vec{e}_0)_{MCRF}$

$\vec{p} = m\vec{U}$ (four momentum) $\vec{p} \to_O (E, p^1, p^2, p^3)$
$\vec{p} := \sum_i \vec{p}_{(i)},\ \sum_i \vec{p}_(i) \to_{CM} (E_{tot}, 0, 0, 0)$ (center of momentum frame (CM))

$\vec{A}^2 := -(A^0)^2 + (A^1)^2 + (A^2)^2 + (A^3)^2$ (mag of $\vec{A}$) $\to$ frame independent
$\vec{A}^2 > 0 \implies \vec{A}$ spacelike, $\vec{A}^2 < 0 \implies$ timelike, $\vec{A}^2 = 0$ null-like
$\vec{A} \cdot \vec{B} := -A^0B^0 + A^1B^1 + A^2B^2 + A^3B^3$

$\vec{e}_\alpha \cdot \vec{e}_\beta = \eta_{\alpha\beta},\ \eta_{00} = -1;\ \eta_{0i} = \eta_{j0} = 0,\ \eta_{\alpha\beta} = \delta_{\alpha\beta} for \alpha,\ \beta \neq 0$
$ds^2 = d\vec{x} \cdot d\vec{x} = -dt^2 + dx^2 + dy^2 + dz^2,\ (d\tau)^2 = -d\vec{x} \cdot d\vec{x}$

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(374) $\underline{\vec{U} \overset{\star}{=} \frac{d\vec{x}}{d\tau}}$
$\frac{d\vec{x}}{d\tau} \cdot \frac{d\vec{x}}{d\tau} = \frac{d\vec{x} \cdot d\vec{x}}{d\tau^2} = -\frac{d\tau^2}{d\tau^2} = -1,\ d\vec{x} \to_{MCRF, d\tau = dt} (dt, 0, 0, 0)$
$\frac{d\vec{x}}{d\tau} \to_{MCRF} (1, 0, 0, 0),\ \frac{d\vec{x}}{d\tau} = (\vec{e}_0)_{MCRF}$
$\therefore \vec{U} = \frac{d\vec{x}}{d\tau}$

<u>Note:</u> $\vec{v} = v^\alpha \vec{e}_\alpha = \Lambda^\alpha_{\beta'} V^{\beta'} \Lambda^{\mu'}_\alpha \vec{e}_{\mu'} = \Lambda^\alpha_{\beta'} \Lambda^{\mu'}_\alpha V^{\beta'} \vec{e}_{\mu'} = \delta^{\mu'}_{\beta'} V^{\beta'} \vec{e}_{\mu'}$
$= v^{\mu'} \vec{e}_{\mu'} = \vec{v}$, Lorentz invariant

---

(375) $\underline{-\vec{p} \cdot \vec{U}_{obs} = \bar{E}}$
$\vec{p} \cdot \vec{p} = m^2 \vec{U} \cdot \vec{U} = -m^2$
$\vec{p} \cdot \vec{p} = -E^2 + (p^1)^2 + (p^2)^2 + (p^3)^2 = -m^2$
$\implies E^2 = m^2 + \sum_{i=1}^3 (p^i)^2$
$\vec{p} \cdot \vec{U}_{obs} = \vec{p} \cdot \vec{e}_{\bar{0}},\ \vec{p} \to_{\bar{O}} (\bar{E}, p^{\bar{1}}, p^{\bar{2}}, p^{\bar{3}})$
$\therefore -\vec{p} \cdot \vec{U}_{obs} = -\bar{E} \vec{e}_{\bar{0}} \cdot \vec{e}_{\bar{0}} = \bar{E}$

<u>Photons</u>
$d\vec{x} \cdot d\vec{x} = 0$ (null like)
- no MCRF for photons

- no tangent vectors to the world line of a photon with nonzero magnitudes

- -

(376) $\underline{\vec{A}\cdot\vec{B} = g_{\alpha\beta}A^\alpha B^\beta}$

$\vec{A} = A^\alpha\vec{e}_\alpha;\ \vec{B} = B^\beta\vec{e}_\beta$

$\vec{A}\cdot\vec{B} = (A^\alpha\vec{e}_\alpha)\cdot(B^\beta\vec{e}_\beta) = A^\alpha B^\beta\vec{e}_\alpha\cdot\vec{e}_\beta = g_{\alpha\beta}A6\alpha B^\beta$

- -

Linearity

First Argument

$(\alpha\vec{A})\cdot\vec{B} = \alpha(\vec{A}\cdot\vec{B})$

$(\vec{A}+\vec{B})\cdot\vec{C} = \vec{A}\cdot\vec{C}+\vec{B}\cdot\vec{C}$

Second Arugument

$\vec{A}\cdot(\beta\vec{B}) = \beta(\vec{A}\cdot\vec{B})$

$\vec{A}\cdot(\vec{B}+\vec{C}) = \vec{A}\cdot\vec{B}+\vec{A}\cdot\vec{C}$

- -

(377) $\underline{g(\vec{A},\vec{B}) := \vec{A}\cdot\vec{B}}$

$g(\alpha\vec{A}+\beta\vec{B},\vec{C}) = \alpha g(\vec{A},\vec{C})+\beta g(\vec{B},\vec{C})$ (linear)

(similar argument for second argument)

$g(\vec{e}_\alpha,\vec{e}_\beta) = \vec{e}_\alpha\cdot\vec{e}_\beta = \eta_{\alpha\beta}$ or $g_{\alpha\beta}$ in general

- -

properties

$$\begin{cases}\tilde{s} = \tilde{p}+\tilde{q}\\ \tilde{r} = \alpha\tilde{p}\end{cases} \implies \begin{cases}\tilde{s}(\vec{A} = \tilde{p}\vec{A})+\tilde{q}(\vec{A})\\ \tilde{r}(\vec{A}) = \alpha\tilde{p}(\vec{A})\end{cases}$$

definition: $p_\alpha := \tilde{p}(\vec{e}_\alpha)$

$\tilde{p}(\vec{A} = \tilde{p}(A^\alpha\vec{e}_\alpha) = A^\alpha\tilde{p}(\vec{e}_\alpha) = A^\alpha p_\alpha$

- -

(378) $\underline{p_{\bar\beta} = \Lambda^\alpha_{\bar\beta}p_\alpha}$

$p_{\bar\beta} = \tilde{p}(\vec{e}_{\bar\beta}) = \tilde{p}(\Lambda^\alpha_{\bar\beta}\vec{e}_\alpha) = \Lambda^\alpha_{\bar\beta}\tilde{p}(\vec{e}_\alpha) = \Lambda^\alpha_{\bar\beta}p_\alpha$

compare with $\vec{e}_{\bar\beta} = \Lambda_{\bar\beta^\alpha\vec{e}_[\alpha}$

- -

(379) $\underline{ds^2 = dt^2 - a(t)^2(\frac{dr^2}{1-kr^2}+r^2d\theta^2+r^2\sin^2\theta d\phi^2)}$

$ds^2 = dt^2 - (\frac{dr^2}{1-\frac{r^2}{a^2}}+r^2d\theta^2+r^2\sin^2\theta d\phi^2)$

$= dt^2 - a^2(\frac{d(\frac{r}{a})^2}{1-(\frac{r}{a})^2}+(\frac{r}{a}^2d\theta^2+(\frac{r}{a})^2\sin^2\theta d\phi^2)$

$\frac{r}{a}\to r$

$\implies ds^2 = dt^2 - a^2(\frac{dr^2}{1-r^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2)$
$\therefore ds^2 = dt^2 - a^2(\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2)$

- -

Note: $T(k,\ell)\ S(m,n),\ T\otimes S \to (k+m, \ell+n)$

- -

(380) $\underline{A^{\bar{\alpha}} p_{\bar{\alpha}} = A^\beta p_\beta}$
$A^{\bar{\alpha}} p_{\bar{\alpha}} = (\Lambda^{\bar{\alpha}}_\beta A^\beta)(\Lambda^\mu_{\bar{\alpha}} p_\mu) = \Lambda^{\bar{\alpha}}_\beta \Lambda^\mu_{\bar{\alpha}} A^\beta p_\mu = \delta^\mu_\beta A^\beta p_\mu = A^\beta p_\beta$

- -

$\{\tilde{\omega}^\alpha\} dual to \{\vec{e}_\alpha\} \implies \tilde{p} = p_\alpha \tilde{\omega}^\alpha$

- -

(381) $\underline{\tilde{\omega}^\alpha(\vec{e}_\beta) = \delta^\alpha_\beta}$
$\tilde{p}(\vec{A}) = \tilde{p}(A^\alpha \vec{e}_\alpha) = A^\alpha \tilde{p}(\vec{e}_\alpha|) = A^\alpha p_\alpha$
$= p_\alpha \tilde{\omega}^\alpha(\vec{A}) = A^\beta p_\alpha \tilde{\omega}^\alpha(\vec{e}_\beta)$
$\implies \tilde{\omega}^\alpha(\vec{e}_\beta) = \delta^\alpha_\beta$

- -

(382) $\underline{\tilde{\omega}^{\bar{\alpha}} = \Lambda^{\bar{\alpha}}_\beta \tilde{\omega}^\beta \ p_\beta = \Lambda^{\bar{\alpha}}_\beta p_{\bar{\alpha}}}$
$\tilde{p} = p_{\bar{\alpha}} \tilde{\omega}^{\bar{\alpha}} = p_\beta \tilde{\omega}^\beta \implies p_{\bar{\alpha}} \tilde{\omega}^{\bar{\alpha}} = \Lambda^{\bar{\alpha}}_\beta p_{\bar{\alpha}} \tilde{\omega}^\beta$
$\therefore \tilde{\omega}^{\bar{\alpha}} = \Lambda^{\bar{\alpha}}_\beta \tilde{\omega}^\beta$

- -

$[t = t(\tau), x = x(\tau), y = y(\tau), z = z(\tau)]$
$\vec{U} \to (\frac{dt}{d\tau}, \frac{dx}{d\tau}, \dots), \phi(\tau) = \phi[t(\tau, x(\tau), y(\tau), z(\tau)]$
$\frac{d\phi}{d\tau} = \frac{\partial\phi}{\partial t}\frac{dt}{d\tau} + \frac{\partial\phi}{\partial x}\frac{dx}{d\tau} + \frac{\partial\phi}{\partial y}\frac{dy}{d\tau} + \frac{\partial\phi}{\partial z}\frac{dz}{d\tau}$
$= \frac{\partial\phi}{\partial t} U^t + \frac{\partial\phi}{\partial z} U^x + \frac{\partial\phi}{\partial y} U^y + \frac{\partial\phi}{\partial z} U^z$
one form has components $(\frac{\partial\phi}{\partial t}, \frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z})$
$\tilde{d}\phi \to_O (\frac{\partial\phi}{\partial t}, \frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z})$ gradient of ϕ

- -

(383) $\underline{(\tilde{d}\phi)_{\bar{\alpha}} = \Lambda^\beta_{\bar{\alpha}} (\tilde{d}\phi)_\beta,\ \frac{\partial x^\beta}{\partial x^{\bar{\alpha}}} = \Lambda^\beta_{\bar{\alpha}}}$
$(\tilde{d}\phi)_{\bar{\alpha}} = \Lambda^\beta)_{\bar{\alpha}} (\tilde{d}\phi)_{b\eta}$
$\frac{\partial\phi}{\partial x^{\bar{\alpha}}} = \frac{\partial x^\beta}{\partial x^{\bar{\alpha}}} \frac{\partial\phi}{\partial x^\beta} \implies (\tilde{d}\phi)_{\bar{\alpha}} = \frac{\partial x^\beta}{\partial x^{\bar{\alpha}}} (\tilde{d}\phi)_\beta$
recall: $x^\beta = \Lambda^\beta_{\bar{\alpha}} x^{\bar{\alpha}} \implies \frac{\partial x^\beta}{\partial x^{\bar{\alpha}}} = \Lambda^\beta_{\bar{\alpha}}$

- -

<u>Definition:</u> $\frac{\partial\phi}{\partial x^\alpha} := \phi_{,\alpha},\ x^\alpha_{,\beta} \equiv \delta^\alpha_\beta$

$\tilde{\omega}^\alpha(\vec{e}_\beta) = \delta^\alpha_\beta \implies \tilde{d}x^\alpha := \tilde{\omega}^\alpha \implies \tilde{d}f = \frac{\partial f}{\partial x^\alpha}\tilde{d}x^\alpha$

- -

<u>definition:</u> $f_{\alpha\beta} := f(\vec{e}_\alpha, \vec{e}_\beta)$ (general $\begin{pmatrix}0\\2\end{pmatrix}$ tensor)

<u>aside:</u> $f(\vec{A}, \vec{B}) = f(A^\alpha\vec{e}_\alpha, B^\alpha\vec{e}_\beta = A^\alpha B^\beta f(\vec{e}_\alpha, \vec{e}_\beta) = A^\alpha B^\beta f_{\alpha\beta}$

- -

(384) $\underline{f = f_{\alpha\beta}\tilde{\omega}^\alpha \otimes \tilde{\omega}^\beta}$

$f = f_{\alpha\beta}\tilde{\omega}^{\alpha b\eta} \implies f_{\mu\nu} = f(\vec{e}_\mu, \vec{e}_\nu) = f_{\alpha\beta}\tilde{\omega}^{\alpha\beta}(\vec{e}_\mu, \vec{e}_\nu)$

$\tilde{\omega}^{\alpha\beta}(\vec{e}_\mu, \vec{e}_\nu) = \delta^\alpha_\mu\delta^\beta_\nu \implies \tilde{\omega}^{\alpha\beta} = \tilde{\omega}^\alpha \otimes \tilde{\omega}^\beta$

$\therefore f = f_{\alpha\beta}\tilde{\omega}^\alpha \otimes \tilde{\omega}^\beta$

- -

(385) $\underline{h_{(\alpha\beta)} := \frac{1}{2}(h_{\alpha\beta} + h_{\beta\alpha})}$

$f(\vec{A}, \vec{B}) = f(\vec{B}, \vec{A}),\ \forall\vec{A}, \vec{B}$ (symmettric)

$\vec{A} = \vec{e}_\alpha,\ \vec{B} = \vec{e}_\beta$

$\implies f_+\alpha\beta = f_{\beta\alpha}$

Arbitrary $\begin{pmatrix}0\\2\end{pmatrix}$ can be symmetric

$h_{(s)}(\vec{A}, \vec{B}) = \frac{1}{2}h(\vec{A}, \vec{B}) + \frac{1}{2}h(\vec{B}, \vec{A})$

$\implies h_{(s)\alpha\beta} = \frac{1}{2}(h_{\alpha\beta} + h_{\beta\alpha})$

$\implies h_{(\alpha\beta)} := \frac{1}{2}(h_{\alpha\beta} + h_{\beta\alpha})$

- -

(386) $\underline{h_{[\alpha\beta]} = \frac{1}{2}(h_{\alpha\beta} - h_{\beta\alpha})}$

$f(\vec{A}, \vec{B}) = -f(\vec{B}, \vec{A})\forall\vec{A}, \vec{B}$ (antisymmetric)

$\implies f_{\alpha\beta} = -f_{\beta\alpha}$

$h_{(A)}(\vec{A}, \vec{B}) = \frac{1}{2}h(\vec{A}, \vec{B}) - \frac{1}{2}h(\vec{B}, \vec{A})$

$\implies h_{(A)\alpha\beta} = \frac{1}{2}(h_{\alpha\beta} - h_{\beta\alpha})$

$\therefore h_{[\alpha\beta]} = \frac{1}{2}(h_{\alpha\beta} - h_{\beta\alpha})$

<u>Note:</u> $h_{\alpha\beta} = h_{(\alpha\beta)} + h_{[\alpha\beta]};\ g(\vec{A}, \vec{B}) = g(\vec{B}, \vec{A})$

- -

(387) $\underline{V_\alpha = \eta_{\alpha\beta}V^\beta}$

$g(\vec{V},) := \tilde{V}(),\ \tilde{V}(\vec{A}) = g(\vec{V}, \vec{A}) = \vec{V}\cdot\vec{A},\ g(, \vec{V}) := \tilde{V}()$

$V_\alpha = \tilde{V}(\vec{e}_\alpha) = \vec{V}\cdot\vec{e}_\alpha = V^\beta\vec{e}_\alpha\cdot\vec{e}_\beta = g_{\alpha\beta}V^{\beta\therefore V_[\alpha} = \eta_{\alpha b\eta}V^\beta$

$if\vec{V} \to (a, b, c, d) \implies \tilde{V} \to (-a, b, c, d)$

$V_\alpha = \eta_{\alpha\beta}V^\beta$ invertible $\implies A^\alpha := \eta^{\alpha\beta}A_\beta$

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(388) $\underline{\tilde{p}^2 = \eta^{\alpha\mu}p_\mu p_\alpha}$
$\tilde{p}^2 = \vec{p}^2 = \eta_{\alpha\beta}p^\alpha p^\beta$
$\implies \tilde{p}^2 = \eta_{\alpha\beta}(\eta^{\alpha\mu}p_\mu)(\eta^{\beta\nu}p_\nu) \implies \eta_{\alpha\beta}\eta^{\beta\nu} = \delta^\nu_\alpha$
$\therefore \tilde{p}^2 = \eta^{\alpha\mu}p_\mu p_\alpha$

$\eta^{00} = -1,\ \eta^{0i} = 0, \eta^{ij} = \delta^{ij} \implies \tilde{p}^2 = -(p_0)^2 + (p_1)^3 + (p_2)^2 + (p_3)^2$
$\tilde{p}\cdot\tilde{q} := \frac{1}{2}[(\tilde{p}+\tilde{q})^2 - \tilde{p}^2 - \tilde{q}^2] \implies \tilde{p}\cdot\tilde{q} = -p_0q_0 + \sum_{n=1}^3 p_nq_n$
$\vec{V}(\tilde{p}) \equiv \tilde{p}(\vec{V}) \equiv p_\alpha V^\alpha \equiv \langle\tilde{p},\vec{V}\rangle$
$An \begin{pmatrix} M \\ N \end{pmatrix}$ tensor is a linear function of M one-forms and N vectors into the real numbers.
$R^{\bar{\alpha}}{}_{\bar{\beta}} = R(\tilde{\omega}^{\bar{\alpha}}; \vec{e}_{\bar{\beta}}) = R(\Lambda^{\bar{\alpha}}_\mu\tilde{\omega}^\mu; \Lambda^\nu_{\bar{\beta}}\vec{e}_\nu) = \Lambda^{\bar{\alpha}}_\mu\Lambda^\nu_{\bar{\beta}}R^\mu_\nu$
upper indices are contravariant and lower ones are covariant

$\begin{pmatrix} 2 \\ 1 \end{pmatrix} \leftarrow \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ ex. $T^\alpha_{\beta\gamma} := \eta_{\beta\mu}T^{\alpha\mu}_\gamma$ (mapping 2^{nd} one form)
ex. $T^\beta_{\alpha\gamma} := \eta_{\alpha\mu}T^{\mu\beta}_\gamma$ (mapping on first index)
$\begin{pmatrix} 3 \\ 0 \end{pmatrix}$ tensor $T^{\alpha\beta\gamma} := \eta^{\gamma\mu}T^{\alpha\beta}_\mu$
$\eta^\alpha_\beta \equiv \eta^{\alpha\mu}\eta_{\mu\beta} \equiv \delta^\alpha_\beta$

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(389) $\underline{d\tilde{T}/d\tau = (T^\alpha_{\beta,\gamma}\tilde{\omega}^\beta \otimes \vec{e}_\alpha)U^\gamma,\ \nabla\tilde{T} := (T^\alpha_{\beta,\gamma}\tilde{\omega}^\beta \otimes \tilde{\omega}^{J\gamma} \otimes \vec{e}_\alpha)}$
$\tilde{T} = T^\alpha_\beta\tilde{\omega}^\beta \otimes \vec{e}_\alpha$
$\frac{d\tilde{T}}{d\tau} = \lim_{\Delta\tau\to0}\frac{\tilde{T}(\tau+\Delta\tau)-\tilde{T}(\tau)}{\Delta\tau},\ \tilde{\omega}^\alpha(\tau+\Delta\tau) = \tilde{\omega}^\alpha(\tau)$
$\implies \frac{d\tilde{T}}{d\tau} = (\frac{dT^\alpha_\beta}{d\tau})\tilde{\omega}^\beta \otimes \vec{e}_\alpha$
$\frac{d\tilde{T}}{d\tau} = (T^\alpha_{\beta,\gamma}\tilde{\omega}^\beta \otimes \vec{e}_\alpha)U^\gamma = \nabla_{\vec{U}}\tilde{T}, \nabla_{\vec{U}}\tilde{T} \to \{T^\alpha_{\beta,\gamma}U^\gamma\}$
$\nabla\tilde{T} := (T^\alpha_{\beta,\gamma}\tilde{\omega}^\beta \otimes \tilde{\omega}^\gamma \otimes \vec{e}_\alpha)$

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(390) $\underline{\tilde{p}^2 = \vec{p}^2}$
$\tilde{p}^2 = (p_\alpha\tilde{\omega}^\alpha)(p_\beta\tilde{\omega}^\beta) = \eta^{\alpha\beta}p_\alpha p_\beta = \eta^{\alpha\beta}(\eta_{\alpha\mu}p^\mu)(\eta_{\beta\nu}p^\nu)$
$= \eta^{\alpha\beta}\eta_{\beta\nu}\eta_{\alpha\mu}p^\mu p^\nu = \delta^\alpha_\nu\eta_{\alpha\mu}p^\mu p^\nu = \eta_{\alpha\mu}p^\mu p^\alpha = \eta_{\alpha\mu}p^\alpha p^\mu$

$\vec{p}^2 = (p^\alpha \vec{e}_\alpha)(p^\beta \vec{e}_\beta) = \vec{e}_\alpha \cdot \vec{e}_\beta p^\alpha p^\beta = \eta_{\alpha\beta} p^\alpha p^\beta$
$\therefore \tilde{p}^2 = \vec{p}^2$

– –

Chapter 4

$n :=$ number density in the MCRF of the element.
$\frac{n}{\sqrt{1-v^2}} = \{$ number density in frame in which particles have velocity $v\}$
$(\mathrm{flux})^{\bar{\mathrm{x}}} = \frac{\mathrm{nv}}{\sqrt{1-\mathrm{v}^2}}$
$\tilde{\mathrm{N}} = \mathrm{n}\tilde{\mathrm{U}}, \tilde{\mathrm{U}} \rightarrow_{\mathrm{O}} (\frac{1}{\sqrt{1-\mathrm{v}^2}}, \frac{\mathrm{v}^{\mathrm{x}}}{\sqrt{1-\mathrm{v}^2}}, \frac{\mathrm{v}^{\mathrm{y}}}{\sqrt{1-\mathrm{v}^2}}, \frac{\mathrm{v}^{\mathrm{z}}}{\sqrt{1-\mathrm{v}^2}})$
$\tilde{\mathrm{N}} \cdot \tilde{\mathrm{N}} = -\mathrm{n}^2,\ \mathrm{n} = (-\tilde{\mathrm{N}} \cdot \tilde{\mathrm{N}})^{1/2}$

– –

$\phi(t,x,y,z) = const.,\ \tilde{n} := \frac{\tilde{d}\phi}{|\tilde{d}\phi|}$ (unit normal one- form)
$|\tilde{d}\phi| is mag of \tilde{d}\phi : |\tilde{d}\phi| = |\eta^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta}|^{1/2} compare with \eta^{\alpha\beta}p_\alpha p_\beta = \tilde{p} \cdot \tilde{p}$
$\tilde{n} dx^\alpha dx^\beta dx^\gamma$

– –

$-\tilde{d\bar{t}} = U_\alpha = \eta_{\alpha\beta} U^\beta$ (four velocity as one form), $U_0 = -1,\ U_i = 0$
$E = \langle \tilde{d}t, \vec{p} \rangle = p^0 = -\vec{p} \cdot \vec{U}$

– –

$\rho :=$ energy density in the MCRF $= mn$
$\frac{\rho}{1-v^2} = \{$ energy density in a frame which particles have velocity $\vec{v}\}$

– –

$\tilde{T}(\tilde{d}x^\alpha, \tilde{d}x^\beta) = T^{\alpha\beta} := \{$ flux of α momentum across a surface of constant $x^\beta\}$
$T^{00} =$ energy density
$T^{0i} =$ energy flux across x^i surface
$T^{i0} = i$ momentum density
$T^{ij} =$ flux i momentum across j surface
in MCRF $(T^{00})_{MCRF} = \rho = mn;\ (T^{i0})_{MCRF} = (T^{0i})_{MCRF} = (T^{ij})_{MCRF} = 0$
Dust: $\tilde{T} = \vec{p} \otimes \vec{N} = mn\vec{U} \otimes \vec{U} = \rho\vec{U} \otimes \vec{U}$

- -

<u>aside:</u> $T^{\alpha\beta} = \tilde{T}(\tilde{\omega}^\alpha, \tilde{\omega}^\beta) = \rho\vec{U}(\tilde{\omega}^\alpha)\vec{U}(\tilde{\omega}^\beta) = \rho U^\alpha U^\beta$

- -

(391) $\underline{d\rho - (\rho + p)\frac{dn}{n} = nTdS,\ \Delta q = TdS}$
$\Delta E = \Delta Q - P\Delta V,\ or \Delta Q = \Delta E + P\Delta V$
particles conserved, $V = \frac{N}{n} \implies \Delta V = -\frac{N}{n^2}\Delta n$
$E = \rho V = \rho\frac{N}{n} \implies \Delta E = \rho\Delta V + V\Delta\rho$
$\implies \Delta Q = \frac{N}{n}\Delta\rho - N(\rho + P)\frac{\Delta n}{n^2}$
$\implies n\frac{\Delta Q}{N} \implies n\Delta q = \Delta\rho - \frac{\rho+P}{n}\Delta n,$
$, q := \frac{Q}{N}$
$\implies d\rho - (\rho. + P)\frac{dn}{n}$
$\implies d\rho - (\rho + P)\frac{dn}{n} \equiv AdB$
$\therefore d\rho - (\rho + P)\frac{dn}{n} = nTdS \implies \Delta q = T\Delta S$

- -

If S is a surface of constant x^i, then T^{ij} for fluid element A is $F^i/A,\ S$ has area A

- -

(392) $\underline{T^{ij} = T^{ji}}$ (symmetric; stuck)
<u>Proof</u>
Only need to show components symmetric in one frame $\implies$ $\forall \tilde{r}, \tilde{q},\ \tilde{T}(\tilde{r}, \tilde{q}) = \tilde{T}(\tilde{q}, \tilde{r}) \implies$ symmetry in any frame
(see diagram)
(a) $F_1^i = T^{ix}\ell^2$ (force exerted on neighbor)
$F_2^i = T^{iy}\ell^2,\ F_3^i = -T^{ix}\ell^2,\ F_4^i = -T^{iy}\ell^2$
as $\ell \to 0$ acceleration becomes infinite
unless, $F_3^i \approx -F_1^i,\ F_2^i \approx -F_4^i$
assume forces acti on center of face
$\tau_z^1 = -xF_1^y = -\frac{1}{2}\ell T^{yx}\ell^2$($F_1^y$ is exerted on neighbor $-F_1^y$ is exerted on self)
$\tau_z^2 = -yF^x_{2(self)}$ (Since if $F^x_{2(swlf)}$ is positive then τ_z^2 is negative)
$= yF_2^x = \frac{\ell}{2}T^{xy}\ell^2$
(take note of weird coordinate system too)
$\tau_z^3 = x_3F^y_{3(self)} = -\frac{\ell}{2}T^{yx}\ell^2$
$\tau_z^4 = y_4F^y_{4(self)} = \frac{\ell}{2}T^{xy}\ell^2$
total torque
$\implies \tau_z = \sum_{i=1}^4 \tau_z^i = \ell^2(T^{xy} - T^{yx}) = I\alpha$
$\implies \alpha = \frac{\tau_z}{I} since I depends on m and m \to 0 \tau_z or else \alpha \to \infty$

$\therefore \tau_z = 0$
$\implies T^{xy} = T^{yx}$

- -

(393) $\underline{T^{\alpha\beta}_{\ ,\beta} = 0}$
flow of energy: (4): $\ell^2 T^{0x}(x=0)$, $(2): -\ell^2 T^{0x}(x=\ell)$
flow in $y: \ell^2 T^{0y}(y=0) - \ell^2 T^{0y}(y=\ell)$
$\implies \frac{\partial}{\partial t}(T^{00}\ell^3) = \ell^2[T^{0x}(x=0) - T^{0x}(x=\ell) + T^{0y}(y=0)$
$- T^{0y}(y=\ell) + T^{0z}(z=0) - T^{0z}(z=\ell)$
$\implies \frac{\partial}{\partial t}T^{00} = -\frac{\partial}{\partial x}T^{0x} - \frac{\partial}{\partial y}T^{0y} - \frac{\partial}{\partial z}T^{0z}$
$\implies T^{00}_{\ ,0} + T^{0x}_{\ ,x} + T^{0y}_{\ ,y} + T^{0z}_{\ ,z} = T^{0\beta}_{\ ,\beta} = 0$
same analysis for momentum
$\therefore T^{\alpha\beta}_{\ ,\beta} = 0$

- -

(394) $\underline{N^{\alpha}_{\ ,\alpha} = (nU^{\alpha})_{,\alpha} = 0}$ (conservation of particles)
$\frac{\partial}{\partial t}N^0 = -\frac{\partial}{\partial x}N^x - \frac{\partial}{\partial y}N^y - \frac{\partial}{\partial z}N^z$

- -

perfect fluid - no heat conduction, no viscosity no heat conduction - $T^{0i} = T^{i0} = 0$

- -

No viscosity - $T^{ij} = 0$ unless $i = j \implies T^{ij}$ diagonal
$\implies T^{ij} = P\delta^{ij}$

- -

$$(T^{\alpha\beta}) = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix} (MCRF)$$

$\implies T^{\alpha\beta} = (\rho + P)U^{\alpha}U^{\beta} + P\eta^{\alpha\beta}$
$\therefore \tilde{T} = (\rho + P)\vec{U} \otimes \vec{U} + P\tilde{g}^{-1}$ (stress energy tensor for perfect fluid)

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(395) $\underline{\eta_{\alpha\gamma} = \eta_{\gamma\alpha}}$

- -

(396) $\underline{U^{\alpha}_{\ ,\beta}U_{\alpha} = 0}$
<u>Proof</u>
$\vec{U} \cdot \vec{U} = \eta_{\alpha\beta}U^{\alpha}U^{\beta} = U^{\alpha}U_{\alpha} = -1 \implies (U^{\alpha}U_{\alpha})_{,\beta} = 0$

$\implies (U^\alpha U^\gamma \eta_{\alpha\gamma})_{,\beta} = (U^\alpha U^\gamma)_{,\beta}\eta_{\alpha\gamma} = U^\alpha_{,\beta}U^\alpha\eta_{\alpha\gamma} + U^\alpha U^\gamma_{,\beta}\eta_{\alpha\gamma}$
$= 2U^\alpha_{,\beta}U^\gamma\eta_{\alpha\gamma} = 2U^\alpha_{,\beta}U_\alpha = 0$ Since: $U^\alpha_{,\beta}U^\gamma\eta_{\alpha\beta} = U^\alpha U^\gamma_{,\beta}\eta_{\alpha\gamma}$
$\therefore U^\alpha_{,\beta}U_\alpha = 0$

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(397) $\underline{U^\alpha S_{,\alpha} = \frac{dS}{d\tau} = 0}$
$T^{\alpha\beta}_{,\beta} = [(\rho+P)U^\alpha U^\beta + P\eta^{\alpha\beta}]_{,\beta} = 0$
assume $(nU^\beta)_{,\beta} = 0$, First term: $[(\rho+P)U^\alpha U^\beta]_{,\beta} = [\frac{\rho+P}{n}U^\alpha nU^\beta]_{,\beta}$
$= \frac{(\rho+P)}{n}U^\alpha[nU^\beta]_{,\beta} + nU^\beta[\frac{\rho+P}{n}U^\alpha]_{,\beta} = nU^\beta(\frac{\rho+P}{n}U^\alpha)_{,\beta}$
Second term $[P\eta^{\alpha\beta}]_{,\beta} = P_{,\beta}\eta^{\alpha\beta} + P\eta^{\alpha\beta}_{,\gamma} = P_{,\beta}\eta^{\alpha\beta}$
Use $U^\alpha_{,\beta}U_\alpha = 0$, $1^{\text{st}} + 2^{\text{nd}} = 0$
$\implies nU^\beta(\frac{\rho+P}{n}U^\alpha)_{,\beta} + P_{,\beta}\eta^{\alpha\beta} = 0 \implies nU^\beta U_\alpha(\frac{\rho+P}{n}U^\alpha_{,\beta}) + P_{,\beta}\eta^{\alpha\beta}U_\alpha = 0$
last term $p_{,\beta}U^\beta$, using $U^\alpha_{,\beta}U_\alpha = 0$, $U^\alpha U_\alpha = -1$
$\implies U^\beta[-n(\frac{\rho+P}{n})_{,\beta} + P_{,\beta}] = 0$
$\implies -U^\beta[\rho_{,\beta} - \frac{\rho+P}{n}n_{,\beta}] = 0$
$\implies \frac{d\rho}{d\tau} - \frac{\rho+P}{n}\frac{dn}{d\tau} = 0$, (boost into rest frame)
<u>note:</u> $\frac{d\rho}{d\tau} = U^\beta\rho_{,\beta}$
<u>recall:</u> $d\rho - (\rho+p)\frac{dn}{n} = nTdS$
$\frac{d\rho}{d\tau} - \frac{(\rho+p)}{n}\frac{dn}{d\tau} = nT\frac{dS}{d\tau} = 0$
$\therefore U^\alpha S_{,\alpha} = \frac{dS}{d\tau} = 0$

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from $T^{\alpha\beta}_{,\beta} = 0$ and $N^\alpha{}_{,\alpha} = (nU^\alpha)_\alpha = 0$
$\implies \int V^\alpha_{,\alpha}d^4x = \oint V^\alpha n_\alpha d^3S$ (gauss's law)

<u>Chapter 5</u> $\frac{E'}{E} = \frac{h\nu'}{h\nu} = \frac{m}{m+mgh+O(\vec{v}^4)} = 1 - gh + O(v^4)$

$r = (x^2+y^2)^{1/2}$, $x = r\cos\theta$
$\theta = \tan^{-1}(y/x)$,
$, y = r\sin\theta$
$\Delta r = \frac{\partial r}{\partial x}\Delta x + \frac{\partial r}{\partial y}\Delta y = \frac{x}{r}\Delta x + \frac{y}{r}\Delta y = \cos\theta\Delta x + \sin\theta\Delta y$
$\Delta\theta = \frac{\partial\theta}{\partial x}\Delta x + \frac{\partial\theta}{\partial y}\Delta y = -\frac{y}{r^2}\Delta x + \frac{x}{r^2}\Delta y = -\frac{1}{r}\sin\theta\Delta x + \frac{1}{r}\cos\theta\Delta y$

<u>in general</u>
$\xi = \xi(x,y)$, $\Delta\xi = \frac{\partial\xi}{\partial x}\Delta x + \frac{\partial\xi}{\partial y}\Delta y$
$\eta = \eta(x,y)$, $\Delta\eta = \frac{\partial\eta}{\partial x}\Delta x + \frac{\partial\eta}{\partial y}\Delta y$

$\implies \begin{pmatrix} \Delta\xi \\ \Delta\eta \end{pmatrix} = \begin{pmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \xi}{|}\partial y \\ \frac{\partial \eta}{\partial x} & \frac{\partial \eta}{\partial y} \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix}$

$\Delta\xi = \Delta\eta = 0 \implies \Delta x = \Delta y = 0$

$\implies det \begin{pmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \xi}{\partial y} \\ \frac{\partial \eta}{\partial x} & \frac{\partial \eta}{\partial y} \end{pmatrix} \neq 0 (Jacobian)$

$\begin{pmatrix} \Delta\xi \\ \Delta\eta \end{pmatrix} = \begin{pmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \xi}{\partial y} \\ \frac{\partial \eta}{\partial x} & \frac{\partial \eta}{\partial y} \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} \implies \Lambda^{\alpha'}_{\beta} = \begin{pmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \xi}{\partial y} \\ \frac{\partial \eta}{\partial x} & \frac{\partial \eta}{\partial y} \end{pmatrix}$

unprimed $\to (x, y)$; primed $\to (\xi, \eta)$

$V^{\alpha'} = \Lambda^{\alpha'}_{\beta} V^{\beta}$ (transformation of arbitrary $\vec{V}$)

$\tilde{d}\phi \to (\frac{\partial \phi}{\partial \xi}, \frac{\partial \phi}{\partial \eta})$

$\frac{\partial \phi}{\partial \xi} = \frac{\partial x}{\partial \xi}\frac{\partial \phi}{\partial x} + \frac{\partial y}{\partial \xi}\frac{\partial \phi}{\partial y}, \ \frac{\partial \phi}{\partial \eta} = \frac{\partial x}{\partial \eta}\frac{\partial \phi}{\partial x} + \frac{\partial y}{\partial \eta}\frac{\partial \phi}{\partial y}$

$\implies (\frac{\partial \phi}{\partial \xi}, \frac{\partial \phi}{\partial \eta}) = (\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y})(\begin{pmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{pmatrix}$

$\implies \Lambda^{\alpha}_{\beta'} = \begin{pmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{pmatrix}, \ (\tilde{d}\phi)_{\beta'} = \Lambda^{\alpha}_{\beta'} (\tilde{d}\phi)_{\alpha}$

$(\tilde{d}\phi)_\xi = \frac{\partial \phi}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial \phi}{\partial y}\frac{\partial y}{\partial \xi} = \frac{\partial \phi}{\partial \xi} = \frac{\partial \phi}{\partial x}(\tilde{d}x)_\xi + \frac{\partial \phi}{\partial y}(\tilde{y})_\xi$

$(\tilde{d}\phi)_\eta = \frac{\partial \phi}{\partial \eta} = \frac{\partial \phi}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial \phi}{\partial y}\frac{\partial y}{\partial \eta} = \frac{\partial \phi}{\partial x}(\tilde{d}x)_\beta + \frac{\partial \phi}{\partial y}(\tilde{d}y)_\eta$

$\implies \tilde{d}\phi = \frac{\partial \phi}{\partial x}\tilde{d}x + \frac{\partial \phi}{\partial y}\tilde{d}y$

$\Lambda^{\alpha}_{\beta'}, \ \Lambda$

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(398) $\underline{(\rho + p)a_i + p_{,i} = 0}$

$nU^{\beta}(\frac{\rho+p}{n}U^{\alpha})_{,\beta} + p_{,\beta}\eta^{\alpha\beta} = 0$ (previous derivation)

In MCRF $U^i = 0, \ U^i_{,\beta} \neq 0$

$\implies nU^{\beta}(\frac{\rho+p}{n}U^{i})_{,\beta} + p_{,\beta}\eta^{i\beta} = 0; \ U^i = 0$

$\implies (\rho + p)U^i_{,\beta}U^{\beta} + p_{,i} = 0; \ a_i \equiv U_{i,\beta}U^{\beta}$

$\therefore (\rho + p)a_i + p_{,i} = 0$

$\begin{pmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \xi}{\partial y} \\ \frac{\partial \eta}{\partial x} & \frac{\partial \eta}{\partial y} \end{pmatrix} \begin{pmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{pmatrix}$

$= \begin{pmatrix} \frac{\partial \xi}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial \xi}{\partial y}\frac{\partial y}{\partial \xi} & \frac{\partial \xi}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial \xi}{\partial y}\frac{\partial y}{\partial \eta} \\ \frac{\partial \eta}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial \eta}{\partial y}\frac{\partial y}{\partial \xi} & \frac{\partial \eta}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial \eta}{\partial y}\frac{\partial y}{\partial \eta} \end{pmatrix}$

$$\implies \begin{pmatrix} \frac{\partial\xi}{\partial\xi} & \frac{\partial\xi}{\partial\eta} \\ \frac{\partial\eta}{\partial\xi} & \frac{\partial\eta}{\partial\eta} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

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a path is what we would normally think is a curve but a curve is a parameterized path
Curve : $\{\xi = f(s),\ \eta = g(s),\ a \le s \le b\}$
can also change parameter, say $s' = s'(S)$
$\implies \{\xi = f'(s'), \eta = g'(s'), a' \le s' \le b'\}$
$\frac{d\phi}{ds} = \langle \tilde{d}\phi, \vec{V}\rangle, \vec{V}$ has components $(\frac{d\xi}{ds}, \frac{d\eta}{ds})$
Vector - the thing that produces $\frac{d\phi}{ds}$ given ϕ

$$\begin{pmatrix} \frac{d\xi}{ds} \\ \frac{d\eta}{ds} \end{pmatrix} = \begin{pmatrix} \frac{\partial\xi}{\partial x} & \frac{\partial\xi}{\partial y} \\ \frac{\partial\eta}{\partial x} & \frac{\partial\eta}{\partial y} \end{pmatrix} \begin{pmatrix} \frac{dx}{ds} \\ \frac{dy}{ds} \end{pmatrix}$$

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1. Chapter 8

(399) $\underline{g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta},\ |h_{\alpha\beta}| << 1}$

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(400) $\underline{h_{\bar\alpha\bar\beta} := \Lambda^{\mu}_{\bar\alpha}\Lambda^{\nu}_{\bar\beta}h_{\mu\nu}}$ (transforms like a tensor)
$g_{\bar\alpha\bar\beta} = \Lambda^{\mu}_{\bar\alpha}\Lambda^{\nu}_{\bar\beta}g_{\mu\nu} = \Lambda^{\mu}_{\bar\alpha}\Lambda^{\nu}_{\bar\beta}\eta_{\mu\nu} + \Lambda^{\mu}_{\bar\alpha}\Lambda^{\mu}_{\bar\beta}h_{\mu\nu}$
$= \eta_{\bar\alpha\bar\beta} + h_{\bar\alpha\bar\beta}$
$\therefore h_{\bar\alpha\bar\beta} = \Lambda^{\mu}_{\bar\alpha}\Lambda^{\nu}_{\bar\beta}h_{\mu\nu}$
<u>Note:</u> under the infinitesimal coordinate transformation below, $\eta_{\alpha\beta}$ is not invariant under the transformation, I always thought that it was invariant under a coordinate transformation, but it may have to do with the fact that $h_{\alpha\beta}$ is not tensor

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(401) $\underline{h_{\alpha\beta} \to h_{\alpha\beta} - \xi_{\alpha,\beta} - \xi_{\beta,\alpha}}$
$x^{\alpha'} = x^{\alpha} + \xi^{\alpha}(x^{\beta}),\ |\xi^{\alpha}_{,\beta}| << 1$
$\Lambda^{\alpha'}_{\beta} = \frac{\partial x^{\alpha'}}{\partial x^{\beta}} = \delta^{\alpha}_{\beta} + \xi^{\alpha}_{,\beta}$
$x^{\alpha'} = x^{\alpha} + \xi^{\alpha} \implies x^{\alpha} = x^{\alpha'} - \xi^{\alpha'}_{,\beta'}$
$\implies \Lambda^{\alpha}_{\beta'} = \frac{\partial x^{\alpha}}{\partial x^{\beta'}} = \delta^{\alpha'}_{\beta'} - \xi^{\alpha'}_{,\beta'}$
$g_{\alpha'\beta'} = \Lambda^{\alpha}_{\alpha'}\Lambda^{\beta}_{\beta'}g_{\alpha\beta} = (\delta^{\alpha}_{\alpha'} - \xi^{\alpha}_{,\alpha'})(\delta^{\beta}_{\beta'} - \xi^{\beta}_{,\beta'})(\eta_{\alpha\beta} + h_{\alpha\beta})$
$= (\delta^{\alpha}_{\alpha'}\delta^{\beta}_{\beta'} - \delta^{\alpha}_{\alpha'}\xi^{\beta}_{,\beta'} - \xi^{\alpha}_{,\alpha'}\delta^{\beta}_{\beta'} + \xi^{\alpha}_{,\alpha'}\xi^{\beta}_{,\beta'})(\eta_{\alpha\beta} + h_{\alpha\beta})$
$= \delta^{\alpha}_{\alpha'}\delta^{\beta}_{\beta'}\eta_{\alpha\beta} + \delta^{\alpha}_{\alpha'}\delta^{\beta}_{\beta'}h_{\alpha\beta} - \delta^{\alpha}_{\alpha'}\xi^{\beta}_{,\beta'}\eta_{\alpha\beta} - \xi^{\alpha}_{,\alpha'}\delta^{\beta}_{\beta'}\eta_{\alpha\beta}$
$= \eta_{\alpha'\beta'} + h_{\alpha'\beta'} - \xi^{\beta}_{,\beta'}\eta_{\alpha'\beta} - \xi^{\alpha}_{,\alpha'}\eta_{\alpha\beta'}$
$= \eta_{\alpha'\beta'} + h_{\alpha'\beta'} - \xi_{\alpha',\beta'} - \xi_{\beta',\alpha'}$

$\Longrightarrow\ h_{\alpha\beta} \to h_{\alpha\beta} - \xi_{\alpha,\beta} - \xi_{\beta,\alpha}$

Shortened: Consider $x^{\alpha'} = x^\alpha + \xi^\alpha(x^\beta)$ calculate $\Lambda^{\alpha'}_{\beta})$ and expand as Taylor series to first order, calculate $g_{\alpha'\beta'}$ to see what $h_{\alpha'\beta'}$ becomes.

$\underline{R_{\mu\nu} = \frac{1}{2}(h^\sigma_{\mu,\nu\sigma} + h^\sigma{}_{\nu,\mu\sigma} - h_{,\nu\mu} - \Box h_{\mu\nu})}$

recall: $R_{\mu\nu\rho\sigma} = \frac{1}{2}(h_{\mu\sigma,\nu\rho} + h_{\nu\rho,\mu\sigma} - h_{\mu\rho,\nu\sigma} - h_{\nu\sigma,\mu\rho})$

$R_{\mu\nu}$ is obtained by contracting μ and ρ

$R_{\mu\nu\rho\sigma} = \eta_{\lambda\mu} R^\lambda_{\nu\rho\sigma}$

$R^\lambda_{\nu\rho\sigma} = \frac{1}{2}(h^\lambda_{\sigma,\nu\rho} + h_{\nu\rho,}{}^\lambda{}_\sigma - h^\lambda{}_{\rho,\nu\sigma} - h_{\nu\sigma,}{}^\lambda{}_\rho)$

$\Longrightarrow\ R^\lambda{}_{\nu\lambda\sigma} = \frac{1}{2}(h^\lambda{}_{\sigma,\nu\lambda} + h_{\nu\lambda,}{}_\sigma - h^\lambda{}_{\lambda,\nu\sigma} - h_{\nu\sigma,}{}^\lambda{}_{\lambda Z})$

$h_{\nu\lambda,}{}^\lambda{}_\sigma = \eta_{\lambda\beta}\eta^{\lambda\alpha} h_\nu{}^\beta{}_{,\alpha\sigma} = h_\mu{}^\alpha{}_{,\alpha\sigma} = h^\alpha{}_{\nu,\alpha\sigma}$

$= \frac{1}{2}(h^\lambda{}_{\sigma,\nu\lambda} + h^\alpha_{\nu,\alpha\sigma} - h_{,\nu\sigma} - \Box h_{\nu\sigma})$

$\Box$

(402) $\underline{R_{\alpha\beta\mu\nu} = \frac{1}{2}(h_{\alpha\nu,\beta\mu} + h_{\beta\mu,\alpha\nu} - h_{\alpha\mu,\beta\nu} - h_{\beta\nu,\alpha\mu})}$

recall: $R_{\alpha\beta\mu\nu} = \frac{1}{2}(g_{\alpha\nu,\beta\mu} - g_{\alpha\mu,\beta\nu} + g_{\beta\mu,\alpha\nu} - g_{\beta\nu,\alpha\mu});\ g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$ (what allows us to use this?)

$\Longrightarrow\ R_{\alpha\beta\mu\nu} = \frac{1}{2}(h_{\alpha\nu,\beta\mu} - h_{\alpha\mu,\beta\nu} + h_{\beta\mu,\alpha\nu} - h_{\beta\nu,\alpha\mu})$

Definitions: $h^\mu_\beta := \eta^{\mu\alpha} h_{\alpha\beta};\ h^{\mu\nu} := \eta^{\nu\beta} h^\mu_{\beta;};\ h := h^\alpha_\alpha$

$\underline{\text{reverse trace with } \bar{h} := \bar{h}^\alpha_\alpha = -h};\ \bar{h}^{\alpha\beta} := h^{\alpha\beta} - \frac{1}{2}\eta^{\alpha\beta} h$

$\bar{h}^\mu_\beta = \eta^{\mu\alpha}\bar{h}_{\alpha\beta};\ \bar{h}_{\alpha\beta} = \eta_{\alpha\mu}\eta_{\beta\nu}\bar{h}^{\mu\nu} = \eta_{\alpha\mu}\eta_{\beta\nu}(h^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu} h)$

$= h_{\alpha\beta} - \frac{1}{2}\eta_{\alpha\mu}\delta^\mu_\beta h = h_{\alpha\beta} - \frac{1}{2}\eta_{\alpha\beta} h$

$\Longrightarrow\ \bar{h}^\mu_\beta = \eta^{\mu\alpha}\bar{h}_{\alpha\beta} = \eta^{\mu\alpha}(h_{\alpha\beta} - \frac{1}{2}\eta_{\alpha\beta} h) = h^\mu_\beta - \frac{1}{2}\delta^\mu_\beta h$

$\Longrightarrow\ \bar{h}^\mu_\mu = h - \frac{1}{2}(4)h = -h;\ \delta^\mu_\mu = 4$

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(403) $\underline{G_{\alpha\beta} = -\frac{1}{2}[\bar{h}_{\alpha\beta,\mu}{}^{,\mu} + \eta_{\alpha\beta}\bar{h}_{\mu\nu}{}^{,\mu\nu} - \bar{h}_{\alpha\mu,\beta}{}^{,\mu} - \bar{h}_{\beta\mu,\alpha}{}^{,\mu} + O(h_{\alpha\beta})^2]}$

Note: $f^{,\mu} := \eta^{\mu\nu} f_{,\nu}$

recall: $G_{\alpha\beta} = R^{\alpha\beta} - \frac{1}{2} g^{\alpha\beta} R;\ R = g^{\beta\nu} g^{\alpha\mu} R_{\alpha\beta\mu\nu}$

$R_{\alpha\beta} = R^\mu_{\alpha\mu\beta}$

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(404) $\underline{G^{\alpha\beta} = -\frac{1}{2}\Box\bar{h}^{\alpha\beta}}$ (Lorentz gauge), $\Box\bar{h}^{\mu\nu} = -16\pi T^{\mu\nu}$; $\bar{h}^{\mu\nu}_{,\nu} = 0$
recall: $G^{\alpha\beta} = -\frac{1}{2}[\bar{h}^{\alpha\beta}{}_{,\mu}{}^{,\mu} + \eta^{\alpha\beta}\bar{h}_{\mu\nu}{}^{,\mu\nu} - \bar{h}^{\alpha}{}_{\mu}{}^{,\beta\mu} - \bar{h}^{\beta}{}_{\mu}{}^{,\alpha\mu}]$
1st term: $\bar{h}^{\alpha\beta}{}_{,\mu}{}^{,\nu} = \eta^{\nu\sigma}\bar{h}^{\alpha\beta}{}_{,\mu\sigma}$
2nd term: $\eta^{\alpha\beta}\bar{h}_{\mu\nu}{}^{;\sigma\lambda} = \eta^{\alpha\beta}\eta^{\gamma\sigma}\eta^{\xi\lambda}\eta_{\mu\ell}\eta_{\nu\zeta}\bar{h}^{\ell\zeta}{}_{,\gamma\xi}$
$\sigma = \mu, \lambda = \nu \implies \eta^{\alpha\beta}\eta^{\gamma\mu}\eta^{\xi\nu}\eta_{\mu\ell}\eta_{\nu\eta}\bar{h}^{\ell\zeta}{}_{,\gamma\xi}$
$= \eta^{\alpha\beta}\delta^{\gamma}_{\ell}\delta^{\xi}_{\zeta}\bar{h}^{\ell\zeta}{}_{,\gamma\xi} = \eta^{\alpha\beta}\bar{h}^{\ell\zeta}{}_{,\ell\zeta} = 0$ (used $\bar{h}^{\mu\nu}_{,\nu} = 0$)
3rd term: $\bar{h}^{\alpha}{}_{\mu}{}^{,\beta\sigma} = \bar{h}^{\alpha}{}_{\mu}{}^{,\sigma\beta} = \eta_{\mu\lambda}\eta^{\sigma\gamma}\bar{h}^{\alpha\lambda}{}_{,\gamma}{}^{,b}$
$\sigma = \mu \implies \eta_{\mu\lambda}\eta^{\mu\gamma}\bar{h}^{\alpha\lambda}{}_{,\gamma}{}^{,\beta} = \delta^{\gamma}_{\lambda}\bar{h}^{\alpha\lambda}{}_{,\gamma}{}^{,\beta} = \bar{h}^{\alpha\lambda}{}_{,\lambda}{}^{,\beta} = 0$
4th term Same logic as third term $\bar{h}^{\beta}{}_{\mu}{}^{,\alpha\mu} = 0$
$\therefore G^{\alpha\beta} = -\frac{1}{2}\bar{h}^{\alpha\beta}{}_{,\mu}{}^{,\mu} = -\frac{1}{2}\Box\bar{h}^{\alpha\beta}$
recall: $G^{\alpha\beta} = 8\pi T^{\alpha\beta} \implies \Box\bar{h}^{\alpha\beta} = -16\pi T^{\alpha\beta}$

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(405) $\underline{\nabla^2\phi = 4\pi\rho;\ ds^2 = -(1+2\phi)dt^2 + (1-2\phi)(dx^2+dy^2+dz^2)}$
recall: $\Box\bar{h}^{\mu\nu} = -16\pi T^{\mu\nu}$
$|\phi| << 1 \implies |\vec{v}| << 1$ (gravitational field cannot produce near light speeds)
$|T^{00}| >> |T^{0i}| >> |T^{ij}|$
$\implies |\bar{h}^{00}| >> |\bar{h}^{0i}| >> |\bar{h}^{ij}|$
$\implies \Box\bar{h}^{00} = -16\pi T^{00} = -16\pi T^{00} = -16\pi\rho;\ T^{00} = \rho + O(\rho v^2)$
(dont understand)
$\frac{\partial^2}{\partial t^2} = \frac{\partial}{\partial t}(\frac{\partial x}{\partial t}\frac{\partial}{\partial x}) = \frac{\partial x}{\partial t}\frac{\partial}{\partial x}\frac{\partial x}{\partial t}\frac{\partial}{\partial x} = v^2\frac{\partial^2}{\partial x^2}; |v| << 1 \implies \Box = -\frac{\partial^2}{\partial t^2} + \nabla^2 = -v^2\nabla^2 + \nabla^2 \approx \nabla^2 \implies \nabla^2\bar{h}^{00} = -16\pi\rho \implies \nabla^2(-\frac{\bar{h}^{00}}{4}) = 4\pi\rho$
compare with $\nabla^2\phi = 4\pi\rho \implies \bar{h}^{00} = -4\phi$
all other $\bar{h}^{\alpha\beta}$ negligible
$\implies h = h^{\alpha}_{\alpha} = -\bar{h}^{\alpha}_{\alpha} = -\eta_{\alpha\nu}\bar{h}^{\alpha\nu} = \bar{h}^{00}$ (all other components negligible)
recall: $\bar{h}^{\alpha\beta} = h^{\alpha\beta} - \frac{1}{2}\eta^{\alpha\beta}h$
$\implies \bar{h}^{00} = h^{00} + \frac{1}{2}h = h^{00} + \frac{1}{2}\bar{h}^{00} = h^{00} - 2\phi$
$\implies h^{00} = -2\phi$
$\bar{h}^{ii} = h^{ii} - \frac{1}{2}(-4\phi); \implies |\bar{h}^{ij}|$ small $\implies |\bar{h}^{ij}| \approx 0$ even if i=j
$\implies h^{ii} = -2\phi$
$h^{ij} = 0$ since $\eta^{ij} = 0; j \neq i$
$ds^2 = g_{\alpha\beta}dx^{\alpha}dx^{\beta} = g_{00}dt^2 + g_{ii}(dx^2+dy^2+dz^2)$
$= (\eta_{00} + h_{00})dt^2 + (\eta_{ii} + h_{ii})(dx^2+dy^2+dz^2)$
$\therefore ds^2 = -(1+2\phi)dt^2 + (1-2\phi)(dx^2+dy^2+dz^2)$

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(406) $\underline{(\phi)_{relativisticfarfield} := -\frac{1}{4}(\bar{h}^{00})_{farfield}}$
recall: $\Box\bar{h}^{\mu\nu} = -16\pi T^{\mu\nu}$

assume $T^{\mu\nu}$ is independent of time
$\implies$ assume $h^{\mu\nu}$ independent of time
T^{00} is zero since $\rho = 0$ far from the source
$T^{0i} = 0$ and T^{ij} is zero since $T^{\mu\nu}$ is stationary.
$\implies \nabla^2 \bar{h}^{\mu\nu} = 0$ assuming no angular dependence
$\implies \nabla^2 \bar{h}^{\mu\nu} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 \frac{\partial \bar{h}^{\mu\nu}}{\partial r}) = 0$
$\implies \bar{h}^{\mu\nu} = \frac{A^{\mu\nu}}{r} + O(r^{-2})$ ($O(r^{-2})$ since we made a bunch of approximations)
$A^{\mu\nu}$ const
<u>recall:</u> $\bar{h}^{\mu\nu}_{,\nu} = 0$, (Lorentz gauge) $\bar{h}^{\mu 0}_{,0} = 0$ assumed $\bar{h}^{\mu\nu}$ is ind. of time
$\bar{h}^{\mu\nu}_{,\nu} = \bar{h}^{\mu j}_{,j} = A^{\mu j}\frac{\partial}{\partial x^j} r^{-1} + O(r^{-3})$
$= -A^{\mu j}\frac{\partial r}{\partial x^j} r^{-2}$; $\frac{\partial r}{\partial x^j} = \frac{\partial}{\partial x^j}\sqrt{\delta_{ik}x^i x^k} = \frac{x_j}{r}$
set $n_j = \frac{x_j}{r} \implies \bar{h}^{\mu j}_{,j} = -A^{\mu j}\frac{n_j}{r^2} + O(r^{-3}) = 0$
(Lorentz condition)(dont understand n_j)
$A^{\mu j} = 0$ since otherwise x^i would have to be fixed at zero
Since $\bar{h}^{\mu\nu} = \frac{A^{\mu\nu}}{r} + O(r^{-2})$
$|\bar{h}^{00}| >> |\bar{h}^{ij}|$, $|\bar{h}^{00}| >> |\bar{h}^{00}|$ (why not just >)
$\implies \nabla^2 \bar{h}^{00} = 0$
this justifies the identification
$(\phi)_{relativisticfarfield} := -\frac{1}{4}(\bar{h}^{00})_{farfield}$

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(407) $\underline{ds^2 = -[1 - 2\frac{M}{r} + O(r^{-2})]dt^2 + [1 + 2\frac{M}{r} + O(r^{-2})](dx^2 + dy^2 + dz^2)}$
$\nabla^2 \phi = 4\pi\rho \implies (\phi)_{Newtonianfarfield} = -\frac{M}{r} + O(r^{-2})$
<u>recall:</u> $(\phi)_{relativisticfarfield} = -\frac{1}{4}(\bar{h}^{00})_{farfield}$;
$\bar{h}^{00} = \frac{A^{00}}{r}$ set $A^{00} = 4M$
$\implies (\phi)_{relativisticfarfield} = -\frac{M}{r} + O(r^{-2})$
<u>recall:</u> $ds^2 = -(1 + 2\phi)dt^2 + (1 - 2\phi)(dx^2 + dy^2 + dz^2)$
$\therefore ds^2 = -[1 - \frac{2M}{r} + O(r^{-2})]dt^2 + [1 + \frac{2M}{r} + O(r^{-2})](dx^2 + dy^2 + dz^2)$

Chapter 9

⋆
(408) $\underline{k^\nu k_\nu = 0,\ \bar{h}^{\alpha\beta} = A^{\alpha\beta}\exp(ik_\nu x^\nu)}$ (Newtonian field equation, solution; i.e. solution to the wave equation)
<u>recall:</u> $(-\frac{\partial^2}{\partial t^2} + \nabla^2)\bar{h}^{\alpha\beta} = \eta^{\mu\nu}\bar{h}^{\alpha\beta}_{,\mu\nu} = 0 \implies \bar{h}^{\alpha\beta} = A^{\alpha\beta}\exp(ik_v x^v)$
$\implies \bar{h}^{\alpha\beta}_{,\mu} = ik_\nu \frac{\partial x^\nu}{\partial x^\mu}\bar{h}^{\alpha\beta} = ik_\nu \delta^\nu_\mu \bar{h}^{\alpha\beta} = ik_\mu \bar{h}^{\alpha\beta}$
$\implies \eta^{\mu\nu}\bar{h}^{\alpha\beta}_{,\mu\nu} = -\eta^{\mu\nu}k_\mu k_\nu \bar{h}^{\alpha\beta} = 0$

$\therefore \eta^{\mu\nu}k_\mu k_\nu = k^\nu k_\nu = 0$

<u>Note:</u> If $k_\alpha x^\alpha = const. \implies \bar{h}^{\alpha\beta}$ is const on hyper-surface (gravitational wave)

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(409) $\underline{k_\mu x^\mu(\lambda) = k_\mu \ell^\mu = const.}$ (photon)
a photon travels in direction of null vector
$\vec{k} \implies x^\mu(\lambda) = k^\mu\lambda + \ell^\mu$, since $k^\nu k_\nu = 0$
$\implies k_\mu x^\mu(\lambda) = k_\mu k^\mu \lambda + \ell^\mu k_\mu = k_\mu \ell^\mu = const.$
this is a wave whose phase is the same as the gravitational wave
$\implies$ photon travels with gravitational wave.

<u>Note:</u> $\vec{k} \implies (\omega, \vec{k}), k_\alpha x^\alpha = k_0 x^0 + \vec{k}\cdot\vec{x} = \eta_{0\mu}k^\mu x^0 + \vec{k}\cdot\vec{x}$
$= -\omega t + \vec{k}\cdot\vec{x}$

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(410) $\underline{\omega^2 = |\vec{k}|^2}$ (dispersion)
$k_\alpha k^\alpha = k_0 k^0 + |\vec{k}|^2 = 0 = -\omega^2 + |\vec{k}|^2$
$\implies \omega^2 = |\vec{k}|^2$

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(411) $\underline{A^{\alpha\beta}k_\beta = 0}$ (apply gauge condition)
<u>recall:</u> $(-\frac{\partial^2}{\partial t^2} + \nabla^2)\bar{h}^{\alpha\beta} = 0$
Einstein's field equations only have this form if the gauge condition is imposed
$\bar{h}^{\alpha\beta}_{,\beta} = 0 \implies \bar{h}^{\alpha\beta} = A^{\alpha\beta}\exp(ik_\lambda x^\lambda)$
$\frac{\partial \bar{h}^{\alpha\beta}}{\partial x^\beta} = ik_\lambda \delta^\lambda_\beta A^{\alpha\beta}\exp(ik_\lambda x^\lambda)$
$= ik_\beta A^{\alpha\beta}\exp(ik_\lambda x^\lambda) = 0$
$\implies k_\beta A^{\alpha\beta} = 0$

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(412) $\underline{A^{(new)}_{\alpha\beta} = A^{(old)}_{\alpha\beta} - iB_\alpha k_\beta - iB_\beta k_\alpha + i\eta_{\alpha\beta}B^\mu k_\mu}$
<u>recall:</u> $(-\frac{\partial^2}{\partial t^2} + \nabla^2)\bar{h}^{\alpha\beta} = 0$
Einstein's field equations only have this form if the gauge condition is imposed
$\bar{h}^{\alpha\beta}_{,\beta} = 0$

$\Box\xi^{\mu} = \bar{h}^{(old)\mu\nu}_{,\nu}$; $\bar{h}^{(old)\mu\nu}_{,\nu} = (A^{\mu\nu}\exp(ik_{\lambda}x^{\lambda}))_{,\nu} = 0$
$\implies (-\frac{\partial^2}{\partial t^2} + \nabla^2)\xi_{\alpha} = 0 \implies \xi_{\alpha} = B_{\alpha}\exp(ik_{\mu}x^{\mu})$
$\underline{recall\ :}h^{(new)}_{\alpha\beta} = h^{(old)}_{\alpha\beta} - \xi_{\alpha,\beta} - \xi_{\beta,\alpha}$
$\implies \bar{h}^{(new)}_{\alpha\beta} = \bar{h}^{(old)}_{\alpha\beta} - \xi_{\alpha,\beta} - \xi_{\beta,\alpha} + \eta_{\alpha\beta}\xi^{\mu}_{,\mu}$
$A^{(new)}_{\alpha\beta} = A^{(old)}_{\alpha\beta} - iB_{\alpha}k_{\beta} - iB_{\beta}k_{\alpha} + i\eta_{\alpha\beta}B^{\mu}k_{\mu}$

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(413) $\underline{\Gamma^{\alpha}_{00} = \frac{1}{2}\eta^{\alpha\beta}(h_{\beta 0,0} + h_{\beta 0,0} - h_{00,\beta})}$
<u>recall:</u> $\Gamma^{\alpha}_{\mu\nu} = \frac{1}{2}g^{\alpha\beta}(g_{\beta\mu,\nu} + g_{\beta\nu,\mu} - g_{\mu\nu,\beta})$;
$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$
$\implies \Gamma^{\alpha}_{\mu\nu} = \frac{1}{2}(\eta^{\alpha\beta} + h^{\alpha\beta})(\eta_{\beta\mu,\nu} + \eta_{\beta\nu,\mu} - \eta_{\mu\nu,\beta})$
$+ \frac{1}{2}(\eta^{\alpha\beta} + h^{\alpha\beta})(h_{\beta\mu,\nu} + h_{\beta\nu,\mu} - h_{\mu\nu,\beta})$
$= \frac{1}{2}\eta^{\alpha\beta}(\eta_{\beta\mu,\nu} + \eta_{\beta\nu,\mu} - \eta_{\mu\nu,\beta}) + \frac{1}{2}h^{\alpha\beta}(\eta_{\beta\mu,\nu} + \eta_{\beta\nu,\mu} - \eta_{\mu\nu,\beta}) + \frac{1}{2}\eta^{\alpha\beta}(h_{\beta\mu,\nu} + h_{\beta\nu,\mu} - h_{\mu\nu,\beta})j + O(|h_{\alpha\beta}|^2)$
<u>Note:</u> $\eta_{\beta\mu,\nu} = 0$; $h^{00} = 0$
$\implies \Gamma^{\alpha}_{00} = \frac{1}{2}\eta^{\alpha\beta}(h_{\beta 0,0} + h_{\beta 0,0} - h_{00,\beta})$

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(414) $\underline{(\frac{dU^{\alpha}}{d\tau}) = 0}$ (TT gauge) particle initially in wave free region and at rest, equations of motion for free particle
<u>recall:</u> $U^{\beta}V^{\alpha}_{;\beta} \implies \frac{d}{d\lambda}(\frac{dx^{\alpha}}{d\lambda}) + \Gamma^{\alpha}_{\mu\beta}\frac{dx^{\mu}}{d\lambda}\frac{dx^{\beta}}{d\lambda} = 0$
(geodesic equaiton)
$\implies \frac{d}{d\tau}U^{\alpha} + \Gamma^{\alpha}_{\mu\nu}U^{\mu}U^{\nu} = 0$
initially at rest $\implies \vec{U} = (U^0, \vec{0})$ but $\vec{U}\cdot\vec{U} = -1 \implies U^0 = 1$
$\implies$ initial acceeleration of particle
$\implies (\frac{dU^{\alpha}}{d\tau})_0 = -\Gamma^{\alpha}_{\mu\nu}U^{\mu}U^{\nu} = -\Gamma^{\alpha}_{00}$
<u>recall:</u> $\Gamma^{\alpha}_{00} = \frac{1}{2}\eta^{\alpha\beta}(h_{\beta 0,0} + h_{\beta 0,0} - h_{00,\beta})$
<u>Note:</u> h is in TT so $h_{\alpha\beta} = h^{TT}_{\alpha\beta}$ here and $h_{\beta 0} = h_{00} = 0$
$\implies \Gamma^{\alpha}_{00} = 0$ (recall the matrix equation$A^{TT}_{\alpha\beta}$ on pg. 206)
$\therefore (\frac{d\vec{U}}{d\tau})_0 = 0$ (initially)
Since initial acceleration is 0, then the particle will still be at rest a moment later
$(\frac{dU^{\alpha}}{d\tau})_0 = 0 \implies U^{\alpha}$(moment later) $= 0$
$\implies (\frac{dU^{\alpha}}{d\tau})_{momentlater} = 0$
$\implies$ TTgauge is a coordinate system that is " attached" to particles, giving the illusion that the particle does not move (i.e. coordinate distance does not change)
particle

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(415) $\underline{\Delta\ell \approx [1 + \frac{1}{2}h^{TT}_{xx}(x=0)]\epsilon}$
$1 \sim (x_0, y_0, z_0) = (0,0,0)$; particle $2 \sim (\epsilon, 0, 0)$
$\Delta\ell \equiv \int |ds|^{1/2} = \int |g_{\alpha\beta}dx^\alpha dx^\beta|^{1/2} = \int_0^\epsilon |g_{xx}|^{1/2}dx$
$\approx |g_{xx}(x=0)|^{1/2}\Delta x = |g_{xx}(x=0)|^{1/2}\epsilon$
recall: $g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta} \implies g_{xx} = 1 + h^{TT}_{xx}(x=0)$
$(1+y)^n \approx \sum_{k=0}^n \begin{pmatrix} n \\ k \end{pmatrix} y^{n-k}$
$(1+y)^n \approx \sum_{k=0}^\infty \begin{pmatrix} n \\ k \end{pmatrix} y^{k}$
Note: do not use $\sum_{k=0}^\infty \begin{pmatrix} n \\ k \end{pmatrix} y^{n-k}$
$(1+y)^n \approx \begin{pmatrix} n \\ 0 \end{pmatrix} y^0 + \begin{pmatrix} n \\ 1 \end{pmatrix} y = 1 + \frac{n!}{(n-1)!}y = 1 + \frac{1}{2}y$
$\implies |g_{xx}(x=0)|^{1/2}\epsilon \approx [1 + \frac{1}{2}h^{TT}_{xx}(x=0)]\epsilon$
$\therefore$ proper distance changes with time even though in the TT gauge it is standing still.

Consider two freely falling particles with connecting vector ξ^α

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(416) $\underline{\frac{\partial^2}{\partial t^2}\xi^x = \frac{1}{2}\epsilon\frac{\partial^2}{\partial t^2}h^{TT}_{xx},\ \frac{\partial^2}{\partial t^2}\xi^y = \frac{1}{2}\epsilon\frac{\partial^2}{\partial t^2}h^{TT}_{xy}}$ (initially separated by ϵ in x direction);

(417) $\underline{\frac{\partial^2}{\partial t^2}\xi^y = \frac{1}{2}\epsilon\frac{\partial^2}{\partial t^2}h^{TT}_{yy} = -\frac{1}{2}\epsilon\frac{\partial^2}{\partial t^2}h^{TT}_{xx}, \frac{\partial^2}{\partial t^2}\xi^x = \frac{1}{2}\epsilon\frac{\partial^2}{\partial^2}h^{TT}_{xy}}$
coord dist = proper dist
recall: $\nabla_V\nabla_V\xi^\alpha = R^\alpha_{\mu\nu\beta}V^\mu V^\nu\xi^\beta$
$\implies \nabla_U\nabla_U\xi^\alpha = R^\alpha_{\mu\nu\beta}U^\mu U^\nu\xi^\beta$
$\vec{U} \to (1,0,0,0)$ and $\vec{\xi} \to (0,\epsilon,0,0)$ initially
$\frac{d^2}{d\tau^2}\xi^\alpha = \frac{d}{d\tau}\frac{dt}{d\tau}\frac{\partial}{\partial t}\xi^\alpha = \gamma^2\frac{\partial^2}{\partial t^2}\xi^\alpha = \frac{\partial^2}{\partial t^2}\xi^\alpha$
$= R^\alpha_{00\beta}U^0U^0\xi^\beta = R^\alpha_{001}U^0U^0\xi^1 = R^\alpha_{001}\epsilon = -R^\alpha_{0x0}\epsilon$
recall: $R^\alpha_{\beta\mu\nu} = -R^\alpha_{\beta\nu\mu}$
recall: $R_{\alpha\beta\mu\nu} = \frac{1}{2}(h_{\alpha\nu,\beta\mu} + h_{\beta\mu,\alpha\nu} - h_{\alpha\mu,\beta\nu} - h_{\beta\nu,\alpha\mu})$
$\implies R^x_{0x0} = \eta^{x\mu}R_{\mu 0x0} = R_{x0x0}$
$= \frac{1}{2}(h^{TT}_{x0,0} + h^{TT}_{0x,x0} - h^{TT}_{xx,00} - h^{TT}_{00,xx})$
recall: $A^{TT}_{\alpha\beta} = 0$ unless $\alpha = x, yand\beta = x, y$
$\implies R_{x0x0} = -\frac{1}{2}h^{TT}_{xx,00}$
$R^y_{0x0} = \eta^{\mu y}R_{\mu 0x0} = R_{y0x0}$
$= \frac{1}{2}(h^{TT}_{y0,0x} + h^{TT}_{0x,y0} - h^{TT}_{yx,00} - h^{TT}_{00,yx})$

$= -\frac{1}{2}h^{TT}_{yx,00} = -\frac{1}{2}h^{TT}_{xy,00}$
$R^y_{0y0} = R_{y0y0} = \frac{1}{2}(h_{y0,0y} + h_{0y,yo} - h_{yy,00} - h_{00,yy})$
$= -\frac{1}{2}h^{T}_{yy,00}T = -R^x_{0x0}$
$\frac{d^2\epsilon^\alpha}{d\tau^2} = -\epsilon R^\alpha_{0x0}$
$\implies \frac{d^2\epsilon^0}{d\tau^2} = -\epsilon R^0_{0x0} = 0;\ \frac{d^2\epsilon^x}{d\tau^2} = -\epsilon R^x_{0x0} = \frac{\epsilon}{2}\frac{\partial^2}{\partial t^2}h^{TT}_{xx}$
$\frac{d^2\epsilon^y}{d\tau^2} = -\epsilon R^y_{0x0} = \frac{\epsilon}{2}\frac{\partial^2}{\partial t^2}h^{TT}_{xy}$

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Same analysis can be performed for particles initially separated in the y direction
$\implies \frac{\partial^2}{\partial t^2}\xi^y = \frac{1}{2}\epsilon\frac{\partial^2}{\partial t^2}h^{TT}_{yy} = -\frac{1}{2}\epsilon\frac{\partial^2}{\partial t^2}h^{TT}_{xx}$
$\frac{\partial^2}{\partial t^2}\xi^x = \frac{1}{2}\epsilon\frac{\partial^2}{\partial t^2}h^{TT}_{xy}$

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(418) $\underline{\frac{\partial^2}{\partial t^2}\xi^i = -R^i_{0j0}\xi^j + \frac{1}{m_B}F^i_B - \frac{1}{m_A}F^i_A}$
recall: $\frac{d^2}{d\tau^2}\xi^\alpha = R^\alpha_{\mu\nu\beta}U^\mu U^\nu \xi^\beta = \gamma^2\frac{\partial^2}{\partial t^2}\xi^\alpha = \frac{\partial^2}{\partial t^2}\xi^\alpha$
(ξ^α is the separation vector between A and B)
$\implies \frac{\partial^2}{\partial t^2}\xi^i = R^i_{00\beta}\xi^\beta = -R^i_{0\beta 0}\xi^\beta - R^i_{0j0}\xi^j$
(i.e., they are not separated in time)
particle B experiences force
$\implies \frac{\partial^2}{\partial t^2}\xi^i = -R^i_{0j0}\xi^j + \frac{1}{m_B}F^i_B$
Note: the second term affects separation which is why it is added.
A experiences a force
$\implies \frac{\partial^2}{\partial t^2}\xi^i = -R^i_{0j}\xi^j + \frac{1}{m_B}F^i_B - \frac{1}{m_A}F^i_A$
(don't understand the negative)
A note on the negative, lets say the force on A and the force on B are in the same direction and parallel to ξ and ξ points from A to B, then the force on A will cause $\dot{\xi}$ to decrease while the force on B would cause $\dot{\xi}$ to increase, so they should have the opposite sign.

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Skipped: 9.32 - 9.63

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<u>recall:</u> $(-\frac{\partial^2}{\partial t^2}+\nabla^2)\bar{h}_{\mu\nu} = -16T_{\mu\nu}$
Assume $T_{\mu\nu}$ is sinusoidal in time
$\implies T_{\mu\nu} = S_{\mu\nu}(x^i)e^{-i\omega t}$ (real part)
$\implies \bar{h}_{\mu\nu} = B_{\mu\nu}(x^i)e^{-i\Omega t}$
$\implies \Box\bar{h}_{\mu\nu} = \Omega^2\bar{h}_{\mu\nu} + e^{-i\Omega t}\nabla^2 B_{\mu\nu}(x^i) = -16S_{\mu\nu}(x^i)e^{-i\Omega t}$
$\implies (\nabla^2+\Omega^2)B_{\mu\nu} = -16\pi S_{\mu\nu}$
outside source $S_{\mu\nu}(x^i) = 0$
<u>recall:</u> $\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial}{\partial r})$
$\implies (\nabla^2+\Omega^2)B_{\mu\nu} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial}{\partial r}B_{\mu\nu}) + \Omega^2 B_{\mu\nu}$
$= \frac{1}{r^2}(2r\frac{\partial}{\partial r}B_{\mu\nu}) + \frac{1}{r^2}r^2\frac{\partial^2}{\partial r^2}B_{\mu\nu} + \Omega^2 B_{\mu\nu} = \frac{2}{r}B'_{\mu\nu}(r) + B''_{\mu\nu} + \Omega B_{\mu\nu} =$ 0 (outside source)
$\implies rB''_{\mu\nu} + 2B'_{\mu\nu} + r\Omega B_{\mu\nu} = 0$
$\implies B_{\mu\nu} = \frac{A_{\mu\nu}}{r}e^{i\Omega r} + \frac{Z_{\mu\nu}}{r}e^{-i\Omega r}$
$A_{\mu\nu},\ Z_{\mu\nu} \sim const,\ e^{-i\Omega r}$ (ingoing), $e^{i\Omega r}$ (outgoing)
we want to consider emitted waves (outgoing) $\implies Z_{\mu\nu} = 0$
source nonzero inside $R = \epsilon << \frac{2\pi}{\Omega}$ Lets integrate over this sphere
$(\nabla^2+\Omega^2)B_{\mu\nu} = -16\pi S_{\mu\nu}$
$\implies \int\nabla^2 B_{\mu\nu}d^3x + \Omega^2\int B_{\mu\nu}d^3x = -16\pi\int S_{\mu\nu}d^3x$
$|\int B_{\mu\nu}d^3x| \le \int|B_{\mu\nu}|d^3x \le |B_{\mu\nu}|_{max}V = |B_{\mu\nu}|_{max}\frac{4\pi}{3}\epsilon^3$
$\int\nabla^2 B_{\mu\nu}d^3x = \int\nabla\cdot\nabla B_{\mu\nu}d^3x = \int\nabla B_{\mu\nu}\cdot\hat{n}dS$
<u>recall:</u> $\nabla B_{\mu\nu} = \frac{\partial}{\partial r}B_{\mu\nu}\hat{r} + \frac{1}{r}\frac{\partial}{\partial\theta}B_{\mu\nu}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}B_{\mu\nu}\hat{\phi} = \frac{\partial}{\partial r}B_{\mu\nu}\hat{r}$
$\implies \int\nabla B_{\mu\nu}\cdot\hat{r}dS \approx (\frac{\partial B_{\mu\nu}}{\partial r})_{r=\epsilon}\int dS = 4\pi\epsilon^2(\frac{\partial B_{\mu\nu}}{\partial r})_{r=\epsilon}$
$(\frac{\partial B_{\mu\nu}}{\partial r})_{r=\epsilon} = (-\frac{A_{\mu\nu}}{r^2}e^{i\Omega r} + \frac{A_{\mu\nu}}{r}i\Omega e^{i\Omega r})_{r=\epsilon}$
$= -\frac{A_{\mu\nu}}{\epsilon^3 2}e^{i\Omega\epsilon} + \frac{A_{\mu\nu}}{\epsilon}i\Omega e^{i\Omega\epsilon}$
$\implies \int\nabla B_{\mu nu}\cdot\hat{n}dS = -4\pi A_{\mu\nu}e^{i\Omega\epsilon} + 4\pi\epsilon i\Omega A_{\mu\nu}e^{i\Omega\epsilon} \approx -4\pi A_{\mu\nu}e^{i\omega\epsilon} \approx$ $-4\pi A_{\mu\nu}$ (Since $\epsilon << \frac{2\pi}{\Omega}$
$J_{\mu\nu} \equiv \int S_{\mu\nu}d^3x$
$\implies -4\pi A_{\mu\nu} = -16\pi J_{\mu\nu} \implies A_{\mu\nu} = 4J_{\mu\nu}$
$\therefore \bar{h}_{\mu\nu} = B_{\mu\nu}e^{-i\Omega t} \approx \frac{A_{\mu\nu}}{r}e^{i\Omega r} = 4\frac{J_{\mu\nu}}{r}e^{i\Omega(r-t)}$
$J_{\mu\nu} = \int S_{\mu\nu}d^3x;\ e^{i\Omega t}J_{\mu\nu} = S_{\mu\nu}$
$\implies e^{-i\Omega t}J_{\mu\nu} = \int T_{\mu\nu}d^3x$
$\implies J^{\mu\nu}e^{-i\Omega t} = \int T^{\mu\nu}d^3x$
<u>Note:</u> $J^{\mu\nu}_{,0} = \int S^{\mu\nu}(x^i)_{,0}d^3x = 0$
$w/\nu = 0$
$\implies -i\Omega J^{\mu 0}e^{-i\Omega t} = \int T^{\mu 0}_{,0}d^3x$
<u>recall :</u> $T^{\mu\nu}_{,\nu} = 0 \implies T^{\mu 0}_{,00} = -T^{|muk}_{,k}$
$\implies i\Omega J^{\mu 0}e^{-i\Omega t} = \int T^{\mu k}_{,k}d^3x$
$= \oint T^{\mu k}\cdot d\vec{S} = \oint T^{\mu k}n_k dS$

Let S extend outside the source $\implies T^{\mu\nu} = 0$ there
$\Omega \neq 0 \implies i\Omega J^{\mu 0} e^{-i\Omega t} = 0 \implies J^{\mu 0} = 0$
recall: $\bar{h}^{\mu\nu} = 4J^{\mu\nu} e^{i\Omega(r-t)}/r$
$\implies \bar{h}^{\mu 0} = 0$
recall: $\frac{\partial^2}{\partial t^2} \int T^{00} x^\ell x^m d^3x = 2 \int T^{\ell m} d^3x$ (4.10 ex. 23)
recall: $T^{00} \approx \rho$ (chapter 7)
$\implies I^{\ell m} := \int T^{00} x^\ell x^m d^3x = \int S^{00}(x^i) e^{-i\Omega t} x^\ell x^m d^3x$
$= \int S^{00}(x^i) x^\ell x^m d^3x e^{-i\Omega t} = D^{\ell m} e^{-i\Omega t}$
recall :$\bar{h}_{jk} = 4J_{jk} e^{i\Omega(r-t)}/r; J_{jk} = e^{i\Omega t} \int T_{jk} d^3x; \frac{d^2}{dt^2} \int T^{00} x^\ell x^m d^3x =$
$2 \int T^{\ell m} d^3x$
$\implies \bar{h}_{jk} = 4(e^{i\Omega t} \int T_{jk} d^3x) e^{i\Omega(r-t)}/r$
$= 4(e^{i\Omega t} \frac{1}{2} \frac{d^2}{dt^2} \int T^{--} x_j x_k d^3x) e^{i\Omega(r-t)}/r$
$= 2e^{i\Omega t} \frac{d^2}{dt^2} (D_{jk} e^{-i\Omega t}) e^{i\Omega(r-t)}/r$
$= 2e^{i\Omega rt} D_{jk} (-i\Omega)^2 e^{-i\Omega t}/r$
$= -2\Omega^2 D_{jk} e^{i\Omega(r-t)}/r$
Quadrupole approximation

Chapter 10

(419) $\underline{ds^2 = g_{00} dt^2 + 2g_{0r} dt dr + g_{rr} dr^2 + r^2 d\Omega^2}$
general metric for spherical symmetry
minkowski metric in spherical coordinates:
$ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$
a surface of constant t and r is given by a sphere
$ds^2 = r^2(d\theta^2 + \sin^2\theta d\phi^2) = r^2 d\Omega^2$
in general, spherical symmetry
$\implies d\ell^2 = f(r', t)(d\theta^2 + \sin^2\theta d\phi^2)$
a line with const t, θ, ϕ is orthogonal to the concentric two spheres centered at 0 with radius r and $r + dr \implies g_{r\theta} = g_{r\phi} = 0$
since $\vec{e}_r \cdot \vec{e}_\theta = 0$
$\implies ds^2 = g_{\mu\nu} dx^\mu dx^\nu = g_{00} dt^2 + 2g_{0r} dr dt + 2g_{0\theta} d\theta dt + 2g_{0\phi} d$
$phidt + g_{rr} dr^2 + r^2 d\Omega^2$
If we consider line of const r, θ, ϕ then $g_{0\theta} = g_{0\phi} = 0$
$\implies ds^2 = g_{00} dt^2 + 2g_{0r} dr dt + g_{rr} dr^2 + r^2 d\Omega^2$
spherically symmetric spacetime is not necessarily static so $g_{0r} \neq 0$ since space could be expanding

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<u>static</u>
(i) metric components independent of time
(ii) geometry unchanged by $t \to -t$
these are not equivalent conditions, consider for example a rotating star
(ii) means it looks the same when played backwards
if (i) is satisfied but not (ii)
$\implies$ stationary

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$$\underline{ds^2 = -e^{2\Phi}dt^2 + e^{2\Lambda}dr^2 + r^2 d\Omega^2} \tag{420}$$

(ii) $\implies (t, r, \theta, \phi) \to (-t, r, \theta, \phi)$
$\implies \Lambda^{\bar{0}}_{0} = -1,\ \Lambda^{i}_{j} = \delta^{i}_{j}$
$\implies g_{\bar{0}\bar{0}} = (\Lambda^{0}_{\bar{0}})^2 g_{00} = g_{00}$
$g_{\bar{0}\bar{r}} = \Lambda^{0}_{\bar{0}}\Lambda^{r}_{\bar{r}} g_{0r} = -g_{0rr}$
$g_{\bar{r}\bar{r}} = (\Lambda^{r}_{\bar{r}})^2 g_{rr} = g_{rr}$
geometry unchanged $\implies g_{\bar{\alpha}\bar{\beta}} = g_{\alpha\beta} but g_{\bar{0}\bar{r}} \neq g_{0r}$
$\implies g_{0r} = g_{r0} = 0$
<u>recall:</u> $ds^2 = g_{00}dt^2 + 2g_{0r}drdt + g_{rr}dr^2 + r^2 d\Omega^2$
$\therefore ds^2 = -e^{2\Phi}dt^2 + e^{2\Lambda}dr^2 + r^2 d\Omega^2$
this metric works inside stars but not inside black holes.

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Chapter 11

need to construct metric for cosmological model. Must be homogeneous (independent of space) and isotropic (same in every direction) It can expand, no random motion, t is proper time for each galaxy. At time $t = t_0$
$\implies d\ell^2(t_0) = h_{ij}(t_0)dx^i dx^j$
At time $t_1 \implies d\ell^2(t_1) = f(t_1, t_0)h_{ij}(t_0)dx^i dx^j = h_{ij}(t_1)dx^i dx^j$
this guarantees all $h'_{ij}s$ increase at same rate (isotropy)
$\implies d\ell^2(t) = R^2(t)h_{ij}dx^i dx^j,\ R(t_0) = 1 h_{ij}(const$
$R \sim$ scale factor sometimes denoted by a
$\implies ds^2 = -dt^2 + 2g_{0i}dtdx^i + R^2(t)h_{ij}dx^i dx^j$
$g_{00} = -1$ because t is proper time or $dx^i = 0$
def of simultaneity must agree with lorentz frame attached to galaxy $\implies g_{0i} = 0$
$\implies ds^2 = -dt^2 + R^2(t)h_{ij}dx^i dx^j$
isotropic $\implies$ spherical symmetry

$\implies d\ell^2 = e^{2\Lambda(r)}dr^2 + r^2 d\Omega^2$
$\implies$ only isotropy about one point

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This metric implies isotropy about one point. We want it to be homogeneous. A condition that satisfies this is the Ricci scalar must have same value at every point in space (R^i_i)
$G_{rr} = -\frac{1}{r^2}e^{2\Lambda}(1-e^{-2\Lambda});\ G_{\theta\theta} = -re^{-2\Lambda}\Lambda;\ G_{\phi\phi} = \sin^2\theta G_{\theta\theta}$

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R^i_i at every point $\implies G$ has same value at every point
is $g^{\alpha\beta}$ inverse of $g_{\alpha\beta}$?
$G = G_{ij}g^{ij} = G_{rr}g^{rr} + G_{\phi\phi}g^{\phi\phi} + G_{\theta\theta}g^{\theta\theta}$
$= -\frac{1}{r^2}e^{2\Lambda}(1-e^{-2\Lambda})e^{-2\Lambda} + \sin^2\theta(-re^{-2\Lambda}\Lambda')\frac{r^{0-2}}{\sin^2\theta} + (-re^{-2\Lambda}\Lambda')r^{-2}$
$= -\frac{1}{r^2}(1-e^{-2\Lambda}) - \frac{e^{-2\Lambda}\Lambda'}{r}0 - \frac{e^{-2\Lambda}\Lambda'}{r}$
$= -\frac{1}{r^2} + \frac{e^{-2\Lambda}}{r^2} - \frac{2e^{-2\Lambda}\Lambda'}{r}$
$= -\frac{1}{r^2}[1-(re^{-2\Lambda})'] = \kappa =$ constant
$\implies \kappa r^2 + 1 = (re^{-2\Lambda})'$
$\implies \int(1+\kappa r^2)dr = re^{-2\Lambda} + A$
$\implies r + \frac{1}{3}\kappa r^3 = re^{-2\Lambda} + A$
$\implies 1 + \frac{1}{3}\kappa r^2 - \frac{a}{r} = e^{-2\Lambda}$
$\implies e^{2\Lambda} = \frac{1}{1+\frac{1}{3}\kappa r^2 - \frac{A}{r}} = g_{rr}$
demand local flatness $\implies g_{rr}(r=0) = 1 \implies \frac{1}{3}\kappa r^2 - \frac{A}{r} = 0$
$\implies A = 0,$ define $k = -\frac{\kappa}{3}$
$\implies g_{rr} = \frac{1}{1-kr^2}$
$\implies d\ell^2 = \frac{dr^2}{1-kr^2} + r^2 d\Omega^2 \implies$ curvature scalar homogeneous

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$\implies ds^2 = -dt^2 + R^2(t)[\frac{dr^2}{1-kr^2} + r^2 d\Omega^2]$
is homogeneous and isotropic for any value k

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Notice that if $k = -3 then define \tilde{r} = \sqrt{3} r and \tilde{R} = \frac{1}{\sqrt{3}}R$
$\implies ds^2 = -dt^2 + \tilde{R}^2(t)[\frac{d\tilde{r}^2}{1-\tilde{r}^2} + \tilde{r}^2 d\Omega^2]$
$\therefore$ only need to consider 3 values $k = (-1, 0, 1)$

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$k = 0$
$\implies d\ell^2 = R^2(t_0)[dr^2 + r^2 d\Omega^2] = d(r')^2 + (r')^2 d\Omega$
$r' = R(t_0)r \implies$ flat Robertson-Walker Universe

$k = 1$
$d\chi^2 = \frac{dr^2}{1-r^2}\chi = 0 when r = 0$
$\implies r = \sin\chi$
$\implies d\ell^2 = R^2(t_0)[d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta d\phi^2)$
$\implies$ spherical Robertson- walker universe
$k = -1$
$\implies d\ell^2 = R^2(t_0)(d\chi^2 + \sinh^2\chi d\Omega^2) \sim$ hyperbolic
Since universe is expanding, the distance is fuzzy since it is measured on the speed of light, a better way is to measure distances based on redshift, z. For large scale the random velocity of galaxies can be ignored compared to the velocity from expansion.

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(421) $\underline{1 + z = \frac{R(t_0)}{R(t)}}$
consider photon on $\theta = \text{const.}$, $\phi = \text{const.}$
$\implies 0 = -dt^2 + R^2(t)d\chi^2$
t is proper time?
$\implies$ energy measured by local observer at rest on trajectory is $-p^0$
p_χ constant. $\implies p^0 \propto \frac{1}{R(t)}$
$\implies \lambda \propto R(t)$?
$z \sim$ redshift ?
$\implies 1 + z = \frac{R(t_0)}{R(t)}$

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<u>Note:</u> $H(t) = \frac{\dot{R}(t)}{R(t)} \implies R(t) = R_0\exp[\int_{t_0}^{t} H(t')dt']$
taylor expand
$\int_{t_0}^{t} H(t')dt' = 0 + H_0(t - t_0) + \frac{1}{2}\dot{H}_0(t - t_0)^2 + \cdots$
$\implies R(t) = R_0\exp(H_0(t - t_0) + \frac{1}{2}\dot{H}_0(t - t_0)^2)$
$= R_0\exp(H_0(t - t_0))\exp(\frac{1}{2}\dot{H}_0(t - t_0)^2)$
$= R_0(1 + H_0(t - t_0) + \frac{1}{2}H_0^2(t - t_0)^2 + \cdots)(1 + \frac{1}{2}\dot{H}_0(t - t_0)^2 + \cdots)$
$= R_0(1 + \frac{1}{2}\dot{H}_0(t - t_0)^2 + H_0(t - t_0) + \frac{1}{2}H_0^2(t - t_0)^2 + \cdots)$
$= R_0(1 + H_0(t - t_0) + \frac{1}{2}(H_0^2 + \dot{H}_0)(t - t_0)^2 + \cdots)$

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$1 + z(t) = \exp[-\int_{t_0}^{t} H(t')dt']$

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(422) $\underline{H(t) = -\frac{\dot{z}}{1+z}}$
<u>recall:</u> $1 + z(t) = \exp[-\int_{t_0}^{t} H(t')dt']$

$\implies -\ln(1+z) = \int_{t_0}^{t} H(t')dt'$
$\implies = -\frac{1}{1+z}\dot{z} = H(t)$
$\therefore H(t) = -\frac{\dot{z}}{1+z}$

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(423) $\underline{d_L = (\frac{L}{4\pi F})^{1/2}}$ spose we know the flux and distance oof a star
$\frac{J}{sm^2}$
to get lumosity $(\frac{J}{s})$
$\implies L = 4\pi d^2 f$
solve for $d \implies d_L = (\frac{L}{4\pi F})^{1/2}$

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(424) $\underline{F = \frac{L}{A(1+z)^2},\ d_L = R_0 r(1+z)}$
object L at t_0 what flux do we recieve at t_0 photon frequency ν_e at t_e
small δt_e
\# photons emitted in $\delta t_e = N = \frac{energyemitted}{energyperphoton} = \frac{L\delta t_e}{h\nu_e}$
suppose object at origin and we sit at r the area the number of photons rest on is proper area
recall: $ds^2 = -dt^2 + R^2(rt)[\frac{dr^2}{1-kr^2} + r^2 d\Omega^2]$
$dt = dr = 0 \implies A = 4\pi R_0^2 r^2$
ν_e redshifts to $\nu_0 by(1+z) = R_0/R(t_e)$
$\implies h\nu_0 = h\frac{\nu_e}{1+z}$
δt_0 grows due to redshift
$\implies \delta t_0 = \delta t_e(1+z)$
$F_0 = \frac{Nh\nu_0}{A\delta t_0} = \frac{Nh\nu_0}{A\delta t_e(1+z)} = \frac{Nh\nu_e}{A\delta t_e(1+z)^2}$
but $L = \frac{h\nu_e N}{\delta t_e} \implies F = \frac{L}{A(1+z)^2}$
recall: $d_L = (\frac{L}{4\pi F})^{1/2} = (\frac{A(1+z)^2}{4\pi})^{1/2} = (R_0^2 r^2(1+z)^2)^{1/2}$
$= R_0 r(1+z)$

$d_A = \frac{D}{\theta} D$ is transverse diameter (arclength) of object θ is the angle of object and d_A is distance to object

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the metric only depends on one time dependent quantity, R(t) (scale factor, not Ricci scalar)

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(425) $\underline{\frac{d}{dt}(\rho R^3) = -p\frac{d}{dt}(R^3)}$

<u>recall:</u> $T^{\mu\nu}_{;\nu} = 0$; $V^{\alpha}_{;\beta} = V^{\alpha}_{,\beta} + V^{\mu}\Gamma^{\alpha}_{\mu\beta}$; $G^{\alpha\beta} = 8\pi T^{\alpha\beta} \implies \frac{1}{8\pi}G^{\alpha\beta} = T^{\alpha\beta}$; $ds^2 = -dt^2 + R^2(t)[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)]$

$$\implies g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{R^2(t)}{1-kr^2} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix}$$

use program to compute spacial component of $T^{\mu\nu}$ are zero, only time component matters

$T^{0\nu}_{;\nu} = \dot{\rho} + \frac{3(p+\rho)\dot{R}}{R}$

$\implies \frac{1}{R}(R\dot{\rho} + 3\rho\dot{R}) + \frac{3p}{R}\dot{R} = 0$

$\implies \frac{1}{R^3}(R^3\dot{\rho} + 3R^2\dot{R}\rho) = -3\frac{p}{R}$

$\implies \frac{1}{R^3}\frac{d}{dt}(R^3\rho) = -3\frac{p}{R}$

$\implies \frac{d}{dt}(R^3\dot{\rho}) = -3pR^2\dot{R} = -p\frac{d}{dt}R^3$