(893) $\underline{\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}) = 0}$

$L = \int \mathcal{L} d^3x$, $\mathcal{L}$ is called the Lagrangian density, where $\mathcal{L} = \mathcal{L}(\phi, \partial_\mu \phi)$

$S = \int L dt = \int \mathcal{L} d^4x$

$\delta S = \int (\frac{\partial \mathcal{L}}{\partial \phi}\delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta(\partial_\mu \phi))d^4$

$\delta(\partial_\mu \phi) = \partial_\mu \delta\phi$

$= \int [\frac{\partial \mathcal{L}}{\partial \phi}\delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\partial_\mu(\delta\phi)]d^4x$

$\partial_\mu(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta\phi)$

$= \partial_\mu(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)})\delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\partial_\mu(\delta\phi)$

$= \int [\frac{\partial \mathcal{L}}{\partial \phi}\delta\phi + \partial_\mu(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta\phi) - \partial_\mu(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)})\delta\phi]d^4x$

$= \int [\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)})]\delta\phi d^4x$

$\implies \frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}) = 0$

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(894) $\underline{\hat{H} = \sum_{k=1}^N \hbar\omega_k(a_k^\dagger a_k + \frac{1}{2})}$

<u>recall:</u> $\hat{H} = \sum_j \frac{\hat{p}_j^2}{2m} + \frac{1}{2}k(\hat{x}_{j+1} - \hat{x}_j)^2$ (Hamiltonian for 1-D lattice connected by springs)

replace w/ Fourier transforms also Note that $\tilde{x} \to \hat{x}$ (Notation)

$\hat{H} = \sum_k [\frac{1}{2m}\hat{p}_k\hat{p}_{-k} + \frac{1}{2}m\omega_k^2\hat{x}_k\hat{x}_{-k}]$

Notice similarity from QM $H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2x^2$

$\implies \begin{cases} \hat{a}_k = \sqrt{\frac{m\omega_k}{2\hbar}}(\hat{x}_k + \frac{i}{m\omega_k}\hat{p}_k) \\ \hat{a}_k^\dagger = \sqrt{\frac{m\omega_k}{2\hbar}}(\hat{x}_k - \frac{i}{m\omega_k}\hat{p}_k) \end{cases}$

invert and insert (check this)

$\therefore \hat{H} = \sum_{k=1}^N \hbar\omega_k(a_k^\dagger a_k + \frac{1}{2})$

$\implies \hat{H} = \int d^3p E_p \hat{a}_p^\dagger \hat{a}_p$

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(895) $\underline{L = \int d^3x[\frac{1}{2}\rho(\frac{\partial\phi}{\partial t})^2 - \frac{1}{2}\mathcal{T}(\nabla\phi)^2]}$

<u>recall:</u> $L = \sum_j \frac{p_j^2}{2m} - \frac{1}{2}k(q_{j+1} - q_j)^2$ (easy to obtain from previous Hamiltonian)

$\sum_j \to \frac{1}{\ell}\int dx,\ \ell \sim$ dist between lattice points
$\sum_j \frac{p_j^2}{2m} = \sum_j \frac{1}{2}m(\frac{\partial q_j}{\partial t})^2 \to \frac{1}{\ell}\int dx \frac{1}{2}m(\frac{\partial\phi(x,t)}{\partial t})^2$
$\sum \frac{1}{2}k(q_{j+1}-q_j)^2 \to \frac{1}{2}k\int dx(\frac{q_{j+1}-q_j}{\ell})^2 = \frac{1}{2}\int dx k(\frac{\partial\phi}{\partial x})^2\ell$
$\implies L \to \int d^3x[\frac{1}{2}\rho(\frac{\partial\phi}{\partial t})^2 - \frac{1}{2}\mathcal{T}(\nabla\phi)^2]$

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recall: $\mathcal{L} = \frac{1}{2}\rho(\frac{\partial\phi}{\partial t})^2 - \frac{1}{2}T(\nabla\phi)^2$
T is like the bulk modulus B
$\implies \mathcal{L} = \rho[\frac{1}{2}(\frac{\partial\phi}{\partial t})^2 - \frac{1}{2}v^2(\nabla\phi)^2]$
if $\delta\mathcal{L} = 0 \implies \rho$ doesn't matter
$\implies \mathcal{L} = \frac{1}{2}(\frac{\partial\phi}{\partial t})^2 - \frac{1}{2}(\nabla\phi)^2$
$\implies \mathcal{L} = \frac{1}{2}\partial^\mu\phi\partial_\mu\phi$
this is the wave equation, or in other words it is a Klein gordon equation for a massless scalar field that travels at $c = 1$ (the speed of light) more generally,
recall: $\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2$ (Klein-Gordon $\mathcal{L}$ for a *real* scalar field)

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(896) $\tilde{\phi}(\vec{p}) = \frac{1}{\sqrt{2E_{\vec{p}}}}(a_{\vec{p}} + a^\dagger_{-\vec{p}})$

recall: $a = \frac{1}{2}(\sqrt{2\omega}q + i\sqrt{\frac{2}{\omega}}\pi)$
$q \sim$ position operator, $\pi \sim$ momentum operator
promote them to fields
(why promote them to fourier transforms $\tilde{\phi} of \phi$?)
$\implies a_{\vec{p}} = \frac{1}{2}(\sqrt{2\omega_{\vec{p}}}\tilde{\phi}(\vec{p}) + i\sqrt{\frac{2}{\omega_{\vec{p}}}}\tilde{\pi}(\vec{p}))$
Note: $\tilde{\phi}^\dagger(\vec{p}) = \tilde{\phi}(-\vec{p}) and \tilde{p}^\dagger(\vec{p}) = \tilde{\pi}(-\vec{p})$
(not sure why...)

$$\implies \begin{cases} a_{\vec{p}} = \frac{1}{2}(\sqrt{2\omega_{\vec{p}}}\tilde{\phi}(\vec{p}) + i\sqrt{\frac{2}{\omega_{\vec{p}}}}\tilde{\pi}(\vec{p})) \\ a^\dagger_{-\vec{p}} = \frac{1}{2}(\sqrt{2\omega_{\vec{p}}}\tilde{\phi}(\vec{p}) - i\sqrt{\frac{2}{\omega_{\vec{p}}}}\tilde{\pi}(\vec{p})) \end{cases}$$

Note: I used $\omega_{-\vec{p}} = \omega_{\vec{p}}$
add
$\implies a_{\vec{p}} + a^\dagger_{-\vec{p}} = \sqrt{2\omega_{\vec{p}}}\tilde{\phi}(\vec{p})$
$\therefore \tilde{\phi}(\vec{p}) = \frac{1}{\sqrt{2E_{\vec{p}}}}(a_{\vec{p}} + a^\dagger_{-\vec{p}})$

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(897) $\phi(\vec{x}) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}(a^\dagger_{\vec{p}}e^{-i\vec{p}\cdot\vec{x}} + a_{\vec{p}}e^{i\vec{p}\cdot\vec{x}})$

recall: $\tilde{\phi}(\vec{p}) = \frac{1}{\sqrt{2E_{\vec{p}}}}(a_{\vec{p}} + a^\dagger_{-\vec{p}})$

now Fourier transform
$\phi(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} e^{i\vec{p}\cdot\vec{x}}\tilde{\phi}(\vec{p})$
$= \int \frac{d^3p}{(2\pi)^3} e^{i\vec{p}\cdot\vec{x}} \frac{1}{\sqrt{2E_{\vec{p}}}}(\hat{a}_{\vec{p}} + \hat{a}^\dagger_{-\vec{p}})$
$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} e^{i\vec{p}\cdot\vec{x}}\hat{a}_{\vec{p}} + \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E)_{\vec{p}}}} a^\dagger_{-\vec{p}} e^{i\vec{p}\cdot\vec{x}}$
$\int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} a^\dagger_{-\vec{p}} e^{i\vec{p}\cdot\vec{x}}$
$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} a^\dagger_{-\vec{p}} e^{i\vec{p}\cdot\vec{x}}$
$\vec{p}' = -\vec{p}$
$= \int_{\infty}^{-\infty} \int_{\infty}^{-\infty} \int_{\infty}^{-\infty} \frac{-d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{-\vec{p}}}} a^\dagger_{\vec{p}} e^{-i\vec{p}\cdot\vec{x}}$
$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} a^\dagger_{\vec{p}} e^{-i\vec{p}\cdot\vec{x}}$
$\therefore \phi(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} (a^\dagger_{\vec{p}} e^{-i\vec{p}\cdot\vec{x}} + a_{\vec{p}} e^{i\vec{p}\cdot\vec{x}})$
where we used $E_{\vec{p}} = E_{-\vec{p}}$ since this is the mode expansion for a free particle which only has kinetic energy so $E_{\vec{p}} = \frac{p^2}{2m} = E_{-\vec{p}}$

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(898) $\underline{\mathcal{L} = |\partial\phi|^2 - m^2|\phi|^2}$, $\phi = \frac{\phi_1 + i\phi_2}{\sqrt{2}}$
<u>recall:</u> $\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2;\ \phi \in \mathbb{R}$
$\implies \mathcal{L} = \frac{1}{2\partial_\mu\phi_1}\partial^\mu\phi_1 - \frac{1}{2}m^2\phi_1^2 +]\frac{1}{2}\partial_\mu\phi_2\partial^\mu\phi_2 - \frac{1}{2}m^2\phi_2^2$
$= \frac{1}{2}\begin{pmatrix}\partial_\mu\phi_1 & \partial_\mu\phi_2\end{pmatrix}\begin{pmatrix}\partial^\mu\phi_1 \\ \partial^\mu\phi_2\end{pmatrix} - \frac{1}{2}m^2\begin{pmatrix}\phi_1 & \phi_2\end{pmatrix}\begin{pmatrix}\phi_1 \\ phi_2\end{pmatrix}$
but $\begin{pmatrix}\phi_1 & \phi_2\end{pmatrix}\begin{pmatrix}\phi_1 \\ \phi_2\end{pmatrix} = (\phi_1 + i\phi_2)(\phi_1 + i\phi_2)^*$
$\implies \mathcal{L} = \partial_\mu\frac{(\phi_1 + i\phi_2)}{\sqrt{2}}\partial^\mu\frac{\phi_1 - i\phi_2}{\sqrt{2}}) - m^2|\frac{\phi_1 + i\phi_2}{\sqrt{2}}|^2$
$\phi = \frac{\phi_1 + i\phi_2}{\sqrt{2}}$
$\therefore \mathcal{L} = \partial_\mu\phi\partial^\mu\phi^* - m^2\phi\phi^*$
$= |\partial\phi|^2 - m^2|\phi|^2$

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now lets try to make sense of field operators (say $\psi(x)$); when we originally constructed our Lagrangian (say the Klein Gordon Lagrangian) we did so assuming that the fields contained within it were scalars, this is a classical field theory, however, once you promote the fields to operators, this is what is known as quantization; and it involves expressing the fields in terms of creation and annihilation operators, a process known as mode expansion.

Consider, $\hat{\psi}^\dagger(x) = \frac{1}{\sqrt{V}} \sum_{\vec{p}} \hat{a}^\dagger_{\vec{p}} e^{-i\vec{p}\cdot\vec{x}}$ (field operators)

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(899) <u>Claim:</u> Field operators $\hat{\psi}^\dagger(x)$ and $\hat{\psi}(x)$ create or destroy a particle at a point x.

<u>proof:</u> $|\Psi\rangle = \hat{\psi}^\dagger(x)|0\rangle = \frac{1}{\sqrt{V}} \sum_{\vec{p}} e^{-i\vec{p}\cdot\vec{x}} \hat{a}^\dagger_{\vec{p}}|0\rangle$
$\hat{n}_{\vec{p}} = \hat{a}^\dagger_{\vec{p}} \hat{a}_{\vec{p}}$
$\implies \sum_{\vec{q}} \hat{n}_{\vec{q}}|\Psi\rangle = \sum_{\vec{q}} \hat{a}^\dagger_{\vec{q}} \hat{a}_{\vec{q}} (\frac{1}{\sqrt{V}} \sum_{\vec{p}} e^{-i\vec{p}\cdot\vec{x}} a^\dagger_{\vec{p}}|0\rangle$
$= \frac{1}{\sqrt{V}} \sum_{\vec{p},\vec{q}} e^{-i\vec{p}\cdot\vec{x}} a^\dagger_{\vec{q}} a_{\vec{q}}] a^\dagger_{\vec{p}}|0\rangle$
<u>recall:</u> $\langle 0|\hat{a}_{\vec{q}} \hat{a}^\dagger_{\vec{p}}|0\rangle = \langle \vec{q}|\vec{p}\rangle = \delta_{\vec{q}\vec{p}}$
$\implies \sum_{\vec{q}} \hat{\vec{q}}|\Psi\rangle = \frac{1}{\sqrt{V}} \sum_{\vec{p}} e^{-ip|\vec{\cdot}\vec{x}} a^\dagger_{\vec{p}}|0\rangle = |\Psi\rangle$
$\implies |\Psi\rangle$ contains a single particle; $\hat{a}^\dagger_{\vec{p}}|0\rangle = |\vec{p}\rangle$
$\langle y|\Psi\rangle = \frac{1}{\sqrt{V}} \sum_{\vec{p}} e^{-i\vec{p}\cdot\vec{x}} \langle \vec{y}|\vec{p}\rangle$
$= \frac{1}{V} \sum_{\vec{p}} e^{-i\vec{p}\cdot(\vec{x}-\vec{y})} = \delta^{(3)}(\vec{x}-\vec{y})$
$\implies$ this shows the particle created by $\psi^\dagger(\vec{x})$ can only be found at $y = x$

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(900) $\underline{|\psi_I(t)\rangle = e^{i\hat{H}_0 t}|\psi(t)\rangle,\ \hat{O}_I(t) = e^{i\hat{H}_0 t}\hat{O}e^{-i\hat{H}_0 t}}$
$\langle\phi(t)|\hat{O}|\psi(t)\rangle = \langle\phi(t)|e^{-i\hat{H}_0 t}(e^{i\hat{H}_0 t}\hat{O}e^{-iH_0 t})e^{iH_0 t}|\psi(t)\rangle$
$= \langle\phi_I(t)|\hat{O}_I(t)|\psi_I(t)\rangle$
where $|\psi_I(t)\rangle = e^{i\hat{H}_0 t}|\psi(t)\rangle;\ \hat{O}_I(t) = e^{iH_0 t}\hat{O}e^{-iH_0 t}$

<u>Note:</u> As of now the label "I" is confusing on $|\psi_I(t)\rangle$ but keep in mind $|\psi(t)\rangle = e^{-iH' t}|\psi\rangle$
so $|\psi_I(t)\rangle = e^{i\hat{H}_0 t}|\psi(t)\rangle = e^{i\hat{H}_0 t}e^{-i\hat{H}_0 t}e^{-i\hat{H}_I t}|\psi\rangle = e^{-i\hat{H}_I t}|\psi\rangle$
which motivates the "I" notation

Now, let's transition into the S-matrix. In a scattering experiment we shoot a free particle (free since it does not 'feel' its target yet), it interacts with the target and reaches a free state as $t \to \infty$

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(901) $\underline{S = \lim_{t\to\infty} U_I(t,-t)}$
first take a free initial state at $t = -\infty, |\Psi\rangle$ and scatter it, we want to know the amplitude that it will reach a final state $|\phi\rangle$
$\implies$ scatter $|\Psi\rangle \to \hat{S}|\Psi\rangle$
amplitude $\langle\phi|\hat{S}|\Psi\rangle$
phrased another way, take a free initial state and evolve it through the scatterer (interaction picture) $|\Psi\rangle \to e^{-iH_I t}|\Psi\rangle = \Psi(t)\rangle$
$\implies \langle\phi|\hat{S}|\Psi\rangle = \langle\phi|\Psi(t)\rangle = \langle\phi|U_I(\infty,-\infty)|\Psi\rangle$
$\therefore S = U_I(\infty,-\infty)$

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Assumptions
1. System initially in state $|i\rangle$
$\implies \psi(0)\rangle = \sum_n c_n(0)|\phi_n\rangle = \sum_n c_n(0)|n\rangle = |i\rangle$
$\implies \sum_n c_n(0)\langle i|n\rangle = \sum_n c_n(0)\delta_{in} = c_i(0) = 1$
$\implies c_j(0) = \delta_{ij}$
2. perturbation very weak and applied for a short period of time so coefficients remain nearly unchanged

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(902) $\underline{c_f(t) = \frac{1}{i\hbar}\int_0^t W_{fi}(t')e^{i\omega_{fi}t'}dt'}$ for time dependent perturbation theory
$H = H_0 + W(t)$
$H_0|\phi_n\rangle = E_n|\phi_n\rangle \implies |\psi_n(t)\rangle = |\phi_n\rangle e^{-iE_n t/\hbar}$(unperturbed)

$$\implies H|\psi(t)\rangle = [H_0 + W(t)]|\psi(t)\rangle = i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle \tag{1}$$

$|\psi(t)\rangle = \sum_n c_n(t)|\psi_n(t)\rangle = \sum_n c_n(t)|\phi_n\rangle e^{-iE_n t/\hbar}$ (perturbed)
<u>Note:</u> $|\phi_n\rangle$ are a complete set of kets, meaning any ket in hilbert space can be represented as a linear combination of them, so we can express the perturbed hamiltonian wave functions using the same kets, the only difference is there is a time dependence in the coefficients, other than just the usual $e^{-iE_n t/\hbar}$ insert this expression into 1
$H_0\sum_k c_k(t)|\phi_k\rangle e^{-iE_k t/\hbar} + W(t)\sum_k c_k(t)|\phi_k\rangle e^{-iE_k t/\hbar}$
$= i\hbar\frac{\partial}{\partial t}\sum_k c_k(t)|\phi_k\rangle e^{-iE_k t/\hbar}$
$\implies \sum_k c_k(t)E_k|\phi_k\rangle e^{-iE_k t/\hbar} + \sum_k c_k(t)W(t)|\phi_k\rangle e^{-iE_k t/\hbar}$

$= i\hbar \sum_k \frac{\partial c_k(t)}{\partial t}|\phi_k\rangle e^{-iE_kt/\hbar} + i\hbar \sum_k c_k(t)|\phi_k\rangle(-i\frac{E_k}{\hbar})e^{-iE_kt/\hbar}$
contract with $\langle\phi_n|$
$\implies E_n c_n(t)e^{-iE_nt/\hbar} + \sum_k c_k(t)W_{nk}(t)e^{-iE_kt/\hbar}$
$= i\hbar\frac{\partial c_n(t)}{\partial t}e^{-iE_nt/\hbar} + E_n c_n(t)e^{-iE_nt/\hbar}$
$\implies \sum_k c_k(t)W_{nk}(t)e^{-iE_kt/\hbar} = i\hbar\frac{\partial c_n(t)}{\partial t}e^{-iE_nt/\hbar}$
$\implies \frac{\partial c_N(t)}{\partial t} = \frac{1}{i\hbar}\sum_k c_k(t)W_{nk}(t)e^{-i(E_k-E_n)t/\hbar}$
$= \frac{1}{i\hbar}\sum_k c_k(t)W_{nk}(t)e^{i\omega_{nk}t}$
$\omega_{nk} \equiv \frac{E_n-E_k}{\hbar}$
$\implies \frac{\partial c_n(t)}{\partial t} = \frac{1}{i\hbar}\sum_k c_k(t)W_{nk}(t)e^{\omega_{nk}t}$
assume it is highly peaked around $c_i(t)$ (I assume it starts in the ith state but not sure)
$\approx \frac{1}{i\hbar}\sum_k \delta_{ik}c_k(t)W_{nk}(t)e^{i\omega_{nk}t} = \frac{1}{i\hbar}W_{ni}(t)e^{i\omega_{ni}t}$
$\approx \frac{1}{i\hbar}c_i(t)W_{ni}(t)e^{i\omega_{ni}t}$
$\implies \frac{\partial c_n(t)}{\partial t} = \frac{1}{i\hbar}c_i(t)W_{ni}(t)e^{i\omega_{ni}t} \approx \frac{1}{i\hbar}W_{ni}(t)e^{i\omega_{ni}t}$
$\implies \frac{\partial c_f(t)}{\partial t} = \frac{1}{i\hbar}W_{fi}(t)e^{i\omega_{fi}t}$
$\therefore c_f(t) = \frac{1}{i\hbar}\int_0^f W_{fi}(t')e^{i\omega_{fi}t'}dt'$

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(903) $\underline{\mathcal{P}_{if}(t) = \frac{w_{fi}^2}{\hbar^2}F(t,\omega-\omega_{fi});\ F(t,\omega-\omega_{fi}) = \{\frac{\sin[(\omega_{fi}-\omega)t/2]}{(\omega_{fi}-\omega)/2}\}^2}$
consider $W(t) = 2W\cos(\omega t) = W(e^{i\omega t}+e^{-i\omega t})$ (I think W is an operator)
recall: $c_f(t) = \frac{1}{i\hbar}\int_0^t W_{fi}(t')e^{i\omega_{fi}t'}dt'$
$= \frac{W_{fi}}{i\hbar}\int_0^t(e^{i\omega t'}+e^{-i\omega t'})e^{i\omega_{fi}t'}dt'$
$= \frac{W_{fi}}{i\hbar}\{\frac{e^{i(\omega_{fi}+\omega)t}-1}{i(\omega_{fi}+\omega)} + \frac{e^{i(\omega_{fi}-\omega)t}-1}{i(\omega_{fi}-\omega)}\}$
recall: $\mathcal{P}_{if}(t) = \frac{1}{\hbar^2}|\int_0^t e^{i\omega_{fi}t'}W_{fi}(t')dt'|^2 = |c_f(t)|^2$
$= \frac{W_{fi}^2}{\hbar^2}|\frac{e^{i(\omega_{fi}+\omega)t}-1}{i(\omega_{fi}+\omega)} + \frac{e^{i(\omega_{fi}-\omega)t}-1}{i(\omega_{fi}-\omega)}|^2$
assume$\omega \approx \omega_{fi} \implies |\omega-\omega_{fi}| << |\omega_{fi}|$ (I don't totally understand this part.)
$\implies$ first term negligible
$\implies \mathcal{P}_{if}(t) \approx \frac{W_{fi}^2}{\hbar^2}|\frac{e^{i(\omega_{fi}-\omega)t}-1}{i(\omega_{fi}-\omega)}|^2$
Note: $A_- \equiv \frac{e^{i(\omega_{fi}-\omega)t}-1}{i(\omega_{fi}-\omega)}$
$= e^{i(\omega_{fi}-\omega)t/2}\frac{e^{i(\omega_{fi}-\omega)t/2}-e^{-i(\omega_{fi}-\omega)t/2}|}{i(\omega_{fi}-\omega)}$
$= e^{i(\omega_{fi}-\omega)t/2}\frac{\sin[(\omega_{fi}-\omega)t/2]}{(\omega_{fi}-\omega)/2}$
$\therefore \mathcal{P}_{if}(t) = \frac{W_{fi}^2}{\hbar^2}|\frac{e^{i(\omega_{fi}-\omega)t}-1}{i(\omega_{fi}-\omega)}|^2$

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(904) $\mathcal{P}_{if}(t) = |\langle\phi_f|\psi(t)\rangle|^2 = |c_f(t)|^2 = \frac{1}{\hbar^2}|\int_0^t e^{i\omega_{fi}t'}W_{fi}(t')dt'|^2$
If there is a continuum of final energies we weight it by the density of states $\rho(E)$
$\implies \mathcal{P}(t) = \int_{\{E_{acc}\}} \mathcal{P}_{if}(t)\rho(E)dE$
E goes over all possible final energies allowed for system.

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(905) $\underline{\mathcal{W} = \frac{d\mathcal{P}(t)}{dt} = \frac{2\pi}{\hbar^2}W_{fi}^2\rho(E_{fi})}$ (Fermi's Golden Rule)
recall: $\mathcal{P}(t) = \int_{E_{acc}} \frac{W_{fi}^2}{\hbar^2}\left\{\frac{\sin[(\omega_{fi}-\omega)t/2]}{(\omega_{fi}-\omega)/2}\right\}^2 \rho(E)dE$
$= \frac{W_{fi}^2}{\hbar^2}\int_{\{E_{acc}\}} \frac{W_{fi}^2}{\hbar^2}\left\{\frac{\sin[(\omega_{fi}-\omega)t/2]}{(\omega_{fi}-\omega)/2}\right\}^2 dE$
the factor in curly brackets is sharply peaked around ω_{fi}
$\approx \frac{W_{fi}^2}{\hbar^2}\rho(E_{fi})\int_{\{E_{acc}\}}\left\{\frac{\sin[(\omega_{fi}-\omega)t/2]}{(\omega_{fi}-\omega)/2}\right\}^2 \hbar d\omega$
$x = (\omega_{fi}-\omega)t/2$
$= \frac{W_{fi}^2}{\hbar^2}\rho(E_{fi})\hbar(\frac{2}{t})t^2\int_{-\infty}^{\infty}\frac{\sin^2 x}{x^2}dx$
$= \frac{2\pi}{\hbar}W_{fi}^2\rho(E_{fi})t$
$\implies \mathcal{W} = \frac{d\mathcal{P}}{dt} = \frac{2\pi}{\hbar}W_{fi}^2\rho(E_{fi})$ (Fermi's Golden Rule)

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(906) $\underline{i\hbar\frac{\partial}{\partial t}|\alpha, t_0; t\rangle_I = V_I|\alpha, t_0; t\rangle_I}$
recall: $|\alpha, t_0; t\rangle_I = e^{iH_0t/\hbar}|\alpha, t_0; t\rangle_S$; $\hat{O}_I(t) = e^{i\hat{H}_0t}\hat{O}e^{-i\hat{H}_0t}$
$i\hbar\frac{\partial}{\partial t}|\alpha, t_0; t\rangle_I = i\hbar\frac{\partial}{\partial t}e^{iH_0t/\hbar}|\alpha, t_0; t\rangle_S$
$= i\hbar(\frac{iH_0}{\hbar}e^{iH_0t/\hbar}|\alpha, t_0; t\rangle_S + e^{iH_0t/\hbar}\frac{\partial}{\partial rt}|\alpha, t_0; t\rangle_S)$
$= -e^{-iH_0t/\hbar}H_0|\alpha, t_0; t\rangle_S + e^{iH_0t/\hbar}H|\alpha, t_0; t\rangle_S$
$H = H_0 + B$
$= e^{iH_0t/\hbar}V|\alpha, t_0; t\rangle_S = (e^{iH_0t/\hbar}Ve^{0-iH_0t/\hbar})e^{iH_0t/\hbar}|\alpha, t_0; t\rangle_S$
$= V_I(t)|\alpha, t_0; t\rangle_I$

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Note: $|\alpha, t_0; t\rangle_I = U_I(t, t_0)|\alpha, t_0; t_0\rangle$
so that $i\hbar\frac{\partial}{\partial t}|\alpha, t_0; t\rangle_I = V_I|\alpha, t_0; t\rangle_I$
$\implies i\hbar\frac{\partial}{\partial t}U_I(t, t_0) = V_I(t)U_I(t, t_0)$

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(907) $\underline{U_I(t, t_0) = 1 - \frac{i}{\hbar}\int_{t_0}^t dt' V_I(t')}$
$\underline{+(\frac{-i}{\hbar})^2\int_{t_0}^t dt'\int_{t_0}^{t'} dt'' V_I(t')V_I(t'') + \cdots + (-\frac{i}{\hbar})^n\int_{t_0}^t dt'\int_{t_0}^{t'} dt''}$
$\underline{\cdots \times \int_{t_0}^{t^{(n-1)}} dt^{(n)} V_I(t')V_I(t'')\cdots V_I(t^{(n)}) + \cdots}$ (Dyson series)
recall: $i\hbar\frac{\partial}{\partial t}U_I(t, t_0) = V_I(t)U_I(t, t_0)$

$\implies i\hbar U_I(t,t_0) = \int_{t_0}^{t} V_I(t')U_I(t',t_0)dt' + C$
$\implies U_I(t,t_0) = C - \frac{i}{\hbar}\int_{t_0}^{t} V_I(t')U_I(t',t_0)dt'$
but $U_I(t_0,t_0) = 1$
$\implies C = 1$
$\implies U_I(t,t_0) = 1 - \frac{i}{\hbar}\int_{t_0}^{t} V_I(t')U_I(t',t_0)dt'$
use $U_I(t',t_0) = 1 - \frac{i}{\hbar}\int_{t_0}^{t'} V_I(t'')U_I(t'',t_0)dt''$
$\implies U_I(t,t_0) = 1 - \frac{i}{\hbar}\int_{t_0}^{t} dt' z v_I(t')[1 - \frac{i}{\hbar}\int_{t_0}^{t'} V_I(t'')U_I(t'',t_0)dt'']$
$= 1 - \frac{i}{\hbar}\int_{t_0}^{t} dt' V_I(t') + (-\frac{i}{\hbar})^2 \int_{t_0}^{t} dt' \int_{t_0}^{t'} dt'' V_I(t')V_I(t'')U_I(t'',t_))$
$+\cdots+(-\frac{i}{\hbar})^n \int_{t_0}^{t} dt' \int_{t_0}^{t'} dt'' \cdots \times \int_{t_0}^{t^{(n-1)}} dt^{(n)} V_I(t')V_I(t'') \cdots V_I(t^{(n)}) +$
$\cdots$

Note: In QFT time ordering operators re introduced to simplify this expression into an exponential
Note: also note from the meeting with dane that Grassman numbers are a way to keep the Lagrangian classical while including all the quantum information, theyre numbers such that we simply change the commutative property to anti-commutative

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(908) $U_I(t,t_0) = e^{iH_0t/\hbar}U(t,t_0)e^{-iH_0t_0/\hbar}$
recall: $|\alpha,t_0;t\rangle_I = e^{iH_0t/\hbar}|\alpha,t_0;t\rangle_S$
$= e^{iH_0t/\hbar}U(t,t_0)|\alpha,t_0;t_0\rangle_S = e^{iH_0t/\hbar}U(t,t_0)e^{-iH_0t_0/\hbar}|\alpha,t_0;t_0\rangle_I$
$\therefore U_I(t,t_0) = e^{iH_0t/\hbar}U(t,t_0)e^{-iH_0t_0/\hbar}$

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(909) $c_n(t) = \langle m|U_I(t,t_0)|i\rangle$
assume $|i,t_0;t_0\rangle_S = e^{-iE_it_0/\hbar}|i\rangle$?
, note that this is allowed because the transition amplitude does not care about a phase, this phase is chosen so that $|i,t_0;t_0\rangle_I = |i\rangle$
system is in $|i\rangle$ at t_0($|i\rangle$ eigenket of H_0)
$\implies |i,t_0;t_0\rangle_I = e^{iH_0t_0/\hbar}|i,t_0;t_0\rangle_S = e^{iH_0t_0/\hbar}e^{-iE_it_0/\hbar}|i\rangle = |i\rangle$
since $H_0|i\rangle = E_i|i\rangle$
$\implies |i,t_0;t\rangle_I = U_I(t,t_0)|i\rangle$
compare with $|i,t_0;t\rangle_I = \sum_n c_n(t)|n\rangle$
$\implies \sum_n c_n(t)|n\rangle = U_I(t,t_0)|i\rangle$
$\implies \sum_n c_n(t)\delta_{mn} = \langle m|U_I(t,t_0)|i\rangle$
$\therefore c_m(t) = \langle m|U_I(t,t_0)|i\rangle$

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(910) $\underline{c_n^{(0)} = \delta_{ni}}$

$\underline{c_n^{(1)}(t) = -\frac{i}{\hbar}\int_{t_0}^{t}\langle n|V_I(t')|i\rangle dt' = -\frac{i}{\hbar}\int_{t_0}^{t} e^{i\omega_{ni}t'}V_{ni}(t')dt'}$

$\underline{c_n^{(2)}(t) = (-\frac{i}{\hbar})^2\sum_m \int_{t_0}^{t} dt' \int_{t_0}^{t'} dt'' e^{i\omega_{nm}t'}V_{nm}(t')e^{i\omega_{mi}t''}V_{mi}(t'')}$

recall: Dyson series; $c_n(t) = \langle n|U_I(t,t_0)|i\rangle; \hat{V}_I(t) = e^{i\hat{H}_0 t}\hat{V}e^{-i\hat{H}_0 t}$

$c_n(t) = c_n^{(0)} + c_n^{(1)} + c_n^{(2)} + \cdots$

$\implies \langle n|U_I(t,t_0)|i\rangle$

$= \langle n|(1 - \frac{i}{\hbar}\int_{t_0}^{t} dt' V_I(t') + (-\frac{i}{\hbar})^2\int_{t_0}^{t} dt' \int_{t_0}^{t'} dt'' V_I(t' V_I(t'') + \cdots |i\rangle$

$= \delta_{ni} - \frac{i}{\hbar}\int_{t_0}^{t} dt'\langle n|V_I(t')|i\rangle$

$+ (-\frac{i}{\hbar})^2\int_{t_0}^{t} dt' \int_{t_0}^{t'} dt''\langle n|V_I(t')V_I(t'')|i\rangle + \cdots$

$= \delta_{ni} - \frac{i}{\hbar}\int_{t_0}^{t} dt'\langle n|V_I(t')|i\rangle$

$+ (-\frac{i}{\hbar})^2\sum_m\int_{t_0}^{t} dt' \int_{t_0}^{t'} dt''\langle n|V_I(t')|m\rangle\langle m|V_I(t'')|i\rangle + c\ldots$

$= \delta_{ni} - \frac{i}{\hbar}\int_{t_0}^{t} dt' e^{i\omega_{ni}t'}V_{ni}(t')dt'$

$+ (-\frac{i}{\hbar})^2\sum_m\int_{t_0}^{t} dt \int_{t_0}^{t} dt'' e^{i\omega_{nm}t'}V_{nm}(t')e^{i\omega_{mi}t''}V_{mi}(t'')$

$= c_n^{(0)} + c_n^{(1)} + c_n^{(2)} + \cdots$

since $\langle n|V_I(t')|i\rangle = \langle n|e^{iH_0t'/\hbar}Ve^{-iH_0t/\hbar}|i\rangle$

$= \langle n|e^{iE_nt'/\hbar}Ve^{-iE_it'/\hbar}|i\rangle = e^{i\omega_{ni}t'}V_{ni}(t')$.

We can equate coefficients to obtain desired result.

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(911) $\underline{c_i(t) = e^{-i\Delta_i t/\hbar},\ \frac{\dot{c}_i(t)}{c_i(t)} = -\frac{i}{\hbar}\Delta_i}$

Work with potential $V(t) = e^{\eta t}V$ we will set $\eta \to 0$ later, also $t_0 \to -\infty$

?? I am not sure why $V_{im} = V_{mi}$ and $V_{im}^2 = |V_{im}|^2$

recall: $\begin{cases} c_n^{(0)} = \delta_{ni} \\ c_n^{(1)}(t) = -\frac{i}{\hbar}\int_{t_0}^{t}\langle n|V_I(t')|i\rangle dt' = -\frac{i}{\hbar}\int_{t_0}^{t} e^{i\omega_{ni}t'}V_{ni}(t')dt' \\ c_n^{(2)}(t) = (-\frac{i}{\hbar})^2\sum_m \int_{t_0}^{t} dt' \int_{t_0}^{t'} dt'' e^{i\omega_{nm}t'}V_{nm}(t')e^{i\omega_{mi}t''}V_{mi}(t'') \end{cases}$

$\implies \begin{cases} c_i^{(0)} = 1 \\ c_i^{(1)} = -\frac{i}{\hbar}\lim_{t_0\to-\infty}\int_{t_0}^{t} V_{ii}e^{\eta t'}dt' = -\frac{i}{\hbar\eta}V_{ii}e^{\eta t} \end{cases}$

$c_i^{(2)} = (-\frac{i}{\hbar})^2\sum_m \lim_{t_0\to-\infty}\int_{t_0}^{t} dt' \int_{t_0}^{t'} dt'' e^{i\omega_{nm}t'}V_{nm}(t')e^{i\omega_{mi}t''}V_{mi}(t'')$

$= (-\frac{i}{\hbar})^2\sum_m \lim_{t_0\to-\infty}\int_{t_0}{}^{t} dt' \int_{t_0}{}^{t'} dt'' e^{i\omega_{im}t'}V_{im}e^{\eta t'}e^{i\omega_{mi}t''}V_{mi}(t'')$

$= (-\frac{i}{\hbar})^2\sum_m \lim_{t_0\to-\infty}\int_{t_0}^{t} dt' \int_{t_0}^{t'} dt'' e^{i\omega_{im}t'}V)_{im}e^{\eta t'}e^{i\omega_{mi}t''}V_{mi}e^{\eta t''}$

$= (-\frac{i}{\hbar})^2\lim_{t_0\to-\infty}|V_{ii}|^2\int_{t_0}^{t} dt' e^{\eta t'}\int_{t_0}^{t'} dt'' e^{\eta t''}$

$+ (-\frac{i}{\hbar})^2\sum_{m\neq i}|V_{mi}|^2\lim_{t_0\to-\infty}\int_{t_0}^{t} dt' e^{i\omega_{im}t'+\eta t'}\int_{t_0}^{t'} dt'' e^{i\omega_{mi}t''+\eta t''}$

$= (-\frac{i}{\hbar})^2\sum_{m\neq i}|V_{mi}|^2\lim_{t_0\to-\infty}\int_{t_0}^{t} dt' e^{i\omega_{im}t'+\eta t'}\int_{t_0}^{t'} dt'' e^{i\omega_{mi}t''+\eta t''}$

$= (-\frac{i}{\hbar})^2 |V_{ii}|^2 \frac{e^{2\eta t}}{2\eta^2} + (-\frac{i}{\hbar})^2 \sum_{m \neq i} \frac{|V_{mi}|^2}{(i\omega_{mi}+\eta)2\eta} e^{2\eta t}$
$= (-\frac{i}{\hbar})^2 |V_{ii}|^2 \frac{e^{2\eta t}}{2\eta^2} + (-\frac{i}{\hbar}) \sum_{m \neq i} \frac{|V_{mi}|^2 e^{2\eta t}}{2\eta(E_i - E_m + i\hbar\eta)}$
add $c_i^{(0)},\ c_i^{(1)},\ c_i^{(2)}$
$\implies c_i(t) \approx 1 - \frac{i}{\hbar\eta} V_{ii} e^{\eta t} + (-\frac{i}{\hbar})^2 |V_{ii}|^2 \frac{e^{2\eta t}}{2\eta^2}$
$+ (-\frac{i}{\hbar}) \sum_{m \neq i} \frac{|V_{mi}|^2 e^{2\eta t}}{2\eta(E_i - E_m + i\hbar\eta)}$
take derivative, divide by c_i, set $\eta \to 0$
$\dot{c}_i \approx -\frac{i}{\hbar} V_{ii} e^{\eta t} + (\frac{-i}{\hbar})^2 |V_{ii}|^2 \frac{e^{2\eta t}}{\eta}$
$+ (-\frac{i}{\hbar}) \sum_{m \neq i} \frac{|V_{mi}|^2 e^{2\eta t}}{(E_i - E_m + i\hbar\eta)}$
$\eta \to 0$ in exps
$\implies \frac{\dot{c}_i}{c_i} \approx \frac{-\frac{i}{\hbar} V_{ii} + (-\frac{i}{\hbar})^2 \frac{|V_{ii}|^2}{\eta} + (-\frac{i}{\hbar}) \sum_{m \neq i} \frac{|V_{mi}|^2}{(E_i - E_m + i\hbar\eta)}}{1 - \frac{i}{\hbar} \frac{V_{ii}}{\eta}}$
$\approx (-\frac{i}{\hbar} V_{ii} + (-\frac{i}{\hbar})^2 \frac{|V_{ii}|^2}{\eta} + (-\frac{i}{\hbar}) \sum_{m \neq i} \frac{|V_{mi}|^2}{(E_i - E_m + i\hbar\eta)})(1 - \frac{i}{\hbar} \frac{V_{ii}}{\eta}) \approx$
$-\frac{i}{\hbar} V_{ii} + (-\frac{i}{\hbar}) \sum_{m \neq i} \frac{|V_{mi}|^2}{E_i - E_m + i\hbar\eta} \equiv -\frac{i}{\hbar} \Delta_i$
this is an expansion up to second order in V_{ii} $\frac{\dot{c}_i}{c_i}$ is independent of time
$\implies \dot{c}_i = -\frac{i}{\hbar} \Delta_i c_i \implies c_i(t) = A e^{-i\Delta_i t/\hbar}$
$c_i(0) = 1$ (renormalize, don't understand what justifies this)
$\implies A = 1$
$\therefore \frac{\dot{c}_i(t)}{c_i(t)} = -\frac{i}{\hbar} \Delta_i;\ c_i(t) = e^{-i\Delta_i t/\hbar}$

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(912) $\underline{\lim_{\epsilon \downarrow 0} \frac{1}{x+i\epsilon} = Pr\frac{1}{x} - i\pi\delta(x)}$
First we show,
$\lim_{\epsilon \downarrow 0} \int_{-\infty}^{\infty} dx \frac{\phi(x)}{x+i\epsilon} = -i\pi\phi(0) + \lim_{\epsilon \downarrow 0} \int_{|x|>\epsilon} dx \frac{\phi(x)}{x}$
$= -i\pi\phi(0) + Pr \int dx \frac{\phi(x)}{x}$
Pr is the principal value
$\oint dz \frac{\phi(z)}{x+i\epsilon} = -\int_{-\infty}^{\infty} dx \phi(x) \frac{1}{x+i\epsilon}$
$= \int_R^{\epsilon} dx \frac{\phi(x)}{x+i\epsilon} + \int_{-\epsilon}^{-R} dx \frac{\phi(x)}{x+i\epsilon} + \int_{\gamma_2} dz \frac{\phi(z)}{z+i\epsilon} + \int_{\gamma_4} dz \frac{\phi(z)}{z+i\epsilon}$
Evaluate each piece separately
$\int_{\gamma_2} dz \frac{\phi(z)}{z+i\epsilon} = \int_0^{\pi} dt \frac{i\varepsilon e^{it} \phi(\varepsilon e^{it})}{\varepsilon e^{it} + i\epsilon},\ \epsilon \to 0$
$\approx \int_0^{\pi} dt \frac{i\epsilon}{\epsilon} \frac{e^{it}}{e^{it}} \phi(\epsilon e^{it}) = \int_0^{\pi} dt i\phi(0) = i\pi\phi(0)$ as $\epsilon \to 0$
$\int_{\gamma_4} dz \frac{\phi(z)}{z+i\epsilon} = \int_{\pi}^{2\pi} dt \frac{Rie^{it}\phi(Re^{it})}{Re^{it}+i\epsilon}$
$\to \int_{\pi}^{2\pi} dt \frac{ie^{it}}{e^{it}} \phi(Re^{it})$ for large R
but $\phi(x) = 0$ for $|x| > a$ by assumption
$\implies \phi(Re^{it}) \to 0 \implies \int_{\gamma_4} dz \frac{\phi(z)}{z+i\epsilon} \to 0$
So we are left with

$\int_R^{\epsilon} dx \frac{\phi(x)}{x+i\epsilon} + \int_{-\epsilon}^{-R} dx \frac{\phi(x)}{x+i\epsilon} + i\pi\phi(0)$
$= -\int_{-R}^{-\epsilon} dx \frac{\phi(x)}{x+i\epsilon} - \int_{\epsilon}^{R} dx \frac{\phi(x)}{x+i\epsilon} + i\pi\phi(0)$
$= -\int_{-\infty}^{\infty} dx \phi(x) \frac{1}{x+i\epsilon}$
$\therefore \lim_{\epsilon\downarrow 0} \int_{-\infty}^{\infty} dx \phi(x) \frac{1}{x+i\epsilon} = -i\pi\phi(0) + \lim_{\epsilon\downarrow 0} \int_{|x|>\epsilon} dx \frac{\phi(x)}{x}$
$= \int_{-\infty}^{\infty} \phi(x) \lim_{\epsilon\downarrow 0} \frac{1}{x+i\epsilon} = -i\pi \int_{-\infty}^{\infty} dx \delta(x)\phi(x) + Pr \int_{-\infty}^{\infty} dx \frac{\phi(x)}{x} =$
$\int_{-\infty}^{\infty} dex \phi(x)(-i\pi\delta(x) + \phi(x) Pr \frac{1}{x})$
$\therefore \lim_{\epsilon\downarrow 0} \frac{1}{x+i\epsilon} = Pr \frac{1}{x} - i\pi\delta(x)$

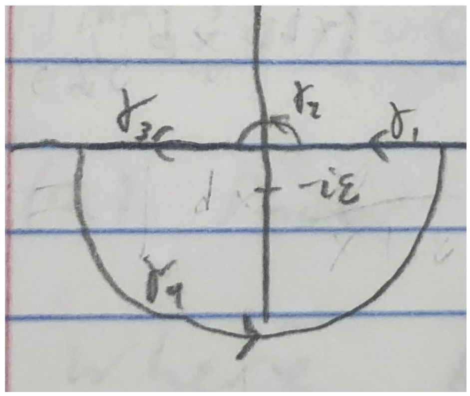

Figure 1. Contour for line integral

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<u>Miscellaneous details between derivations</u>
<u>Note:</u> $e^{-i\Delta_i t/\hbar}|i\rangle \implies e^{-i\Delta_i t/\hbar - iE_i t/\hbar}|i\rangle$
$\implies E_i \to E_i + \Delta_i$ as a result of perturbation
$\implies \Delta_i$ is the level shift of time dependent perturbation theory
<u>recall:</u> $\frac{\dot{c}_i(t)}{c_i(t)} = -\frac{i}{\hbar}\Delta_i \approx -\frac{i}{\hbar}(V_{ii} + \sum_{m\neq i} \frac{|V_{mi}|^2}{E_i - E_m + i\hbar\eta})$
$\Delta_i = \Delta_i^{(1)} + \Delta_i^{(2)} + \cdots$
$\implies \Delta_i^{(1)} = V_{ii};\ \Delta_i^{(2)} = \sum_{m\neq i} \frac{|V_{mi}|^2}{E_i - E_m + i\hbar\eta}$
<u>recall:</u> $\lim_{\epsilon\to 0} \frac{1}{x+i\epsilon} = Pr\frac{1}{x} - i\pi\delta(x)$
$\implies \begin{cases} Re(\Delta_i^{(2)}) = Pr \sum_{m\neq i} \frac{|V_{mi}|^2}{E_i - Em} \\ Im(\Delta_i^{(2)}) = -\sum_{m\neq i} |V_{mi}|^2 \delta(E_i - E_m) \end{cases}$ <u>recall:</u> $\mathcal{W} =$
$\frac{2\pi}{\hbar} W_{fi}^2 \rho(E_{fi})$
$\rho(E_{fi}) \to \delta(E_n - E_i);\ W_{fi}^2 \to V_{ni}|^2$
$\implies \mathcal{W}_{i\to n} = \frac{2\pi}{\hbar}|V_{ni}|^2 \delta(E_n - E_i)$
$\implies \sum_{m\neq i} \mathcal{W}_{i\to n} = \sum_{m\neq i} \frac{2\pi}{\hbar}|V_{mi}|^2 \delta(E_n - E_i) = -\frac{2}{\hbar} Im[\delta_i^{(2)}]$
<u>recall:</u> $e^{-i\Delta_i t/\hbar} = c_i(t)$
$\implies c_i t(t) = e^{-i(Re(\Delta_i) + iIm(\Delta_i))t/\hbar} = e^{-iRe(\Delta_i)t/\hbar + Im(\Delta_i)t/\hbar}$

$|c_i|^2 = e^{2Im(\Delta_i)t/\hbar} = e^{-\Gamma_i t/\hbar}$
$\implies \frac{\Gamma_i}{\hbar} \equiv -\frac{2}{\hbar}Im(\Delta_i)$

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(913) $\underline{Im(\Delta_i^{(2)}) = -\pi \sum_{m\neq i} |V_{mi}|^2 \delta(E_i - E_m)}$
recall: $\lim_{\epsilon\to 0} \frac{1}{x+i\epsilon} = Pr\frac{1}{x} - i\pi\delta(x);\ \Delta_i^{(2)} = \sum_{m\neq i} \frac{|V_{mi}|^2}{(E_i - E_m + i\hbar\eta)}$
Note: Pr means principal value
$\implies \Delta_i^{(2)} = \sum_{m\neq i} |V_{mi}|^2 \lim_{\eta\to 0} \frac{1}{E_i - E_m + i\hbar\eta}$
$- = \sum_{m\neq i} |V_{mi}|^2 (Pr\frac{1}{E_i - E_m} - i\pi\delta(E_i - E_m))$
$= \sum_{m\neq i} |V_{mi}|^2 Pr\frac{1}{E_i - E_m} - i\pi \sum_{m\neq i} |V_{mi}|^2 \delta(E_i - E_m)$
$= Re(\Delta_i^{(2)}) + iIm(\Delta_i^{(2)})$
$\therefore Im(\Delta_i^{(2)}) = -\pi \sum_{m\neq i} |V_{mi}|^2 \delta(E_i - E_m)$

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(914) $\underline{|c_i|^2 = e^{2Im(\Delta_i)t/\hbar} = e^{-\Gamma_i t/\hbar}dt}$
recall: $c_i(t) = e^{-i(Re(\Delta_i) + iIm(\Delta_i))t/\hbar} = e^{-i\Delta_i t/\hbar}$
$\implies |c_i(t)|^2 = e^{2Im(\Delta_i)t/\hbar}$
recall: $\frac{\dot{c}_i(t)}{c_i(t)} = -\frac{i}{\hbar}\Delta_i = -\frac{i}{\hbar}(V_{ii} + \sum_{m\neq i} \frac{|V_{mi}|^2}{E_i - E_m + i\hbar\eta});\ \Delta_i = \Delta_i^{(1)} + \Delta_i^{(2)} + \cdots$
$\implies Im(\Delta_i) \approx Im(\Delta_i^{(2)}) \approx -\pi \sum_{m\neq i} |V_{mi}|^2 \delta(E_i - E_m)$
recall: $\mathcal{W}_{i\to n} \approx \frac{2\pi}{\hbar}|V_{ni}|^2 \delta(E_n - E_i)$
$\implies \sum_{m\neq i} \mathcal{W}_{i\to n} = \frac{2\pi}{\hbar} \sum_{m\neq i} |V_{mi}|^2 \delta(E_i - E_m) = -\frac{2}{\hbar}Im(\Delta_i^{(2)})$
$\therefore |c_i|^2 = e^{2Im(\Delta_i)t/\hbar}$
$\implies \frac{\Gamma_i}{\hbar} \equiv -\frac{2}{\hbar}Im(\Delta_i)$
This also motivates defining the lifetime as $\frac{\hbar}{\Gamma_i} = \tau_i$

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(915) $\underline{f(E) = \frac{\psi(0)}{E - [E_i + Re(\Delta_i)] + i\Gamma/2}}$
Note: $|\psi\rangle = e^{-t/2\tau_i} e^{-iRe(\Delta_i)t/\hbar} e^{-iE_i t/\hbar}|i\rangle; \tau_i = \frac{\hbar}{\Gamma_i}$
$\implies \psi(t) = e^{-\Gamma_i t/2\hbar} e^{-iRe(\Delta_i)t/\hbar} e^{-iE_i t/\hbar}\psi(0)$
$\implies f(E) = \int dt\psi(t)e^{iEt/\hbar} = \int_0^\infty dte^{[i(E - [E_i + Re(\Delta_i)]) - \Gamma_i/2]t/\hbar}\psi(0)$
$= \frac{-\psi(0)}{i(E - [E_i + Re(\Delta_i)]) - \Gamma_i/2}$
$= \frac{\psi(0)}{(E - [E_i + Re(\Delta_i)]) + i\Gamma_i/2}$
$\therefore f(E) = \frac{\psi(0)}{(E - [E_i + Re(\Delta_i)]) + i\Gamma_i/2}$ (Breit-Wigner formula)
$f(E)$ is like the wave function in energy space, the probability that a particle will have energy E is $|f(E)|^2$ (I don't understand why there is only a half Fourier transform and also why it is missing the pre-factor)

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(916) $\sigma = \frac{Number of scattering events}{\rho_A \ell_A \rho_B \ell_B A}$

σ tells us how often particles collide.
Assume we have a fixed bunch of particles A which ρ_A, ℓ_A; and B with ρ_B, ℓ_B. They share a common area A.
It makes sense that the number of scattering events is poroportional to all of these quantties
$\implies N \propto \rho_B \ell_A \rho_B \ell_B A$
the constant of proportionality is σ
$\therefore \sigma = \frac{N}{\rho_A \ell_A \rho_B \ell_B A}$

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Note: the cross section tells us how often a specific process occurs, e.g. $e^+e^- \to \mu^+\mu^-$
$\sigma_{tot} = \sigma_{\mu^+\mu^-} + \sigma_{\tau^+\tau^-} + \cdots ?$

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Note: we often care about not only the process but all the momentum of outgoing particle so $\sigma \to d\sigma$ (since we care about a very narrow range of momentum) so to make it finite we divide by $d^3p_1 \cdots d^3p_n$
$\implies \sigma \to \frac{d\sigma}{d^3p_1 \cdots d^3p_n}$
integrate over $d^3p_1 \cdots d^3p_n$ to obtain how often a particle process happens with a certain range of outgoing momenta.

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(917) $\frac{d\sigma}{d^3p_1 d^3p_2} \to \frac{d\sigma}{d\Omega}$

(assuming 2 species of outgoing particles we can integrate over the 4 constrained momentum components)
Lets prove that there are only 2 unconstrained momentum components.
right now we have 6 unconstrained components
$p_A^\mu + p_B^\mu = p_1^\mu + p_2^\mu$
A fixed $\implies \vec{p}_1 + \vec{p}_2 = \vec{p}_B \to \vec{p}_1 = \vec{p}_B - \vec{p}_2$
$\implies$ 3 unconstrained components
$E_A + E_B = E_1 + E_2 = \sqrt{p_1^2 + m_1^2} + \sqrt{p_2^2 + m_2^2}$
$= \sqrt{p_1^2 + m_1^2} + \sqrt{(\vec{p}_B - \vec{p}_1)^2 + m_2^2}$
can solve for $|\vec{p}_1| \implies 3 \to 2$ unconstrained components
we usually choose to parametrize these components by ϕ, θ

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(918) $d\sigma = \frac{1}{2E_A 2E_B|v_A - v_B|}(\Pi_f \frac{d^3p_f}{(2\pi)^3}\frac{1}{2E_f}) \times |\mathcal{M}(p_A, p_B \to \{p_f\})|^2 (2\pi)^4 \delta^{(4)}(p_A + p_B - \sum p_f)$

(919) $|\phi_A\phi_B\rangle_{in} = \int \frac{d^3k_A}{(2\pi)^3\sqrt{2E_A}}\phi_A(\vec{k}_A)|\vec{k}_A\rangle \int \frac{d^3k_B}{(2\pi)^3\sqrt{2E_B}}\phi_B(\vec{k}_B)e^{-i\vec{b}\cdot\vec{k}_B}|\vec{k}_B\rangle$

$out\langle\phi_1\phi_2\cdots| = (\Pi_f)\frac{d^3p_f}{(2\pi)^3}\frac{\phi_f(\vec{p}_f)}{\sqrt{2E_f}}out\langle\vec{p}_1\vec{p}_2\cdots|$

recall: $\langle\vec{x}|\vec{p}\rangle \propto e^{i\vec{p}\cdot\vec{x}}$ (single free particle k state $\psi(x)$ from QM)
Note: $|\vec{k}\rangle$ is a simultaneous eigenfunction of the free hamiltonian and the momentum operator $\langle 0|\phi(\vec{x})|\vec{p}\rangle = e^{i\vec{p}\cdot\vec{x}}$ (allows us to interpret $\phi(\vec{x})|0\rangle$)
recall: $\phi(\vec{x}) = \int \frac{d^3k}{(2\pi)^3}e^{i\vec{k}\cdot\vec{x}}\phi(\vec{k})$
Note: $|\phi\rangle \equiv \phi(x)|\vec{k}\rangle$ and not $|\phi\rangle = \phi(x)|0\rangle$; this makes sense because in order to obtain the wave function of a free field just hit it with $\langle 0|$
$\phi(\vec{x})$ is basically a particles field at $\vec{x}$ so $\phi(0) = \int \frac{d^3k}{(2\pi)^3}\phi(\vec{k})$ is a particle's wave function evaluated at 0, also $\phi(\vec{x})$ as written from the fourier transform is centered at $\vec{x}' = 0$
Note: $\int \frac{d^3k}{(2\pi)^3}|\phi(\vec{k})|^2 = 1 \implies \langle\phi|\phi\rangle = 1$
$\mathcal{P} = |\langle\phi_1\phi_2\cdots|\phi_A\phi_B\rangle|^2$ 2 species initially (in) and n species finally (out)
$|\phi_A\phi_B\rangle_{in} = |\phi_A\rangle|\phi_B\rangle = \int \frac{d^3k_A}{(2\pi)^3\sqrt{2E_A}}\phi_A(\vec{k}_A)|\vec{k}_A\rangle \int \frac{d^3k_B}{(2\pi)^3\sqrt{2E_B}}\phi_B(\vec{k}_B)e^{-i\vec{b}\cdot\vec{k}_B}|\vec{k}_B\rangle$
Note: the previous line we evaluated $|\phi_A\rangle at\vec{x} = 0$, (it seems like we only care about the transverse direction for some reason??) however, for $|\phi_B\rangle$ we still evaluate it at $\vec{x} = 0$ but we had to shift it, that is $\phi(x) \to \phi'(x) = \phi(x - b)$ so for $x = 0$ we have $\phi(-b)$ which explains the phase factor
$= \int \frac{d^3k_A}{(2\pi)^3}\int \frac{d^3k_B}{(2\pi)^3}\frac{\phi_A(\vec{k}_A)\phi_B(\vec{k}_B)e^{-i\vec{b}\cdot\vec{k}_B}}{\sqrt{(2E_A)(2E_B)}}|\vec{k}_A\vec{k}_B\rangle$
We could write
$out\langle\phi_1\phi_2\cdots| = (\Pi_f)\frac{d^3p_f}{(2\pi)^3}\frac{\phi_f(\vec{p}_f)}{\sqrt{2E_f}}out\langle\vec{p}_1\vec{p}_2\cdots|$

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(920) $\mathcal{P}(AB \to 1\,2\cdots n) = \Pi_f \frac{d^3p_f}{(2\pi)^3}\frac{1}{2E_f}|out\langle p_1p_2\cdots|\phi_A\phi_B\rangle|^2$

recall: $|\vec{p}(t)\rangle = e^{iHt}|\vec{p}\rangle$
$|\vec{k}\rangle \sim -T$, $|\vec{p}\rangle \sim -T$
$\implies$ $\text{out}\langle\vec{p}_1\vec{p}_2\cdots|\vec{k}_A\vec{k}_B\rangle_{in} = \lim_{T\to\infty}\langle\vec{p}_1\vec{p}_2\cdots|\vec{k}_A\vec{k}_B\rangle = \lim_{T\to\infty}\langle\vec{p}_1\vec{p}_2\cdots|e^{-iH(2T)}|\vec{k}_A s\vec{k}_B\rangle$
$\implies out\langle\vec{p}_1\vec{p}_2\cdots|\vec{k}_A\vec{k}_B\rangle_{in} \equiv \langle\vec{p}_1\vec{p}_2|S|\vec{k}_A\vec{k}_B\rangle$
$S = 1 + iT$
define $\langle\vec{p}_1\vec{p}_2\cdots|iT|\vec{k}_A\vec{k}_B\rangle = (2\pi)^4\delta^{(4)}(k_A + k_B - \sum p_f)\cdot iM(k_A, k_B \to p_f)$

$M \sim f$ (f from QM)
recall: $out_{\langle}\phi_1\phi_2\cdots| = (\Pi_f \int \frac{d^3p_f}{(2\pi)^3}\frac{\phi_f(\vec{p}_f)}{\sqrt{2E_f}} out_{\langle} p_1p_2\cdots|$
$\mathcal{P}(AB \to 1\,2\cdots n) = |\langle\phi_1\phi_2\cdots|\phi_A\phi_B\rangle|^2$
$= |\Pi_f \int \frac{d^3p_f}{(2\pi)^3}\frac{\phi_f(\vec{p}_f)}{2E_f} out_{\langle} p_1p_2\cdots|\phi_A\phi_B\rangle_{in}|^2$
$= \Pi_f \int \frac{d^3p_f}{(2\pi)^3(2E_f)} \Pi_{f'}\left[\int \frac{d^3p_{f'}|\phi_{f'}(p_{f'})|^2}{(2\pi)^3(2E_f)}\right] |\langle p_1\cdots|\phi_A\phi_B\rangle|^2$
(the last line requires just a little bit of thought)
$= \Pi_f \frac{d^3p_f}{(2\pi)^3}\frac{1}{2E_f}|out_{\langle} p_1p_2\cdots|\phi_A\phi_B\rangle|^2$
I think the reason $\phi_f(\vec{p}_f)$ disappears is due to $\int \frac{d^3k}{(2\pi)^3}|\phi(\vec{k})|^2 = 1$

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(921) $d\sigma = (\Pi_f \frac{d^3p_f}{(2\pi)^3}\frac{1}{2E_f})(\Pi_{i=A,B}\int \frac{d^3k_i}{(2\pi)^3}\frac{\phi_i(\vec{k}_i)}{\sqrt{2E_i}}\int \frac{d^3\bar{k}_i}{(2\pi)^3}\frac{\phi_i^*(\vec{\bar{k}}_i)}{\sqrt{2\bar{E}_i}})(2\pi)^2\delta^{(2)}(k_B^\perp - \bar{k}_B^\perp)$

$\left[i\mathcal{M}(\{k_i\}\to\{p_f\})(2\pi)^4\delta^{(4)}(\sum k_i - \sum p_f)\right]$
$\left[-i\mathcal{M}^*(\{\bar{k}_i\}\to\{p_f\})(2\pi)^4\delta^{(4)}(\sum \bar{k}_i - \sum p_f)\right]$
single A many B
$N = \sum_{\text{all inident particles i}} \mathcal{P}_i = \int d^2b n_B \mathcal{P}(\vec{b})$
$\vec{b} \sim$ impact parameter, $n_B \sim$ particles/unit area
n_B constant roughly (over the effects of ϕ_A)
$\implies \sigma = \frac{\text{Number of scatter events}}{\rho_A\ell_A\rho_B\ell_B A} = \frac{N}{n_B N_A} = \frac{N}{n_B\cdot 1} = \int d^2b\mathcal{P}(\vec{b})$
$\implies d\sigma = \int d^2b|\langle\phi_1\phi_2\cdots|\phi_A\phi_B\rangle|^2$
$= \int d^2b(\Pi_f \frac{d^3p_f}{(2\pi)^3}\frac{1}{2E_f})|out_{\langle} p_1\cdots p_n|\phi_A\phi_B\rangle_{in}|^2$
$= \int d^2b(\Pi_f \frac{d^3p_f}{(2\pi)^3}\frac{1}{2E_f})\langle p_1\cdots p_n|\phi_A\phi_B\rangle\langle\phi_A\phi_B|p_1\cdots p_n\rangle$
$= \int d^2b(\Pi_f \frac{d^3p_f}{(2\pi)^3}\frac{1}{2E_f})(\Pi_{i=A,B}\int \frac{d^3k_i}{(2\pi)^3}\frac{\phi_i(\vec{k}_i)}{\sqrt{2E_i}}\int \frac{d^3\bar{k}_i}{(2\pi)^3}\frac{\phi_i^*(\bar{k}_i)}{\sqrt{2\bar{E}_i}})$
$\times e^{i\vec{b}\cdot(\vec{\bar{k}}_B - \vec{k}_B)}(out_{\langle}\{\vec{p}_f\}|\{\vec{k}_i\}\rangle_{in})(out_{\langle}\{\vec{p}_f\}|\{\bar{k}_i\}\rangle_{in})^*$
where $\langle\vec{p}_1\cdots\vec{p}_n|\vec{k}_A\vec{k}_B\rangle_{in} \equiv \langle\{\vec{p}\}_f|\{\vec{k}_i\}\rangle_{in}$?
$\int d^2be^{i\vec{b}\cdot(\vec{\bar{k}}_B - \vec{k}_B)} = (2\pi)^2\delta^{(2)}(\vec{k}_B^\perp - \vec{\bar{k}}_B^\perp)$
$(out_{\langle}\{p_f\}|\{k_i\}\rangle_{in}) = i\mathcal{M}(\{k_i\}\to\{p_f\})(2\pi)^4\delta^{(4)}(\sum_i k_i - \sum_f p_f)$
$(out_{\langle}\{p_f\}|\{\bar{k}_i\}\rangle_{in})^* = -i\mathcal{M}^*(\{\bar{k}_i\}\to\{p_f\})(2\pi)^4\delta^{(4)}(\sum \bar{k}_i - \sum p_f)$
x, y transverse, z parallel to beam
$\implies d\sigma = (\Pi_f \frac{d^3p_f}{(2\pi)^3}\frac{1}{2E_f})(\Pi_{i=A,B}\int \frac{d^3k_i}{(2\pi)^3}\frac{\phi_i(\vec{k}_i)}{\sqrt{2E_i}}\int \frac{d^3\bar{k}_i}{(2\pi)^3}\frac{\phi_i^*(\vec{\bar{k}}_i)}{\sqrt{2\bar{E}_i}})(2\pi)^2\delta^{(2)}(k_B^\perp -$
$\bar{k}_B^\perp)\left[i\mathcal{M}(\{k_i\}\to\{p_f\})(2\pi)^4\delta^{(4)}(\sum k_i - \sum p_f)\right]$
$\left[-i\mathcal{M}^*(\{\bar{k}_i\}\to\{p_f\})(2\pi)^4\delta^{(4)}(\sum \bar{k}_i - \sum p_f)\right]$

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(922) $$\frac{\int d^3\bar{k}_A\phi_A^*(\bar{k}_A)\int d^3\bar{k}_B\phi_B^*(\bar{k}_b)\delta^{(4)}(\sum_i\bar{k}_i-\sum p_f)\delta^{(2)}(k_B^\perp-\bar{k}_B^\perp)\delta^{(4)}(\sum_i k_i-\sum p_f)}{=\delta^{(4)}(\sum_i k_i-\sum_f p_f)\phi_A^*(k_A)\phi_B^*(k_B)\frac{1}{|\frac{k_A^z}{E_A}-\frac{k_B^z}{E_B}|}}$$

$$\overline{\int d^3\bar{k}_A\phi_A^*(\bar{k}_A)\int d^3\bar{k}_B\phi_B^*(\bar{k}_b)\delta^{(4)}(\sum_i\bar{k}_i-\sum p_f)\delta^{(2)}(k_B^\perp-\bar{k}_B^\perp)\delta^{(4)}(\sum_i k_i-\sum p_f)}$$

$=\int d^3\bar{k}_A\phi_A^*(\bar{k}_A)\int d\bar{k}_B^z\phi_B^*(k_B^\perp,\bar{k}_B^z)\delta^{(4)}(\sum_i\bar{k}_i-\sum p_f)\delta^{(4)}(\sum_i k_i-\sum p_f)$

$\underline{recall\ :}\delta(x-y)\delta(z-y)=\delta(x-y)\delta(z-x)$

$x=\sum_i k_i;\ y=\sum_f p_f;\ z=\sum_i\bar{k}_i$

$\implies\delta^{(4)}(\sum_i\bar{k}_i-\sum p_f)\delta^{(4)}(\sum_i k_i-\sum p_f)$

$=\delta^{(4)}(\sum_i k_i-\sum_f p_f)\delta^{(4)}(\sum_i\bar{k}_i-\sum_i k_i)$

insert into previous

$\implies\delta^{(4)}(\sum_i k_i-\sum_f p_f)$

$\int d^3\bar{k}_A\phi(\bar{k}_A)\int d\bar{k}_B^z\phi_B^*(k_B^\perp,\bar{k}_B^z)$

$\delta^{(2)}(\bar{k}_A^\perp+k_B^\perp-k_A^\perp-k_B^\perp)\delta(\bar{k}_A^z+\bar{k}_B^z-k_A^z-k_B^z)\delta(\bar{E}_A+\bar{E}_B-E_A-E_B)$

$=\delta^{(4)}(\sum_i k_i-\sum_F p_f)$

$\int d\bar{k}_A^z\phi_A^*(k_A^\perp,\bar{k}_A^z)\int d\bar{k}_B^z\phi^*(k_B^\perp,\bar{k}_B^z)$

$\delta(\bar{k}_A^z+\bar{k}_B^z-k_A^z-k_B^z)\delta(\bar{E}_A+\bar{E}_B-E_A-E_B)$

$=\delta^{(4)}(\sum_i k_i-\sum_f p_f)\int d\bar{k}_A^z\phi_A^*(k_A^\perp,\bar{k}_A^z\phi_B^*(k_B^\perp,\bar{k}_B^z)_{\bar{k}_B^z=\sim}$

$\delta(\sqrt{k_A^{\perp 2}+m_A^2+\bar{k}_A^{z2}}+\sqrt{k_B^{\perp 2}+(k_A^z-\bar{k}_A^z+k_B^z)^2+m_B^2}-E_A-E_B)$

$\delta(f(\bar{k}_A^z))$ has a zero where $\bar{k}_A^z=k_A^z\implies\bar{k}_B^z=k_B^z$

$=\delta^{(4)}(\sum_i k_i-\sum_f p_f)\phi_A^*(k_A)\phi_B^*(k_B)\frac{1}{|\frac{k_A^z}{E_A}-\frac{k_B^z}{E_B}|}$

Note: We used $\int dx\delta(f(x))\phi(x)=\sum_j\frac{1}{|f'(x_j)|}\phi(x_j)$

and $\frac{\bar{k}_A^z}{\bar{E}_A}-\frac{\bar{k}_B^z}{\bar{E}_B}=\frac{k_A^z}{E_A}-\frac{k_B^z}{E_B}$ due to all of the constraints.

$\frac{1}{|\frac{\bar{k}_A^z}{\bar{E}_A}-\frac{\bar{k}_B^z}{\bar{E}_B}|}\equiv\frac{1}{|v_A-v_B|}$

$\frac{\bar{k}_A^z}{\bar{E}_A}\sim\frac{p}{E}=\frac{\gamma mv}{\gamma m}=v$

initial wave packets highly localized around $\vec{p}_A,\vec{p}_B$ which I believe allows us to pull the amplitude out of the integral $\implies$

$d\sigma=(\Pi_f\frac{d^3p_f}{(2\pi)^3}\frac{1}{2E_f})(\Pi_{i=A,B}\int\frac{d^3k_i}{(2\pi)^3}\frac{\phi_i(\vec{k}_i)}{\sqrt{2E_i}}\int\frac{d^3\bar{k}_i}{(2\pi)^3}\frac{\phi_i^*(\vec{\bar{k}}_i)}{\sqrt{2\bar{E}_i}})(2\pi)^2\delta^{(2)}(k_B^\perp-\bar{k}_B^\perp)\left[i\mathcal{M}(\{k_i\}\to\{p_f\})(2\pi)^4\delta^{(4)}(\sum k_i-\sum p_f)\right]$

$\left[-i\mathcal{M}^*(\{\bar{k}_i\}\to\{p_f\})(2\pi)^4\delta^{(4)}(\sum\bar{k}_i-\sum p_f)\right]$

$=(\Pi_f\frac{d^3p_f}{(2\pi)^3}\frac{1}{2E_f})\frac{|\mathcal{M}(p_A,p_B\to\{p_f\})|^2}{2E_A2E_B|v_A-v_B|}\int\frac{d^3k_A}{(2\pi)^3}\int\frac{d^3k_B}{(2\pi)^3}$

$\times|\phi_A(\vec{k}_A)|^2|\phi_B(\vec{k}_B)|^2(2\pi)^4\delta^{(4)}(k_A+k_B-\sum p_f)$

this is main result we just need to go to the center of mass frame

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(923) $\underline{\text{improve this part}}\int d\bar{k}^z_A d\bar{k}^z_B \delta(\bar{k}^z_A+\bar{k}^z_B-\sum p^z_f)\delta(\bar{E}_A+\bar{E}_B-\sum E_f)$
$=\int d\bar{k}^z_A\delta(\bar{E}_A+\bar{E}_B-\sum E_f)|_{\bar{k}^z_B=\sum p^z_f-\bar{k}^z_A}$
$=\int d\bar{k}^z_A\delta(\sqrt{\bar{k}^2_A+m^2_A}+\sqrt{\bar{k}^2_B+m^2_B}-\sum E_f)|_{\bar{k}^z_B=\sum p^z_f-\bar{k}^z_A}$
$=\frac{1}{|\frac{\bar{k}^z_A}{\bar{E}_A}-\frac{\bar{k}^z_B}{\bar{E}_B}|}\equiv\frac{1}{|v_A-v_B|}(???)$
$\frac{\bar{k}^z_A}{\bar{E}_A}\sim\frac{p}{E}=\frac{\gamma m v}{\gamma m}=v$
initial wave packets highly localized around $\vec{p}_A,\vec{p}_B$
needs more steps here, the fact that one integral pops up has to deal with the factor of e^{-ibk} and the integral turning into a delta function
$\implies d\sigma=(\Pi_f\frac{d^3p_f}{(2\pi)^3}\frac{1}{2E_f})\frac{|\mathcal{M}(p_A,p_B\to\{p_f\})|^2}{2E_A2E_B|v_A-v_B|}\int\frac{d^3k_A}{(2\pi)^3}\int\frac{d^3k_B}{(2\pi)^3}$
$\times|\phi_A(\vec{k}_A)|^2|\phi_B(\vec{k}_B)|^2(2\pi)^4\delta^{(4)}(k_A+k_B-\sum p_f)$
this is main result we just need to go to the center of mass frame

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(924) $\underline{(\frac{d\sigma}{d\Omega})_{CM}=\frac{1}{2E_A2E_B|v_A-v_B|}\frac{|\vec{p}_1|}{(2\pi)^4 4E_{CM}}|\mathcal{M}(p_A,p_B\to p_1,p_2)|^2}$
$k_A+k_B\sim p_A+p_B\implies$ take δ out of integral
and we have
$(\int\frac{d^3k_A}{(2\pi)^3}|\phi_A(\vec{k}_A)|^2)(\int\frac{d^3k_B}{(2\pi)^3}|\phi_B(\vec{k}_B)|^2)\approx 1$
$\implies d\sigma=\frac{1}{2E_A2E_B|v_A-v_B|}(\Pi_f\frac{d^3p_f}{(2\pi)^3}\frac{1}{2E_f})$
$\times\mathcal{M}(p_a,p_B\to\{p_f\})|^2(2\pi)^4\delta^{(4)}(p_A+p_B-\sum p_f)$
everything except $\frac{1}{2E_A2E_B|v_A-v_B|}$ is Lorentz invariant
$E_AE_B|v_A-v_B|=|E_Av_AE_B-E_AE_Bv_B|=|p^z_AE_B-E_Ap^z_B|$
since $E_Av_a=\gamma m_Av_A=p^z_A$ since v_A is in z direction
$\implies\frac{1}{E_AE_B|v_A-v_B|}=\frac{1}{|E_Bp^z_A-E_Ap^z_B|}=\frac{1}{|\epsilon_{\mu xy\nu}p^\mu_Ap^\nu_B|}$
$\int d\Pi_n=(\Pi_f\int\frac{d^3p_f}{(2\pi)^3}\frac{1}{2E_f})(2\pi)^4\delta^{(4)}(p_A+p_B-p_1-p_2)$
$=(\int\frac{d^3p_1}{(2\pi)^3}\frac{1}{2E_1})(\sum\frac{d^3p_2}{(2\pi)^3}\frac{1}{2E_2}(2\pi)^4\delta^{(4)}(p_A+p_B-p_1-p_2))$
$=\int\frac{d^3p_1}{(2\pi)^32E_12E_2}(2\pi)\delta(E_A+E_B-p_1-p_2)$
COM, $E_{CM}=E_A+E_B;\ \vec{p}_2=-\vec{p}_1$
$\implies\int\frac{dp_1p_1^2d\Omega}{(2\pi)^32E_12E_2}(2\pi)\delta(E_{cm}-E_1-E_2)$
$\underline{\text{recall:}}\int dx\delta[f(x)]\phi(x)=\frac{1}{|f'(x)|}\phi(x)$
$\implies\int d\Pi_2=\int\frac{d\Omega}{(2\pi)^32E_12E_2}(2\pi)\int dp_1p_1^2\delta(E_{cm}-\sqrt{p_1^2+m_1^2}-\sqrt{p_1^2+m_2^2})$
$\phi(p_1)=p_1^2,\ f(p_1)=E_{cm}-\sqrt{p_1^2+m_1^2}-\sqrt{p_1^2+m_2^2}$

$\implies f'(p_1) = -\frac{p_1}{E_1} - \frac{p_1}{E_2}$
$\implies \int d\Pi_2 = \int \frac{d\Omega}{(2\pi)^3 2E_1 2E_2}(2\pi)(p_1^2|(\frac{p_1}{E_1} + \frac{p_1}{E_2})^{-1}|)$
$= \int d\Omega \frac{1}{16\pi^2}\frac{1}{E_1E_2}p_1^2|(\frac{E_2p_1+p_1E_1}{E_1E_2})^{-1}|$
$= \int d\Omega \frac{1}{16\pi^2}|\frac{1}{E_2p_1+p_1E_1}|p_1^2$
$= \int d\Omega \frac{1}{16\pi^2}\frac{1}{E_{CM}}|\vec{p}_1|$
Notice $\int d\Omega$ cant be performed, why?
integrate out ϕ, since we can assume reaction takes place in a plane.
$\implies \int d\Pi_2 = \int d\cos\theta \frac{1}{16\pi}\frac{2|\vec{p}_1|}{E_{CM}}$ (not important right now)
$\implies d\sigma = \frac{1}{2E_A 2E_B|v_A - v_B|}\frac{d\Omega|\vec{p}_1|}{(2\pi)^2 4E_{CM}}|\mathcal{M}(p_A, p_B \to p_1, p_2)|^2$
$\therefore (\frac{d\sigma}{d\Omega})_{CM} = \frac{1}{2E_A 2E_B|v_A - v_B|}\frac{|\vec{p}_1|}{(2\pi)^2 4E_{CM}}|\mathcal{M}(p_A, p_B \to p_1, p_2)|^2$

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(925) $\underline{(\frac{d\sigma}{d\Omega})_{CM} = \frac{|M|^2}{64\pi^2 E_{cm}^2}}$ (all four masses equal)
recall: $(\frac{d\sigma}{d\Omega})_{CM} = \frac{1}{2E_A 2E_B|v_A - v_B|}\frac{|\vec{p}_1|}{(2\pi)^2 4E_{CM}}|\mathcal{M}(p_A, p_B \to p_1, p_2)|^2$
$m \to 0$, all four masses equal
$|v_A - v_B| = 2,\ E_A = E_B = \frac{1}{2}E_{CM};\ |\vec{p}_1| = \gamma m = E_1 = \frac{E_{CM}}{2}$
$\implies (\frac{d\sigma}{d\Omega})_{CM} = \frac{1}{2E_{CM}^2}\frac{E_{CM}}{2(2\pi)^2 4E_{CM}}|M|^2$
$= \frac{|M|^2}{64\pi^2 E_{CM}^2}$

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(926) $\underline{\int d\bar{k}_A^z \delta(\sqrt{\bar{k}_A^2 + m_A^2} + \sqrt{\bar{k}_B^2 + m_B^2} - \sum E_f)|_{\bar{k}_B^z = \sum p_f^z - \bar{k}_A^z} = \frac{1}{|v_A - v_B|}}$
recall: $\int dx \delta[f(x)]\phi(x) = \sum_j \frac{1}{|f'(x_j)|}\phi(x_j)$ x_j are roots of the function f $f(k_A^z) = \sqrt{\bar{k}_A^2 + m_A^2} + \sqrt{\bar{k}_B^2 + m_B^2} - \sum E_f$
$= \sqrt{(\bar{k}_A^\perp)^2 + (\bar{k}_A^z)^2 + m_A^2} + \sqrt{(\sum p_f^z - k_A^z)^2 + (\bar{k}_B^\perp)^2 + m_B^2} - \sum E_f$
$\implies f'(k_A^z) = \frac{\bar{k}_A^z}{\sqrt{\bar{k}_A^2 + m_A^2}} - \frac{(\sum p_f^z - k_A^z)}{\sqrt{\bar{k}_B^2 + m_B^2}}$
$= \frac{\bar{k}_A^z}{E_A} - \frac{\bar{k}_B^z}{E_B}$
$\frac{\bar{k}_A^z}{E_A} \sim \frac{p}{E} \sim \frac{\gamma m v}{\gamma m} \sim v$
$= v_A - v_B$
$\therefore \int d\bar{k}_A^z \delta(\sqrt{\bar{k}_A^2 + m_A^2} + \sqrt{\bar{k}_B^2 + m_B^2} - \sum E_f)|_{\bar{k}_B^z = \sum p_f^z - \bar{k}_A^z}$
$= \frac{1}{|v_A - v_B|}$

Note: $\Gamma = \frac{\text{Number of decays per unit time}}{\text{Number of A particles present}}$
We assume A is at rest
understand $\frac{d\sigma}{d\Omega} = |f|^2$ next

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(927) $d\Gamma = \frac{1}{2m_A}(\Pi_f \frac{d^3p_f}{(2\pi)^3}\frac{1}{2E_f})(\mathcal{M}(m_A \to \{p_f\})|^2(2\pi)^4\delta^{(4)}(p_A - \sum p_f)$

recall: $\frac{1}{2E_A 2E_B|v_A - v_B|}(\Pi_f \frac{d^3p_f}{(2\pi)^3}\frac{1}{2E_f})$
$\times |\mathcal{M}(p_A, p_B \to \{p_f\}|^2(2\pi)^4\delta^{(4)}(p_A + p_B - \sum p_f) = d\sigma$
remove factors that don't make sense for a one body decay
$\implies d\Gamma = \frac{1}{2m_A}(\Pi_f \frac{d^3p_f}{(2\pi)^3}\frac{1}{2E_f}) \times |\mathcal{M}(m_A \to \{p_f\})|^2(2\pi)^4\delta^{(4)}(p_A - \sum p_f)$

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(928) $\hat{U}_I(t', t) = T\exp(-i\int_t^{t'} d\tau H^I_{int}(\tau))$

Note: $e^{iH_0\Delta}e^{-iH\Delta} = e^{i(H_0 - H)\Delta + O(\Delta^2)} \approx e^{-iH_I(t)\Delta}$
$\hat{U}_I(t', t) = \hat{U}(t', t' - \Delta)\hat{U}(t' - \Delta, t' - 2\Delta)\cdots\hat{U}(t + \Delta, t)$
$= e^{-iH^I_{int}(t'-\Delta)\Delta} \cdot e^{-iH^I_{int}(t'-2\Delta)\Delta} \cdots e^{-iH^I_{int}(t)\Delta}$
$= Te^{-iH^I_{int}(t'-\Delta)\Delta} \cdot e^{-iH^I_{int}(t'-2\Delta)\Delta} \cdots e^{-iH^I_{int}(t)\Delta}$
$= T^{-iiH^I_{int}(t'-\Delta)\Delta - iH^I_{int}(t'-2\Delta)\Delta - \cdots - iH^I_{int}(t)\Delta}$
$= T\exp\left(-i\int_t^{t'} d\tau H^I_{int}(\tau)\right)$

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(929) $U_I(t + \Delta, t) = e^{-iH^I_{int}(t)\Delta}$

recall: $U_I(t', t) = e^{iH_0 t'}e^{-iH(t'-t)}e^{-iH_0 t}$
$\implies U_I(t + \Delta, t) = e^{iH_0(t+\Delta)}e^{-iH(\Delta}e^{-iH_0 t}$
$= e^{iH_0 t}e^{iH_0\Delta}e^{-iH\Delta}e^{-iH_0 t} = e^{iH_0 t}e^{-iH_I(t)\Delta}e^{-iH_0 t}$
expand $e^{-iH\Delta}$ and Note $e^{iH_0}H^n e^{-iH_0 t} = e^{iH_0 t}He^{-iH_0 t}e^{iH_0 t}H\cdots$
$= H_I^n$
$\implies U_I(t + \Delta, t) = e^{-iH^I_{int}(t)\Delta}$

$\mathcal{L}_I = -\frac{\lambda}{4!}\phi^4 \implies \hat{\mathcal{H}}_I = \frac{\lambda}{4!}\hat{\phi}(x)^4$

$\implies \hat{S} = T[\exp(-i\int d^4z\hat{\mathcal{H}}_I(z))]$
$= T[1 - i\int d^4z\hat{\mathcal{H}}_I(z) + \frac{(-i)^2}{2}\int d^4y d^4w\hat{\mathcal{H}}_I(y)\hat{\mathcal{H}}_I(w) + \cdots]$
$= T[1 - \frac{i\lambda}{4!}\int d^4z\hat{\phi}(z)^4 + \frac{(-i)^2}{2!}(\frac{\lambda}{4!})^2\int d^4y d^4w\hat{\phi}(y)^4\hat{\phi}(w)^4 + \cdots]$

$A = \langle q|\hat{S}|p\rangle = (2\pi)^3(2E_q)^{1/2}(2E_p)^{1/2}T[\langle 0|\hat{a}_q\hat{a}^\dagger_p|0\rangle + (\frac{-i\lambda}{4})\int d^4\langle 0|\hat{a}_q\hat{\phi}(z)^4\hat{a}^\dagger_p|0\rangle + \frac{(-i)^2}{2!}(\frac{\lambda}{4!})^2\int d^4y d^y w\hat{\phi}(y)^4\hat{\phi}(w)^4 + \cdots]$

---

(930) $A^{(1)} = -\frac{i\lambda}{4!}\int d^4z[3\langle 0|\hat{a}_q\hat{a}^\dagger_p|0\rangle\langle 0|\hat{\phi}(z)\hat{\phi}(z)|0\rangle\langle 0|\hat{\phi}(z)\hat{\phi}(z)|0\rangle$
$+12\langle 0|\hat{a}_q\hat{\phi}(z)|0\rangle\langle 0|\hat{\phi}(z)\hat{\phi}(z)|0\rangle\langle 0|\hat{\phi}(z)\hat{a}^\dagger_p|0\rangle$

use wicks theorem on $T\langle 0|\hat{a}_q\hat{\phi}(z)^4\hat{a}_p^\dagger|0\rangle$

<u>recall:</u> $T[\hat{A}\hat{B}\cdots\hat{Z}] = N[\hat{A}\hat{B}\hat{C}\cdots\hat{Z}$ + all possible contractions of $\hat{A}\hat{B}\cdots\hat{Z}]$

<u>Note:</u> $\langle 0|N[\hat{a}_q\hat{\phi}(z)^4\hat{a}_p^\dagger]|0\rangle = 0$

there will be two types of contractions, first is kinds like

$\langle 0|a_q\phi\phi\phi\phi a_p^\dagger|0\rangle$

$= \langle n|\hat{a}_q\hat{a}_p^\dagger|0\rangle\langle 0|\hat{\phi}\hat{\phi}|0\rangle\langle 0|\hat{\phi}\hat{\phi}|0\rangle$ there are 3 of these contractions since we can choose 3 other phi's to contract phi with, and there is only one way to contract the a's.

the other term is like

$\langle 0|a_q\phi\phi\phi\phi a_p^\dagger|0\rangle = \langle 0|a_q\phi|0\rangle\langle 0|\phi\phi|0\rangle\langle 0|\phi a_p^\dagger|0\rangle$

does $\langle 0|N[a_q a_p^\dagger]|0\rangle = 0$?

there are 12 ways this contraction can happen, four choices of ϕ to contract a_q with and 3 choices of ϕ to contract $a_p^\dagger$ with.

$\implies A^{(1)} = -\frac{i\lambda}{4!}\int d^4z[3\langle 0|a_q a_p^\dagger|0\rangle\langle 0|\phi\phi|0\rangle\langle 0|\phi\phi|0\rangle + 12\langle 0|a_q\phi|0\rangle\langle 0|\phi\phi|0\rangle\langle 0|\phi a_p^\dagger|0\rangle]$

<u>Note:</u> $A = A^{(0)} + A^{(1)} + A^{(2)} + \cdots$

<u>Note:</u> $\phi(y)\phi(z) = \langle 0|T\hat{\phi}(y)\hat{\phi}^\dagger(z)|0\rangle = \Delta(y-z)$

<u>Note:</u> $\hat{\phi}^\dagger = \hat{\phi}$, $\hat{\phi}$ is real here

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(931) <u>create Feynman diagrams for $A^{(1)}$</u>

Lets Analyze Feynman diagrams for $A^{(1)} = -\frac{i\lambda}{4!}\int d^4z[3\langle 0|a_q a_p^\dagger|0\rangle\langle 0|\phi\phi|0\rangle\langle 0|\phi\phi|0\rangle + 12\langle 0|a_q\phi|0\rangle\langle 0|\phi\phi|0\rangle\langle 0|\phi a_p^\dagger|0\rangle]$

---

(932) $\langle 0|T\hat{\phi}(x)\hat{\phi}^\dagger(y)|0\rangle \equiv \Delta(x,y) = \int\frac{d^3p}{(2\pi)^3(2E_p)}[\theta(x^0-y^0)e^{-ip\cdot(x-y)} + \theta(y^0-x^0)e^{ip\cdot(x-y)}]$

<u>recall:</u> $\Delta(x,y) = \langle 0|T\hat{\phi}(x)\hat{\phi}^\dagger(y)|0\rangle = \theta(x^0-y^0)\langle 0|\phi(x)\phi^\dagger(y)|0\rangle + \theta(y^0-x^0)\langle 0|\phi^\dagger(y)\phi(x)|0\rangle$

<u>*recall :*</u>$\hat{\phi}(x) = \int\frac{d^3p}{(2\pi)^{3/2}}\frac{1}{(2E_p)^{1/2}}(\hat{a}_p e^{-ipx} + \hat{b}_p^\dagger e^{ipx})$

$\implies \hat{\phi}^\dagger(x)|0\rangle = \int\frac{d^3p}{(2\pi)^{3/2}(2E_p)^{1/2}}(\hat{a}_p^\dagger|0\rangle e^{ipyy} + \hat{b}_p|0\rangle e^{-ipx})$

$\hat{a}_p^\dagger|0\rangle = |p\rangle$

$\implies \hat{\phi}^\dagger(y)|0\rangle = \int\frac{d^3p}{(2\pi)^{3/2}(2E_p)^{1/2}}|\vec{p}\rangle e^{ipy}$

$\hat{\phi}^\dagger(y)|0\rangle \to \langle 0|\hat{\phi}(x) = \int\frac{d^3q}{(2\pi)^{3/2}(2E_q)^{1/2}}\langle q|e^{-iqz}$

$\implies \langle 0|\hat{\phi}(x)\hat{\phi}^\dagger(y)|0\rangle = \int\frac{d^3p\,d^3q}{(2\pi)^3(2E_p2E_q)^{1/2}}e^{-iqx+ipy}\delta^{(3)}(\vec{q}-\vec{p})$

$= \int\frac{d^3p}{(2\pi)^3(2E_p)}e^{-ip(x-y)}$

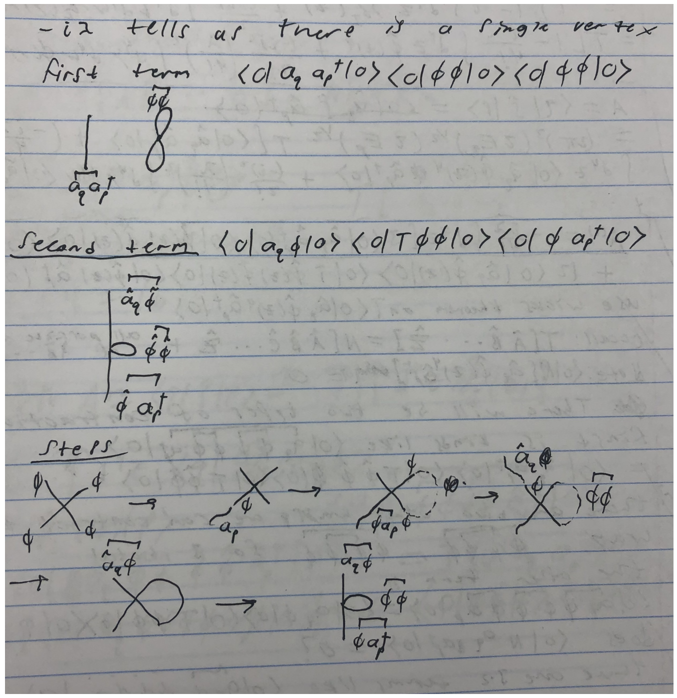

FIGURE 2. constructing Feynman diagrams for $A^{(1)}$

likewise $\langle 0|\hat{\phi}^\dagger(y)\hat{\phi}(x)|0\rangle = \int \frac{d^3p}{(2\pi)^3(2E_p)} e^{ip(x-y)}$

---

(933) $\underline{\Delta(x,y) = \int \frac{d^4p}{(2\pi)^4} e^{-ip(x-y)} \frac{i}{(p^0)^2 - E_p^2 + i\epsilon}}$

recall:$\theta(x^0 - y^0) = i\int_{-\infty}^{\infty} \frac{dz}{2\pi} \frac{e^{-iz(x^0-y^0)}}{z+i\epsilon}$

recall: $\Delta(x,y) = \langle 0|T\hat{\phi}(x)\hat{\phi}^\dagger(y)|0\rangle$

$= \int \frac{d^3p}{(2\pi)^3(2E_p)}[\theta(x^0-y^0)e^{-ip\cdot(x-y)} + \theta(y^0-x^0)e^{ip\cdot(x-y)}]$

$= [\Delta(x,y)]_{(1)} + [\Delta(x,y)]_{(2)}$

$[\Delta(x,y)]_{(1)} \equiv \theta(x^0-y^0)\langle 0|\hat{\phi}(x)\hat{\phi}^\dagger(y)|0\rangle$

$= \theta(x^0-y^0)\int \frac{d^3p}{(2\pi)^3(2E_p)} e^{-iE_p(x^0-y^0)+i\vec{p}\cdot(\vec{x}-\vec{y})}$(signature=2)

$= i\int_{-\infty}^{\infty} \frac{dz d^3p}{(2\pi)^4(2E_p)} \frac{e^{-i(E_p+z)(x^0-y^0)+i\vec{p}\cdot(\vec{x}-\vec{y})}}{z+i\epsilon}$

$z' = z + E_p \implies z = z' - E_p$

$= i\int_{-\infty}^{\infty} \frac{dz' d^3p}{(2\pi)^4(2E_p)} \frac{e^{-iz'(x^0-y^0)+i\vec{p}\cdot(\vec{x}-\vec{y})}}{z'-E_p+i\epsilon}$

$p = (z', \vec{p}) = (p^0, \vec{p})$

$\implies [\Delta(x,y)]_{(1)} = i\int_{-\infty}^{\infty} \frac{d^4p}{(2\pi)^4(2E_p)} \frac{e^{-ip\cdot(x-y)}}{p^0 - E_p + i\epsilon}$

$\underline{Note:} p^0 = E, but E \neq (p^2+m^2)^{1/2} \neq p_0$

but $E_p = (\vec{p}^2 + m^2)^{1/2}$
likewise,
$[\Delta(x,y)]_{(2)} = -i\int_{-\infty}^{\infty} \frac{d^4p}{(2\pi)^4(2E_p)} \frac{e^{-ip\cdot(x-y)}}{p^0+E_p-i\epsilon}$
$\implies \Delta(x,y) = i\int_{-\infty}^{\infty} \frac{d^4p}{(2\pi)^4(2E_p)} e^{-ip\cdot(x-y)}(\frac{1}{p^0-E_p+i\epsilon} - \frac{1}{p^0+E_p-i\epsilon})$
$= \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot(x-y)} \frac{i}{(p^0)^2-E_p^2+i\epsilon}$

Note: $\tilde{\Delta}(p) = \frac{i}{p^2-m^2+i\epsilon}$ (Fourier transform)

$G^{(2)} \sim 2$ pt propagator (what we've been dealing with
$G^{(4)} \sim 4$ pt propagator
$G^{(4)}(x_1,x_2,x_3,x_4) = \langle\Omega|T\hat{\phi}(x_1)\hat{\phi}(x_2)\hat{\phi}^\dagger(x_3)\hat{\phi}^\dagger(x_4)|\Omega\rangle$

---

(934) $\langle 0|\hat{\phi}(z)\hat{a}^\dagger_{\vec{p}}|0\rangle = \frac{1}{(2\pi)^{3/2}}\frac{1}{(2E_p)^{1/2}}\exp(-ip\cdot x)$

recall: $\hat{\phi}(x) = \int \frac{d^3p}{(2\pi)^{3/2}}\frac{1}{(2E_p)^{1/2}}(\hat{a}_p e^{-ipx} + a_p^\dagger e^{ip\cdot x})$
$\implies \langle 0|\int \frac{d^3q}{(2\pi)^{3/2}}\frac{1}{(2E_q)^{1/2}}(\hat{a}_q e^{-iq\cdot x} + \hat{a}_q^\dagger e^{iq\cdot x})\hat{a}_p^\dagger|0\rangle$
$= \int \frac{d^3q}{(2\pi)^{3/2}}\frac{1}{(2E_q)^{1/2}}\langle 0|(\hat{a}_q e^{-iq\cdot x} + \hat{a}_q^\dagger e^{iq\cdot x})|\vec{p}\rangle$
$= \int \frac{d^3q}{(2\pi)^{3/2}}\frac{1}{(2E_1)^{1/2}}\langle\vec{q}|\vec{p}\rangle e^{-iq\cdot x}$
$= \int \frac{d^3q}{(2\pi)^{3/2}}\frac{1}{(2E_q)^{1/2}}\delta^{(3)}(\vec{q}-\vec{p})e^{-iq\cdot x}$
$= \frac{1}{(2\pi)^{3/2}}\frac{1}{(2E_{\vec{q}})^{1/2}}e^{-ip\cdot x}$

Note: in QFTFGA, $|p\rangle = (2\pi)^{3/2}(2E_p)^{1/2}a_p^\dagger|0\rangle$

Note: ignore disconnected diagrams (don't fully understand)

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(935) $A_{12}^{(1)} = -\frac{12}{4!}i\lambda\int d^4z e^{-ip\cdot z}e^{iq\cdot z}\Delta(0)$

recall: $A_{12}^{(1)} = -\frac{i\lambda/2}{4!}\int d^4z\langle 0|\hat{a}_q\hat{\phi}(z)|0\rangle\langle 0|\hat{\phi}(z)\hat{\phi}(z)|0\rangle\langle 0|\hat{\phi}(z)\hat{a}_p^\dagger|0\rangle$
$= -\frac{12}{4!}i\lambda\int d^4z e^{-ip\cdot z}e^{iq\cdot z}$

Feynman rules in position space ϕ^4 theory
translate a Feynman diagram to an equation, as follows
1. each vertex contributes $-i\lambda\int d^4z$
2. each internal line contributes $\Delta(x-y)$ where x, y are locations of vertices
3. External lines: incoming $e^{-ip\cdot x}$; outgoing e^{ipx}

4. symmetry factors

(936) understand symmetry factors

(937) understand ignoring disconnected diagrams
disconnected diagrams contribute to the S-matrix but they dont contribute to the interesting part, generally a term in the s-matrix has a connected diagram that accounts for the disconnected part, but the disconnected part does contribute.

(938) obtain $A_{12}^{(1)}$ by using position space Feynman rules

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(939) $\underline{A_{12}^{(1)} = -\frac{i\lambda}{2}(2\pi)^4\delta^{(4)}(q-p)\int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2-m^2+i\epsilon}}$
recall: $A_{12}^{(1)} = -i\lambda\frac{12}{4!}\int d^4z e^{-ipz}e^{iqz}\int\frac{d^4k}{(2\pi)^4}\frac{i}{(k^0)^2-E_p^2+i\epsilon}$
$= -\frac{i\lambda}{2}(2\pi)^4\delta^{(4)}(q-p)\int\frac{d^4k}{(2\pi)^4}\frac{i}{k^2-m^2+i\epsilon}$

- -

Note: When we integrate position space integrals we convert the amplitude to momentum space and there is corresponding Feynman rules in momentum space

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Feynman rules for ϕ^4 theory in momentum space
1. each vertex gets $-i\lambda$
2. internal lines get a factor of $\frac{i}{q^2-m^2+i\epsilon}$
may need to use conservation of momentum, the total momentum entering and leaving the vertex is 0.
3. conservation of momentum on each vertex.
4. Integrate over unconstrained momentum w/ $\frac{1}{(2\pi)^4}\int d^4q$
5. External lines = 1
6. divide by symmetry factor
7. include overall energy momentum conserving delta function for each diagram

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(940) $\underline{\langle q_{k+1}|f(q)|q_k\rangle = f(\frac{q_{k+1}+q_k}{2})(\Pi_i\int\frac{dp_k^i}{2\pi})\exp[i\sum_i p_k^i(q_{k+1}^i-q_k^i)]}$
Note: $\langle\vec{x}|\vec{x}'\rangle = \delta(\vec{x}-\vec{x}') \implies \langle q_b|q_a\rangle = \Pi_i\delta(q_b^i-q_a^i)$
$\implies \langle q_{k+1}|f(q)|q_k\rangle = f(q_k)\langle q_{k+1}|q_k\rangle = f(q_k)\Pi_i\delta(q_k^i-q_{k+1}^i)$
recall: $\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}e^{ikx}dk$
$\implies \delta(q_k^i-q_{k+1}^i) = \frac{1}{2\pi}\int dp_k^i e^{ip_k^i(q_k^i-q_{k+1})}$

$\implies \langle q_{k+1}|f(q)|q_k\rangle = f(q_k)[\Pi_i \int \frac{dp_k^i}{2\pi} \exp(ip_k^i(q_{k+1}^i - q_k^i)]$

- -

(941) $\underline{\langle q_{k+1}|f(p)|q_k\rangle = (\Pi_i \int \frac{dp_k^i}{2\pi})f(p_k)\exp[i\sum_i p_k^i(q_{k+1}^i - q_k^i)]}$
<u>recall:</u> $1 = \int d\vec{x}|\vec{x}\rangle\langle\vec{x}| = \Pi\int dx^i|\vec{x}\rangle\langle\vec{x}|$
$\implies 1 = (\Pi_i \int dq_k^i)|q_k\rangle\langle q_k|$
$\implies \langle q_{k+1}|f(p)|q_k\rangle = \langle q_{k+1}|f(p)(\Pi_i \int dp_k^i)|p_k\rangle\langle p_k||q_k\rangle$
$= (\Pi_i \int dp_k^i)f(p_k)\langle q_{k+1}|p_k\rangle\langle p_k|q_k\rangle$
<u>recall:</u> $\langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}}\exp(ipx/\hbar)$ This might be only for 1-D
$\implies \langle q_{k+1}|f(p)|q_k\rangle = (\Pi_i \int dp_k^i)f(p_k)\exp(ip_kq_{k+1})\exp(-ip_kq_k)\frac{1}{2\pi}$
$= (\Pi_i \int \frac{dp_k^i}{2\pi})f(p_k)\exp[i\sum_i p_k^i(q_{k+1}^i - q_k^i)]$
(I don't understand why $\frac{1}{2\pi}$ gets grouped with Π symbol)

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(942) $\underline{\langle q_{k+1}|H(q,p)|q_k\rangle = (\Pi_i \int \frac{dp_k^i}{2\pi})H(\frac{q_{k+1}+q_k}{2}, p_k)\exp[i\sum_i p_k^i(q_{k+1}^i - q_k^i)]}$
Assume $H(q,p)$ only consists of terms of the form $f(p)$ and $f(q)$, e.g. $H(q,p) = f(p) + g(q)$
<u>Note:</u> As far as I can see, if each term has the form $f(p)$ or $g(q)$ then the most general form allowed is $H(q,p) = \sum_i f_i(p) + \sum_i g_i(q) = f(p) + g(q)$
$\implies \langle q_{k+1}|H(q,p)|q_k\rangle = \langle q_{k+1}|f(p)|q_k\rangle + \langle q_{k+1}|g(q)|q_k\rangle$
$= (\Pi_i \int \frac{dp_k^i}{2\pi})f(p_k)\exp[i\sum_i p_k^i(q_{k+1}^i - q_k^i)]$
$+ g(\frac{q_{k+1}+q_k}{2})(\Pi_k \int \frac{dp_k^i}{2\pi})\exp[i\sum_i p_k^i(q_{k+1}^i - q_k^i)]$
$= (\Pi_i \int \frac{dp_k^i}{2\pi})[f(p_k) + g(\frac{q_{k+1}+q_k}{2})]\exp[i\sum_i p_k^i(q_{k+1}^i - g_k^i)]$
$\therefore \langle q_{k+1}|H(q,p)|q_k\rangle = (\Pi_i \int \frac{dp_k^i}{2\pi})H(\frac{q_{k+1}+q_k}{2}, p_k)\exp[i\sum_i p_k^i(q_{k+1}^i - q_k^i)]$

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$H(q,p)$ doesn't have to be of the form $f(p) + g(q)$ in general this formula works if the terms are Weyl ordered.
<u>Example:</u> $H(q,p) = (q^2p^2 + 22qp^2q + p^2q^2)$
Weyl ordered since q's appear symetrically on both sides.

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<u>Note:</u> $\langle q_{k+1}|e^{-iH\epsilon}|q_k\rangle = U(q_k, q_{k+1}; \epsilon)$
(I don't udnerstand why $U(q_a, q_b; T)$ instead of $U(q_b, q_a; T)$, it may have something to do with the fact that we are evolving coordinates in space, not time

- -

(943) $\underline{U(q_a,q_b;T) = (\Pi_i \int \mathcal{D}q^i(t)\mathcal{D}p^i(t)) \exp[i\int_0^T dt(\sum_i p^i\dot{q}^i - H(q,p))]}$

$\langle q_{k+1}|e^{-i\epsilon H}|q_k\rangle = \langle q_{k+1}|1 - i\epsilon H|q_k\rangle$

$= \langle q_{k+1}|q_k\rangle - i\epsilon\langle q_{k+1}|H|q_k\rangle$

<u>recall:</u> $\langle q_{k+1}H(q,p)|q_k\rangle = (\Pi_i \int \frac{dp_k^i}{2\pi})H(\frac{q_{k+1}+q_k}{2}, p_k)\exp(i\sum_i(q_{k+1}^i - q_k^i))$

$\langle q_{k+1}|q_k\rangle = \Pi_i\delta(q_k^i - q_{k+1}^i) = (\Pi_i \int \frac{dp_k^i}{2\pi})\exp(i\sum_i p_k^i(q_{k+1}^i - q_k^i))$

$\implies \langle q_{k+1}|e^{-i\epsilon H}|q_k\rangle = (\Pi_i \int \frac{dp_k^i}{2\pi})\exp[i\sum_i p_k^i(q_{k+1}^i - q_k^i)]$

$- i\epsilon(\Pi_i \int \frac{dp_k^i}{2\pi})H(\frac{q_{k+1}+q_k}{2}, p_k)\exp[i\sum_i p_k^i(q_{k+1}^i - q_k^i)]$

$= (\Pi_i \int \frac{dp_k^i}{2\pi})(1 - i\epsilon H)\exp[i\sum_i p_k^i(q_{k+1}^i - q_k^i)]$

$= (\Pi_i \int \frac{dp_k^i}{2\pi})\exp[-i\epsilon H(\frac{q_{k+1}+q_k}{2}, p_k)]\exp[i\sum_i p_k^i(q_{k+1}^i - q_k^i)]$

$U(q_0,q_N;T) = \Pi_k \int dq_k^i U(q_k, q_{k+1};\epsilon)$

$= \Pi_k \int dq_k^i \langle q_{k+1}|e^{-i\epsilon H}|q_k\rangle$ (step is skechy)

$= (\Pi_{i,k} \int dq_k^i \int \frac{dp_k^i}{2\pi})\times\exp[i\sum_k(\sum_i p_k^i(q_{k+1}^i - q_k^i) - \epsilon H(\frac{q_{k+1}+q_k}{2}, p_k)]$

$\therefore U(q_a,q_b;T) = (\Pi_i \int \mathcal{D}q^i(t)\mathcal{D}p^i(t))\exp[i\int_0^T dt(\sum_i p^i\dot{q}^i - H(q,p))]$

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<u>Note:</u> $\int \mathcal{D}q(t)\mathcal{D}p(t) = \Pi_i \int \mathcal{D}q^i(t)\mathcal{D}p^i(t) \to \lim_{N\to\infty}\Pi_{k=1}^{N-1}\Pi_i \int \frac{dq_k^i dp_k^i}{2\pi\hbar}$

- -

(944) $\underline{U(q_a,q_b;T) = \left(\frac{1}{C(\epsilon)}\Pi_k \int \frac{dq_k}{C(\epsilon)}\right)\exp[i\sum_k(\frac{m}{2}\frac{(q_{k+1}-q_k)^2}{\epsilon} - \epsilon V(\frac{q_{k+1}-q_k}{2})]}$

This formula explains why in non-relativistic quantum mechanics, there is no integral over momentum. We work in 1-D

<u>recall:</u> $U(q_0,q_N;T) = (\Pi_{i,k} \int dq_k^i \int \frac{dp_k^i}{2\pi})\times\exp[i\sum_k(\sum_i p_k^i(q_{k+1}^i - q_k) - \epsilon H(\frac{q_{k+1}+q_k}{2}, p_k))]$

$= (\Pi_k \int dq_k \int \frac{dp_k}{2\pi})\times\exp[i\sum_k(p_k(q_{k+1} - q_k) - \epsilon H(\frac{q_{k+1}+q_k}{2}, p_k))]$

$= (\Pi_k \int dq_k \int \frac{dp_k}{2\pi}\exp[ip_k(q_{k+1} - q_k) - \epsilon H(\frac{q_{k+1}+q_k}{2}, p_k)])$

$H = \frac{p^2}{2m} + V(q)$

$\implies \int \frac{dp_k}{2\pi}\exp[i(p_k(q_{k+1} - q_k) - \epsilon p_k^2/2m)]$

complete square or mathematica

$= \frac{1}{2}\frac{1}{\sqrt{\pi}}(1-i)\sqrt{\frac{m}{\epsilon}}\exp[im(q_{k+1} - q_k)^2/2\epsilon]$

<u>Note:</u> $\sqrt{-i} = i\sqrt{i} = i(e^{i\pi/2})^{1/2} = i(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i) = \frac{\sqrt{2}}{2}(1-i)$

$\implies \sqrt{-i}\frac{2}{\sqrt{2}} = (1-i)$

$\implies U(q_0,q_N;T) = \sqrt{\frac{-im}{2\pi\epsilon}}\exp[\frac{im}{2\epsilon}(q_{k+1} - q_k)^2]$

$= \frac{1}{C(\epsilon)}\exp[\frac{im}{2\epsilon}(q_{k+1} - q_k)^2]$

$\therefore U(q_a,q_b;T) = (\frac{1}{C(\epsilon)}\Pi_k \int \frac{dq_k}{C(\epsilon)})\exp[i\sum_k(\frac{m}{2}\frac{(q_{k+1}-q_k)^2}{2\epsilon} - \epsilon V(\frac{q_{k+1}+q_k}{2}))]$

$C(\epsilon)$ for every time slice, including t=0 slice
(don't fully understand why)

- -

(945) $\underline{(-k^2 g_{\mu\nu} + (1 - \frac{1}{\xi} k_\mu k_\nu)\tilde{D}_F^{\nu\rho}(k) = i\delta_\mu^\rho}$

$\underline{\text{recall:}}$ $S = \frac{1}{2}\int d^4x A_\mu(x)(\partial^2 g^{\mu\nu} - \partial^\mu\partial^\nu)A_\nu(x)$
$\implies S = \frac{1}{2}\int d^4x[A_\mu(x)(\partial^2 g^{\mu\nu} - \partial^\mu\partial^\nu)A_\nu(x) - \frac{1}{\xi}(\partial^\mu A_\mu)^2]?$
$(\partial^\mu A_\mu)^2 = (\partial^\mu A_\mu)(\partial^\nu A_\nu)$
$\partial^\mu(A_\mu\partial^\nu A_\nu) = (\partial^\mu A_\mu)(\partial^\nu A_[\nu) + A_\mu\partial^\mu\partial^\nu A_\nu$
$\implies (\partial^\mu A_\mu)^2 = \partial^\mu(A_\mu\partial^\nu A_\nu) - A_\mu\partial^\mu\partial^\nu A_\nu$
$\implies \int d^4x \frac{1}{2\xi}(\partial^\mu A_\mu)^2 = -\frac{1}{2\xi}\int d^4x A_\mu\partial^\mu\partial^\nu A_\nu$
$\implies S = \frac{1}{2}\int d^4x[A_\mu(x)(\partial^2 g^{\mu\nu} - (1 - \frac{1}{\xi})\partial^\mu\partial^\nu)A_\nu(x)$
$= \frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\tilde{A}_\mu(k)(-k^2 g^{\mu\nu} + (1 - \frac{1}{\xi})k^\mu k^\nu)\tilde{A}_\nu(k)$
$\therefore (-k^2 g_{\mu\nu} + (1 - \frac{1}{\xi})k_\mu k_\nu)\tilde{D}_F^{\nu\rho}(k) = i\delta_\mu^\rho$

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(946) $\underline{1 = \int \mathcal{D}\alpha(x)\delta(G(A^\alpha))det(\frac{\delta G(A^\alpha)}{\delta\alpha})}$

$1 = \int dg\delta^{(n)}(g) = \Pi_i \int dg_i\delta(g_i) = (\Pi_i \int dg_i)\delta^{(n)}(g)$
$= (\Pi_i \int da_i)\delta^{(n)}(g(a))det(\frac{\partial g_i}{\partial a_j})$
$\implies 1 = \int \mathcal{D}\alpha(x)\delta(G(A^\alpha))det(\frac{\delta G(A^\alpha)}{\delta\alpha})$
don't understand why $A^\alpha \to \alpha(x)$

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(947) $\underline{\text{photon progagator}}$

$\underline{\text{recall:}}$ $\int \mathcal{D}ae^{iS[A]}$
$S = \int d^4x[-\frac{1}{4}(F_{\mu\nu})^2]$
$= -\frac{1}{4}\int d^4x(\partial_\mu A_\nu 0 - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu)$
$= -\frac{1}{4}\int d^4x[\partial_\mu A_\nu\partial^\mu A^\nu - \partial_\mu A_\nu\partial^\nu A^\mu - \partial_\nu A_\mu\partial^\mu A^\nu + \partial_\nu A_\mu\partial^\nu A^\mu]$
$= -\frac{1}{4}\int d^4x[2\partial_\mu A_\nu\partial^\nu A^\nu - 2\partial_\mu A_\nu\partial^\nu A^\mu]$
$= -\frac{1}{2}\int d^4x[\partial_\mu A_\nu\partial^\mu A^\nu - \partial_\mu A_\nu\partial^\nu A^\mu]$
$\partial_\mu(A_\nu\partial^\mu A^\nu) = \partial_\mu A_\nu\partial^\mu A^\nu + A_\nu\partial_\mu\partial^\mu A^\nu$
$\implies \partial_\mu A_\nu\partial^\mu A^\nu = \partial_\mu(A_\nu\partial^\mu A^\nu) - A_\nu\partial_\mu\partial^\mu A^\nu$
$\partial_\mu(A_\nu\partial^\nu A^\mu) = A_\nu\partial_{\mu\partial^\nu A^\mu + \partial_P\mu}A_\nu\partial^\nu A^\mu$
$\implies \partial_\mu A_\nu.\partial^\nu A^\mu = \partial_\mu(A_\nu\partial^\nu A^\mu) - A_\nu\partial_\mu\partial^\nu A^\mu$
plug in
$\implies -\frac{1}{2}\int d^4x[-A_\nu\partial_\mu\partial^\mu A^\nu + A_\nu\partial_\mu\partial^\nu A^\mu] = \frac{1}{2\int d^4x[A_\mu\partial_\mu\partial^\mu A^\nu - A_\nu\partial_\mu\partial^\nu A^\mu] = \frac{1}{2}\int d^4x A_\mu[\partial_\ni\partial^\nu A^\mu - \partial_\nu\partial^\mu A}$
$= \frac{1}{2}\int d^4x A_\mu[\partial^2 g^{\mu\sigma} - \partial_\nu\partial^\mu g^{\nu\sigma}]A_\sigma$
$= \frac{1}{2}\int d^4x A_\mu[\partial^2 g^{\mu\sigma} - \partial^\sigma\partial^\mu]A_\sigma$
$= \frac{1}{2}\int d^5x A_\mu(x)[\partial^2 g^{\mu\nu} - \partial^\mu\partial^\nu]A_\nu(x)$

$A_\nu(-k)$ comes from the fact that we will get $\int d^4xe^{-i(k+k')x} \propto \delta(k+k')$
and $\int d^4k'\delta(k+k')A(k') = A(-k)$
$A_\nu = \int \frac{d^4k}{(2\pi)^4}e^{-ikx}\tilde{A}_\nu(k)$
$\implies S = \frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\tilde{A}_\mu(k)(-k^2g^{\mu\nu} + k^\mu k^\nu)\tilde{A}_\nu(-k)$
divergent due to including infinitely many gauge equivalent terms since there is no gaussian damping in $\int \mathcal{D}Ae^{iS[A]}$. So lets fix our gauge and retry $G(A)$ is our gauge condition is $G(A) = 0$ Let's enforce this condition by δ function $\delta(G(A))$, $G(A) = 0\forall x$
<u>recall:</u> $1 = (\pi_i \int da_i)\delta^{(a)}(g(a))det(\frac{\partial g_i}{\partial a_j})$
$\implies 1 = \int \mathcal{D}\alpha(x)\delta(G(A^\alpha))det(\frac{\delta G(A^\alpha)}{\delta\alpha})$
$G(A) = \partial_\mu A^\mu;\ A^\alpha_\mu(x) \propto A_\mu + \frac{1}{e}\partial_\mu\alpha(x)$
$\implies G(A^\alpha) = \partial_\mu A^{\alpha\mu} = \partial_\mu A^\mu + \frac{1}{e}\partial_\mu\partial^\mu\alpha(x)$
$= \partial_\mu A^\mu + \frac{1}{e}\partial^2\alpha(x)$
insert into $\int \mathcal{D}Ae^{iS[A]}$
$= \det(\frac{\delta G(a^\alpha)}{\delta\alpha})\int \mathcal{D}\alpha\int \mathcal{D}Ae^{iS[A]}\delta(G(A))$
<u>Note:</u> determinant does not depend on α type up and spend more time on this.
$A^\alpha \to A,\ \ \alpha$ dummy $S[A] = S[A^\alpha],\ \mathcal{D}A = \mathcal{D}a^\alpha$
$\implies \int \mathcal{D}Ae^{iS[A]} = det(\frac{\delta G(A^\alpha)}{\delta\alpha})\int \mathcal{D}\alpha\int \mathcal{D}Ae^{iS[A]}\delta(G(A))$
choose $G(A) = \partial^\mu A_\mu(x) - \omega(x)$
need to understand functional determinants before we can understand this next step
$\det(\frac{\delta G(A^\alpha)}{\delta\alpha}) = det(\frac{\partial^2}{e})$
$\implies \int \mathcal{D}Ae^{iS[A]} = det(\frac{1}{e}\partial^2)(\int \mathcal{D}\alpha)\int \mathcal{D}Ae^{iS[A]}\delta(\partial^\mu A_\mu - \omega(x))$
integrate over all ω and normalize with gaussian weight
$\implies N(\xi)\int \mathcal{D}\omega exp[-i\int d^4x\frac{\omega^2}{2\xi}]det(\frac{1}{e\partial^2})(\int \mathcal{D}\alpha)(\int \mathcal{D}Ae^{iS[A]}\delta(\partial^\mu A_\mu - \omega(x)))$
$= N(\xi)\delta(\frac{1}{e}\partial^2)(\int \mathcal{D}\alpha)\int \mathcal{D}A\exp[i\int d^4x(\mathcal{L} - \frac{1}{2\xi}(\partial^\mu A_\mu)^2]$
$\therefore \mathcal{L} \to \mathcal{L} - \frac{1}{2\xi}(\partial^\mu A_\mu)^2$
<u>recall:</u> $\langle\Omega|T\mathcal{O}(A)|\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)}\frac{\int \mathcal{D}A\mathcal{O}(A)\exp[i\int_{-T}^{T}d^4x\mathcal{L}]}{\int \mathcal{D}A\exp[i\int_{-T}^{T}d^4x\mathcal{L}]}$
$\implies \langle\Omega|T\mathcal{O}(A)|\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)}\frac{\int \mathcal{D}A\mathcal{O}(A)\exp[i\int_{-T}^{T}d^4x[\mathcal{L}-\frac{1}{2\xi}(\partial^\mu A_\mu)^2]}{\int}\mathcal{D}A\exp[i\int_{-T}^{T}d^4x[\mathcal{L}-\frac{1}{2\xi}(\partial^\mu A_\mu)^2]$