4.3 自回归过程

$$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \cdots + \phi_p Y_{t-p} + e_t, \qquad \text{其中 } e_t \sim WN(0, \sigma_e^2).$$

- 假设e_t 与 $\{Y_{t-1}, Y_{t-2}, \cdots, Y_1\}$ 是不相关的。

- 当$p = 1$时，**AR(1)过程：** $Y_t = \phi Y_{t-1} + e_t$.

  如果Y_t序列平稳，那么

  - 均值 $\mu = \frac{0}{1-\phi} = 0$, 因为$E(Y_t) = \phi E(Y_{t-1}) + E(e_t)$;

  - 方差 $\gamma_0 = \frac{\sigma_e^2}{1-\phi^2} \geq 0$, 因为$Var(Y_t) = \phi^2 Var(Y_{t-1}) + Var(e_t)$;

    ①. **平稳性的必要条件是** $|\phi| < 1$.

  - k阶协方差

$$\begin{aligned} Cov(Y_t, Y_{t-k}) &= Cov(\phi Y_{t-1} + e_t, Y_{t-k}) \\ \gamma_k &= \phi\gamma_{k-1} \\ &= \phi \cdot \phi\gamma_{k-2} = \cdots \\ &= \phi^k \gamma_0 \end{aligned}$$

  - 自相关函数 $\rho_k = \phi^k, k = 0, 1, 2, \cdots$

  - AR(1)的一般线性表示

$$\begin{array}{lrcll} & Y_t &=& \phi Y_{t-1} + e_t & \\ \phi\cdot & Y_{t-1} &=& \phi Y_{t-2} + e_{t-1} & \cdot\phi \\ \phi^2\cdot & Y_{t-2} &=& \phi Y_{t-3} + e_{t-2} & \cdot\phi^2 \\ & & \cdots & & \\ \phi^{k-1}\cdot & Y_{t-(k-1)} &=& \phi Y_{t-k} + e_{t-(k-1)} & \cdot\phi^{k-1} \end{array}$$

    求和可得 $Y_t = e_t + \phi e_{t-1} + \phi^2 e_{t-2} + \cdots + \phi^{k-1} e_{t-(k-1)} + \phi^k Y_{t-k}$.

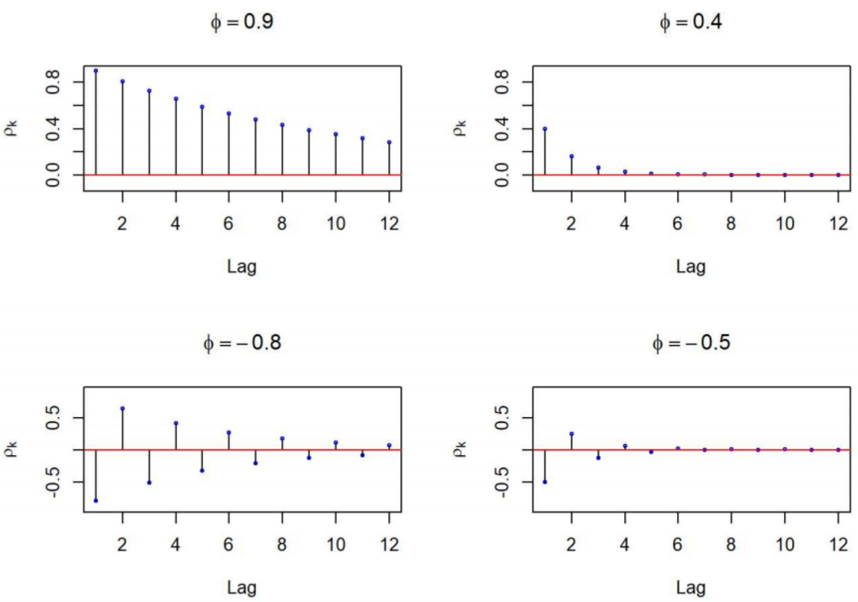
    如果$|\phi| < 1$, 并且$k \to \infty$, 那么

$$Y_t = e_t + \phi e_{t-1} + \phi^2 e_{t-2} + \phi^3 e_{t-3} + \cdots$$

    ②. **平稳性的充分条件是** $|\phi| < 1$.

  结论：**AR(1)是平稳的充分必要条件是$|\phi| < 1$，自相关函数$\rho_k = \phi^k$随着时滞k的增加而指数衰减。**

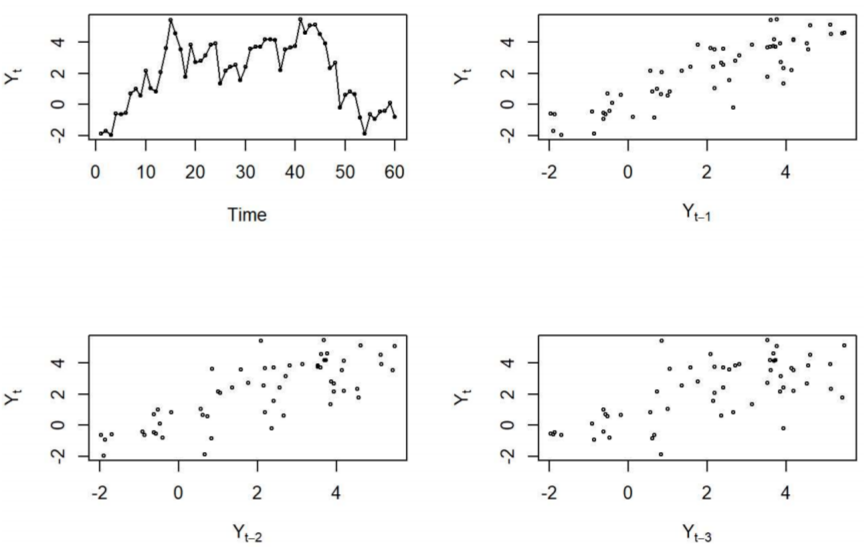
```
> x<-seq(1:12);
> y1<-0.9^x; y2<-0.4^x; y3<-(-0.8)^x; y4<-(-0.5)^x; opar=par(mfrow=c(2,2))
> plot(x,y1,type='h', ylim=c(-0.1,0.9), xlab='Lag', ylab=expression(rho[k]), main=expression(phi==0.9));
> points(x,y1,col='blue',cex=0.5); abline(h=0,col='red');
> plot(x,y2,type='h', ylim=c(-0.1,0.9),  xlab='Lag', ylab=expression(rho[k]), main=expression(phi==0.4));
> points(x,y2,col='blue',cex=0.5); abline(h=0,col='red');
> plot(x,y3,type='h', ylim=c(-0.9,0.9),  xlab='Lag', ylab=expression(rho[k]), main=expression(phi==-0.8));
> points(x,y3,col='blue',cex=0.5); abline(h=0,col='red');
> plot(x,y4,type='h', ylim=c(-0.9,0.9),  xlab='Lag', ylab=expression(rho[k]), main=expression(phi==-0.5));
> points(x,y4,col='blue',cex=0.5); abline(h=0,col='red');
```



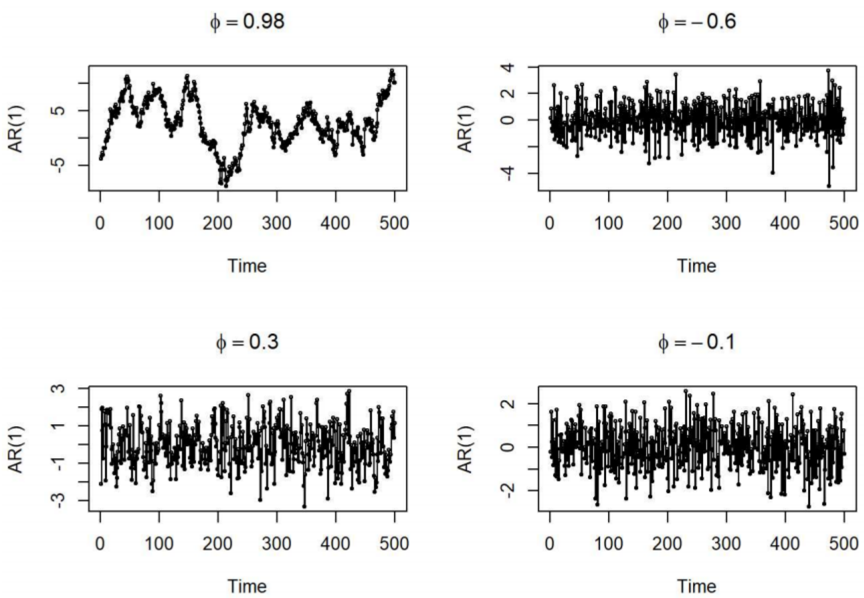

```
> par(opar)
```

例如：AR(1)序列 $\phi=0.9, \rho_1=0.9, \rho_2=0.81, \rho_3=0.729\cdots$

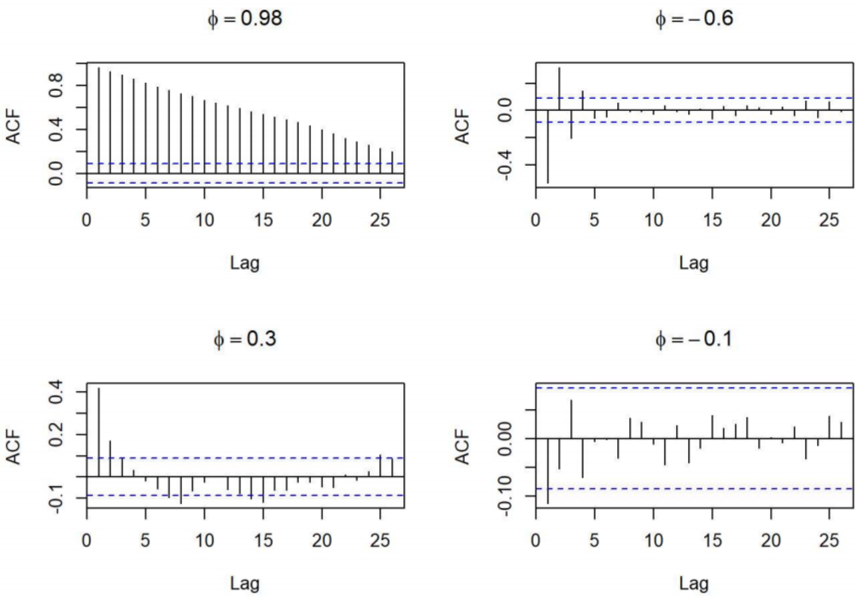
```
> data(ar1.s)
>
> opar=par(mfrow=c(2,2))
> # Exhibit 4.13
> plot(ar1.s, ylab=expression(Y[t]), type='o', cex=0.5)
> # Exhibit 4.14
> plot(y=ar1.s, x=zlag(ar1.s), ylab=expression(Y[t]), xlab=expression(Y[t-1]), type='p', cex=0.5)
> # Exhibit 4.15
> plot(y=ar1.s, x=zlag(ar1.s,2), ylab=expression(Y[t]), xlab=expression(Y[t-2]), type='p', cex=0.5)
> # Exhibit 4.16
> plot(y=ar1.s, x=zlag(ar1.s,3), ylab=expression(Y[t]), xlab=expression(Y[t-3]), type='p', cex=0.5)
```




```
>
>
> # An AR(1) series with AR coefficient equal to 0.98 and of length n=100.
> set.seed(12345); y1=arima.sim(model=list(ar=c(0.98)),n=500)
> set.seed(1234); y2=arima.sim(model=list(ar=c(-0.6)),n=500)
> set.seed(54321); y3=arima.sim(model=list(ar=c(0.3)),n=500)
> set.seed(4321); y4=arima.sim(model=list(ar=c(-0.1)),n=500)
> # Note that the R convention for the AR model formula is same as the book.
>
> plot(y1, type='o', xlab='Time', ylab='AR(1)', main=expression(phi==0.98), cex=0.5);
> plot(y2, type='o', xlab='Time', ylab='AR(1)', main=expression(phi==-0.6), cex=0.5);
> plot(y3, type='o', xlab='Time', ylab='AR(1)', main=expression(phi==0.3), cex=0.5);
> plot(y4, type='o', xlab='Time', ylab='AR(1)', main=expression(phi==-0.1), cex=0.5);
```



```
>
> acf(y1, xlab='Lag', ylab='ACF', main=expression(phi==0.98));
> acf(y2, xlab='Lag', ylab='ACF', main=expression(phi==-0.6));
> acf(y3, xlab='Lag', ylab='ACF', main=expression(phi==0.3));
> acf(y4, xlab='Lag', ylab='ACF', main=expression(phi==-0.1));
```




```
> par(opar)
```

- 当$p=2$时，**AR(2)过程：** $Y_t=\phi_1Y_{t-1}+\phi_2Y_{t-2}+e_t$.

  如果Y_t序列平稳，那么

  - 均值 $\mu=\frac{0}{1-\phi_1-\phi_2}=0$, 因为$E(Y_t)=\phi_1E(Y_{t-1})+\phi_2E(Y_{t-2})+E(e_t)$;

  - 方差 $Var(Y_t)=\phi_1^2Var(Y_{t-1})+\phi_2^2Var(Y_{t-2})+Var(e_t)+2\phi_1\phi_2Cov(Y_{t-1},Y_{t-2})$, 则

$$(1-\phi_1^2-\phi_2^2)\gamma_0=2\phi_1\phi_2\gamma_1+\sigma_e^2.$$

  - 一阶协方差 $(1-\phi_2)\gamma_1=\phi_1\gamma_0$, 因为

$$\begin{aligned}Cov(Y_t,Y_{t-1})&=Cov(\phi_1Y_{t-1}+\phi_2Y_{t-2}+e_t,Y_{t-1})\\&=\phi_1Var(Y_{t-1})+\phi_2Cov(Y_{t-1},Y_{t-2})\end{aligned}$$

  解方程组可得

$$\begin{cases}\gamma_0&=&\sigma_e^2\frac{1-\phi_2}{(1+\phi_2)[(1-\phi_2)^2-\phi_1^2]}\\\gamma_1&=&\sigma_e^2\frac{\phi_1}{(1+\phi_2)[(1-\phi_2)^2-\phi_1^2]}\end{cases}$$

  - 一阶自相关函数 $\rho_1=\frac{\phi_1}{1-\phi_2}$

  - k阶协方差/自相关函数

$$\mathbf{Yule-Walker}\text{方程}\qquad\begin{aligned}Cov(Y_t,Y_{t-k})=Cov(\phi_1Y_{t-1}+\phi_2Y_{t-2}+e_t,Y_{t-k})\\\gamma_k=\phi_1\gamma_{k-1}+\phi_2\gamma_{k-2}\\\rho_k=\phi_1\rho_{k-1}+\phi_2\rho_{k-2}\end{aligned}$$

  求解二阶齐次线性差分方程$\rho_{k+2}-\phi_1\rho_{k+1}-\phi_2\rho_k=0$，**初值条件为**$\rho_0=1$，$\rho_{-k}=\rho_k$.

  -①. 参考"差分方程讲解"http://www.homepage.zjut.edu.cn/yjq/ (http://www.homepage.zjut.edu.cn/yjq/)第19页，

  当λ是常数时，$\rho_k=\lambda^k$和它的各阶差商有倍数关系，因此不妨设解为$\rho_k=\lambda^k$，那么

$$\lambda^{k+2}-\phi_1\lambda^{k+1}-\phi_2\lambda^k=0$$

  该齐次差分方程的**特征方程**为

$$\lambda^2-\phi_1\lambda-\phi_2=0$$

  特征方程的根(**特征根**)为

$$\lambda_1=\frac{\phi_1-\sqrt{\phi_1^2+4\phi_2}}{2},\qquad\lambda_2=\frac{\phi_1+\sqrt{\phi_1^2+4\phi_2}}{2}$$

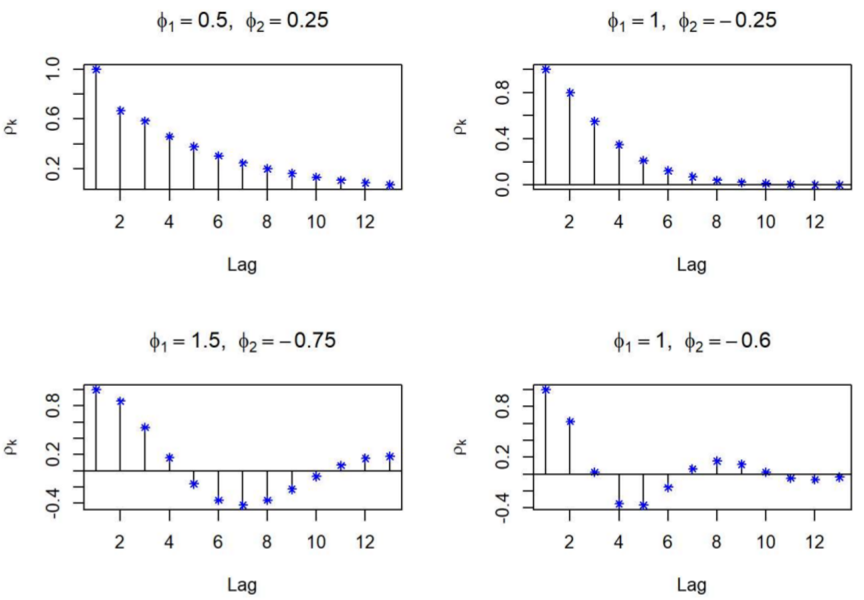
特征根的情况	差分方程的通解
当 $\phi_1^2+4\phi_2>0$ 时，$\lambda_1\neq\lambda_2$,	$\rho_k=C_1\lambda_1^k+C_2\lambda_2^k$
当 $\phi_1^2+4\phi_2=0$ 时，$\lambda_1=\lambda_2$,	$\rho_k=(C_1+C_2k)\lambda_1^k$
当 $\phi_1^2+4\phi_2<0$ 时，$\lambda_{1,2}=\alpha+\beta i$,	$\rho_k=(C_1cos(k\theta)+C_2sin(k\theta))R^k$
	$R=\sqrt{\alpha^2+\beta^2}=\sqrt{-\phi_2}, tan\theta=\dfrac{\beta}{\alpha}$

将条件$\rho_0=1,\rho_1=\rho_{-1}$代入通解，可以确定参数$C_1,C_2$，写出$k$阶自相关函数的表达式。

$$\rho_k=\begin{cases}\frac{(1-\lambda_2^2)\lambda_1^{k+1}-(1-\lambda_1^2)\lambda_2^{k+1}}{(\lambda_1-\lambda_2)(1+\lambda_1\lambda_2)}, & \text{当}\phi_1^2+4\phi_2>0\text{时}, & \text{过阻尼状态}\\ (1+\frac{1+\phi_2}{1-\phi_2}k)(\frac{\phi_1}{2})^k, & \text{当}\phi_1^2+4\phi_2=0\text{时}, & \text{临界阻尼状态}\\ \frac{\sin(\Theta k+\Phi)}{\sin(\Phi)}R^k, & \text{当}\phi_1^2+4\phi_2<0\text{时}. & \text{欠阻尼状态}\\ & \cos(\Theta)=\frac{\phi_1}{2\sqrt{-\phi_2}},\quad \tan(\Phi)=\frac{1-\phi_2}{1+\phi_2} & \end{cases}$$

结论：AR(2)的自相关函数 ρ_k 随着滞后阶数 k 的增加而指数衰减。 在复数特征根的情况下，ρ_k 显示为阻尼正弦波动曲线，具有阻尼因子$R(0\leq R<1)$、频率 Θ、相位 Φ。下图给出了自相关函数可能的几种图形，左上是两个不相等的实数特征根的情形，右上是两个相等的实数特征根情形，第二排的图都是复数特征根的情况。

```
> ###AR(2)的自相关函数的理论值 #Exhibit 4.18
> Rho1 <- ARMAacf(ar=c(0.5,0.25), lag.max = 12, pacf=FALSE)
> Rho2 <- ARMAacf(ar=c(1.0,-0.25), lag.max = 12, pacf=FALSE)
> Rho3 <- ARMAacf(ar=c(1.5,-0.75), lag.max = 12, pacf=FALSE)
> Rho4 <- ARMAacf(ar=c(1.0,-0.6), lag.max = 12, pacf=FALSE)
>
> opar=par(mfrow=c(2,2))
> plot(Rho1, type='h',  xlab='Lag', ylab=expression(rho[k]), main=expression(paste(phi[1]==0.5, ",  ", phi[2]==0.25)));
> points(Rho1, pch=8, col='blue', cex=0.8); abline(h=0);
> plot(Rho2, type='h',  xlab='Lag', ylab=expression(rho[k]), main=expression(paste(phi[1]==1.0, ",  ", phi[2]==-0.25)));
> points(Rho2, pch=8, col='blue', cex=0.8); abline(h=0);
> plot(Rho3, type='h',  xlab='Lag', ylab=expression(rho[k]), main=expression(paste(phi[1]==1.5, ",  ", phi[2]==-0.75)));
> points(Rho3, pch=8, col='blue', cex=0.8); abline(h=0);
> plot(Rho4, type='h',  xlab='Lag', ylab=expression(rho[k]), main=expression(paste(phi[1]==1.0, ",  ", phi[2]==-0.6)));
> points(Rho4, pch=8, col='blue', cex=0.8); abline(h=0);
```




```
> par(opar)
```

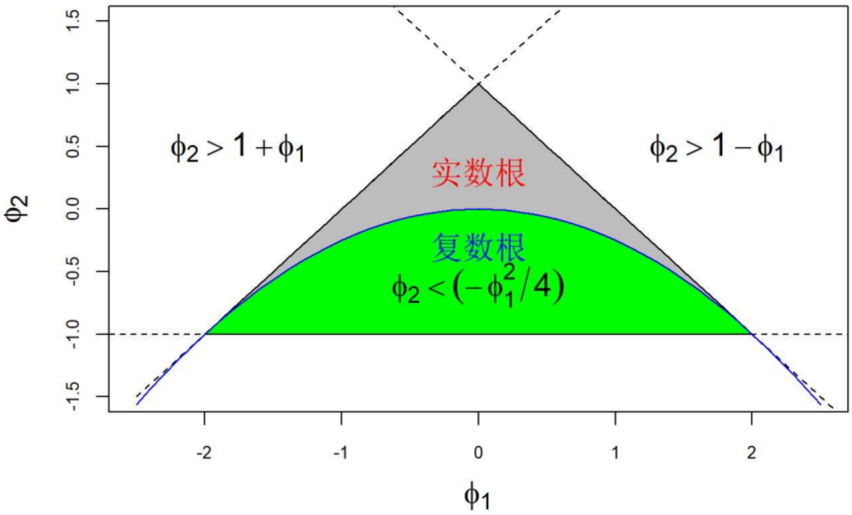
-②. **差分方程稳定性的条件是：** 特征根在单位圆内，$|\lambda_{1,2}|<\mathbf{1}$.

利用特征方程根与系数的关系$\begin{cases}\lambda_1+\lambda_2 &= \phi_1,\\ \lambda_1\cdot\lambda_2 &= -\phi_2\end{cases}$，可得

$$\phi_2+\phi_1=-\lambda_1\lambda_2+\lambda_1+\lambda_2=1-(1-\lambda_1)(1-\lambda_2)$$
$$\phi_2-\phi_1=-\lambda_1\lambda_2-\lambda_1-\lambda_2=1-(1+\lambda_1)(1+\lambda_2)$$

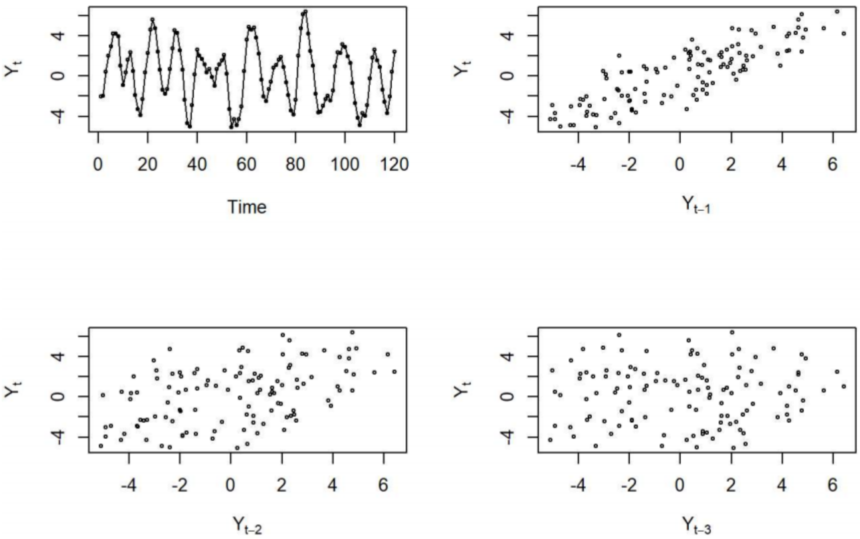
差分方程稳定性的等价条件是： $\boldsymbol{\phi_2+\phi_1<1,\quad \phi_2-\phi_1<1,\quad |\phi_2|<1.}$

```
> ###AR(2)的平稳参数区域 #Exhibit 4.17
> phi1 <- seq(from = -2.5, to = 2.5, length = 51)
> plot(phi1,1+phi1,lty="dashed",type="l",xlab="",ylab="",cex.axis=0.8,ylim=c(-1.5,1.5))
> abline(a = -1, b = 0, lty="dashed")
> abline(a = 1, b = -1, lty="dashed")
> title(ylab=expression(phi[2]),xlab=expression(phi[1]),cex.lab=1.5)
> polygon(x = phi1[6:46], y = 1-abs(phi1[6:46]), col="gray")
> polygon(x = phi1[6:46], y = -phi1[6:46]^2/4, col="green")
> lines(phi1, -phi1^2/4, col="blue")
> text(0,-.6,expression(phi[2]< (-phi[1]^2/4)),cex=1.5)
> text(0,0.3,"实数根", col='red', cex=1.5)
> text(0,-0.3,"复数根", col='blue', cex=1.5)
> text(1.75,.5,expression(phi[2]>1-phi[1]),cex=1.5)
> text(-1.75,.5,expression(phi[2]>1+phi[1]),cex=1.5)
```

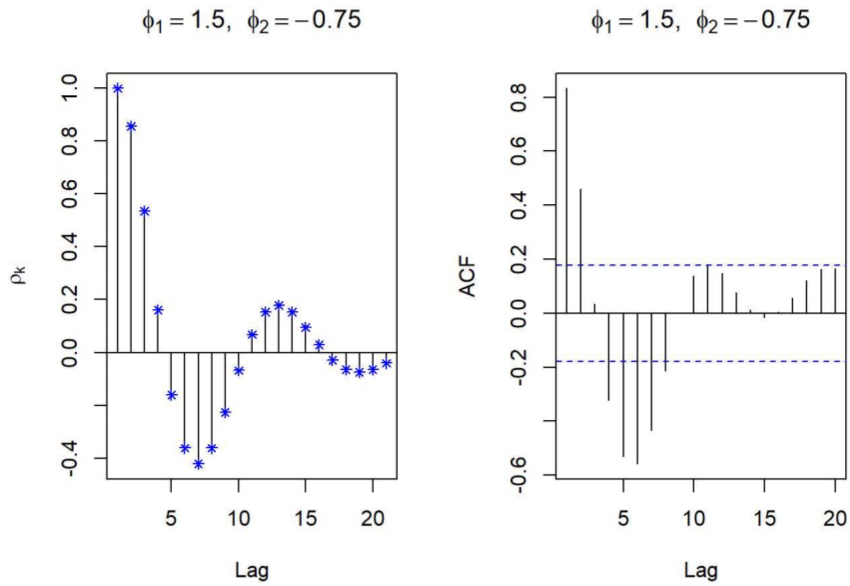


例如，AR(2)模型$\phi_1 = 1.5, \phi_2 = -0.75, \phi_1^2 + 4\phi_2 = -\frac{3}{4} < 0$ 特征方程有两个不相等的复数根。k阶自相关函数为$R^k \frac{\sin(\Theta k + \Phi)}{\sin(\Phi)}$，其中，阻尼因子$R = \sqrt{-\phi_2} = \frac{\sqrt{3}}{2} \simeq 0.866, \cos(\Theta) = \frac{\phi_1}{2\sqrt{-\phi_2}} = \frac{3/2}{2\cdot\sqrt{3}/2} = \frac{\sqrt{3}}{2}$, 频率 $\Theta = \pi/6$, 周期 $f = \frac{2\pi}{\Theta} = \frac{2\pi}{\pi/6} = 12$, $\tan(\Phi) = \frac{1-\phi_2}{1+\phi_2} = \frac{1+3/4}{1-3/4} = 7$, 相位 $\Phi = \arctan(7) \simeq 81.87^o$. (习题4.9 on P.59)

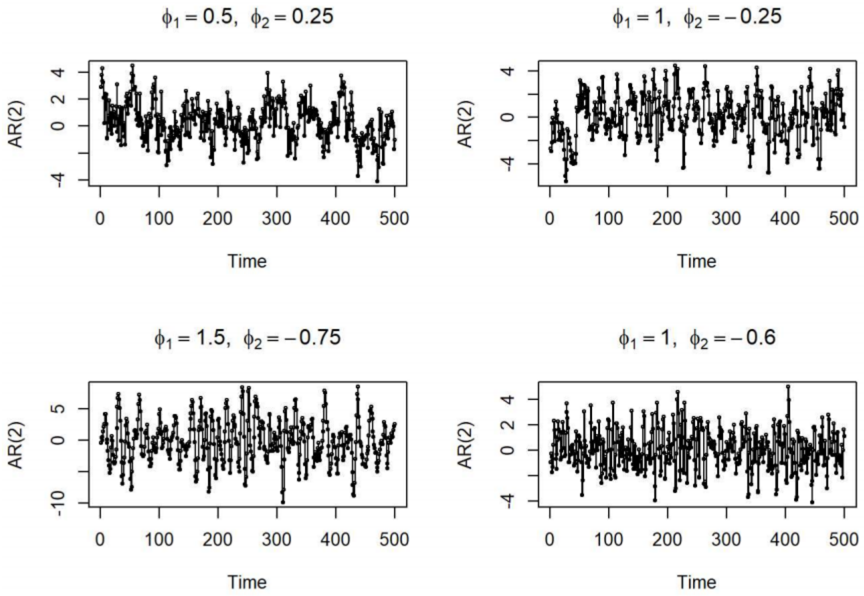
```
> ##polyroot(c(1,2,3,4))  多项式方程1+2x+3x^2+4x^3=0求根
> polyroot(c(0.75,-1.5,1)) #特征方程求根
> atan(7)*360/(2*pi)
>
> #Exhibit 4.19
> data(ar2.s) ##phi_1=1.5, phi_2=-0.75
> Rho <- ARMAacf(ar=c(1.5,-0.75), lag.max = 20, pacf=FALSE); Rho
>
> opar=par(mfrow=c(2,2))
> plot(ar2.s, ylab=expression(Y[t]), type='o', cex=0.5)
> plot(y=ar2.s, x=zlag(ar2.s), ylab=expression(Y[t]), xlab=expression(Y[t-1]), type='p', cex=0.5)
> plot(y=ar2.s, x=zlag(ar2.s,2), ylab=expression(Y[t]), xlab=expression(Y[t-2]), type='p', cex=0.5)
> plot(y=ar2.s, x=zlag(ar2.s,3), ylab=expression(Y[t]), xlab=expression(Y[t-3]), type='p', cex=0.5)
```



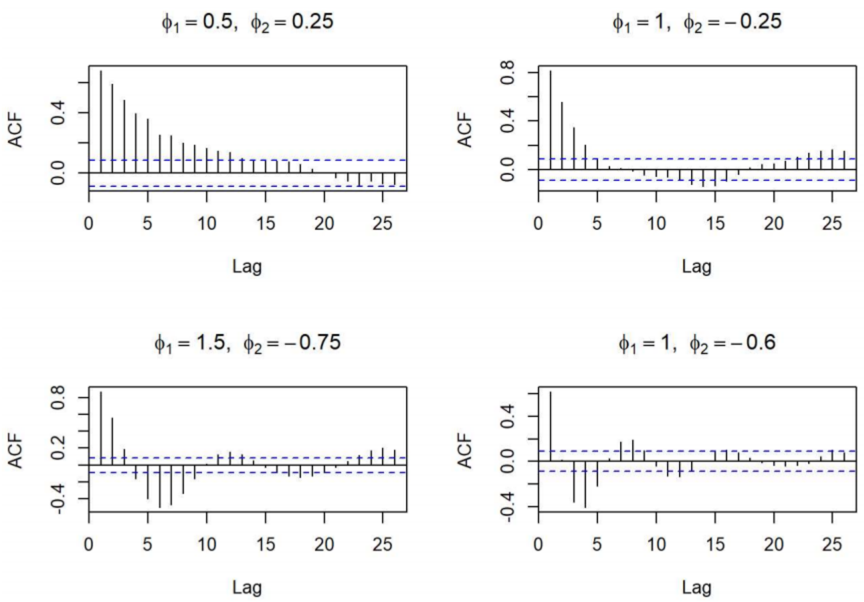

```
> par(opar)
>
> opar=par(mfrow=c(1,2))
> plot(Rho, type='h',  xlab='Lag', ylab=expression(rho[k]), main=expression(paste(phi[1]==1.5, ",  ", phi[2]==-0.75)));
> points(Rho, pch=8, col='blue', cex=0.8); abline(h=0);
> acf(ar2.s, xlab='Lag', ylab='ACF', main=expression(paste(phi[1]==1.5, ",  ", phi[2]==-0.75)))
```




```
> par(opar)
>
> set.seed(12345); y1=arima.sim(model=list(ar=c(0.5, 0.25)),n=500)
> set.seed(1234); y2=arima.sim(model=list(ar=c(1.0,-0.25)),n=500)
> set.seed(54321); y3=arima.sim(model=list(ar=c(1.5,-0.75)),n=500)
> set.seed(4321); y4=arima.sim(model=list(ar=c(1.0,-0.6)),n=500)
>
> opar=par(mfrow=c(2,2))
> plot(y1, type='o', xlab='Time', ylab='AR(2)', main=expression(paste(phi[1]==0.5, ",  ", phi[2]==0.25)), cex=0.5);
> plot(y2, type='o', xlab='Time', ylab='AR(2)', main=expression(paste(phi[1]==1.0, ",  ", phi[2]==-0.25)), cex=0.5);
> plot(y3, type='o', xlab='Time', ylab='AR(2)', main=expression(paste(phi[1]==1.5, ",  ", phi[2]==-0.75)), cex=0.5);
> plot(y4, type='o', xlab='Time', ylab='AR(2)', main=expression(paste(phi[1]==1.0, ",  ", phi[2]==-0.6)), cex=0.5);
```




```
>
> acf(y1, xlab='Lag', ylab='ACF', main=expression(paste(phi[1]==0.5, "   ", phi[2]==0.25)));
> acf(y2, xlab='Lag', ylab='ACF', main=expression(paste(phi[1]==1.0, "   ", phi[2]==-0.25)));
> acf(y3, xlab='Lag', ylab='ACF', main=expression(paste(phi[1]==1.5, "   ", phi[2]==-0.75)));
> acf(y4, xlab='Lag', ylab='ACF', main=expression(paste(phi[1]==1.0, "   ", phi[2]==-0.6)));
```




```
> par(opar)
[1] 0.75+0.4330127i 0.75-0.4330127i
[1] 81.8699
          0           1           2           3           4           5
 1.00000000  0.85714286  0.53571429  0.16071429 -0.16071429 -0.36160714
          6           7           8           9          10          11
-0.42187500 -0.36160714 -0.22600446 -0.06780134  0.06780134  0.15255301
         12          13          14          15          16          17
 0.17797852  0.15255301  0.09534563  0.02860369 -0.02860369 -0.06435830
         18          19          20
-0.07508469 -0.06435830 -0.04022394
```

- 引入滞后算子(延迟算子) B，即 $Y_{t-k} = B^k Y_t$, 则一阶向后差分 $\nabla Y_t = Y_t - Y_{t-1} = (1-B)Y_t$， 而"差分方程讲解"中使用的是一阶向前差分 $\Delta Y_t = Y_{t+1} - Y_t$。那么 k 阶向后差分可以表示为

$$\nabla^k Y_t = \nabla^{k-1}(Y_t - Y_{t-1}) = \cdots = (1-B)^k Y_t = \sum_{j=0}^{k}(-1)^j \binom{k}{j} B^j Y_t = \sum_{j=0}^{k}(-1)^j \binom{k}{j} Y_{t-j}.$$

AR(2)过程 $Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + e_t$ 可以写成

$$(1-\phi_1 B - \phi_2 B^2)Y_t = \Phi(B)Y_t = e_t$$

$\Phi(z) = 1 - \phi_1 z - \phi_2 z^2$ 称为**自回归系数多项式**，$\Phi(z)=0$ 称为**AR特征方程**（自回归系数方程，差分方程的逆反特征方程），它的根和差分方程的特征根互为倒数 （习题4.22 on P.60）

$$\begin{aligned} 1-\phi_1 z - \phi_2 z^2 &= 0 \\ \lambda^2 - \phi_1\lambda - \phi_2 &= 0 \end{aligned} \qquad z_{1,2} = \frac{1}{\lambda_{1,2}}$$

AR(2)过程平稳性的条件是：$\mathbf{AR}$特征方程的根在单位圆外，$|\mathbf{z_{1,2}}| > \mathbf{1}$.

- AR(2)的一般线性表示

如果存在两个不相等的实数特征根$\lambda_1 \neq \lambda_2$的话，那么

$$\begin{aligned} \Phi(z) &= -\phi_2(z-z_1)(z-z_2) \\ &= -\phi_2(z-\frac{1}{\lambda_1})(z-\frac{1}{\lambda_2}) \\ &= \frac{-\phi_2}{\lambda_1\lambda_2}(\lambda_1 z-1)(\lambda_2 z-1) \\ &= (1-\lambda_1 z)(1-\lambda_2 z). \end{aligned} \qquad z_{1,2} = \frac{1}{\lambda_{1,2}} \qquad \lambda_1\lambda_2 = -\phi_2$$

记 $\Phi^{-1}(z) = \frac{1}{\Phi(z)}$, 则 $Y_t = \Phi^{-1}(B)e_t$.

$$\Phi^{-1}(B) = \frac{1}{(1-\lambda_1 B)(1-\lambda_2 B)} = \frac{a_1}{1-\lambda_1 B} + \frac{a_2}{1-\lambda_2 B}$$

其中，$a_1(1-\lambda_2 B) + a_2(1-\lambda_1 B) = 1$, 即$a_1 = \frac{\lambda_1}{\lambda_1-\lambda_2}$, $\quad a_2 = \frac{-\lambda_2}{\lambda_1-\lambda_2}$.

利用泰勒展开$\frac{1}{1-x} = 1 + x + x^2 + \cdots$, 我们有

$$\begin{aligned} \Phi^{-1}(B) &= a_1(1+\lambda_1 B + \lambda_1^2 B^2 + \cdots) + a_2(1+\lambda_2 B + \lambda_2^2 B^2 + \cdots) \\ &= \sum_{j=0}^{\infty}(a_1\lambda_1^j + a_2\lambda_2^j)B^j \\ \Psi(B) &\triangleq \sum_{j=0}^{\infty}\psi_j B^j \end{aligned}$$

AR(2)的逆转形式为$Y_t = \Phi^{-1}(B)e_t = \Psi(B)e_t = \sum_{j=0}^{\infty}\psi_j e_{t-j}$. 类似的推导可以证明(证明略)

$$\psi_j = \begin{cases} \frac{\lambda_1^{j+1}-\lambda_2^{j+1}}{\lambda_1-\lambda_2}, & \text{当}\phi_1^2+4\phi_2 > 0\text{时}, \\ (1+j)\phi_1^j, & \text{当}\phi_1^2+4\phi_2 = 0\text{时}, \\ R^j\frac{\sin[(j+1)\Theta]}{\sin\Theta}, & \text{当}\phi_1^2+4\phi_2 < 0\text{时}, \end{cases}$$

- Green函数递推公式

将AR(2)的逆转形式 $Y_t = \Psi(B)e_t$ 代入 传递形式 $\Phi(B)Y_t = e_t$,

$$\begin{aligned} \Phi(B)\Psi(B)e_t &= e_t \\ (1-\phi_1 B - \phi_2 B^2)(\psi_0 + \psi_1 B + \psi_2 B^2 + \cdots)e_t &= e_t \end{aligned}$$

$$\begin{aligned} &\psi_0 - \phi_1\psi_0 B - \phi_2\psi_0 B^2 \\ &\quad +\psi_1 B - \phi_1\psi_1 B^2 - \phi_2\psi_1 B^3 \\ &\qquad \psi_2 B^2 - \phi_1\psi_2 B^3 - \phi_2\psi_2 B^4 \\ &\qquad\quad \cdots \end{aligned}$$

左右比较e_j的系数，可以得到

$$\begin{cases} \psi_0 & = 1, \\ \psi_1 - \phi_1\psi_0 & = 0, \\ \psi_2 - \phi_1\psi_1 - \phi_2\psi_0 & = 0, \\ \psi_3 - \phi_1\psi_2 - \phi_2\psi_1 & = 0, \\ & \vdots \\ \psi_k - \phi_1\psi_{k-1} - \phi_2\psi_{k-2} & = 0, \end{cases}$$

解方程组可得

$$\begin{cases} \psi_0 & = & 1, \\ \psi_1 & = & \phi_1, \\ \psi_2 & = & \phi_1^2 + \phi_2, \\ & \vdots & \\ \psi_k & = & \phi_1\psi_{k-1} + \phi_2\psi_{k-2}, \end{cases}$$

- 一般自回归过程AR(p)

$$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \cdots + \phi_p Y_{t-p} + e_t, \qquad \text{其中 } e_t \sim WN(0, \sigma_e^2).$$

类似AR(2)的分析过程，AR(p)自回归系数多项式

$$\Phi(z) = 1 - \phi_1 z - \phi_2 z^2 - \cdots - \phi_p z^p$$

$\underline{AR(p)\text{特征方程}1 - \phi_1 z - \phi_2 z^2 - \cdots - \phi_p z^p = 0\text{的根落在单位圆之外时，序列是平稳的。}}$

利用多项式根与系数的关系， 可以得到满足平稳性的**必要(非充分)条件**:

$$\left.\begin{array}{r} \phi_1 + \phi_2 + \cdots + \phi_p < 1 \\ |\phi_p| < 1 \end{array}\right\}$$

如果Y_t序列平稳，那么

- 均值 $\mu = \frac{0}{1-\phi_1-\phi_2-\cdots-\phi_p} = 0$, 因为$E(Y_t) = \phi_1 E(Y_{t-1}) + \phi_2 E(Y_{t-2}) + \cdots + \phi_p E(Y_{t-p}) + E(e_t)$;
- 方差 $\gamma_0 = \frac{\sigma_e^2}{1-\phi_1\rho_1-\phi_2\rho_2-\cdots-\phi_p\rho_p}$. 因为

$$\begin{aligned} Var(Y_t) &= Cov(\phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \cdots + \phi_p Y_{t-p} + e_t,\ Y_t) \\ \gamma_0 &= \phi_1\gamma_1 + \phi_2\gamma_2 + \cdots + \phi_p\gamma_p + \sigma_e^2 \\ \gamma_0 &= \phi_1\rho_1\gamma_0 + \phi_2\rho_2\gamma_0 + \cdots + \phi_p\rho_p\gamma_0 + \sigma_e^2 \end{aligned}$$

- k阶协方差/自相关函数 ($k = 1, 2, \cdots$)

$$\textbf{Yule} - \textbf{Walker}\text{方程} \qquad \begin{aligned} Cov(Y_t, Y_{t-k}) &= Cov(\phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \cdots + \phi_p Y_{t-p} + e_t, Y_{t-k}) \\ \gamma_k &= \phi_1\gamma_{k-1} + \phi_2\gamma_{k-2} + \cdots + \phi_p\gamma_{k-p} \\ \rho_k &= \phi_1\rho_{k-1} + \phi_2\rho_{k-2} + \cdots + \phi_p\rho_{k-p} \end{aligned}$$

$\underline{\text{差分方程的特征方程}\lambda^p - \phi_1\lambda^{p-1} - \phi_2\lambda^{p-2} - \cdots - \phi_p = 0\text{的根落在单位圆之内}|\lambda_{1,\cdots,p}| < 1\text{时，序列是平稳的。}}$

特征根的情况	差分方程的通解
当存在p个不相等的实数根$\lambda_1 \neq \cdots \neq \lambda_p$时,	$\rho_k = C_1\lambda_1^k + C_2\lambda_2^k + \cdots + C_p\lambda_p^k$
当存在相等的实根$\lambda_{1,\cdots,d}$和不等的实根$\lambda_{3,\cdots,p}$时,	$\rho_k = (C_1 + C_2 k + \cdots + C_d k^{d-1})\lambda_1^k + C_{d+1}\lambda_{d+1}^k + \cdots + C_p\lambda_p^k$
当存在复数根$\lambda_{1,2}$和不相等的实数根$\lambda_{3,\cdots,p}$时,	$\rho_k = R^k(C_1 cos(k\theta) + C_2 sin(k\theta)) + C_3\lambda_3^k + \cdots + C_p\lambda_p^k$

将条件$\rho_0 = 1, \rho_k = \rho_{-k}$代入通解，可以确定参数$C_1, C_2, \cdots, C_p$, 写出$k$阶自相关函数的表达式。

结论：当$|\lambda_{1,\cdots,p}| < 1$时，平稳AR(p)的ACF图像呈现减幅的正弦、余弦和指数衰减的混合形式，具体形式取决于特征根的性质。

另一方面，根据 $\rho_0 = 1, \quad \rho_{-k} = \rho_k$, 得到一般的**Yule-Walker方程组**

$$\begin{cases} \rho_1 = \phi_1 + \phi_2\rho_1 + \phi_3\rho_2 + \cdots + \phi_p\rho_{p-1} \\ \rho_2 = \phi_1\rho_1 + \phi_2 + \phi_3\rho_1 + \cdots + \phi_p\rho_{p-2} \\ \quad\vdots \\ \rho_p = \phi_1\rho_{p-1} + \phi_2\rho_{p-2} + \phi_3\rho_{p-3} + \cdots + \phi_p \end{cases}$$

给定 $\phi_1, \phi_2, \cdots, \phi_p$的值，令$\phi_{p+1} = \phi_{p+2} = \cdots = \phi_{2p} = 0$,

$$\begin{cases} (\phi_2 - 1)\rho_1 + \phi_3\rho_2 + \phi_4\rho_3 + \cdots + \phi_{p-1}\rho_{p-2} + \phi_p\rho_{p-1} + \underline{\phi_{p+1}}\rho_p \\ (\phi_1 + \phi_3)\rho_1 + (\phi_4 - 1)\rho_2 + \phi_5\rho_3 + \cdots + \phi_p\rho_{p-2} + \underline{\phi_{p+1}}\rho_{p-1} + \underline{\phi_{p+2}}\rho_p \\ \quad\vdots \\ (\phi_{k-1} + \phi_{k+1})\rho_1 + (\phi_{k-2} + \phi_{k+2})\rho_2 + \cdots + (\phi_1 + \phi_{2k-1})\rho_{k-1} + (\phi_{2k} - 1)\rho_k + \phi_{2k+1}\rho_{k+1} + \cdots + \underline{\phi_{p+k-1}}\rho_{p-1} + \underline{\phi_{p+k}}\rho_p \\ \quad\vdots \\ (\phi_{p-1} + \underline{\phi_{p+1}})\rho_1 + (\phi_{p-2} + \underline{\phi_{p+2}})\rho_2 + (\phi_{p-3} + \underline{\phi_{p+3}})\rho_3 + \cdots + (\phi_2 + \underline{\phi_{2p-2}})\rho_{p-2} + (\phi_1 + \underline{\phi_{2p-1}})\rho_{p-1} + (\underline{\phi_{2p}} - 1)\rho \end{cases}$$

可以利用Cramer法则，$\rho_i = \frac{|D_i|}{|D|}$, 其中$D$是系数矩阵，求解该线性方程组得到 $\rho_1, \rho_2, \cdots, \rho_p$ 的值,

然后利用递归关系$\rho_k = \phi_1\rho_{k-1} + \phi_2\rho_{k-2} + \cdots + \phi_p\rho_{k-p}$求得任意高阶$(k > p)$时的$\rho_k$.

- Green函数递推公式

将AR(p)的逆转形式 $Y_t = \Psi(B)e_t$ 代入 传递形式 $\Phi(B)Y_t = e_t$,

$$\begin{aligned} \Phi(B)\Psi(B)e_t &= e_t \\ (1 - \phi_1 B - \phi_2 B^2 - \cdots - \phi_p B^p)(\psi_0 + \psi_1 B + \psi_2 B^2 + \cdots)e_t &= e_t \end{aligned}$$

$$\begin{aligned} &\psi_0 - \phi_1\psi_0 B - \phi_2\psi_0 B^2 - \phi_3\psi_0 B^3 - \cdots - \phi_p\psi_0 B^p \\ &\quad + \psi_1 B - \phi_1\psi_1 B^2 - \phi_2\psi_1 B^3 - \cdots - \phi_{p-1}\psi_1 B^p - \phi_p\psi_1 B^{p+1} \\ &\qquad \psi_2 B^2 - \phi_1\psi_2 B^3 - \cdots - \phi_{p-2}\psi_2 B^p - \phi_{p-1}\psi_2 B^{p+1} - \phi_p\psi_2 B^{p+2} \\ &\qquad\quad \cdots\cdots \end{aligned}$$

左右比较e_j的系数，可以得到

$$\begin{cases} \psi_0 & = 1, \\ \psi_1 - \phi_1\psi_0 & = 0, \\ \psi_2 - \phi_1\psi_1 - \phi_2\psi_0 & = 0, \\ \psi_3 - \phi_1\psi_2 - \phi_2\psi_1 - \phi_3\psi_0 & = 0, \\ & \vdots \\ \psi_k - \phi_1\psi_{k-1} - \phi_2\psi_{k-2} - \cdots - \phi_k\psi_0 & = 0, \quad (k < p) \\ & \vdots \\ \psi_p - \phi_1\psi_{p-1} - \phi_2\psi_{p-2} - \cdots - \phi_p\psi_0 & = 0, \\ \psi_{p+1} - \phi_1\psi_p - \phi_2\psi_{p-1} - \cdots - \phi_p\psi_1 & = 0, \\ & \vdots \\ \psi_m - \phi_1\psi_{m-1} - \phi_2\psi_{m-2} - \cdots - \phi_p\psi_{m-p} & = 0, \quad (m > p) \\ & \vdots \end{cases}$$

解方程组可得

$$\begin{cases} \psi_0 & = & 1, \\ \psi_1 & = & \phi_1\psi_0 = \phi_1, \\ \psi_2 & = & \phi_1\psi_1 + \phi_2\psi_0 = \phi_1^2 + \phi_2, \\ \psi_3 & = & \phi_1\psi_2 + \phi_2\psi_1 + \phi_3\psi_0 = \phi_1^3 + 2\phi_1\phi_2 + \phi_3, \\ & \vdots & \\ \psi_k & = & \phi_1\psi_{k-1} + \phi_2\psi_{k-2} + \cdots + \phi_k\psi_0, \quad \text{if } k \le p \\ & \vdots & \\ \psi_m & = & \phi_1\psi_{m-1} + \phi_2\psi_{m-2} + \cdots + \phi_p\psi_{m-p}, \quad \text{if } m > p \end{cases}$$