第四章: 平稳时间序列模型

4.1 滑动平均过程

$$Y_t = e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2} - \cdots - \theta_q e_{t-q}, \qquad \text{其中 } e_t \sim WN(0, \sigma_e^2).$$

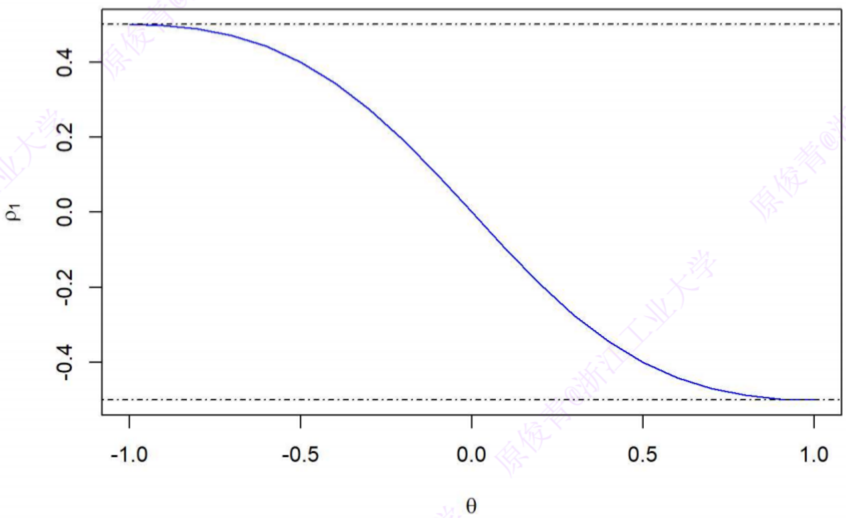
- 当$q=1$时，**MA(1)过程**：$Y_t = e_t - \theta e_{t-1}$.

  均值 $E(Y_t)=0$, 方差 $Var(Y_t)=(1+\theta^2)\sigma_e^2$, 协方差及自相关函数

$$\begin{aligned} Cov(Y_t, Y_{t-1}) &= Cov(e_t - \theta e_{t-1}, e_{t-1} - \theta e_{t-2}) = -\theta\sigma_e^2. \\ Cov(Y_t, Y_{t-2}) &= Cov(e_t - \theta e_{t-1}, e_{t-2} - \theta e_{t-3}) = 0. \\ Cov(Y_t, Y_{t-k}) &= Cov(e_t - \theta e_{t-1}, e_{t-k} - \theta e_{t-k-1}) = 0. \text{ if } k > 2. \end{aligned} \qquad \rho_k = \begin{cases} 1, & \text{if } k=0 \\ -\frac{\theta}{1+\theta^2}, & \text{if } k=1 \\ 0, & \text{if } k \ge 2 \end{cases}$$

  结论：MA(1)是平稳的，自相关函数在滞后 1 阶之后"截尾"。

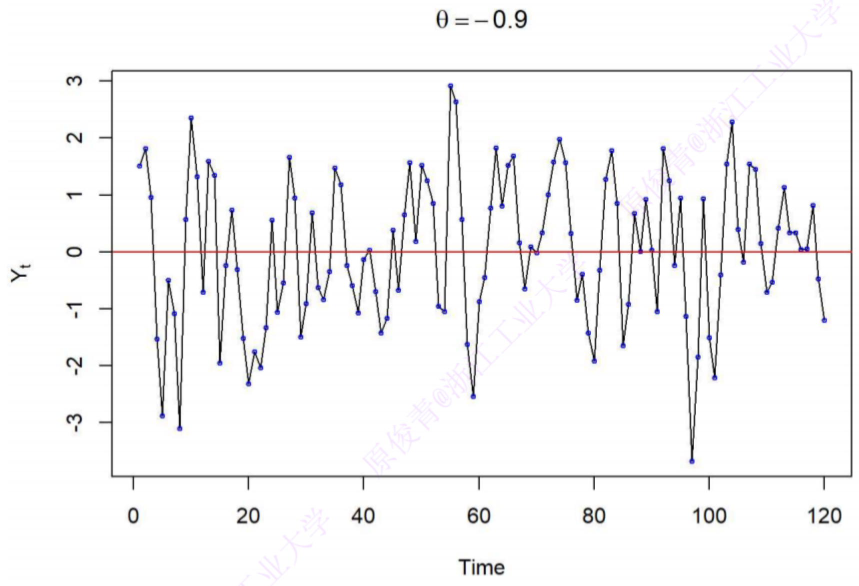
```
> theta <- seq(-1,1,by=0.1)
> rho_1 <- (-theta)/(1+theta^2)
> data.frame(theta=theta, rho_1=rho_1)
>
> plot(theta, rho_1, col='blue', type='l', ylab=expression(rho[1]), xlab=expression(theta))
> abline(h=0.5, lty=4); abline(h=-0.5, lty=4)
```



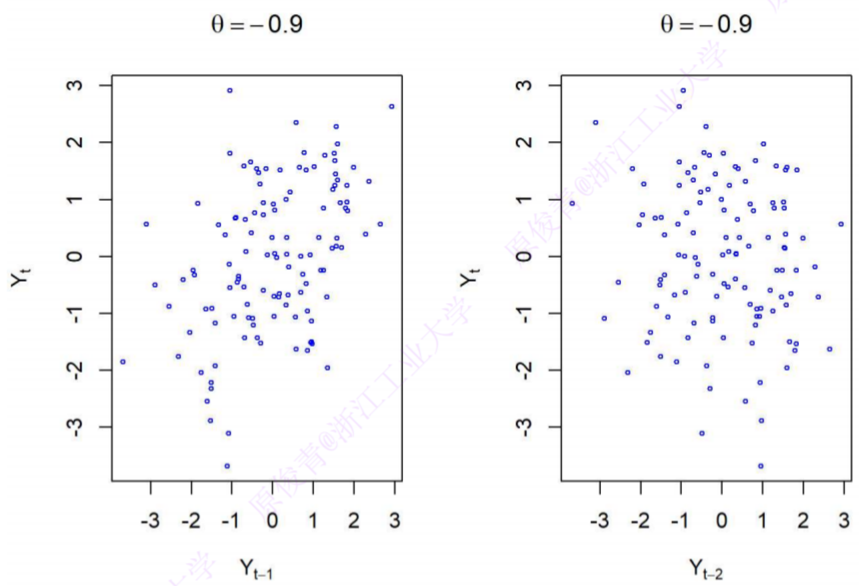

```
   theta      rho_1
1   -1.0  0.5000000
2   -0.9  0.4972376
3   -0.8  0.4878049
4   -0.7  0.4697987
5   -0.6  0.4411765
6   -0.5  0.4000000
7   -0.4  0.3448276
8   -0.3  0.2752294
9   -0.2  0.1923077
10  -0.1  0.0990099
11   0.0  0.0000000
12   0.1 -0.0990099
13   0.2 -0.1923077
14   0.3 -0.2752294
15   0.4 -0.3448276
16   0.5 -0.4000000
17   0.6 -0.4411765
18   0.7 -0.4697987
19   0.8 -0.4878049
20   0.9 -0.4972376
21   1.0 -0.5000000
```

例如：MA(1)序列$\theta=-0.9, \rho_1=0.4972$, 和$\theta=0.9, \rho_1=-0.4972$。

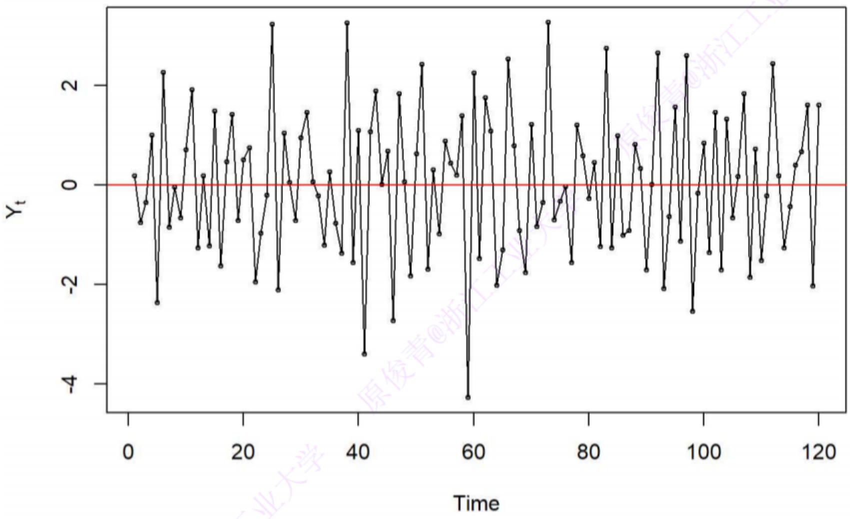
```
> # Exhibit 4.2
> data(ma1.2.s)  #theta=-0.9
> plot(ma1.2.s, ylab=expression(Y[t]), type='l', main=expression(theta==-0.9))
> points(ma1.2.s, col='blue', cex=0.5); abline(h=0, col='red')
```



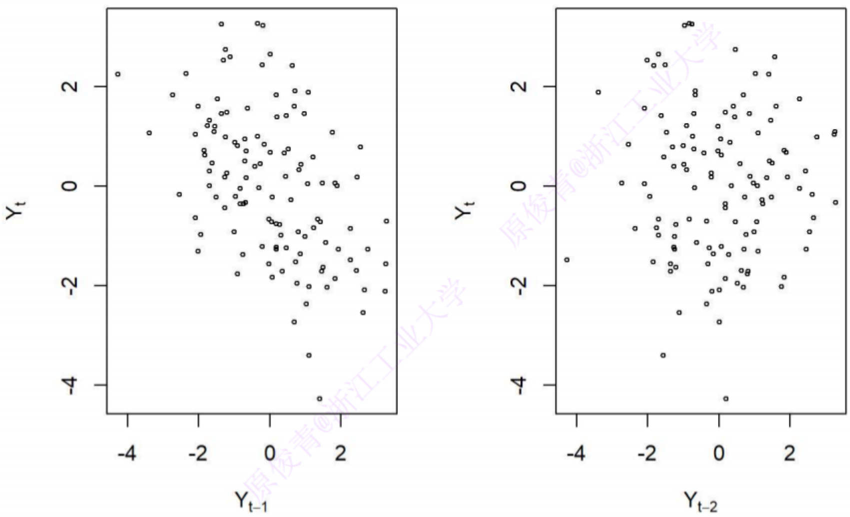

```
> opar=par(mfrow=c(1,2))
> # Exhibit 4.3
> plot(y=ma1.2.s, x=zlag(ma1.2.s), col='blue', ylab=expression(Y[t]), xlab=expression(Y[t-1]), type='p', main=expression(theta==-0.9), cex=0.5)
> # Exhibit 4.4
> plot(y=ma1.2.s, x=zlag(ma1.2.s,2), col='blue', ylab=expression(Y[t]), xlab=expression(Y[t-2]), type='p', main=expression(theta==-0.9), cex=0.5)
```



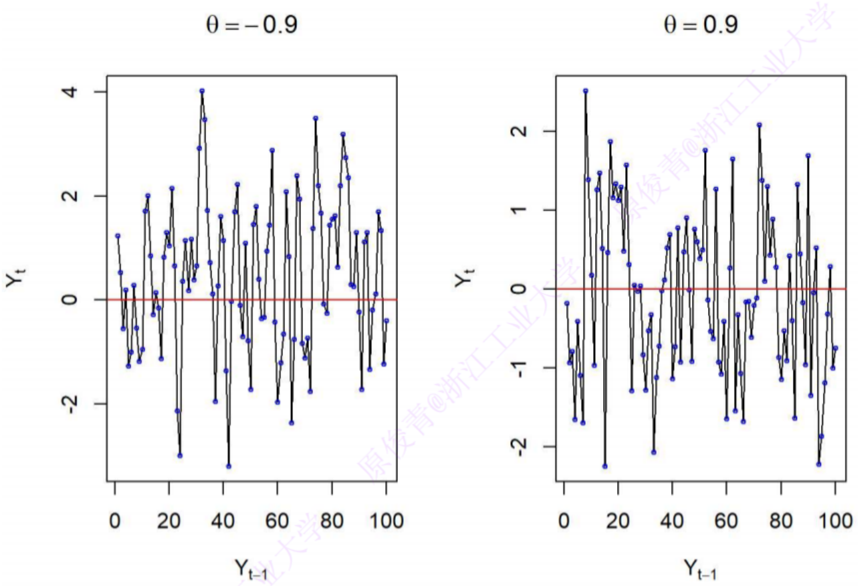

```
> par(opar)
>
> # Exhibit 4.5
> data(ma1.1.s)
> plot(ma1.1.s,ylab=expression(Y[t]),type='o', cex=0.5); abline(h=0, col='red')
```




```
> opar=par(mfrow=c(1,2))
> # Exhibit 4.6
> plot(y=ma1.1.s, x=zlag(ma1.1.s), ylab=expression(Y[t]), xlab=expression(Y[t-1]), type='p', cex=0.5)
> # Exhibit 4.7
> plot(y=ma1.1.s,x=zlag(ma1.1.s,2), ylab=expression(Y[t]), xlab=expression(Y[t-2]), type='p', cex=0.5)
```




```
>
>
> # An MA(1) series of length n=100 with MA coefficient equal to -0.9 and 0.9 respectively.
> set.seed(12345); y1=arima.sim(model=list(ma=-c(-0.9)),n=100)
> set.seed(54321); y2=arima.sim(model=list(MA=-c(0.9)), n=100)
> # Note that R uses the plus convention in the MA model formula so the additional minus sign.
> plot(y1, ylab=expression(Y[t]), xlab=expression(Y[t-1]), type='l', main=expression(theta==-0.9))
> points(y1, col='blue', cex=0.5); abline(h=0, col='red')
> plot(y2, ylab=expression(Y[t]), xlab=expression(Y[t-1]), type='l', main=expression(theta==0.9))
> points(y2, col='blue', cex=0.5); abline(h=0, col='red')
```




```
> par(opar)
```

- 当$q=2$时，**MA(2)过程：** $Y_t=e_t-\theta_1 e_{t-1}-\theta_2 e_{t-2}$.

  均值 $E(Y_t)=0$, 方差 $Var(Y_t)=(1+\theta_1^2+\theta_2^2)\sigma_e^2$, 协方差及自相关函数

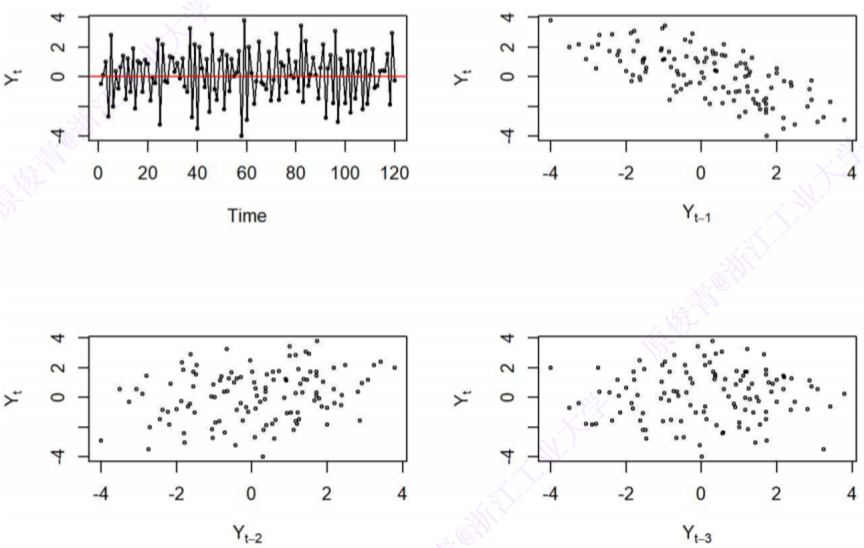
$$Cov(Y_t,Y_{t-1})=Cov(e_t-\theta_1 e_{t-1}-\theta_2 e_{t-2},e_{t-1}-\theta_1 e_{t-2}-\theta_2 e_{t-3})=(-\theta_1+\theta_1\theta_2)\sigma_e^2.$$
$$Cov(Y_t,Y_{t-2})=Cov(e_t-\theta_1 e_{t-1}-\theta_2 e_{t-2},e_{t-2}-\theta_1 e_{t-3}-\theta_2 e_{t-4})=-\theta_2\sigma_e^2.$$
$$Cov(Y_t,Y_{t-k})=Cov(e_t-\theta_1 e_{t-1}-\theta_2 e_{t-2},e_{t-k}-\theta_1 e_{t-k-1}-\theta_2 e_{t-k-2})=0.\text{ if } k\geq 3.$$

$$\rho_k=\begin{cases}1, & \text{if } k=0\\ \frac{-\theta_1+\theta_1\theta_2}{1+\theta_1^2+\theta_2^2}, & \text{if } k=1\\ \frac{-\theta_2}{1+\theta_1^2+\theta_2^2}, & \text{if } k=2\\ 0, & \text{if } k\geq 3\end{cases}$$

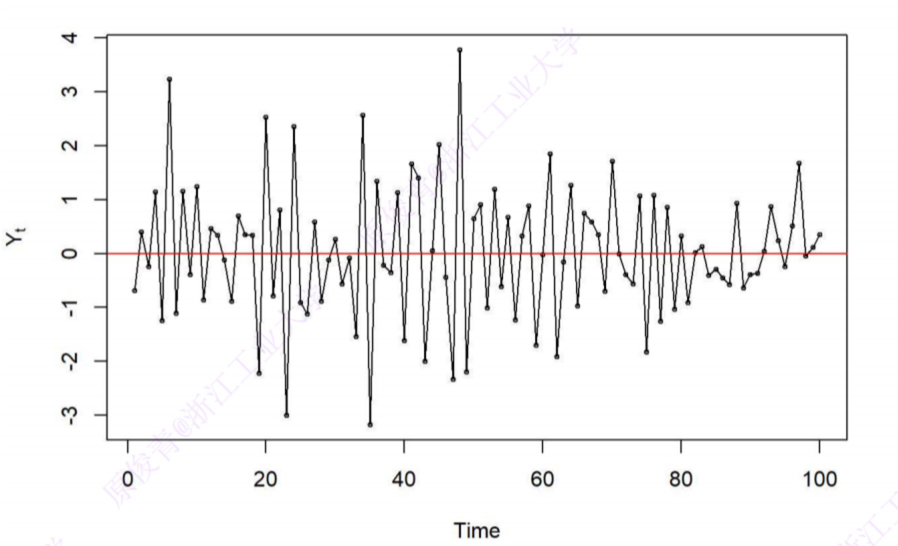
结论：MA(2)是平稳的，自相关函数在滞后 2 阶之后“截尾”。

例如：MA(2)序列$\theta_1=1,\theta_2=-0.6$。$\rho_1=-0.678,\rho_2=0.254.$

```
>
> data(ma2.s) ##theta_1=1, theta_2=-0.6
>
> opar=par(mfrow=c(2,2))
> # Exhibit 4.8
> plot(ma2.s, ylab=expression(Y[t]), type='o', cex=0.5); abline(h=0, col='red')
> # Exhibit 4.9
> plot(y=ma2.s, x=zlag(ma2.s), ylab=expression(Y[t]), xlab=expression(Y[t-1]), type='p', cex=0.5)
> # Exhibit 4.10
> plot(y=ma2.s, x=zlag(ma2.s,2), ylab=expression(Y[t]), xlab=expression(Y[t-2]), type='p', cex=0.5)
> # Exhibit 4.11
> plot(y=ma2.s, x=zlag(ma2.s,3), ylab=expression(Y[t]), xlab=expression(Y[t-3]), type='p', cex=0.5)
```



```
> par(opar)
>
> # An MA(2) series with MA coefficients equal to 1 and -0.6 and of length n=100.
> y=arima.sim(model=list(ma=-c(1, -0.6)), n=100)
> # Note that R uses the plus convention in the MA model formula so the additional minus sign.
> plot(y, ylab=expression(Y[t]), type='o', cex=0.5); abline(h=0, col='red')
```



- **MA(q)过程**：$Y_t = e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2} - \cdots - \theta_q e_{t-q}$

  均值 $E(Y_t) = 0$, 方差 $Var(Y_t) = (1 + \theta_1^2 + \theta_2^2 + \cdots + \theta_q^2)\sigma_e^2$, 协方差及自相关函数

$$Cov(Y_t, Y_{t-k}) = \begin{cases} \sigma_e^2(-\theta_k + \sum_{i=1}^{q-k} \theta_i\theta_{i+k}), & \text{if } 1 \le k \le q \\ 0. \text{ if } k > q. \end{cases} \qquad \rho_k = \begin{cases} 1, & \text{if } k = 0 \\ \frac{-\theta_k + \sum_{i=1}^{q-k} \theta_i\theta_{i+k}}{1 + \sum_{i=1}^{q} \theta_i^2}, & \text{if } 1 \le k \le q \\ 0, & \text{if } k > q \end{cases}$$

  结论：MA(q)是平稳的，自相关函数在滞后 q 阶之后“截尾”。

4.2 一般线性过程

$\{Y_t\}$ 表示观测到的时间序列，$\{e_t\}$ 表示未观测到的白噪声。

$$Y_t = \psi_0 e_t + \psi_1 e_{t-1} + \psi_2 e_{t-2} + \cdots = \sum_{i=0}^{\infty} \psi_i e_{t-i}.$$

- 期望为零 $E(Y_t) = 0$. 一般性假设$\psi_0 = 1$.
- 方差有限 $Var(Y_t) = \sigma_e^2(\sum_{i=0}^{\infty} \psi_i^2) < \infty$.
- 一阶协方差

$$\begin{aligned} Cov(Y_t, Y_{t-1}) &= Cov(\psi_0 e_t + \psi_1 e_{t-1} + \psi_2 e_{t-2} + \cdots, \psi_0 e_{t-1} + \psi_1 e_{t-2} + \psi_2 e_{t-3} + \cdots) \\ &= (\psi_0\psi_1 + \psi_1\psi_2 + \psi_2\psi_3 + \cdots)\sigma_e^2 = \sigma_e^2(\sum_{i=0}^{\infty} \psi_i\psi_{i+1}) \end{aligned}$$

- 二阶协方差

$$\begin{aligned} Cov(Y_t, Y_{t-2}) &= Cov(\psi_0 e_t + \psi_1 e_{t-1} + \psi_2 e_{t-2} + \cdots, \psi_0 e_{t-2} + \psi_1 e_{t-3} + \psi_2 e_{t-4} + \cdots) \\ &= (\psi_0\psi_2 + \psi_1\psi_3 + \psi_2\psi_4 + \cdots)\sigma_e^2 = \sigma_e^2(\sum_{i=0}^{\infty} \psi_i\psi_{i+2}) \end{aligned}$$

- k阶协方差

$$Cov(Y_t, Y_{t-k}) = \sigma_e^2(\sum_{i=0}^{\infty} \psi_i\psi_{i+k}), \qquad k \ge 0.$$

- k阶自相关函数

$$Corr(Y_t, Y_{t-k}) = \frac{\sum_{i=0}^{\infty} \psi_i\psi_{i+k}}{\sum_{i=0}^{\infty} \psi_i^2}, \quad k \ge 0.$$

例如，**一般线性过程**$\psi_j = \phi^j, |\phi| < 1$.

$$Y_t = e_t + \phi e_{t-1} + \phi^2 e_{t-2} + \cdots, \qquad |\phi| < 1.$$

  - 期望为零 $E(Y_t) = 0$, 方差有限 $Var(Y_t) = \sigma_e^2(\sum_{i=0}^{\infty} \phi^{2i}) = \frac{1}{1-\phi^2}\sigma_e^2$.

- k阶协方差 $Cov(Y_t, Y_{t-k}) = \sigma_e^2(\sum_{i=0}^{\infty}\phi^i\phi^{i+k}) = \frac{\phi^k}{1-\phi^2}\sigma_e^2.$
- k阶自相关函数 $Corr(Y_t, Y_{t-k}) = \phi^k, \quad k \geq 0.$
