4.4 ARMA(p,q)模型

$$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \cdots + \phi_p Y_{t-p} + e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2} - \cdots - \theta_q e_{t-q}, \qquad 其中\ e_t \sim WN(0, \sigma_e^2).$$

- 用滞后算子B表示

$$\begin{aligned}
\Phi(B)Y_t &= \Theta(B)e_t \\
其中，\Phi(B) &= 1 - \phi_1 B - \phi_2 B^2 + \cdots - \phi_p B^p, \\
\Theta(B) &= 1 - \theta_1 B - \theta_2 B^2 + \cdots - \theta_q B^q,
\end{aligned}$$

- 当自回归系数多项式的零点（$\Phi(z) = 0$的根）位于单位圆外时，ARMA(p,q)序列是平稳的。
- 当$p = q = 1$时，ARMA(1,1)过程

$$Y_t = \phi Y_{t-1} + e_t - \theta e_{t-1}, \qquad |\phi| < 1.$$

-均值 $\mu = \frac{0}{1-\phi} = 0$，因为$E(Y_t) = \phi E(Y_{t-1}) + E(e_t) - \theta E(e_{t-1})$.

-方差

$$\begin{aligned}
Cov(e_t, Y_t) &= Cov(e_t, \phi Y_{t-1} + e_t - \theta e_{t-1}) = \sigma_e^2 \\
Cov(e_{t-1}, Y_t) &= Cov(e_{t-1}, \phi Y_{t-1} + e_t - \theta e_{t-1}) = (\phi - \theta)\sigma_e^2 \\
\gamma_0 = Cov(Y_t, Y_t) &= Cov(\phi Y_{t-1} + e_t - \theta e_{t-1}, Y_t) \\
&= \phi Cov(Y_{t-1}, Y_t) + Cov(e_t, Y_t) - \theta Cov(e_{t-1}, Y_t) \\
&= \phi\gamma_1 + \sigma_e^2 - \theta(\phi - \theta)\sigma_e^2
\end{aligned}$$

-自协方差/自相关函数($k = 1$时)

$$\begin{aligned}
\gamma_1 = Cov(Y_t, Y_{t-1}) &= Cov(\phi Y_{t-1} + e_t - \theta e_{t-1}, Y_{t-1}) \\
&= \phi\gamma_0 - \theta\sigma_e^2
\end{aligned}$$

解二元线性方程组，

$$\begin{cases} \gamma_0 &= \phi\gamma_1 + \sigma_e^2 - \theta(\phi - \theta)\sigma_e^2 \\ \gamma_1 &= \phi\gamma_0 - \theta\sigma_e^2 \end{cases}$$

可得 $\gamma_0 = \frac{1-2\phi\theta+\theta^2}{1-\phi^2}\sigma_e^2$，$\gamma_1 = \frac{(\phi-\theta)(1-\phi\theta)}{1-\phi^2}\sigma_e^2$，从而可得一阶自相关函数为 $\rho_1 = \frac{\gamma_1}{\gamma_0} = \frac{(\phi-\theta)(1-\phi\theta)}{1-2\phi\theta+\theta^2}$

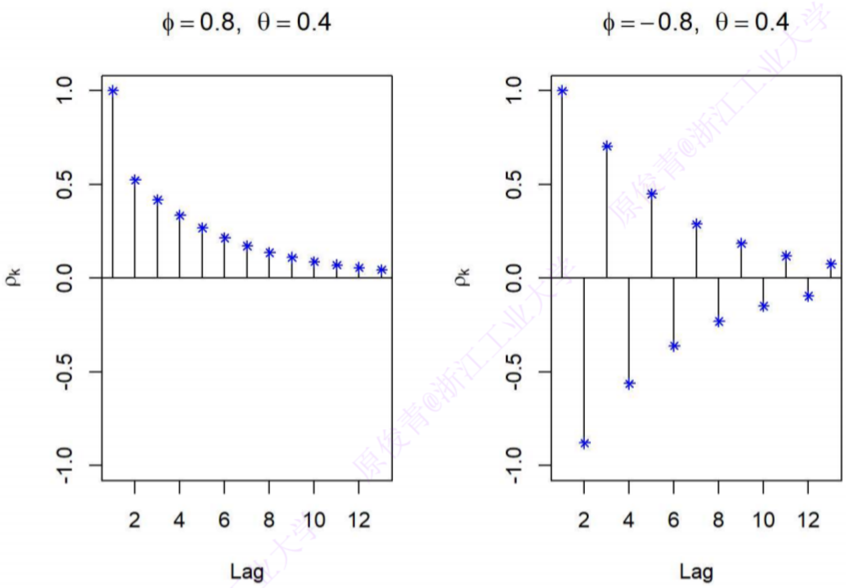
-自协方差/自相关函数, $\rho_k = \phi^{k-1}\rho_1 = \frac{(\phi-\theta)(1-\phi\theta)}{1-2\phi\theta+\theta^2}\phi^{k-1}, \quad k \geq 1.$

$$\begin{aligned}
\gamma_2 &= Cov(Y_t, Y_{t-2}) = Cov(\phi Y_{t-1} + e_t - \theta e_{t-1}, Y_{t-2}) = \phi\gamma_1 \\
\gamma_3 &= Cov(Y_t, Y_{t-3}) = Cov(\phi Y_{t-1} + e_t - \theta e_{t-1}, Y_{t-3}) = \phi\gamma_2 = \phi^2\gamma_1 \\
&\vdots \\
\gamma_k &= Cov(Y_t, Y_{t-k}) = Cov(\phi Y_{t-1} + e_t - \theta e_{t-1}, Y_{t-k}) = \phi\gamma_{k-1} = \phi^{k-1}\gamma_1
\end{aligned}$$

结论：**ARMA(1,1)是平稳的充分必要条件也是$|\phi| < 1$，自相关函数$\rho_k = \phi^{k-1}\rho_1$随着时滞$k$的增加而指数衰减。**阻尼因子是 ϕ，但递减开始于 ρ_1 （依赖于θ）；这与AR(1)的自相关函数$\rho_k \overset{AR(1)}{=} \phi^k\rho_0$不同.

例如，ARMA(1,1)序列$\phi = 0.8, \theta = 0.4$, 则$\rho_1 = 0.523, \rho_2 = 0.418, \rho_3 = 0.335$.

```
> ###ARMA(2)的自相关函数的理论值 #Exhibit 4.18
> Rho1 <- ARMAacf(ar=c(0.8),ma=-c(0.4), lag.max = 12, pacf=FALSE);Rho1
> Rho2 <- ARMAacf(ar=c(-0.8),ma=-c(0.4), lag.max = 12, pacf=FALSE);Rho2
>
> opar=par(mfrow=c(1,2))
> plot(Rho1, type='h', xlab='Lag', ylab=expression(rho[k]), main=expression(paste(phi==0.8, ",  ", theta==0.4)), ylim=c(-1.0,
1.0));
> points(Rho1, pch=8, col='blue', cex=0.8); abline(h=0);
> plot(Rho2, type='h', xlab='Lag', ylab=expression(rho[k]), main=expression(paste(phi==-0.8, ",  ", theta==0.4)), ylim=c(-1.0,
1.0));
> points(Rho2, pch=8, col='blue', cex=0.8); abline(h=0);
```




```
> par(opar)
         0          1          2          3          4          5          6 
1.00000000 0.52307692 0.41846154 0.33476923 0.26781538 0.21425231 0.17140185 
         7          8          9         10         11         12 
0.13712148 0.10969718 0.08775775 0.07020620 0.05616496 0.04493197 
          0           1           2           3           4           5 
 1.00000000 -0.88000000  0.70400000 -0.56320000  0.45056000 -0.36044800 
          6           7           8           9          10          11 
 0.28835840 -0.23068672  0.18454938 -0.14763950  0.11811160 -0.09448928 
         12 
 0.07559142 
```

- ARMA(1,1)的一般线性表示

$$
\begin{array}{rrcl}
 & Y_t & = & \phi Y_{t-1} + e_t - \theta e_{t-1} \\
\phi\cdot & Y_{t-1} & = & \phi Y_{t-2} + e_{t-1} - \theta e_{t-2} \qquad \cdot\phi \\
\phi^2\cdot & Y_{t-2} & = & \phi Y_{t-3} + e_{t-2} - \theta e_{t-3} \qquad \cdot\phi^2 \\
 & & \cdots & \\
\phi^{k-1}\cdot & Y_{t-(k-1)} & = & \phi Y_{t-k} + e_{t-(k-1)} - \theta e_{t-k} \qquad \cdot\phi^{k-1}
\end{array}
$$

求和可得 $Y_t = e_t + (\phi-\theta)e_{t-1} + (\phi-\theta)\phi e_{t-2} + \cdots + (\phi-\theta)\phi^{k-2}e_{t-(k-1)} - \theta\phi^{k-1}e_{t-k} + \phi^k Y_{t-k}$.

如果$|\phi| < 1$, 并且$k \to \infty$, 那么

$$Y_t = e_t + (\phi-\theta)(e_{t-1} + \phi e_{t-2} + \phi^2 e_{t-3} + \cdots) = e_t + (\phi-\theta)\sum_{i=1}^{\infty}\phi^{i-1}e_{t-i}.$$

- ARMA(p,q)过程的一般线性表示

  若$\Phi(z)$的零点在单位圆之外，Y_t 为平稳可逆的过程。设ARMA(p,q)的线性表示为$Y_t = \Psi(B)e_t = 1 + \psi_1 B + \psi_2 B^2 + \cdots$,

$$
\begin{aligned}
\Phi(B)Y_t &= \Theta(B)e_t \\
\Phi(B)\Psi(B)e_t &= \Theta(B)e_t \\
(1 - \phi_1 B - \phi_2 B^2 - \cdots)(1 + \psi_1 B + \psi_2 B^2 + \cdots)e_t &= (1 - \theta_1 B - \theta_2 B^2 - \cdots)e_t
\end{aligned}
$$

等式左边展开得,

$$
\begin{aligned}
&\psi_0 - \phi_1\psi_0 B - \phi_2\psi_0 B^2 - \phi_3\psi_0 B^3 - \cdots - \phi_p\psi_0 B^p \\
&\quad + \psi_1 B - \phi_1\psi_1 B^2 - \phi_2\psi_1 B^3 - \cdots - \phi_{p-1}\psi_1 B^p - \phi_p\psi_1 B^{p+1} \\
&\qquad \psi_2 B^2 - \phi_1\psi_2 B^3 - \cdots - \phi_{p-2}\psi_2 B^p - \phi_{p-1}\psi_2 B^{p+1} - \phi_p\psi_2 B^{p+2} \\
&\qquad \cdots\cdots
\end{aligned}
$$

左右比较e_j的系数，可以得到

$$\begin{cases}\psi_0 & =1,\\ \psi_1-\phi_1\psi_0 & =-\theta_1,\\ \psi_2-\phi_1\psi_1-\phi_2\psi_0 & =-\theta_2,\\ \psi_3-\phi_1\psi_2-\phi_2\psi_1-\phi_3\psi_0 & =-\theta_3,\\ & \vdots\\ \psi_k-\phi_1\psi_{k-1}-\phi_2\psi_{k-2}-\cdots-\phi_k\psi_0 & =-\theta_k,\quad (k<p)\\ & \vdots\\ \psi_p-\phi_1\psi_{p-1}-\phi_2\psi_{p-2}-\cdots-\phi_p\psi_0 & =-\theta_p,\\ \psi_{p+1}-\phi_1\psi_p-\phi_2\psi_{p-1}-\cdots-\phi_p\psi_1 & =-\theta_{p+1},\\ & \vdots\\ \psi_m-\phi_1\psi_{m-1}-\phi_2\psi_{m-2}-\cdots-\phi_p\psi_{m-p} & =-\theta_m,\quad (m>p)\\ & \vdots\end{cases}$$

令$\theta_j=0$(当$j>q$时)，解方程组可得

$$\begin{cases}\psi_0 & = & 1,\\ \psi_1 & = & -\theta_1+\phi_1\psi_0,\\ \psi_2 & = & -\theta_2+\phi_1\psi_1+\phi_2\psi_0,\\ \psi_3 & = & -\theta_3+\phi_1\psi_2+\phi_2\psi_1+\phi_3\psi_0,\\ & \vdots & \\ \psi_k & = & -\theta_k+\phi_1\psi_{k-1}+\phi_2\psi_{k-2}+\cdots+\phi_k\psi_0,\quad \text{if } k\le p\\ & \vdots & \\ \psi_m & = & -\theta_m+\phi_1\psi_{m-1}+\phi_2\psi_{m-2}+\cdots+\phi_p\psi_{m-p},\quad \text{if } m>p\end{cases}$$

- ARMA(p,q)过程的自相关函数

$$Y_t=\sum_{j=0}^{\infty}\psi_j e_{t-j}=\sum_{i=1}^{p}\phi_i Y_{t-i}-\sum_{j=0}^{q}\theta_j e_{t-j},\qquad \theta_0=-1.$$

-均值 $\mu=0$

-自协方差/自相关函数

$$Cov(e_{t-j},Y_{t-k})=Cov(e_{t-j},\sum_{m=0}^{\infty}\psi_m e_{t-k-m})=\begin{cases}\psi_{j-k}\sigma_e^2, & \text{if } j\ge k,\\ 0, & \text{if } j<k.\end{cases}$$

$$\begin{aligned}Cov(Y_t,Y_{t-k})&=Cov(\sum_{i=1}^{p}\phi_i Y_{t-i}-\sum_{j=0}^{q}\theta_j e_{t-j},Y_{t-k})\\ &=\sum_{i=1}^{p}\phi_i Cov(Y_{t-i},Y_{t-k})-\sum_{j=0}^{q}\theta_j Cov(e_{t-j},Y_{t-k})\\ \gamma_k&=\begin{cases}\sum_{i=1}^{p}\phi_i\gamma_{i-k}-\sigma_e^2\sum_{j=k}^{q}\theta_j\psi_{j-k}, & \text{if } 0\le k\le q,\\ \sum_{i=1}^{p}\phi_i\gamma_{i-k}, & \text{if } k>q.\end{cases}\end{aligned}$$

根据上式，$\gamma_{-k}=\gamma_k$, 求解$q+1$元线性方程组得$\gamma_0,\gamma_1,\cdots,\gamma_q$，然后利用 $\rho_i=\frac{\gamma_i}{\gamma_0}$ 计算 $\rho_1,\cdots,\rho_q$,

$$\rho_k=\phi_1\rho_{k-1}+\phi_2\rho_{k-2}+\cdots+\phi_p\rho_{k-p},\qquad \text{if } k>q.$$

4.5 MA(q)的可逆性

- 比较两个MA(1)过程：$Y_t=e_t-\theta e_{t-1}$ 和 $X_t=e_t-\frac{1}{\theta}e_{t-1}$

  自相关函数分别为

$$\rho_k^Y=\begin{cases}1, & \text{if } k=0\\ -\frac{\theta}{1+\theta^2}, & \text{if } k=1\\ 0, & \text{if } k\ge 2\end{cases}\qquad \rho_k^X=\begin{cases}1, & \text{if } k=0\\ -\frac{1/\theta}{1+(1/\theta)^2}=-\frac{\theta}{1+\theta^2}, & \text{if } k=1\\ 0, & \text{if } k\ge 2\end{cases}$$

  参数互为倒数的两个MA(1)过程有相同的自相关函数。 (习题4.4 on P.58)

- 比较两个MA(2)过程：$Y_t=e_t-\frac{1}{6}e_{t-1}-\frac{1}{6}\theta_2 e_{t-2}$ 和 $X_t=e_t+e_{t-1}-6e_{t-2}$.

  自相关函数分别为

$$\rho_k^Y=\begin{cases}1, & \text{if } k=0\\ \frac{-\theta_1+\theta_1\theta_2}{1+\theta_1^2+\theta_2^2}=\frac{-\frac{1}{6}+\frac{1}{6}\frac{1}{6}}{1+\frac{1}{36}+\frac{1}{36}}=\frac{-5}{38}, & \text{if } k=1\\ \frac{-\theta_2}{1+\theta_1^2+\theta_2^2}=\frac{-\frac{1}{6}}{1+\frac{1}{36}+\frac{1}{36}}=\frac{-6}{38}, & \text{if } k=2\\ 0, & \text{if } k\ge 3\end{cases}\qquad \rho_k^X=\begin{cases}1, & \text{if } k=0\\ \frac{-(-1)+(-1)\cdot 6}{1+(-1)^2+6^2}=\frac{-5}{38}, & \text{if } k=1\\ \frac{-6}{1+(-1)^2+6^2}=\frac{-6}{38}, & \text{if } k=2\\ 0, & \text{if } k\ge 3\end{cases}$$

  不同参数的这两个MA(2)过程却有相同的自相关函数。 (习题4.12 on P.59)

  相应的特征多项式分别为

$$\Theta^Y(z)=1-\frac{1}{6}z-\frac{1}{6}z^2=-\frac{1}{6}(z+3)(z-2)=(1+\frac{z}{3})(1-\frac{z}{2})$$
$$\Theta^X(w)=1+w-6w^2=(1+3w)(1-2w)$$

这两个MA(2)的特征多项式的根互为倒数， $z_{1,2}=\frac{1}{w_{1,2}}$.

```
> ARMAacf(ma=-c(1/6,1/6),lag.max=6)
> ARMAacf(ma=-c(-1,6),lag.max=6)
        0          1          2         3         4         5         6
1.0000000 -0.1315789 -0.1578947 0.0000000 0.0000000 0.0000000 0.0000000
        0          1          2         3         4         5         6
1.0000000 -0.1315789 -0.1578947 0.0000000 0.0000000 0.0000000 0.0000000
```

- 将MA(1)过程改写为$AR(\infty)$过程

$$\begin{array}{lcl} & Y_t=e_t-\theta e_{t-1} & \\ \theta\cdot & Y_{t-1}=e_{t-1}-\theta e_{t-2} & \cdot\theta \\ \theta^2\cdot & Y_{t-2}=e_{t-2}-\theta e_{t-3} & \cdot\theta^2 \\ & \cdots & \\ \theta^{k-1}\cdot & Y_{t-(k-1)}=e_{t-(k-1)}-\theta e_{t-k} & \cdot\theta^{k-1} \end{array}$$

求和可得 $Y_t+\theta Y_{t-1}+\theta^2Y_{t-2}+\cdots+\theta^{k-1}Y_{t-(k-1)}=e_t-\theta^k e_{t-k}$.

如果$|\theta|<1$, 并且$k\to\infty$, 那么

$$Y_t=e_t-\theta Y_{t-1}-\theta^2Y_{t-2}-\cdots-\theta^kY_{t-k}-\cdots$$

结论：$\underline{MA(1)\text{可逆} \iff |\theta|<1 \iff MA\text{系数多项式}\Theta(z)\text{的零点在单位圆之外。}}$

- MA(q)或者ARMA(q)模型的可逆性

MA系数多项式 $\Theta(z)=1-\theta_1z--\theta_2z^2-\cdots-\theta_qz^q$

MA特征方程为 $1-\theta_1z--\theta_2z^2-\cdots-\theta_qz^q=0$

类似于AR模型的平稳性条件，$\underline{MA(q)\text{模型可逆} \iff MA\text{系数多项式的零点}(MA\text{特征方程的根})}$在单位圆之外。

$$Y_t=\pi_1Y_{t-1}+\pi_2Y_{t-2}+\pi_3Y_{t-3}+\cdots+e_t$$

$\boxed{\text{对于一般的}ARMA(p,q)\text{模型，要求同时满足平稳性和可逆性条件。}}$