MATH 3283W Latex Project

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Problem: 2.2.32

A relation R on a set A is called **circular** if for all $a, b, c \in A, a\mathrm{R}b$ and $b\mathrm{R}c$ imply $c\mathrm{R}a$. Prove: A relation is an equivalence relation if and only if it is reflexive and circular.

Proof in the forward direction:

We must show that the relation R is symmetric and transitive. The reflexive requirement of an equivalence relation has already been fulfilled by the given proof statement, so we can assume that R is reflexive.

To show R is symmetric:

Given elements a, b of a set A, suppose $a\mathrm{R}b$. By the reflexive property provided, we know $b\mathrm{R}b$ to be a valid relation on A. By the circular relation provided in the proof statement:

$$a\mathrm{R}b,\ b\mathrm{R}c \Rightarrow c\mathrm{R}a$$

Since we know the reflexive property applies to R, we can let the second relation to be $b\mathrm{R}b$:

$$a\mathrm{R}b,\ b\mathrm{R}b \Rightarrow b\mathrm{R}a$$

This shows that, through applying the reflexive property, $a\mathrm{R}b \Rightarrow b\mathrm{R}a$; in other words, the symmetric property. So we can see that through the reflexive and circular properties of R, R must also be a symmetric relation.

To show R is transitive:

We will follow similar logic to show the transitivity of R.

Given elements a, b, c of a set A, suppose $a\mathrm{R}b$, $b\mathrm{R}c$. Then, again by the circular property defined above, we can see:

$$a\mathrm{R}b,\ b\mathrm{R}c \Rightarrow c\mathrm{R}a$$

By the symmetric property we have shown above, we can directly see that for any elements d, e of a set A, $d\mathrm{R}e$ implies $e\mathrm{R}d$. So we can apply that to our own relation of $c\mathrm{R}a$ to arrive at the result $a\mathrm{R}c$. So we have shown the transitive property to be valid on this reflexive circular relation as well.

Thus, by showing the relation R that is defined as both reflexive and circular to also be symmetric and transitive, we have shown it to be a valid equivalence relation.

Proof in the reverse direction:

We must show that a relation R is reflexive and circular if it is an equivalence relation. So let us assume R is an equivalence relation.

To show R is reflexive:

This is implied directly through the proof statement. Since R is an equivalence relation, we know it must be reflexive, given that equivalence relations must be transitive, symmetric, and reflexive.

To show R is circular:

Let the elements a, b, c be of a set A, and R is an equivalence relation on A. Then it follows that,

By the reflexive property:

$$a\mathrm{R}a \Rightarrow a\mathrm{R}a$$

By the symmetric property:

$$a\mathrm{R}b \Rightarrow b\mathrm{R}a$$

By the transitive property:

$$a\mathrm{R}b,\ b\mathrm{R}c \Rightarrow a\mathrm{R}c$$

Let $c = a$. Then we have $a\mathrm{R}b$, $b\mathrm{R}c \Rightarrow a\mathrm{R}c$. Since $a = c$, then the equivalent expression would be $a\mathrm{R}a$, which is valid under the reflexive property. This relation adheres to the defined circular relation in the problem, and as such we have proven R to be circular.

Thus, we have shown that if a relation R is an equivalence relation, it must also be reflexive and circular.