Conic section using Python

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Problem

Find the equation of the common tangent to the curves

$$y^2 = 8x \tag{1}$$
$$xy = -1 \tag{2}$$

Solution

(1) and (2) can be expressed in vector forms

$$\begin{aligned} & \mathbf{x}^\top \mathbf{V_1} \mathbf{x} + 2\mathbf{u_1}^\top \mathbf{x} + f_1 = 0 \\ \implies & \quad \mathbf{x}^\top \mathbf{V_1} \mathbf{x} + 2\mathbf{u_1}^\top \mathbf{x} = 0 \end{aligned} \tag{3}$$

$$\begin{aligned} & \mathbf{x}^\top \mathbf{V_2} \mathbf{x} + 2\mathbf{u_2}^\top \mathbf{x} + f_2 = 0 \\ \implies & \quad \mathbf{x}^\top \mathbf{V_2} \mathbf{x} + f_2 = 0 \end{aligned} \tag{4}$$

where,

$$\mathbf{V_1} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{u_1} = \begin{pmatrix} -8 \\ 0 \end{pmatrix}, \quad f_1 = 0$$
$$\mathbf{V_2} = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}, \quad \mathbf{u_2} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad f_2 = 1.$$

In projective plane (3) and (4) becomes

$$\mathbf{X}^\top \mathbf{M_1} \mathbf{X} = 0 \tag{5}$$
$$\mathbf{X}^\top \mathbf{M_2} \mathbf{X} = 0 \tag{6}$$

where,

$$\mathbf{M_1} = \begin{pmatrix} \mathbf{V_1} & \mathbf{u_1} \\ \mathbf{u_1}^\top & f_1 \end{pmatrix},$$
$$\mathbf{M_2} = \begin{pmatrix} \mathbf{V_2} & \mathbf{u_2} \\ \mathbf{u_2}^\top & f_2 \end{pmatrix}, \text{ and } \mathbf{X} = \begin{pmatrix} \mathbf{x} \\ 1 \end{pmatrix}$$

Here, $\mathbf{M_1}$ and $\mathbf{M_2}$ are symmetric, and $\mathbf{X}$ is called the homogeneous coordinate of $\mathbf{x}$.

The dual conics of (5) and (6) is given by

$$\mathbf{X}^\top \left|\mathbf{M_1}\right| \mathbf{M_1^{-1}} \mathbf{X} = 0 \tag{7}$$
$$\mathbf{X}^\top \left|\mathbf{M_2}\right| \mathbf{M_2^{-1}} \mathbf{X} = 0 \tag{8}$$

The dual conics (7) and (8) intersect at most at 4 points $\mathbf{x_i}$. The corresponding common tangent(s) is given by

$$\begin{pmatrix} \mathbf{x_i}^\top & 1 \end{pmatrix} \mathbf{X} = 0 \tag{9}$$

$$\left|\mathbf{M_1}\right| \mathbf{M_1^{-1}} \text{ can be written as } \begin{pmatrix} \mathbf{N_1} & \mathbf{w_1} \\ \mathbf{w_1}^\top & k_1 \end{pmatrix}$$
$$\left|\mathbf{M_2}\right| \mathbf{M_2^{-1}} \text{ can be written as } \begin{pmatrix} \mathbf{N_2} & \mathbf{w_2} \\ \mathbf{w_2}^\top & k_2 \end{pmatrix}$$

Let's take,

$$\mathbf{N} = \mathbf{N_1} + \mu \mathbf{N_2} \quad \text{and} \quad \mathbf{w} = \mathbf{w_1} + \mu \mathbf{w_2}.$$

The intersection of dual conics is given by

$$\left|\mathbf{M_1}\right| \mathbf{M_1^{-1}} + \mu \left|\mathbf{M_2}\right| \mathbf{M_2^{-1}} = 0 \tag{10}$$

The intersection (10) represents a pair of straight lines if

$$\left|\left|\mathbf{M_1}\right| \mathbf{M_1^{-1}} + \mu \left|\mathbf{M_2}\right| \mathbf{M_2^{-1}}\right| = 0$$
$$\text{and } \left|\mathbf{N_1} + \mu \mathbf{N_2}\right| < 0$$

On solving, we get $\mu = -16$. The point of intersection of two lines of the pair of straight lines represented by (10), point H, is given by

$$\mathbf{h} = -\mathbf{N}^{-1} \mathbf{w} \tag{11}$$

The normal vectors of the two lines of (10) are given by

$$\mathbf{n_1} = \mathbf{P} \begin{pmatrix} \sqrt{|\lambda_1|} \\ \sqrt{|\lambda_2|} \end{pmatrix} \text{ and}$$
$$\mathbf{n_2} = \mathbf{P} \begin{pmatrix} \sqrt{|\lambda_1|} \\ -\sqrt{|\lambda_2|} \end{pmatrix}$$

where $\lambda_i, \mathbf{P}$ are the eigenparameters of $\mathbf{N}$. Then, the direction vectors of lines of (10) are given by

$$\mathbf{m_1} = \mathbf{R}_{\frac{\pi}{2}} \mathbf{n_1} \text{ and } \mathbf{m_2} = \mathbf{R}_{\frac{\pi}{2}} \mathbf{n_2} \tag{12}$$

where,

$$\mathbf{R}_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Therefore, the lines of (10) are given by

$$\mathbf{x} = \mathbf{h} + \kappa\mathbf{m_1} \quad \text{and} \quad \mathbf{x} = \mathbf{h} + \kappa\mathbf{m_2}$$

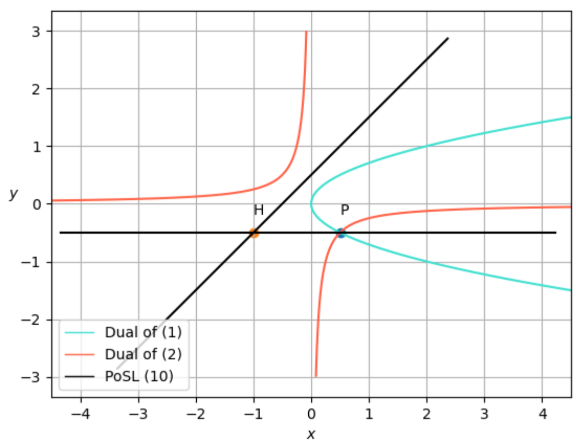

Fig. 1: Intersection of the Dual Curves

The points of intersection of the lines $\mathbf{x} = \mathbf{h} + \kappa\mathbf{m_k}$ with a dual conic, say $\mathbf{X}^\top \left|\mathbf{M_1}\right| \mathbf{M_1^{-1}X}$, which are also the point(s) intersection of the dual conics, are given by

$$\mathbf{x_j} = \mathbf{h} + \kappa_j\mathbf{m_k} \tag{13}$$

where,

$$\kappa_j = \frac{1}{\mathbf{m_1}^T\mathbf{N_1m_1}}\left(-\mathbf{m_1^\top}\left(\mathbf{N_1h}+\mathbf{w_1}\right)\right.$$
$$\left.\pm\sqrt{\left[\mathbf{m_1^\top}\left(\mathbf{N_1h}+\mathbf{w_1}\right)\right]^2 - \left(\mathbf{h^\top N_1h} + 2\mathbf{w_1^\top h} + k_1\right)\left(\mathbf{m_1^\top N_1m_1}\right)}\right)$$

Upon substituting respective values we find that the dual conics (7) and (8) inersect at a single point, $P = \mathbf{a} = \begin{pmatrix} 0.5 \\ -0.5 \end{pmatrix}$.

Then from (9),

$$\begin{pmatrix}\mathbf{a}^\top & 1\end{pmatrix}\mathbf{X} = 0$$
$$\mathbf{a}^\top\mathbf{x} + 1 = 0$$
$$\begin{pmatrix}0.5 & -0.5\end{pmatrix}\mathbf{x} = -1 \tag{14}$$

Thus, (14) is the equation of common tangent to the curves (1) and (2).

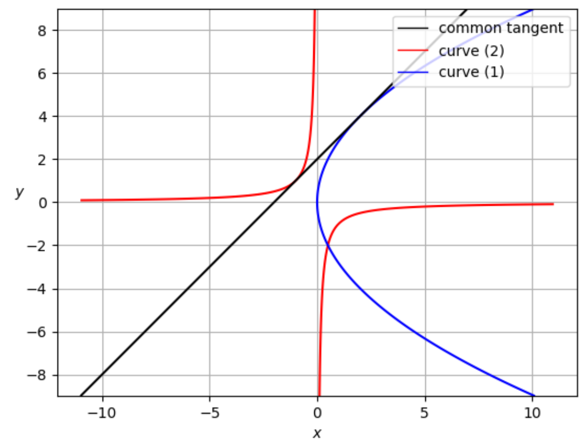

Fig. 2: Common tangent to the given curves