Optimization Assignment

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Problem

Show that the greatest triangle inscribed in an ellipse has area of $\frac{3\sqrt{3}}{4}ab$, where a and b are the semi-axes.

Using Calculus

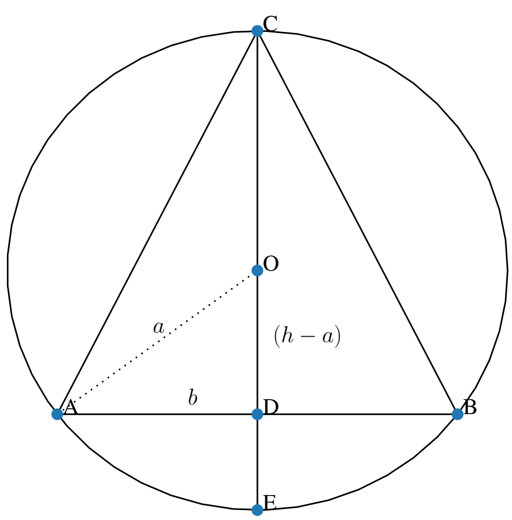
Let's take given ellipse as

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \tag{1}$$

We can write (1) as

$$x^2+\frac{y^2}{\left(\frac{b}{a}\right)^2}=a^2$$
$$x^2+Y^2=a^2 \tag{2}$$

Scaling along y-axis by a factor of $\frac{b}{a}$, we get new Y-axis. Here, (1) becomes (2), which is a circle with radius a.



Let CE be perpendicular to AB. Then $CD=h$. CD will pass through the center O of the circle, since h should be maximum for a given chord, for $\triangle ABC$ to have maximum area. Also $AD=DB=b$. Therefore,

$$R=ar(\triangle ABC)=\frac{1}{2}(2b)h=bh \tag{3}$$

From $\triangle ADO$, we have

$$\begin{aligned} b^2 &= a^2-(h-a)^2 \\ &= 2ah-h^2 \end{aligned} \tag{4}$$

Substituting (4) in R^2 from (3),

$$\begin{aligned} R^2 &= (2ah-h^2)h^2 \\ &= 2ah^3-h^4 \end{aligned} \tag{5}$$
$$\implies f(x)=2ax^3-x^4 \tag{6}$$
$$f'(x)=6ax^2-4x^3 \tag{7}$$
$$f''(x)=12ax-12x^2 \tag{8}$$

On solving for maxima of $f(x)$ from (7) and (8), we get $x=\frac{3}{2}a$. Upon substitution in (6), we have

$$R^2=\frac{27}{16}a^4$$
$$\implies R=\frac{3\sqrt{3}}{4}a^2 \tag{9}$$

On scaling back to the original axes, we get the area of the ellipse (1), S. (9) becomes

$$S=\frac{3\sqrt{3}}{4}a^2\left(\frac{b}{a}\right)$$
$$S=\frac{3\sqrt{3}}{4}ab \tag{10}$$

Gradient Ascent

Since R is area, R will be maximum if R^2 is also maximum. Using gradient ascent method we can find the maximum of R^2, i.e. (6).

$$x_{n+1}=x_n+\alpha\nabla f(x_n) \tag{11}$$
$$x_{n+1}=x_n+\alpha\left(6ax^2-4x^3\right) \tag{12}$$

Suppose the semi-axes of the ellipse are $a=5$ and $b=3$. Taking $x_0=1$, $\alpha=0.0001$ and precision $=0.000000001$, values obtained using python are:

$$\boxed{\text{Maxima Point}=7.499999956932471} \tag{13}$$
$$\boxed{\text{Maxima}=1054.687499999999} \tag{14}$$

The corresponding value of S obtained using (13) and (14) is,

$$\boxed{S_{opt} = 19.48557158514986}$$

The corresponding value of S obtained using (10) is,

$$\boxed{S = 19.48557158514987}$$

Since, $S_{opt} = S$ it is verified that the greatest triangle inscribed in an ellipse has area of $\frac{3\sqrt{3}}{4}ab$.