Optimization Assignment

Ahilan R - FWC22090

Problem

Find the maximum area of a triangle which can be inscribed in an ellipse.

Using Calculus

Let's take an ellipse,

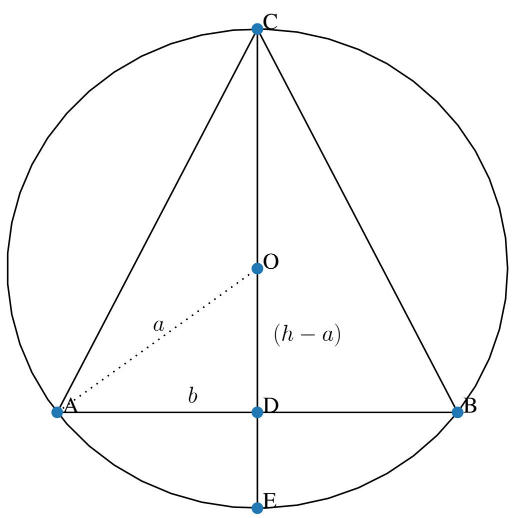
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{1}$$

We can write (1) as

$$x^2 + \frac{y^2}{\left(\frac{b}{a}\right)^2} = a^2$$

$$x^2 + Y^2 = a^2 \tag{2}$$

Scaling along y-axis by a factor of $\frac{b}{a}$, we get new Y-axis. Here, (1) becomes (2), which is a circle with radius a.



Let CE be perpendicular to AB. Then $CD = h$. CD will pass through the center O of the circle, since h should be maximum for a given chord, for $\triangle ABC$ to have maximum area. Also $AD = DB = b$. Therefore,

$$R = ar(\triangle ABC) = \frac{1}{2}(2b)h = bh \tag{3}$$

From $\triangle ADO$, we have

$$\begin{aligned} b^2 =& a^2 - (h-a)^2 \\ =& 2ah - h^2 \end{aligned} \tag{4}$$

Substituting (4) in R^2 from (3),

$$\begin{aligned} R^2 &= (2ah - h^2)h^2 \\ &= 2ah^3 - h^4 \end{aligned} \tag{5}$$

$$\implies f(x) = 2ax^3 - x^4 \tag{6}$$

$$f'(x) = 6ax^2 - 4x^3 \tag{7}$$

$$f''(x) = 12ax - 12x^2 \tag{8}$$

On solving for maxima of $f(x)$ from (7) and (8), we get $x = \frac{3}{2}a$. Upon substitution in (6), we have

$$\begin{aligned} R^2 &= \frac{27}{16}a^4 \\ \implies R &= \frac{3\sqrt{3}}{4}a^2 \end{aligned} \tag{9}$$

On scaling back to the original axes, we get the area of the ellipse (1), S. (9) becomes

$$S = R\left(\frac{b}{a}\right) \tag{10}$$

$$S = \frac{3\sqrt{3}}{4}ab \tag{11}$$

Gradient Ascent

Since area is a positive quantity, R will be maximum if R^2 is also maximum. So, the optimization problem is,

$$\begin{aligned} \max_{x} \quad & 2ax^3 - x^4 \\ \text{s.t.} \quad & x > 0, \\ & x < 2a \end{aligned} \tag{12}$$

Using gradient ascent method we can find the maximum of R^2, i.e. (6).

$$\begin{aligned} x_{n+1} &= x_n + \alpha \nabla f(x_n) \\ &= x_n + \alpha\left(6a{x_n}^2 - 4{x_n}^3\right) \end{aligned} \tag{13}$$

Suppose the semi-axes of the ellipse are $a = 5$ and $b = 3$. Taking $x_0 = 1$, $\alpha = 0.0001$ and precision $= 0.000000001$, values obtained using python are:

$$\boxed{\text{Maxima Point} = 7.499999956932471} \tag{14}$$

$$\boxed{\text{Maxima} = 1054.687499999999} \tag{15}$$

The corresponding value of S obtained using (10) and (15) is,

$$\boxed{S_{opt} = 19.48557158514986}$$

The corresponding value of S obtained using (11) is,

$$\boxed{S = 19.48557158514987}$$

$$\implies S_{opt} = S$$

Hence, we have found the maximum area of a triangle that can be inscribed in an ellipse.