Conic section using Python

Ahilan R - FWC22090

Problem

Find the equation of the common tangent to the curves

$$y^2 = 8x \tag{1}$$
$$xy = -1 \tag{2}$$

Solution

(1) and (2) can be expressed in vector forms

$$\mathbf{x}^\top \mathbf{V_1}\mathbf{x} + 2\mathbf{u_1}^\top \mathbf{x} + f_1 = 0 \tag{3}$$
$$\mathbf{x}^\top \mathbf{V_2}\mathbf{x} + 2\mathbf{u_2}^\top \mathbf{x} + f_2 = 0 \tag{4}$$

where,

$$\mathbf{V_1} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{u_1} = \begin{pmatrix} -8 \\ 0 \end{pmatrix}, \quad f_1 = 0 \tag{5}$$
$$\mathbf{V_2} = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}, \quad \mathbf{u_2} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad f_2 = 1. \tag{6}$$

In projective plane (3) and (4) becomes

$$\mathbf{X}^\top \mathbf{M_1}\mathbf{X} = 0 \tag{7}$$
$$\mathbf{X}^\top \mathbf{M_2}\mathbf{X} = 0 \tag{8}$$

where,

$$\mathbf{M_1} = \begin{pmatrix} \mathbf{V_1} & \mathbf{u_1} \\ \mathbf{u_1}^\top & f_1 \end{pmatrix}, \tag{9}$$
$$\mathbf{M_2} = \begin{pmatrix} \mathbf{V_2} & \mathbf{u_2} \\ \mathbf{u_2}^\top & f_2 \end{pmatrix}, \text{ and } \mathbf{X} = \begin{pmatrix} \mathbf{x} \\ 1 \end{pmatrix} \tag{10}$$

Here, $\mathbf{M_1}$ and $\mathbf{M_2}$ are symmetric, and $\mathbf{X}$ is called the homogeneous coordinate of $\mathbf{x}$.
The dual conics of (7) and (8) is given by

$$\mathbf{X}^\top \left|\mathbf{M_1}\right| \mathbf{M_1^{-1}}\mathbf{X} = 0 \tag{11}$$
$$\mathbf{X}^\top \left|\mathbf{M_2}\right| \mathbf{M_2^{-1}}\mathbf{X} = 0 \tag{12}$$

The dual conics (11) and (12) intersect at most at 4 points $\mathbf{x_i}$. The corresponding common tangent(s) is given by

$$\begin{pmatrix} \mathbf{x_i}^\top & 1 \end{pmatrix} \mathbf{X} = 0 \tag{13}$$

$$\left|\mathbf{M_1}\right| \mathbf{M_1^{-1}} \text{ can be written as } \begin{pmatrix} \mathbf{N_1} & \mathbf{w_1} \\ \mathbf{w_1}^\top & k_1 \end{pmatrix} \tag{14}$$

$$\left|\mathbf{M_2}\right| \mathbf{M_2^{-1}} \text{ can be written as } \begin{pmatrix} \mathbf{N_2} & \mathbf{w_2} \\ \mathbf{w_2}^\top & k_2 \end{pmatrix} \tag{15}$$

Let's take,

$$\mathbf{N} = \mathbf{N_1} + \mu\mathbf{N_2} \quad \text{and} \quad \mathbf{w} = \mathbf{w_1} + \mu\mathbf{w_2}. \tag{16}$$

The intersection of dual conics is given by

$$\left|\mathbf{M_1}\right| \mathbf{M_1^{-1}} + \mu \left|\mathbf{M_2}\right| \mathbf{M_2^{-1}} = 0 \tag{17}$$

The intersection (17) represents a pair of straight lines if

$$\left|\left|\mathbf{M_1}\right| \mathbf{M_1^{-1}} + \mu \left|\mathbf{M_2}\right| \mathbf{M_2^{-1}}\right| = 0 \tag{18}$$
$$\text{and } \left|\mathbf{N_1} + \mu\mathbf{N_2}\right| < 0 \tag{19}$$

On solving, we get $\mu = -16$. The point of intersection of two lines of the pair of straight lines represented by (17), point H, is given by

$$\mathbf{h} = -\mathbf{N}^{-1}\mathbf{w} \tag{20}$$

The normal vectors of the two lines of (17) are given by

$$\mathbf{n_1} = \mathbf{P} \begin{pmatrix} \sqrt{|\lambda_1|} \\ \sqrt{|\lambda_2|} \end{pmatrix} \text{ and} \tag{21}$$
$$\mathbf{n_2} = \mathbf{P} \begin{pmatrix} \sqrt{|\lambda_1|} \\ -\sqrt{|\lambda_2|} \end{pmatrix} \tag{22}$$

where $\lambda_i, \mathbf{P}$ are the eigenparameters of $\mathbf{N}$. Then, the direction vectors of lines of (17) are given by

$$\mathbf{m_1} = \mathbf{R}_{\frac{\pi}{2}}\mathbf{n_1} \text{ and } \mathbf{m_2} = \mathbf{R}_{\frac{\pi}{2}}\mathbf{n_2} \tag{23}$$

where,

$$\mathbf{R}_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Therefore, the lines of (17) are given by

$$\mathbf{x} = \mathbf{h} + \kappa\mathbf{m_1} \quad \text{and} \quad \mathbf{x} = \mathbf{h} + \kappa\mathbf{m_2} \tag{24}$$

The points of intersection of the lines $\mathbf{x} = \mathbf{h} + \kappa\mathbf{m_k}$ with a dual conic, say $\mathbf{X}^\top \left|\mathbf{M_1}\right| \mathbf{M_1^{-1}X}$, which are also the point(s) intersection of the dual conics, are given by

$$\mathbf{x_i} = \mathbf{h} + \kappa_j \mathbf{m_k} \tag{25}$$

where,

$$\kappa_j = \frac{1}{\mathbf{m_1}^T\mathbf{N_1}\mathbf{m_1}} \Big(-\mathbf{m_1}^\top\left(\mathbf{N_1}\mathbf{h} + \mathbf{w_1}\right) \\ \pm \sqrt{\left[\mathbf{m_1}^\top\left(\mathbf{N_1}\mathbf{h} + \mathbf{w_1}\right)\right]^2 - \left(\mathbf{h}^\top\mathbf{N_1}\mathbf{h} + 2\mathbf{w_1}^\top\mathbf{h} + k_1\right)\left(\mathbf{m_1}^\top\mathbf{N_1}\mathbf{m_1}\right)}\Big) \tag{26}$$

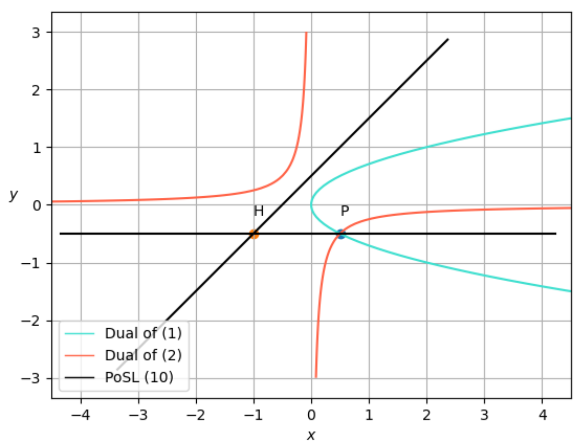

Fig. 1: Intersection of the Dual Curves

Upon substituting respective values we find that the dual conics (11) and (12) inersect at a single point, $P = \mathbf{p} = \begin{pmatrix} 0.5 \\ -0.5 \end{pmatrix}$.

Then from (13),

$$\begin{pmatrix} \mathbf{p}^\top & 1 \end{pmatrix} \mathbf{X} = 0 \tag{27}$$

$$\mathbf{p}^\top \mathbf{x} + 1 = 0 \tag{28}$$

$$\begin{pmatrix} 0.5 & -0.5 \end{pmatrix} \mathbf{x} = -1 \tag{29}$$

Thus, (29) is the equation of common tangent to the curves (1) and (2).

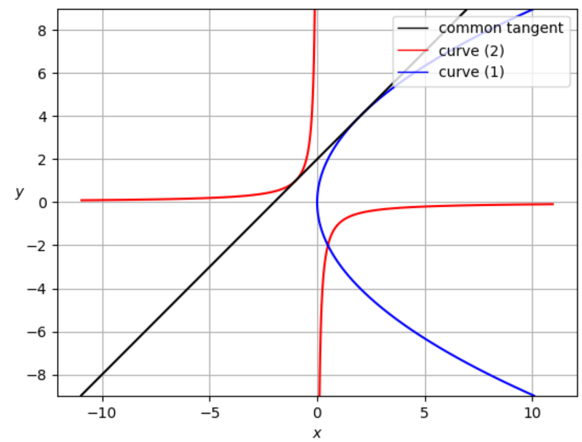

Fig. 2: Common tangent to the given curves