Circles using Python

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Problem

Let C_1 and C_2 be two circles with C_2 lying inside C_1. A circle C lying inside C_1 touches C_1 internally and C_2 externally. Identify the locus of the center of C.

Solution

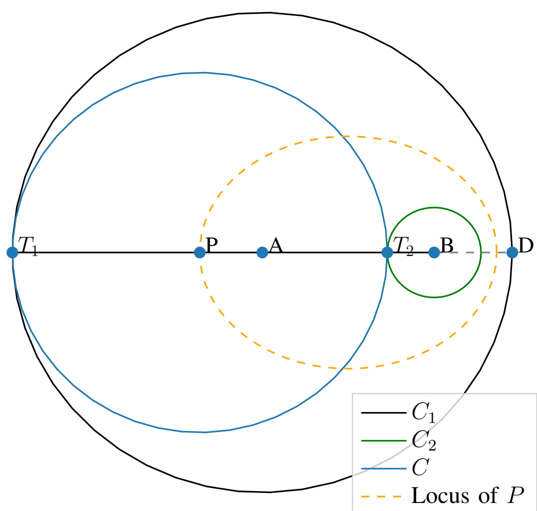

Fig. 1: Rough sketch of given problem

Symbol	**Description**
A	Center of C_1
B	Center of C_2
P	Center of C
r_1	Radius of C_1
r_2	Radius of C_2
r	Radius of C

From the diagram we can deduce,

$$
\begin{aligned}
AP &= AT_1 - PT_1 \\
&= r_A - r
\end{aligned} \tag{1}
$$

$$
\begin{aligned}
BP &= BT_2 + PT_2 \\
&= r - r_A
\end{aligned} \tag{2}
$$

Adding (1) and (2) we get,

$$
\begin{aligned}
AP + BP &= (r_A - r) + (r - r_B) \\
&= r_A + r_B = c
\end{aligned} \tag{3}
$$

Since R.H.S of (3) is constant, we can say that the locus of P is ellipse, by the defintion of ellipse.

Construction

Let the circles C_1, C_2 and C be represented by following equations respectively,

$$
\begin{aligned}
\mathbf{x}^\top \mathbf{I}\mathbf{x} + 2\mathbf{u_1}^\top \mathbf{x} + f_2 = 0 \\
\mathbf{x}^\top \mathbf{I}\mathbf{x} + 2\mathbf{u_2}^\top \mathbf{x} + f_2 = 0 \\
\mathbf{x}^\top \mathbf{I}\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0
\end{aligned}
$$

where,

Center of C_1	$\mathbf{A}$	$-\mathbf{u_1}$
Center of C_2	$\mathbf{B}$	$-\mathbf{u_2}$
Center of C	$\mathbf{P}$	$-\mathbf{u}$

For the sake of simplicity, let's assume $\mathbf{A} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

$$
\begin{aligned}
AP &= \|\mathbf{A} - \mathbf{P}\| \\
&= \|\mathbf{u}\| \\
BP &= \|\mathbf{B} - \mathbf{P}\| \\
&= \|\mathbf{u_2} - \mathbf{u}\|
\end{aligned}
$$

Now, (3) becomes

$$
\begin{aligned}
\|\mathbf{u_2} - \mathbf{u}\| + \|\mathbf{u}\| &= c \\
\|\mathbf{u_2} - \mathbf{u}\|^2 &= (c - \|\mathbf{u}\|)^2 \\
\left(\mathbf{u_2} - \mathbf{u}^\top\right)(\mathbf{u_2} - \mathbf{u}) &= c^2 + \mathbf{u}^\top \mathbf{u} - 2c\,\|\mathbf{u}\| \\
\mathbf{u_2}^\top \mathbf{u_2} + \mathbf{u}^\top \mathbf{u} - 2\mathbf{u_2}^\top \mathbf{u} &= c^2 + \mathbf{u}^\top \mathbf{u} - 2c\,\|\mathbf{u}\| \\
-2\mathbf{u_2}^\top \mathbf{u} &= c^2 - \mathbf{u_2}^\top \mathbf{u_2} - 2c\,\|\mathbf{u}\|
\end{aligned}
$$

Taking $\mathbf{u_2}^\top \mathbf{u_2} - c^2 = k$,

$$
\begin{aligned}
\left(k - 2\mathbf{u_2}^\top \mathbf{u}\right)^2 &= (-2c\,\|\mathbf{u}\|)^2 \\
k^2 + 4\left(\mathbf{u_2}^\top \mathbf{u}\right)\left(\mathbf{u_2}^\top \mathbf{u}\right) - 4k\mathbf{u_2}^\top \mathbf{u} &= 4c^2 \mathbf{u}^\top \mathbf{u} \\
4c^2 \mathbf{u}^\top \mathbf{u} - 4\mathbf{u}^\top \left(\mathbf{u_2}\mathbf{u_2}^\top\right)\mathbf{u} + 4k\mathbf{u_2}^\top \mathbf{u} - k^2 &= 0 \\
\implies \quad \mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{w}^\top \mathbf{x} + g &= 0
\end{aligned} \tag{4}
$$

(4) is the equation of the ellipse, the locus of P, where,

$$\mathbf{V} = 4c^2\mathbf{I} - 4\mathbf{u_2}\mathbf{u_2}^\top,$$
$$\mathbf{w} = 2k\mathbf{u_2} \quad \text{and} \quad g = -k^2.$$

Construction Parameters Used

Symbol	Description	Value
A	Center of C_1	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$
B	Center of C_2	$\begin{pmatrix} 11 \\ 0 \end{pmatrix}$
P	Center of C	$\begin{pmatrix} -4 \\ 0 \end{pmatrix}$
r_1	Radius of C_1	16
r_2	Radius of C_2	3
r	Radius of C	12
r	Radius of C	12
T_1D	Diameter of C_1	
gap	Length of BD	2

TABLE I: Circles parameters

On substituting respective values taken from I, the equation of the locus of, P, (4) will be

$$\mathbf{x}^\top \begin{pmatrix} 60 & 0 \\ 0 & 90.25 \end{pmatrix} \mathbf{x} + 2 \begin{pmatrix} 330 & 0 \end{pmatrix} \mathbf{x} - 3600 = 0 \tag{5}$$

Symbol	Description	Value
a	Semi-major axis	9.5
b	Semi-minor axis	7.746
c	Center of Ellipse	$\begin{pmatrix} 5.5 \\ 0 \end{pmatrix}$

TABLE II: Ellipse parameters from (5)