Description of the methods used in the SDFC package

Robin Y.

October 7, 2020

Abstract

Contents

Introduction

Our goal is to find the parameters of a probability distribution, e.g. for a Normal distribution $\mathcal{N}(\mu, \sigma)$ to find μ and σ (see. Sec. 2 for the definition). We place ourselves in a very general context: we assumes that μ and σ can vary with time t, and depend of "true" parameters θ. Mathematically, we write:

$$\begin{pmatrix} \mu_t \\ \sigma_t \end{pmatrix} = \mathcal{L}(\theta, X_t) = \begin{pmatrix} \mathcal{L}_\mu(\theta, X_t) \\ \mathcal{L}_\sigma(\theta, X_t) \end{pmatrix}$$

The variable $\theta \in \mathbb{R}^p$ is the parameter that we want to infer, X_t is called a *co-variable*, which describes the time dependance. The function $\mathcal{L}$ is called the *link function*.

Notations

$$Z = \frac{Y - \mu}{\sigma}$$
$$\mathcal{Z} = 1 + \xi Z$$

1 Generic estimation

1.1 Maximum likelihood estimation

1.2 Bayesian estimation

2 Normal distribution

2.1 Definition

Definition 2.1 (Normal distribution)**.** A random variable Y follows the *Normal distribution*, noted $Y \sim \mathcal{N}(\mu, \sigma)$, $\mu \in \mathbb{R}$, $\sigma > 0$ if

$$\mathbb{P}(Y \leq y) := \int_{-\infty}^{y} f(\tilde{y}).\mathrm{d}\tilde{y}$$
$$f(y) := \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}Z^2\right]$$

The function f is the density of the Normal distribution, μ is called the *location* parameter, and σ is called the *scale* parameter.

2.2 Likelihood function

For a time series of observations $(Y_k)_{k=1\ldots K}$, the negative log-likelihood function is given by:

$$\mathbf{NL}(Y_k) := \sum_{k=1}^{K} \frac{1}{2} Z_k^2 + \log(\sigma_k)$$

Proposition 2.1. *The gradient of the negative log-likelihood of the Normal distribution is given by*

$$\partial_{\theta_j}\mathbf{NL}(Y_k) = \sum_{k=1}^{K} \partial_{\theta_j}\mu \left[-\frac{Z_k}{\sigma_k}\right] + \partial_{\theta_j}\sigma \left[-\frac{Y_k Z_k}{\sigma_k^2} + \frac{\mu_k Z_k}{\sigma_k^2} + \frac{1}{\sigma_k}\right]$$

Proof. Start with the gradient in the direction θ_j

$$\partial_{\theta_j}\mathbf{NL}(Y_k) = \sum_{k=1}^{K} \partial_{\theta_j}\left[Z_k\right] Z_k + \frac{\partial_{\theta_j}\sigma_k}{\sigma_k}$$

We have

$$\partial_{\theta_j} Z_k = -\frac{Y_k}{\sigma_k^2}\partial_{\theta_j}\sigma_k - \frac{\sigma_k\partial_{\theta_j}\mu_k - \mu_k\partial_{\theta_j}\sigma_k}{\sigma_k^2}$$

Consequently

$$\begin{aligned}\partial_{\theta_j}\mathbf{NL}(Y_k) &= \sum_{k=1}^{K}\left[-\frac{Y_k}{\sigma_k^2}\partial_{\theta_j}\sigma_k - \frac{\partial_{\theta_j}\mu_k}{\sigma_k} + \frac{\mu_k}{\sigma_k^2}\partial_{\theta_j}\sigma_k\right] Z_k + \frac{\partial_{\theta_j}\sigma_k}{\sigma_k} \\ &= \partial_{\theta_j}\mu\left[-\frac{Z_k}{\sigma_k}\right] + \partial_{\theta_j}\sigma\left[-\frac{Y_k Z_k}{\sigma_k^2} + \frac{\mu_k Z_k}{\sigma_k^2} + \frac{1}{\sigma_k}\right]\end{aligned}$$

□

3 GEV distribution

3.1 Definition

Definition 3.1 (Normal distribution)**.** A random variable Y follows the *Generalized Extreme Value* distribution (GEV), noted $Y \sim \mathrm{GEV}(\mu, \sigma, \xi)$, $\mu, \xi \in \mathbb{R}$, $\sigma > 0$ if

$$\begin{aligned}\mathbb{P}(Y \leq y) &:= \exp\left[-\mathcal{Z}^{-1/\xi}\right] = \int_{-\infty}^{y} f(\tilde{y}).\mathrm{d}\tilde{y} \\ f(y) &:= \frac{1}{\sigma}\mathcal{Z}^{-1-1/\xi}\exp\left[\mathcal{Z}^{-1/\xi}\right]\end{aligned}$$

The function f is the density of the GEV distribution, μ is called the *location* parameter, σ is called the *scale* parameter and ξ is called the *shape* parameter.

3.2 Likelihood function

For a time series of observations $(Y_k)_{k=1\ldots K}$, the negative log-likelihood function is given by:

$$\mathbf{NL}(Y_k) := \sum_{k=1}^{K}\left(1 + \frac{1}{\xi_k}\right)\log(\mathcal{Z}_k) + \mathcal{Z}_k^{-1/\xi_k} + \log(\sigma_k)$$

Proposition 3.1. *The gradient of the negative log-likelihood of the GEV distribution is given by*

$$
\begin{aligned}
\partial_{\theta_j}\mathbf{NL}(Y_k) &= \sum_{k=1}^{K} \partial_{\theta_j}\mu\left[-\frac{\xi_k}{\sigma_k}\kappa_k\right] + \partial_{\theta_j}\sigma\left[\frac{1}{\sigma_k} - \xi_k\frac{Z_k}{\sigma_k}\kappa_k\right] + \partial_{\theta_j}\xi\left[\log(\mathcal{Z}_k)\frac{\mathcal{Z}_k^{-1/\xi_k}-1}{\xi_k^2} + Z_k\kappa_k\right] \\
\kappa_k &= \frac{1}{\mathcal{Z}_k}\left(1+\frac{1}{\xi_k}\right) - \frac{\mathcal{Z}_k^{-1/\xi_k-1}}{\xi_k}
\end{aligned}
$$

Proof. Start with the gradient in the direction θ_j

$$
\begin{aligned}
\partial_{\theta_j}\mathbf{NL}(Y_k) &= \sum_{k=1}^{K} \partial_{\theta_j}\left[\left(1+\frac{1}{\xi_k}\right)\log(\mathcal{Z}_k)\right] + \partial_{\theta_j}\left[\mathcal{Z}_k^{-1/\xi_k}\right] + \partial_{\theta_j}\log(\sigma_k) \\
&= \sum_{k=1}^{K}\left[-\frac{\partial_{\theta_j}\xi_k}{\xi_k^2}\log(\mathcal{Z}_k) + \left(1+\frac{1}{\xi_k}\right)\frac{\partial_{\theta_j}\mathcal{Z}_k}{\mathcal{Z}_k}\right] + \partial_{\theta_j}\left[\mathcal{Z}_k^{-1/\xi_k}\right] + \frac{\partial_{\theta_j}\sigma_k}{\sigma_k}
\end{aligned}
$$

The term $\partial_{\theta_k}\mathcal{Z}_k$ is given by

$$
\begin{aligned}
\partial_{\theta_j}\mathcal{Z}_k &= \partial_{\theta_j}\left[1+\xi_k\frac{Y_k-\mu_k}{\sigma_k}\right] \\
&= Z_k\partial_{\theta_j}\xi_k + \xi_k\partial_{\theta_j}\frac{Y_k-\mu_k}{\sigma_k} \\
&= Z_k\partial_{\theta_j}\xi_k + \xi_k\left[-Y_k\frac{\partial_{\theta_j}\sigma_k}{\sigma_k^2} - \frac{\sigma_k\partial_{\theta_j}\mu_k - \mu_k\partial_{\theta_j}\sigma_k}{\sigma_k^2}\right] \\
&= \partial_{\theta_j}\mu_k\left[-\frac{\xi_k}{\sigma_k}\right] + \partial_{\theta_j}\sigma_k\left[\xi_k\frac{\mu_k-Y_k}{\sigma_k^2}\right] + \partial_{\theta_j}\xi_k\left[Z_k\right]
\end{aligned}
$$

And the term $\partial_{\theta_k}\mathcal{Z}_k^{-1/\xi_k}$ is given by

$$
\begin{aligned}
\partial_{\theta_j}\left[\mathcal{Z}_k^{-1/\xi_k}\right] &= \partial_{\theta_j}\exp\left[-\frac{1}{\xi_k}\log\left[\mathcal{Z}_k\right]\right] \\
&= \partial_{\theta_j}\left[-\frac{1}{\xi_k}\log\left[\mathcal{Z}_k\right]\right]\mathcal{Z}_k^{-1/\xi_k} \\
&= \left[\frac{\partial_{\theta_j}\xi_k}{\xi_k^2}\log\left[\mathcal{Z}_k\right] - \frac{1}{\xi_k}\frac{\partial_{\theta_j}\mathcal{Z}_k}{\mathcal{Z}_k}\right]\mathcal{Z}_k^{-1/\xi_k} \\
&= \partial_{\theta_j}\xi_k\left[\log\left[\mathcal{Z}_k\right]\frac{\mathcal{Z}_k^{-1/\xi_k}}{\xi_k^2}\right] - \frac{\mathcal{Z}_k^{-1/\xi_k-1}}{\xi_k}\partial_{\theta_j}\mathcal{Z}_k
\end{aligned}
$$

Thus, we find

$$
\begin{aligned}
\partial_{\theta_j}\mathbf{NL}(Y_k) &= \sum_{k=1}^{K} -\frac{\partial_{\theta_j}\xi_k}{\xi_k^2}\log(\mathcal{Z}_k) + \left(1+\frac{1}{\xi_k}\right)\frac{\partial_{\theta_j}\mathcal{Z}_k}{\mathcal{Z}_k} + \partial_{\theta_j}\left[\mathcal{Z}_k^{-1/\xi_k}\right] + \frac{\partial_{\theta_j}\sigma_k}{\sigma_k} \\
&= \sum_{k=1}^{K} \partial_{\theta_j}\xi_k\left[\log\left[\mathcal{Z}_k\right]\frac{\mathcal{Z}_k^{-1/\xi_k}-1}{\xi_k^2}\right] + \underbrace{\left[\frac{1}{\mathcal{Z}_k}\left(1+\frac{1}{\xi_k}\right) - \frac{\mathcal{Z}_k^{-1/\xi_k-1}}{\xi_k}\right]}_{=\kappa_k}\partial_{\theta_j}\mathcal{Z}_k + \frac{\partial_{\theta_j}\sigma_k}{\sigma_k} \\
&= \sum_{k=1}^{K} \partial_{\theta_j}\mu_k\left[-\frac{\xi_k}{\sigma_k}\kappa_k\right] + \partial_{\theta_j}\sigma_k\left[\frac{1}{\sigma_k} - \xi_k\frac{Z_k}{\sigma_k}\kappa_k\right] + \partial_{\theta_j}\xi_k\left[\log\left[\mathcal{Z}_k\right]\frac{\mathcal{Z}_k^{-1/\xi_k}-1}{\xi_k^2} + Z_k\kappa_k\right]
\end{aligned}
$$

□

3.3 L-moments estimator

See Hosking et al. (1985).

Summary

References

Hosking, J. R. M. et al. (1985). "Estimation of the Generalized Extreme-Value Distribution by the Method of Probability-Weighted Moments". In: *Technometrics* 27.3, pp. 251–261.

Tabulars

Figures