

# MATH 211

## Online Asynchronous Survey in Calculus and Analytical Geometry

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# L'Hospital's Rule

A limit of the form

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)}$$

where both

$$\lim_{x \rightarrow a} f(x) = 0$$

and

$$\lim_{x \rightarrow a} g(x) = 0$$

is called **indeterminate form of type  $\frac{0}{0}$** .

Often cancellation of common factors helps:

$$\lim_{x \rightarrow 1} \frac{x^2 - x}{x^2 - 1} = \lim_{x \rightarrow 1} \frac{(x - 1)x}{(x - 1)(x + 1)} = \lim_{x \rightarrow 1} \frac{x}{x + 1} = \frac{1}{2}$$

But not for examples like:

$$\lim_{x \rightarrow 0} \frac{\sin x}{x}$$

and

$$\lim_{x \rightarrow 1} \frac{\ln x}{x - 1}$$

# L'Hospital's Rule

A limit of the form

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)}$$

where both

$$\lim_{x \rightarrow a} f(x) = \pm\infty$$

and

$$\lim_{x \rightarrow a} g(x) = \pm\infty$$

is called **indeterminate form of type**  $\frac{\infty}{\infty}$ .

Often helps to divide by highest power of  $x$  in the denominator:

$$\lim_{x \rightarrow \infty} \frac{x^2 - 1}{2x^2 + 1} = \lim_{x \rightarrow \infty} \frac{1 - \frac{1}{x^2}}{2 + \frac{1}{x^2}} = \frac{1}{2}$$

But not for examples like:

$$\lim_{x \rightarrow \infty} \frac{\ln x}{x - 1}$$

# L'Hospital's Rule

## L'Hospital's Rule

Suppose  $f$  and  $g$  are differentiable and  $g'(x) \neq 0$  near  $a$ , and

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)}$$

is an indeterminate form of type  $\frac{0}{0}$  or  $\frac{\infty}{\infty}$ . Then

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$

if the limit on the right side exists or is  $-\infty$  or  $\infty$ .

(near  $a$  = on an open interval containing  $a$  except possibly  $a$  itself)

Before applying L'Hospital's Rule it is important to verify that:

$$\lim_{x \rightarrow a} f(x) = 0 \quad \text{and} \quad \lim_{x \rightarrow a} g(x) = 0$$

or

$$\lim_{x \rightarrow a} f(x) = \pm\infty \quad \text{and} \quad \lim_{x \rightarrow a} g(x) = \pm\infty$$

# L'Hospital's Rule

Find

$$\lim_{x \rightarrow 1} \frac{\ln x}{x - 1}$$

We have

$$\lim_{x \rightarrow 1} \ln x = \ln 1 = 0$$

and

$$\lim_{x \rightarrow 1} (x - 1) = 0$$

and hence we can apply l'Hospital's Rule:

$$\lim_{x \rightarrow 1} \frac{\ln x}{x - 1} = \lim_{x \rightarrow 1} \frac{\frac{d}{dx} \ln x}{\frac{d}{dx} (x - 1)} = \lim_{x \rightarrow 1} \frac{(\frac{1}{x})}{1} = \lim_{x \rightarrow 1} \frac{1}{x} = 1$$

# L'Hospital's Rule

Find

$$\lim_{x \rightarrow 0} \frac{\sin x}{x}$$

We have

$$\lim_{x \rightarrow 0} \sin x = 0$$

and

$$\lim_{x \rightarrow 0} x = 0$$

Hence we can apply l'Hospital's Rule:

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = \lim_{x \rightarrow 0} \frac{\frac{d}{dx} \sin x}{\frac{d}{dx} x} = \lim_{x \rightarrow 0} \frac{\cos x}{1} = 1$$

# L'Hospital's Rule

Find

$$\lim_{x \rightarrow \infty} \frac{e^x}{x^2}$$

We have

$$\lim_{x \rightarrow \infty} e^x = \infty$$

and

$$\lim_{x \rightarrow \infty} x^2 = \infty$$

Hence we can apply l'Hospital's Rule:

$$\lim_{x \rightarrow \infty} \frac{e^x}{x^2} = \lim_{x \rightarrow \infty} \frac{\frac{d}{dx} e^x}{\frac{d}{dx} x^2} = \lim_{x \rightarrow \infty} \frac{e^x}{2x}$$

Again we have:

$$\lim_{x \rightarrow \infty} e^x = \infty$$

and

$$\lim_{x \rightarrow \infty} 2x = \infty$$

So we can again use l'Hospital's Rule:

$$\lim_{x \rightarrow \infty} \frac{e^x}{x^2} = \lim_{x \rightarrow \infty} \frac{e^x}{2x} = \lim_{x \rightarrow \infty} \frac{\frac{d}{dx} e^x}{\frac{d}{dx} 2x} = \lim_{x \rightarrow \infty} \frac{e^x}{2} = \infty$$

# L'Hospital's Rule

Find

$$\lim_{x \rightarrow \infty} \frac{\ln x}{\sqrt[3]{x}}$$

We have

$$\lim_{x \rightarrow \infty} \ln x = \infty$$

and

$$\lim_{x \rightarrow \infty} \sqrt[3]{x} = \infty$$

Hence we can apply l'Hospital's Rule:

$$\lim_{x \rightarrow \infty} \frac{\ln x}{\sqrt[3]{x}} = \lim_{x \rightarrow \infty} \frac{\frac{d}{dx} \ln x}{\frac{d}{dx} \sqrt[3]{x}} = \lim_{x \rightarrow \infty} \frac{\left(\frac{1}{x}\right)}{\frac{1}{3}x^{-\frac{2}{3}}} = \lim_{x \rightarrow \infty} \frac{3}{\sqrt[3]{x}} = 0$$



# L'Hospital's Rule

Find

$$\lim_{x \rightarrow \pi^-} \frac{\sin x}{1 - \cos x}$$

If we were to apply l'Hospital's Rule:

$$\lim_{x \rightarrow \pi^-} \frac{\sin x}{1 - \cos x} = \lim_{x \rightarrow \pi^-} \frac{\cos x}{\sin x} = -\infty$$

However, this is wrong!

We have  $\lim_{x \rightarrow \pi^-} (1 - \cos x) = 1 - (-1) = 2$ .

$$\lim_{x \rightarrow \pi^-} \frac{\sin x}{1 - \cos x} = \frac{0}{1 - (-1)} = 0$$

Before applying l'Hospital's Rule, always check that the limit is an indeterminate form  $\frac{0}{0}$  or  $\frac{\infty}{\infty}$ .

# L'Hospital's Rule

L'Hospital's Rule is valid for one-sided limits and limits at infinity:

$$\lim_{x \rightarrow a^-} \frac{f(x)}{g(x)}$$

$$\lim_{x \rightarrow a^+} \frac{f(x)}{g(x)}$$

$$\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)}$$

$$\lim_{x \rightarrow -\infty} \frac{f(x)}{g(x)}$$

# L'Hospital's Rule

A limit of the form

$$\lim_{x \rightarrow a} (f(x)g(x))$$

where

$$\lim_{x \rightarrow a} f(x) = 0 \quad \text{and} \quad \lim_{x \rightarrow a} g(x) = \pm\infty$$

is called **indeterminate form of type**  $0 \cdot \infty$ .

We then rewrite the limit as:

$$\lim_{x \rightarrow a} (f(x)g(x)) = \lim_{x \rightarrow a} \frac{f(x)}{1/g(x)}$$

an indeterminate form of type  $\frac{0}{0}$ , or as

$$\lim_{x \rightarrow a} (f(x)g(x)) = \lim_{x \rightarrow a} \frac{g(x)}{1/f(x)}$$

an indeterminate form of type  $\frac{\infty}{\infty}$ .

# L'Hospital's Rule

Evaluate the limit

$$\lim_{x \rightarrow 0^+} x \ln x$$

We have

$$\lim_{x \rightarrow 0^+} x = 0$$

and

$$\lim_{x \rightarrow 0^+} \ln x = -\infty$$

Thus we can choose for rewriting to:

$$\lim_{x \rightarrow 0^+} \frac{x}{1/\ln x}$$

or

$$\lim_{x \rightarrow 0^+} \frac{\ln x}{1/x}$$

We choose the 2nd since the derivatives are easier:

$$\lim_{x \rightarrow 0^+} x \ln x = \lim_{x \rightarrow 0^+} \frac{\ln x}{1/x} = \lim_{x \rightarrow 0^+} \frac{1/x}{-1/x^2} = \lim_{x \rightarrow 0^+} (-x) = 0$$

# L'Hospital's Rule

A limit of the form

$$\lim_{x \rightarrow a} (f(x) - g(x))$$

where

$$\lim_{x \rightarrow a} f(x) = \infty \quad \text{and} \quad \lim_{x \rightarrow a} g(x) = \infty$$

is called **indeterminate form of type**  $\infty - \infty$ .

We then rewrite the limit as a **quotient**.

# L'Hospital's Rule

Evaluate the limit

$$\lim_{x \rightarrow (\pi/2)^-} (\sec x - \tan x)$$

Then  $\lim_{x \rightarrow (\pi/2)^-} \sec x = \infty$  and  $\lim_{x \rightarrow (\pi/2)^-} \tan x = \infty$

We use a common denominator:

$$\lim_{x \rightarrow (\pi/2)^-} (\sec x - \tan x) = \lim_{x \rightarrow (\pi/2)^-} \frac{1 - \sin x}{\cos x}$$

Now  $\lim_{x \rightarrow (\pi/2)^-} (1 - \sin x) = 0$  and  $\lim_{x \rightarrow (\pi/2)^-} \cos x = 0$

Hence we can apply l'Hospital's Rule:

$$\begin{aligned} \lim_{x \rightarrow (\pi/2)^-} (\sec x - \tan x) &= \lim_{x \rightarrow (\pi/2)^-} \frac{1 - \sin x}{\cos x} \\ &= \lim_{x \rightarrow (\pi/2)^-} \frac{-\cos x}{-\sin x} = 0 \end{aligned}$$

# L'Hospital's Rule

A limit of the form

$$\lim_{x \rightarrow a} [f(x)]^{g(x)}$$

is an indeterminate form

- ▶ **of type**  $0^0$  if  $\lim_{x \rightarrow a} f(x) = 0$  and  $\lim_{x \rightarrow a} g(x) = 0$
- ▶ **of type**  $\infty^0$  if  $\lim_{x \rightarrow a} f(x) = \infty$  and  $\lim_{x \rightarrow a} g(x) = 0$
- ▶ **of type**  $1^\infty$  if  $\lim_{x \rightarrow a} f(x) = 1$  and  $\lim_{x \rightarrow a} g(x) = \infty$

Each of these cases can be treated by writing the limit as:

$$\begin{aligned}\lim_{x \rightarrow a} [f(x)]^{g(x)} &= \lim_{x \rightarrow a} e^{\ln([f(x)]^{g(x)})} \\ &= \lim_{x \rightarrow a} e^{g(x) \ln f(x)} = e^{\lim_{x \rightarrow a} (g(x) \ln f(x))}\end{aligned}$$

Other types are **not** indeterminate forms:  $0^\infty$ ,  $1^0$  and  $\infty^1$ .

# L'Hospital's Rule

Evaluate the limit

$$\lim_{x \rightarrow 0^+} x^x$$

Then  $\lim_{x \rightarrow 0^+} x = 0$ .

We write the limit as:

$$\begin{aligned}\lim_{x \rightarrow 0^+} x^x &= \lim_{x \rightarrow 0^+} e^{\ln x^x} \\ &= e^{\lim_{x \rightarrow 0^+} (x \ln x)} \\ &= e^0 \\ &= 1\end{aligned}$$



# L'Hospital's Rule

Evaluate the limit  $\lim_{x \rightarrow 0^+} (1 + \sin 4x)^{\cot x}$

Then  $\lim_{x \rightarrow 0^+} (1 + \sin 4x) = 1$  and  $\lim_{x \rightarrow 0^+} \cot x = \infty$

We write the limit as:

$$\begin{aligned}\lim_{x \rightarrow 0^+} (1 + \sin 4x)^{\cot x} &= \lim_{x \rightarrow 0^+} e^{\ln(1 + \sin 4x)^{\cot x}} \\ &= e^{\lim_{x \rightarrow 0^+} (\cot x \cdot \ln(1 + \sin 4x))}\end{aligned}$$

$$\lim_{x \rightarrow 0^+} (\cot x \cdot \ln(1 + \sin 4x)) = \lim_{x \rightarrow 0^+} \frac{\ln(1 + \sin 4x)}{\tan x}$$

Now  $\lim_{x \rightarrow 0^+} \ln(1 + \sin 4x) = 0$  and  $\lim_{x \rightarrow 0^+} \tan x = 0$

Hence we can apply l'Hospital's Rule:

$$\lim_{x \rightarrow 0^+} \frac{\ln(1 + \sin 4x)}{\tan x} = \lim_{x \rightarrow 0^+} \frac{\frac{4 \cos 4x}{1 + \sin 4x}}{(\sec x)^2} = \frac{\left(\frac{4}{1}\right)}{1} = 4$$

Thus  $\lim_{x \rightarrow 0^+} (1 + \sin 4x)^{\cot x} = e^4$