# MATH 211
## Online Asynchronous Survey in Calculus and Analytical Geometry

Dr. Ahmed Kaffel

Department of Mathematical Sciences
University of Wisconsin Milwaukee

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**UNIVERSITY of WISCONSIN**
**UWMILWAUKEE**

# Differentiation Rules: Product Rule

Lets $f$ and $g$ be linear functions:

$$f(x) = ax + b \qquad\qquad g(x) = cx + d$$

What is the derivative of $f \cdot g$?

$$\begin{aligned}
(f \cdot g)'(x) &= \frac{d}{dx}[f(x) \cdot g(x)] \\
&= \frac{d}{dx}[(ax + b) \cdot (cx + d)] \\
&= \frac{d}{dx}[acx^2 + adx + bcx + bd] \\
&= 2acx + ad + bc \\
&= a(cx + d) + c(ax + b) \\
&= f'(x) \cdot g(x) + g'(x) \cdot f(x)
\end{aligned}$$

We will now see that this also holds for general $f$ and $g$.

# Differentiation Rules: Product Rule

Assume that *f* and *g* are differentiable at *x*, and define

$$h(x) = f(x) \cdot g(x)$$

We try to find the derivative of *h* at *x*:

$$h'(x) = \lim_{\Delta x \to 0} \frac{\Delta h}{\Delta x} \qquad \text{where} \qquad \begin{aligned} \Delta h &= h(x + \Delta x) - h(x) \\ \Delta f &= f(x + \Delta x) - f(x) \\ \Delta g &= g(x + \Delta x) - g(x) \end{aligned}$$

Then

$$\begin{aligned} \Delta h &= f(x + \Delta x) \cdot g(x + \Delta x) - f(x) \cdot g(x) \\ &= (f(x) + \Delta f) \cdot (g(x) + \Delta g) - f(x) \cdot g(x) \\ &= \Delta f \cdot g(x) + f(x) \cdot \Delta g + \Delta f \cdot \Delta g \end{aligned}$$

## Differentiation Rules: Product Rule

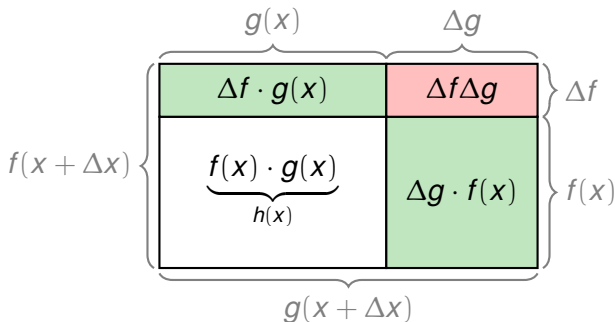$$h'(x) = \lim_{\Delta x \to 0} \frac{\Delta h}{\Delta x} \qquad \text{where} \qquad \begin{aligned} \Delta h &= h(x + \Delta x) - h(x) \\ \Delta f &= f(x + \Delta x) - f(x) \\ \Delta g &= g(x + \Delta x) - g(x) \end{aligned}$$

Then

$$\Delta h = \Delta f \cdot g(x) + f(x) \cdot \Delta g + \Delta f \cdot \Delta g$$

## Differentiation Rules: Product Rule

$$h'(x) = \lim_{\Delta x \to 0} \frac{\Delta h}{\Delta x} \qquad \Delta h = \Delta f \cdot g(x) + f(x) \cdot \Delta g + \Delta f \cdot \Delta g$$

We compute the limit:

$$
\begin{aligned}
h'(x) &= \lim_{\Delta x \to 0} \frac{\Delta h}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta f \cdot g(x) + f(x) \cdot \Delta g + \Delta f \cdot \Delta g}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\Delta f \cdot g(x)}{\Delta x} + \lim_{\Delta x \to 0} \frac{f(x) \cdot \Delta g}{\Delta x} + \lim_{\Delta x \to 0} \frac{\Delta f \cdot \Delta g}{\Delta x} \\
&= g(x) \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} + f(x) \lim_{\Delta x \to 0} \frac{\Delta g}{\Delta x} + \lim_{\Delta x \to 0} \left( \frac{\Delta f}{\Delta x} \cdot \Delta g \right) \\
&= g(x) f'(x) + f(x) g'(x) + \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} \cdot \lim_{\Delta x \to 0} \Delta g \\
&= g(x) f'(x) + f(x) g'(x) + f'(x) \cdot 0 \\
&= g(x) f'(x) + f(x) g'(x)
\end{aligned}
$$

# Differentiation Rules: Product Rule

### Product Rule

If *f* and *g* are both differentiable, then

$$\frac{d}{dx}[f(x) \cdot g(x)] = f(x)\frac{d}{dx}(g(x)) + g(x) \cdot \frac{d}{dx}(f(x))$$

In different notation

$$(f \cdot g)'(x) = f(x) \cdot g'(x) + f'(x) \cdot g(x)$$

In words:

> The derivative of the product of two function is the first function times the derivative of the second function plus the second function times the derivative of the first.

## Differentiation Rules: Product Rule

Let $f(x) = xe^x$. Find $f'(x)$.

$$\begin{aligned}
f'(x) &= \frac{d}{dx}(x \cdot e^x) \\
&= x\frac{d}{dx}(e^x) + e^x\frac{d}{dx}(x) \\
&= xe^x + e^x \\
&= (x+1)e^x
\end{aligned}$$

Let $f(x) = xe^x$. Find the $n$-th derivative $f^{(n)}(x)$.

$$f''(x) = \frac{d}{dx}(xe^x + e^x) = (x+1)e^x + e^x = (x+2)e^x$$

$$f'''(x) = \frac{d}{dx}(xe^x + 2e^x) = (x+1)e^x + 2e^x = (x+3)e^x$$

Thus obviously we have

$$f^{(n)}(x) = (x+n)e^x$$

## Differentiation Rules: Product Rule

Differentiate $f(t) = \sqrt{t}(a + bt)$.

$$f'(t) = \sqrt{t}\frac{d}{dt}(a + bt) + (a + bt)\frac{d}{dt}(\sqrt{t})$$

$$= \sqrt{t}b + (a + bt)\frac{1}{2}t^{-\frac{1}{2}})$$

$$= b\sqrt{t} + \frac{a + bt}{2\sqrt{t}}$$

$$= \frac{2bt}{2\sqrt{t}} + \frac{a + bt}{2\sqrt{t}}$$

$$= \frac{a + 3bt}{2\sqrt{t}}$$

Alternative solution: first simplify $f(t) = \sqrt{t}(a + bt) = at^{\frac{1}{2}} + bt^{\frac{3}{2}}$.
Then compute the derivative.

## Differentiation Rules: Product Rule

Let $f(x) = \sqrt{x} \cdot g(x)$ where $g(4) = 2$ and $g'(4) = 3$. Find $f'(4)$.

$$f'(x) = \sqrt{x} \cdot \frac{d}{dx}g(x) + g(x) \cdot \frac{d}{dx}\sqrt{x}$$
$$= \sqrt{x} \cdot g'(x) + g(x) \cdot \frac{1}{2}x^{-\frac{1}{2}}$$
$$= \sqrt{x} \cdot g'(x) + g(x) \cdot \frac{1}{2\sqrt{x}}$$

Thus

$$f'(4) = \sqrt{4} \cdot g'(4) + g(4) \cdot \frac{1}{2\sqrt{4}}$$
$$= 2 \cdot 3 + 2 \cdot \frac{1}{2 \cdot 2}$$
$$= 6 + \frac{1}{2}$$
$$= \frac{13}{2}$$

## Differentiation Rules: Quotient Rule

Assume that $f$ and $g$ are differentiable at $x$, and define

$$h(x) = \frac{f(x)}{g(x)}$$

$$\Delta h = h(x + \Delta x) - h(x)$$
$$\Delta f = f(x + \Delta x) - f(x)$$
$$\Delta g = g(x + \Delta x) - g(x)$$

We try to find the derivative of $h$ at $x$:

$$\Delta h = h(x + \Delta x) - h(x) = \frac{f(x + \Delta x)}{g(x + \Delta x)} - \frac{f(x)}{g(x)} = \frac{f(x) + \Delta f}{g(x) + \Delta g} - \frac{f(x)}{g(x)}$$

$$= \frac{(f(x) + \Delta f) \cdot g(x) - (g(x) + \Delta g) \cdot f(x)}{(g(x) + \Delta g) \cdot g(x)} = \frac{g(x)\Delta f - f(x)\Delta g}{(g(x) + \Delta g) \cdot g(x)}$$

$$h'(x) = \lim_{\Delta x \to 0} \frac{\Delta h}{\Delta x} = \lim_{\Delta x \to 0} \frac{\frac{g(x)\Delta f - f(x)\Delta g}{(g(x) + \Delta g) \cdot g(x)}}{\Delta x} = \lim_{\Delta x \to 0} \frac{g(x)\frac{\Delta f}{\Delta x} - f(x)\frac{\Delta g}{\Delta x}}{(g(x) + \Delta g) \cdot g(x)}$$

$$= \frac{g(x) \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} - f(x) \lim_{\Delta x \to 0} \frac{\Delta g}{\Delta x}}{\lim_{\Delta x \to 0}(g(x) + \Delta g) \cdot g(x)} = \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}$$

# Differentiation Rules: Quotient Rule

## Quotient Rule

If *f* and *g* are both differentiable, then

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x) \cdot \frac{d}{dx}(f(x)) - f(x) \cdot \frac{d}{dx}(g(x))}{[g(x)]^2}$$

In different notation

$$\left(\frac{f}{g}\right)'(x) = \frac{g(x) \cdot f'(x) - f(x) \cdot g'(x)}{g(x)^2}$$

In words:

> The derivative of a quotient is the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator, all divided by the square of the denominator.

# Differentiation Rules: Quotient Rule

$$\left(\frac{f}{g}\right)'(x) = \frac{g(x) \cdot f'(x) - f(x) \cdot g'(x)}{g(x)^2}$$

Let

$$f(x) = \frac{x^2 + x - 2}{x^3 + 6}$$

Then

$$f'(x) = \frac{(x^3 + 6) \cdot \frac{d}{dx}(x^2 + x - 2) - (x^2 + x - 2) \cdot \frac{d}{dx}(x^3 + 6)}{(x^3 + 6)^2}$$

$$= \frac{(x^3 + 6) \cdot (2x + 1) - (x^2 + x - 2) \cdot 3x^2}{(x^3 + 6)^2}$$

$$= \frac{(2x^4 + x^3 + 12x + 6) - (3x^4 + 3x^3 - 6x^2)}{(x^3 + 6)^2}$$

$$= \frac{-x^4 - 2x^3 + 6x^2 + 12x + 6}{(x^3 + 6)^2}$$

## Differentiation Rules: Quotient Rule

$$\left(\frac{f}{g}\right)'(x) = \frac{g(x) \cdot f'(x) - f(x) \cdot g'(x)}{g(x)^2}$$

Find an equation to the tangent line to

$$f(x) = \frac{e^x}{1 + x^2}$$

at point $(1, \frac{e}{2})$. We have

$$f'(x) = \frac{(1 + x^2) \cdot \frac{d}{dx}(e^x) - e^x \frac{d}{dx}(1 + x^2)}{(1 + x^2)^2} = \frac{(1 + x^2)e^x - e^x \cdot 2x}{(1 + x^2)^2}$$

$$= \frac{x^2 e^x - 2xe^x + e^x}{(1 + x^2)^2} = \frac{(x - 1)^2 e^x}{(1 + x^2)^2}$$

Thus the slope of the tangent is $f'(1) = 0$. Hence the tangent is

$$y = \frac{e}{2}$$

# Differentiation Rules: Quotient Rule

Sometimes it is easier to simplify than apply the quotient rule:
$$f(x) = \frac{3x^2 + 2\sqrt{x}}{x}$$
Instead of applying the quotient rule, we can simplify to
$$f(x) = 3x + 2x^{-\frac{1}{2}}$$
which is easier to differentiate.

## Differentiation Rules: Chain Rule

Suppose we want to differentiate

$$f(x) = \sqrt{x^2 + 1}$$

The rules, we have seen so far, do not help.

However, we know how to differentiate the functions:

$$g(x) = \sqrt{x} \qquad\qquad h(x) = x^2 + 1$$

We can write $f$ as:

$$f(x) = g(h(x))$$

That is:

$$f = g \circ h$$

We need a rule that gives us $f'$ from $g'$ and $h'$...

# Differentiation Rules: Chain Rule

### Chain Rule

If $g$ is differentiable at $x$ and $f$ at $g(x)$, then

$$h = f \circ g \qquad \text{or equivalently} \qquad h(x) = f(g(x))$$

is differentiable at $x$ and

$$h'(x) \,=\, (f \circ g)'(x) \,=\, f'(g(x)) \cdot g'(x)$$

$$\frac{d}{dx} \quad \underbrace{f}_{\substack{\text{outer} \\ \text{function}}} \quad \underbrace{(\ g(x)\ )}_{\substack{\text{evaluated} \\ \text{at inner} \\ \text{function}}} = \underbrace{f'}_{\substack{\text{derivative} \\ \text{of outer} \\ \text{function}}} \underbrace{(\ g(x)\ )}_{\substack{\text{evaluated} \\ \text{at inner} \\ \text{function}}} \cdot \underbrace{g'(x)}_{\substack{\text{derivative} \\ \text{of inner} \\ \text{function}}}$$

In words:

> The derivative of the composition of $f$ and $g$ is the derivative of $f$ at $g(x)$ times the derivative of $g$ at $x$.

# Differentiation Rules: Chain Rule

### Chain Rule

If $g$ is differentiable at $x$ and $f$ at $g(x)$, then

$$h = f \circ g \qquad \text{or equivalently} \qquad h(x) = f(g(x))$$

is differentiable at $x$ and

$$h'(x) = (f \circ g)'(x) = f'(g(x)) \cdot g'(x)$$

Intuition with rates of change:

- If $g'(x) = N$. Then $g(x)$ changes $N$ times as much as $x$.
- If $f'(g(x)) = M$. Then $f(x)$ changes $M$ times as much as $g(x)$.
- Thus $(f \circ g)(x) = f(g(x))$ changes $N \cdot M$ times as much as $x$.

# Differentiation Rules: Chain Rule

$$(f \circ g)'(x) \;=\; f'(g(x)) \cdot g'(x)$$

Let $f(x) = \sqrt{x^2 + 1}$. Find $f'(x)$.

We have that

$$f(x) = g(h(x)) \quad \text{where} \quad g(x) = \sqrt{x} \quad h(x) = x^2 + 1$$

and

$$g'(x) = \frac{1}{2\sqrt{x}} \qquad\qquad h'(x) = 2x$$

Hence:

$$f'(x) = (g \circ h)'(x) = \frac{1}{2\sqrt{x^2 + 1}} \cdot 2x = \frac{x}{\sqrt{x^2 + 1}}$$

# Differentiation Rules: Chain Rule

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$$

Differentiate $f(x) = (x^3 - 1)^{100}$.

We have that

$$f(x) = g(h(x)) \quad \text{where} \quad g(x) = x^{100} \quad h(x) = x^3 - 1$$

and

$$g'(x) = 100x^{99} \qquad\qquad h'(x) = 3x^2$$

Hence:

$$f'(x) = (g \circ h)'(x) = 100(x^3 - 1)^{99} \cdot 3x^2$$
$$= 300x^2 \cdot (x^3 - 1)^{99}$$

## Differentiation Rules: Chain Rule

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$$

In general (combining the power and chain rule) we have:

$$\frac{d}{dx}[g(x)]^n = n \cdot [g(x)]^{n-1} \cdot g'(x)$$

if $g(x)$ is differentiable.

Differentiate

$$f(x) = \frac{1}{\sqrt[3]{x^2 + x + 1}}$$

We have

$$f(x) = (x^2 + x + 1)^{-\frac{1}{3}}$$
$$f'(x) = -\frac{1}{3} \cdot (x^2 + x + 1)^{-\frac{4}{3}} \cdot (2x + 1)$$

# Differentiation Rules: Chain Rule

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$$

Differentiate

$$f(x) = \left( \frac{x-2}{2x+1} \right)^9$$

We have

$$
\begin{aligned}
f'(x) &= 9 \left( \frac{x-2}{2x+1} \right)^8 \frac{d}{dx} \frac{x-2}{2x+1} \\
&= 9 \left( \frac{x-2}{2x+1} \right)^8 \frac{(2x+1) \cdot 1 - (x-2) \cdot 2}{(2x+1)^2} \\
&= 9 \left( \frac{x-2}{2x+1} \right)^8 \frac{5}{(2x+1)^2} \\
&= 45 \frac{(x-2)^8}{(2x+1)^{10}}
\end{aligned}
$$

# Differentiation Rules: Chain Rule

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$$

Differentiate

$$f(x) = (2x+1)^5 \cdot (x^3 - x + 1)^4$$

We have

$$f'(x) = (2x+1)^5 \cdot \frac{d}{dx}[(x^3 - x + 1)^4]$$
$$+ (x^3 - x + 1)^4 \cdot \frac{d}{dx}[(2x+1)^5]$$
$$= (2x+1)^5 \cdot 4(x^3 - x + 1)^3 \cdot (3x^2 - 1)$$
$$+ (x^3 - x + 1)^4 \cdot 5(2x+1)^4 \cdot 2$$

# Differentiation Rules: Chain Rule

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$$

Use

$$\frac{d}{dx} e^x = e^x$$

and the chain rule to prove

$$\frac{d}{dx} a^x = \ln a \cdot a^x$$

We have

$$a^x = (e^{\ln a})^x = e^{\ln a \cdot x}$$

and $f = g \circ h$ where $g(x) = e^x$ and $h(x) = \ln a \cdot x$ . Thus

$$f'(x) = g'(h(x)) \cdot h'(x) = e^{\ln a \cdot x} \cdot \ln a = \ln a \cdot a^x$$

# Summary of Differentiation Rules

$$\frac{d}{dx}(c) = 0 \qquad\qquad \frac{d}{dx}(x^r) = r\, x^{r-1}$$

$$\frac{d}{dx}(e^x) = e^x \qquad\qquad \frac{d}{dx}(a^x) = \ln a \cdot a^x$$

$$(f + g)' = f' + g' \qquad\qquad (f - g)' = f' - g'$$

$$(cf)' = cf'$$

$$(fg)' = f'g + fg' \qquad\qquad \left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$$