1 圆累积量

$$F^{(k)}(s,t)\equiv\left\langle\exp\left(se^{-i\varphi}\right)\right\rangle^{(k)}=\left\langle\sum_{m=0}^{\infty}\frac{s^m}{m!}e^{-im\varphi}\right\rangle^{(k)},$$

$$F^{(k)}(s,t)\equiv\sum_{m=0}^{\infty}A_m^{(k)}(t)\frac{s^m}{m!},\ A_m^{(k)}(t)=\frac{\partial^m}{\partial s^m}\left.F^{(k)}(s,t)\right|_{s=0}.$$

$$\#:\frac{\partial}{\partial t}F^{(k)}=-\left(i\omega_0+\gamma\right)s\frac{\partial}{\partial s}F^{(k)}-h_k s\frac{\partial^2}{\partial s^2}F^{(k)}+{h_k}^{*}sF^{(k)}+r_{j\to k}\frac{m_j}{m_k}\left(F^{(j)}-F^{(k)}\right).$$

$$\Psi^{(k)}(s,t)\equiv s\frac{\partial}{\partial s}\ln F^{(k)}(s,t)=\frac{s}{F^{(k)}}\frac{\partial}{\partial s}F^{(k)}\equiv\sum_{m=1}^{\infty}\varkappa_m^{(k)}(t)\,s^m,\varkappa_m^{(k)}(t)=\frac{1}{m!}\frac{\partial^m}{\partial s^m}\left.\Psi^{(k)}(s,t)\right|_{s=0}.$$

简化记号：F默认为$F^{(k)}$，$F^{(j)}$总是保留，$A=i\omega_0+\gamma,B=h_k,C={h_k}^{*},D=r_{j\to k}\frac{m_j}{m_k}$,

$$\partial_t F=-As\partial_s F-Bs\partial_s^2F+CsF+D\left(F^{(j)}-F\right),$$

交换求导顺序大大化简计算：$\partial_t\Psi=s\partial_t\left(\frac{\partial_s F}{F}\right)=\frac{s}{F}\partial_t\partial_s F-\frac{s}{F^2}\partial_t F\partial_s F=s\partial_s\left(\frac{\partial_t F}{F}\right)$,

$$1:\partial_s F=\frac{\Psi}{s}F,2:\partial_s^2F=\partial_s\left(\frac{\Psi}{s}F\right)=\frac{\partial_s\Psi s-\Psi}{s^2}F+\frac{\Psi}{s}\frac{\Psi}{s}F=\frac{F}{s^2}\left(s\partial_s\Psi-\Psi+\Psi^2\right)$$

$$\frac{\partial_t F}{F}=-A\Psi-B\left(\partial_s\Psi-\frac{\Psi}{s}+\frac{\Psi^2}{s}\right)+Cs+D\left(\frac{F^{(j)}}{F^{(k)}}-1\right)$$

$$\#:\partial_t\Psi=-As\partial_s\Psi-Bs\partial_s\left(\partial_s\Psi-\frac{\Psi}{s}+\frac{\Psi^2}{s}\right)+Cs+Ds\partial_s\left(\frac{F^{(j)}}{F^{(k)}}\right).$$

$$\partial_s\left(\frac{F^{(j)}}{F^{(k)}}\right)=\frac{\frac{\Psi^{(j)}F^{(j)}}{s}F^{(k)}-F^{(j)}\frac{\Psi^{(k)}F^{(k)}}{s}}{\left(F^{(k)}\right)^2}=\frac{1}{s}\frac{F^{(j)}}{F^{(k)}}\left(\Psi^{(j)}-\Psi^{(k)}\right),\partial_s\Psi-\frac{\Psi}{s}=s\partial_s\left(\frac{\Psi}{s}\right)$$

$$\#:\partial_t\Psi=-As\partial_s\Psi-Bs\partial_s\left(s\partial_s\left(\frac{\Psi}{s}\right)+\frac{\Psi^2}{s}\right)+Cs+D\frac{F^{(j)}}{F^{(k)}}\left(\Psi^{(j)}-\Psi^{(k)}\right).$$

方程代入级数匹配$\sum_{m=1}^{\infty}c_n s^m$形式的系数$c_n$：

$$\Psi^{(k)}=\sum_{m=1}^{\infty}\varkappa_m^{(k)}s^m,F^{(k)}=\sum_{m=0}^{\infty}A_m^{(k)}\frac{s^m}{m!},$$

$$\frac{\Psi}{s}=\frac{\partial}{\partial s}\ln F,F(0,t)=1,\ln F(0,t)=0,$$

$$\ln F(\sigma,t)\mid_{\sigma=0}^{\sigma=s}=\int_0^s\frac{\Psi(\sigma,t)}{\sigma}d\sigma,\Psi(\sigma,t)=\sum_{m=1}^{\infty}\varkappa_m\sigma^m,$$

$$\ln F(s,t)=\int_0^s\sum_{m=1}^{\infty}\varkappa_m\sigma^{m-1}d\sigma=\sum_{m=1}^{\infty}\frac{\varkappa_m}{m}s^m,$$

$$F(s,t)=\exp\left(\sum_{m=1}^{\infty}\frac{\varkappa_m}{m}s^m\right),\Delta\varkappa_m=\varkappa_m^{(j)}-\varkappa_m^{(k)},\Delta\varkappa_0=0$$

$$\Psi^{(j)}-\Psi^{(k)}\equiv\Delta V=\sum_{m=1}^{\infty}\Delta\varkappa_m s^m=\sum_{m=0}^{\infty}\Delta\varkappa_m s^m,$$

$$\frac{F^{(j)}}{F^{(k)}} \equiv R\left(s,t\right) \equiv \sum_{m=0}^{\infty} \rho_m s^m = \exp\left(\sum_{m=0}^{\infty} \frac{\Delta \varkappa_m}{m} s^m\right) \equiv \mathrm{e}^{H(s,t)}.$$

$$s\frac{\partial}{\partial s}R = sR\frac{\partial}{\partial s}H = R\sum_{m=0}^{\infty} \Delta\varkappa_m s^m = R\Delta V,$$

$$s\frac{\partial}{\partial s}R = \sum_{m=1}^{\infty} m\rho_m s^m, R\Delta V = \left(\sum_{i=0}^{\infty}\rho_i s^i\right)\left(\sum_{j=0}^{\infty}\Delta\varkappa_j s^j\right) = \sum_{n=1}^{\infty}\left(\sum_{k=0}^{n-1}\rho_k\Delta\varkappa_{n-k}\right)s^n,$$

$$m\rho_n = \sum_{k=0}^{m-1} \rho_k \Delta\varkappa_{m-k}, m-k=q,$$

$$\text{通过让}s=0\text{提取出}m=0\text{的常数项}:\rho_0 = \frac{A_0^{(j)}}{A_0^{(k)}} = 1,$$

$$s\partial_s\left(\frac{F^{(j)}}{F^{(k)}}\right) = \frac{F^{(j)}}{F^{(k)}}\left(\Psi^{(j)} - \Psi^{(k)}\right) = R\Delta V = s\frac{\partial}{\partial s}R = \sum_{m=1}^{\infty}\left(m\rho_m\right)s^m.$$

$$A = i\omega_0 + \gamma, B = h_k, C = {h_k}^*, D = r_{j\to k}\frac{m_j}{m_k},$$

$$\#: \dot{\varkappa}_m^{(k)} = -Am\varkappa_m^{(k)} - Bm\big(m\varkappa_{m+1}^{(k)} + \sum_{l=1}^{m}\varkappa_l^{(k)}\varkappa_{m+1-l}^{(k)}\big) + C\delta_{m,1} + \ Dm\rho_m,$$

$$\text{where } m\rho_m = \sum_{q=1}^{m}\rho_{m-q}\Delta\varkappa_q,\ \rho_0 = 1,\ \Delta\varkappa_q = \varkappa_q^{(j)} - \varkappa_q^{(k)}.$$

$$\begin{cases} \rho_1 = \Delta\varkappa_1 \\ 2\rho_2 = (\Delta\varkappa_1)^2 + \Delta\varkappa_2 \\ 3\rho_3 = \frac{1}{2}(\Delta\varkappa_1)^3 + \frac{3}{2}\Delta\varkappa_1\Delta\varkappa_2 + \Delta\varkappa_3 \\ \dots \end{cases}.$$

$$m\rho_m = \eta_m : \eta_m = \sum_{q=1}^{m}\frac{\eta_{m-q}}{m-q}\Delta\varkappa_q, \eta_0 = 1.$$

$$\sum_{m=0}^{\infty} A_m\frac{s^m}{m!} = \exp\left(\sum_{m=1}^{\infty}\frac{\varkappa_m}{m}s^m\right) = \exp\left(\sum_{m=1}^{\infty}(m-1)!\varkappa_m\frac{s^m}{m!}\right),$$

$$\varkappa_m = \frac{A_m}{(m-1)!} - \sum_{j=1}^{m-1}\varkappa_{m-j}\frac{A_j}{j!}, m \geqslant 1.$$