

$$F^{(k)}(s, t) \equiv \langle \exp(se^{-i\varphi}) \rangle^{(k)} = \left\langle \sum_{m=0}^{\infty} \frac{s^m}{m!} e^{-im\varphi} \right\rangle^{(k)} \equiv \sum_{m=0}^{\infty} A_m^{(k)}(t) \frac{s^m}{m!},$$

$$\# : \frac{\partial}{\partial t} F^{(k)} = - (i\omega_0 + \gamma) s \frac{\partial}{\partial s} F^{(k)} - h_k s \frac{\partial^2}{\partial s^2} F^{(k)} + h_k^* s F^{(k)} + r_{j \rightarrow k} \frac{m_j}{m_k} (F^{(j)} - F^{(k)}).$$

$$\Psi^{(k)}(s, t) \equiv s \frac{\partial}{\partial s} \ln F^{(k)}(s, t) = \frac{s}{F^{(k)}} \frac{\partial}{\partial s} F^{(k)} \equiv \sum_{m=1}^{\infty} \varkappa_m^{(k)}(t) s^m, \quad \varkappa_m^{(k)}(t) = \frac{1}{m!} \frac{\partial^m}{\partial s^m} \Psi^{(k)}(s, t)|_{s=0}.$$

简化记号: F 默认为 $F^{(k)}$, $F^{(j)}$ 总是保留, $A = i\omega_0 + \gamma$, $B = h_k$, $C = h_k^*$, $D = r_{j \rightarrow k} \frac{m_j}{m_k}$,

$$\partial_t F = -As\partial_s F - Bs\partial_s^2 F + CsF + D(F^{(j)} - F),$$

$$\text{交换求导顺序大大化简计算: } \partial_t \Psi = s\partial_t \left(\frac{\partial_s F}{F} \right) = \frac{s}{F} \partial_t \partial_s F - \frac{s}{F^2} \partial_t F \partial_s F = s\partial_s \left(\frac{\partial_t F}{F} \right),$$

$$1: \partial_s F = \frac{\Psi}{s} F, 2: \partial_s^2 F = \partial_s \left(\frac{\Psi}{s} F \right) = \frac{\partial_s \Psi s - \Psi}{s^2} F + \frac{\Psi}{s} \frac{\Psi}{s} F = \frac{F}{s^2} (s\partial_s \Psi - \Psi + \Psi^2)$$

$$\frac{\partial_t F}{F} = -A\Psi - B \left(\partial_s \Psi - \frac{\Psi}{s} + \frac{\Psi^2}{s} \right) + Cs + D \left(\frac{F^{(j)}}{F^{(k)}} - 1 \right)$$

$$\#: \partial_t \Psi = -As\partial_s \Psi - Bs\partial_s \left(\partial_s \Psi - \frac{\Psi}{s} + \frac{\Psi^2}{s} \right) + Cs + Ds\partial_s \left(\frac{F^{(j)}}{F^{(k)}} \right).$$

$$\partial_s \left(\frac{F^{(j)}}{F^{(k)}} \right) = \frac{\frac{\Psi^{(j)} F^{(j)}}{s} F^{(k)} - F^{(j)} \frac{\Psi^{(k)} F^{(k)}}{s}}{(F^{(k)})^2} = \frac{1}{s} \frac{F^{(j)}}{F^{(k)}} (\Psi^{(j)} - \Psi^{(k)}), \quad \partial_s \Psi - \frac{\Psi}{s} = s\partial_s \left(\frac{\Psi}{s} \right)$$

$$\#: \partial_t \Psi = -As\partial_s \Psi - Bs\partial_s \left(s\partial_s \left(\frac{\Psi}{s} \right) + \frac{\Psi^2}{s} \right) + Cs + D \frac{F^{(j)}}{F^{(k)}} (\Psi^{(j)} - \Psi^{(k)}).$$

方程代入级数匹配 $\sum_{m=1}^{\infty} c_n s^m$ 形式的系数 c_n :

$$\Psi^{(k)} = \sum_{m=1}^{\infty} \varkappa_m^{(k)} s^m, \quad F^{(k)} = \sum_{m=0}^{\infty} A_m^{(k)} \frac{s^m}{m!},$$

$$\frac{\Psi}{s} = \frac{\partial}{\partial s} \ln F, \quad F(0, t) = 1, \quad \ln F(0, t) = 0,$$

$$\ln F(\sigma, t) |_{\sigma=0}^{\sigma=s} = \int_0^s \frac{\Psi(\sigma, t)}{\sigma} d\sigma, \quad \Psi(\sigma, t) = \sum_{m=1}^{\infty} \varkappa_m \sigma^m,$$

$$\ln F(s, t) = \int_0^s \sum_{m=1}^{\infty} \varkappa_m \sigma^{m-1} d\sigma = \sum_{m=1}^{\infty} \frac{\varkappa_m}{m} s^m,$$

$$\begin{aligned}
F(s, t) &= \exp \left(\sum_{m=1}^{\infty} \frac{\varkappa_m}{m} s^m \right), \Delta \varkappa_m = \varkappa_m^{(j)} - \varkappa_m^{(k)}, \Delta \varkappa_0 = 0 \\
\Psi^{(j)} - \Psi^{(k)} &\equiv \Delta V = \sum_{m=1}^{\infty} \Delta \varkappa_m s^m = \sum_{m=0}^{\infty} \Delta \varkappa_m s^m, \\
\frac{F^{(j)}}{F^{(k)}} &\equiv R(s, t) \equiv \sum_{m=0}^{\infty} \rho_m s^m = \exp \left(\sum_{m=0}^{\infty} \frac{\Delta \varkappa_m}{m} s^m \right) \equiv e^{H(s, t)}. \\
s \frac{\partial}{\partial s} R &= s R \frac{\partial}{\partial s} H = R \sum_{m=0}^{\infty} \Delta \varkappa_m s^m = R \Delta V, \\
s \frac{\partial}{\partial s} R &= \sum_{m=1}^{\infty} m \rho_m s^m, R \Delta V = \left(\sum_{i=0}^{\infty} \rho_i s^i \right) \left(\sum_{j=0}^{\infty} \Delta \varkappa_j s^j \right) = \sum_{n=1}^{\infty} \left(\sum_{k=0}^{n-1} \rho_k \Delta \varkappa_{n-k} \right) s^n \\
m \rho_m &= \sum_{k=0}^{m-1} \rho_k \Delta \varkappa_{m-k}, \\
\text{通过让 } s = 0 \text{ 提取出 } m = 0 \text{ 的常数项: } \rho_0 &= \frac{A_0^{(j)}}{A_0^{(k)}} = 1, \\
s \partial_s \left(\frac{F^{(j)}}{F^{(k)}} \right) &= \frac{F^{(j)}}{F^{(k)}} (\Psi^{(j)} - \Psi^{(k)}) = R \Delta V = s \frac{\partial}{\partial s} R = \sum_{m=1}^{\infty} (m \rho_m) s^m \\
\begin{cases} \dot{\varkappa}_m^{(k)} = -A m \varkappa_m^{(k)} - B m (\varkappa_{m+1}^{(k)} + \sum_{l=1}^m \varkappa_l^{(k)} \varkappa_{m+1-l}^{(k)}) + C \delta_{m,1} + D m \rho_m \\ \rho_m = \frac{1}{m} \sum_{q=1}^m \rho_{m-q} \Delta \varkappa_q, \rho_0 = 1, \Delta \varkappa_q = \varkappa_q^{(j)} - \varkappa_q^{(k)} \end{cases} \\
\begin{cases} \rho_0 = 1 \\ \rho_1 = \Delta \varkappa_1 \\ \rho_2 = \frac{1}{2} (\Delta \varkappa_1)^2 + \frac{1}{2} \Delta \varkappa_2 \\ \rho_3 = \frac{1}{6} (\Delta \varkappa_1)^3 + \frac{1}{2} \Delta \varkappa_1 \Delta \varkappa_2 + \frac{1}{3} \Delta \varkappa_3 \\ \dots \end{cases}
\end{aligned}$$