1 累积量

$$F^{(k)}(s,t) = \left\langle \exp\left(se^{-i\varphi}\right)\right\rangle^{(k)} = \left\langle \sum_{m=0}^{\infty} \frac{s^m}{m!} e^{-im\varphi}\right\rangle^{(k)} \equiv \sum_{m=0}^{\infty} A_m{}^{(k)}(t)\frac{s^m}{m!}, A_m{}^{(k)}(t) = \frac{\partial^m}{\partial s^m} F^{(k)}(s,t)\Big|_{s=0}$$

$$\# : \frac{\partial}{\partial t}F^{(k)} = -\left(i\omega_0+\gamma\right)s\frac{\partial}{\partial s}F^{(k)} - h_k s\frac{\partial^2}{\partial s^2}F^{(k)} + h_k{}^* sF^{(k)} + r_{j\to k}\frac{p_j}{p_k}\left(F^{(j)}-F^{(k)}\right).$$

$$\text{圆累积量：} \Psi^{(k)}(s,t) = s\frac{\partial}{\partial s}\ln F^{(k)}(s,t) \equiv \sum_{m=1}^{\infty} \varkappa_m{}^{(k)}(t)\, s^m, \varkappa_m{}^{(k)}(t) = \frac{1}{m!}\frac{\partial^m}{\partial s^m}\Psi^{(k)}(s,t)\Big|_{s=0}.$$

$$\text{累积量：} \Phi^{(k)}(s,t) \equiv \ln F^{(k)}(s,t) \equiv \sum_{m=1}^{\infty} K_m{}^{(k)}(t)\frac{s^m}{m!}, K_m{}^{(k)}(t) = \frac{\partial^m}{\partial s^m}\Phi^{(k)}(s,t)\Big|_{s=0}.$$

简化记号：$F\Phi K_m$默认为$F^{(k)}\Phi^{(k)}K_m^{(k)}$，$F^{(j)}\Phi^{(j)}K_m{}^{(j)}$总是保留，$A = i\omega_0+\gamma, B = h_k, C = h_k{}^*, D = r_{j\to k}\frac{m_j}{m_k}$,

$$\partial_t F = -As\partial_s F - Bs\partial_s^2 F + CsF + D\left(F^{(j)}-F\right),$$

$$F = \mathrm{e}^{\Phi}, \partial_s F = \mathrm{e}^{\Phi}\partial_s\Phi, \partial_s^2 F = \mathrm{e}^{\Phi}\left(\left(\partial_s\Phi\right)^2+\partial_s^2\Phi\right),$$

$$\partial_t\Phi = \frac{\partial_t F}{F} = -As\frac{\partial_s F}{F} - Bs\frac{\partial^2{}_s F}{F} + Cs + D\left(\frac{F^{(j)}}{F}-1\right),$$

$$\# : \partial_t\Phi = -As\partial_s\Phi - Bs\left(\left(\partial_s\Phi\right)^2+\partial_s^2\Phi\right) + Cs + D\left(e^{\Phi^{(j)}-\Phi^{(k)}}-1\right).$$

$$\Phi = \sum_{m=1}^{\infty} K_m\frac{s^m}{m!} = K_1 s + K_2\frac{s^2}{2!} + K_3\frac{s^3}{3!} + \cdots,$$

$$\Phi^{(k)} = \sum_{m=1}^{\infty} K_m^{(k)}\frac{s^m}{m!}, \Phi^{(j)} = \sum_{m=1}^{\infty} K_m^{(j)}\frac{s^m}{m!},$$

$$e^{\Phi^{(j)}-\Phi^{(k)}} = \sum_{m=0}^{\infty} E_m\frac{s^m}{m!},$$

$$1 : \partial_t\Phi = \sum_{m=1}^{\infty}\dot{K}_m\frac{s^m}{m!}. \partial_s\Phi = \sum_{m=1}^{\infty} K_m\frac{s^{m-1}}{(m-1)!} = \sum_{n=0}^{\infty} K_{n+1}\frac{s^n}{n!}, 2 : s\partial_s\Phi = \sum_{m=1}^{\infty} mK_m\frac{s^m}{m!}.$$

$$\left(\sum_{n=0}^{\infty} a_n\frac{s^n}{n!}\right)\left(\sum_{n=0}^{\infty} b_n\frac{s^n}{n!}\right) = \sum\nolimits_{n=0}^{\infty} c_n\frac{s^n}{n!}, \text{ where } c_n = \sum_{r=0}^{n}\binom{n}{r} a_r b_{n-r},$$

$$\left(\partial_s\Phi\right)^2 = \left(\sum_{n=0}^{\infty} K_{n+1}\frac{s^n}{n!}\right)\left(\sum_{n=0}^{\infty} K_{n+1}\frac{s^n}{n!}\right) = \sum_{n=0}^{\infty}\left[\sum_{r=0}^{n}\binom{n}{r} K_{r+1}K_{n-r+1}\right]\frac{s^n}{n!},$$

$$3\text{：} s\left(\partial_s\Phi\right)^2 = \sum_{m=1}^{\infty}\left[m\sum_{r=0}^{m-1}\binom{m-1}{r} K_{r+1}K_{m-r}\right]\frac{s^m}{m!}.$$

$$4 : s\partial_s^2\Phi = s\sum_{m=2}^{\infty} K_m\frac{s^{m-2}}{(m-2)!} = \sum_{n=2}^{\infty} K_n\frac{s^{n-1}}{(n-2)!} = \sum_{m=1}^{\infty} mK_{m+1}\frac{s^m}{m!}.$$

$$5 : \mathrm{CompleteBellPolynomials} :$$

$$\mathrm{Define} H = \Phi^{(j)} - \Phi^{(k)} = \sum_{m=1}^{\infty}\Delta K_m\frac{s^m}{m!}, \mathrm{e}^{H} = \sum_{m=0}^{\infty} E_m\frac{s^m}{m!},$$

$$\partial_s \mathrm{e}^H = H\prime \cdot \mathrm{e}^H, H\prime = \sum_{m=1}^{\infty} \Delta K_m \frac{s^{m-1}}{(m-1)!}$$

$$H\prime \cdot \mathrm{e}^H = \left(\sum_{n=0}^{\infty} \Delta K_{n+1} \frac{s^n}{n!}\right)\left(\sum_{m=0}^{\infty} E_m \frac{s^m}{m!}\right) = \sum_{j=0}^{\infty}\left(\sum_{k=0}^{j}\begin{pmatrix} j \\ k \end{pmatrix}\Delta K_{k+1} E_{j-k}\right)\frac{s^j}{j!},$$

$$\partial_s \mathrm{e}^H = \sum_{m=1}^{\infty} E_m \frac{s^{m-1}}{(m-1)!} = \sum_{j=0}^{\infty} E_{j+1}\frac{s^j}{j!},$$

$$E_{j+1} = \sum_{k=0}^{j}\begin{pmatrix} j \\ k \end{pmatrix}\Delta K_{k+1} E_{j-k}, j+1 = m, r = m-1-k$$

$$E_m = \sum_{r=0}^{m-1}\begin{pmatrix} m-1 \\ r \end{pmatrix} E_r \cdot \Delta K_{m-r}, E_0 = 1,$$

$$\mathrm{e}^{\Phi^{(j)} - \Phi^{(k)}} - 1 = \sum_{m=1}^{\infty} E_m \frac{s^m}{m!}.$$

$$\# : \dot{K}_m = -mAK_m - mB\left[\sum_{r=0}^{m-1}\begin{pmatrix} m-1 \\ r \end{pmatrix} K_{r+1}K_{m-r} + K_{m+1}\right] + C\delta_{m,1} + DE_m,$$

$$\text{where } E_m = \sum_{r=0}^{m-1}\begin{pmatrix} m-1 \\ r \end{pmatrix} E_r \Delta K_{m-r},\ E_0 = 1,\ \Delta K_{m-r} = K_{m-r}^{(j)} - K_{m-r}^{(k)}.$$

$$\begin{cases} E_1 = \Delta K_1, \\ E_2 = \Delta K_2 + (\Delta K_1)^2, \\ E_3 = \Delta K_3 + 3\Delta K_1 \Delta K_2 + (\Delta K_1)^3 \\ \qquad\qquad \cdots \end{cases}$$

$$\text{Define } \kappa_m = \frac{K_m}{(m-1)!},$$

$$\begin{pmatrix} m \\ r \end{pmatrix} = \frac{m!}{r!\,(m-r)!}, \begin{pmatrix} m-1 \\ r \end{pmatrix} = \frac{(m-1)!}{r!\,(m-1-r)!},$$

$$\kappa_{m-r} = \frac{K_{m-r}}{(m-r-1)!}, E_m = \sum_{r=0}^{m-1}\frac{E_r}{r!}\left(\kappa_{m-r}^{(j)} - \kappa_{m-r}^{(k)}\right)$$

$$\# : \dot{\kappa}_m = -mA\kappa_m - mB\left(\sum_{r=0}^{m-1}\kappa_{r+1}\kappa_{m-r} + m\kappa_{m+1}\right) + C\delta_{m,1} + DE_m,$$

$$\text{where } E_m = \sum_{r=0}^{m-1}\frac{E_r}{r!}\Delta\kappa_{m-r}, \Delta\kappa_{m-r} = \kappa_{m-r}^{(j)} - \kappa_{m-r}^{(k)}, E_0 = 1.$$

$$\begin{cases} E_1 = \Delta\kappa_1, \\ E_2 = \Delta\kappa_2 + (\Delta\kappa_1)^2 \\ E_3 = \Delta\kappa_3 + \frac{2}{3}\Delta\kappa_1\Delta\kappa_2 + \frac{1}{2}(\Delta\kappa_1)^3 \\ \qquad\qquad \cdots \end{cases}$$

$$\mathcal{E}_m = \frac{E_m}{(m-1)!}$$

$$\mathcal{E}_m\,(m-1)! = \sum_{r=0}^{m-1} \frac{\mathcal{E}_r}{r} \Delta\kappa_{m-r}$$