# Analysis III - CheatSheet

IN BA3 - Martin Werner Licht

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*An Analysis III Cheatsheet has been authorized for the upcoming exam, and I'm sharing a copy of mine for anyone interested. It provides a concise summary of the key concepts and techniques covered in the course. While it's not yet complete, I plan to update it soon, especially to include additional trigonometric identities and a step-by-step guide to solving differential equations using Fourier methods. When printing, you can select the last two pages, as only one A4 recto-verso page is allowed. For any updates or suggestions, feel free to reach out to me on Telegram at* `elazdi_al` *or via EPFL email at* `ali.elazdi@epfl.ch`.

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**Regular Curve.** A continuously differentiable map $\gamma : [a,b] \to \mathbb{R}^n$ is *regular* if $\gamma'(t) \neq 0$ for all $t \in [a,b]$.

**Simple Curve.** A continuous map $\gamma : [a,b] \to \mathbb{R}^n$ is *simple* if it does not intersect itself (except possibly at endpoints): $\gamma(t_1) = \gamma(t_2) \implies t_1 = t_2$

**Simply Connected Domain.** An open set $\Omega \subseteq \mathbb{R}^n$ is *simply connected* if for any two continuous curves $\gamma_0, \gamma_1 : [a,b] \to \Omega$ with $\gamma_0(a) = \gamma_1(a)$ and $\gamma_0(b) = \gamma_1(b)$, there exists a continuous homotopy $\Gamma : [a,b] \times [0,1] \to \Omega$ that deforms $\gamma_0$ into $\gamma_1$ within $\Omega$ while keeping the endpoints fixed.

**Curve in** $\mathbb{R}^n$: Given $\gamma : [a,b] \to \mathbb{R}^n$, $\int_\gamma F \cdot d\mathbf{l} = \int_a^b \langle F(\gamma(t)), \gamma'(t) \rangle \, dt$, $\quad \int_\gamma f(\mathbf{x}) \, ds = \int_a^b f\big(\gamma(t)\big) \, \|\gamma'(t)\| \, dt.$

**Surface in** $\mathbb{R}^3$: Given $\sigma : D \subset \mathbb{R}^2 \to \mathbb{R}^3$, $\sigma(u,v)$,

$$\iint_\sigma F \cdot d\mathbf{S} = \iint_D \big\langle F(\sigma(u,v)), \sigma_u(u,v) \times \sigma_v(u,v) \big\rangle \, du \, dv. \quad \iint_\sigma f(\mathbf{x}) \, ds = \iint_D f\big(\sigma(u,v)\big) \, \|\sigma_u(u,v) \times \sigma_v(u,v)\| \, du \, dv,$$

**Conservative Fields and Path Independence**

A map $F : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^n$ is *conservative* if there exists a differentiable function $\varphi : \Omega \to \mathbb{R}$ such that $\nabla f = F$.

$F$ is conservative on $\Omega \iff \forall \Gamma_1, \Gamma_2 \subseteq \Omega$ reg. curves from $A$ to $B$, $\int_{\Gamma_1} F \cdot d\mathbf{l} = \int_{\Gamma_2} F \cdot d\mathbf{l}$, $\iff \forall \Gamma \subseteq \Omega$ reg. closed curve, $\int_\Gamma F \cdot d\mathbf{l} = 0$.

**is F Conservative over $\Omega$ ?**

**1 - Compute** curl **F.**
- If curl $\mathbf{F} \neq \mathbf{0}$, **F** is **not** conservative.
- If curl $\mathbf{F} = \mathbf{0}$, proceed to Step 2.

**2 - is $\Omega$ Simply Connected ?**
- If *yes*, **F** is conservative.
- If *no*, proceed to Step 3.

**3 - Circulation Method:**
1. Select closed curves $\Gamma$ around each hole in $\Omega$.
2. Compute $\oint_\Gamma \mathbf{F} \cdot d\mathbf{l}$.
3. If any integral $\neq 0$, **F** is **not** conservative.
4. If all integrals $= 0$, proceed to Step 4.

**4 - Finding the Potential**

$$\varphi(x,y,z) = \int_{x_0}^{x} F_1(t,y,z) \, dt + \alpha(y,z)$$

1. Determine $\alpha(y,z)$ such that $\nabla \varphi = \mathbf{F}$.
2. If successful, $\varphi$ is the potential function for **F**, confirming **F** is conservative.

**Vector Calculus Theorems**

**Green's Theorem (Plane)** Let $\Omega \subseteq \mathbb{R}^2$ be a regular domain with a positively oriented boundary $\partial\Omega$, and $F \in C^1(\overline{\Omega}, \mathbb{R}^2)$. Then

$$\iint_\Omega \text{curl}(F(x,y)) \, dx \, dy = \int_{\partial\Omega} F \cdot dl$$

**Divergence Theorem (Space)** Let $\Omega \subseteq \mathbb{R}^3$ be a regular domain, $n : \partial\Omega \to \mathbb{R}^3$ a continuous outward unit normal vector field, and $F \in C^1(\overline{\Omega}, \mathbb{R}^3)$. Then

$$\iiint_\Omega \text{div}\, F(x,y,z) \, dx \, dy \, dz = \iint_{\partial\Omega} F \cdot n \, ds = \iint_A \langle F(\sigma(u,v)); \sigma_u(u,v) \wedge \sigma_v(u,v) \rangle \, du \, dv$$

**Divergence Theorem (Plane)** Let $\Omega \subseteq \mathbb{R}^2$ be a regular domain with a positively oriented boundary $\partial\Omega$, and $F \in C^1(\overline{\Omega}, \mathbb{R}^2)$. Then

$$\iint_\Omega \text{div}\, F(x,y) \, dx \, dy = \int_{\partial\Omega} F \cdot n \, dl = \int_a^b \langle F(\gamma(t)), (\gamma_2'(t), -\gamma_1'(t)) \rangle \, dt.$$

**Stokes' Theorem** Let $\Omega \subseteq \mathbb{R}^3$ be an open set, $\Sigma \subseteq \Omega$ a piecewise smooth orientable surface, and $F \in C^1(\Omega, \mathbb{R}^3)$, then

$$\iint_\Sigma \text{curl}\, F \, ds = \int_{\partial\Sigma} F \cdot dl$$

**Polar (2D)**

$$|\det J| = r$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix};$$

$$r \geq 0, \ \theta \in [0, 2\pi].$$

**Cylindrical (3D)**

$$|\det J| = r$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} r\cos\theta \\ r\sin\theta \\ z \end{bmatrix};$$

$$r \geq 0, \ \theta \in [0, 2\pi), \ z \in (-\infty, \infty).$$

**Spherical (3D)**

$$|\det J| = \rho^2 \sin\phi$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \rho\sin\phi\cos\theta \\ \rho\sin\phi\sin\theta \\ \rho\cos\phi \end{bmatrix};$$

$$\rho \geq 0, \ \phi \in [0, \pi], \ \theta \in [0, 2\pi].$$

**Distribution Theory**

Let $\mathcal{D}'$ be the set of distributions over $\mathbb{R}$, the set of linear continuous functionals over $\mathcal{D}$, $\mathcal{D}' = \{T \colon \mathcal{D} \to \mathbb{R} \mid T \text{ is linear and continuous}\}$. For a distribution $T \in \mathcal{D}'$ and a test function $\varphi \in \mathcal{D}$, the pairing is defined by $\langle T, \varphi \rangle = \int_\Omega f(x)\varphi(x) \, dx$.

A distribution $T \in \mathcal{D}'$ satisfies:

**Support**
$\text{supp}(T) = \overline{\{ x \in \mathbb{R} \mid T(\varphi) \neq 0 \}}$.

**Derivative**
$\varphi \in \mathcal{D}$, $T'(\varphi) = -T(\varphi')$.

**Boundedness** For every $\psi \in \mathcal{D}$, $|T(\psi)|$ is finite.

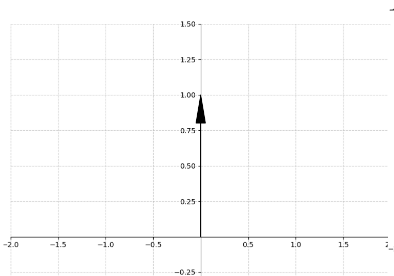
**Linearity** For all scalars $\alpha, \beta \in \mathbb{R}$ and test functions $\psi, \varphi \in \mathcal{D}$, $T(\alpha\psi + \beta\varphi) = \alpha T(\psi) + \beta T(\varphi)$.

**Continuity**

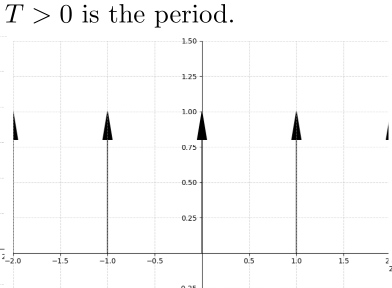
$\forall [a,b] \subseteq \mathbb{R}$, there exist constants $C > 0$ and $k \in \mathbb{N}_0$, such that $\forall \varphi \in \mathcal{D}$, $\text{supp}(\varphi) \subseteq [a,b] \implies |T(\varphi)| \leq C \sum_{0 \leq i \leq k} \max_{x \in \mathbb{R}} \left| \partial^i \varphi(x) \right|$.

**Higher-Order Derivatives** $\forall n \in \mathbb{N}$, $T^{(n)}(\varphi) = (-1)^n T(\varphi^{(n)})$.

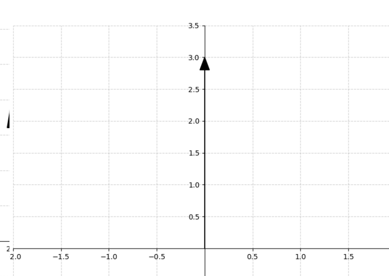
**Dirac Delta**
$\delta(\varphi) = \varphi(0)$ for all $\varphi \in \mathcal{D}$.



**Dirac Comb**
$\Delta_T(x) = \sum_{n \in \mathbb{Z}} \delta(x - nT)$,
$T > 0$ is the period.

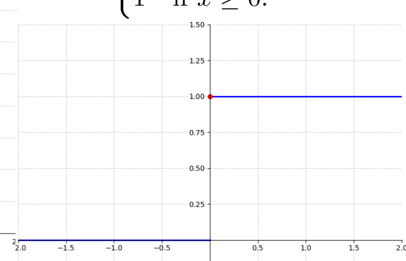


**Scaled Dirac Delta**
$a \in \mathbb{R}$, $(a\delta)(\varphi) = a \cdot \varphi(0)$.



**Heaviside Step**

$$H(x) = \begin{cases} 0 & \text{if } x < 0, \\ 1 & \text{if } x \geq 0. \end{cases}$$



**Piecewise Continuity & Differentiability**
$f : [a, b] \to \mathbb{R}$ is *piecewise continuous* if there is a partition

$$a = a_0 < a_1 < \cdots < a_n = b$$

such that $\lim_{x\to a_i^-} f(x)$ and $\lim_{x\to a_i^+} f(x)$ exist (finite).
Similarly, $f$ is *piecewise* $C^1$ if it is continuously differentiable on each open subinterval and the one-sided derivatives at boundaries exist.

**Euler's Formulas**

$$e^{x+iy} = e^x\big(\cos y + i \sin y\big),\ \sin x = \frac{e^{ix} - e^{-ix}}{2i},\ \cos x = \frac{e^{ix} + e^{-ix}}{2}.$$

**Orthogonality (Sine/Cosine Products)**
For $n, m \in \mathbb{N}_{\geq 1}$ and period $T > 0$:

$$\frac{2}{T}\int_0^T \cos\Big(\tfrac{2\pi n}{T}x\Big)\cos\Big(\tfrac{2\pi m}{T}x\Big)\,dx = \begin{cases}1 & n = m,\\ 0 & n \neq m\end{cases}$$

(Same for $\sin\sin$, and $\cos\sin$ integrates to 0.)

**Integration Over One Period**
If $f$ is $T$-periodic and piecewise continuous, then for any $a \in \mathbb{R}$:

$$\int_a^{a+T} f(x)\,dx = \int_0^T f(x)\,dx.$$

**Dirichlet's Theorem (Pointwise Convergence)**
Let $f : \mathbb{R} \to \mathbb{R}$ be $T$-periodic and piecewise $C^1$. Then, for all $x \in \mathbb{R}$,

$$Ff(x) = \lim_{t\to 0}\frac{f(x-t) + f(x+t)}{2}.$$

**Real Fourier Series**
For $f : \mathbb{R} \to \mathbb{R}$, $T$-periodic, piecewise $C^1$, the real Fourier series is

$$Ff(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\Big[a_n \cos\big(\tfrac{2\pi n}{T}x\big) + b_n \sin\big(\tfrac{2\pi n}{T}x\big)\Big].$$

**Fourier Coefficients:**

$$a_n = \frac{2}{T}\int_0^T f(x)\cos\Big(\tfrac{2\pi n}{T}x\Big)\,dx, \quad b_n = \frac{2}{T}\int_0^T f(x)\sin\Big(\tfrac{2\pi n}{T}x\Big)\,dx,$$

$$a_0 = \frac{2}{T}\int_0^T f(x)\,dx.$$

**Parity:** If $f$ is even, $b_n = 0$; if $f$ is odd, $a_n = 0$.

**Convolution Product**
Let $f, g : \mathbb{R} \to \mathbb{R}$ such that $\int_{-\infty}^{+\infty}|f(x)|\,dx < +\infty$ and $\int_{-\infty}^{+\infty}|g(x)|\,dx < +\infty$.

$$(f * g)(x) = \int_{-\infty}^{+\infty} f(x-t)\,g(t)\,dt = \int_{-\infty}^{+\infty} f(t)\,g(x-t)\,dt$$

**Term-by-Term Differentiation**
If $f$ is $T$-periodic, continuous, and piecewise $C^1$, then

$$\frac{d}{dx}\big[Ff(x)\big] = \sum_{n=1}^{\infty}\frac{2\pi n}{T}\big[-a_n \sin(\tfrac{2\pi n}{T}x) + b_n \cos(\tfrac{2\pi n}{T}x)\big]$$

$$= \lim_{t\to 0}\frac{f'(x-t) + f'(x+t)}{2}.$$

**1) Poisson on** $[a, b]$:

$$\begin{cases}-u''(x) = f(x),\\ u(a) = g_a, \quad u(b) = g_b,\end{cases} \qquad L = b - a$$

$$u^g(x) = \frac{g_b - g_a}{b-a}\,x \;+\; \frac{b\,g_a - a\,g_b}{b-a}.$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\Big(\frac{n\pi x}{L}\Big), \quad b_n = \frac{2}{L}\int_0^L f(t)\,\sin\Big(\frac{n\pi t}{L}\Big)\,dt,$$

$$u^f(x) = \sum_{n=1}^{\infty} b_n\,\frac{L^2}{\pi^2 n^2}\,\sin\Big(\frac{n\pi x}{L}\Big).$$

$$u(x) = u^g(x) + u^f(x).$$

**2) Poisson with mass term on** $\mathbb{R}$:

$$-u''(x) + k^2\,u(x) = f(x), \quad \widehat{u}(\alpha) = \frac{\widehat{f}(\alpha)}{\alpha^2 + k^2}.$$

$$g(x) = \frac{1}{2k}\,e^{-k|x|}, \quad \widehat{g}(\alpha) = \frac{1}{\alpha^2 + k^2}, \quad u(x) = (g*f)(x) = \frac{1}{2k}\int_{-\infty}^{\infty} f(y)\,e^{-k|\,x-y\,|}\,dy.$$

**Complex Fourier Coefficient**
Let $f : \mathbb{R} \to \mathbb{R}$ be $T$-periodic and piecewise continuous. The complex Fourier coefficients are:

$$c_n = \frac{1}{T}\int_0^T f(x)\,e^{-i\frac{2\pi}{T}nx}\,dx, \quad Ff(x) = \sum_{n=-\infty}^{\infty} c_n\,e^{i\frac{2\pi n}{T}x}.$$

For $\phi : \mathbb{R} \to \mathbb{C}$,

$$\int_a^b \phi(x)\,dx = \int_a^b \mathrm{Re}(\phi(x))\,dx + i\int_a^b \mathrm{Im}(\phi(x))\,dx.$$

**Relation to** $(a_n, b_n)$

$$c_n = \tfrac{1}{2}(a_n - i\,b_n),\ c_{-n} = \tfrac{1}{2}(a_n + i\,b_n),\ c_0 = \tfrac{a_0}{2}.$$

**Fourier Series on** $[0, L]$
For $f : [0, L] \to \mathbb{R}$ (piecewise $C^1$):

$$F_c f(x) = \tfrac{\tilde{a}_0}{2} + \sum_{n=1}^{\infty}\tilde{a}_n \cos\Big(\tfrac{\pi n}{L}x\Big), \quad \tilde{a}_n = \frac{2}{L}\int_0^L f(x)\cos\Big(\tfrac{\pi n}{L}x\Big)\,dx.$$

$$F_s f(x) = \sum_{n=1}^{\infty}\tilde{b}_n \sin\Big(\tfrac{\pi n}{L}x\Big), \quad \tilde{b}_n = \frac{2}{L}\int_0^L f(x)\sin\Big(\tfrac{\pi n}{L}x\Big)\,dx.$$

**Parseval's Identity (Periodic Case)**
If $f$ is $T$-periodic (piecewise $C^1$),

$$\frac{1}{T}\int_0^T f^2(x)\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2) = \sum_{n=-\infty}^{\infty}|c_n|^2.$$

**Plancherel Theorem**
Let $f \in L^2(\mathbb{R})$. Then its Fourier transform $\hat{f}$ is also in $L^2(\mathbb{R})$, and:

$$\int_{-\infty}^{\infty}|f(x)|^2\,dx = \int_{-\infty}^{\infty}|\hat{f}(\xi)|^2\,d\xi.$$

**The Fourier Transform**
If $f : \mathbb{R} \to \mathbb{R}$ with $\int_{-\infty}^{\infty}|f(x)|\,dx < \infty$, its (unitary) Fourier transform is

$$\widehat{f}(\alpha) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\,e^{-i\,\alpha x}\,dx,$$

**Inverse Transform**
If $\varphi(\alpha)$ is similarly integrable,

$$\mathcal{F}^{-1}(\varphi)(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\varphi(\alpha)\,e^{i\,\alpha x}\,d\alpha.$$

**Fourier Transform of** $f * g$
Let $f$ and $g$ be piecewise continuous and absolutely integrable. Then

$$\int_{-\infty}^{+\infty}|(f * g)(x)|\,dx < +\infty, \quad \mathcal{F}[f * g](\alpha) = \sqrt{2\pi}\,\hat{f}(\alpha)\,\hat{g}(\alpha).$$

**Scaling**
$\mathcal{F}\{f(ax)\} = \frac{1}{|a|}F\big(\frac{\alpha}{a}\big)$

**Shifting**
$\mathcal{F}\{f(x - x_0)\} = e^{-i\,\alpha x_0}\,F(\alpha)$

**Modulation**
$\mathcal{F}\{e^{i\,\omega_0 x}f(x)\} = F(\alpha - \omega_0)$

**Differentiation**
$\mathcal{F}\Big\{\dfrac{d^n}{dx^n}f(x)\Big\} = (i\alpha)^n F(\alpha)$

**Integration**
$\mathcal{F}\Big\{\int_{-\infty}^{x} f(\xi)\,d\xi\Big\} = \dfrac{1}{i\alpha}\,F(\alpha),\ \alpha \neq 0$

**Convolution**
$\mathcal{F}\{f * g\}(\alpha) = F(\alpha)\,G(\alpha)$

**Multiplication**
$\mathcal{F}\{f(x)\,g(x)\}(\alpha) = \dfrac{1}{2\pi}\int_{-\infty}^{\infty} F(\kappa)\,G(\alpha - \kappa)\,d\kappa$

**Real/Imag Parts**
$\mathcal{F}\{\mathrm{Re}[\,f(x)\,]\} = \dfrac{F(\alpha) + F^*(-\alpha)}{2}, \quad \mathcal{F}\{\mathrm{Im}[\,f$
$\dfrac{F(\alpha) - F^*(-\alpha)}{2i}$

**Important Trigonometric Identities**

$\sin(2x) = 2\sin x \cos x,$
$\cos(2x) = 2\cos^2 x - 1 = 1 - 2\sin^2 x = \cos^2 x - \sin^2 x,$
$\cos(a \pm b) = \cos a \cos b \mp \sin a \sin b,$
$\sin(a \pm b) = \sin a \cos b \pm \cos a \sin b.$

$\cos a \cos b = \frac{1}{2}\big[\cos(a-b) + \cos(a+b)\big],$
$\sin a \sin b = \frac{1}{2}\big[\cos(a-b) - \cos(a+b)\big],$
$\sin a \cos b = \frac{1}{2}\big[\sin(a+b) + \sin(a-b)\big],$
$\cos a \sin b = \frac{1}{2}\big[\sin(a+b) - \sin(a-b)\big].$