Analysis III - CheatSheet

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Regular Curve. A continuously differentiable map $\gamma : [a, b] \to \mathbb{R}^n$ is *regular* if $\gamma'(t) \neq 0$ for all $t \in [a, b]$.

Simple Curve. A continuous map $\gamma : [a, b] \to \mathbb{R}^n$ is *simple* if it does not intersect itself (except possibly at endpoints): $\gamma(t_1) = \gamma(t_2) \implies t_1 = t_2$

Simply Connected Domain. An open set $\Omega \subseteq \mathbb{R}^n$ is *simply connected* if for any two continuous curves $\gamma_0, \gamma_1 : [a, b] \to \Omega$ with $\gamma_0(a) = \gamma_1(a)$ and $\gamma_0(b) = \gamma_1(b)$, there exists a continuous homotopy $\Gamma : [a, b] \times [0, 1] \to \Omega$ that deforms γ_0 into γ_1 within Ω while keeping the endpoints fixed.

Curve in $\mathbb{R}^n$: Given $\gamma : [a, b] \to \mathbb{R}^n, \int_\gamma F \cdot d\mathbf{l} = \int_a^b \langle F(\gamma(t)), \gamma'(t) \rangle\, dt, \quad \int_\gamma f(\mathbf{x})\, ds = \int_a^b f(\gamma(t)) \, \|\gamma'(t)\|\, dt.$

Surface in $\mathbb{R}^3$: Given $\sigma : D \subset \mathbb{R}^2 \to \mathbb{R}^3$, $\sigma(u, v)$,

$$\iint_\sigma F \cdot d\mathbf{S} = \iint_D \langle F(\sigma(u, v)), \sigma_u(u, v) \times \sigma_v(u, v) \rangle\, du\, dv. \quad \iint_\sigma f(\mathbf{x})\, ds = \iint_D f(\sigma(u, v))\, \|\sigma_u(u, v) \times \sigma_v(u, v)\|\, du\, dv,$$

Conservative Fields and Path Independence

A map $F : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^n$ is *conservative* if there exists a differentiable function $\varphi : \Omega \to \mathbb{R}$ such that $\nabla f = F$.

F is conservative on $\Omega \iff \forall \Gamma_1, \Gamma_2 \subseteq \Omega$ reg. curves from A to B, $\int_{\Gamma_1} F \cdot d\mathbf{l} = \int_{\Gamma_2} F \cdot d\mathbf{l}, \iff \forall \Gamma \subseteq \Omega$ reg. closed curve, $\int_\Gamma F \cdot d\mathbf{l} = 0$.

is F Conservative over Ω ?

1 - Compute $\operatorname{curl} \mathbf{F}$.
- If $\operatorname{curl} \mathbf{F} \neq \mathbf{0}$, $\mathbf{F}$ is **not** conservative.
- If $\operatorname{curl} \mathbf{F} = \mathbf{0}$, proceed to Step 2.

2 - is Ω Simply Connected ?
- If *yes*, $\mathbf{F}$ is conservative.
- If *no*, proceed to Step 3.

3 - Circulation Method:
1. Select closed curves Γ around each hole in Ω.
2. Compute $\oint_\Gamma \mathbf{F} \cdot d\mathbf{l}$.
3. If any integral $\neq 0$, $\mathbf{F}$ is **not** conservative.
4. If all integrals $= 0$, proceed to Step 4.

4 - Finding the Potential

$$\varphi(x, y, z) = \int_{x_0}^x F_1(t, y, z)\, dt + \alpha(y, z)$$

1. Determine $\alpha(y, z)$ such that $\nabla \varphi = \mathbf{F}$.
2. If successful, φ is the potential function for $\mathbf{F}$, confirming $\mathbf{F}$ is conservative.

Vector Calculus Theorems

Green's Theorem (Plane) Let $\Omega \subseteq \mathbb{R}^2$ be a regular domain with a positively oriented boundary $\partial\Omega$, and $F \in C^1(\overline{\Omega}, \mathbb{R}^2)$. Then

$$\iint_\Omega \operatorname{curl}(F(x, y))\, dx\, dy = \int_{\partial\Omega} F \cdot dl$$

Divergence Theorem (Plane) Let $\Omega \subseteq \mathbb{R}^2$ be a regular domain with a positively oriented boundary $\partial\Omega$, and $F \in C^1(\overline{\Omega}, \mathbb{R}^2)$. Then

$$\iint_\Omega \operatorname{div} F(x, y)\, dx\, dy = \int_{\partial\Omega} F{\cdot}n\, dl = \int_a^b \langle F(\gamma(t)), (\gamma_2'(t), -\gamma_1'(t)) \rangle\, dt.$$

Divergence Theorem (Space) Let $\Omega \subseteq \mathbb{R}^3$ be a regular domain, $n : \partial\Omega \to \mathbb{R}^3$ a continuous outward unit normal vector field, and $F \in C^1(\overline{\Omega}, \mathbb{R}^3)$. Then

Stokes' Theorem Let $\Omega \subseteq \mathbb{R}^3$ be an open set, $\Sigma \subseteq \Omega$ a piecewise smooth orientable surface, and $F \in C^1(\Omega, \mathbb{R}^3)$, then

$$\iint_\Sigma \operatorname{curl} F\, ds = \int_{\partial\Sigma} F \cdot dl$$

$$\iiint_\Omega \operatorname{div} F(x, y, z)\, dx\, dy\, dz = \iint_{\partial\Omega} F{\cdot}n\, ds = \iint_A \langle F(\sigma(u, v)); \frac{\partial\sigma}{\partial u}(u, v) \wedge \frac{\partial\sigma}{\partial v}(u, v) \rangle\, du\, dv$$

Polar (2D)	**Cylindrical (3D)**	**Spherical (3D)**
$\|\det J\| = r$	$\|\det J\| = r$	$\|\det J\| = \rho^2 \sin\phi$
$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix};$	$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} r\cos\theta \\ r\sin\theta \\ z \end{bmatrix};$	$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \rho\sin\phi\cos\theta \\ \rho\sin\phi\sin\theta \\ \rho\cos\phi \end{bmatrix};$
$r \geq 0,\ \theta \in [0, 2\pi].$	$r \geq 0,\ \theta \in [0, 2\pi),\ z \in (-\infty, \infty).$	$\rho \geq 0,\ \phi \in [0, \pi],\ \theta \in [0, 2\pi].$

Distribution Theory

Let $\mathcal{D}'$ be the set of distributions over $\mathbb{R}$, the set of linear continuous functionals over $\mathcal{D}$, $\mathcal{D}' = \{T : \mathcal{D} \to \mathbb{R} \mid T \text{ is linear and continuous}\}$. For a distribution $T \in \mathcal{D}'$ and a test function $\varphi \in \mathcal{D}$, the pairing is defined by $\langle T, \varphi \rangle = \int_\Omega f(x)\varphi(x)\, dx$.

A distribution $T \in \mathcal{D}'$ satisfies:

Boundedness For every $\psi \in \mathcal{D}$, $|T(\psi)|$ is finite.

Linearity For all scalars $\alpha, \beta \in \mathbb{R}$ and test functions $\psi, \varphi \in \mathcal{D}$, $T(\alpha\psi + \beta\varphi) = \alpha T(\psi) + \beta T(\varphi)$.

Continuity

$\forall [a, b] \subseteq \mathbb{R}$, there exist constants $C > 0$ and $k \in \mathbb{N}_0$, such that $\forall \varphi \in \mathcal{D}$, $\operatorname{supp}(\varphi) \subseteq [a, b] \implies |T(\varphi)| \leq C \sum_{0 \leq i \leq k} \max_{x \in \mathbb{R}} |\partial^i \varphi(x)|.$

Higher-Order Derivatives $\forall n \in \mathbb{N}$, $T^{(n)}(\varphi) = (-1)^n T(\varphi^{(n)})$.

Support	**Derivative**
$\operatorname{supp}(T) = \overline{\{x \in \mathbb{R} \mid T(\varphi) \neq 0\}}.$	$\varphi \in \mathcal{D},\ T'(\varphi) = -T(\varphi').$

Dirac Delta
$\delta(\varphi) = \varphi(0)$ for all $\varphi \in \mathcal{D}$.

Dirac Comb
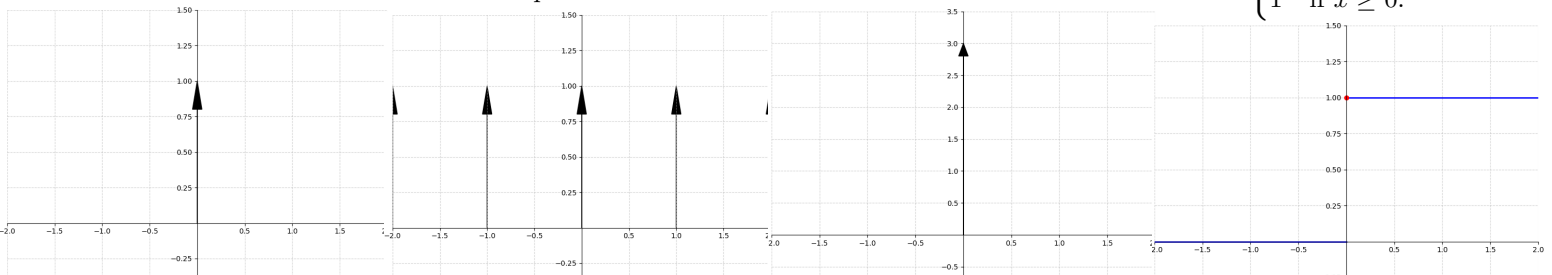$\Delta_T(x) = \sum_{n \in \mathbb{Z}} \delta(x - nT)$, $T > 0$ is the period.

Scaled Dirac Delta
$a \in \mathbb{R}$, $(a\delta)(\varphi) = a \cdot \varphi(0)$.

Heaviside Step
$$H(x) = \begin{cases} 0 & \text{if } x < 0, \\ 1 & \text{if } x \geq 0. \end{cases}$$

Piecewise Continuity & Differentiability

$f : [a, b] \to \mathbb{R}$ is *piecewise continuous* if there is a partition

$$a = a_0 < a_1 < \cdots < a_n = b$$

such that $\lim_{x \to a_i^-} f(x)$ and $\lim_{x \to a_i^+} f(x)$ exist (finite).

Similarly, f is *piecewise C^1* if it is continuously differentiable on each open subinterval and the one-sided derivatives at boundaries exist.

Euler's Formulas

$$e^{x+iy} = e^x(\cos y + i \sin y), \ \sin x = \frac{e^{ix} - e^{-ix}}{2i}, \ \cos x = \frac{e^{ix} + e^{-ix}}{2}.$$

Orthogonality (Sine/Cosine Products)

For $n, m \in \mathbb{N}_{\geq 1}$ and period $T > 0$:

$$\frac{2}{T} \int_0^T \cos\left(\frac{2\pi n}{T} x\right) \cos\left(\frac{2\pi m}{T} x\right) dx = \begin{cases} 1 & n = m, \\ 0 & n \neq m \end{cases}$$

(Same for $\sin \sin$, and $\cos \sin$ integrates to 0.)

Integration Over One Period

If f is T-periodic and piecewise continuous, then for any $a \in \mathbb{R}$:

$$\int_a^{a+T} f(x) \, dx = \int_0^T f(x) \, dx.$$

Dirichlet's Theorem (Pointwise Convergence)

Let $f : \mathbb{R} \to \mathbb{R}$ be T-periodic and piecewise C^1. Then, for all $x \in \mathbb{R}$,

$$Ff(x) = \lim_{t \to 0} \frac{f(x-t) + f(x+t)}{2}.$$

Real Fourier Series

For $f : \mathbb{R} \to \mathbb{R}$, T-periodic, piecewise C^1, the real Fourier series is

$$Ff(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{2\pi n}{T} x\right) + b_n \sin\left(\frac{2\pi n}{T} x\right)\right].$$

Fourier Coefficients:

$$a_n = \frac{2}{T} \int_0^T f(x) \cos\left(\frac{2\pi n}{T} x\right) dx, \quad b_n = \frac{2}{T} \int_0^T f(x) \sin\left(\frac{2\pi n}{T} x\right) dx,$$

$$a_0 = \frac{2}{T} \int_0^T f(x) \, dx.$$

Parity: If f is even, $b_n = 0$; if f is odd, $a_n = 0$.

Convolution Product

Let $f, g : \mathbb{R} \to \mathbb{R}$ such that $\int_{-\infty}^{+\infty} |f(x)| \, dx < +\infty$ and $\int_{-\infty}^{+\infty} |g(x)| \, dx < +\infty$.

$$(f * g)(x) = \int_{-\infty}^{+\infty} f(x - t) \, g(t) \, dt = \int_{-\infty}^{+\infty} f(t) \, g(x - t) \, dt$$

Term-by-Term Differentiation

If f is T-periodic, continuous, and piecewise C^1, then

$$\frac{d}{dx}\left[Ff(x)\right] = \sum_{n=1}^{\infty} \frac{2\pi n}{T}\left[-a_n \sin(\tfrac{2\pi n}{T} x) + b_n \cos(\tfrac{2\pi n}{T} x)\right]$$

$$= \lim_{t \to 0} \frac{f'(x-t) + f'(x+t)}{2}.$$

1) Poisson on $[a, b]$:

$$\begin{cases} -u''(x) = f(x), \\ u(a) = g_a, \quad u(b) = g_b, \end{cases} \qquad L = b - a$$

$$u^g(x) = \frac{g_b - g_a}{b - a} x + \frac{b \, g_a - a \, g_b}{b - a}.$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right), \quad b_n = \frac{2}{L} \int_0^L f(t) \sin\left(\frac{n\pi t}{L}\right) dt,$$

$$u^f(x) = \sum_{n=1}^{\infty} b_n \frac{L^2}{\pi^2 n^2} \sin\left(\frac{n\pi x}{L}\right).$$

$$u(x) = u^g(x) + u^f(x).$$

2) Poisson with mass term on $\mathbb{R}$:

$$-u''(x) + k^2 u(x) = f(x), \quad \hat{u}(\alpha) = \frac{\hat{f}(\alpha)}{\alpha^2 + k^2}.$$

$$g(x) = \frac{1}{2k} e^{-k|x|}, \quad \hat{g}(\alpha) = \frac{1}{\alpha^2 + k^2}, \quad u(x) = (g*f)(x) = \frac{1}{2k} \int_{-\infty}^{\infty} f(y) e^{-k|x-y|} \, dy.$$

Complex Fourier Coefficient

Let $f : \mathbb{R} \to \mathbb{R}$ be T-periodic and piecewise continuous. The complex Fourier coefficients are:

$$c_n = \frac{1}{T} \int_0^T f(x) e^{-i\frac{2\pi}{T} n x} \, dx, \quad Ff(x) = \sum_{n=-\infty}^{\infty} c_n e^{i\frac{2\pi n}{T} x}.$$

For $\phi : \mathbb{R} \to \mathbb{C}$,

$$\int_a^b \phi(x) \, dx = \int_a^b \text{Re}(\phi(x)) \, dx + i \int_a^b \text{Im}(\phi(x)) \, dx.$$

Relation to (a_n, b_n)

$$c_n = \frac{1}{2}(a_n - i \, b_n), \ c_{-n} = \frac{1}{2}(a_n + i \, b_n), \ c_0 = \frac{a_0}{2}.$$

Fourier Series on $[0, L]$

For $f : [0, L] \to \mathbb{R}$ (piecewise C^1):

$$F_c f(x) = \frac{\tilde{a}_0}{2} + \sum_{n=1}^{\infty} \tilde{a}_n \cos\left(\frac{\pi n}{L} x\right), \quad \tilde{a}_n = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{\pi n}{L} x\right) dx.$$

$$F_s f(x) = \sum_{n=1}^{\infty} \tilde{b}_n \sin\left(\frac{\pi n}{L} x\right), \quad \tilde{b}_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{\pi n}{L} x\right) dx.$$

Parseval's Identity (Periodic Case)

If f is T-periodic (piecewise C^1),

$$\frac{2}{T} \int_0^T f^2(x) \, dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) = 2 \sum_{n=-\infty}^{\infty} |c_n|^2.$$

Plancherel Theorem

Let $f \in L^2(\mathbb{R})$. Then its Fourier transform $\hat{f}$ is also in $L^2(\mathbb{R})$, and:

$$\int_{-\infty}^{\infty} |f(x)|^2 \, dx = \int_{-\infty}^{\infty} |\hat{f}(\xi)|^2 \, d\xi.$$

The Fourier Transform

If $f : \mathbb{R} \to \mathbb{R}$ with $\int_{-\infty}^{\infty} |f(x)| \, dx < \infty$, its (unitary) Fourier transform is

$$\hat{f}(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\alpha x} \, dx,$$

Inverse Transform

If $\varphi(\alpha)$ is similarly integrable,

$$\mathcal{F}^{-1}(\varphi)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \varphi(\alpha) e^{i\alpha x} \, d\alpha.$$

Fourier Transform of $f * g$

Let f and g be piecewise continuous and absolutely integrable. Then

$$\int_{-\infty}^{+\infty} |(f * g)(x)| \, dx < +\infty, \quad \mathcal{F}[f * g](\alpha) = \sqrt{2\pi} \, \hat{f}(\alpha) \, \hat{g}(\alpha).$$

Scaling
$\mathcal{F}\{f(ax)\} = \frac{1}{|a|} F\left(\frac{\alpha}{a}\right)$

Shifting
$\mathcal{F}\{f(x - x_0)\} = e^{-i\alpha x_0} F(\alpha)$

Modulation
$\mathcal{F}\{e^{i\omega_0 x} f(x)\} = F(\alpha - \omega_0)$

Differentiation
$\mathcal{F}\left\{\frac{d^n}{dx^n} f(x)\right\} = (i\alpha)^n F(\alpha)$

Integration
$\mathcal{F}\left\{\int_{-\infty}^x f(\xi) \, d\xi\right\} = \frac{1}{i\alpha} F(\alpha), \ \alpha \neq 0$

Multiplication
$\mathcal{F}\{f(x) \, g(x)\}(\alpha) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\kappa) \, G(\alpha - \kappa) \, d\kappa$

Real/Imag Parts
$$\mathcal{F}\{\text{Re}[\, f(x) \,]\} = \frac{F(\alpha) + F^*(-\alpha)}{2},$$
$$\mathcal{F}\{\text{Im}[\, f(x) \,]\} = \frac{F(\alpha) - F^*(-\alpha)}{2i}$$

Important Trigonometric Identities

$\sin(2x) = 2 \sin x \cos x$,
$\cos(2x) = 2\cos^2 x - 1 = 1 - 2\sin^2 x = \cos^2 x - \sin^2 x$,
$\cos(a \pm b) = \cos a \cos b \mp \sin a \sin b$,
$\sin(a \pm b) = \sin a \cos b \pm \cos a \sin b$.
$\cos a \cos b = \frac{1}{2}\left[\cos(a - b) + \cos(a + b)\right]$,
$\sin a \sin b = \frac{1}{2}\left[\cos(a - b) - \cos(a + b)\right]$,
$\sin a \cos b = \frac{1}{2}\left[\sin(a + b) + \sin(a - b)\right]$,
$\cos a \sin b = \frac{1}{2}\left[\sin(a + b) - \sin(a - b)\right]$.