Analysis III - CheatSheet

IN BA3 - Martin Werner Licht

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An Analysis III Cheatsheet has been authorized for the upcoming exam, and I'm sharing a copy of mine for anyone interested. It provides a concise summary of the key concepts and techniques covered in the course. While it's not yet complete, I plan to update it soon, especially to include additional trigonometric identities and a step-by-step guide to solving differential equations using Fourier methods. When printing, you can select the last two pages, as only one A4 recto-verso page is allowed. For any updates or suggestions, feel free to reach out to me on Telegram at `elazdi_al` *or via EPFL email at* `ali.elazdi@epfl.ch`.

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Regular Curve. A continuously differentiable map $\gamma : [a, b] \to \mathbb{R}^n$ is *regular* if $\gamma'(t) \neq 0$ for all $t \in [a, b]$.

Simple Curve. A continuous map $\gamma : [a, b] \to \mathbb{R}^n$ is *simple* if it does not intersect itself (except possibly at endpoints): $\gamma(t_1) = \gamma(t_2) \implies t_1 = t_2$

Simply Connected Domain. An open set $\Omega \subseteq \mathbb{R}^n$ is *simply connected* if for any two continuous curves $\gamma_0, \gamma_1 : [a, b] \to \Omega$ with $\gamma_0(a) = \gamma_1(a)$ and $\gamma_0(b) = \gamma_1(b)$, there exists a continuous homotopy $\Gamma : [a, b] \times [0, 1] \to \Omega$ that deforms γ_0 into γ_1 within Ω while keeping the endpoints fixed.

Curve in $\mathbb{R}^n$: Given $\gamma : [a, b] \to \mathbb{R}^n$, $\int_\gamma F \cdot d\mathbf{l} = \int_a^b \langle F(\gamma(t)), \gamma'(t)\rangle \, dt$, $\quad \int_\gamma f(\mathbf{x})\, ds = \int_a^b f\big(\gamma(t)\big) \, \|\gamma'(t)\| \, dt$.

Surface in $\mathbb{R}^3$: Given $\sigma : D \subset \mathbb{R}^2 \to \mathbb{R}^3$, $\sigma(u, v)$,

$$\iint_\sigma F \cdot d\mathbf{S} = \iint_D \left\langle F(\sigma(u, v)), \frac{\partial \sigma}{\partial u} \times \frac{\partial \sigma}{\partial v} \right\rangle du\, dv. \qquad \iint_\sigma f(\mathbf{x})\, ds = \iint_D f\big(\sigma(u, v)\big) \left\| \frac{\partial \sigma}{\partial u} \times \frac{\partial \sigma}{\partial v} \right\| du\, dv.$$

Conservative Fields and Path Independence

A map $F : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^n$ is *conservative* if there exists a differentiable function $\varphi : \Omega \to \mathbb{R}$ such that $\nabla f = F$.

F is conservative on $\Omega \iff \forall \Gamma_1, \Gamma_2 \subseteq \Omega$ reg. curves from A to B, $\int_{\Gamma_1} F \cdot d\mathbf{l} = \int_{\Gamma_2} F \cdot d\mathbf{l}$, $\iff \forall \Gamma \subseteq \Omega$ reg. closed curve, $\int_\Gamma F \cdot d\mathbf{l} = 0$.

is F Conservative over Ω ?

1 - Compute curl F.
- If $\operatorname{curl} \mathbf{F} \neq \mathbf{0}$, $\mathbf{F}$ is **not** conservative.
- If $\operatorname{curl} \mathbf{F} = \mathbf{0}$, proceed to Step 2.

$$\boxed{\textbf{2D:} \nabla \times \mathbf{F} = \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}}$$

2 - is Ω Simply Connected ?
- If *yes*, $\mathbf{F}$ is conservative.
- If *no*, proceed to Step 3.

$$\boxed{\textbf{3D: } \nabla \times \mathbf{F} = \begin{bmatrix} \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \\ \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \\ \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \end{bmatrix}}$$

3 - Circulation Method:
1. Select closed curves Γ around each hole in Ω.
2. Compute $\oint_\Gamma \mathbf{F} \cdot d\mathbf{l}$.
3. If any integral $\neq 0$, $\mathbf{F}$ is **not** conservative.
4. If all integrals $= 0$, proceed to Step 4.

4 - Finding the Potential

$$\varphi(x, y, z) = \int_{x_0}^{x} F_1(t, y, z)\, dt + \alpha(y, z)$$

1. Determine $\alpha(y, z)$ such that $\nabla \varphi = \mathbf{F}$.
2. If successful, φ is the potential function for $\mathbf{F}$, confirming $\mathbf{F}$ is conservative.

$$\gamma(t) = \big(x(t), y(t)\big),\ \mathbf{n}(t) = \frac{(y'(t), -x'(t))}{\|\gamma'(t)\|} \qquad \sigma(u, v) = \big(x(u, v), y(u, v), z(u, v)\big),\ \mathbf{n}(u, v) = \frac{\frac{\partial \sigma}{\partial u} \times \frac{\partial \sigma}{\partial v}}{\left\| \frac{\partial \sigma}{\partial u} \times \frac{\partial \sigma}{\partial v} \right\|}, \ \text{!not necessary outward}$$

Vector Calculus Theorems

Green's Theorem (Plane) Let $\Omega \subseteq \mathbb{R}^2$ be a regular domain with a positively oriented boundary $\partial\Omega$, and $F \in C^1(\overline{\Omega}, \mathbb{R}^2)$. Then

$$\iint_\Omega \operatorname{curl}(F(x, y))\, dx\, dy = \int_{\partial\Omega} F \cdot dl$$

Divergence Theorem (Space) Let $\Omega \subseteq \mathbb{R}^3$ be a regular domain, $n : \partial\Omega \to \mathbb{R}^3$ a continuous outward unit normal vector field, and $F \in C^1(\overline{\Omega}, \mathbb{R}^3)$. Then

$$\iiint_\Omega \operatorname{div} F(x, y, z)\, dx\, dy\, dz = \iint_{\partial\Omega} F \cdot n\, ds = \iint_A \langle F(\sigma); \frac{\partial \sigma}{\partial u} \wedge \frac{\partial \sigma}{\partial v} \rangle\, du\, dv$$

Divergence Theorem (Plane) Let $\Omega \subseteq \mathbb{R}^2$ be a regular domain with a positively oriented boundary $\partial\Omega$, and $F \in C^1(\overline{\Omega}, \mathbb{R}^2)$. Then

$$\iint_\Omega \operatorname{div} F(x, y)\, dx\, dy = \int_{\partial\Omega} F \cdot n\, dl = \int_a^b \langle F(\gamma(t)), (\gamma_2'(t), -\gamma_1'(t)) \rangle\, dt.$$

Stokes' Theorem Let $\Omega \subseteq \mathbb{R}^3$ be an open set, $\Sigma \subseteq \Omega$ a piecewise smooth orientable surface, and $F \in C^1(\Omega, \mathbb{R}^3)$, then

$$\iint_\Sigma \operatorname{curl} F \cdot n\, ds = \int_{\partial\Sigma} F \cdot dl = \iint_A \left\langle \operatorname{curl} F(\sigma(u, v)), \frac{\partial \sigma}{\partial u} \wedge \frac{\partial \sigma}{\partial v} \right\rangle du\, dv$$

Polar (2D)

$$|\det J| = r$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix};$$

$$r \geq 0,\ \theta \in [0, 2\pi].$$

Cylindrical (3D)

$$|\det J| = r$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} r\cos\theta \\ r\sin\theta \\ z \end{bmatrix};$$

$$r \geq 0,\ \theta \in [0, 2\pi),\ z \in (-\infty, \infty).$$

Spherical (3D)

$$|\det J| = \rho^2 \sin\phi$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \rho\sin\phi\cos\theta \\ \rho\sin\phi\sin\theta \\ \rho\cos\phi \end{bmatrix};$$

$$\rho \geq 0,\ \phi \in [0, \pi],\ \theta \in [0, 2\pi].$$

Distribution Theory

Let $\mathcal{D}'$ be the set of distributions over $\mathbb{R}$, the set of linear continuous functionals over $\mathcal{D}$, $\mathcal{D}' = \{T \colon \mathcal{D} \to \mathbb{R} \mid T \text{ is linear and continuous}\}$.
For a distribution $T \in \mathcal{D}'$ and a test function $\varphi \in \mathcal{D}$, the pairing is defined by $\langle T, \varphi \rangle = \int_\Omega f(x)\varphi(x)\, dx$.

A distribution $T \in \mathcal{D}'$ satisfies:

Support
$\operatorname{supp}(T) = \overline{\{ x \in \mathbb{R} \mid T(\varphi) \neq 0 \}}$.

Derivative
$\varphi \in \mathcal{D}$, $T'(\varphi) = -T(\varphi')$.

Boundedness For every $\psi \in \mathcal{D}$, $|T(\psi)|$ is finite.

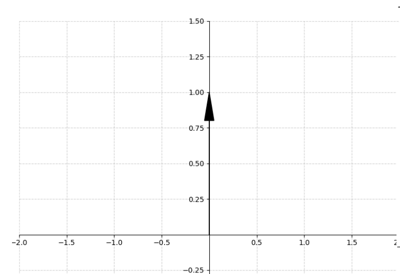
Linearity For all scalars $\alpha, \beta \in \mathbb{R}$ and test functions $\psi, \varphi \in \mathcal{D}$, $T(\alpha\psi + \beta\varphi) = \alpha T(\psi) + \beta T(\varphi)$.

Continuity

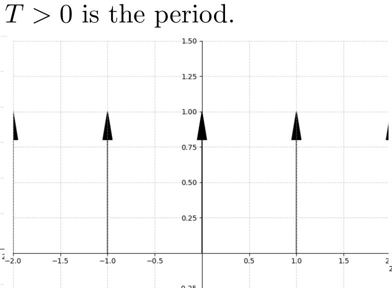
$\forall [a, b] \subseteq \mathbb{R}$, there exist constants $C > 0$ and $k \in \mathbb{N}_0$, such that $\forall \varphi \in \mathcal{D}$, $\operatorname{supp}(\varphi) \subseteq [a, b] \implies |T(\varphi)| \leq C \sum_{0 \leq i \leq k} \max_{x \in \mathbb{R}} \left|\partial^i \varphi(x)\right|$.

Higher-Order Derivatives $\forall n \in \mathbb{N}$, $T^{(n)}(\varphi) = (-1)^n T(\varphi^{(n)})$.

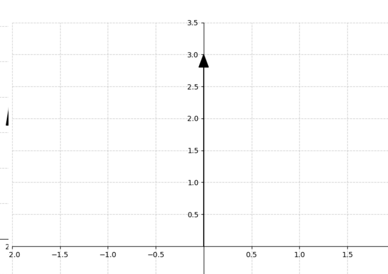
Dirac Delta
$\delta(\varphi) = \varphi(0)$ for all $\varphi \in \mathcal{D}$.



Dirac Comb
$\Delta_T(x) = \sum_{n \in \mathbb{Z}} \delta(x - nT)$,
$T > 0$ is the period.

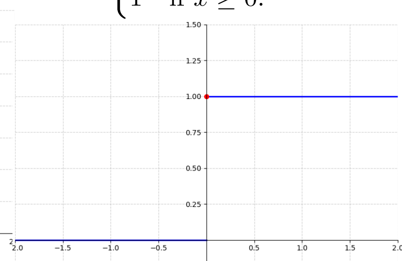


Scaled Dirac Delta
$a \in \mathbb{R}$, $(a\delta)(\varphi) = a \cdot \varphi(0)$.



Heaviside Step

$$H(x) = \begin{cases} 0 & \text{if } x < 0, \\ 1 & \text{if } x \geq 0. \end{cases}$$



Piecewise Continuity & Differentiability
$f:[a,b]\to\mathbb{R}$ is *piecewise continuous* if there is a partition
$$a=a_0<a_1<\cdots<a_n=b$$
such that $\lim_{x\to a_i^-} f(x)$ and $\lim_{x\to a_i^+} f(x)$ exist (finite).
Similarly, f is *piecewise* C^1 if it is continuously differentiable on each open subinterval and the one-sided derivatives at boundaries exist.

Euler's Formulas
$$e^{x+iy}=e^x\big(\cos y+i\sin y\big),\ \sin x=\frac{e^{ix}-e^{-ix}}{2i},\ \cos x=\frac{e^{ix}+e^{-ix}}{2}.$$

Orthogonality (Sine/Cosine Products)
For $n,m\in\mathbb{N}_{\geq 1}$ and period $T>0$:
$$\frac{2}{T}\int_0^T \cos\Big(\tfrac{2\pi n}{T}x\Big)\cos\Big(\tfrac{2\pi m}{T}x\Big)\,dx=\begin{cases}1 & n=m,\\ 0 & n\neq m\end{cases}$$
(Same for $\sin\sin$, and $\cos\sin$ integrates to 0.)

Integration Over One Period
If f is T-periodic and piecewise continuous, then for any $a\in\mathbb{R}$:
$$\int_a^{a+T} f(x)\,dx=\int_0^T f(x)\,dx.$$

Dirichlet's Theorem (Pointwise Convergence)
Let $f:\mathbb{R}\to\mathbb{R}$ be T-periodic and piecewise C^1. Then, for all $x\in\mathbb{R}$,
$$Ff(x)=\lim_{t\to 0}\frac{f(x-t)+f(x+t)}{2}.$$

Real Fourier Series
For $f:\mathbb{R}\to\mathbb{R}$, T-periodic, piecewise C^1, the real Fourier series is
$$Ff(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\Big[a_n\cos(\tfrac{2\pi n}{T}x)+b_n\sin(\tfrac{2\pi n}{T}x)\Big].$$
Fourier Coefficients:
$$a_n=\frac{2}{T}\int_0^T f(x)\cos\Big(\tfrac{2\pi n}{T}x\Big)\,dx,\quad b_n=\frac{2}{T}\int_0^T f(x)\sin\Big(\tfrac{2\pi n}{T}x\Big)\,dx,$$
$$a_0=\frac{2}{T}\int_0^T f(x)\,dx.$$
Parity: If f is even, $b_n=0$; if f is odd, $a_n=0$.

Term-by-Term Differentiation
If f is T-periodic, continuous, and piecewise C^1, then
$$\frac{d}{dx}\big[Ff(x)\big]=\sum_{n=1}^{\infty}\frac{2\pi n}{T}\big[-a_n\sin(\tfrac{2\pi n}{T}x)+b_n\cos(\tfrac{2\pi n}{T}x)\big]$$
$$=\lim_{t\to 0}\frac{f'(x-t)+f'(x+t)}{2}.$$

Term-by-Term Integration
If f is T-periodic, continuous, and piecewise C^1, then
$$\int Ff(x)\,dx=\sum_{n=1}^{\infty}\frac{2}{Tn\pi}\left[a_n\sin\left(\tfrac{2\pi n}{T}x\right)-b_n\cos\left(\tfrac{2\pi n}{T}x\right)\right]+C$$
$$=\lim_{h\to 0}\frac{1}{2h}\int_{x-h}^{x+h}Ff(t)\,dt,$$
where C is the constant of integration.

Poisson on $[a,b]$
$$\begin{cases}-u''(x)=f(x),\\ u(a)=g_a,\quad u(b)=g_b,\end{cases}\qquad L=b-a$$
$$u^g(x)=\frac{g_b-g_a}{b-a}\,x\;+\;\frac{b\,g_a-a\,g_b}{b-a}.$$
$$f(x)=\sum_{n=1}^{\infty}b_n\,\sin\Big(\frac{n\pi x}{L}\Big),\quad b_n=\frac{2}{L}\int_0^L f(t)\,\sin\Big(\frac{n\pi t}{L}\Big)\,dt,$$
$$u^f(x)=\sum_{n=1}^{\infty}b_n\,\frac{L^2}{\pi^2 n^2}\,\sin\Big(\frac{n\pi x}{L}\Big).$$
$$u(x)=u^g(x)+u^f(x)\quad\text{(superposition principle).}$$

Poisson with mass term on $\mathbb{R}$
$$-u''(x)+k^2u(x)=f(x),\quad \widehat{u}(\alpha)=\frac{\widehat{f}(\alpha)}{\alpha^2+k^2}.$$
$$g(x)=\frac{1}{2k}\,e^{-k|x|},\quad \widehat{g}(\alpha)=\frac{1}{\alpha^2+k^2},$$
$$u(x)=(g*f)(x)=\frac{1}{2k}\int_{-\infty}^{\infty}f(y)\,e^{-k|\,x-y\,|}\,dy.$$

Complex Fourier Coefficient
Let $f:\mathbb{R}\to\mathbb{R}$ be T-periodic and piecewise continuous. The complex Fourier coefficients are:
$$c_n=\frac{1}{T}\int_0^T f(x)\,e^{-i\frac{2\pi}{T}nx}\,dx,\quad Ff(x)=\sum_{n=-\infty}^{\infty}c_n\,e^{i\frac{2\pi n}{T}x}.$$
For $\phi:\mathbb{R}\to\mathbb{C}$,
$$\int_a^b\phi(x)\,dx=\int_a^b\mathrm{Re}(\phi(x))\,dx+i\int_a^b\mathrm{Im}(\phi(x))\,dx.$$

Relation to (a_n,b_n)
$$c_n=\tfrac{1}{2}(a_n-i\,b_n),\ c_{-n}=\tfrac{1}{2}(a_n+i\,b_n),\ c_0=\tfrac{a_0}{2}.$$

Fourier Series on $[0,L]$
For $f:[0,L]\to\mathbb{R}$ (piecewise C^1):
$$F_cf(x)=\tfrac{\tilde{a}_0}{2}+\sum_{n=1}^{\infty}\tilde{a}_n\cos\Big(\tfrac{\pi n}{L}x\Big),\quad \tilde{a}_n=\frac{2}{L}\int_0^L f(x)\cos\Big(\tfrac{\pi n}{L}x\Big)\,dx.$$
$$F_sf(x)=\sum_{n=1}^{\infty}\tilde{b}_n\sin\Big(\tfrac{\pi n}{L}x\Big),\quad \tilde{b}_n=\frac{2}{L}\int_0^L f(x)\sin\Big(\tfrac{\pi n}{L}x\Big)\,dx.$$

Parseval's Identity (Periodic Case)
If f is T-periodic (piecewise C^1),
$$\frac{2}{T}\int_0^T f^2(x)\,dx=\frac{a_0^2}{2}+\sum_{n=1}^{\infty}(a_n^2+b_n^2)=2\sum_{n=-\infty}^{\infty}\big|c_n\big|^2.$$

Plancherel Theorem
Let $f\in L^2(\mathbb{R})$. Then its Fourier transform $\hat{f}$ is also in $L^2(\mathbb{R})$, and:
$$\int_{-\infty}^{\infty}|f(x)|^2\,dx=\int_{-\infty}^{\infty}|\hat{f}(\xi)|^2\,d\xi.$$

The Fourier Transform
If $f:\mathbb{R}\to\mathbb{R}$ with $\int_{-\infty}^{\infty}|f(x)|\,dx<\infty$, its (unitary) Fourier transform is
$$\widehat{f}(\alpha)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x)\,e^{-i\,\alpha x}\,dx,$$
Inverse Transform
If $\varphi(\alpha)$ is similarly integrable,
$$\mathcal{F}^{-1}(\varphi)(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\varphi(\alpha)\,e^{i\,\alpha x}\,d\alpha.$$

Convolution Product
Let $f,g:\mathbb{R}\to\mathbb{R}$ such that $\int_{-\infty}^{+\infty}|f(x)|\,dx<+\infty$, $\int_{-\infty}^{+\infty}|g(x)|\,dx<+\infty$.
$$(f*g)(x)=\int_{-\infty}^{+\infty}f(x-t)\,g(t)\,dt=\int_{-\infty}^{+\infty}f(t)\,g(x-t)\,dt$$

Scaling
$\mathcal{F}\{f(ax)\}=\frac{1}{|a|}\hat{f}\big(\frac{\alpha}{a}\big)$

Shifting
$\mathcal{F}\{f(x-x_0)\}=e^{-i\alpha x_0}\,\hat{f}(\alpha)$

Modulation
$\mathcal{F}\{e^{i\omega_0 x}f(x)\}=\hat{f}(\alpha-\omega_0)$

Convolution
$\mathcal{F}[f*g](\alpha)=\sqrt{2\pi}\,\hat{f}(\alpha)\,\hat{g}(\alpha).$

Differentiation
$\mathcal{F}\Big\{\dfrac{d^n}{dx^n}f(x)\Big\}=(i\alpha)^n\hat{f}(\alpha)$

Integration
$\mathcal{F}\Big\{\int_{-\infty}^{x}f(\xi)\,d\xi\Big\}=\dfrac{1}{i\alpha}\,\hat{f}(\alpha),\ \alpha\neq 0$

Multiplication
$\mathcal{F}\{f(x)\cdot g(x)\}=\frac{1}{\sqrt{2\pi}}\left(\mathcal{F}\{f(x)\}*\mathcal{F}\{g(x)\}\right)$

Important Trigonometric Identities
$\sin(2x)=2\sin x\cos x,$
$\cos(2x)=2\cos^2 x-1=1-2\sin^2 x=\cos^2 x-\sin^2 x,$
$\cos(a\pm b)=\cos a\cos b\mp\sin a\sin b,$
$\sin(a\pm b)=\sin a\cos b\pm\cos a\sin b,$
$\cos a\cos b=\frac{1}{2}\big[\cos(a-b)+\cos(a+b)\big],$
$\sin a\sin b=\frac{1}{2}\big[\cos(a-b)-\cos(a+b)\big],$
$\sin a\cos b=\frac{1}{2}\big[\sin(a+b)+\sin(a-b)\big],$
$\cos a\sin b=\frac{1}{2}\big[\sin(a+b)-\sin(a-b)\big].$