Regular Curve. A continuously differentiable map $\gamma : [a, b] \to \mathbb{R}^n$ is *regular* if $\gamma'(t) \neq 0$ for all $t \in [a, b]$.

Simple Curve. A continuous map $\gamma : [a, b] \to \mathbb{R}^n$ is *simple* if it does not intersect itself (except possibly at endpoints): $\gamma(t_1) = \gamma(t_2) \implies t_1 = t_2$

Simply Connected Domain. An open set $\Omega \subseteq \mathbb{R}^n$ is *simply connected* if for any two continuous curves $\gamma_0, \gamma_1 : [a, b] \to \Omega$ with $\gamma_0(a) = \gamma_1(a)$ and $\gamma_0(b) = \gamma_1(b)$, there exists a continuous homotopy $\Gamma : [a, b] \times [0, 1] \to \Omega$ that deforms γ_0 into γ_1 within Ω while keeping the endpoints fixed.

Curve in $\mathbb{R}^n$: Given $\gamma : [a, b] \to \mathbb{R}^n$, $\int_\gamma F \cdot d\mathbf{l} = \int_a^b \langle F(\gamma(t)), \gamma'(t) \rangle \, dt$, $\quad \int_\gamma f(\mathbf{x}) \, ds = \int_a^b f\big(\gamma(t)\big) \, \|\gamma'(t)\| \, dt$.

Surface in $\mathbb{R}^3$: Given $\sigma : D \subset \mathbb{R}^2 \to \mathbb{R}^3$, $\sigma(u, v)$,

$$\iint_\sigma F \cdot d\mathbf{S} = \iint_D \big\langle F(\sigma(u, v)), \sigma_u(u, v) \times \sigma_v(u, v) \big\rangle \, du \, dv. \qquad \iint_\sigma f(\mathbf{x}) \, ds = \iint_D f\big(\sigma(u, v)\big) \, \|\sigma_u(u, v) \times \sigma_v(u, v)\| \, du \, dv,$$

Conservative Fields and Path Independence

A map $F : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^n$ is *conservative* if there exists a differentiable function $\varphi : \Omega \to \mathbb{R}$ such that $\nabla f = F$.

F is conservative on Ω $\iff$ $\forall \Gamma_1, \Gamma_2 \subseteq \Omega$ reg. curves from A to B, $\int_{\Gamma_1} F \cdot d\mathbf{l} = \int_{\Gamma_2} F \cdot d\mathbf{l}$, $\iff$ $\forall \Gamma \subseteq \Omega$ reg. closed curve, $\int_\Gamma F \cdot d\mathbf{l} = 0$.

is F Conservative over Ω ?

1 - Compute curl $\mathbf{F}$.
- If curl $\mathbf{F} \neq \mathbf{0}$, $\mathbf{F}$ is **not** conservative.
- If curl $\mathbf{F} = \mathbf{0}$, proceed to Step 2.

2 - is Ω Simply Connected ?
- If *yes*, $\mathbf{F}$ is conservative.
- If *no*, proceed to Step 3.

3 - Circulation Method:
1. Select closed curves Γ around each hole in Ω.
2. Compute $\oint_\Gamma \mathbf{F} \cdot d\mathbf{l}$.
3. If any integral $\neq 0$, $\mathbf{F}$ is **not** conservative.
4. If all integrals $= 0$, proceed to Step 4.

4 - Finding the Potential

$$\varphi(x, y, z) = \int_{x_0}^{x} F_1(t, y, z) \, dt + \alpha(y, z)$$

1. Determine $\alpha(y, z)$ such that $\nabla \varphi = \mathbf{F}$.
2. If successful, φ is the potential function for $\mathbf{F}$, confirming $\mathbf{F}$ is conservative.

Vector Calculus Theorems

Green's Theorem (Plane) Let $\Omega \subseteq \mathbb{R}^2$ be a regular domain with a positively oriented boundary $\partial\Omega$, and $F \in C^1(\overline{\Omega}, \mathbb{R}^2)$. Then

$$\iint_\Omega \operatorname{curl}(F(x, y)) \, dx \, dy = \int_{\partial\Omega} F \cdot dl$$

Divergence Theorem (Plane) Let $\Omega \subseteq \mathbb{R}^2$ be a regular domain with a positively oriented boundary $\partial\Omega$, and $F \in C^1(\overline{\Omega}, \mathbb{R}^2)$. Then

$$\iint_\Omega \operatorname{div} F(x, y) \, dx \, dy = \int_{\partial\Omega} F \cdot n \, dl = \int_a^b \langle F(\gamma(t)), (\gamma_2'(t), -\gamma_1'(t)) \rangle \, dt.$$

Divergence Theorem (Space) Let $\Omega \subseteq \mathbb{R}^3$ be a regular domain, $n : \partial\Omega \to \mathbb{R}^3$ a continuous outward unit normal vector field, and $F \in C^1(\overline{\Omega}, \mathbb{R}^3)$. Then

$$\iiint_\Omega \operatorname{div} F(x, y, z) \, dx \, dy \, dz = \iint_{\partial\Omega} F \cdot n \, ds = \iint_A \langle F(\sigma(u, v)); \sigma_u(u, v) \wedge \sigma_v(u, v) \rangle \, du \, dv$$

Stokes' Theorem Let $\Omega \subseteq \mathbb{R}^3$ be an open set, $\Sigma \subseteq \Omega$ a piecewise smooth orientable surface, and $F \in C^1(\Omega, \mathbb{R}^3)$, then

$$\iint_\Sigma \operatorname{curl} F \, ds = \int_{\partial\Sigma} F \cdot dl$$

Polar (2D)

$$|\det J| = r$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix}; \quad r \geq 0, \ \theta \in [0, 2\pi].$$

Cylindrical (3D)

$$|\det J| = r$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} r\cos\theta \\ r\sin\theta \\ z \end{bmatrix}; \quad r \geq 0, \ \theta \in [0, 2\pi), \ z \in (-\infty, \infty).$$

Spherical (3D)

$$|\det J| = \rho^2 \sin\phi$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \rho\sin\phi\cos\theta \\ \rho\sin\phi\sin\theta \\ \rho\cos\phi \end{bmatrix}; \quad \rho \geq 0, \ \phi \in [0, \pi], \ \theta \in [0, 2\pi].$$

Distribution Theory

Let $\mathcal{D}'$ be the set of distributions over $\mathbb{R}$, the set of linear continuous functionals over $\mathcal{D}$, $\mathcal{D}' = \{T : \mathcal{D} \to \mathbb{R} \mid T \text{ is linear and continuous}\}$.
For a distribution $T \in \mathcal{D}'$ and a test function $\varphi \in \mathcal{D}$, the pairing is defined by $\langle T, \varphi \rangle = \int_\Omega f(x)\varphi(x) \, dx$.

A distribution $T \in \mathcal{D}'$ satisfies:

Support $\operatorname{supp}(T) = \overline{\{x \in \mathbb{R} \mid T(\varphi) \neq 0\}}$.

Derivative $\varphi \in \mathcal{D}$, $T'(\varphi) = -T(\varphi')$.

Boundedness For every $\psi \in \mathcal{D}$, $|T(\psi)|$ is finite.

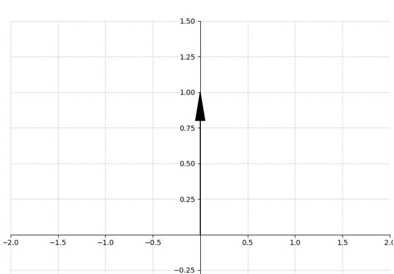
Linearity For all scalars $\alpha, \beta \in \mathbb{R}$ and test functions $\psi, \varphi \in \mathcal{D}$, $T(\alpha\psi + \beta\varphi) = \alpha T(\psi) + \beta T(\varphi)$.

Continuity

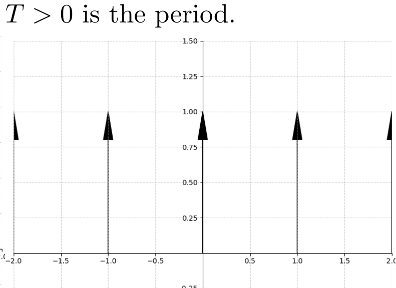
$\forall [a, b] \subseteq \mathbb{R}$, there exist constants $C > 0$ and $k \in \mathbb{N}_0$, such that $\forall \varphi \in \mathcal{D}$, $\operatorname{supp}(\varphi) \subseteq [a, b] \implies |T(\varphi)| \leq C \sum_{0 \leq i \leq k} \max_{x \in \mathbb{R}} \left|\partial^i \varphi(x)\right|$.

Higher-Order Derivatives $\forall n \in \mathbb{N}$, $T^{(n)}(\varphi) = (-1)^n T(\varphi^{(n)})$.

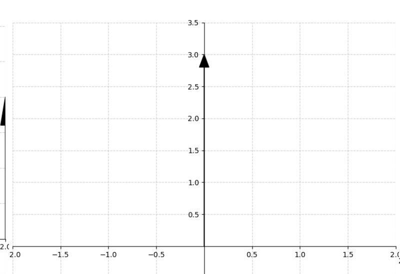
Dirac Delta
$\delta(\varphi) = \varphi(0)$ for all $\varphi \in \mathcal{D}$.



Dirac Comb
$\Delta_T(x) = \sum_{n \in \mathbb{Z}} \delta(x - nT)$,
$T > 0$ is the period.

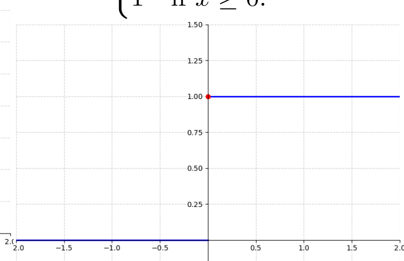


Scaled Dirac Delta
$a \in \mathbb{R}$, $(a\delta)(\varphi) = a \cdot \varphi(0)$.



Heaviside Step

$$H(x) = \begin{cases} 0 & \text{if } x < 0, \\ 1 & \text{if } x \geq 0. \end{cases}$$



Piecewise Continuity & Differentiability
A function $f : [a, b] \to \mathbb{R}$ is *piecewise continuous* if there is a partition

$$a = a_0 < a_1 < \cdots < a_n = b$$

such that $\lim_{x\to a_i^-} f(x)$ and $\lim_{x\to a_i^+} f(x)$ exist (finite).
Similarly, f is *piecewise* C^1 if it is continuously differentiable on each open subinterval and the one-sided derivatives at boundaries exist.

Euler's Formulas

$$e^{x+iy} = e^x\big(\cos y + i\sin y\big),\ \sin x = \frac{e^{ix} - e^{-ix}}{2i},\ \cos x = \frac{e^{ix} + e^{-ix}}{2}.$$

Orthogonality (Sine/Cosine Products)
For $n, m \in \mathbb{N}_{\geq 1}$ and period $T > 0$:

$$\frac{2}{T}\int_0^T \cos\Big(\tfrac{2\pi n}{T}x\Big)\cos\Big(\tfrac{2\pi m}{T}x\Big)\,dx = \begin{cases} 1 & n = m, \\ 0 & n \neq m \end{cases}$$

(same for $\sin\sin$, and $\cos\sin$ integrates to 0). **Integration Over One Period**
If f is T-periodic and piecewise continuous, then for any $a \in \mathbb{R}$:

$$\int_a^{a+T} f(x)\,dx = \int_0^T f(x)\,dx.$$

Dirichlet's Theorem (Pointwise Convergence)
Let $f : \mathbb{R} \to \mathbb{R}$ be T-periodic and piecewise C^1. Then, for all $x \in \mathbb{R}$,

$$Ff(x) = \lim_{t\to 0}\frac{f(x-t) + f(x+t)}{2}.$$

Real Fourier Series
For $f : \mathbb{R} \to \mathbb{R}$, T-periodic, piecewise C^1, the real Fourier series is

$$Ff(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\big[a_n\cos(\tfrac{2\pi n}{T}x) + b_n\sin(\tfrac{2\pi n}{T}x)\big].$$

Fourier Coefficients:

$$a_n = \frac{2}{T}\int_0^T f(x)\cos\Big(\tfrac{2\pi n}{T}x\Big)dx, \quad b_n = \frac{2}{T}\int_0^T f(x)\sin\Big(\tfrac{2\pi n}{T}x\Big)dx,$$

$$a_0 = \frac{2}{T}\int_0^T f(x)\,dx.$$

Parity: If f even, $b_n = 0$; if f odd, $a_n = 0$.

Convolution Product
Let $f, g : \mathbb{R} \to \mathbb{R}$ such that $\int_{-\infty}^{+\infty}|f(x)|dx < +\infty$ and $\int_{-\infty}^{+\infty}|g(x)|dx < +\infty$.
The convolution product of f and g is defined by:

$$(f * g)(x) = \int_{-\infty}^{+\infty} f(x-t)g(t)dt = \int_{-\infty}^{+\infty} f(t)g(x-t)dt$$

Complex Fourier Coefficient
Let $f : \mathbb{R} \to \mathbb{R}$ be a T-periodic and piecewise continuous function. The complex Fourier coefficients are defined as

$$c_n = \frac{1}{T}\int_0^T f(x)e^{-i\frac{2\pi}{T}nx}\,dx, \quad \forall n \in \mathbb{Z}. \quad Ff(x) = \sum_{n=-\infty}^{\infty} c_n\, e^{i\frac{2\pi n}{T}x}.$$

For a function $\phi : \mathbb{R} \to \mathbb{C}$, we have

$$\int_a^b \phi(x)\,dx = \int_a^b \mathrm{Re}(\phi(x))\,dx + i\int_a^b \mathrm{Im}(\phi(x))\,dx.$$

Relation to (a_n, b_n)

$$c_n = \tfrac{1}{2}(a_n - i\,b_n), \quad c_{-n} = \tfrac{1}{2}(a_n + i\,b_n), \quad c_0 = \tfrac{a_0}{2}.$$

Fourier Series on $[0, L]$
For $f : [0, L] \to \mathbb{R}$ (piecewise C^1):

$$F_c f(x) = \tfrac{\tilde{a}_0}{2} + \sum_{n=1}^{\infty}\tilde{a}_n\cos\Big(\tfrac{\pi n}{L}x\Big), \quad \tilde{a}_n = \frac{2}{L}\int_0^L f(x)\cos\Big(\tfrac{\pi n}{L}x\Big)dx.$$

$$F_s f(x) = \sum_{n=1}^{\infty}\tilde{b}_n\sin\Big(\tfrac{\pi n}{L}x\Big), \quad \tilde{b}_n = \frac{2}{L}\int_0^L f(x)\sin\Big(\tfrac{\pi n}{L}x\Big)dx.$$

Parseval's Identity (Periodic Case)
If f is T-periodic (piecewise C^1),

$$\frac{1}{T}\int_0^T f^2(x)\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2) = \sum_{n=-\infty}^{\infty}\big|c_n\big|^2.$$

The Fourier Transform
If $f : \mathbb{R} \to \mathbb{R}$ with $\int_{-\infty}^{\infty}|f(x)|\,dx < \infty$, its (unitary) Fourier transform is

$$\widehat{f}(\alpha) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\,e^{-i\,\alpha x}\,dx.$$

Inverse Transform
If $\varphi(\alpha)$ is similarly integrable,

$$\mathcal{F}^{-1}(\varphi)(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\varphi(\alpha)\,e^{i\,\alpha x}\,d\alpha, \quad f(x) = \mathcal{F}^{-1}\big(\widehat{f}\big)(x).$$

Fourier Transform of $f * g$
Let f and g be piecewise continuous on $\mathbb{R}$ such that $\int_{-\infty}^{+\infty}|f(x)|dx < +\infty$ and $\int_{-\infty}^{+\infty}|g(x)|dx < +\infty$. Then:

$$\int_{-\infty}^{+\infty}|(f * g)(x)|dx < +\infty \quad \text{and} \quad \mathcal{F}[f * g](\alpha) = \sqrt{2\pi}\hat{f}(\alpha)\cdot\hat{g}(\alpha)$$

Term-by-Term Differentiation
If f is T-periodic, continuous, and piecewise C^1, then

$$\frac{d}{dx}\Big[\tfrac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos(\tfrac{2\pi n}{T}x) + b_n\sin(\tfrac{2\pi n}{T}x))\Big] = \sum_{n=1}^{\infty}\frac{2\pi n}{T}\big[-a_n\sin(\tfrac{2\pi n}{T}x) + b_n\cos(\tfrac{2\pi n}{T}x)\big]$$

$$= \lim_{t\to 0}\frac{f'(x-t) + f'(x+t)}{2}.$$

Term-by-Term integration

Let f be a T-periodic and piecewise C^1 function with zero mean,

$$\int_0^T f(t)\,dt = 0.$$