Office of Sustainability
Amirkabir University of Technology
(Tehran Polytechnic)

Computer Engineering && IT Department

Amirkabir University of Technology

Statistical Pattern Recognition

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Contents

1 Problem 1

Two normal distribution are characterized by:

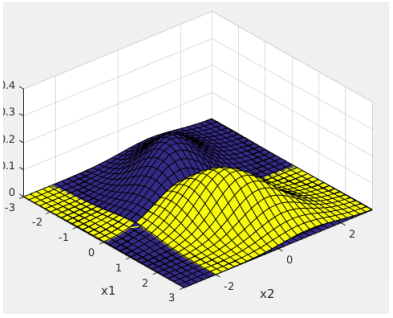
$$P_1 = P_2 = 0.5$$
$$M_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \Sigma_1 = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix}$$
$$M_2 = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \quad \Sigma_2 = \begin{bmatrix} 1 & -0.5 \\ -0.5 & 1 \end{bmatrix}$$

1. Draw the Bayes decision boundary to minimize the probability of error.

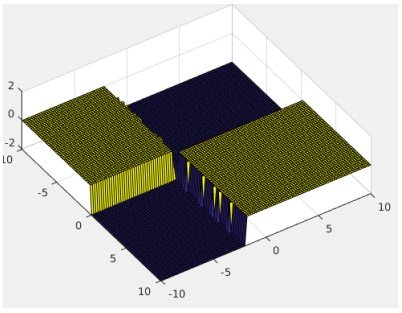
   The bayesian decision function is:

$$P(X|w_1) \cdot P(w_1) \underset{<w_2}{>^{w_1}} P(X|w_2) \cdot P(w_2)$$
$$\frac{P(X|w_1) \cdot P(w_1)}{P(X|w_2) \cdot P(w_2)} \underset{<w_2}{>^{w_1}} 1 \rightarrow \frac{P(X|w_1)}{P(X|w_2)} \underset{<w_2}{>^{w_1}} 1$$

   The figure 1 displays the data distributions and decision boundaries of bayesian decision function without considering costs.



(a) Distribution of data. The distribution of class 1 is displayed in blue color and the distribution of class 2 is displayed in yellow.



(b) Decision Boundary. The decision boundary of class 1 is displayed in blue color and the that of class 2 is displayed in yellow.

Figure 1: Bayesian Decision Boundary without considering decision costs

Also, it is possible to transform the data to a one dimensional space first and make decision in the new space using a single threshold as bellow:

$$
\begin{aligned}
h(X) =& \frac{1}{2}(X-M_1)^T\Sigma_1^{-1}(X-M_1) - \frac{1}{2}(X-M_2)^T\Sigma_2^{-1}(X-M_2) \\
&+ \frac{1}{2}ln\frac{|\Sigma_1|}{|\Sigma_2|} \underset{<_{w_2}}{\overset{>^{w_1}}{\cdot}} ln\frac{P(W_1)}{P(W_2)} \\
&\rightarrow h(X) = \frac{1}{2}(X-\begin{bmatrix}1\\0\end{bmatrix})^T \begin{bmatrix}1 & 0.5\\0.5 & 1\end{bmatrix}^{-1}(X-\begin{bmatrix}1\\0\end{bmatrix}) \\
&- \frac{1}{2}(X-\begin{bmatrix}-1\\0\end{bmatrix})^T \begin{bmatrix}1 & -0.5\\-0.5 & 1\end{bmatrix}^{-1}(X-\begin{bmatrix}-1\\0\end{bmatrix}) \\
&+ \frac{1}{2}ln\frac{|\begin{bmatrix}1 & 0.5\\0.5 & 1\end{bmatrix}|}{|\begin{bmatrix}1 & -0.5\\-0.5 & 1\end{bmatrix}|} \underset{<_{w_2}}{\overset{>^{w_1}}{\cdot}} 0 \\
&\rightarrow h(X) = \frac{1}{2}(X-\begin{bmatrix}1\\0\end{bmatrix})^T \begin{bmatrix}1 & 0.5\\0.5 & 1\end{bmatrix}^{-1}(X-\begin{bmatrix}1\\0\end{bmatrix}) \\
&- \frac{1}{2}(X-\begin{bmatrix}-1\\0\end{bmatrix})^T \begin{bmatrix}1 & -0.5\\-0.5 & 1\end{bmatrix}^{-1}(X-\begin{bmatrix}-1\\0\end{bmatrix}) \underset{<_{w_2}}{\overset{>^{w_1}}{\cdot}} 0
\end{aligned}
$$

---

2. Draw the Bayes decision boundary to minimize the cost with $c_{11} = c_{22} = 0$ and $c_{12} = 2c_{21}$

   The Bayesian decision boundary when considering costs of decisions turns to the form bellow:

$$\frac{p_1(X)}{p_2(X)} \underset{<_{w_2}}{\overset{>^{w_1}}{\cdot}} \frac{(c_{12}-c_{22})P_2}{(c_{21}-c_{11})P_1} \tag{1}$$

   Figure 2 displays the decision boundary in this case.

---

3. Assume that $c_{11} = c_{22} = 0$ and $c_{12} = c_{21}$:

   (a) Plot the operating characteristics.

   As equation (1) yields, with $c_{11} = c_{22} = 0$ and $c_{12} = c_{21}$ the decision boundary is not different from that of section 1. So the figure 1 is displaying the required decision boundary

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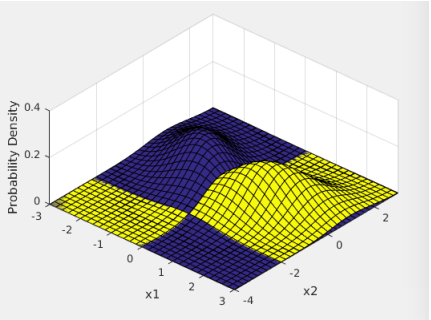
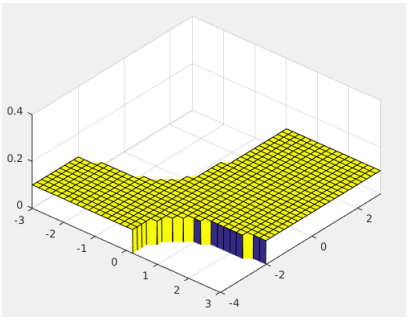
   (b) Find the total error when Neyman-Pearson test is performed with $\epsilon_1 = 0.05$



(a) Distribution of data. The distribution of class 1 is displayed in yellow color and the distribution of class 2 is displayed in blue.



(b) Decision Boundary. The decision boundary of class 1 is displayed in yellow color and the that of class 2 is displayed in blue.

Figure 2: Bayesian Decision Boundary considering decision costs $c_{11} = c_{22} = 0$ and $c_1 2 = 2c_{21}$

We should estimate μ such that the following function r be minimized.

$$r = \mu(\epsilon_1 - 0.05) + \epsilon_2 \tag{2}$$

After finding appropriate μ, the decision boundary is derivable from:

$$\frac{p_1(X)}{p_2(X)} \underset{<w_2}{\overset{>w_1}{\cdot}} \mu \tag{3}$$

To find such μ an script will iterate on following equation to reach the best fitting value.

$$\epsilon_1 = \int_{\mu}^{\infty} p_h(h|W-1)dh = 0.05 \tag{4}$$

The $\mu = 1.0795e - 78$ is the best value estimated with $\epsilon_1 = 0.051$ error rate. Figure 3 illustrates the resulted decision boundary.

(c) Find the threshold value and total error for the minimax test.
The threshold in minimax test, when $c_{11} = c_{22}$ and $c_{12} = c_{21}$, should satisfy the following equation:

$$\int_{R_1} p_1(X)dx = \int_{R_2} p_2(X)dx \tag{5}$$

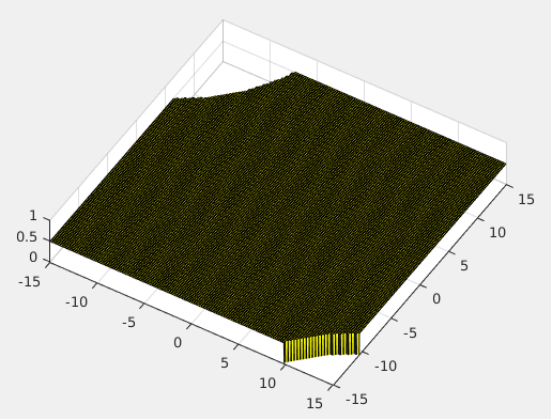


Figure 3: Decision boundary resulted from Neyman-Pearson test with $\epsilon_1 = 0.05$

So:

$$\int_{R_1} N(\mu_1, \Sigma_1)dx = \int_{R_2} N(\mu_2, \Sigma_2)dx$$
$$\rightarrow \int_{R_1} \frac{e^{-0.5(x-\mu_1)^T\Sigma_1^{-1}(x-\mu_1)}}{\sqrt{2\pi}|\Sigma_1|^{\frac{1}{2}}}dx = \int_{R_2} \frac{e^{-0.5(x-\mu_2)^T\Sigma_2^{-1}(x-\mu_2)}}{\sqrt{2\pi}|\Sigma_2|^{\frac{1}{2}}}dx$$

As $|\Sigma_1| = |\Sigma_2|$:

$$\int_{R_1} e^{-0.5(x-\mu_1)^T\Sigma_1^{-1}(x-\mu_1)}dx = \int_{R_2} e^{-0.5(x-\mu_2)^T\Sigma_2^{-1}(x-\mu_2)}dx$$

The solution of the above equation is not easy, so taking information about data given in problem and the figure 2, the solution of the equation is $X_1 = 0$ and $X_2 = -2$, which makes the error of both classes equal.
The other solution could be first projecting the data to a one-dimensional space using appropriate V and v_0 such that:

$$V^T X + v_0 \underset{<w_2}{\overset{>w_1}{.}} 0 \tag{6}$$

The process of finding such parameters has been fully described in problem 2. The best parameters are computed as $V = \begin{bmatrix} -0.5 \\ -0.25 \end{bmatrix}$ and $v_0 = 0.5$ with total error $\epsilon = 0.0227$.

(d) Plot the error-reject curve. The error-reject curve which displays the error rate according to regions has been displayed in figure 4. In this diagram, the parameter s represent the acceptance region.

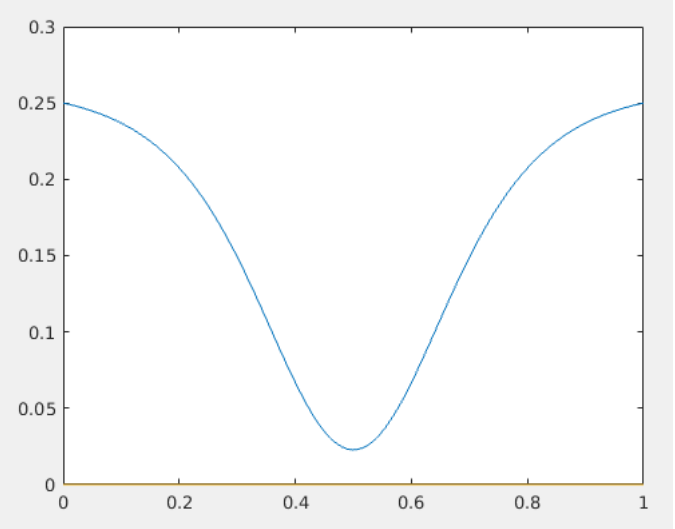


Figure 4: The error-reject curve for the data given in problem 1

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2 Problem 2

Two normal distribution are characterized by:

$$P_1 = P_2 = 0.5$$
$$M_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \Sigma_1 = \begin{bmatrix} 4 & -3 \\ -3 & 4 \end{bmatrix}$$
$$M_2 = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \quad \Sigma_2 = \begin{bmatrix} 4 & 3 \\ 3 & 4 \end{bmatrix}$$

1. Find the linear discriminant function which maximize the Fisher criterion and minimize the error by adjusting the threshold.

$$f = \frac{(\mu_1 - \mu_2)^2}{\sigma_1^2 + \sigma_2^2} \rightarrow \frac{\partial f}{\partial \sigma_1^2} = \frac{\partial f}{\partial \sigma_2^2} = \frac{-(\mu_1 - \mu_2)^2}{(\sigma_1^2 + \sigma_2^2)^2}$$

$$\rightarrow s = \frac{\frac{\partial f}{\partial \sigma_1^2}}{\frac{\partial f}{\partial \sigma_1^2} + \frac{\partial f}{\partial \sigma_2^2}} = 0.5$$

$$\rightarrow V = [0.5\Sigma_1 + 0.5\Sigma_2]^{-}1(\mu_1 - \mu_2) = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}^{-1} \begin{bmatrix} -2 \\ 0 \end{bmatrix} = \frac{1}{16} \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} -2 \\ 0 \end{bmatrix}$$

$$\rightarrow V = \begin{bmatrix} -0.5 \\ 0 \end{bmatrix} \tag{7}$$

Equation (7) illustrates the best mapping vector V to project distribution over, therefore the LDA will be in the form of $V^t x \geq \alpha$, in which α should be find from equation (8).

$$\frac{\partial f}{\partial \mu_1} + \frac{\partial f}{\partial \mu_2} = 0 \tag{8}$$

which results:

$$\frac{2\mu_1 - 2\mu_2}{\sigma_1^2 + \sigma_2^2} + \frac{-2\mu_1 + 2\mu_2}{\sigma_1^2 + \sigma_2^2} = 0 \rightarrow 0 = 0$$

Therefore, the best α is not derivable analytical. In order to estimate the best α a MATLAB script using bootstrapping has been written by me, which is appended to the homework submission zip file. The bootstrapping process, iterates 1000 number of times in each iteration generating 1000 multivariate random variables regarding given means and variances for each class. We attempt to estimate best α in each iteration using grid search in the range of $[-4, 4]$ with step length 0.01. To measure the accuracy of chosen α we use the ratio $\frac{tp+tn-fp-fn}{N}$ as the measurement function. After all iterations, the mean of all best estimated αs is reported as the best α. Experiments resulted the $\alpha = 0.23$ as the best value separating two classes from each other. So the LDA function is being reformed as bellow:

$$\begin{bmatrix} -0.5 \\ 0 \end{bmatrix}^T X \geq^{w_1} 0.23 \tag{9}$$

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2. Find the optimum linear discriminant function which minimize the probability of error.

The iterative process has been coded in MATLAB and its code is attached to the submitted homework zip file.
The parameter s has been estimated using a grid search in rage $[0, 1]$ with step length 0.01. the error probability *epsilon* in each iteration with respect to selected s has been calculated. The figure 5 illustrates the behaviour of error probability w.r.t. selected s. The best value of s is in 0 or 1.

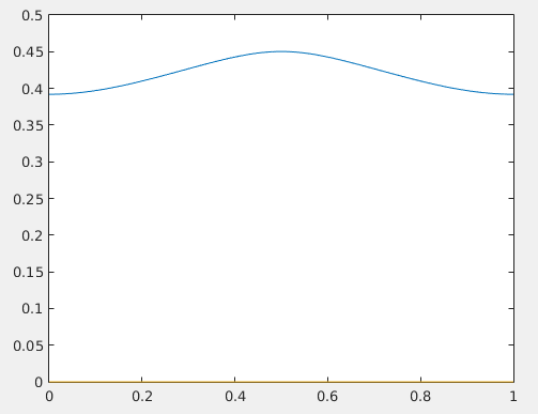
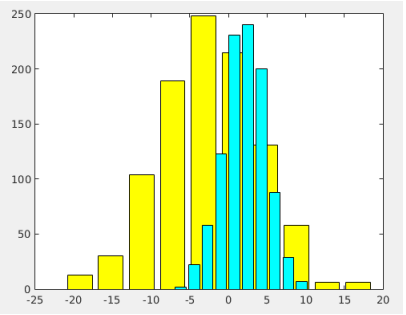
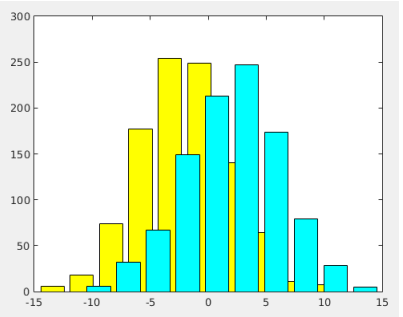


Figure 5: The behaviour of error probability ϵ w.r.t. chosen s



(a) Projected distribution of data. The projected distribution of class 1 w.r.t. mapping vector V is displayed in blue color and the that of class 2 is displayed in yellow when $s = 1$ (BEST CASE).



(b) Projected distribution of data. The projected distribution of class 1 w.r.t. mapping vector V is displayed in blue color and the that of class 2 is displayed in yellow when $s = 0.5$ (WORST CASE).

Figure 6: A comparison between projected distributions over matrices V w.r.t. selected s

To through a light over reality, the projection of data in two cases of $s = 0.5$ and $s = 1$ has been displayed in the figure 6, respectively the worst- and the

best-case. As its obviously clear, the separation of classes is more suitable in case $s = 1$ rather than $s = 0.5$, hence the lower error probability will be gained when $s = 1$.

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3 Problem 3

We consider a classification problem in dimension $d = 2$, with $k = 3$ classes where: $p(x|w_i) \propto N(\mu_i, \Sigma_i), i = 1, 2, 3$, with:

$$\mu_1 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \mu_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \mu_3 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \Sigma_i = \Sigma = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{3} \end{bmatrix}$$

1. Calculate the discriminant function $g_i(x)$ for each class.
   Generating a discriminant function between all pairs of classes can be a good solution. Assuming $P(W_1) = P(W_2) = P(W_3)$, the Bayes classifier is a good choice:

   $$d_{ij} : \frac{P(X|W_i)}{P(X|W_j)} \underset{<W_j}{\overset{>W_i}{\cdot}} 1$$

   Defining such classifiers, the feature function $g_i(X)$ can be defined as :

   $$g_1(X) = \Pi_{j=2,3} sgn(\frac{P(X|W_1)}{P(X|W_j)} - 1)$$
   $$g_2(X) = \Pi_{j=2,3} sgn(\frac{P(X|W_2)}{P(X|W_j)} - 1)$$
   $$g_3(X) = \Pi_{j=2,3} sgn(\frac{P(X|W_3)}{P(X|W_j)} - 1)$$

   In which the positive value of each function $g_i(X)$ indicates that X belongs to W_i and vice versa.

   ---

2. Express your discriminant functions in the form of linear discriminant functions.

The d_{ij}s could be expressed as linear functions as perpendicular bisectors of linking line between the center points of each class:

$$d_{12} = X_2 + 3X_1 + 3$$
$$d_{13} = X_2 + \frac{1}{2}X_1 - \frac{3}{4}$$
$$d_{23} = X_2 + 2X_1 - \frac{9}{2}$$

Defining $d_{ij}(X) = -d_{ji}(X)$, the vector X belongs to class W_i iff $\forall j \neq i, d_{ij}(X) > 0$

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3. Determine and plot the decision boundaries. Figure 7 displays the distribution of data given mean vectors and covariance matrices.

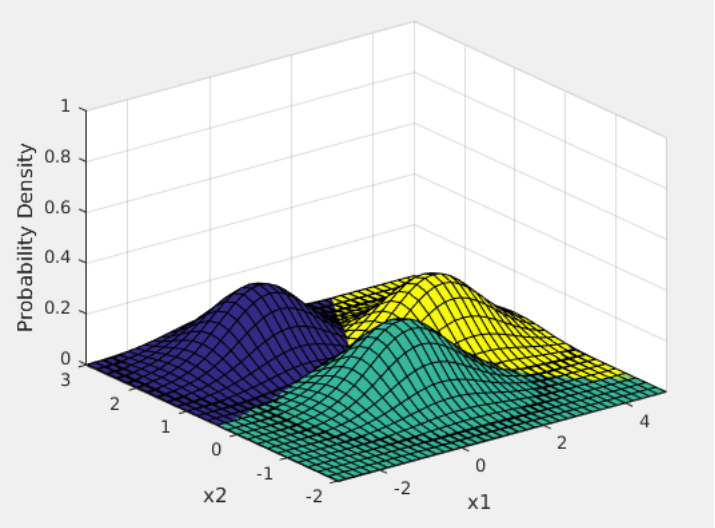
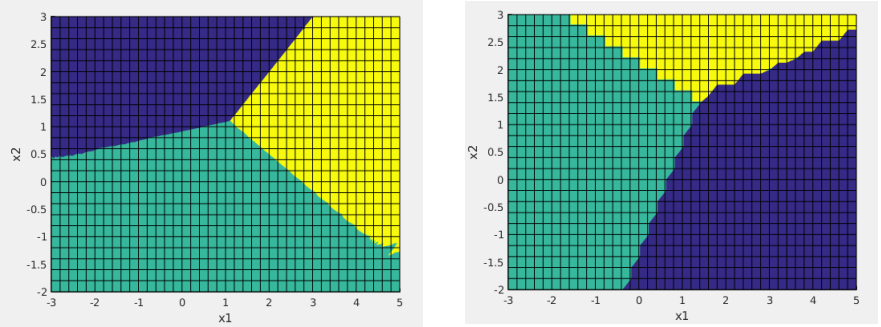


Figure 7: Data distribution w.r.t. means and covariance matrices

Figure 8 displays the data distribution boundary and deriven boundaries using two-class discriminant functions defined in section 2 of this problem.



(a) Projected distribution of data on $X1 - X2$ coordinates

(b) Boundaries generated by discriminant functions

Figure 8: A comparison between projected distributions over matrices V w.r.t. selected s

4 Problem 4

Consider the following 2-class classification problem involving a single feature x. Assume equal class priors and 0-1 loss function.

$$p(x|w_1) = \begin{Bmatrix} 2x & 0 \leq x \leq 1 \\ 0 & otherwise \end{Bmatrix}, p(x|w_2) = \begin{Bmatrix} 2-2x & 0 \leq x \leq 1 \\ 0 & otherwise \end{Bmatrix},$$

1. Sketch the two densities Since the class priors are equal:

$$p(w_1|x) = \frac{p(x|w_1)p(w_1)}{\Sigma_{i=1}^{2} p(x|w_i)p(w_i)} = \frac{0.5(2x)}{1} = x$$
$$p(w_2|x) = \frac{p(x|w_2)p(w_2)}{\Sigma_{i=1}^{2} p(x|w_i)p(w_i)} = \frac{0.5(2-2x)}{1} = 1 - x$$

---

2. State the Bayes decision rule and show the decision boundary.

$$p(w_1|x) \underset{<_{w_2}}{\overset{>^{w_1}}{}} p(w_2|x) \rightarrow x \underset{<_{w_2}}{\overset{>^{w_1}}{}} 1 - x \rightarrow 2x - 1 \underset{<_{w_2}}{\overset{>^{w_1}}{}} 0$$

Figure 9 displays the data distribution, decision boundary, false positive and false negative errors. The area coloured blue and green are displaying correct classifications for class w_1 and w_2 respectively. The yellow and red areas are illustrating ϵ_1 and ϵ_2 respectively.

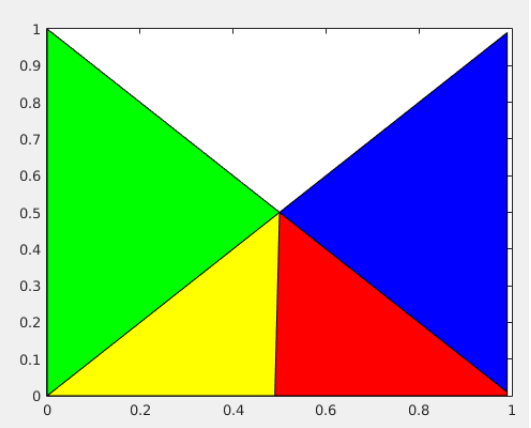


Figure 9: Bayes decision boundary for proposed data in problem 4 when $p(w_1) = p(w_2) = 0.5$

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3. What is the Bayes classification error? The classification error is being calculated as a weighted sum of each misclassified point w.r.t. its probability. Therefore,

$$\epsilon = 0.5 * \epsilon_1 + 0.5 * \epsilon_2 = 0.5 * \int_0^0 .5xdx + 0.5 * \int_0 .5^1 1 - xdx = \frac{1}{16} + \frac{1}{16} = \frac{1}{8}$$

---

4. How will the decision boundary change if the prior for class w_2 is increased to 0.6?

$$p(w_1|x) = \frac{p(x|w_1)p(w_1)}{\Sigma_{i=1}^{2} p(x|w_i)p(w_i)} = \frac{0.4(2x)}{1} = 0.8x$$
$$p(w_2|x) = \frac{p(x|w_2)p(w_2)}{\Sigma_{i=1}^{2} p(x|w_i)p(w_i)} = \frac{0.6(2-2x)}{1} = 0.6 - 0.6x$$
$$p(w_1|x) \underset{<w_2}{>^{w_1}} p(w_2|x) \rightarrow 0.4x \underset{<w_2}{>^{w_1}} 0.6 - 0.6x \rightarrow x - 0.6 \underset{<w_2}{>^{w_1}} 0$$

Figure 10 displays the data distribution, decision boundary, false positive and false negative errors. The area coloured blue and green are displaying correct classifications for class w_1 and w_2 respectively. The yellow and red areas are illustrating ϵ_1 and ϵ_2 respectively.

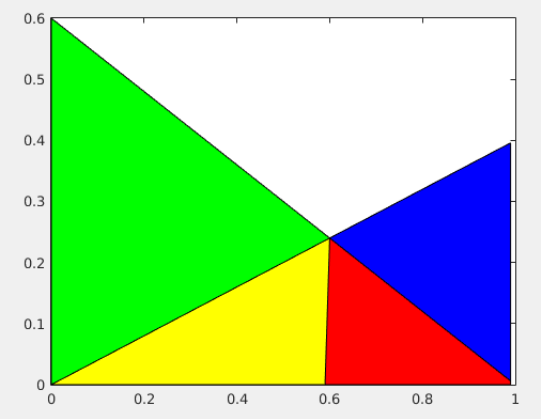

Figure 10: Bayes decision boundary for proposed data in problem 4 when $p(w_1) = 0.4, p(w_2) = 0.6$

The classification error is being calculated as a weighted sum of each misclassified point w.r.t. its probability. Therefore,

$$\epsilon = 0.4 * \epsilon_1 + 0.6 * \epsilon_2 = 0.4 * \int_0^0 .60.4xdx + 0.6 * \int_0 .6^1 0.6 - 0.6xdx = 0.0768$$

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5 Problem 5

Consider a two-category classification problem in two dimensions with:

$$p(X|W_1) \propto N(0, I), P(X|W_2) \propto N(\begin{bmatrix}1\\0\end{bmatrix}, I), P(W_1) = P(W_2) = 0.5.$$

1. Calculate the Bayes decision boundary.
   As $\Sigma_1 = \Sigma_2 = \Sigma = I$, a one dimensional function $h(x)$ can be defined as bellow discriminating two classes linearly:

$$h(x) = (M_2 - M_1)^T \Sigma^{-1} X + \frac{1}{2}(M_1^T \Sigma^{-1} M_1 - M_2^T \Sigma^{-1} M_2) \gtrless^{w_1}_{w_2} ln\frac{P_1}{P_2}$$

So the appropriate function is as follows:

$$h(x) = \begin{bmatrix}1\\0\end{bmatrix}^T X + \frac{1}{2}(-\begin{bmatrix}1\\0\end{bmatrix}^T \begin{bmatrix}1\\0\end{bmatrix}) \gtrless_{w_2}^{w_1} 0$$
$$\rightarrow h(x) = \begin{bmatrix}1\\0\end{bmatrix}^T X - \frac{1}{2} \gtrless_{w_2}^{w_1} 0$$

---

2. Calculate the Bhattacharyya error bound.
The Bhattacharyya error bound is in the form of following equation:

$$\epsilon = \sqrt{P(W_1)P(W_2)}e^{-\mu(0.5)} \tag{10}$$

Where $\mu(0.5)$ for normal distribution is:

$$\mu(0.5) = \frac{1}{8}(M_2 - M_1)^T(\frac{\Sigma_1 + \Sigma_2}{2})^{-1}(M_2 - M_1) + \frac{1}{2}ln\frac{|\frac{\Sigma_1+\Sigma_2}{2}|}{\sqrt{|\Sigma_1||\Sigma_2|}} \tag{11}$$

Considering $\Sigma_1 = \Sigma_2 = I$:

$$\mu(0.5) = \frac{1}{8}(M_2 - M_1)^T(M_2 - M_1) = \frac{1}{8}$$

So:

$$\epsilon = \sqrt{P(W_1)P(W_2)}e^{-\mu(0.5)} = \frac{1}{2}e^{-\frac{1}{8}}$$

---

3. Repeat the above for the same prior probabilities, but

$$p(X|W_1) \propto N(0, \begin{bmatrix}2 & 0.5\\0.5 & 2\end{bmatrix}), p(X|W_2) \propto N(\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}5 & 2\\2 & 5\end{bmatrix})$$

As $\Sigma_1 \neq \Sigma_2$, the one dimensional discriminant function $h(x)$ is formed as

bellow:

$$h(x) = \frac{1}{2}(x - M_1)^T \Sigma_1^{-1}(x - M_1) - \frac{1}{2}(x - M_2)^T \Sigma_2^{-1}(x - M_2) + \frac{1}{2} ln \frac{|\Sigma_1|}{|\Sigma_2|}$$

$$\rightarrow h(x) = \frac{1}{3.75} x^T \begin{bmatrix} 2 & -0.5 \\ -0.5 & 2 \end{bmatrix} x - \frac{1}{21}(x - \begin{bmatrix} 1 \\ 0 \end{bmatrix})^T \begin{bmatrix} 5 & -2 \\ -2 & 5 \end{bmatrix} (x - \begin{bmatrix} 1 \\ 0 \end{bmatrix})$$

$$+ \frac{1}{2} ln \frac{|\begin{bmatrix} 2 & 0.5 \\ 0.5 & 2 \end{bmatrix}|}{|\begin{bmatrix} 5 & 2 \\ 2 & 5 \end{bmatrix}|}$$

$$\rightarrow h(x) = \frac{1}{3.75} x^T \begin{bmatrix} 2 & -0.5 \\ -0.5 & 2 \end{bmatrix} x - \frac{1}{21}(x - \begin{bmatrix} 1 \\ 0 \end{bmatrix})^T \begin{bmatrix} 5 & -2 \\ -2 & 5 \end{bmatrix} (x - \begin{bmatrix} 1 \\ 0 \end{bmatrix}) + \frac{3.75}{42}$$

$$\rightarrow h(x) = \frac{1}{3.75} x^T \begin{bmatrix} 2 & -0.5 \\ -0.5 & 2 \end{bmatrix} x - \frac{1}{21}(x^T \begin{bmatrix} 5 & -2 \\ -2 & 5 \end{bmatrix} x - x^T \begin{bmatrix} 5 & -2 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$- \begin{bmatrix} 1 \\ 0 \end{bmatrix}^T \begin{bmatrix} 5 & -2 \\ -2 & 5 \end{bmatrix} x - \begin{bmatrix} 1 \\ 0 \end{bmatrix}^T \begin{bmatrix} 5 & -2 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix})$$

$$\rightarrow h(x) = \frac{1}{78.75} x^T \begin{bmatrix} 23.25 & -3 \\ -3 & 23.25 \end{bmatrix} x + \frac{1}{21} x^T \begin{bmatrix} 5 \\ -2 \end{bmatrix} + \frac{1}{21} \begin{bmatrix} 5 \\ -2 \end{bmatrix}^T x + \frac{1}{21} \begin{bmatrix} 29 \\ -20 \end{bmatrix}^T I + \frac{3.75}{42}$$

$$\rightarrow h(x) = \frac{1}{78.75} x^T \begin{bmatrix} 23.25 & -3 \\ -3 & 23.25 \end{bmatrix} x + \frac{1}{21} \begin{bmatrix} 10 \\ -4 \end{bmatrix}^T x + \frac{21.75}{42}$$

So the discriminant function is such as bellow:

$$h(x) = \frac{1}{78.75} x^T \begin{bmatrix} 23.25 & -3 \\ -3 & 23.25 \end{bmatrix} x + \frac{1}{21} \begin{bmatrix} 10 \\ -4 \end{bmatrix}^T x + \frac{21.75}{42} \underset{<w_2}{\overset{>w_1}{.}} 0$$

The Bhattacharyya error bound is as bellow:

$$\mu(0.5) = \frac{1}{8}(M_2 - M_1)^T (\frac{\Sigma_1 + \Sigma_2}{2})^{-1}(M_2 - M_1) + \frac{1}{2} ln \frac{|\frac{\Sigma_1 + \Sigma_2}{2}|}{\sqrt{|\Sigma_1||\Sigma_2|}}$$

$$\mu(0.5) = \frac{1}{8} \begin{bmatrix} 1 \\ 0 \end{bmatrix}^T (\frac{\begin{bmatrix} 2 & 0.5 \\ 0.5 & 2 \end{bmatrix} + \begin{bmatrix} 5 & 2 \\ 2 & 5 \end{bmatrix}}{2})^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \frac{1}{2} ln \frac{|\frac{\begin{bmatrix} 2 & 0.5 \\ 0.5 & 2 \end{bmatrix} + \begin{bmatrix} 5 & 2 \\ 2 & 5 \end{bmatrix}}{2}|}{\sqrt{|\begin{bmatrix} 2 & 0.5 \\ 0.5 & 2 \end{bmatrix}||\begin{bmatrix} 5 & 2 \\ 2 & 5 \end{bmatrix}|}}$$

$$\mu(0.5) = \frac{3.5}{85.5} + 0.4948 = 0.5357$$

6 Problem 6

Consider a two-category classification problem in two dimensions with $p(W_1) = p(W_2)$, assume that the class-conditional densities are Gaussian with mean μ_1 and co-variance Σ_1 under class 1, and mean μ_2 and co-variance Σ_2 under class 2. Further, assume that $\mu_1 = \mu_2$. (Note that in the problem set, there was stated that $\mu 0 = \mu 1$, which I guessed that the correct form is $\mu_1 = \mu_2$ rather than $\mu_0 = \mu_1$!)

$$\Sigma_1 = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}, \Sigma_2 = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$$

For the following case, draw contours of the level sets of the class conditional densities. Also, draw the decision boundaries obtained using the Bayes optimal classifier in each case and indicate the regions where the classifier will predict class 1 and where it will predict class 2.

Calculating the Bayes decision boundary yields:

$$h(X) = \frac{1}{2}(X-\mu_1)^T\Sigma_1^{-1}(X-\mu_1) - \frac{1}{2}(X-\mu_2)^T\Sigma_2^{-1}(X-\mu_2) + \frac{1}{2}ln\frac{|\Sigma_1|}{|\Sigma_2|} \underset{<_{w_2}}{\overset{>^{w_1}}{.}} ln\frac{P(W_1)}{P(W_2)}$$

Considering the assumptions of the problem in addition to taking $\mu_1 = \mu_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ yields:

$$h(X) = \frac{1}{2}X^T \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}^{-1} X - \frac{1}{2}X^T \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}^{-1} X + \frac{1}{2}ln\frac{|\begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}|}{|\begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}|} \underset{<_{w_2}}{\overset{>^{w_1}}{.}} 0$$

$$\rightarrow h(X) = \frac{1}{8}X^T \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} X - \frac{1}{8}X^T \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} X \underset{<_{w_2}}{\overset{>^{w_1}}{}} 0$$

$$\rightarrow h(X) = \frac{1}{8}\begin{bmatrix} 1 \\ 4 \end{bmatrix} X^TX - \frac{1}{8}\begin{bmatrix} 4 \\ 1 \end{bmatrix} X^TX \underset{<_{w_2}}{\overset{>^{w_1}}{.}} 0$$

$$\rightarrow h(X) = \frac{3}{8}\begin{bmatrix} -1 \\ 1 \end{bmatrix}^T (X^TX) \underset{<_{w_2}}{\overset{>^{w_1}}{.}} 0$$

$$\rightarrow h(X) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}^T (X^TX) \underset{<_{w_2}}{\overset{>^{w_1}}{.}} 0$$

A MATLAB script has been written to draw the required boundaries and regions. Figure 11 displays the result of this script.

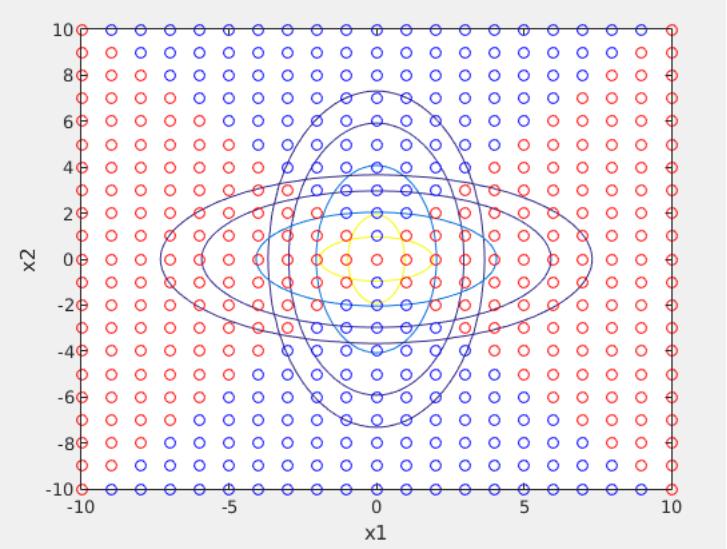


Figure 11: Bayesian decision boundaries and class acceptance regions for data described in problem 6

Contours of two classes are drawn in figure 11 and the acceptance region for each class has been displayed by coloured circles. The blue region is dedicated to the class W_1 and the red region is dedicated to class W_1.

---

7 Problem 7

Consider the two-dimensional data points from two classes W_1 and W_2 below, and each of them come from a Gaussian distribution $p(x|W_k) \propto N(\mu_k, \Sigma_k)$. (Data table is not reported here)

1. What is the prior probability for each class, i.e. $p(W_1)$ and $p(W_2)$.

   Assuming that the given table represents the distribution of original data, the probability of each class is proportional to the ratio of observed data from that class in the given small dataset. So, the $P(W_1) = \frac{6}{14} = 0.4286$ and therefore $P(W_2) = 1 - P(W_1) = 0.5714$.

---

2. Calculate the mean and covariance matrix for each class.
   Estimating sample mean and sample covariance matrix from data is according to the following equations:

$$\hat{\mu_i} = \frac{1}{N_i}\Sigma_{j=1}^{N_i} X_j \tag{12}$$

For the covariance matrix $\Sigma^k = [\sigma_{ij}^k]$, in which k denotes class number:

$$\hat{\sigma_{ij}^k} = \frac{1}{N_k - 1}\Sigma_{l=1}^{N_k}(x_{li} - \bar{x_i})(x_{lj} - \bar{x_j}) \tag{13}$$

So according to these equations:

$$\hat{\mu_1} = \begin{bmatrix} 1.5 \\ 1.3333 \end{bmatrix}, \hat{\mu_2} = \begin{bmatrix} 8.25 \\ 8.625 \end{bmatrix} \tag{14}$$

and

$$\hat{\Sigma_1} = \begin{bmatrix} 1.1 & 1 \\ 1 & 1.8667 \end{bmatrix}, \hat{\Sigma_2} = \begin{bmatrix} 2.7857 & 0.25 \\ 0.25 & 1.4107 \end{bmatrix} \tag{15}$$

---

3. Derive the equation for the decision boundary that separates these two classes, and plot the boundary. (Hint: you may want to use the posterior probability)
   Estimating class means and covariance matrices, the Bayesian decision boundary can be computed using the equations used in previous problems.

$$\frac{P(W_1|X)}{P(W_2|X)} \underset{<_{w_2}}{>^{w_1}} \frac{P(W_2)}{P(W_1)}$$

$$\frac{P(W_1|X)}{P(W_2|X)} = \frac{P(X|W_1).P(W_1)}{P(X|W_2).P(W_2)} = \frac{N(\begin{bmatrix} 1.5 \\ 1.3333 \end{bmatrix}, \begin{bmatrix} 1.1 & 1 \\ 1 & 1.8667 \end{bmatrix}).0.4286}{N(\begin{bmatrix} 8.25 \\ 8.625 \end{bmatrix}, \begin{bmatrix} 2.7857 & 0.25 \\ 0.25 & 1.4107 \end{bmatrix}).0.5714} \underset{<_{w_2}}{>^{w_1}} \frac{0.5714}{0.4286}$$

In which $N(\mu, \Sigma)$ represents the Gaussian normal function for each class and with corresponding mean vector and covariance matrix.

---

4. Think of the case that the penalties for misclassification are different for the two classes (i.e. not zero-one loss), will it affect the decision boundary, and

how?
Of course it will affect the decision boundary. Generally, the boundary is being tightened where the penalty of misclassification is less than other locations in favour of the locations in which the penalty of misclassification is larger. In other words, decision boundary will be adjusted in the way that make less misclassifications in the regions in which the penalty of misclassification is large, and prefer to make misclassifications in regions with less penalty.

---

8 Problem 8

Consider a classification problem with 2 classes and a single real-valued feature vector X. For class 1, $p(x|c_1)$ is uniform $U(a,b)$ with $a = 2$ and $b = 4$. For class 2, $p(x|c_2)$ is exponential with density $\lambda exp(-\lambda x)$ where $\lambda = 1$. Let $p(c_1) = p(c_2) = 0.5$.

1. Determine the location of the optimal decision regions

$$\begin{aligned} P(W_1|X) \underset{<w_2}{\overset{>w_1}{}} P(W_2|X) &\rightarrow P(X|W_1)P(W_1) \underset{<w_2}{\overset{>w_1}{}} P(X|W_2)P(W_2) \\ &\rightarrow \frac{1}{4-2} . \underset{<w_2}{\overset{>w_1}{}} e^{-x} \rightarrow -ln(2) . \underset{<w_2}{\overset{>w_1}{}} -x \\ &\rightarrow x \underset{<w_2}{\overset{>w_1}{}} ln(2) \end{aligned}$$

   On the other hands, as $p(X|W_1) \propto U(2,4)$, data from class 1 are not defined in $R - [2,4]$. So there exists some implicit boundary conditions, $\forall X \in W_1, X \in [2,4]$. Comparing the probability densities in $X = ln(2)$,$X = 2$, $X = 4$ and everywhere $X > 4$, results choosing the region $[2,4]$ as the acceptance region for class 1 and every other region as the acceptance region for class 2.
   This above-mentioned fact, is obviously clear in the figure 12, which represents the class densities and acceptance regions of classes.

---

2. Draw a sketch of the two class densities multiplied by $P(c_1)$ and $P(c_2)$ respectively, as a function of x, clearly showing the optimal decision boundary (or boundaries)
   The figure 12, displays class densities multiplied by class priors along side optimal decision boundaries extracted from section 1 of this current problem.
   The yellow area, is acceptance region for class 1. As it is clear, the class 1 is not defined in $R - [2,4]$ and in its domain, its density multiplied by its prior is

greater than that of class 2. So the yellow region is an optimal boundary for this class. On the other hands, in $R - [2, 4]$, where class 1 is not defined, the optimal decision for data is obviously assigning that to class 2. The blue region in figure 12 displays optimal boundary for this class 2. The purple region displays the region of class 2 misclassifications.

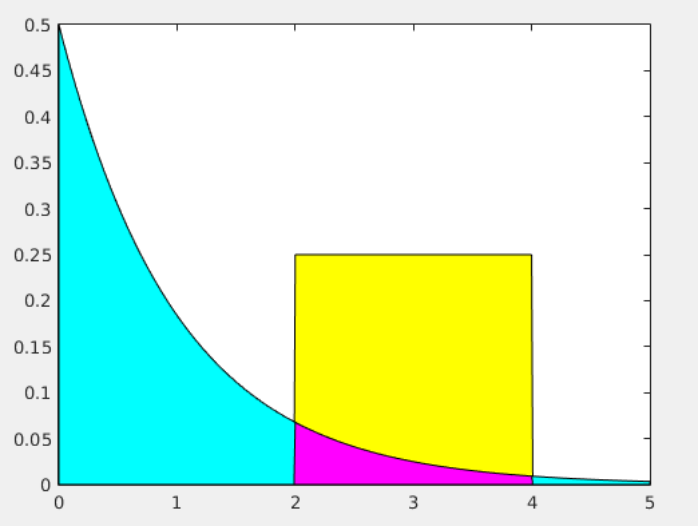


Figure 12: Class densities multiplied by class priors for each class given their conditional distributions in problem 8 when $dom\, W_1 = [2, 4]$

---

3. Compute the Bayes error rate for this problem within 3 decimal places of accuracy
   The Bayes error rate is computable as bellow:

$$\epsilon = \int_2^4 0.5exp(-x)dx = -0.5exp(-4) + 0.5exp(-2) = -0.5 * 0.018 + 0.5 * 0.135 = 0.058$$

---

4. Answer the questions above but now with $a = 2$ and $b = 22$. Optimal decision boundary:

$$\begin{aligned} P(W_1|X) \underset{<w_2}{\overset{>w_1}{}} P(W_2|X) &\rightarrow P(X|W_1)P(W_1) \underset{<w_2}{\overset{>w_1}{}} P(X|W_2)P(W_2) \\ &\rightarrow \frac{1}{22-2} . \underset{<w_2}{\overset{>w_1}{}} e^{-x} \rightarrow -ln(20) . \underset{<w_2}{\overset{>w_1}{}} -x \\ &\rightarrow x \underset{<w_2}{\overset{>w_1}{}} ln(20) \rightarrow x \underset{<w_2}{\overset{>w_1}{}} 2.9957 \end{aligned}$$

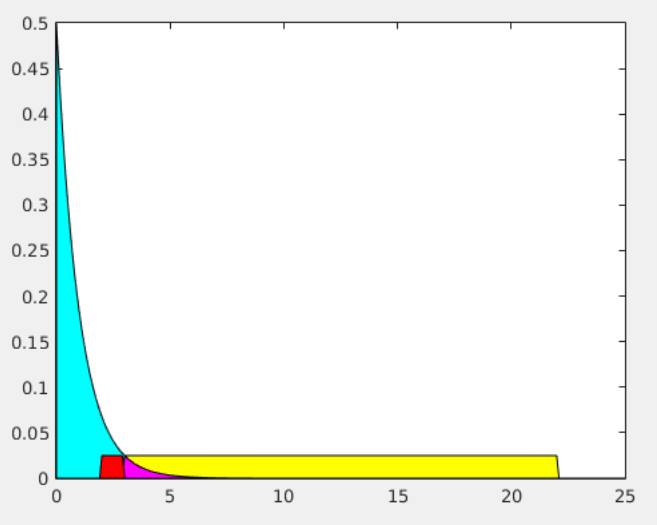


Figure 13: Class densities multiplied by class priors for each class given their conditional distributions in problem 8 when $dom\, W_1 = [2, 22]$

Since the $x = 2.9957$ is in domain of class 1, the optimal point is itself and there is no need to compare it with boundary points of class 1.
Figure 13 displays the decision boundary for this properties. As it is clearly obvious, when $2 \leq X \leq ln(20)$, the optimum decision for X is class 2 and when $ln(20) \leq X \leq 22$ the optimum decision is class 1 and after that when $X > 22$, as the class 1 is not defined there, the optimum decision is class 2 again.

Now, there is two types of error such that:

$$\epsilon = \in_2^{ln(20)} 0.5\frac{1}{22-2}dx + \int_{ln(20)}^{2} 20.5exp(-x)dx = \frac{1}{40}(ln(20) - 2) + \frac{1}{2}(exp(-22) - \frac{1}{20})$$

So, this is important to check the domain of each class in the cases in which the distribution of classes are finite and limited to a specific region.

9 Problem 9

Consider a 2-class classification problem with d-dimensional real-valued inputs x, where the class- conditional densities, $p(x|c_1)$ and $p(x|c_2)$ are multivariate Gaussian with different means μ_1 and μ_2 and a common covariance matrix Σ , with class probabilities $P(c_1)$ and $P(c_2)$.

1. Write the discriminant functions for this problem in the form of $g_1(x) = logp(x|c_1) + logp(c_1)$ (same for $g_2(x)$).

   The typical discriminant function could be specified as equation (16).

   $$P(W_1|X) \gtrless_{w_2}^{w_1} P(W_2|X) \rightarrow \frac{P(X|W_1)P(W_1)}{P(X)} \gtrless_{w_2}^{w_1} \frac{P(X|W_2)P(W_2)}{P(X)} \tag{16}$$

   So with the same $P(X)$ for two classes, it will be in the form of equation (17)

   $$P(X|W_1)P(W_1) \gtrless_{w_2}^{w_1} P(X|W_2)P(W_2) \tag{17}$$

   Also, with respect to $log()$ function properties, it is possible to rewrite the equation (**??**) in the form of equation (18).

   $$\begin{gathered} log(P(X|W_1)P(W_1)) \gtrless_{w_2}^{w_1} log(P(X|W_2)P(W_2)) \\ \rightarrow log(P(X|W_1)) + log(P(W_1)) \gtrless_{w_2}^{w_1} log(P(X|W_2)) + log(P(W_2)) \end{gathered} \tag{18}$$

   So defining $g_i(X)$ as follows yields good discriminant functions for each class.

   $$g_i(X) = log(P(X|W_i)) + log(P(W_i)) \tag{19}$$

   For the testing a new data, it is reasonable according to (18) and (19) to just compare the results of functions $g_1(X)$ and $g_2(X)$ and assign the X to the class which results larger value.

---

2. Prove that the optimal decision boundary, at $g(x) = g_1(x) - g_2(x) = 0$, can be written in the form of a linear discriminant, $w^T x + w_0 = 0$, where w is a d-dimensional weight vector and w_0 is a scalar, and clearly indicate what are w and w_0 are in terms of parameters of the classification model.

10 Problem 10

Consider the two-dimensional data points from two classes W_1 and W_2. (The data table is not reported here)

1. Determine and plot the optimal projection line in a single dimension using Fisher linear discriminant method.

   The sample mean vector and covariance matrix for each given data set has been calculated.

   $$\mu_1 = \begin{bmatrix}1.8334\\1.8334\end{bmatrix}, \mu_2 = \begin{bmatrix}3.8334\\3\end{bmatrix}, \Sigma_1 = \begin{bmatrix}1.1 & 1\\1 & 1.8667\end{bmatrix}, \Sigma_2 = \begin{bmatrix}2.7857 & 0.25\\0.25 & 1.4107\end{bmatrix}$$

   So:

   $$V = (0.5\Sigma_1 + 0.5\Sigma_2)^{-1}(\mu_2 - \mu_1) = \frac{11.1725}{2}\begin{bmatrix}3.8857 & 1.25\\1.25 & 3.2774\end{bmatrix}\begin{bmatrix}2\\1.1667\end{bmatrix}$$

   $$\rightarrow V = \frac{11.1725}{2}\begin{bmatrix}9.223\\6.3234\end{bmatrix} = \begin{bmatrix}51.5220\\35.3241\end{bmatrix}$$

   And to compute optimum v_0, the iterative process should be accomplished. The script is implemented in MATLAB and the result is $v_0 = 1.8969$ with $\epsilon = 0.1667$.

---

2. Show the mapping of the points to the line as well as the Bayes discriminant assuming a suitable distribution.

   Figure 14, displays required diagram.

---

11 Problem 11

[Computer Experiment] In this problem, you will be classifying the Iris dataset from the UCI Machine Learning Repository. The original data describes 3 classes of Iris flowers and contains various measurements about them:

1. sepal length in cm

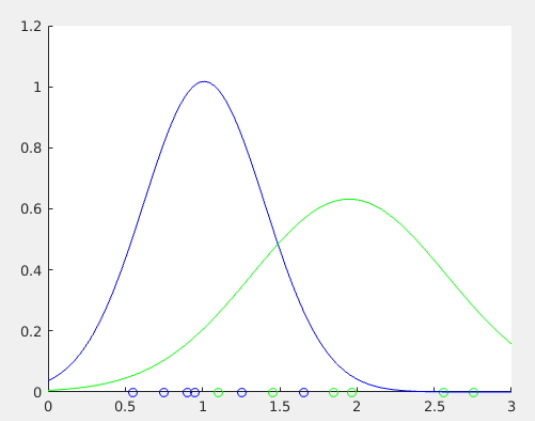


Figure 14: Mapped data points and the best fitting distributions for mapped data, problem 10.

2. sepal width in cm
3. petal length in cm
4. petal width in cm

To make things a little easier, we left out one of the flower types and only included the Iris Setosa (class 0) and Iris Versicolor (class 1). There are two data files for this assignment, a training data set (70 data items) and a test data set (30 data items).

Every line in each file describes one data item; the 5 numbers in each row contain the 4 measurements in above order plus the class label. Use only the second measurement (sepal width) for classification and ignore all others.

The script of this problem is attached to the submitted homework zip file. In this report just the answers have been reported.

---

1. Load the training data. Compute a feature histogram for each class (see histc and bar in Matlab), using edges between the bins at 0.2 intervals between 1 and 5; i.e. $1 : 0.2 : 5$. Show the histograms either in two figures using subplot in Matlab, and make sure both plots have the same x-axis range.

   Figure 15 displays the required subplot for feature histograms of both classes.

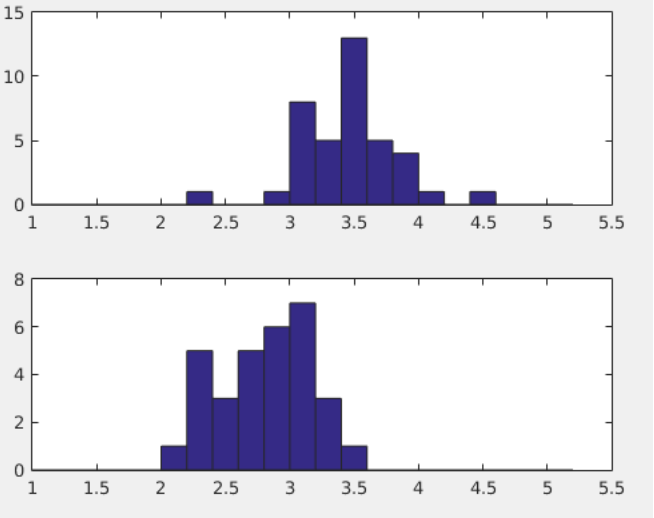


Figure 15: Histogram of features of two classes in train data set, problem 11.

---

2. Estimate the class priors from the training data. Report the prior probabilities of a flower being an Iris Setosa or being an Iris Versicolor.

   $P(W_0) = 0.5571$ and $P(W_1) = 0.4429$.

---

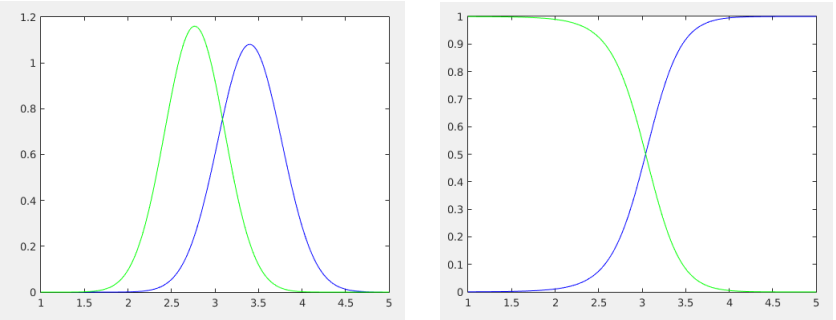
3. Assume that we can model the class-conditional probability density of each class using a univariate Gaussian (remember to only use the sepal width). What are the mean and variance of both class-conditional densities?

---

4. Plot the estimated class-conditional densities in a single diagram. Make a second figure plotting the class posteriors for both classes. Write down the equation you used to compute the class posteriors .
   Figure 16 illustrates the class-conditional densities.

   To compute the class posterior the following equation has been used.

$$P(W_i|X) = \frac{P(X|W_i)P(W_i)}{\Sigma_{l=1}^{2} P(X|W_l)P(W_l)} \tag{20}$$



(a) Estimated class-conditional densities

(b) Estimated class posterior densities

Figure 16: Estimated class-conditional and class posterior densities, problem 11

5. Compute the Bayes classification of the training data. How many Setosas do you mistakenly classify as Versicolor and how many Versicolors do you mistakenly classify as Setosa? What is the overall training error (i.e. percentage of incorrectly classified training data items)?

   The error rate for class 0, which is the number of class 0 data misclassified as class 1 is 0.0513 and the error rate for class 1, which is the number of class 1 data misclassified as class 2 is 0.0645. The total error rate for training data is 0.0571.

---

6. Now classify the test data. How many Setosas do you mistakenly classify as Versicolor and how many Versicolors do you mistakenly classify as Setosa? Report the number of cases that are incorrectly classified as above as well as the test error.

   The error rate for class 0, which is the number of class 0 data misclassified as class 1 is 0.0256 and the error rate for class 1, which is the number of class 1 data misclassified as class 2 is 0.0645. The total error rate for testing data is 0.0428.
