Fano schemes of singular symmetric matrices

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Introducing the problem

Fix $n \geq 3$ and let S_n be the vector space of all symmetric matrices over an algebraically closed field of characteristic not equal to 2. We are interested in the following question:

> What are vector subspaces of S_n such that every element is a singular matrix?

It turns out that the space parametrizing all such subspaces (henceforth called **singular subspaces**) exhibits a rich geometry as we will investigate.

Examples

Let $0 \leq s \leq \frac{n-1}{2}$ be an integer. The following is an example of a singular subspace of S_n:

$$\begin{array}{r c} & \begin{array}{ccc} s & n-2s-1 & s+1 \end{array} \\ \begin{array}{r} s \\ n-2s-1 \\ s+1 \end{array} & \begin{pmatrix} * & * & * \\ * & * & 0 \\ * & 0 & 0 \end{pmatrix} \end{array} \tag{1}$$

Here the entries in the star blocks can either vary freely over the field or they could have linear dependencies. The maximum dimension of such a subspace occurs when every entry in the star blocks varies freely and we denote this maximum dimension by $\kappa(s)+1$.

The general linear group $\mathrm{GL}(n)$ acts on S_n via APA^t for every $A \in \mathrm{GL}(n)$ and $P \in S_n$. This action takes a singular subspace to a singular one. For a fixed s, we call any $\mathrm{GL}(n)$ translate of a subspace of the above form an ***s*-compression space** [3].

An example of a non-compression space is the 3-dimensional singular subspace.

$$Q = \begin{pmatrix} 0 & B \\ B^t & 0 \end{pmatrix}_{6\times 6}, \quad \text{where} \quad B = \begin{pmatrix} 0 & z_0 & z_1 \\ -z_0 & 0 & z_2 \\ -z_1 & -z_2 & 0 \end{pmatrix}, \quad \text{and } z_i \text{ are free variables.}$$

Background

We understand compression spaces very well, and two interesting questions are

> For a fixed n,
> 1. What is the maximum dimension of a singular subspace of S_n?
> 2. What are the linear dimensions ℓ such that every singular subspace of dimension ℓ is a compression space?

A theorem due to Meshulam answers the first question.

> **Theorem 1 (Meshulam [4]).** The maximum linear dimension of a singular subspace of S_n is $\binom{n}{2}$.

The next relevant result is that of Loewy and Radwan which characterizes the singular subspaces attaining the above maximum dimension. They showed every singular subspace of maximum dimension is a compression space. Pazzis generalized this result to dimensions near the maximum dimension:

> **Theorem 2 (Pazzis [2]).** Let P be an ℓ-dimensional singular subspace of S_n. If
> $$\ell > \max\{\kappa(1)+1, \kappa(d-1)+1\},$$
> where $d = \lfloor \frac{n-1}{2} \rfloor$ then P is either a 0- or d-compression space.

This project aims to revisit these results from a geometric perspective to provide shorter proofs that are easier to understand and in the case of Pazzis' result, we would like to try to improve the range of values ℓ for which the statement holds. The geometric approach is via Fano schemes. This approach was taken by Ilten and Chan in their study of subspaces of rectangular matrices of bounded rank [1].

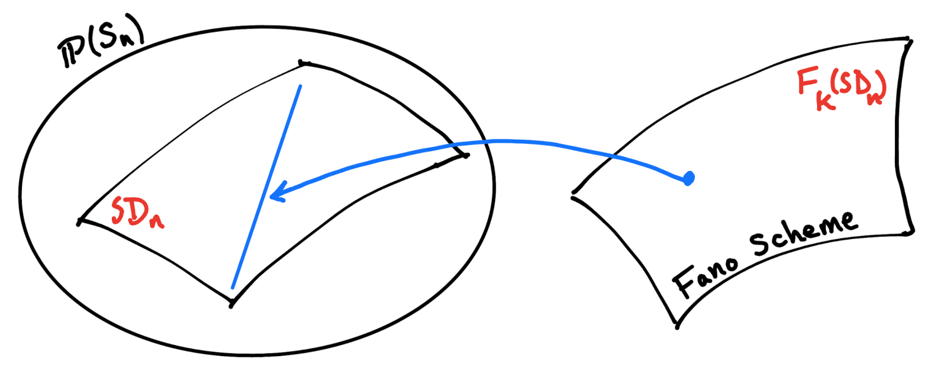
Geometric perspective

Let $\mathbb{P}^{\binom{n+1}{2}-1} = \mathbb{P}(S_n)$ be the space of all symmetric $n \times n$ matrices up to scalar multiplication. Define SD_n to be the hypersurface in $\mathbb{P}(S_n)$ given by the vanishing of the determinant of

$$\begin{pmatrix} x_{1,1} & x_{1,2} & \cdots & x_{1,n} \\ x_{1,2} & x_{2,2} & \cdots & x_{2,n} \\ \vdots & & \ddots & \vdots \\ x_{1,n} & x_{2,n} & \cdots & x_{n,n} \end{pmatrix}.$$

Here $x_{i,j}$ are the homogeneous coordinates on $\mathbb{P}(S_n)$. Fixing $k > 0$, a $(k+1)$-dimensional singular subspace of S_n corresponds to a k-plane lying on the hypersurface SD_n. The Grassmannian $\mathrm{Gr}(k+1, S_n)$ parametrizes all k-planes in $\mathbb{P}(S_n)$ and we are interested in the locus

$$\mathbf{F}_k(SD_n) = \{k\text{-planes lying on } SD_n\} \subset \mathrm{Gr}(k+1, S_n).$$



This locus is closed and there is a natural way of defining a scheme structure (not discussed here) on it that comes from the defining equation of SD_n. With this scheme structure, we call $\mathbf{F}_k(SD_n)$, the **Fano scheme** of k-planes on SD_n.

> **Examples of Fano schemes:** Let $n = 3$ and $k = 2$. The Fano scheme $\mathbf{F}_2(SD_3) \subset \mathrm{Gr}(3,6)$ parametrizes 3-dimensional singular subspaces of 3×3 symmetric matrices:
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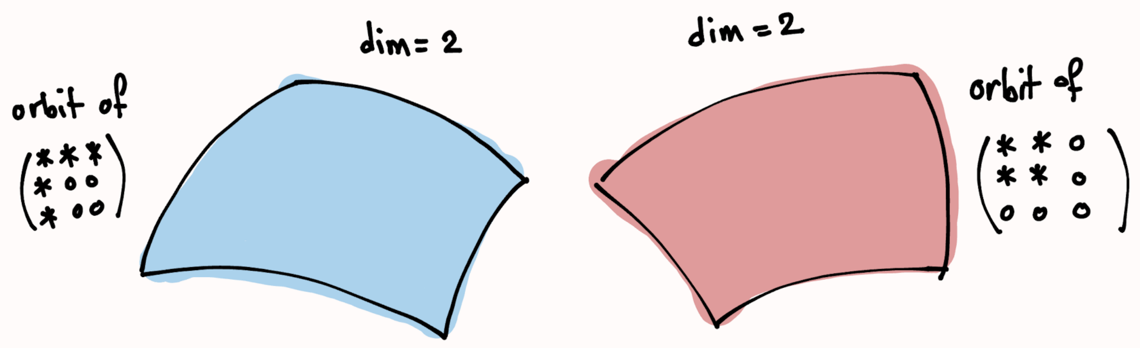
> 
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> The scheme $\mathbf{F}_2(SD_3)$ has two disjoint irreducible components each of dimension 2.

The goals of this project is that given n, k we would like to know the following for $\mathbf{F}_k(SD_n)$:

1. Is it irreducible? If not irreducible, what are (some of) the components?
2. Is it smooth?
3. Is it connected?

Answering these question (even partial answers) translates back to results on singular subspaces of S_n such as recovering results of Meshulam and Pazzis.

Analyzing the geometry: Fixed points

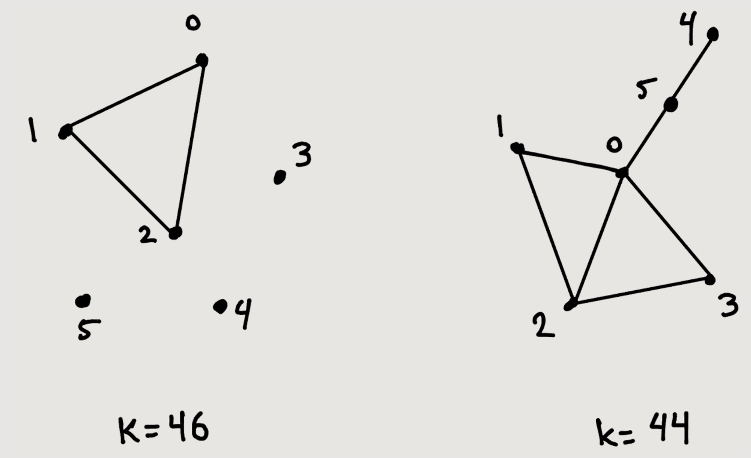
By Borel fixed point theorem, the subgroup $B_n \subset \mathrm{GL}(n)$ consisting of all upper triangular matrices has the property that if it acts on a complete variety, it will have a fixed point, called a **Borel fixed point**. This is the case with Fano schemes, therefore we have at least one fixed point on every irreducible component of $\mathbf{F}_k(SD_n)$ and on the non-empty intersection of every two components. We gain much information by characterizing Borel fixed points of $\mathbf{F}_k(SD_n)$.

> **Proposition (M.):** The Borel fixed points of $\mathbf{F}_k(SD_n)$ are exactly singular subspaces of S_n of the form (1) where each entry in the star blocks is either zero for every matrix or varies freely. Moreover, if an entry is zero, all entries to the right and bottom are zero too.
> **Example:** Let $n = 3$ and $k = 1$. The subspace on the left is a Borel fixed point but the one on the right is not (each z_i is a free variable).
> $$\begin{pmatrix} z_0 & z_1 & 0 \\ z_1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 & z_1 & 0 \\ z_1 & z_0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Characterizing connectedness

Theorem (M.): Given n and k there is a combinatorial characterization of connectedness for $\mathbf{F}_k(SD_n)$. Build a graph as follows: Draw a vertex for each integer $0 \leq s \leq \frac{n-1}{2}$ if the scheme has a Borel fixed point that is an s-compression space. Two vertices s, s' are adjacent if there is a Borel fixed point that is both an s- and an s'-compression space. Then the set of connected components of $\mathbf{F}_k(SD_n)$ is in bijection with the set of connected components of this graph.

Example: Let $n = 12$, the graphs for two values of k are as follows:



We see that for $k = 44$, the scheme is connected, but when $k = 46$, there are 4 connected components.

Characterizing irreducibility

Theorem (M.): The scheme $\mathbf{F}_k(SD_n)$ is irreducible if and only if the graph associated to the scheme (defined above) has only one vertex. In this case, every point of the scheme is a 0-compression space.

Smoothness

Theorem (M.): Fix n. For each integer $0 \leq s < \frac{n-1}{2}$, the scheme $\mathbf{F}_{\kappa(s)}(SD_n)$ has an irreducible component that is generically non-reduced. In particular, the scheme is not smooth.

We suspect this pathology happens for every k, that is we conjecture the following

Conjecture: The scheme $\mathbf{F}_k(SD_n)$ always has an irreducible component that is generically non-reduced, in particular the scheme is never smooth.

Fano scheme of lines

The following result gives a complete list of irreducible components of $F_1(SD_n)$.

Theorem (M.): The Fano scheme $\mathbf{F}_1(SD_n)$ has $\lfloor \frac{n-1}{2} \rfloor + 1$ irreducible components, each of dimension $n^2 - 5$. For each integer $0 \leq s \leq \lfloor \frac{n-1}{2} \rfloor$, the s-compression spaces form a component. These components intersect pairwise and the graph of the scheme is the complete graph on $\lfloor \frac{n-1}{2} \rfloor + 1$ vertices.

References

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