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FanoScheme

MAGMA Package | Author Ahmad Mokhtar | Updated Feb 26, 2024

FanoScheme is a package in MAGMA for computation with Fano schemes of embedded projective varieties. Let $X \subset \mathbb{P}^n$ be an embedded projective variety. Then the Fano scheme $\mathbf{F}_k(X)$ of k-planes in X is the fine moduli space that parametrizes those k-planes contained in X. The scheme $\mathbf{F}_k(X)$ is a subscheme of the Grassmannian $\mathbb{G}(k, n)$.

Moreover, a Grassmannian $\mathbb{G}(k, n)$ is the same as the Fano scheme $\mathbf{F}_k(\mathbb{P}^n)$.

`FanoScheme(X, k, grassAmbient) : Sch,RngIntElt,Prj -> Sch`

Returns the Fano scheme $\mathbf{F}_k(X)$ as a subscheme of a Grassmannian $\mathbb{G}(k, r)$ embedded in the projective space `grassAmbient`. The dimension of `grassAmbient` should be equal to $\binom{r+1}{k+1}$ where r is the dimension of the ambient projective space of X. The returned Fano scheme is a subscheme of `grassAmbient`.

`FanoScheme(X , k) : Sch,RngIntElt -> Sch`

Returns the Fano scheme $\mathbf{F}_k(X)$ as a subscheme of a Grassmannian $\mathbb{G}(k, r)$ embedded in a projective space containing $\mathbb{G}(k, r)$. It creates a projective space `ambientSpace` of dimension $\binom{r+1}{k+1}$ and then calls `FanoScheme(X, k, grassAmbient)`.

> **Example 1** : The famous Cayley-Salmon theorem asserts that a smooth cubic surface in $\mathbb{P}^3$ contains exactly 27 lines. We will use `FanoScheme` to demonstrate the theorem.

```
> KK:=Rationals();
> KK;
Rational Field
> P<x,y,z,w>:=ProjectiveSpace(KK,3);
> P;
Projective Space of dimension 3 over Rational Field
Variables: x, y, z, w
> grassAmbient<p_0,p_1,p_2,p_3,p_4,p_5>:=ProjectiveSpace(KK,5);
> grassAmbient;
Projective Space of dimension 5 over Rational Field
Variables: p_0, p_1, p_2, p_3, p_4, p_5
> X:=Scheme(P, x^3+y^3+z^3+w^3);
> X;
Scheme over Rational Field defined by
x^3 + y^3 + z^3 + w^3
> Y:=FanoScheme(X,1,grassAmbient);
> Dimension(Y);
0
> Degree(Y);
27
```

Example 2 : The smooth quadric $X \subset \mathbb{P}^3$ defined by $xy - zw = 0$ has two disjoint family of lines, namely its two sets of rulings. Let's examine the Fano scheme $\mathbf{F}_1(X)$. We will see that the Fano scheme $\mathbf{F}_1(X)$ has two irreducible components. They are curves of degree 2. Upon inspecting the equations for each compnent, we see that they are two disjoint conics in the Grassmannian $\mathbb{G}(1,3)$.

```
KK;
Rational Field
> P<x,y,z,w>:=ProjectiveSpace(KK,3);
> P;
Projective Space of dimension 3 over Rational Field
Variables: x, y, z, w
> grassAmbient<p_0,p_1,p_2,p_3,p_4,p_5>:=ProjectiveSpace(KK,5);
> grassAmbient;
Projective Space of dimension 5 over Rational Field
Variables: p_0, p_1, p_2, p_3, p_4, p_5
X:=Scheme(P, x*y-z*w);
X;
Scheme over Rational Field defined by
x*y - z*w
> Y:=FanoScheme(X,1,grassAmbient);
for component in IrreducibleComponents(Y) do
    component;
    printf "Dimension of component = %o\n", Dimension(component);
    printf "Degree of component = %o\n", Degree(component);
    print "-----";
end for;
Scheme over Rational Field defined by
p_2*p_3 + p_5^2,
p_0 - p_5,
p_1,
p_4
Dimension of component = 1
Degree of component = 2
-----
Scheme over Rational Field defined by
p_1*p_4 + p_5^2,
p_0 + p_5,
p_2,
p_3
Dimension of component = 1
Degree of component = 2
-----
```

`Grassmannian(k, n, grassAmbient) : RngIntElt,RngIntElt,Prj -> Sch`

Returns the Grassmannian $\mathbb{G}(k, r)$ of k-planes in an n-projective space P. It works by calling `FanoScheme(P, k, grassAmbient)`. The returned Grassmannian is a subscheme of the ambient projective space `grassAmbient` which must have dimension $\binom{n+1}{k+1} - 1$, otherwise an error occurs.

`Grassmannian(k, P) : RngIntElt,Prj -> Sch`

Returns the Grassmannian $\mathbb{G}(k, P)$ of k-planes in the n-projective space P by calling `FanoScheme(P, k)`. The returned Grassmannian is a subscheme of an ambient projective space of dimension $\binom{n+1}{k+1} - 1$, otherwise an error occurs.

`Grassmannian(k, P, grassAmbient) : RngIntElt,Prj,Prj -> Sch`

Returns the Grassmannian $\mathbb{G}(k, P)$ of k-planes in the n-projective space P by calling `FanoScheme(P, k, grassAmbient)`. The returned Grassmannian is a subscheme of the ambient projective space `grassAmbient` which must have dimension $\binom{n+1}{k+1} - 1$, otherwise an error occurs.

> **Example 3** : We create the Grassmannian $\mathbb{G}(1, 3)$ and display its plücker relation.
>
> ```
> > KK:=Rationals();
> > KK;
> Rational Field
> > grassAmbient<p_0,p_1,p_2,p_3,p_4,p_5>:=ProjectiveSpace(KK,5);
> > grassAmbient;
> Projective Space of dimension 5 over Rational Field
> Variables: p_0, p_1, p_2, p_3, p_4, p_5
> > G:=Grassmannian(1,3,grassAmbient);
> > G;
> Scheme over Rational Field defined by
> p_2*p_3 - p_1*p_4 + p_0*p_5
> ```