Portfolio Allocation Using Wavelet Transform

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ABSTRACT

To apply the mean-variance portfolio theory in real life, an investor needs to estimate a set of statistical characteristics from the underlying securities in the portfolio of interest, as well as the weight assigned to each portfolio; however, the noise present in the underlying securities may distort the estimated statistical characteristics of securities and in turn the resulting portfolio allocation strategy. In this paper, I investigate the effect of such noise on the statistical characteristics of financial series and on the portfolio allocation decision. I rely on the wavelet transform to minimize the effect of noise in the financial series. Different combination of smooth and non smooth series are employed to estimated the optimal portfolio weights, where each combination leads to different risk and return for investor. From these estimations, I observed that the allocation decision with highly smoothed variance matrix and non smoothed mean provides the highest Sharpe ratio. Overall, the obtained results show that the wavelet transform is an effective tool for smoothing the financial series and can lead to an improved investor allocation strategy.

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1. Introduction

The goal of any investor from individuals to large corporations is to maximize profit, while minimizing any risk associated with their investment. One approach is to invest in portfolios that are formed as a combination of risky and risk-free securities. To further benefit from risk diversification, investors usually choose securities that are less dependent on each other. Although, such approaches may seem logical and even mundane, half a century ago when Markowitz (1952) published his *Portfolio Selection* paper, it was a new idea that has left a lasting mark on the financial world.

Before the works of Markowitz (1959) and (1952) on portfolio theory, investors were only concerned with maximizing the discounted return of the portfolio. Markowitz showed that diversification can not be achieved by maximizing discounted future returns. He then argued for a new approach in selection of a portfolio and notes that an investor should :

"*Consider expected return a desirable thing and variance of return an undesirable thing.*"

His theory, which is better known as the *Mean Variance Portfolio Theory*, provides the investor with the optimal portfolio based on the expected return of each security, variance of securities, and correlation among securities. The best approach for estimating expected return and variance of securities is based on historical mean and variance of the underlying securities. Estimation of correlation coefficients among securities is more difficult, due to the fact that calculating the correlation coefficients in large portfolios is very inefficient, and leads to a very huge correlation matrix. For example for a portfolio consisting of 300 stocks, there are 4,4850 correlation coefficients to be computed. To alleviate this problem, Elton and Gruber (1973) propose several efficient models for computing correlation coefficients between securities. Briefly the peculiarities of the models are:

- Full historical model : Past data is used to estimate future correlation coefficients.
- Index model *: As the trend of change in securities is affected by the overall market direction, the market index is also included in the structure of the securities return.
- Multi-index model: In addition to the overall market direction, a number of economical subgroups that affect the movement of securities are also considered.
- Average correlation model: Past means of each pairwise correlation coefficient is used for future estimations.

The above mentioned models brought more efficiency in estimating the correlation coefficients among a set of securities, however Laloux, Cizeau, Bouchaud, and Potters (1999) brought to light new issues that renewed the quest for better modeling. They show that high amounts of noise is present in covariance and correlation matrix estimations and argue that as the amount of noise is high, we can treat an empirical correlation as a random matrix. Hence, as they note, one should take into account the effect of noise present in the empirical correlation and covariance of the financial model; e.g. the mean variance portfolio theory. To that end, one could minimize the noise present in the financial series using well studies smoothing technique such as the wavelet transform, and obtain more accurate financial models. In fact the use of wavelet transform for smoothing financial series was first proposed by Ramsey (2002).

In this paper I study the effect of noise reduction on the mean, correlation and variance matrices of the financial series. In addition I evaluate the effect of such noise reduction on the allocation of the portfolios based on the mean-variance portfolio theory. I rely on the wavelet transform to smooth the financial series, therefor minimizing the noise. Furthermore, to investigate the main source of improvement in allocation of portfolios, different combinations of raw and smooth series are used to obtain the mean and variance consequently used in estimating the optimal portfolio. For the sake of simplicity, I denote the mean obtained

*The Idea of index model was first developed by Sharpe (1963).

from smoothed series as smooth mean. Similarly, variance obtained from the smoothed series is denoted as smooth variance. The studied models include:

- Raw mean and variance: In estimating of the optimal portfolio weight, both mean and variance are estimated from the raw series.
- Smooth mean and variance: Both mean and variance are estimated from smooth series.
- Raw mean and smooth variance: Raw series is used to estimate the mean and the smooth series is used to estimate the variance.
- Smooth mean and raw variance: Smooth series is used to estimate mean and raw series is used to estimate the variance.

Results reveal that the correlation and standard deviation estimates are changed after reducing the noise present in the financial series. Also Results from different studied models reveal that in the case of raw mean and smooth variance while series are smoothed until third level the portfolio Sharpe ratio is higher than the the Sharpe ratio obtained from the other three cases. Over all, I observe that the wavelet smoothing will improve the allocation of the portfolios and in turn increases the investors return, although the improvement or the lack of it, depends on the the level of smoothing used.

This paper is organized as follows. Section 2 briefly discusses wavelets, and review relevant financial literature employing wavelets. Section 3 describes the methodology in employing the wavelet transform to increase the portfolio return. Section 4 describes the data sets used in work as well as the evaluation results followed by a conclusion in Section5.

2. Wavelet theory

It has been well established that by representing time series in other domains (i.e. frequency, wavelet, Z transform, laplacian, etc.), certain characteristics that are not visible in the time domain are highlighted. Such characteristics may be used to better understand the underlying time series. For example, the superposition of a few sine series could lead to a complex time series, which is difficult to describe. However, by representing this complex time series in the frequency domain, we could simply observe the frequency of the sinusoidal components, which make up such a series. Hence, one is able to represent this complex time series with a few simple components in the frequency domain. As an illustration, Figure 1 plots such a time series in the top panel, and its Fourier transform in the bottom panel. Where the time series x(t), is generated by the superposition of a number of sine waves with different frequencies:

$$x(t) = sin(t) + sin(2t) + sin(.5t) + sin(5t), \qquad t = 0, 1, 2, 3..., 12. \tag{1}$$

The bottom panel of Figure 1, presents the time series x(t) in the fourier domain, from which we observe four peaks corresponding to the four frequency components making up x(t).

Frequency domain representation are obtained through the Fourier transform, first proposed by Joseph Fourier in the 19th century. He showed that any periodic function could be presented by complex exponential functions. In simpler terms, Fourier transform decomposes the time series into a combination of sines and cosines with different frequency and amplitude values. Nevertheless, there are limitations to the Fourier transform. First, it assumes that the time series is periodic, which is not true in many cases. Secondly, it operates on the entire time series, not taking into account changes in local time (i.e. no time resolution). For example, in Figure 2 top panels, two distinct time series are plotted. The left panel series is generated by:

$$x(t) = \begin{cases} sin(t) & 0 \leq t < 4\Pi \\ sin(4t) & \Pi \leq t < 6\Pi \\ sin(8t) & \Pi \leq t < 7\Pi \end{cases}$$

and the right panel series is generated by:

$$x(t) = \begin{cases} sin(t) & \Pi \leq t < 7\Pi \\ sin(4t) & \Pi \leq t < 6\Pi \\ sin(8t) & 0 \leq t < 4\Pi \end{cases}$$

The two panels at the bottom of Figure 2 plot the Fourier representation for each time series above them. It is clearly observed that although the two time series are easily distinguishable, the frequency representation of the two are the same. To capture frequency changes over time, Gabor (1946) proposes the windowed Fourier transform, in which the time series is segmented into a set of smaller components by a fixed window. Each segment is then transformed to the frequency domain thus resulting in a representation, which has a component in time as well as frequency. Figure 3 schematically shows the differences between Fourier transform and windowed Fourier transform. Panel (a) represents the Fourier transform (i.e. no time resolution), while Panel (b) represents the windowed Fourier transform (i.e. time and frequency resolution). Although windowed Fourier transform introduces time dependency in the frequency analysis, it is difficult to accurately define a window. For example, what would be the optimal window width and shape?

About a century after Joseph Fourier proposed the Fourier transform, in 1909 Alfred Haar introduced the orthogonal base function on the footnote of his thesis as an alternate basis for decomposing time series. His work later led to the development of wavelet transforms. In the 1980s, Morlet and Grossman showed that the wavelet process is lossless, hence one could transform the time series to the wavelet domain and then back with out any loss of information. In fact, Jean Morlet was the first to name his basis function a "wavelet", later known as the Morlet Wavelet. Also, in the late 1980's the work on multiresolution analysis by Stephane Mallat and Yves Meyer was introduced, where a time series is represented at different times and scales resolutions. [†]

Unlike Fourier transform, where sine is the only basis function, there are several wavelet basis functions with different shapes; although all have finite energy [‡] and are compactly supported. Figure 4 shows the difference between a wave and a wavelet. These characterizes of wavelets allow the wavelet transform to deal with nonstationary and transient series, as well as the ability to decompose time series to different components at different scales. In the wavelet domain, a basis function is called the mother wavelet and other bases are obtained from the translation (location) and the dilation (size) of the mother wavelet. Some well known mother wavelets are shown in Figure 5. Panel (a) plots the Haar wavelet, which is an orthogonal, symmetric, and discontinuous wavelet. Panel (b) plots the Daubechie2 wavelet, which is an orthogonal, asymmetric, and compacted wavelet. Panel (c) plots the orthogonal symmetric Meyer wavelet, and Panel (d) plots an Symlet wavelet, which is orthogonal and nearly symmetric.

In the following section, I discuss a range of issues concerning wavelets and their application in financial economics. In Section 2.1 the continuous wavelet transform and in Section 2.2 the discrete wavelet transform are reviewed. In Section 2.3 the multiresolution analysis with the help of wavelet transform is discussed. Section 2.4 reviews filter banks and their relation to the wavelet transform. Finally Section 2.5 reviews related literature in economics and finance, which make use of the wavelet transform.

[†]In the wavelet literature the notion of scale is used instead of frequency, where $scale = \frac{1}{frequency}$

[‡]Signal energy is defined as the area under the signal squared.

2.1. Continuous wavelet transform

As discussed earlier in this section, wavelet transform converts a time series to the frequency domain with the ability to represent the time series at different time and scale resolutions. For a continues time series, one would employ the continuous wavelet transform (CWT). Genay, Seluk, and Whitcher (2001) represent the continuous wavelet transform, $W(u,s)$ as:

$$W(u,s) = \int_{-\infty}^{\infty} x(t)\psi_{u,s}(t)dt, \tag{2}$$

and the continuous basis (mother) wavelet, $\psi_{u,s}$, as:

$$\psi_{u,s} = \frac{1}{\sqrt{s}}\psi(\frac{t-u}{s}). \tag{3}$$

For example, the Morlet and Mexican-Hat are two continuous mother wavelets, Figure (6).

As we can see from equation (2), continuous wavelet transform $W(u,s)$, which is a projection of time series $x(t)$ on to the basis wavelet (i.e. $\psi_{u,s}$), is a function of two continuous variables s and u. Where s is a parameter for dilation (size of wavelet) and u is a parameter for translation (location) of the wavelet. By obtaining the wavelet with different dilation and translation values, the wavelet transform decomposes the time series into the different scale and time resolution components.

When the continuous wavelet satisfies the admissibility condition, the initial time series could be reconstructed from their respective wavelet coefficients without any loss of information. Genay, Seluk, and Whitcher (2001) represent the admissibility condition as:

$$C_\psi = \int_0^\infty \frac{|\Psi(f)|}{f} df < \infty, \tag{4}$$

where $\Psi(f)$ is the fourier representation of $f(t)$. In order to have the above condition satisfied, we should have $\Psi(0) = 0$, which means that $\int_{-\infty}^{\infty} \psi(t)dt = 0$. Given that this condition is satisfied, the original time series can be reconstructed from the wavelet coefficients by:

$$x(t) = \frac{1}{C_\psi} \int_0^\infty \int_{-\infty}^{\infty} W(u,s)\psi_{u,s}(t)du\frac{ds}{s^2} \ . \tag{5}$$

2.2. Discrete wavelet transform

The previous section reviews the continuous wavelet transform, but as most financial series are in discrete form, discrete wavelet transform is more applicable to my work. Similar to that of the continuous wavelet transform, in the discrete wavelet transform, the mother wavelet $\psi_{j,k}(t)$ is defined as:

$$\psi_{j,k}(t) = 2^{j/2}\psi(2^j - k) \qquad j,k \in Z, \tag{6}$$

where k is the index for translation (location) of the wavelet and j is the index for dilation (size) of the wavelet. For example, as j is increased the wavelet becomes more compact (i.e. smaller in length), hence the time resolution will increase since smaller time durations are analyzed.

The set of two dimensional *discrete wavelet transform* coefficients, $d_{j,k}$ can be obtained by the inner product of series $x(t)$ and mother wavelets $\psi_{j,k}(t)$:

$$d_{j,k} = \langle f(t), \psi_{j,k}(t) \rangle = \int_{-\infty}^{\infty} x(t)\psi_{j,k}(t)dt, \tag{7}$$

The original time series is obtained from the wavelet coefficients through the following formulation:

$$x(t) = \sum_{k}\sum_{j} d_{j,k}\psi_{j,k}(t). \tag{8}$$

For example, Haar wavelet a discrete wavelet function is defined as:

$$\psi(t) = \begin{cases} 1 & 0 \leq t < \frac{1}{2} \\ -1 & \frac{1}{2} \leq t < 1 \\ 0 & \text{otherwise} \end{cases}$$

Figure 7 shows the haar wavelets and its translated and dilated versions of it.

2.3. Multiresolution analysis

So far I have discussed transforming the time series to the wavelet domain. Furthermore, wavelet transform has the ability to represent a time series at different resolutions (i.e., time and scale). Building on the wavelet transform, Mallat (1989) proposes the multiresolution analysis, with which a time series is decomposed to an approximation and a detailed components at different resolutions. To be able to represent multiresolution analysis of a time series, in addition to the mother wavelet function, which capture the detailed component there is a need for another function to capture the approximation component. This function, $\varphi_{j,k}$, is usually called the scaling function, and is represented by Burrus and Gopinath (1997) as:

$$\varphi_{j,k}(t) = 2^{j/2}\varphi(2^j - k) \qquad j,k \in Z \tag{9}$$

Scaling functions always satisfy the following condition:

$$\int \varphi_{j,k}(t) = 1 \tag{10}$$

So any square integrable function , $g(t) \in L^2(\mathbf{R})$, can be express as a combination of the scaling functions and the mother wavelets.

$$g(t) = \sum_k c_{j0}(k)\varphi_{j0,k(t)} + \sum_k \sum_{j=j0}^{\infty} d_j(k)\psi_{j,k}(t) \tag{11}$$

If the bases functions are orthogonal, the coefficients of the wavelet expansion can be easily obtained by the following inner products:

$$c_j(k) = \langle g(t), \varphi_{j,k}(t) = \int g(t)\varphi_{j,k}(t)dt \tag{12}$$

$$d_j(k) = \langle g(t), \psi_{j,k}(t) = \int g(t)\psi_{j,k}(t)dt, \tag{13}$$

where $c_j(k)$ donates the approximation coefficients (smooth) and $d_j(k)$ denotes the detailed coefficients(noisy).

2.4. Wavelet and filter banks

The discrete wavelet transform is usually applied through filter banks. This approach was first proposed by Mallat (1989), employing the pyramid algorithm and *quadrature mirror filters*. In general, a signal is filtered by convolving it by the filter coefficients. In the context of the discrete wavelet transforms, a time series will pass through two filters, a low pass and a high pass filter. Afterwards, the output of each filter is down-sampled. Through this process, the original time series is decomposed into two components, each half the size of the original time series. The component obtained through the high pass filter, contains the details from the original time series. The component obtained through the low pass filter represents the smoothed version of the original time series. This filtering and down-sampling process can be reiterated, each time feeding the smoothed time series as the input to the next stage of low and high pass filters. Each iteration is also called a level, for example a two level decomposition would require two iterations of the described process. Figure 8 shows a three level decomposition of a time series.

Based on the representation used by Burrus and Gopinath (1997), the high pass filter is obtained from the wavelet equation by:

$$\psi(t) = \sum_n h_1(n)\sqrt{2}\varphi(2t-n) \qquad n \in Z. \tag{14}$$

where the wavelet equation relates the scaling function to the wavelet function. The low pass filter is obtained from the dilation equation :

$$\varphi(t) = \sum_n h(n)\sqrt{2}\varphi(2t-n) \qquad n \in Z. \tag{15}$$

where the dilation equation is the equation for relating the scaling function to itself. The high pass filter, $h_1(n)$, and the low pass filter, $h(n)$, are obtained as:

$$h_1(n) = \frac{1}{\sqrt{2}} \int \psi(t)\varphi(2t-n). \tag{16}$$

$$h(n) = \frac{1}{\sqrt{2}} \int \varphi(t)\varphi(2t-n). \tag{17}$$

The inverse of the discrete wavelet transform could be implemented using filter banks as well. This is essentially reversing the decomposition process discussed earlier. Specifically, one can initially up-sample the last level smoothed and detailed coefficients, and then convolve the up-sampled smoothed coefficients by the low pass filter and the up-sampled detailed coefficients by the high pass filter. The resulting signal is then used as the smoothed signal to the next level of filters. After reiterating this process, the original time series is reconstructed. The number of iterations depends on the level of decomposition conducted on the original time series. Figure 9 presents the reconstruction process by filter banks for a time series that was initially decomposed by a three level filter bank.

So far I have conducted a brief review of wavelet transforms and their implementation using filter banks. In the remainder of this section, I conduct a review of the related economics and financial literatures that have employed wavelets as part of their research work.

2.5. Literature review

Wavelet analysis is a well-known subject in fields such as mathematics, physics, and signal processing. Ramsey (1999), one of the first adapters of the wavelet analysis to the field of economics, identifies four areas whereby researchers can benefit from this form of analysis. These areas are, time scale versus frequency, density estimation and local inhomogeneity, time scale decomposition, and forecasting. The vast majority of literature in economics, which employ wavelets are published after Ramsey's (1999) canonical paper.

In what follows, first I review literatures in which the wavelet transform is used for time scale decomposition. Afterwards, I cover works in which the wavelet transform is used as a tool to pre-process data sets. Such pre-processing can be done with the goal of smoothing, denoising, or deterending the original data sets.

2.5.1. *Time scale decomposition*

Estimating the relation between variables in economic and financial models has always been of great importance. With the ability to decompose the time series into different components, wavelet transform can help researchers understand the relation between variables in the short term and the long term.

Ramsey and Lampart (1998) use wavelet decomposition to investigate the relation between consumption and income between the period of 1960 to 1998 at each scale. Their study found that the relation between money and income varies from scale to scale; therefore indicating that the wavelet transform is beneficial in capturing these variations.

Gencay, Selcukb, and Whitcherc (2001) apply wavelets to the foreign exchange rates to investigate the properties of foreign exchange volatilities at different levels of decomposition. They observe that the correlations between volatility of two exchange rates are constant at the one day interval as well as longer intervals, while they increased in intra-day intervals. They conclude that as the foreign exchange rate volatilities differ at different levels of decomposition, converting short horizons risk measures to long horizons risk measures is not appropriate.

Jamdee and Cornelis A. Los (2003) use wavelet analysis to investigate the behavior of the interest rate term structure. They analyze the Treasury bill quoted from secondary markets and the constant maturity Treasuries using multiresolution analysis. The authors find that each period (scale) has different high or low risk values and that using a simple affine model may cause some misunderstanding in the behavior of the interest rate term structure. They show that the basic assumption of an affine model such as stationary and gaussian distribution is too simplistic. Hence they conclude that the affine model is not the best model to explain interest rates movement.

In another work, Gencay, Selcuk, and Whitcher (2005) use wavelets to estimate systematic risk (i.e. beta of an asset in capital asset pricing model (CAPM)) at different wavelet decomposition levels. At each level of decomposition the variance and covariance are estimated between individual stock premium and the market premium. By using data sets from the U.S., Germany, and United kingdom, the authors conclude that at higher decomposition levels the relation between the return of a portfolio and its beta is higher. This observation needs to be considered when the CAPM model is used.

Kim and In (2005) use the multiresolution analysis to propose a new look at the Fisher hypothesis. The Fisher hypothesis states that a positive relation between nominal stock return and inflation exists. They observe that the nominal interest rate and inflation have a positive relation in the short horizon, first level smooth component, as well as in the long horizon, seventh level smooth component. Furthermore, they find that the correlation of stock returns with inflation is positive in the first and seventh level smooth component, while for the rest of the studied wavelet decomposition levels this relation is negative.

Shrestha and Tan (2005), apply the wavelet transform to analyze the deep relation between short-run and long-run real interest rate in the G-7 countries. They perform a multiresolution analysis with seven level decomposition using the maximal overlap discrete wavelet transform. They estimate regressions between the U.S. and other six countries of G-7 countries at each scale and observe that there are strong relations between the U.S. real interest rates and those of the other six G-7 countries in the short-run, some of which had not been observed previously with other models

2.5.2. *Wavelet a tool for data preparation*

Ramsey (2002) notes that wavelets are a proper tool for noise smoothing and denoising. Denoising is done by threshholding the wavelet coefficients before reconstructing the time series. This is in contrast to smoothing, in which case only the smooth coefficients are used to reconstruct the time series, which the detailed coefficients are discarded. Ramsey notes that denoising would be the proper approach over smoothing when the underlying time series includes regime shifts and discontinuities. Below I review a few related works in which either wavelet denoising or smoothing is used.

Capobianco (1999) investigates the effect of denoising on the prediction of volatility. His experiments are done using the Nikkie daily stock index for the period of May 17,1949 to July 31,1996, and employs the GARCH model. The author concludes that denoising the financial series with wavelet threshholding improves the volatility prediction power of the GARCH(1,2) model.

Lee (2004) investigate the effect of the U.S. markets on the Korean markets by using multiresolution analysis. This is achieved by reconstructing the time series from the detailed wavelet coefficients. The author found that trends in the U.S. stock market, affect the trend in the Korean market.

Mitsui and Tabata (2006) employ wavelet transform as a tool to reduce noise in the correlation matrix estimation. They use wavelet threshholding to denoise the noisy empirical correlation matrix, and find that the wavelet denoising can be helpful in decreasing the noise in the correlation matrix.

3. Methodology

In the previous section I discussed the wavelet transform. In this section I discuss the methodology used to evaluate the effect of noise reduction through wavelet smoothing on the allocation of portfolios. First I review the framework of mean-variance portfolio theory, and the procedure of obtaining the optimal portfolio weights. In fact there are a number of models I consider for estimating the parameters used to calculate the portfolio weight, which I review. In addition to the estimation model employed, I need to consider the parameters required for the smoothing operation as well as the level of smoothing employed. In what follows Section 3.1 reviews mean-variance portfolio theory and the models used for estimating the parameters of interest. Section 3.2 describe the methodology used for smoothing financial series.

3.1. Mean-variance portfolio theory

The Mean-variance portfolio theory was introduced by Markowitz (1952), where he argues for investors to maximize expected return while minimizing variance of the portfolio. (weight should be estimated)A number of researchers (e.g, see Levy and Markowitz (1979), Kroll, Levy, and Markowitz (1984), and Dupacova, Stepan, and Hurt (2002)), have shown that the optimal portfolio in the framework of mean variance portfolio theory could be obtained by solving an optimization problem, assuming that asset returns are normally distributed or using a quadratic utility function. The optimization problem is defined as[§]:

$$\max_{x} \quad EU(R_p) \qquad\qquad s.t. \;\; \acute{W}I = 1. \tag{18}$$

where R_p is the portfolio return, and is formed by combination of risky assets and a risk-free asset:

[§]The notations are adopted from Okhrin and Schmid (2007)

$$R_p = \acute{W}(\mathbf{X} - r_f\mathbf{I}) + r_f, \tag{19}$$

where $\mathbf{X}$ denotes the K-dimensional vector of asset returns with a normal distribution of $\mathbf{X} \sim N(\mu, \mathbf{\Sigma})$. r_f is a risk-free asset, and $\acute{W}$ denotes the vector of the portfolio weights. By using quadratic utility function, the maximization problem of (18) is transformed to:

$$\max_x \quad E(R_p) - \frac{\gamma}{2} Var(R_p), \qquad s.t. \;\; \acute{W}I = 1. \tag{20}$$

where $\gamma > 0$ is the risk aversion coefficient. By substituting R_p from equation (19) to equation (21), then the mean variance maximization problem can be written as:

$$\max_x \quad \acute{W}(\mu - r_f\mathbf{I}) - \frac{\gamma}{2} \acute{W}\Sigma W, \qquad s.t. \;\; \acute{W}I = 1. \tag{21}$$

therefore, the optimal weight of a risky asset is:

$$W_{Op} = \gamma^{-1}\mathbf{\Sigma}^{-1}(\mu - r_f\mathbf{I}). \tag{22}$$

and the weight of a risk-free asset is $W_{r_f} = 1 - W_{op}$. As we observe from equation (22), the optimal portfolio weight depends on the inverse of the mean and the covariance matrix of assets. Hence, different estimations of the mean and the covariance matrix of assets will effect on the optimal weight. Below I explain three different models I use in this work to estimate the mean and covariance matrix.

3.1.1. *Benchmark model*

A simple benchmark model is set up to estimate the portfolio weights and, consequently the portfolio returns. For estimation of the optimal weights in this model, the historical mean (sample mean) of each series is used, as well as the sample mean of the risk-free asset. The covariance matrix is also obtained from the same samples. I use this simple model for comparative purposes.

3.1.2. *Mean varying financial series*

In the previous model, the sample mean and the sample covariance of securities are used. But in the real world, mean and variance of financial series vary over time. Hence, to include this variation in my analysis, I employ an autoregressive model of order one (AR(1)) to estimate conditional mean and variance. The AR(1) model is represented by D.Hamilton (1994) as:

$$R_{t,i} = C_{t,i} + \varphi R_{t-1,i} + \varepsilon_{t,i} \qquad i = 1,2,..N \quad t = 1,2,..T, \tag{23}$$

where $R_{t,i}$ is a time series of portfolio returns at time t, $\varepsilon_{t,i}$ is a random variable and is normally distributed. The covariance matrix is named Σ and is estimated from the regression residual. Given the estimates, the conditional mean is given by:

$$\mu_{t,i} = \hat{C_{t,i}} + \hat{\varphi} R_{t-1,i} \qquad i = 1,2,..N \quad t = 1,2,..T. \tag{24}$$

3.1.3. *Time varying financial series*

In this section, the Generalized Autoregressive Conditional Heteroscedasticity (GARCH) model, which was introduced by Bollerslev (1986), is employed. More specifically in this research the AR(1)-GARCH(1,1) model is used to estimate the conditional mean and conditional variance matrices. Let $R_{t,i}$ be the time series of returns of security i at time t, then the AR(1)-GARCH(1,1) model is defined as:

$$R_{t,i} = C + \varphi R_{t-1,i} + \sigma_{t,i}\varepsilon_{t,i} \qquad i = 1,2,..N; \quad t = 1,2,..T. \tag{25}$$

$$\sigma^2_{t,i} = k + a\varepsilon^{i2}_{t-1} + g\sigma^2_{t-1,i} \qquad i = 1,2,..N; \quad t = 1,2,..T. \tag{26}$$

where $\varepsilon_{t,i}$ is an *i.i.d.* random variable with a Gaussian distribution. $\sigma^2_{t,i}$ is conditional variance, which depends on the past variance as well as the past residual. The conditional mean of security i at time t, $\mu_{t,i}$, is given by:

$$\mu_{i,t} = \hat{C} + \hat{\varphi} R_{i,t-1} \qquad i = 1,2.....,6 \tag{27}$$

covariance matrix of securities, in GARCH(1,1) is obtained as:

$$\Sigma = Q * (\sigma' * \sigma), \tag{28}$$

where Q is the estimated correlation matrix of the securities and is obtained from the residuals, and σ is the estimated standard deviations vector obtained from equation (26).

3.2. Smoothing financial series.

As discussed earlier, the wavelet transform can be used as a tool for smoothing or denoising time series. I should note, that I employ wavelet smoothing in this work instead of wavelet de-noising. This selection is due to the fact that with wavelet denoising additional parameters need to be estimated (i.e. denoising threshold), which would complicate my analysis. I have experimented with different bases wavelets such as Haar, Daubechies and Symlet, where better results are obtained with the Haar wavelet. Similar observations were made by Kaplan (2001), who recommends the selection of Haar wavelets for financial series as a good choice, since they are not inherently smooth and are usually jagged. Using the Haar wavelet, the time series are first decomposed to detail and approximate coefficients, and then the wavelet synthesis function is used to reconstruct the series from only the approximate coefficients. In other words, I remove the second part of the equation (11), which represents the detail (noise) coefficients.

One note of importance is that the discrete wavelet transform requires the time series it operates on to be of dyadic length (i.e. $length = 2^i$, where $i = 1 : N$), but this requirement is not always met. For example in this work, as I am forecasting the portfolio return, the time series I operate on grows by one point at each iteration of forecasting. Hence, it will not always be of a dyadic length. A common way to deal with this issue is to extend the length of the time series so that it would become of dyadic length. This is done by padding the time series with zeros, sample mean of series, repetition of only the last value, or repeating the values of the series in a periodic form. In fact, through experimentation, I have found that the periodic padding approach provides the best results in my work.

In this section, I discussed the approach used in this paper to obtain the smooth series as well as the the models used to estimate the portfolio weights. In the next section I will explain the data set employed and the obtained empirical results.

4. Data set and empirical analysis

In the following section, I first introduce the data set used in my experiments in Section 4.1. Afterwards, in Section 4.2, I report the obtained results based on the evaluation methodology discussed in the previous section.

4.1. Data description and summary statistics

The data sets are obtained from the online data library of French (2006). This library contains a set of useful data sets, which offer a different range of portfolio returns and are appropriate for our goal of analyzing the effect of smoothing on portfolio allocation. Six monthly portfolio returns are chosen, which cover the period of July 1926 through end of August 2006, consisting of a total of 962 monthly observation for each series. Also a 1-month T-Bill is chosen as the risk-free asset and its rate downloaded for the same period as the portfolio returns from the online data library.

Each portfolio is constructed based on a set of rules and assumptions with consideration of the underlying assets. French (2006) defined the construction of the portfolios as:

> "The portfolios, which are constructed at the end of each June, are the intersections of 2 portfolios formed on size (market equity, ME) and 3 portfolios formed on the ratio of book equity to market equity (BE/ME). The size breakpoint for year t is the median NYSE market equity at the end of June of year t. BE/ME for June of year t is the book equity for the last fiscal year end in t-1 divided by ME for December of t-1. The BE/ME breakpoints are the 30th and 70th NYSE percentiles. The portfolios for July of year t to June of t+1 include all NYSE, AMEX, and NASDAQ stocks for which we have market equity data for December of t-1 and June of t, and (positive) book equity data for t-1."

Panel A of Table 1 reports the summary statistics for the monthly portfolio returns. The mean of all monthly portfolio returns are positive. Except the return of the Big/Low, which is negatively skewed the skewness statics of all other returns are positive. Figure 10 plots the times series of six monthly portfolio returns. Panel (a) plots three portfolios, which are constructed in Small size at three levels of Low, Medium and High Book-to-market equity ratios and panel (b) plots the three portfolios, which are constructed in Big size at three level of Low, Medium and High Book-to-market equity ratio.

4.2. Empirical analysis

In this section, I discuss the effect of wavelet smoothing on the statistics of my data set and then investigate the effect of wavelet smoothing on the allocation of portfolios.

4.2.1. *Wavelet decomposition of financial series*

As discussed in Section 3.2 the Haar wavelet is employed as the basis function in order to decompose the raw data into detail and approximate coefficients. Afterwards, the smooth series are reconstructed by using only the approximate coefficients. To better understand the impact of the level of smoothing, the smooth series are reconstructed from the approximate coefficients obtained at different levels of decomposition. More specifically, the following three cases are studied:

- One Level Smooth: Raw series are decomposed by one level, and the smooth series are reconstructed from only the approximate coefficients.
- Two Levels Smooth: Wavelet decomposition is applied to the approximate coefficients obtained from the first level decomposition resulting in the second level detail and approximate coefficients. Smooth data are then reconstructed from the second level approximation coefficients.

- Three levels Smooth: Wavelet decomposition is applied to the approximate coefficients obtained from the second level decomposition resulting in the third level details and approximation coefficients. Smooth series are then reconstructed from the third level approximation coefficients.

For example, in Figure 11, the first panel plots the raw Small/Low returns. The second panel plots the one level smooth Small/Low returns, and the last panel plots the residuals (i.e. the extracted noise at the level one of decomposition). The sample covers period of July 1926 until September 2006

The second level smoothing for the Small/Low monthly portfolio returns is in Figure 12. Where the first (top), second, third, and fourth (bottom) panels plot in respective order; the raw Small/Low return series, the two level smooth series, the residual from the first level decomposition, and the residual from the second level decomposition.

Lastly, the third level smoothing for the Small/Low monthly portfolio returns is in Figure 13. Where the first (top), second, third, fourth, and fifth (bottom) panels plot in respective order; the raw Small/Low return series, the three level smooth series, the residual from the first level decomposition, the residual from the second level decomposition, and the residual from the third level decomposition. Similar plots are obtained for the remaining series.

Figure 14 plots the Small/Low, Small/Middle and Small/High for the raw series, as well as the one, two, and three level smooth series. Similarly, Figure plots the Big/Low, Big/Middle and Big/High for raw series and the one, two, and three level smooth series. From both figures, I observe the reduction of oscillations in the series as the level of smoothing increases.

Next, the impact of noise reduction on statistical characteristics of the financial series is investigated. Panel B, C, and D of Table 1, reports summary statistics for the one, two, and three level of smooth series respectively. It is observed that the sample mean for all six returns in all three cases are positive and remain unchanged with respect to the sample mean of the raw series. This is not surprising, as noise generally has a zero mean and therefore its reduction does not impact the mean of the series. Also, from Table 1 it is observed that as the data is further smoothed (i.e. from the first to the third level smoothing), the estimated standard deviations decreas further. For example, standard deviation for raw Small/Middle series is 0.071, while for the one, second, and three level smooth Small/Middle series the standard deviation decreases to 0.055, 0.042, and 0.027. This clearly points to a reduction of noise in the data as it is further smoothed .

As presented in Panel A of Table 1, except for the Big/Low monthly returns the skewness statistics of all raw monthly returns are positive, implying that these series are skewed to the right. From Panel B and C, I observe that the skewness statistics of all monthly returns becomes positive for one and two level smoothing, pointing to a right skewed series. For the three level smooth, skewness statistics of Small/Low, Small/Middle and Small/High returns remain positive, while the skewness statistics of Big/Low, Big/Middle and Big/High returns are changed to negative.

Table 2 reports the estimates for the sample correlations of the monthly portfolio returns. From Panel A, the Small/Middle, and Small/High series with a correlation coefficient of 0.9627 are the most correlated. The Small/High and Big/Low series, with a correlation coefficient of 0.7836, are the least correlated. Panels B, C, and D of Table 2 provide the estimates for the sample correlation of the one, two, and three level smooth series respectively. It is observed that the noise reduction has an impact on the correlation coefficients, although there is no direct relation between the reduced noise and the estimated correlation coefficients. For example, the correlation coefficient in the case of the Big/High and Big/Middle when considering the raw series is 0.9356. This value increases to 0.9419, and 0.9519 with the one, and two level smooth series. But then the correlation coefficient falls to 0.9438 when considering the three level smooth series. It is also observed that with all the three levels of smoothing considered, the Small/Middle and Small/High series, and the Small/High and Big/Low series are the most and least correlated series respectively.

The AR(1) model is estimated for the period of July 1926 through August 2006 using both the raw and smooth series. Table 3 presents the estimated correlation matrices from this model. Panel A shows that the Small/Middle and Small/High series are the most correlated series with a correlation coefficient of 0.960, and the Small/High and Big/Low series are the least correlated with a correlation coefficient of 0.782. Panel B presents the results for the one level smooth series, where the Small/Middle and Small/High series are again the most correlated series with a correlation coefficient of 0.965. The Small/Low and Big/High series are the least correlated. Similarly Panels C and D present the estimated correlation matrices for the two and three level smooth series. Comparing the results from Panel C to that of Panels A and B, I observed that the correlation coefficients for the two level smooth series are higher than those estimated from the raw and one level smooth series. Although similar conclusion from Panel D are not possible.

Next, I estimate the AR(1)-GARCH(1,1) model for the period of July 1926 through August 2006 using both the raw and smooth series. As noted in Section 3.1.3, the covariance matrix obtained from the AR(1)-GARCH(1,1) model consists of two components; the correlation matrix, and the conditional standard deviations of each series. Therefore to better understand the effect of smoothing on each of these components, I present the estimated values for each component independently in Tables 4 and 5. Table 4 presents the estimated correlation matrices of six monthly portfolio returns for the raw and smooth series. According to Panel A and B all correlation coefficients for the one level smooth returns are higher than the correlation coefficients for the raw returns, except for the correlation coefficient of Small/Low and Big/Middle returns. From Panel C, it is observed that in the case of the two level smoothing, correlation coefficients are higher than those estimated for the raw series. The only exception is with the Big/Middle and Big/Low series. Table 5 presents the summary statistics of the estimated conditional standard deviations. These results reveal that for the three level smooth series, the average of the estimated standard deviations are lower than those obtained from the raw series, as well as the one and two level smooth series.

The results in this section indicate that smoothing the financial series does indeed affect the estimated variance, correlation, and skewness of the financial series. Hence one should consider such effects and take them into account when using the estimated variance and correlation to make financial decisions. In the next section, I investigate the effect of smoothing on the optimal portfolio weights, and the allocation of portfolios.

4.2.2. *Effect of wavelet smoothing on portfolio allocation*

In the previous section, the effect of smoothing on the statistics of financial series was investigated. In the following, I expand on those results and investigate the impact of the adjusted statistics on the allocation decision.

The portfolio chosen in this research is a combination of six portfolios (each portfolio contains assets of different stocks) and a risk-free asset. Also, I assume that riskless lending and short selling are permitted. To estimate the parameters required to obtain the optimal portfolio weights (i.e. expected rates of portfolio return, correlation matrix of underlying portfolio returns, and standard deviations of portfolio returns) three models are considered: a benchmark model, a mean varying model, and a time varying conditional variance model. In all mentioned models, weights are rescaled by the sum of the absolute value of the estimated weights.

Table 6 reports the optimal portfolio weights estimated for the benchmark mean-variance portfolio model using the raw series. The investor assigns 75 percent of his wealth to the risk free-asset and the rest to the other six risky portfolios. The Big/Low portfolio has an assigned weight of 21 percent, which is the highest among the risky portfolios.

Next I employ the AR(1) model, with which the portfolio weights are obtained from the estimated conditional mean and variance matrix. Then, the out-of-sample portfolio return is forecasted based on the in-sample portfolio weights through the following steps:

1. The first 400 data points from each monthly portfolio return, starting from July 1926, is used to estimate the in-sample AR(1) model parameters, resulting in the estimated conditional mean and covariance matrix.
2. The optimal portfolio weights are computed for the last month of the in-sample series, based on the estimated conditional mean and covariance matrix.
3. Using the above estimated portfolio weights, and the monthly portfolio returns for the first month of the out-of-sample data, the portfolio return is forecasted for the first month of the out-of-sample data.
4. Steps 1 through 3 are reiterated, where each time the in-sample size is increased by 1 point.

Table 7 presents summary statics for the optimal weights obtained from the AR(1) model. In this model the investor on average assigns 96 percent of his wealth to risk-free asset, where this is a much higher allocation than that obtained from the benchmark model. The Small/High portfolio has a positive weight of 5 percent, while its weight is negative with the benchmark model. On the other hand, the Big/Middle portfolio has a negative weight in both the AR(1) and benchmark models.

The final model which I have employe is the AR(1)-GARCH(1,1) model, with which both raw and smooth series are considered. The out-of-sample portfolio return is estimated using the same four steps discussed above for AR(1) model. Furthermore, the estimated conditional mean and conditional variance values are obtained independently from both the raw and smooth series, resulting in four cases of interest:

- Case one: both the conditional mean and the conditional variance are estimated from raw series.

- Case two: estimation of the conditional mean and the conditional variance are achieved from the smooth series.
- Case three: raw series are used to estimate the mean, while smooth series are used to estimate the variance.
- Case four: smooth series are used to estimate the mean while raw data is used to estimate the variance.

Table 8 reports summary statics for the estimated portfolio weights considering the above four cases, using the one level smooth series. Results reveal that the allocation of portfolios differ for each of the four cases studied. For example in the first case, on average, 87 percent of wealth is assigned to risk-free asset, and the rest is allocated to the risky portfolios. While in the second case, the share of risk-free asset from wealth is larger, at 93 percent on average. As another example, the average weight of the Big/High portfolio is positive, negative, negative, and then positive again for the first, second, third, and fourth cases respectively. Similar results are obtained with the two and three level smooth series in Tables 9 and 10.

Lastly, I investigate whether changes on the allocation of wealth based on the above results, improves the investor‘s overall return. Table 11 reports the summary statistics for the excess returns of the portfolio based on the benchmark, AR(1), and AR(1)-GARCH(1,1) models. The excess return ER_t at time t, is computed as:

$$ER_t = R_{pt} - R_{ft} \qquad t = 1, 2, .., n. \tag{29}$$

where R_{pt} is the forecasted portfolio return at time t, and R_{ft} is the return of risk-free asset at time t.

The ratio of annual mean to annual standard deviation is also reported in Table 11, where the standard deviations are annualized by multiplying them with $\sqrt{12}$, and the means are annualized by multiplying them with 12. From Panel A and B the benchmark and AR(1) models, have a sharpe ration of 0.43 and 0.57. With the AR(1)-GARCH(1,1) models, I observe that the sharpe ratio value is lower with the one and two level smooth series, compared to the sharpe ratio value obtained from the raw series. Nevertheless, I observe that the highest sharpe ratio value of 0.93, is obtained when the raw mean and three level smooth conditional variance are employed, Panel F.

5. Conclusion

In order to apply the mean-variance portfolio theory in real life, an investor would need to estimate a set of statistical characteristics from the underlying securities in the portfolio of interest. But it has been shown that the noise present in the underlying securities may affect the estimated statistical characteristics and in turn the resulting portfolio allocation strategy.

In this paper I investigated the effect of noise, as noted above, through a set of empirical experiments. I employ the wavelet transform to minimize the effect of noise in the financial series before estimating their statistical characteristics. Afterwards, the adjusted statistics are used to estimate the optimal portfolio weights. Furthermore, as part of my investigation I evaluated the effect of noise reduction when the mean and/or variance matrices are obtained from smooth and/or raw data sets independently. More specifically I investigated four cases of raw variance/raw mean, smooth variance/smooth mean, raw variance/ smooth mean, and smooth variance/raw mean.

My results reveal that the correlation and standard deviation estimates are changed after reducing the noise present in the financial series. In fact, as I observe in the Section 4.2.2 these adjusted statistics affect the allocation of the portfolio and resulting excess return. Over all, I observe that the wavelet smoothing will improve the allocation of the portfolios and in turn increases the investors return, although the improvement or the lack of it, depends on the the level of smoothing used. In the best case, I observe that using the AR(1)-GARCH(1,1) model with the variance estimated from the third level smooth series and the mean estimated from the raw series, results in an excess return of 0.93. On the other hand, using mean and variance values obtained from the raw series results in an excess return of 0.87. Hence with smoothing I have achieved a 6.9 percent increase in the excess return of the portfolio studied. Clearly, this indicates that the accuracy of our variance estimates has improved by the smoothing operation.

Table 1
Summary statistics of the portfolio returns

Entries report the summary statistics of raw and smooth monthly portfolio returns, which are formed on size and ratio of book to market equity. The Returns are smoothed by employing wavelet analysis. Panel A reports the summary statistics of raw monthly portfolio returns. Panel B reports the summary statistics of the one level smooth monthly portfolio returns. Panel C reports the summary statistics of the two level smooth monthly portfolio returns, and Panel D reports summary statistics of the three level smooth monthly portfolio returns. The samples are from July 1926 to August 2006, with a total of 962 observations for each series.

	Small			Big		
	Low	Middle	High	Low	Middle	High
A. Raw						
Mean	0.010	0.013	0.015	0.009	0.010	0.012
St. Dev	0.078	0.071	0.083	0.054	0.058	0.072
Skewness	1.040	1.361	2.248	-0.094	1.363	1.656
Kurtosis	13.46	18.15	25.32	8.220	20.73	21.69
Max	0.656	0.641	0.851	0.325	0.515	0.682
Min	-0.321	-0.309	-0.330	-0.275	-0.280	-0.354
B. One Level Smooth						
Mean	0.010	0.013	0.015	0.009	0.010	0.012
St.Dev	0.061	0.055	0.067	0.039	0.045	0.056
Skewness	1.642	1.928	3.232	0.318	2.694	3.164
Kurtosis	16.97	19.44	34.75	9.97	35.81	39.81
Max	0.497	0.450	0.732	0.294	0.496	0.650
Min	-0.173	-0.150	-0.176	-0.142	-0.178	-0.193
C. Two Level Smooth						
Mean	0.010	0.013	0.015	0.009	0.010	0.012
St.Dev	0.046	0.042	0.049	0.028	0.032	0.041
Skewness	2.115	2.409	2.604	0.049	1.629	2.202
Kurtosis	19.52	23.70	22.43	7.769	21.16	20.00
Max	0.384	0.371	0.410	0.155	0.257	0.318
Min	-0.121	-0.106	-0.117	-0.122	-0.152	-0.128
D. Three Level Smooth						
Mean	0.010	0.013	0.015	0.009	0.010	0.012
St.Dev	0.030	0.027	0.030	0.020	0.023	0.027
Skewness	0.649	0.374	0.399	-0.899	-1.132	-0.209
Kurtosis	8.903	8.631	7.595	5.780	12.167	7.686
Max	0.172	0.150	0.160	0.061	0.104	0.122
Min	-0.081	-0.072	-0.085	-0.079	-0.116	-0.100

Table 2

Correlation matrix of the portfolio returns

Entries report the sample correlation matrices for six monthly portfolio returns, which are formed on the size and ratio of book to market equity. Panel A reports the sample correlation matrix for the six raw monthly portfolio returns. Panel B reports the sample correlation matrix for the six monthly portfolio returns smoothed by one level wavelet decomposition. Panel C reports the sample correlation matrix for the six monthly portfolio returns smoothed by two level wavelet decomposition, and Panel D reports sample correlation matrix for the six monthly portfolio returns smoothed by three level wavelet decomposition. The samples cover the period of July 1926 to August 2006, a total of 962 observations for each series.

	Small			Big		
	Low	Middle	High	Low	Middle	High
A. Raw						
	1					
	0.9516	1				
	0.8995	0.9628	1			
	0.8477	0.8397	0.7836	1		
	0.8163	0.8817	0.8739	0.8905	1	
	0.8038	0.8913	0.9143	0.8208	0.9356	1
B. One Level Smooth						
	1					
	0.9573	1				
	0.9009	0.9656	1			
	0.8516	0.8494	0.80216	1		
	0.8176	0.8893	0.9001	0.8878	1	
	0.8113	0.8976	0.9366	0.8228	0.9419	1
C. Two Level Smooth						
	1					
	0.95971	1				
	0.91434	0.97178	1			
	0.85865	0.85524	0.81605	1		
	0.83743	0.9056	0.91509	0.87762	1	
	0.8219	0.90464	0.94224	0.82875	0.95196	1
D. Three Level Smooth						
	1					
	0.9511	1				
	0.9022	0.9645	1			
	0.8395	0.8211	0.7825	1		
	0.8124	0.8838	0.8933	0.8609	1	
	0.8027	0.8852	0.9250	0.8166	0.9438	1

Table 3

Correlation matrix estimated from the AR(1) model

Entries report the estimated correlation matrices obtained from an AR(1) model, for six monthly portfolio returns, which are formed on the size and ratio of book to the market equity. Panel A reports the estimated correlation matrix for the six raw monthly portfolio returns. Panel B reports the estimated correlation matrix for the six monthly portfolio returns, smoothed by one level wavelet decomposition. Panel C reports the estimated correlation matrix for the six monthly portfolio returns smoothed by two level wavelet decomposition, and Panel D reports the estimated correlation matrix for the six monthly portfolio returns smoothed by three level wavelet decomposition. The samples cover the period of July 1926 to August 2006, a total of 962 observations for each series.

	Small			Big		
	Low	Middle	High	Low	Middle	High
A. Raw						
	1					
	0.9493	1				
	0.8965	0.9600	1			
	0.8489	0.8410	0.7820	1		
	0.8118	0.8778	0.8655	0.8910	1	
	0.8007	0.8889	0.9082	0.8220	0.9345	1
B. One Level Smooth						
	1					
	0.9585	1				
	0.9006	0.9653	1			
	0.8502	0.8579	0.8121	1		
	0.8105	0.8869	0.8996	0.8997	1	
	0.8072	0.8960	0.9344	0.8359	0.9432	1
C. Two Level Smooth						
	1					
	0.9662	1				
	0.9272	0.9770	1			
	0.8837	0.8795	0.8508	1		
	0.8645	0.9208	0.9361	0.8902	1	
	0.8453	0.9174	0.9513	0.8510	0.9642	1
D.Three Level Smooth						
	1					
	0.9496	1				
	0.8894	0.9580	1			
	0.8738	0.8432	0.8256	1		
	0.7884	0.8646	0.9001	0.8472	1	
	0.7706	0.8652	0.9193	0.8143	0.9465	1

Table 4
Correlation matrix estimated from the AR(1)-GARCH(1,1)

Entries report the estimated correlation matrices obtained from an AR(1)-GARCH(1,1) model, for six monthly portfolio returns, which are formed on the size and ratio of book to the market equity. Panel A reports the estimated correlation matrix for the six raw monthly portfolio returns. Panel B reports the estimated correlation matrix for the six monthly portfolio returns, smoothed by one level wavelet decomposition. Panel C reports the estimated correlation matrix for the six monthly portfolio returns smoothed by two level wavelet decomposition, and Panel D reports the estimated correlation matrix for the six monthly portfolio returns smoothed by three level wavelet decomposition. The samples cover the period of July 1926 to August 2006, a total of 962 observations for each series.

	Small			Big		
	Low	Middle	High	Low	Middle	High
A. Raw						
	1					
	0.9498	1				
	0.8971	0.9604	1			
	0.8495	0.8416	0.7831	1		
	0.8169	0.8811	0.8698	0.8906	1	
	0.8035	0.8902	0.9102	0.8212	0.9354	1
B. One Level Smooth						
	1					
	0.9584	1				
	0.9007	0.9655	1			
	0.8506	0.8570	0.8109	1		
	0.8118	0.8875	0.8994	0.8979	1	
	0.8079	0.8963	0.9345	0.8342	0.9430	1
C. Two Level Smooth						
	1					
	0.9650	1				
	0.9247	0.9751	1			
	0.8788	0.8760	0.8432	1		
	0.8596	0.9186	0.9310	0.8888	1	
	0.8415	0.9154	0.9494	0.8470	0.9619	1
D. Three Level Smooth						
	1					
	0.9498	1				
	0.8906	0.9584	1			
	0.8689	0.8398	0.8170	1		
	0.7890	0.8647	0.8972	0.8436	1	
	0.7565	0.8467	0.9039	0.7858	0.9363	1

Table 5

Summary statistics of the standard deviations obtained from the AR(1)-GARCH(1,1) model

Entries report the summary statistics of the estimated standard deviations from the AR(1)-GARCH(1,1) model, while the raw and the smoothed monthly portfolio returns are used. Panel A reports the summary statistics of the estimated standard deviations, obtained from the raw monthly portfolio returns. Panel B reports the summary statistics of the estimated standard deviations, obtained from the one level smooth monthly portfolio returns. Panel C reports the summary statistics of the estimated standard deviations, obtained from the two level smooth monthly portfolio returns. Panel D reports the summary statistics of the estimated standard deviations, obtained from the three level smooth monthly portfolio returns. The samples cover the period of July 1926 to August 2006, a total of 962 observations for each series.

	Small			Big		
	Low	Middle	High	Low	Middle	High
A. Raw						
Mean	0.0715	0.0621	0.0697	0.0508	0.0498	0.0599
St.Dev	0.0337	0.0332	0.0436	0.0208	0.0295	0.0386
Min	0.0346	0.0322	0.0329	0.0260	0.0254	0.0298
Max	0.2971	0.2937	0.3489	0.1695	0.2653	0.3352
Auto	0.9665	0.9743	0.9782	0.9660	0.9785	0.9782
B. One Level Smooth						
Mean	0.0468	0.0403	0.0473	0.0312	0.0309	0.0387
St.Dev	0.0239	0.0238	0.0363	0.0139	0.0229	0.0298
Min	0.0233	0.0209	0.0204	0.0181	0.0164	0.0183
Max	0.2095	0.2131	0.3394	0.1466	0.2525	0.2906
Auto	0.9686	0.9718	0.9748	0.9466	0.9604	0.9756
C. Two Level Smooth						
Mean	0.0283	0.0244	0.0274	0.0170	0.0167	0.0216
St.Dev	0.0169	0.0160	0.0214	0.0071	0.0124	0.0167
Min	0.0165	0.0141	0.0144	0.0118	0.0103	0.0124
Max	0.1802	0.1713	0.2138	0.0720	0.1214	0.1579
Auto	0.9454	0.9583	0.9573	0.9432	0.9608	0.9621
D. Three Level Smooth						
Mean	0.0139	0.0119	0.0133	0.0089	0.0095	0.0115
St.Dev	0.0037	0.0039	0.0057	0.0023	0.0050	0.0099
Min	0.0102	0.0078	0.0072	0.0067	0.0053	0.0068
Max	0.0334	0.0308	0.0366	0.0237	0.0449	0.1553
Auto	0.9732	0.9815	0.9874	0.9640	0.9802	0.3833

Table 6

Portfolio weights computed from the benchmark model

This table reports the estimated portfolio weights for the six monthly portfolios (i.e. they are formed on the size and ratio of book to the market equity) and a risk free-asset (i.e. T-Bill), where the benchmark model is used. Optimal weights are estimated by:

$$W = \gamma^{-1}\Sigma^{-1}(\mu_i - r_f I)$$

where μ_i denotes the mean, and Σ denotes the covariance matrix. In this model μ_i and Σ are substituted by sample mean and sample covariance matrix respectively. r_f is return of the risk free-asset and is substituted by sample mean of the risk-free asset. γ is the risk aversion coefficient and is assumed to be equal to the summation of the absolute value of the weights.

Small			Big			T.Bill
Low	Middle	High	Low	Middle	High	
-0.1615	0.2383	-0.0965	0.2123	0.1741	-0.1172	0.7506

Table 7
Summary statistics of the portfolio weights computed from the AR(1)
This table reports the estimated portfolio weights for the six monthly portfolios (i.e. they are formed on the size and ratio of book to the market equity) and a risk free-asset (i.e. T-Bill). Optimal weights are estimated by:

$$W = \gamma^{-1}\Sigma^{-1}(\mu_i - r_f I)$$

where μ_i denotes the mean, and Σ denotes the covariance matrix. Where they are obtained form the AR(1) model. r_f is return of the risk free-asset. γ is the risk aversion coefficient and is assumed to be equal to the summation of the absolute value of the weights.

	Small			Big			T.Bill
	Low	Middle	High	Low	Middle	High	
Mean	-0.1136	0.1362	0.0495	0.0428	-0.0549	-0.0219	0.9619
Std	0.2221	0.2006	0.1249	0.1763	0.2132	0.1407	0.0614
Max	0.5824	0.5564	0.3741	0.4639	0.5744	0.4843	1.1647
Min	-0.4771	-0.4388	-0.3299	-0.4017	-0.5286	-0.4120	0.6893

Table 8
Summary statistics of the portfolio weights, computed form the AR(1)-GARCH(1,1) model and one level smooth series
This table reports the estimated portfolio weights for the six monthly portfolios (i.e. they are formed on the size and ratio of book to the market equity) and a risk free-asset (i.e. T-Bill). Optimal weights are estimated by:

$$W = \gamma^{-1}\Sigma^{-1}(\mu_i - r_f I)$$

where μ_i denotes the mean, and Σ denotes the covariance matrix. The AR(1)-GARCH(1) model is employed to estimate the conditional mean and the conditional covariance matrix while using both the raw monthly portfolio returns and the one level smooth returns. r_f is return of risk free-asset. γ is the risk aversion coefficient and is assumed to be summation of absolute value of the weights. The weights are estimated with different combinations of raw mean, raw variance and smooth mean, smooth variance.

	Small			Big			T.Bill
	Low	Middle	High	Low	Middle	High	
A. Raw Mean and Variance							
Mean	-0.1669	0.1115	0.1207	0.0661	-0.0677	0.0584	0.8778
St.Dev	0.1503	0.2278	0.1614	0.1135	0.1679	0.1862	0.0743
Max	0.3383	0.5833	0.5428	0.4116	0.4916	0.6441	1.1737
Min	-0.4887	-0.4471	-0.3732	-0.3548	-0.5868	-0.4037	0.6081
B. Smooth Mean and Variance							
Mean	-0.0974	0.0939	0.0731	0.0328	-0.0076	-0.0313	0.9364
St.Dev	0.1392	0.2321	0.2046	0.1347	0.2090	0.1983	0.0791
Max	0.3645	0.5586	0.5138	0.4538	0.5793	0.4994	1.1697
Min	-0.3915	-0.4688	-0.3752	-0.3501	-0.4823	-0.5053	0.6692
C. Smooth Mean, Raw Variance							
Mean	-0.1122	0.0965	0.0883	0.0274	0.0045	-0.0363	0.9317
St.Dev	0.1534	0.2183	0.1754	0.1509	0.2188	0.1954	0.0809
Max	0.3570	0.5479	0.5253	0.4798	0.5689	0.5325	1.1790
Min	-0.4715	-0.4598	-0.4503	-0.4889	-0.5070	-0.5037	0.6902
D. Raw Mean, Smooth Variance							
Mean	-0.1433	0.1185	0.0803	0.0634	-0.0721	0.0618	0.8913
St.Dev	0.1425	0.2240	0.1998	0.1085	0.1678	0.2001	0.0774
Max	0.4245	0.5692	0.5215	0.3735	0.4397	0.5693	1.1626
Min	-0.4301	-0.4701	-0.3879	-0.2959	-0.4709	-0.3624	0.6128

Table 9
Summary statistics of the portfolio weights, computed form the AR(1)-GARCH(1,1) model and two level smooth series
This table reports the estimated portfolio weights for the six monthly portfolios (i.e. they are formed on the size and ratio of book to the market equity) and a risk free-asset (i.e. T-Bill). Optimal weights are estimated by:

$$W = \gamma^{-1}\Sigma^{-1}(\mu_i - r_f I)$$

where μ_i denotes the mean, and Σ denotes the covariance matrix. The AR(1)-GARCH(1) model is employed to estimate the conditional mean and the conditional covariance matrix while using both the raw monthly portfolio returns and the two level smooth returns. r_f is return of risk free-asset. γ is the risk aversion coefficient and is assumed to be summation of absolute value of the weights. The weights are estimated with different combinations of raw mean, raw variance and smooth mean, smooth variance.

	Small			Big			T.Bill
	Low	Middle	High	Low	Middle	High	
A. Raw Mean and Variance							
Mean	-0.1669	0.1115	0.1207	0.0662	-0.0677	0.0584	0.8778
St.Dev	0.1503	0.2278	0.1614	0.1135	0.1679	0.1862	0.0743
Max	0.3383	0.5833	0.5428	0.4116	0.4916	0.6441	1.1737
Min	-0.4887	-0.4471	-0.3732	-0.3548	-0.5868	-0.4037	0.6081
B. Smooth Mean and Variance							
Mean	-0.1335	0.0570	0.07618	0.0963	-0.0403	0.0057	0.9386
St.Dev	0.1397	0.1818	0.16416	0.1457	0.2464	0.1999	0.0938
Max	0.3310	0.4795	0.47533	0.5904	0.5284	0.5406	1.2214
Min	-0.4713	-0.4742	-0.45761	-0.4947	-0.5392	-0.4419	0.5643
C. Smooth Mean, Raw Variance							
Mean	-0.1381	0.0649	0.0964	0.1149	-0.0935	0.0290	0.9264
St.Dev	0.1246	0.1972	0.1352	0.1744	0.2141	0.1783	0.0787
Max	0.2643	0.5346	0.4366	0.5376	0.5713	0.5139	1.2443
Min	-0.4432	-0.5189	-0.3099	-0.4994	-0.5116	-0.4923	0.5855
D. Raw Mean, Smooth Variance							
Mean	-0.1518	0.0900	0.0926	0.0340	0.0201	0.0147	0.9004
St.Dev	0.1782	0.2167	0.2008	0.0972	0.2052	0.2094	0.0924
Max	0.4637	0.5512	0.5221	0.4289	0.7404	0.5447	1.1730
Min	-0.4769	-0.4626	-0.4736	-0.3935	-0.5858	-0.3810	0.4841

Table 10

Summary statistics of the portfolio weights, computed form the AR(1)-GARCH(1,1) model and three level smooth series

This table reports the estimated portfolio weights for the six monthly portfolios (i.e. they are formed on the size and ratio of book to the market equity) and a risk free-asset (i.e. T-Bill). Optimal weights are estimated by:

$$W = \gamma^{-1}\Sigma^{-1}(\mu_i - r_f I)$$

where μ_i denotes the mean, and Σ denotes the covariance matrix. The AR(1)-GARCH(1) model is employed to estimate the conditional mean and the conditional covariance matrix while using both the raw monthly portfolio returns and the three level smooth returns. r_f is return of risk free-asset. γ is the risk aversion coefficient and is assumed to be summation of absolute value of the weights. The weights are estimated with different combinations of raw mean, raw variance and smooth mean, smooth variance.

	Small			Big			T.Bill
	Low	Middle	High	Low	Middle	High	
A. Raw Mean and Variance							
Mean	-0.1669	0.1115	0.1207	0.0662	-0.0677	0.0584	0.8778
St.Dev	0.1503	0.2278	0.1614	0.1135	0.1679	0.1862	0.0743
Max	0.3384	0.5833	0.5429	0.4116	0.4916	0.6442	1.1737
Min	-0.4887	-0.4471	-0.3732	-0.3548	-0.5868	-0.4037	0.6081
B. Smooth Mean and Variance							
Mean	-0.1781	0.1158	0.0987	0.1174	-0.0243	-0.0204	0.8909
St.Dev	0.1393	0.1888	0.1547	0.1427	0.1794	0.1789	0.0903
Max	0.3654	0.5457	0.4560	0.4484	0.5085	0.5744	1.2395
Min	-0.4405	-0.4209	-0.4038	-0.3190	-0.4496	-0.4854	0.5958
C. Smooth Mean, Raw Variance							
Mean	-0.1592	0.0339	0.1321	0.1407	-0.1019	0.0223	0.9318
St.Dev	0.1196	0.1874	0.1445	0.1497	0.1841	0.1550	0.0705
Max	0.2956	0.5246	0.4548	0.4091	0.48543	0.4663	1.2626
Min	-0.4953	-0.5364	-0.4287	-0.3733	-0.4767	-0.4906	0.6284
D. Raw Mean, Smooth Variance							
Mean	-0.1567	0.1420	0.0934	0.0269	0.0224	0.0208	0.8510
St.Dev	0.1698	0.21097	0.1887	0.1097	0.1763	0.2091	0.0900
Max	0.4197	0.57761	0.5576	0.4212	0.7691	0.6980	1.1392
Min	-0.4418	-0.43759	-0.5371	-0.3891	-0.3902	-0.3229	0.4617

Table 11
Summary statistics of excess returns of the portfolio
Entries report the summary statistics of excess returns, ER. Where ER is computed as:

$$ER_t = R_{pt} - R_{ft} \quad t = 1, 2, .., n.$$

R_{pt} denotes the portfolio returns and is forecasted from the benchmark, AR(1) and AR(1)-GARCH(1,1) models. Panel A reports summary statistics of excess returns when the portfolio weights are obtained from the benchmark model. Panel B reports summary statistics of excess returns when the portfolio weights are obtained from the AR(1) model. Panel C reports summary statistics of excess returns when the portfolio weights are obtained from the AR(1)-GARCH(1,1) model. Panels D, E, F reports summary statistics of excess returns when the portfolio weights are obtained from the AR(1)-GARCH(1,1) model using the one, two, and three level smooth returns. The ratio of annual mean to annual standard deviation is also reported,where the standard deviations are annualized by multiplying them with $\sqrt{12}$, and the means are annualized by multiplying them with 12. Furthermore, I have denoted Raw as R and Smooth as S.

Model	Mean	St.D	Skew	Kurt	Max	Min	An. Mean	An.St.D	Sharpe
A. Benchmark:	0.0015	0.0119	-0.0856	9.717	0.0932	-0.0691	0.0181	0.0415	0.4367
B. AR(1)	0.0016	0.0098	1.0772	17.763	0.0917	-0.0540	0.0195	0.0340	0.5734
C.1 AR-GARCH.Raw:									
R mean/R Var	0.0026	0.0101	2.0694	28.048	0.1129	-0.0390	0.0307	0.0349	0.8775
C.2 AR-GARCH.One level:									
S Mean/S Var	0.0019	0.0096	-0.9205	12.69	0.0605	-0.0629	0.0234	0.0333	0.7030
R Mean/S Var	0.0022	0.0095	2.2796	28.14	0.1063	-0.0295	0.0269	0.0329	0.8192
S Mean/R Var	0.0023	0.0109	-0.2603	15.32	0.0877	-0.0719	0.0276	0.0378	0.7321
C.3 AR-GARCH.Two level:									
S Mean/S Var	0.0014	0.0091	-0.1133	6.652	0.0522	-0.0452	0.0176	0.0317	0.5557
Raw Mean/S Var	0.0019	0.0092	0.1319	4.421	0.0415	-0.0309	0.0239	0.0318	0.7530
S Mean/R Var	0.0016	0.0098	-1.0861	16.867	0.0679	-0.0688	0.0203	0.0342	0.5960
C.4 AR-GARCH.Three level:									
S Mean/S Var	0.0023	0.00911	0.0362	5.6042	0.0403	-0.0370	0.0285	0.0315	0.9049
R Mean/S Var	0.0026	0.0098	0.3156	6.9502	0.0577	-0.0355	0.0316	0.0339	0.9301
S Mean/R Var	0.0018	0.0088	-0.2093	8.2988	0.0441	-0.0539	0.0222	0.0308	0.7213

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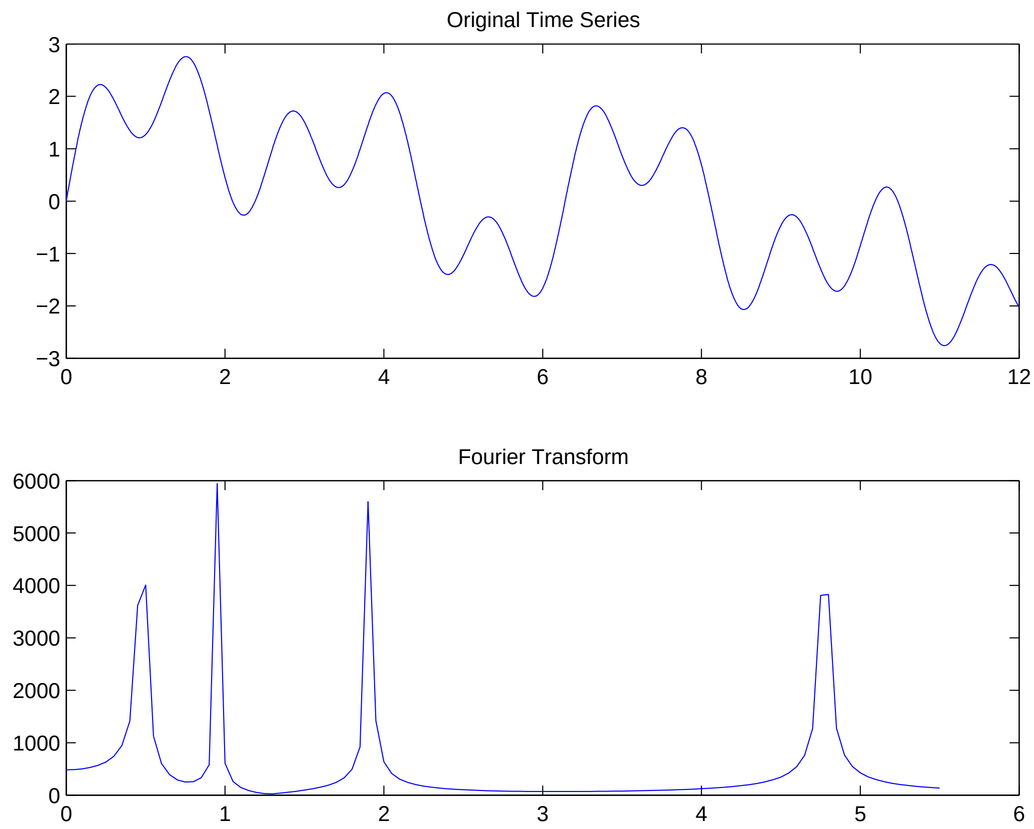


Figure 1
Fourier transform of a complex time series.
Top panel plots the time series x(t), which is generated by the superposition of a number of sine waves with different frequencies::

$$x(t) = sin(t) + sin(2t) + sin(.5t) + sin(5t), \qquad t = 0, 1, 2, 3..., 12.$$

The bottom panel presents the time series x(t) in the fourier domain, from which we observe four peaks corresponding to the four frequency components making up x(t).

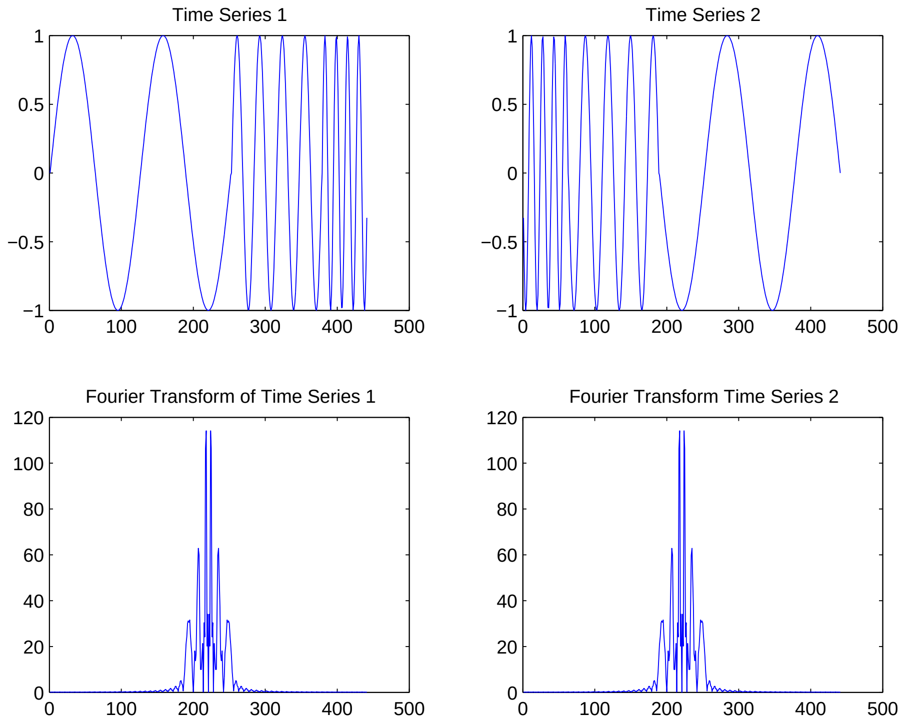


Figure 2

Different time series with same frequency representation.

Top left side panel time series is generated by:

$$x(t) = \begin{cases} sin(t) & 0 \leq t < 4\Pi \\ sin(4t) & \Pi \leq t < 6\Pi \\ sin(8t) & \Pi \leq t < 7\Pi \end{cases}$$

and top right panel time series is generated by:

$$x(t) = \begin{cases} sin(t) & \Pi \leq t < 7\Pi \\ sin(4t) & \Pi \leq t < 6\Pi \\ sin(8t) & 0 \leq t < 4\Pi \end{cases}$$

The two panels at the bottom plot the Fourier representation for each time series above them. It is clearly observed that although the two time series are easily distinguishable, the frequency representation of the two are the same.

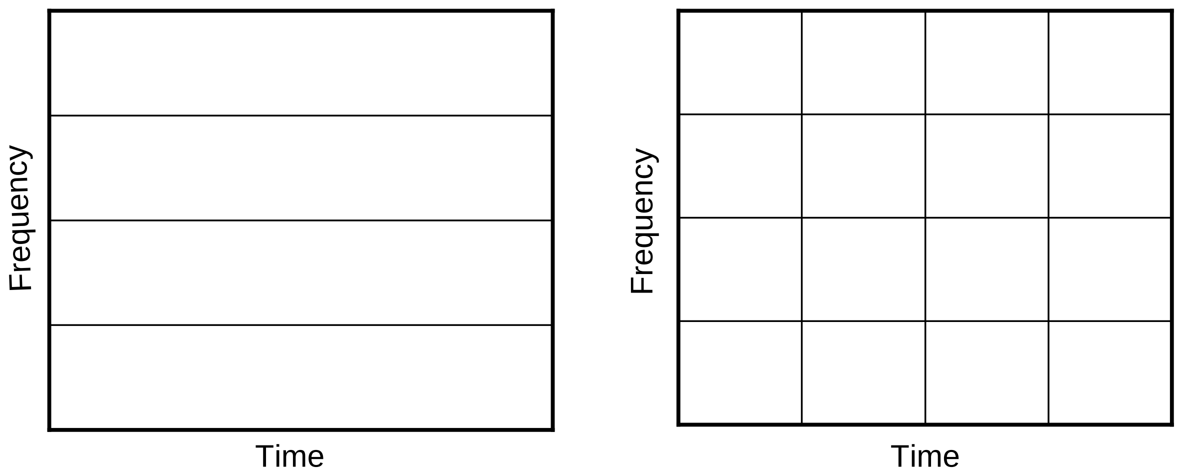
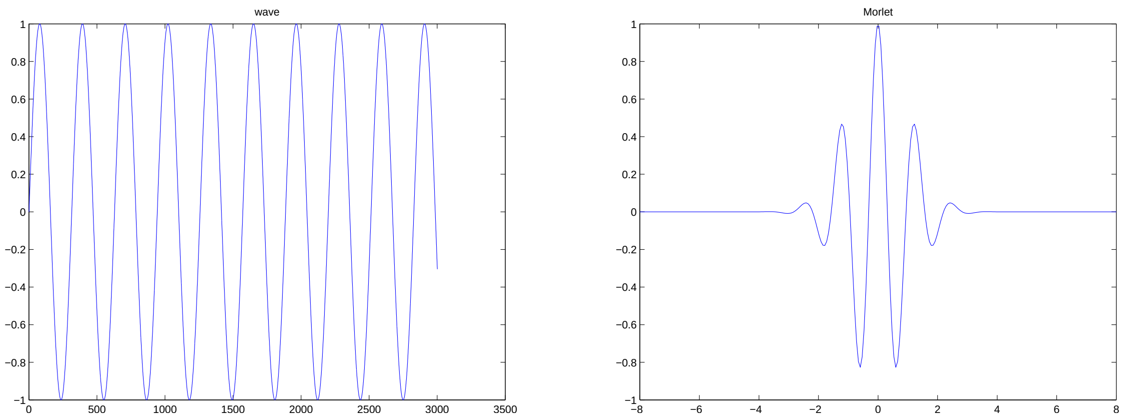


Figure 3

Fourier transform vs. windowed Fourier transform

(a) Represents the frequency domain after applying the Fourier transform to a times series, in which there is perfect frequency resolution. **(b)** Represents the frequency domain of the windowed Fourier transform, in which there is time and frequency resolution.



(a) (b)

Figure 4
Wave vs. wavelet
(a) Plots a wave with infinite energy while **(b)** Plots a wavelet, with finite energy.

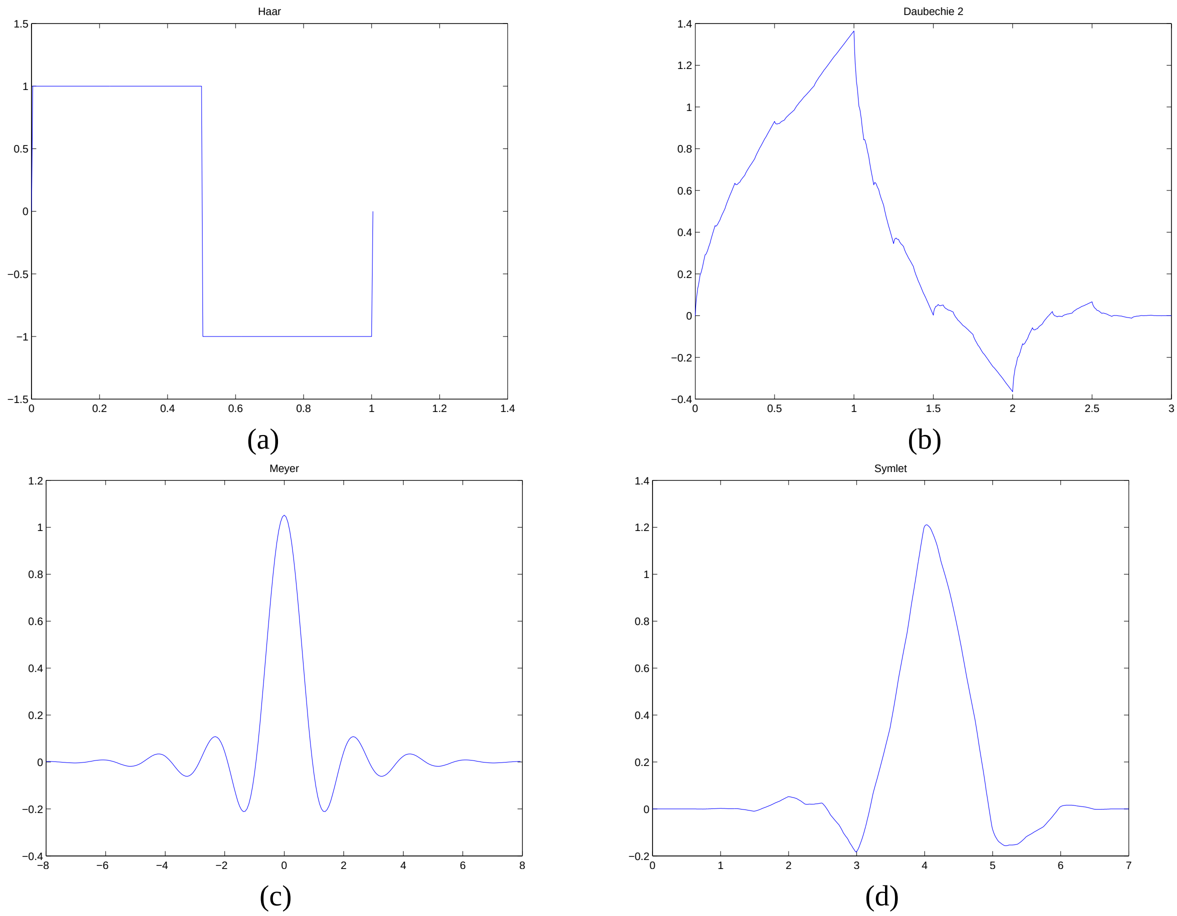


Figure 5
Mother wavelets
(a) Plots the Haar wavelet, which is an orthogonal, symmetric, and discontinuous wavelet.**(b)** Plots the Daubechie2 wavelet, which is an orthogonal, asymmetric, and compacted wavelet.**(c)** Plots the orthogonal symmetric Meyer wavelet, and **(d)** plots the Symlet wavelet which is orthogonal and nearly symmetric.

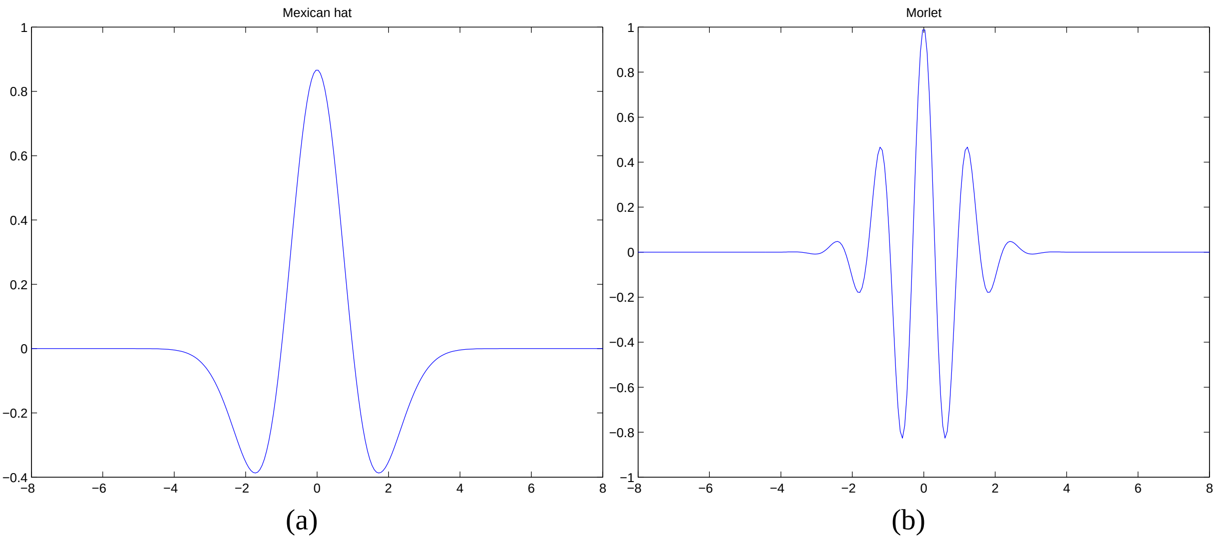


Figure 6
Continuous mother wavelets
(a) The Morlet wavelet. **(b)**The Mexican Hat wavelet.

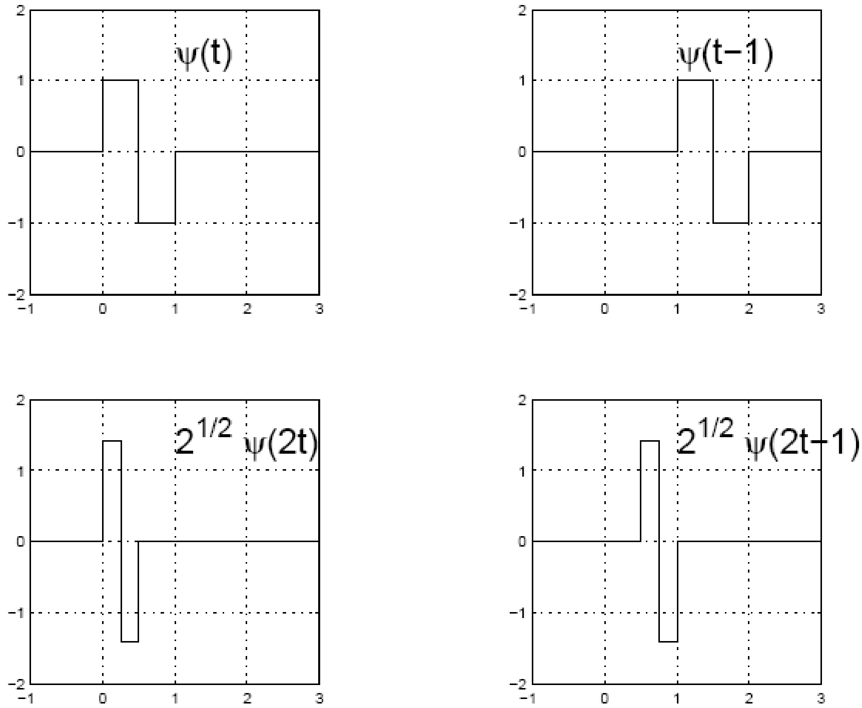


Figure 7

Translation and dilation version of the Haar wavelet

These figures represent different versions of the Haar wavelet obtained by translation and dilation of the Haar mother wavelet.(Selesnick 2005)

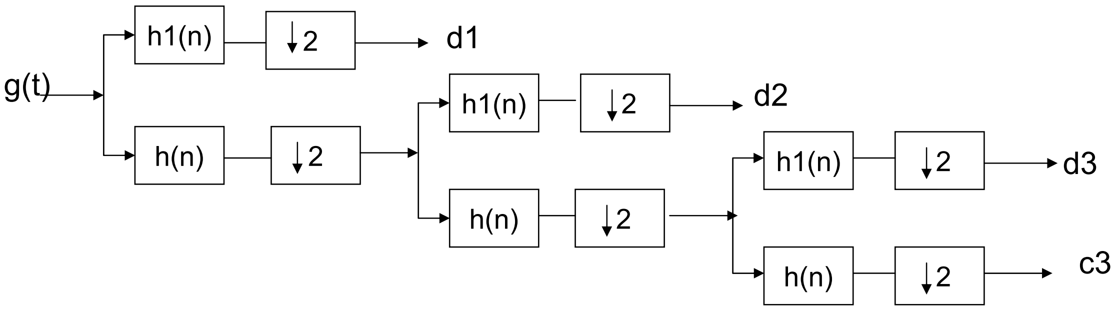


Figure 8

Decomposition of a time series with filter banks.

A time series $g(t)$, is passed through two filters, a low pass and a high pass filter denoted by $h1(n)$ and $h(n)$ respectively. Afterwards, the output of each filter is down-sampled. Through this process, the original time series is decomposed into two components, each half the size of the original time series. The component obtained through the high pass filter, denoted by $d1$, contains the details from the original time series. The component obtained through the low pass filter represents the smoothed version of the original time series. This filtering and down-sampling process is reiterated, each time feeding the smoothed time series as the input to the next stage of low and high pass filters. Each iteration is also called a level. Hence a three level decomposition of a time series is shown here. Where c3 denotes the approximate coefficients at level three and d1, d2, and d3 denote the detail coefficients at levels one, two and three in that order.

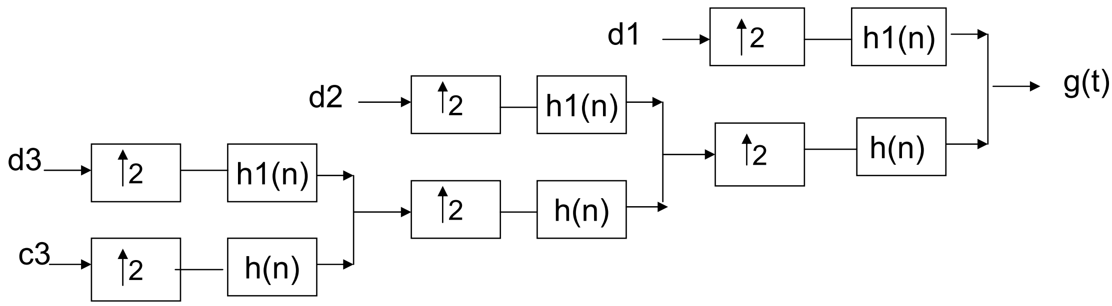


Figure 9

Reconstruction of a time series with filter banks.

Reversing the decomposition process, the last level approximate, $c3$, and detail, $d3$, coefficients are fist up-sampled and then convolved with the low pass filter h(n) and high pass filter h1(n) in that order. The resulting signal is then used as the smooth signal to the next level of up-sampling operations and filters. After reiterating this process, the original time series is reconstructed. The number of iterations depends on the level of decomposition conducted on the original time series $g(t)$. This Figure presents the reconstruction process by filter banks for a time series that was initially decomposed by a three level filter bank.

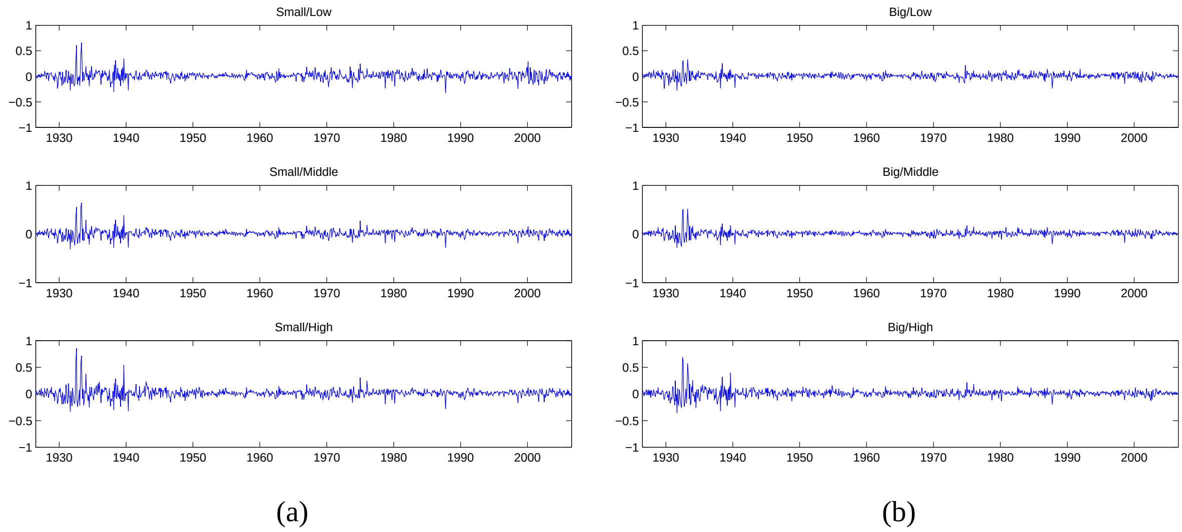


Figure 10
Monthly portfolio returns
(a) Top panel plots the Small/Low monthly portfolio returns. Second panel plots the f Small/Middle monthly portfolio returns. Bottom panel plots the Small/High monthly portfolio returns. **(b)** Top panel plots the Big/Low monthly portfolio returns. Second panel plots the Big/Middle monthly portfolio returns and bottom panel plots the Small/High monthly monthly returns. All samples are from the period of July 1926 through August 2006, a total of 962 observations.

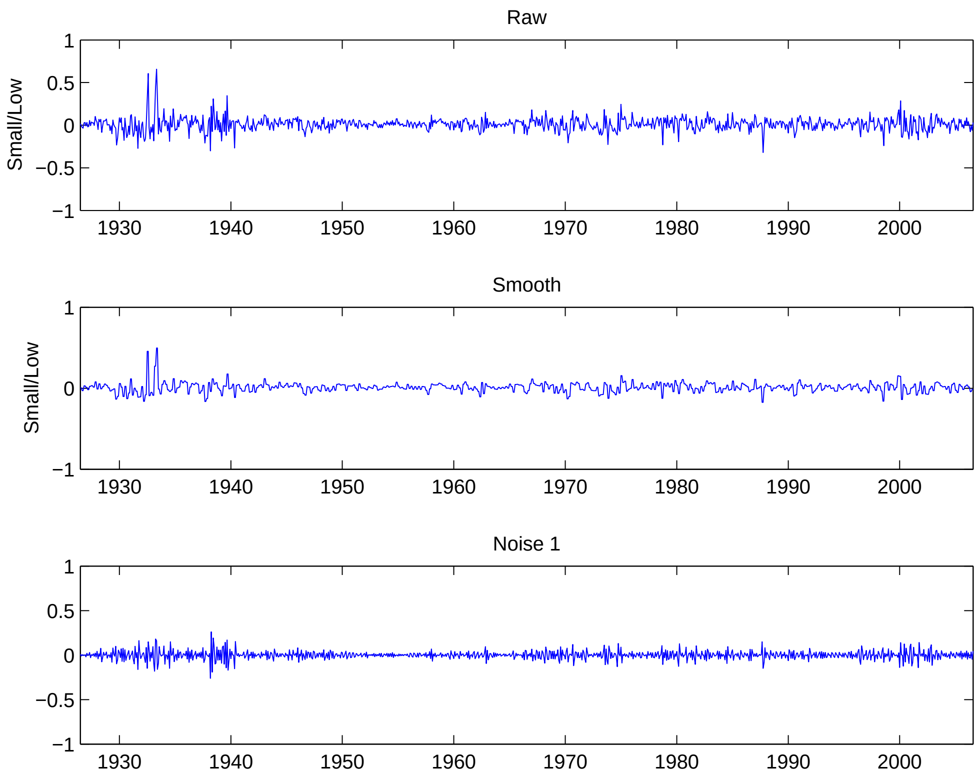


Figure 11
Multiresolution analysis of the Small/Low monthly portfolio returns at one level wavelet decomposition
Multiresolution analysis of the Small/Low monthly portfolio returns using the Haar wavelet for the period of July 1926 until September 2006. The first panel plots the raw Small/Low monthly portfolio returns, and the second panel plots the one level smooth Small/Low monthly portfolio returns. The last panel plots the noise extracted from the raw series (i.e. the residual).

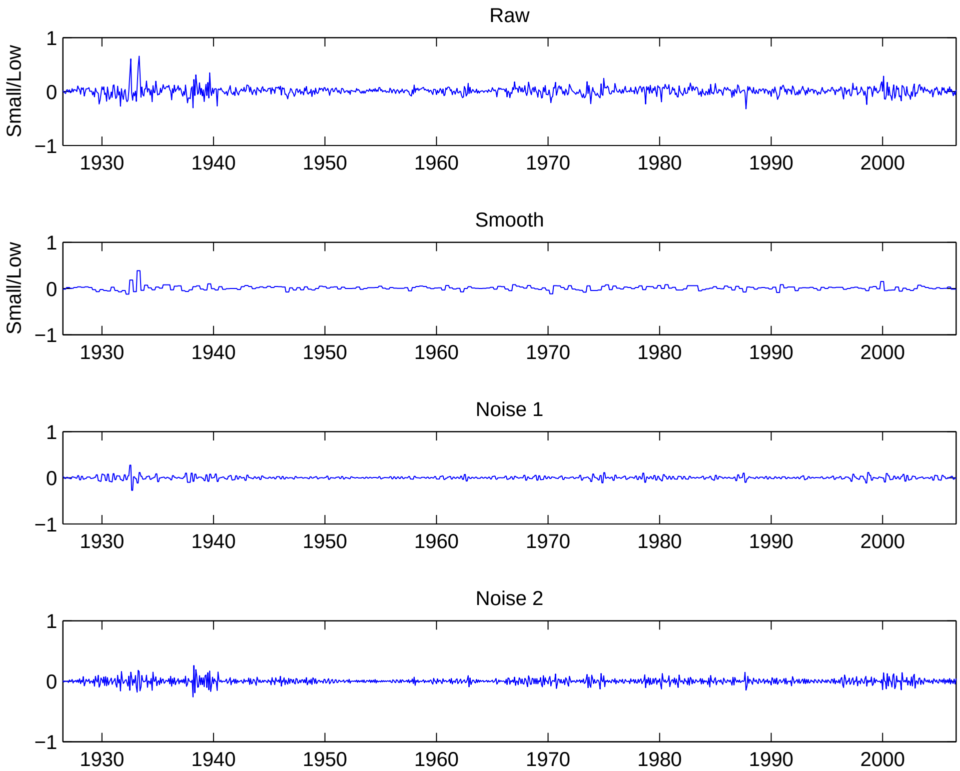


Figure 12
Multiresolution analysis of the Small/Low monthly portfolio returns at two level wavelet decomposition
Multiresolution analysis of the Small/Low monthly portfolio returns using the Haar wavelet for the period of July 1926 till September 2006. Top panel plots the raw Small/Low monthly portfolio return, and second panel plots two level smooth monthly portfolio returns, the last two panels plots noise extracted (i.e. the residual), at the first and second levels.

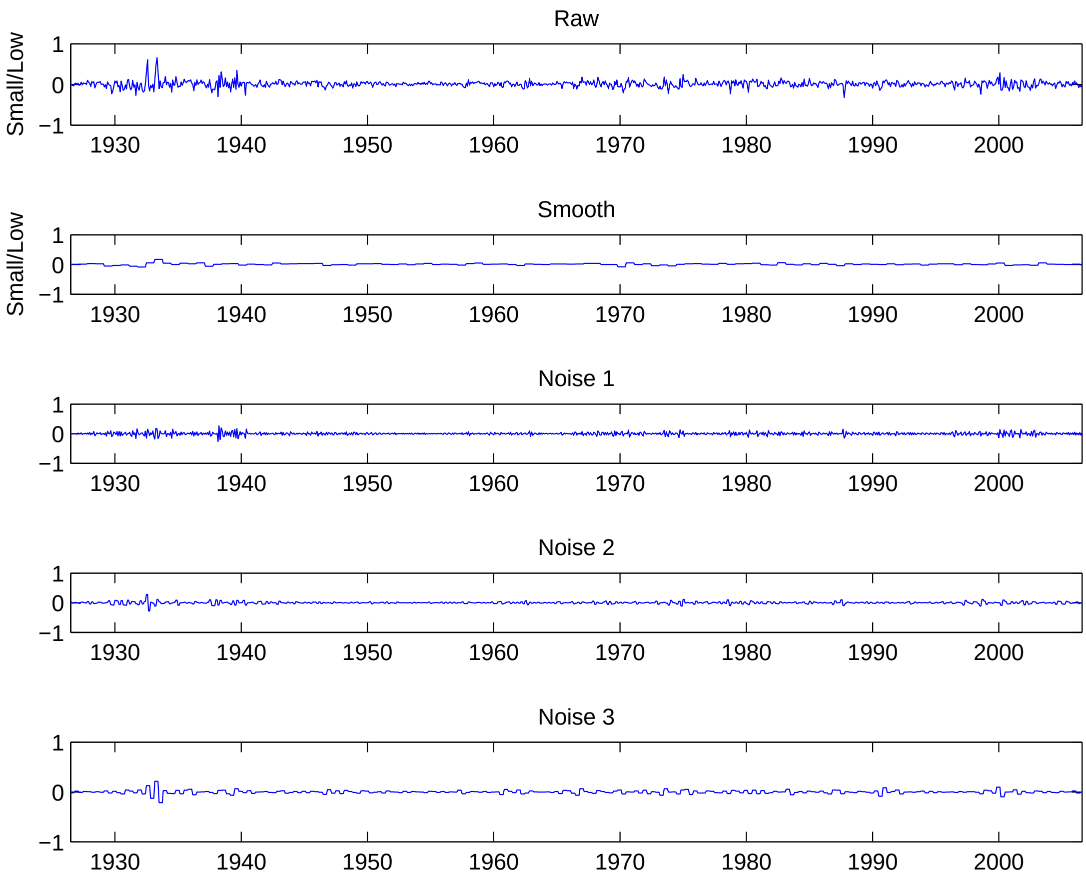
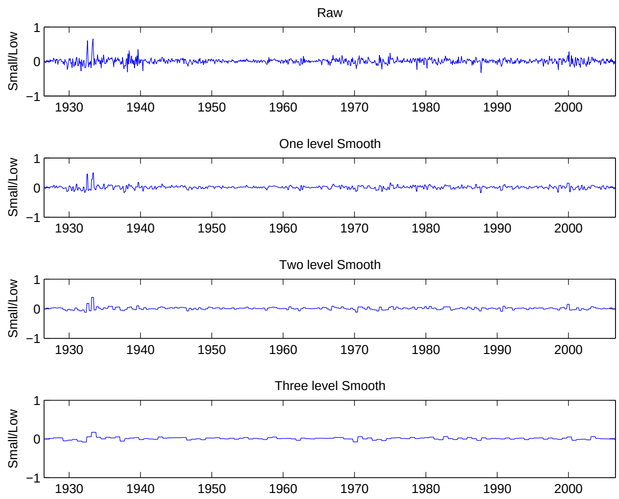


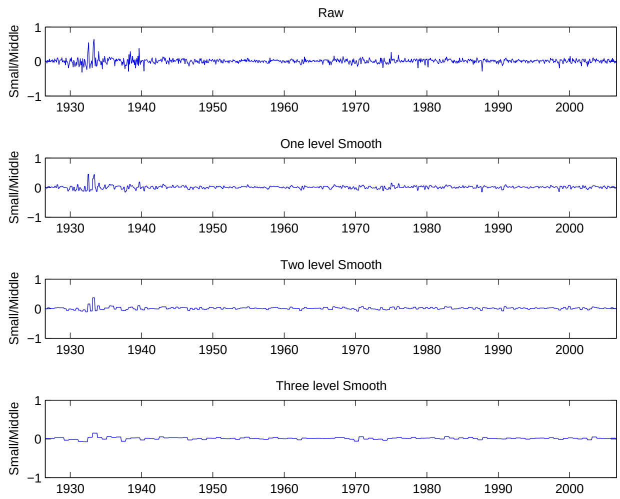
Figure 13

Multiresolution analysis of the Small/Low monthly portfolio returns at three level wavelet decomposition

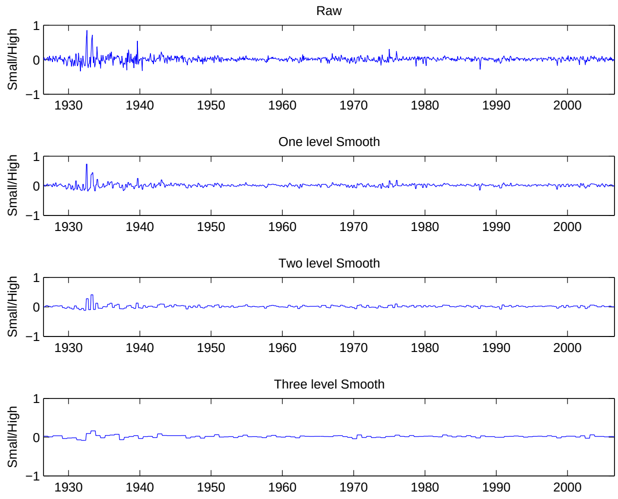
Multiresolution analysis of the Small/Low monthly portfolio returns using the Haar wavelet for the period of July 1926 till September 2006. Top panel plots the the raw monthly portfolio returns, the second panel plots the third level smooth monthly portfolio returns. The third panel plots the noise extracted (i.e. the residual) from the first level of decomposition, the fourth panel plots the noise extracted from the second level of decomposition , and the last panel plots noise extracted from the third level of decomposition.



(a)



(b)



(c)

Figure 14

Raw and smooth Small monthly portfolio returns

(a) Top panel plots the raw Small/Low monthly portfolio returns. Second panel plots the one level smooth Small/Low monthly portfolio returns. Third panel plots the two level smooth Small/Low monthly returns and the last panel plots the three level smooth Small/High monthly portfolio returns.**(b)** Top panel plots the raw Small/Middle monthly portfolio returns. Second panel plots the one level smooth Small/Middle monthly portfolio returns. Third panel plots the two level smooth Small/Middle monthly portfolio returns and last panel plots the three level smooth Small/Middle monthly portfolio returns. **(c)** Top panel plots the raw Small/High monthly portfolio returns. Second panel plots the one level smooth Small/Middle monthly portfolio returns. Third panel plots the two level smooth Small/High monthly portfolio returns and last panel plots the three level smooth Small/High monthly portfolio returns. All samples cover the period of July 1926 through August 2006, a total of 962 observations.

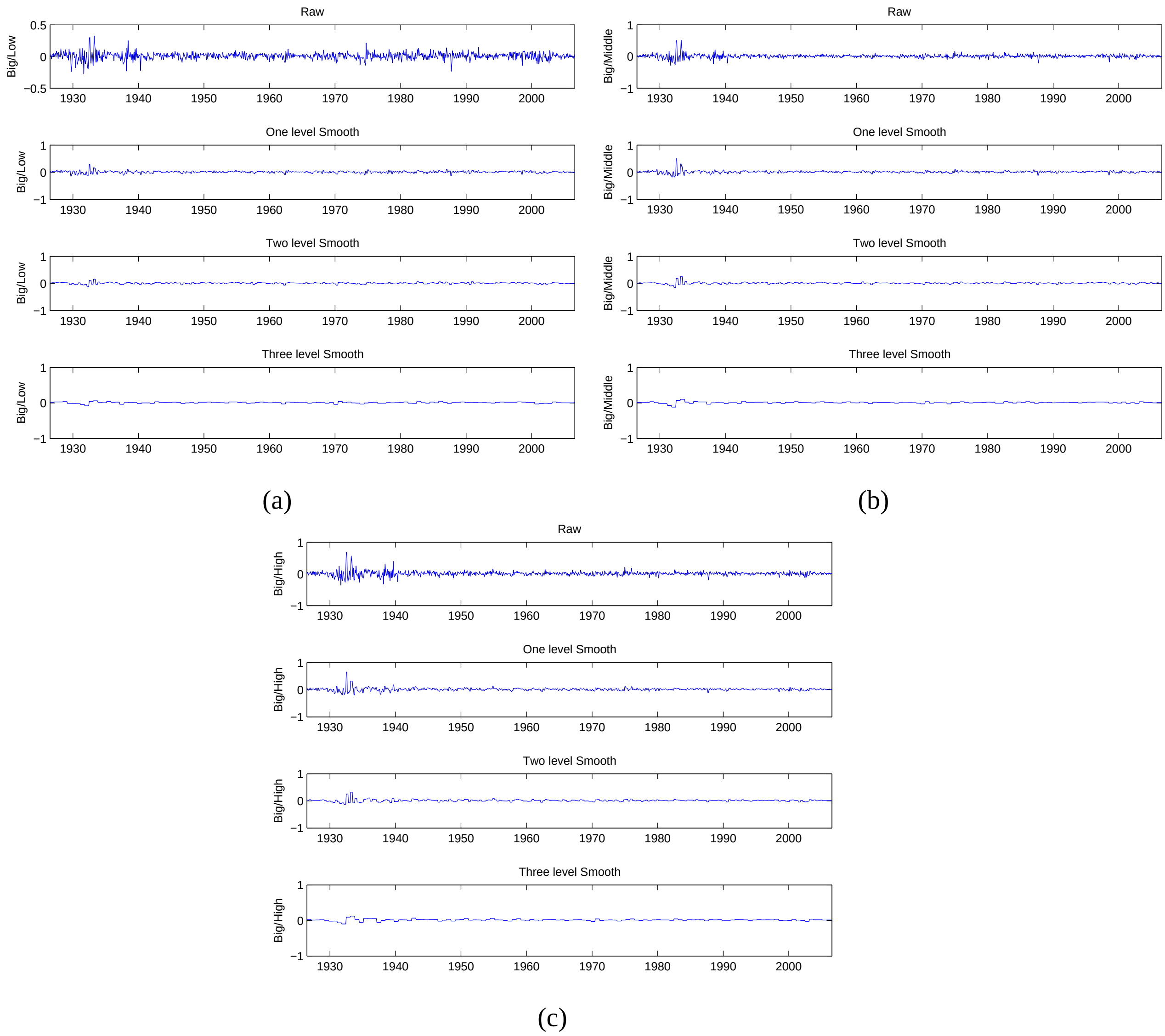


Figure 15

Raw and smooth Big monthly portfolio returns

(a) Top panel plots the raw Big/Low monthly portfolio returns. Second panel plots the one level smooth Big/Low monthly portfolio returns. Third panel plots the two level smooth Big/Low monthly portfolio returns and the last panel plots the three level smooth Big/High monthly portfolio returns.**(b)**Top panel plots the raw Big/Middle monthly returns. Second panel plots the one level smooth Big/Middle monthly portfolio returns. Third panel plots the two level smooth Big/Middle monthly portfolio returns and last panel plots the three level smooth Big/Middle monthly portfolio returns. **(c)** Top panel plots the raw Big/High monthly portfolio returns. Second panel plots the one level smooth Big/Middle monthly portfolio returns. Third panel plots the two level smooth Big/High monthly portfolio returns and last panel plots the three level smooth Big/High monthly portfolio returns. All samples cover the period of July 1926 through August 2006, a total of 962 observations.