

# Wavelet analysis of stock returns and aggregate economic activity

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## Abstract

The relationship between stock market returns and economic activity is investigated using signal decomposition techniques based on wavelet analysis. After the application of the maximum overlap discrete wavelet transform (MODWT) to the DJIA stock price index and the industrial production index for the US over the period 1961:1–2006:10 wavelet variance and cross-correlations analyses are used to investigate the scaling properties of the series and the lead/lag relationship between them at different time scales. The results show that stock market returns tend to lead the level of economic activity, but only at the highest scales (lowest frequencies) corresponding to periods of 16 months and longer, and that the leading period increases as the wavelet time scale increases.

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**Keywords:** Stock markets; Industrial production; Wavelet variance; Wavelet cross-correlation

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## 1. Introduction

According to the discounted cash flow valuation model firms' stock prices tend to reflect investors' expectations about future corporate earnings. Thus, if corporate profits reflect overall economic conditions, i.e. they are highly procyclical, at aggregate level stock price changes may be expected to be related to future economic activity. Following the predictions of the discounted cash flow valuation model many empirical studies have analyzed the relationship between real stock returns and growth rates of industrial production (see Barro, 1990; Fama, 1981, 1990; Schwert, 1990; Chen, 1991; Lee, 1992; Choi et al., 1999; Binswanger, 2000; Hassapis and Kalyvitis, 2002).

The results of the above mentioned studies, mainly coming from univariate or multivariate time series approaches, indicate that real stock returns have a predictive power over real economic activity, that is they tend to lead real output, and that the degree of correlation tends to increase over longer production growth rate horizons (i.e. from monthly, to quarterly and to annual growth rates). According to Fama (1990) such evidence may depend on a type of measurement-error problems stemming from the fact that if the information about the production of a given period is spread over many previous periods and affects the stock returns of many previous periods, the proportion of the variation in production due to information in stock returns becomes larger as the time horizon gets longer. As a consequence, regressions of production on stock returns (and vice versa) obtain better results using annual or quarterly rather than monthly observations.

While earlier studies employed OLS regressions of growth rates of real economic activity on contemporaneous and lagged real stock returns (or real stock returns on future growth rates of real economic activity), recent empirical

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studies employ multivariate statistical approaches such as vector autoregressive (Lee, 1992; Gjerde and Saettem, 1999; Rapach, 2001) and vector error correction (Cheung and Ng, 1998; Nasseh and Strauss, 2000) models. These studies, analyzing the interactions between the stock market and aggregate economic activity, may examine either their short run or long run relationships, as the time series methodologies employed (usually cointegration analysis) may separate out just two time periods (or time scales) in economic time series, i.e. the short run and the long run. But the stock market provides an example of a market in which the agents involved consist of heterogeneous investors making decisions over different time horizons (from minutes to years) and operating at each moment on different time scales (from speculative to investment activity). In this way, the nature of the relationship between stock returns and growth rates of industrial production may well vary across time scales according to the investment horizon of the traders, as small time scales may be linked to speculative activity and coarse scales to investment activity. Thus, for example, if we think that big institutional investors have long term horizons and, consequently, follow macroeconomic fundamentals, we expect the relationship between stock returns and economic activity to be stronger at intermediate and coarsest time scales than at the finest ones.

In such a context, where both the time horizons of economic decisions and the strength and direction of economic relationships between variables may differ according to the time scale of the analysis (Ramsey and Lampart, 1998a), a useful analytical tool may be represented by wavelet analysis. Wavelets are particular types of function that are localized both in time and frequency domain and used to decompose a function  $f(x)$ , i.e. a surface, a series, etc., in more elementary functions which include different information about the same  $f(x)$ . Wavelet analysis is just one among a wide variety of available methods of statistical signal extraction and filtering (Pollock, 2006) as well as denoising (Aminghafari et al., 2006). The main advantage of wavelet analysis is its ability to decompose macroeconomic time series, and data in general, into their time scale components. Several applications of wavelet analysis in economics and finance have been provided by Ramsey and Lampart (1998a, b), Ramsey (2002) and Duchesne (2006) among the others, but only one attempt has been made to apply this methodology to the analysis of the relationship between stock returns and overall economic activity (Kim and In, 2003). In their paper Kim and In (2003) analyze the lead/lag relationship between financial variables and real economic activity using the Granger causality test on wavelet details and signals, while in this paper we analyze the lead/lag relationship between stock returns and real economic activity applying wavelet correlation and cross-correlation analyses to wavelet coefficients.

This paper extends the existing literature on the relationship between real stock returns and output growth by reconsidering the positive relationship between the two variables on a scale-by-scale basis. In particular, we apply the maximum overlap discrete wavelet transform (MODWT) to the Dow Jones Industrial Average stock price index and the industrial production index for the US over the period 1961:1–2006:10, and investigate the scaling properties of the series as well as the lead/lag relationship between stock prices and industrial production at different time scales.

Our time scale results provide evidence on the different scaling behavior of stock market returns and output growth and on the varying time scale relationship between the two series. In particular, we find that the relation, weak at lower scales (higher frequencies), gets stronger at higher scales (lower frequencies), with stock market returns leading output growth at periods of 8 months and longer. Such a result accords quite well with the economic rationale which suggests that mainly investors with longer term horizons are to be linked to macroeconomic fundamentals in their investment activity. In addition, we are able to provide an explanation for the finding of a time frame-dependent relationship between real stock returns and economic growth, as it may be considered the outcome of estimating a time scale averaged relationship. Indeed, as in a time scale perspective aggregate data may be considered the result of a time scale aggregation procedure over all time scales, the relationship between the aggregates is likely to represent a relationship averaged over all time scales.

The structure of the paper is as follows. Section 2 describes briefly the methodology employed, i.e. wavelet analysis, while Section 3 presents the empirical results from wavelet variance and cross-correlation analyses. Section 4 concludes the paper.

## 2. Methodology

The series were filtered using wavelet transform, a relatively new (at least for economists) mathematical tool that, roughly speaking, decomposes a given series according to scale (time components) instead of frequencies, as in the Fourier approach. The comparison with Fourier analysis may be useful firstly because wavelets use a similar strategy, as they employ some basic functions (wavelets instead of sines and cosines) and use them to decompose the series, and

secondly because they have been by far the most important signal processing tool in many fields (including economics) for many years. Said that, we have to stress the main difference between the two tools. Wavelet analysis, in contrast to Fourier analysis, does not need any stationary assumption in order to decompose the series, as spectral decomposition methods perform a global analysis whereas, the wavelets method acts locally in time (and so do not need stationary cyclical components).

In this section we first give a short review of the Fourier transform and its revised version, i.e. the short-term Fourier transform, and then provide a brief discussion on the properties of the wavelet transform, as general introductions to wavelet analysis can be found in [Strang \(1989\)](#) and [Percival and Walden \(2000\)](#). After that we go into the details of discrete wavelet transform (DWT) and maximal overlap discrete wavelet transform (MOWDT), and finally introduce the definition of wavelet variance, wavelet covariance and wavelet cross-correlation.

### 2.1. Fourier vs wavelet transform

The Fourier transform has long been applied for analysis of continuous and discrete signals and systems in many different fields. It decomposes a signal or a function into a sum of harmonic components of different frequencies via a linear combination of Fourier basic functions (sines and cosines). The Fourier transform is given by a pair of functions, the transform pair, represented by forward and inverse Fourier transform (FFT and IFT). Thus, the Fourier transform is a frequency domain representation of a signal or a function containing the same information of the original function, but summarized as a function of frequency. As a consequence, it may be interpreted as a decomposition of a signal on a frequency-by-frequency basis.

An unsuitable feature of the Fourier transform is that it allows analysis of signals or functions under the main assumption that the observed signal is stationary over the time period of analysis, an assumption not valid for many practical signals and, in particular, for most interesting time series. In order to overcome this main limitation of the Fourier transform, i.e. its inability to deal with non-stationary signals, a modified time-dependent version of it has been developed, i.e. the Gabor transform or short-time (or time-variable) Fourier transform (STFT) (see [Gabor, 1946](#)). The short-time Fourier transform uses a fixed window function with respect to frequency and applies the Fourier transform to the windowed signal. The original signal is partitioned into small enough segments such that these portions of the non-stationary signal can be assumed to be stationary over the duration of the window function. The choice of the window length is based on the trade-off between the desired frequency resolution, which depends inversely on the duration of the window function, and the assumption of short-time stationarity. Once the window function is determined, both the time as well as frequency resolutions become fixed for all frequencies and times, respectively. As a consequence, the short-time Fourier transform does not allow any change in resolutions in terms of time or frequency.

An alternative to the short-time Fourier transform for analysis of non-stationary signals is represented by the wavelet transform. The term wavelets literally means small waves, as they have finite length (compactly supported) and oscillatory behavior. They are particular types of basic functions that are used to decompose a function  $f(t)$ , i.e. a signal, a surface, a series, etc., in more elementary functions which include information about  $f(t)$ . These basis windows functions used in transforming the signal are derived from a single prototype function, called the mother wavelet, using its scaled (i.e. dilated and compressed) and translated version, where scaling refers to the level of detail in the analysis of the signal and translation refers to the location of the window.

Thus, in contrast to the fixed time-frequency partition of the short-time Fourier transform the wavelet transform analyzes the signal at different frequencies with different resolutions using a multiresolution analysis. The multiresolution analysis approach may overcome the resolution problem as it adaptively partitions the time-frequency plane, using short windows at high frequencies and long windows at low frequencies, and thus letting both time and frequency resolutions to vary in the time-frequency plane.

[Fig. 1](#) shows the time-frequency resolution properties of the short-time Fourier transform (upper panel) and the wavelet transform (lower panel). It explains that, unlike the short-time Fourier transform which has constant resolution at all times and frequencies, the wavelet transform, through the adaptive partition of the time-frequency plane, provides good frequency resolution (and poor time resolution) at low frequencies and good time resolution (and poor frequency resolution) at high frequencies.

The wavelet transform (filter) depends on two parameters, scale (or frequency) and time, that provide the time and frequency information simultaneously, hence providing the so-called time-scale or time-frequency representation of the signal, where the scale factor is inversely related to the frequency of the wavelet. The scale or dilation factor  $j$  controls

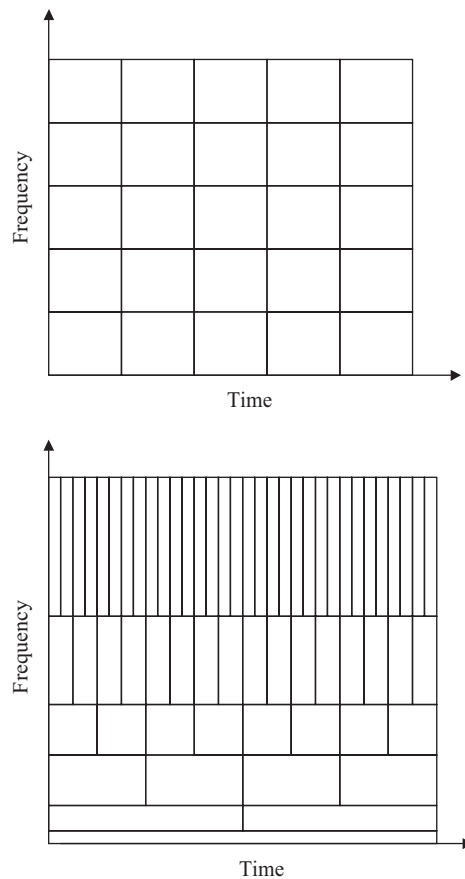


Fig. 1. The short-time Fourier transform (upper panel) and the wavelet transform (lower panel) partitioning of the time–frequency plane.

the length of the wavelet (window), while the translation or location parameter  $k$  refers to the location and indicates the nonzero portion of each wavelet basis vector. The basis wavelet function is stretched (or compressed) according to the scale parameter to extract frequency information and moved on the time line (from the beginning to end) to extract time information from the signal in question. As pointed out in Fig. 1 a narrow window yields information on low frequency movements, while a wide window yields information on high frequency movements.

The wavelet transform decomposes a signal into sets of coefficients where each set of coefficients is associated with a spatial scale and each coefficient in a set is associated with a particular location. In wavelet terminology each single coefficient is called an “atom” and the set of coefficients for each scale a “crystal”. The wavelet coefficients, the output of the wavelet transform, are obtained through a projection of the signal onto shifted and translated versions of mother and father wavelets and represent the underlying smooth behavior of the data at the coarsest scale (the scaling coefficients) and the coarse scale deviations from it (the wavelet coefficients), respectively.

## 2.2. Wavelet transforms: DWT and MODWT

The wavelet transform maps a function  $f(t)$  from its original representation in the time domain into an alternative representation in the time-scale domain by recursively applying two basis wavelet filter functions: the father wavelet and the mother wavelet. The first integrates to 1 and reconstructs the smooth and low frequency parts of a signal, whereas the latter integrates to zero and describes the detailed and high-frequency parts of a signal. The function may be continuous or discrete. As time series are observed at regular intervals and are constituted by a finite-length vector of observations, we restrict our presentation exclusively to the discrete wavelet transformation.

There are several families of wavelet filters available, such as Haar (discrete), symmlets and coiflets (symmetric), daublets (asymmetric), etc., differing by the characteristics of the transfer function of the filter and by filter lengths. In particular, the choice of the filter length depends on a trade-off between leakage and boundary-affected coefficients: a longer length makes the filter closer to an ideal high-pass filter, but reduces the number of boundary-unaffected coefficients. Different wavelet families make different trade-offs between the degree of localization and the degree of approximation of high-pass filters (Lindsay et al., 1996). Daubechies (1992) has developed a family of compactly supported wavelet filters of various lengths, the least asymmetric family of wavelet filters (LA), which is particularly useful in wavelet analysis of time series because it allows the most accurate alignment in time between wavelet coefficients at various scales and the original time series.

With  $\{h_l\}_{l=0}^{L-1}$  (or just  $\{h_l\}$ ) and  $\{g_l\}_{l=0}^{L-1}$  (or just  $\{g_l\}$ ) we denote the set of all filter coefficients of a Daubechies compactly supported wavelet, where  $L$  is the even integer width of the filters. Given a stochastic process  $\{X\}$  whose sample size is a power of two, an orthonormal DWT may be obtained by recursively applying the level  $j$  wavelet filter  $h_l$  and scaling filter  $g_l$  to the original process. The wavelet filter coefficients  $h_l$  and the scaling filter coefficients  $g_l$  represent the high- and low-pass filters, corresponding to the mother and father wavelet filters respectively, and are related to each other through a quadrature mirror filter relationship, that is  $h_l = (-1)^l g_{L-1-l}$  for  $l = 0, \dots, L-1$ . The filter coefficients of a wavelet filter must satisfy three properties:

(i) have zero mean

$$\sum_{l=0}^{L-1} h_l = 0,$$

(ii) have unit energy

$$\sum_{l=0}^{L-1} h_l^2 = 1,$$

(iii) be orthogonal to their even shifts

$$\sum_{l=0}^{L-1} h_l h_{l+2k} = 0.$$

The DWT is implemented via a filtering/decimation scheme where the wavelet filter  $h_{j,l}$  is used in association with the scaling filter  $g_{j,l}$  in a pyramid algorithm (Mallat, 1989) to obtain, at each level  $j = 1, \dots, J$ , the wavelet and scaling coefficients  $w_{j,t}$  and  $v_{j,t}$ . The expressions for DWT (and in what follows for MODWT) wavelet and scaling coefficients in the case of functions defined over the entire real axis, that is  $t \in \mathfrak{R}$  as in this case  $X_t = X_{t \bmod N}$  when  $t < 0$ , are

$$w_{j,t} = \sum_{l=0}^{L-1} h_{j,l} X_{t-l}$$

and

$$v_{j,t} = \sum_{l=0}^{L-1} g_{j,l} X_{t-l},$$

where  $h_{j,l}$  and  $g_{j,l}$  are the  $j$ th level wavelet and scaling filters and, due to downsampling by  $2^j$ , we have  $N/2^j$  scaling and wavelet coefficients. At each  $j$ th level the inputs to the wavelet and scaling filters are the scaling coefficients from the previous level ( $j-1$ ) and the outputs are the  $j$ th level wavelet and scaling coefficients, with the only exception at the unit level ( $j=1$ ) in which wavelet and scaling filters are applied to the original data.

With regard to the spectral interpretation of DWT wavelet coefficients, as the wavelet filter  $h_{j,l}$  for scales  $1 \geq j \geq J$  approximates an ideal high-pass with passband frequencies  $f \in [1/2^{j+1}, 1/2^j]$ , the  $j$ th level scale wavelet coefficients are associated to periods  $[2^j, 2^{j+1}]$ .

However, the orthonormal DWT has two main drawbacks: the dyadic length requirement (i.e. a sample size divisible by  $2^J$ ) and the fact that the wavelet and scaling coefficients are not shift invariant due to their sensitivity to circular shifts because of the decimation operation. An alternative to DWT is represented by a non-orthogonal variant of DWT: the maximal overlap DWT (MODWT). The MODWT goes under several names in the wavelet literature, such as the “non-decimated DWT”, “stationary DWT” (Nason and Silverman, 1995), “translation-invariant DWT” (Coifman and Donoho, 1995) and “time-invariant DWT”.

In the orthonormal DWT the wavelet coefficients are related to non-overlapping differences of weighted averages from the original observations that are concentrated in space. More information on the variability of the signal could be obtained considering all possible differences at each scale, that is considering overlapping differences, and this is precisely what the maximal overlap algorithm does: it computes all possible shifted time intervals. As a consequence, the orthogonality of the transform is lost, but the number of wavelet and scaling coefficients at every scale is the same as the number of observations. Thus, the maximal overlap DWT coefficients may be considered the result of a simple modification in the pyramid algorithm used in computing DWT coefficients through not downsampling the output at each scale and inserting zeros between coefficients in the wavelet and scaling filters. The DWT coefficients may be considered a subset of the MODWT coefficients. Indeed, for a sample size power of two the MODWT may be rescaled and subsampled to obtain an orthonormal DWT. In particular, the MODWT wavelet and scaling coefficients  $\tilde{w}_{j,t}$  and  $\tilde{v}_{j,t}$  are given by

$$\tilde{w}_{j,t} = \frac{1}{2^{j/2}} \sum_{l=0}^{L-1} \tilde{h}_{j,l} X_{t-l}$$

and

$$\tilde{v}_{j,t} = \frac{1}{2^{j/2}} \sum_{l=0}^{L-1} \tilde{g}_{j,l} X_{t-l},$$

where the MODWT wavelet and scaling filters  $\tilde{h}_{j,l}$  and  $\tilde{g}_{j,l}$  are obtained by rescaling the DWT filters as follows:

$$\tilde{h}_{j,l} = \frac{h_{j,l}}{2^{j/2}}$$

and

$$\tilde{g}_{j,l} = \frac{g_{j,l}}{2^{j/2}}.$$

Thus, whereas DWT filters have unit energy, MODWT filters have half energy, that is  $\sum_{l=0}^{L-1} \tilde{h}_{j,l}^2 = \sum_{l=0}^{L-1} \tilde{g}_{j,l}^2 = \frac{1}{2^j}$ . MODWT provides the usual functions of DWT, such as multiresolution decomposition analysis and variance analysis based on wavelet transform coefficients, but unlike the classical DWT it

- can handle any sample size;
- is translation invariant, as a shift in the signal does not change the pattern of wavelet transform coefficients;
- provides increased resolution at coarser scales;
- and produces a more asymptotically efficient wavelet variance estimator than DWT.

Moreover, unlike the classical DWT which has fewer coefficients at coarser scales, MODWT has a number of coefficients equal to the sample size at each scale, and thus is over-sampled at coarse scales.

### 2.3. Wavelet variance, covariance and correlation

In addition to multiresolution decomposition analysis wavelet methods can provide an alternative representation of the variability and association structure of certain stochastic processes on a scale-by-scale basis. Given a stationary stochastic process  $\{X\}$  with variance  $\sigma_X^2$  and the wavelet variance definition at scale  $j$  as  $\sigma_{X,j}^2$ , the following



relationship holds:

$$\sum_{j=1}^{\infty} \sigma_{X,j}^2 = \sigma_X^2.$$

Thus, as  $\sigma_{X,j}^2$  represents the contribution of the changes at scale  $j$  to the overall variability of the process, this relationship says that wavelet variance provides an exact decomposition of the variance of a time series into components that are associated to different time scales. The wavelet variance decomposes the variance of a stationary process with respect to the scale at  $j$ th level just as the spectral density decomposes the variance of the original series with respect to frequency  $f$ , that is

$$\sum_{j=1}^{\infty} \sigma_{X,j}^2 = \sigma_X^2 = \int_{-1/2}^{1/2} S_X(f) df,$$

where  $S_{(\cdot)}$  is the spectral density function.

By definition the (time independent) wavelet variance at scale  $j$ ,  $\sigma_{X,j}^2$ , is given by the variance of  $j$ -level wavelet coefficients

$$\sigma_{X,j}^2 = \text{var}\{\tilde{w}_{j,t}\}.$$

A time-independent wavelet variance may be defined not only for stationary processes, but also for non-stationary processes but with stationary  $d$ th order differences, and for non-stationary processes but with local stationarity (see Percival and Walden, 2000). Indeed, as the wavelet filter  $\{h_l\}$  represents the difference between two generalized averages and is related to a difference operator (Whitcher et al., 2000), wavelet variance is time-independent also in case of non-stationary processes, but with stationary  $d$ th order differences, provided that the length  $L$  of the wavelet filter is large enough. Percival and Walden (2000) show that  $L \geq d$  is a sufficient condition to make the wavelet coefficients  $\tilde{w}_{j,t}$  of a stochastic process whose  $d$ th order backward difference is stationary a sample of stationary wavelet coefficients.

As MODWT employs circular convolution, the coefficients generated by both beginning and ending data could be spurious. Thus, if the length of the filter is  $L$ , there are  $(2^j - 1)(L - 1)$  coefficients affected for  $2^{j-1}$ -scale wavelet and scaling coefficients (Percival and Walden, 2000). As shown in Percival (1995), provided that  $N - L_j \geq 0$ , an unbiased estimator of the wavelet variance based on the MODWT may be obtained after removing all coefficients affected by the periodic boundary conditions using

$$\tilde{\sigma}_{X,j}^2 = \frac{1}{\tilde{N}_j} \sum_{t=L_j}^N \tilde{w}_{j,t}^2, \quad (1)$$

where  $\tilde{N}_j = N - L_j + 1$  is the number of maximal overlap coefficients at scale  $j$  and  $L_j = (2^j - 1)(L - 1) + 1$  is the length of the wavelet filter for level  $j$ . Thus, the  $j$ th scale wavelet variance is simply the variance of the non-boundary or interior wavelet coefficients at that level (Percival, 1995; Serroukh et al., 2000). Both DWT and MODWT can decompose the sample variance of a time series on a scale-by-scale basis via its squared wavelet coefficients, but the MODWT-based estimator has been shown to be superior to the DWT-based estimator (Percival, 1995).

Whitcher et al. (1999, 2000) have extended the notion of wavelet variance for the maximal overlap DWT (MODWT) and introduced the definition of wavelet covariance and wavelet correlation between the two processes, along with their estimators and approximate confidence intervals. To determine the magnitude of the association between two series of observations  $X$  and  $Y$  on a scale-by-scale basis the notion of wavelet covariance has to be used. Following Gençay et al. (2001) the wavelet covariance at wavelet scale  $j$  may be defined as the covariance between scale  $j$  wavelet coefficients of  $X$  and  $Y$ , that is  $\gamma_{XY,j} = \text{Cov}[\tilde{w}_{j,t}^X, \tilde{w}_{j,t}^Y]$ . Again, after removing all wavelet coefficients affected by boundary conditions, an unbiased estimator of the wavelet covariance using MODWT may be given by

$$\tilde{\gamma}_{XY,j} = \frac{1}{\tilde{N}_j} \sum_{t=L_{j-1}}^{N-1} \tilde{w}_{j,t}^X \tilde{w}_{j,t}^Y. \quad (2)$$

Then, the MODWT estimator of the wavelet cross-correlation coefficients for scale  $j$  and lag  $\tau$  may be obtained by making use of the wavelet cross-covariance,  $\tilde{\gamma}_{\tau,XY,j}$ , and the square root of their wavelet variances  $\tilde{\sigma}_{X,j}$  and  $\tilde{\sigma}_{Y,j}$  as follows:

$$\tilde{\rho}_{\tau,XY,j} = \frac{\tilde{\gamma}_{\tau,XY,j}}{\tilde{\sigma}_{X,j}\tilde{\sigma}_{Y,j}}. \quad (3)$$

The wavelet cross-correlation coefficients  $\tilde{\rho}_{\tau,XY,j}$ , just as the usual unconditional cross-correlation coefficients, are between 0 and 1 and provide the lead/lag relationships between the two processes on a scale-by-scale basis.

Starting from spectrum  $Sw_{X,j}$  of scale  $j$  wavelet coefficients it is possible to determine the asymptotic variance  $V_j$  of the MODWT-based estimator of the wavelet variance (covariance) and construct a random interval which forms a  $100(1 - 2p)\%$  confidence interval. The formulas for an approximate  $100(1 - 2p)\%$  confidence intervals MODWT estimator robust to non-Gaussianity for  $\tilde{v}_{X,j}^2$  are provided in Gençay et al. (2002, p.254–256). The empirical evidence from the wavelet variance suggest that  $N_j = 128$  is a large enough number of wavelet coefficients for the large sample theory to be a good approximation (Whitcher et al., 2000).

### 3. Empirical analysis

Wavelet analysis is a new and rapidly evolving field that is having growing applications in many different disciplines such as geophysics, engineering, physics, acoustics and, more recently, in economics and finance. The substantial progress in computations and visualization tools that characterize the wide variety of mathematical and statistical packages implementing wavelet analysis algorithms has made extended graphic and signal processing capabilities available to researchers.

Before performing wavelet analysis a number of decisions must be made: which family of wavelet filters to use, what type of wavelet transform to apply, and how boundary conditions at the end of the series are to be handled.

In order to calculate wavelet coefficient values near the end of the series boundary conditions are to be assumed. According to the two main assumptions the series may be extended in a periodic fashion (periodic boundary condition) or in a symmetric fashion (reflecting boundary condition). We apply for wavelet variance analysis the reflecting boundary condition, where the original signal is reflected at its end point to produce a series of length  $2N$  which has the same mean and variance as the original signal. The wavelet and scaling coefficients are then computed by using a periodic boundary condition on the reflected series, resulting in twice as many wavelet and scaling coefficients at each level. Finally, we perform the time scale decomposition analysis using the MODWT because of the practical limitations of DWT, i.e. the dyadic length requirement and non-shift-invariance.

#### 3.1. Time scale decomposition analysis

We are aware that, as the share of the industrial sector along the developmental path is shrinking, industrial production may not be the most reliable indicator of real economic activity in industrial countries. However, we choose to use it mainly because it is the only aggregate output series available on a monthly basis. Thus, the analysis is conducted using monthly data for the Dow Jones Industrial Average stock market index and the industrial production index for the US between 1961:1–2006:10 (sources: yahoo.finance.it and FREDII). Continuously compounded one-month stock returns and growth rates of industrial production are calculated through log-differentiation. We decompose the two transformed series into their time-scale components using the MODWT which is a non-orthogonal variant of the classical discrete wavelet transform that, unlike the orthogonal discrete wavelet transform, is translation invariant, as shifts in the signal do not change the pattern of coefficients. The wavelet filter used in the decomposition is the Daubechies least asymmetric (LA) wavelet filter of length  $L = 8$ , or  $LA(8)$  wavelet filter, based on eight non-zero coefficients (Daubechies, 1992), with periodic boundary conditions. Given that the maximum decomposition level  $J$  is given by  $\log_2(N)$  we apply the MODWT up to a level  $J = 5$  that produces six wavelet and scaling filter sets of coefficients  $v_5, w_5, w_4, w_3, w_2, w_1$ . Given that the level of the transform defines the effective scale  $\lambda_j$  of the corresponding wavelet coefficients, for all families of Daubechies compactly supported wavelets the level  $j$  wavelet coefficients are associated with changes at scale  $2^{j-1}$ . Since scale  $2^{j-1}$  corresponds to frequencies in the interval  $f \in [1/2^{j+1}, 1/2^j]$ , using monthly data scale 1 wavelet coefficients are associated to 2–4 month periods, while scales 2 to 5 are associated to 4–8, 8–16, 16–32 and 32–64 month periods, respectively.



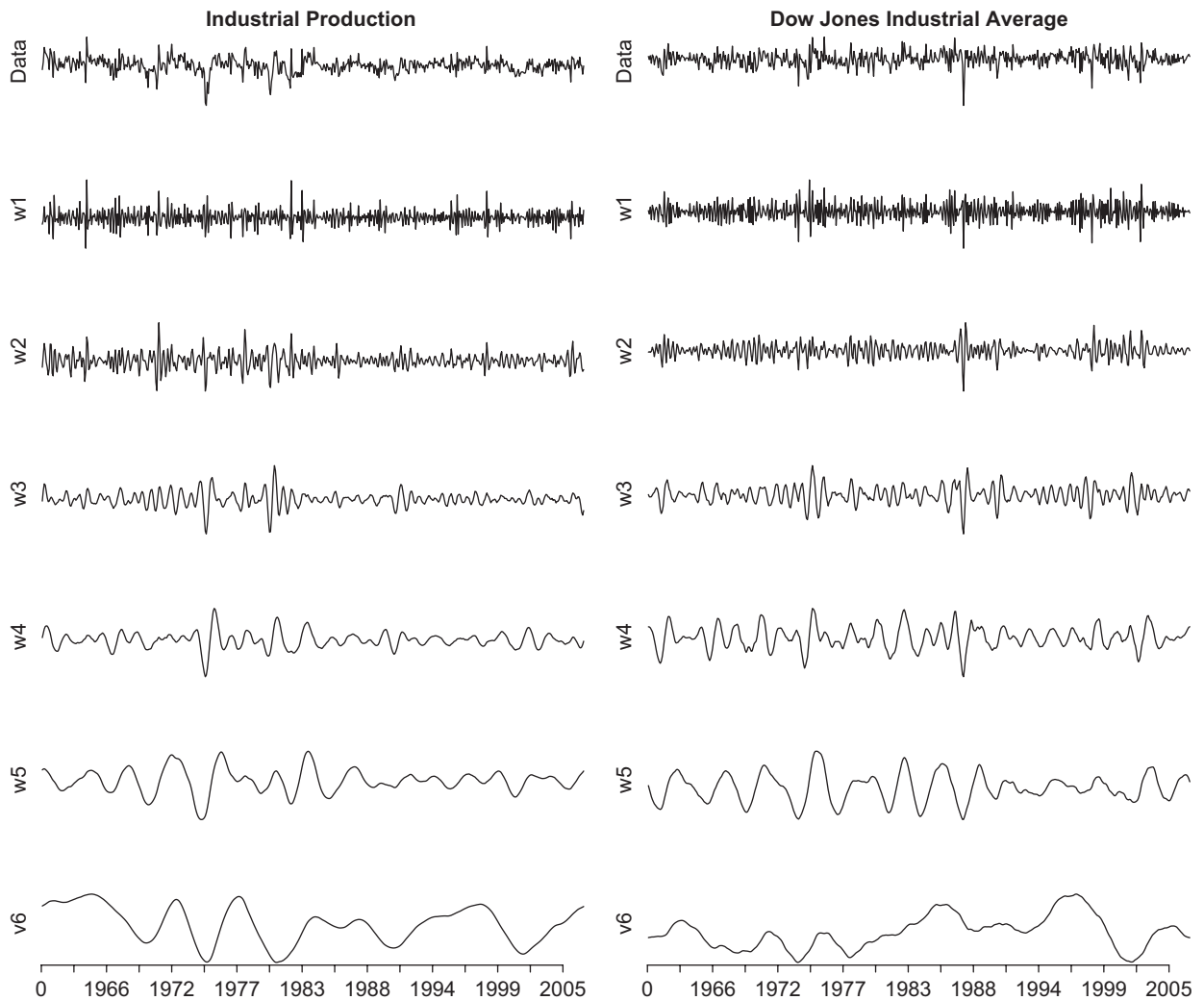


Fig. 2. Industrial production (left) and Dow Jones Industrial Average (right).

All computations have been performed using the *waveslim* package developed by Whitcher (and available at <http://www.cgd.ucar.edu/~whitcher>) running under the R statistical computing environment (R Development Core Team 2006). Fig. 2 shows the MODWT multiresolution decomposition analysis on the log-differenced transformed data for the production index IP and the Dow Jones Industrial Average index DJIA.

### 3.2. Long memory properties and wavelet variance

With wavelet analysis we may disentangle the variance of a process on a scale by scale basis, as the plot of  $\tilde{\sigma}_{X,j}^2$  against scale  $j$  indicates which scales contributes more to the process variance. Fig. 3 shows the MODWT-based variance of the DJIA returns and the IP growth rate plotted on a log-log scale, where the straight line indicates the estimated wavelet variance and the “U” and “L” line the upper and lower bounds for the 95% approximate confidence interval. For the calculation of wavelet variance we apply the reflection boundary condition which reflects the original signal at its end point to produce a series of length  $2N$  which has the same mean and variance as the original signal. In this way we may have a sufficient number of nonboundary coefficients to estimate wavelet variance up to scale 6. An approximate linear relationship emerges between the wavelet variance and the wavelet scale, with decreasing wavelet variance as the wavelet scale increases and the DJIA stock returns being, as expected, more volatile than the IP growth rate.

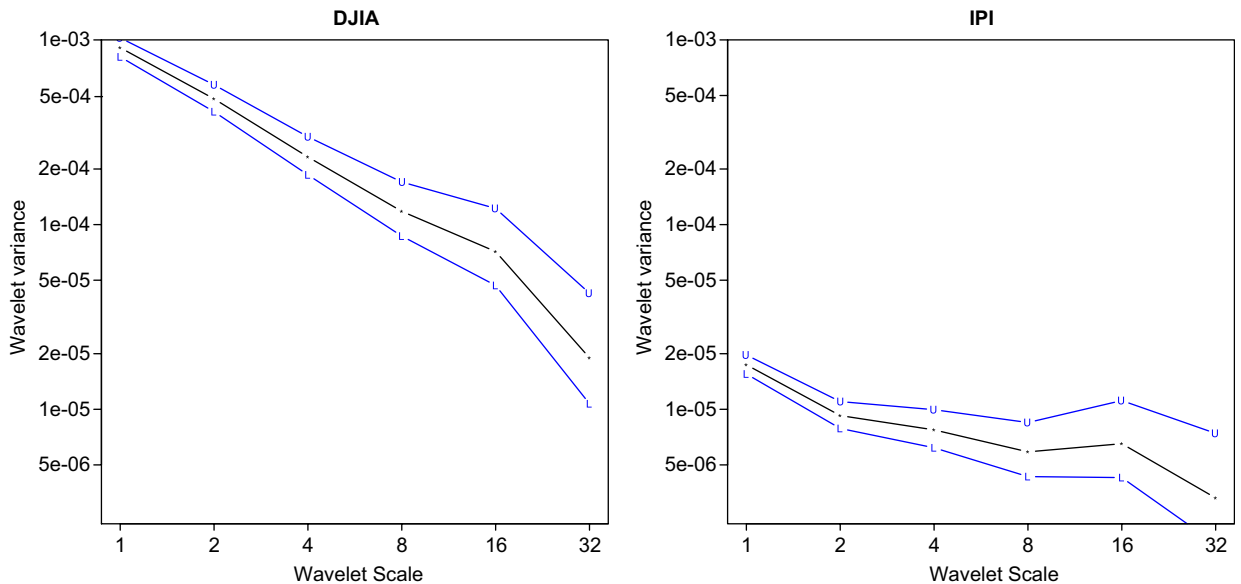


Fig. 3. Production growth rate and stock returns wavelet variance. Wavelet variance for the DJIA index and industrial production index at log–log scale. The (.)'s are estimated variances for each scale. Lines with “U” and “L” denote, respectively, the upper and lower bound for the approximate 95% confidence interval assuming a non-Gaussian process.

Many economic and financial time series exhibit long memory or long-range dependence, thus testing for the presence of such long memory has been a popular research topic in economic and financial research. Recently, with respect to the traditional dichotomy between  $I(0)$  and  $I(1)$  processes, researchers have been focusing on fractionally integrated processes, or  $I(d)$  processes, with the differencing parameter  $d$  being a fractional number (see Granger and Joyeux, 1980, and Hosking, 1981). Such fractionally integrated processes are stationary long memory processes characterized by hyperbolically decaying autocorrelations, in contrast to short memory processes in which the autocorrelations decay at an exponential rate.

Abry et al. (1995), Abry and Veitch (1996) and Jensen (1999) have developed a method that uses wavelet variance to estimate the long memory parameter for a fractionally integrated process via a log–linear relationship between the adjusted estimated variance of the wavelet coefficients at different scales with respect to the scale index  $j$ . The wavelet OLS estimator of the long memory parameter obtained from this log–linear relationship is a consistent estimator of the fractionally differencing parameter which has proven to be superior to other estimators of the long memory parameter, such as the log-periodogram regression estimator proposed by Geweke and Porter-Hudak (1983), due to the short memory contamination problem and small-sample bias (see McCoy and Walden, 1996). In particular, wavelets provide a useful long memory test against two most common causes of spurious long memory estimate results: non-stationarity and aggregation. Indeed, Abry and Veitch (1996) show that from the estimated slope  $\beta$  of the OLS regression  $\log(\hat{\sigma}_{X,j}^2) \approx \xi + \beta \log(j)$  we can estimate the memory parameter estimator  $d$  of a fractional integrated process via  $d = (\beta + 1)/2$ , as the slope  $\beta$  is related to the exponent  $\alpha$  of the approximating power law spectral density function at low frequencies via  $\beta = -\alpha - 1$  and  $d = -\alpha/2$  for a fractionally integrated process.

Table 1 provides a taxonomy of the basic properties of a fractionally differenced process according to the estimated value of the memory parameter. For  $d \in (-0.5, 0.5)$  the process is both stationary and invertible and exhibit a unique kind of dependence, positive or negative, depending whether  $d$  is positive or negative, while for  $d > 0.5$  it will be non-stationary and for  $d < -0.5$  non-invertible. Moreover, the process exhibits long memory or long-range positive interdependence for  $d \in (0, 0.5)$ , and intermediate memory or long-range negative dependence for  $d \in (-0.5, 0)$  (the positive and negative autocorrelations decay hyperbolically to zero as the lag length increases).

As the estimated slopes for DJIA and IP index are  $-1.11$  and  $-0.36$ , respectively, the estimated scaling parameters  $\hat{d}$  are  $-0.05$  for DJIA index and  $0.31$  for IP index. Thus, according to the relationship between the value of the scaling parameter in a pure power law process and the type of the process shown in Table 1 (see Percival and Walden, 2000),

Table 1  
Long-range properties of fractional integrated processes for a range of  $d$  values

Process type	Stationary short memory	White noise process	Stationary long memory	Non-stationary long memory
Fractionally differenced	$-1/2 \leq d \leq 0$	$d = 0$	$0 < d < 1/2$	$d \geq 1/2$

production growth rate may be considered a stationary long memory process, while stock returns a stationary short memory process. Our results are consistent with previous studies testing for long memory in asset returns which generally find little evidence for long memory in the levels of returns (see, among others, [Greene and Fielitz, 1977](#); [Lo, 1991](#); [Crato, 1994](#); [Cheung and Lai, 1995](#)), and with recent findings by [Michelacci and Zaffaroni \(2000\)](#) and [Silverberg and Verspagen \(1999\)](#) on the long memory features of the growth process of aggregate output. On the other hand previous studies on long memory on aggregate output, like [Diebold and Rudebush \(1989\)](#), [Haubrich and Lo \(2001\)](#) and [Sowell \(1992\)](#), provided contrasting evidence on the long memory properties of output as they find significant, no, and small evidence, respectively, of long memory in aggregate output. These findings on the short and long memory processes may have crucial implications for many of the paradigms used in traditional financial economics, like the weak form of the market efficiency hypothesis which is consistent with the short memory feature hypothesis, and the degree of persistence of shocks as the effects of such shocks are likely to be more persistent for output than for stock prices.

### 3.3. Wavelet cross-correlation analysis

In this section we focus on the degree of association between stock returns and the level of economic activity. Common findings in the analysis of the relationship between stock market and the level of economic activity are that stock returns have a predictive power over real economic activity, that is they tend to lead real output, and that the degree of association tends to increase with the time frame (from monthly, to quarterly and to annual growth rates). Wavelet analysis, given its ability to decompose a time series into its time scale components and thus to reveal structure at different time horizons, may be useful in analyzing situations in which the degree of association between two time series is likely to change with the time-horizon. In particular, wavelet cross-correlation analysis, the analogue of the standard time domain measure of association in the time-scale domain, may be used to determine the lead/lag relationship between two time series on a scale-by-scale basis.

In [Fig. 4](#) we report the MODWT-based wavelet correlations and cross-correlation coefficients, with the corresponding approximate confidence intervals, against time leads and lags for all scales, where each scale is associated with a particular time period. For example, scale 1 is associated to 2–4 month periods, scale 2 to 4–8 month periods, scale 3 to 8–16 month periods, and so on. In particular, the correlation coefficient of the growth rate of industrial production at time  $t$  is plotted against the value of stock returns at time  $t - \tau$  and  $t + \tau$  up to 24 months time lags.

At the shortest scales, i.e. scales 1 to 2, the magnitude of the association between the two variables is generally close to zero at all leads and lags, while on the other hand at coarsest scales, particularly at scales 4 and 5, such relationship become stronger. As regards the contemporaneous time scale correlation between production growth rate and stock returns, the values of the wavelet correlation coefficients at lag 0 indicate a generally low magnitude of association between the two series at all scales and an anti-correlation relationship which is stronger at scales 3 and 4. On the other hand, the cross-correlation wavelet coefficients reveal that at the coarsest scales there is a high positive leading relationship between stock market returns and production growth rate (0.5 and 0.7 at scales 4 and 5, respectively), with the leading period increasing as the time scale increases (the largest cross-correlation coefficients occurring at leads 6 for wavelet scale 4, that is 16–32 month periods, and 10 for wavelet scale 5, that is 32–64 month periods).

Thus, the results from wavelet cross-correlation analysis show that stock market returns tend to lead economic activity, but only at scales corresponding to periods of 16 months and longer (lowest frequencies), and that the periods by which stock returns lead output tend to increase as the wavelet time scale increases. Our findings are consistent with those of [Kim and In \(2003\)](#) who, applying the Granger causality test at different time scales to wavelet detail and smooth signals, find that “at longer time scales share prices Granger cause US industrial production” (p.13). Moreover,

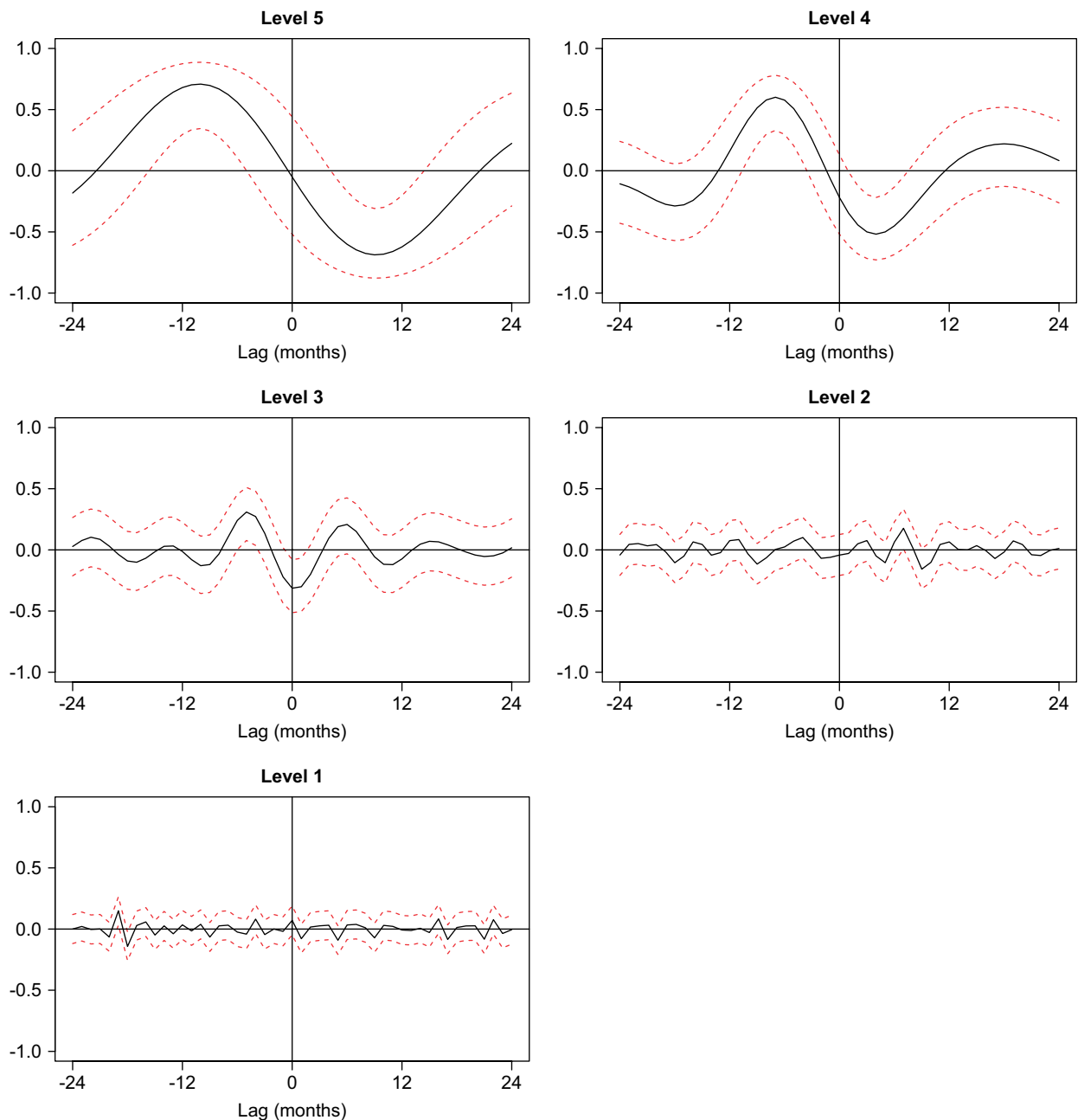


Fig. 4. Wavelet cross-correlations between DJIA stock returns and US Industrial production index.

these results not only accord with the previous evidence on the strong positive (and leading) relationship between stock returns and real activity, but also provide a reliable time scale interpretation of the relationship that try to overcome the time-horizon aggregation problem which is generally omitted in standard time series analysis. Indeed, if traders with longer horizons are more likely to follow fundamentals than traders with shorter horizons, what we should expect is just that the relationship between stock returns and economic activity tends to be stronger at intermediate and coarsest time scales than at the finest ones.

#### 4. Conclusions

The existing evidence provided by the past literature shows that the relationship between stock returns and output is varying with the production growth rate horizon and, in particular, tends to increase as data frequency decreases (from monthly, to quarterly and annual). Wavelet filtering is a tool which, going beyond that of current methodology, provides a useful way to decompose the underlying structure of a relationship across different time scales. Thus, in this paper we apply a multiscaling approach based on a non-decimated discrete wavelet transform in order to investigate the relationship between stock returns and economic activity over different time scales.

We first provide a multiscale decomposition of the variance to identify the scaling properties of the stock returns and the growth rates of industrial production, and then focus on the lead–lag relationship between the two variables on a scale-by-scale basis using wavelet cross-correlation analysis. The main results may be summarized as follows:

- (1) According to wavelet variance analysis the production growth rate exhibits stationary long memory dynamics, while stock returns show stationary short memory dynamics;
- (2) Wavelet contemporaneous correlation is generally close to zero and indicates the presence of a slight anti-correlation;
- (3) Wavelet cross-correlation analysis shows that stock returns are leading aggregate economic activity at scales corresponding to periods of 16 months and longer, i.e. scales 4 and 5.

In short, time-scale decomposition analysis indicates that the relationship between stock returns and output is not fixed over various time scales and, in particular, that stock market returns are leading output at the longest scales, i.e. at scales corresponding to periods of 16 months and longer. Such a result may provide an explanation for the time frame relationship found in the empirical literature and accords quite well with conventional wisdom which suggest that mainly investors with longer term horizons are to be linked to macroeconomic fundamentals in their investment activity.

In conclusion, the application provided here gives an indication of the potential deeper understanding that wavelets may provide in analyzing the relationships that characterize markets where the agents involved consist of heterogeneous agents making decisions over different time horizons and/or operating at each moment on different time scales, as it is the case of financial markets.

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