Introduction to Wavelet Analysis with R

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In an economy or a financial market, people operate across different time horizon. Take the example of a financial market. The market operates from say 9.00 PM to 5 PM, fro monday to friday. But in this market, we have traders who operate on intra-day scales, traders operating on a weekly basis to mutual fund managers who looks for a longer investment horizon. The price formation we see at the end of the day is due to the interaction among the traders of all these groups and it reflects all these activities. Now imagine the following condition. Suppose we want to model market volatility, and we are presented with daily data. We use the ever dependent GARCH model to estimate the conditional volatility and get the volatility behavior at the daily scale.

So far, so good. But what if you are a mutual fund investor? You would like to see how the markets are supposed to be volatile at a monthly scale. One way to solve this problem is to take monthly data and apply the GARCH model. But as we know, once we aggregate the data, it get smoothened and volatility gets less visible.

Now, let us extend this single-market situation to multiple markets. You are an investor and you want to diversify your risk. You will diversify your risk across a) markets b) across time horizons (short, medium and long run investments). If we take the multivariate counterpart of GARCH, that is a DCC GARCH or similar model and apply it to our market data, the same problem persists.

Let me present you with another scenario. You want to see how these markets move together. We take help of the Johanson cointegration method and find the number of co-integrating relationship. But what about the strength of cointegration? If I want to see how the markets are moving together across a monthly scale, JJ method cannot provide me any useful information.

Such questions have always bothered econometricans for a long time and wavelets are one of the emergent solutions.Wavelets mean small waves. They operate in both time and frequency domains.That is, you can get frequency information without loosing the timescale dimension. Another desiered aspect of wavelets are that the Stationarity condtion is not required. The standard time series models are often constrained by the notion of covariance stationarity. Even though there are models such as ARDL and NARDL that overcomes this difficutlty, we are presented with issuesthat were discussed in the previous questions.

Now, some (unavoidable) mathematics !

A wavelet $\psi_{u,s}(t)$ could be defined as a real-valued square integrable function such that

$$\psi_{u,s}(t) = \psi \frac{(t-u)}{\sqrt{s}} \qquad (1)$$

where u is the location, s is the scale and t is the time. A wavelet has zero mean, i.e. $\int_{-\infty}^{\infty} \psi(t)dt = 0$ and it is usually normalized so that $\int_{0}^{\infty} \psi^2(t)dt = 1$. A time series could be reconstructed from its wavelet transform if it satisfies the following admissibility condition.

$$C_\psi = \int_0^\infty \frac{|\psi(f)|^2}{f} df < +\infty \qquad (2)$$

Here, $\psi(f)$ is the Fourier transform of the given wavelet.

Wavelets are classified into:

1. Continuous wavelet Transforms (CWT)
2. Discrete Wavelet Transforms(DWT)

As the name suggests, the CWT analyses the given timeseries across all different frequencies. The DWT on the other hand analyses the time series across dyadic scales (2,4,8,16....). However, the DWT is restricted by the fact that the series under study should be dyadic in length(power of 2). To overcome this, Perciwal and Walden(2000) proposed Maximal Overlap Discreet Wavelet Transform(MODWT).

First let us look into the case of CWT. Here we make use of a wavlet family called Mortlet, that is very much used by researchers in Economics and Finance.

The mortlet wavelet is defined as:

$$\psi(t) = \pi^{-\frac{1}{4}} e^{-i\omega_0 t} e^{\frac{-t^2}{4}}$$

and the wavelet transform of a time series x(t) using mortlet wavelets can be written as :

$$W_x(\tau, s) = \int_{-\infty}^{\infty} x\,(t) \frac{1}{\sqrt{|s|}} \psi^* \left(\frac{t-\tau}{s}\right) \mathrm{d}t,$$

From this, we can construct the wavelet power spectrum, which gives the indication of to what extent a time series is volatile across different timescales.

$$WPS = |W_x(\tau, s)|^2$$

Now, if we want to extend this analysis to bi-variate level, we have two tools, namely wavelet coherence and wavelet cross spectrum.

Wavelet coherence is analogous to correlation. Here we see how the time series are correlated across different timescales. In wavelet cross spectrum, we see how the fluctuations in one time-series affect the other one across different timescales.

The wavelet coherence between two time series X(t) and Y(t) is defined as:

$R_{x,y}^2(\tau,s) = \frac{|S(s^{-1}W_{x_i x_j}(\tau,s))|^2}{S(s^{-1}|W_{x_i}(\tau,s)|^2)\cdot S(s^{-1}|W_{x_j}(\tau,s)|^2)}$, $\backslash R^2 \in [0,1]$

The wavelet cross-spectrum for two time series x(t) and y(t) can be defined as:

$$C_{x,y}(a,b) = X*(a,b)Y(a,b) \quad (3)$$

Where X(a,b) and Y(a,b) are the wavelet coefficients of X(t) and Y(t), and * implies complex conjugate. Equation (10) defines Wavelet cross spectrum at each point in time-scale domain. Since C_{xy} is complex in nature, in the case of a complex wavelet as Mortlet, its absolute value

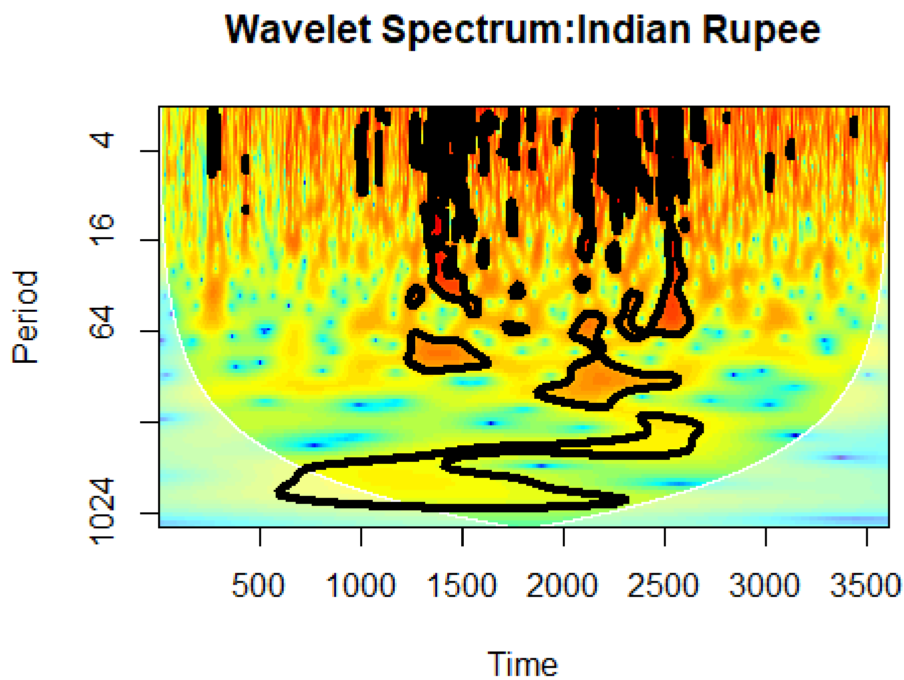
$\left|C_{xy}(a,b)\right|^2 = |\,X*(a,b)Y(a,b)|^2 = |X(a,b)|^2|Y(a,b)|^2$ is used for visualization. Wavelet cross-spectrum is employed to see how the variance in one time series influences the other timeseries across different frequencies.

As we are familar with the terminology, let us see how we can apply this to some real data and get some results! We make use of the r package biwavelet for the forthcoming section. We analyze the dependence structure between indian stock(NSE 50 daily returns) and forex(INR/USD daily returns) market.

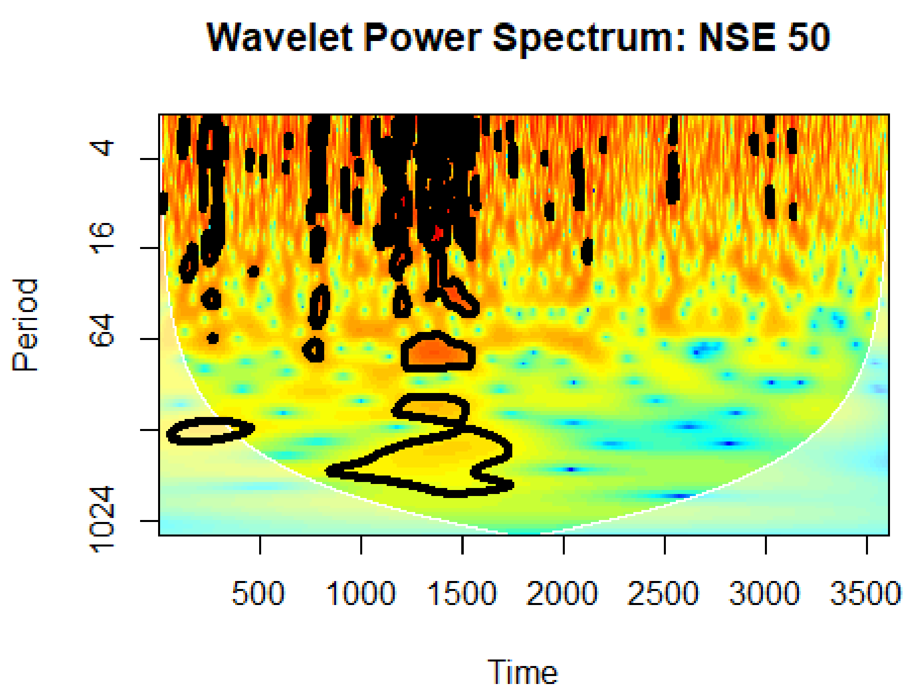
```
#first let us load the data
load("C:/Users/Anoop/Documents/leveldata.RData")
library(biwavelet)

## biwavelet 0.20.11 loaded.

x=leveldata$fret
y=leveldata$nret
inr=data.frame(1:3609,x)
nse=data.frame(1:3609,y)
plot(wt(inr),main="Wavelet Spectrum:Indian Rupee")
```




```
# the command wt() shows how to estimate wavelet power spectrum
plot(wt(nse), main="Wavelet Power Spectrum: NSE 50 ")
```



Let us analyze both of these power spectra. The data ranges from 2003 to 2017. The notable incidents are

1. The General Elections of 2004, 2009 and 2014
2. The 2008 crisis
3. The Euro-Crisis

In the power spectrum, the Y axis shows the frequency/timescale, while the X axis shows the time. Looking at the Indian Rupee power spectrum, we can see that the market was less volatile dutring the 2004 elections. However, there is increased volatility during the period 2008-2014. We had events such as the sub-prime crisis, the 2009 general elections, the Eurozone debit crisis and the 2014 general election. Moreover, between 2014 to 2014, there was some fluctuations in the oil prices that could have resulted in ER volatility. Considering the fact that India is a major oil-importing nation, this is possibility to be considered.

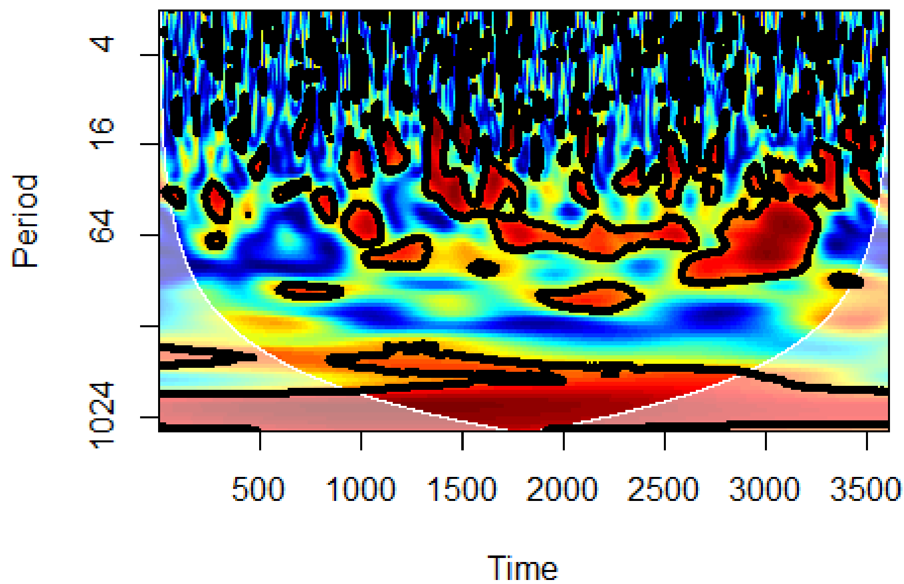
Looking at the power spectrum, we can see that the volatility spilt from High frequency(low-scales) to low-frequency(high-scales) during the 2008 crisis. It is visible from the period 2-4 days (intra-week) to 256-512 (approximately annual-biannual) days. From this, we can infer that the market flcutations affected participants of all investment horizons.

Looking at the NSE 50, we can see that while the market was hugely affected by the 2008 crisis, its reaction on other events are negligible as compared to that of forex market.

Next we see how these markets interact with each other. First we will see the wavelet coherence and then, the wavelet cross spectra.

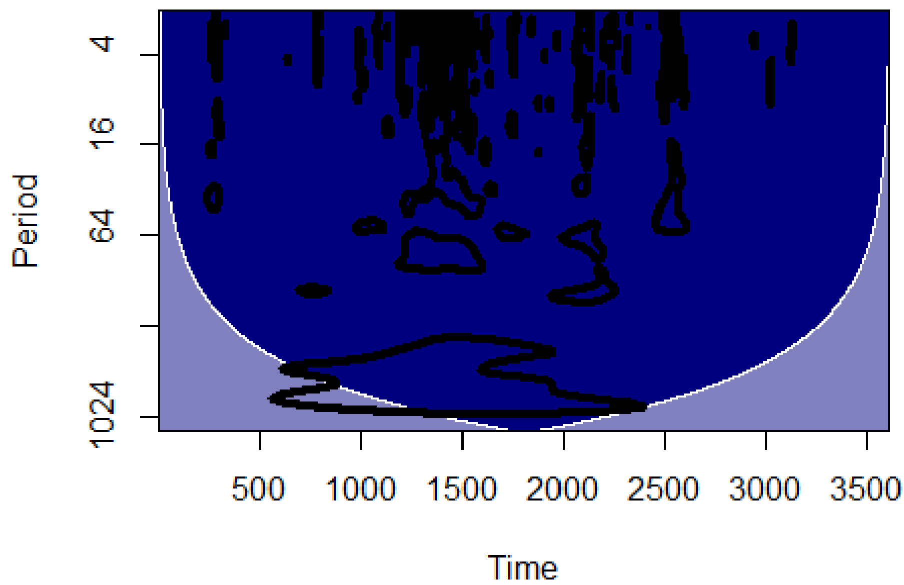
```
plot(wtc(inr,nse))

##
  This step takes some time as there is some simulation involved. Please be patient.
```



From the plot, it is evident that the markets are highly correlated between 2 to 64 days. Next we see how flcutuations in one market affect the other using wavelet cross spectral analysis.

```
plot(xwt(inr,nse))
```



From the plot, we can see that the markets influenced each other during the 2008 crisis,as well as during other events such as the general elections and the Eurozone debt crisis. However, from the heatmap(color of the spectra), the stregth of cross-correlation is weak.

Note: from blue to red, the heat map shows increasing strength of connection between the markets. Blueish--> weak, red---> strong. Same anology can be applied to the wavelet spectrum and wavelet coherence plots.

Here, I have presented a basic idea of how to carry out CWT based wavelet analysis in R. I have skipped a detailed analysis of the results as the idea of this piece is to introduce wavelet analysis in R.The readers are requested to refer to any standard works for a more detailed introduction on wavelets.

One such source is:

An Introduction to Wavelets and Other Filtering Methods in Finance and Economics by Gencay et.al. (2001)

For a detailed explanation of wavelet spectral analysis of financial markets, the readers can refer to my article titled: Fractal market hypothesis: evidence for nine Asian forex markets, Indian Economic Review, 2017. (Bit of self-promotion here!)

Available at :
https://www.researchgate.net/publication/321948632_Fractal_market_hypothesis_evidence_for_nine_Asian_forex_markets

In the next part, I will explain the nuances of DWT and MODWT. Stay tuned!