~~Prove that $N \simeq N \times N$~~

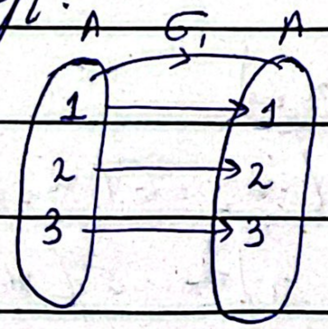
Permutation:

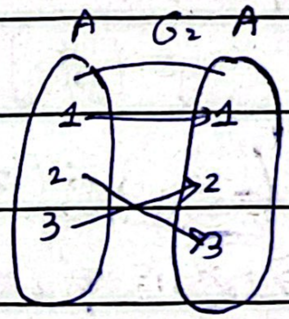
Let A be a non-empty set

Then a set

$$S_n = \left\{ \sigma \;\middle/\; \sigma : A \to A \text{ is bijective function} \right\}$$

is called permutational group.





$\sigma_1(1) = 1$

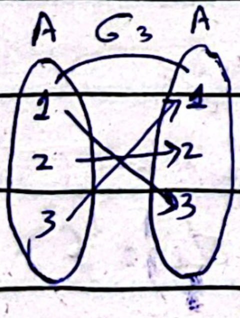
$\sigma_1(2) = 2$

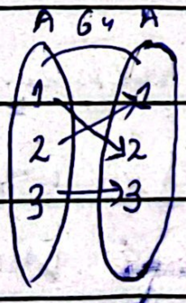
$\sigma_1(3) = 3$

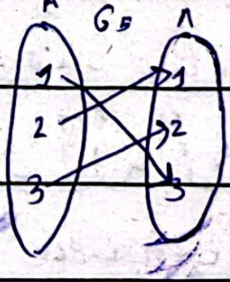
$\sigma_2(1) = 1$

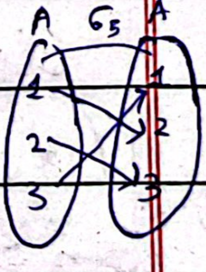
$\sigma_2(2) = 3$

$\sigma_2(3) = 2$









$$S_3 = \{ \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6 \}$$

Prove that $N \approx N \times N$.

Proof:

we defined a mapping

$f: N \longrightarrow N \times N$

as

$f(x) = (x, \sigma(x))$

wher $\sigma: N \to N$ is bijective.

Now we have to show

i) f is 1-1:

$f(x_1) = f(x_2)$

$(x_1, \sigma(x_1)) = (x_2, \sigma(x_2))$

$\Rightarrow x_1 = x_2$ & $\sigma(x_1) = \sigma(x_2)$

$\Rightarrow x_1 = x_2$

$\Rightarrow f$ is 1-1

(ii) f is onto:

Clearly for each image $(x, \sigma(x)) \in N \times N$ we have an element $x \in N$ under f such that

$f(x) = (x, \sigma(x))$

$\Rightarrow f$ is onto

So $f: N \to N \times N$ is bijective
$N \approx N \times N$.

6.6 **show that countable union of countable sets is countable.**

Proof:

Let $\{S_K : K \in I\}$ be a collection disjoint countable sets.

To Prove:

$\bigcup_{K \in N} S_K$ is Countable.

There will be two cases $(\bigcap_{K \in N} S_K = \phi$

Case # I

If S_K are finite set ~~a~~for all $K \in N$.

$\Rightarrow \bigcup_{K \in N} S_K$ is finite

$\Rightarrow \bigcup_{K \in N} S_K$ is countable

Case # II

If $S_K \approx N$ for all $K \in N$.

Then we suppose

$S_1 = \{x_{11}, x_{12}, x_{13}, \dots\}$

$S_2 = \{x_{21}, x_{22}, x_{33}, \dots\}$

$\vdots$

$S_n = \{x_{n1}, x_{n2}, \dots\}$

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$$\bigcup_{K\in N} S_K = S_1 \cup S_2 \cup S_3 \cup \cdots$$

$$= \{x_{11}, x_{12}, \ldots x_{21}, x_{22}, \ldots, x_{ij}, \ldots\}$$

Now we define a maping,

$$f : \bigcup_{K\in N} S_K \longrightarrow N \times N$$

$$f(x_{ij}) = (i, j)$$

Now we have to show that,

i) f is one-one.

$$f(x_{i_1}, j_1) = f(x_{i_2} j_2)$$

$$(i_1, j_1) = (i_2, j_2)$$

$$\Rightarrow i_1 = i_2 \ \& \ j_1 = j_2$$

$$\Rightarrow x_{i_1 j_1} = x_{i_2 j_2}$$

$\Rightarrow f$ is 1-1

ii) f is onto:

Clearly for each image $(i, j) \in N \times N$ we have an element $x_{ij} \in \bigcup_{K\in N} S_K$ under f.

such that

$$f(x_{ij}) = (i,j)$$

$\Rightarrow$ f is onto

So $f : \bigcup_{K \in N} S_K \longrightarrow N \times N$ is bijective

$$\Rightarrow \bigcup_{K \in N} S_K \approx N \times N \text{ --- } ①$$

we know that

$$N \times N \approx N \text{ --- } ②$$

Then by transitive property

① & ② $\Rightarrow \bigcup_{K \in N} S_K \approx N$

$\Rightarrow \bigcup_{K \in N} S_K$ is countable.

6.7 **Show that set rational number Q is countable.**

Proof:

Note that (ya thena countable chahiya)

$$Q = Q^+ \cup \{0\} \cup Q^- \text{ --- } ①$$

we have to show

i) **Q^+ is countable:**

we define a mapping $f : Q^+ \longrightarrow N \times N$.

as $f\left(\frac{p}{q}\right) = (p, q)$

Now we have to show

f is 1-1:

$$f\left(\frac{p_1}{q_1}\right) = f\left(\frac{p_2}{q_2}\right)$$

$\Rightarrow (p_1, q_1) = (p_2, q_2)$

$\Rightarrow p_1 = p_2$ & $q_1 = q_2$

$\Rightarrow \frac{p_1}{q_1} = \frac{p_2}{q_2}$

$\Rightarrow$ f is 1-1

f is onto:

Clearly for each image $(p, q) \in N \times N$ there is an element $p/q \in Q^+$ under f such that

$$f\left(p/q\right) = (p, q)$$

$\Rightarrow$ f is onto

~~Hence~~

so $f : Q^+ \to N \times N$

is bijective

$\Rightarrow Q^+ \approx N \times N$ —— ①

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we know that

$$N \times N \approx N \quad \text{—} \ ②$$

Then by Transitive property

$$① \ \& \ ② \Rightarrow Q^+ \approx N \quad \text{—} \ ③$$

$\Rightarrow Q^+$ is countable.

ii) $\{0\}$ is countable:

since $\{0\}$ is finite

so $\{0\}$ is countable.

iii) Q^- is countable:

we define mapping $g : Q^+ \longrightarrow Q^-$

as

$$g\left(\frac{p}{q}\right) = \frac{-p}{q}$$

now we have to show

g is 1-1:

$$g\left(\frac{p_1}{q_1}\right) = g\left(\frac{p_2}{q_2}\right)$$

$$\frac{-p_1}{q_1} = \frac{-p_2}{q_2}$$

$$\Rightarrow \frac{p_1}{q_1} = \frac{p_2}{q_2}$$

$\Rightarrow g$ is 1-1

g is onto:

Clearly for each image $-\frac{p}{q} \in Q^-$ there is an element $\frac{p}{q} \in Q^+$ under g such that

$$g\left(\frac{p}{q}\right) = -\frac{p}{q}$$

$\Rightarrow g$ is onto

so

$$g : Q^+ \to Q^-$$

is bijective

$\Rightarrow Q^+ \approx Q^- \to (4)$

By symmetric property

$(4) \Rightarrow Q^- \approx Q^+ \to (5)$

By using transitive property

(5) & $(3) \Rightarrow Q^- \approx N$

$\Rightarrow Q^-$ is countable

so

$Q^+ \cup \{0\} \cup Q^-$ is countable

Theorem 6.8:

The unit interval $I = [0, 1]$ is uncountable.

Proof:

Suppose $I = [0, 1]$ is countable

i.e $I = [0, 1] \approx N$

Then Let, $I = \{x_1, x_2, x_3, \dots\}$

~~where~~ Each element of I can be written as

$x_1 = 0, a_{11}, a_{12}, a_{13} \dots\dots \ ; \ a_{11} \neq 0$

$x_2 = 0, a_{21}, a_{22}, a_{23}, \dots\dots \ ; \ a_{22} \neq 0$

$\vdots$

$x_n = 0, a_{n1}, a_{n2}, a_{n3}, \dots\dots \ ; \ a_{nn} \neq 0$

$\vdots$

single digit

where $a_{ij} \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$

For example

$x_n = \frac{1}{n}$

$x_1 = 1 \approx 0.9999 \dots$

~~$x_2 = 0.5 = \frac{1}{2} \approx 0.5099\dots$~~

$x_2 = \frac{1}{2} = 0.5 \approx 0.4999\dots$

Now we contract new number

$y = 0.67\,80\,197\dots \leftarrow \quad y = 0.b_1 b_2 b_3 \dots\dots \in [0, 1] = I$

Choose b_1 so $b_1 \neq a_{11}$ & $b_1 \neq 0$

Choose b_2 so $b_2 \neq a_{22}$ & $b_2 \neq 0 \Rightarrow y \neq x_2$

$\vdots$

Choose b_n so $b_n \neq a_{nn}$ & $b_n \neq 0 \Rightarrow y \neq x_n$

$\vdots$

$\Rightarrow y \notin \{x_1, x_2, \ldots\} = I$

$\Rightarrow y \notin I$

which is contradiction to the fact $y \in I$.

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CH # 6

Cordinal Number:

A set A assign a symbol in such a way that two set A & B are assigned the same symbol if and only if they are equipatent. This symbal is called cardinality or cardinal number of A and it is denated by

$|A|$, $n(A)$ or $card(A)$

Remark:

i) Simply "size of the set A is called cardinality of A."

ii) we use cordinality symbol i.e $card(A)$

Finite cardinal numbers

The cardinal number of finit sets are called finite cardinal number.

Example:

card $\phi = 0$

card $\{0\} = 1$ (Total numbers of set)

card $\{0, 1\} = 2$

$\vdots$

card $\{0, 1, 2, \dots, n-1\} = n$

Transfinite cardinal number:

Cardinal numbers of infinite sets are called infinite or transfinite cardinal number.

There are two types of Transfinite Cardinal number.

Aleph-nought $\aleph_0$:

Aleph-nought $\aleph_0$ is the cardinality of infinite countable set.

Example:

card $(N) = \aleph_0$

card $(Q) = \aleph_0$

Card $(Z) = \aleph_0$

Power of Continuum C:

Power of continum C is the cordinality of un-countable set.

Example

$cord(\mathbb{R}) = C$

$Cord[0,1] = C$

Remember that:

i) $A \approx B \iff cord(A) = cord(B)$

ii) $A \approx \{1, 2, \ldots, n\} \iff cord(A) = cord\{1, 2, 3, \ldots, n\} = n$

iii) $A \approx N \iff cord(A) = cord(N) = \aleph_0$

iv) $A \approx [0, 1] \iff cord(A) = cord[0, 1] = C$

v) $[a, b] \approx [a, b) \approx (a, b] \approx (a, b)$

All interval have same cordinality that is power of continum.

Q Show that

$$card\,[a,b] = card\,[0,1]$$

sol

To prove

$$card\,[a,b] = card\,[0,1]$$

we define a mapping

$$f : [0,1] \longrightarrow [a,b]$$

as

$$f(x) = a + (b-a)$$

Now we have to show

i) f is 1-1:

Consider

$f(x_1) = f(x_2)$ where $x_1, x_2 \in [0,1]$

$$a + (b-a)x_1 = a + (b-a)x_2$$

$$(b-a)x_1 = (b-a)x_2$$

$$x_1 = x_2$$

f is 1-1

ii) f is onto:

as

$$f(0) = a + (b-a)(0) = a$$

$$f(1) = a + (b-a)(1)$$

$$= a + b - a = b$$

$$f(1/2) = a + (b-a)(1/2)$$
$$= a + \frac{1}{2}b - \frac{1}{2}a$$
$$= \frac{1}{2}a + \frac{1}{2}b$$

similarly, every element of interval $[a, b]$ has an eleme in $[0, 1]$ under f.

$\Rightarrow$ f is onto

Hence

$$f : [0,1] \longrightarrow [a,b] \text{ is bijective}$$

$\Rightarrow [0,1] \approx [a,b]$

$\Rightarrow [a,b] \approx [0,1]$

$\Rightarrow$ card $[a,b]$ = card $[0,1]$ (same)

$\Rightarrow$ **show that close interval $[a,b]$ is uncountable.**

Show that close interval $[0, 1]$ is an infinite set

To prove:

$[0,1]$ is an infinite set

we take a proper subset $\left[\frac{1}{3}, \frac{1}{2}\right] \subset [0,1]$ and define a maping

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تاریخ:

$$f : [0,1] \longrightarrow \left[\frac{1}{3}, \frac{1}{2}\right]$$

as

$$f(x) = \frac{1}{3} + \left(\frac{1}{2} - \frac{1}{3}\right) x$$

$$= \frac{1}{3} + \frac{1}{6} x \qquad \text{(same)}$$

Ordering of Cardinal numbers:

Let A & B be sets. Then

$$\text{card}(A) \leq \text{card}(B)$$

if there exist a one-to-one mapping $f : A \to B$ such that either f is not onto or $A \simeq B$.

Exp 6.4 (c)

Prove that $\aleph_0 < C$

Proof:

Suppose,

$$\aleph_0 = \text{card}(\mathbb{N})$$

$$C = \text{card}([0,1])$$

To prove

$$\aleph_0 < C$$

we define a mapping

$$f : N \to [0,1]$$

as

$$f(x) = \frac{1}{x}$$

we show that

f is one-to-one:

Let $x_1, x_2 \in N$

we have

$$f(x_1) = f(x_2)$$

$$\frac{1}{x_1} = \frac{1}{x_2}$$

$\Rightarrow x_2 = x_1$

$\Rightarrow x_1 = x_2$

$\Rightarrow$ f is one-to-one

f is not onto:

Clearly for each element of $[0,1]$ have no element is set N under f such that

$$f(x) = \frac{1}{2}$$

6.7

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$\Rightarrow f$ is not onto

$\text{card}(\mathbb{N}) < \text{card}\,[0,1]$

$\Rightarrow \aleph_0 < C$

Solved Problem

imp

6.7: Let X be a non-empty set and $C(X)$ be the family of characteristic function of X. Then show that $P(X) \approx C(X)$.

Sol:

To Prove $P(X) \approx C(X)$

we define a mapping

$f : P(X) \to C(X)$

as

$$f(A) = \chi_A \quad \text{for all } A \in P(x)$$

$X = \{a, b\}$

$P(X) = \{\varphi, \{a\}, \{b\}, X$

$C(X) = \{\chi_\varphi, \chi_{\{a\}}, \chi_{\{b\}},$

Now we have to show f is

i) **one-to-one:**

Consider

$f(A_1) = f(A_2)$ where $A_1, A_2 \in P(X)$

$\chi_{A_1} = \chi_{A_2}$

$\Rightarrow \chi_{A_1}(x) = \chi_{A_2}(x)$ —— ①

To prove $A_1 = A_2$

Let

$x \in A_1$

$\chi_{A_1}(x) = 1$

① $\Rightarrow \chi_{A_2}(x) = 1$

$\Rightarrow x \in A_2$

Hence $A_1 \subseteq A_2$

similarly

$A_2 \subseteq A_1$

$\Rightarrow A_1 = A_2$

$\Rightarrow f$ is one to one.

ii) **onto**:

Clearly for each

$\chi_A \in C(X)$ we have

an element $A \in P(x)$

under f such that

$f(A) = \chi_A$.

Mon Tue Wed Thu Fri Sat تاریخ:

$\Rightarrow f$ is onto.

Hence $f: P(X) \to C(X)$ is bijective

$\Rightarrow P(X) \approx C(X)$

Cantor's Theorem:

For any set A, we have

$card(A) < card[P(A)]$

Proof:

Let $x \in A \Rightarrow \{x\} \in P(A)$

Define a maping $f: A \to P(A)$

as

$f(x) = \{x\}$ for all $x \in A$.

Now we have to show

(i) f is one-to-one:

Consider

$f(x_1) = f(x_2)$ where $x_1, x_2 \in A$

$\{x_1\} = \{x_2\}$

$\Rightarrow x_1 = x_2$

$\Rightarrow f: A \to P(A)$ is one-to-one.

$\Rightarrow A \precsim P(A)$ precedes

Which is contradiction to fact $g(b) = B$.

Hence

$$\text{cord}(A) \neq \text{cord}(P(A))$$

$$① \Rightarrow \text{cord}(A) < \text{cord}[P(A)]$$

Exponent in cordinals:

Let A and B be sets consider a collection

$$B^{A} = \{f \,/\, f : A \to B\}$$

Then

$$\text{cord}(B^{A}) = [\text{cord}(B)]^{\text{cor}(A)}$$
$$= \beta^{\alpha}$$

is called exponent in cardinals

Example:

Let X be universal set

$$\chi_A = X \to \{0, 1\}$$

$$c(x) = \{\chi_A \mid —\} = \{0, 1\}^{x}$$

$$\text{cord}\{c(x)\} = \text{cord}\{0, 1\}^{x} = [\text{cord}\{0, 1\}]^{\text{cord}\, x}$$
$$= 2^{\alpha}$$

Imp

Continuum Hypothesis:

Prove that $2^{\aleph_0} = c$

Proof

Let $\text{card}(\mathbb{R}) = c$

we know that

$P(\mathbb{Q}) \approx C(\mathbb{Q})$

$\text{card}(P(\mathbb{Q})) = \text{card}(C(\mathbb{Q})) = 2^{\aleph_0}$

Now define a function

$$f : \mathbb{R} \rightarrow P(\mathbb{Q})$$

as

$$f(x) = \{q \in \mathbb{Q} / q < x\}$$

now show that

f is one-to-one:

Let

$x, y \in \mathbb{R}$ and $x \neq y$

$\Rightarrow$ either $x < y$ or $y < x$

if $x < y$

then by using density theorem (property of real numbers) there exist $q \in \mathbb{Q}$ such that

$x < q < y$

$\Rightarrow x < q$ & $q < y$

$q \notin f(x)$ but $q \in f(y)$

$\Rightarrow f(x) \neq f(y)$

$\Rightarrow f$ is one-one

$\Rightarrow \mathbb{R} \precsim P(\mathbb{Q})$

$\Rightarrow \text{card}(\mathbb{R}) \leq \text{card}(P(\mathbb{Q}))$

$c \leq 2^{\aleph_0}$ —— ⓐ

For reverse inequality $2^{\aleph_0} \leq c$

we consider

$\text{card}(C(\mathbb{N})) = 2^{\aleph_0}$ $\boxed{C(\mathbb{N}) = \{\chi_A \mid A \subseteq \mathbb{N}\}}$

$\text{card}([0,1]) = c$

Now define a maping $\phi : C(\mathbb{N}) \to [0,1]$.

$$\phi(\chi_A) = 0.\chi_A(1)\chi_A(2)\chi_A(3)\ldots$$

Now we have to show ~~A = {1, 3, 2, 6, 8, 9}~~

ϕ is one-one.

Let $\phi(\chi_A) = \phi(\chi_B)$

$\Rightarrow 0.\chi_A(1)\chi_A(2)\ldots = 0.\chi_B(1)\chi_B(2)\ldots$

$\Rightarrow \chi_A(1) = \chi_B(1)$

$\Rightarrow \chi_A(2) = \chi_B(2)$

$\vdots$

$\chi_A(x) = \chi_B(x)$

$\forall x \in N$

$\chi_A = \chi_B$

ϕ is one-one

$c(N) \precsim [0,1]$

$card(C(N)) \leq card([0,1])$

$2^{\aleph_0} \leq c$ —— (b)

From (a) & (b) we get

$2^{\aleph_0} = c$

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Theorem 6.10

(Schroeder-Bernstein)

If

$card(X) \leq card(Y)$ $\quad\subseteq$ = subset

& $card(Y) \leq card(X)$

then show that $card(X) = card(Y)$

Proof:

To prove above statement we will prove the following equivalent statement

"Let X, Y, X_1 be sets such that $X_1 \subseteq Y \subseteq X$ ~~then~~ and $X \approx X_1$, then $X \approx Y$."

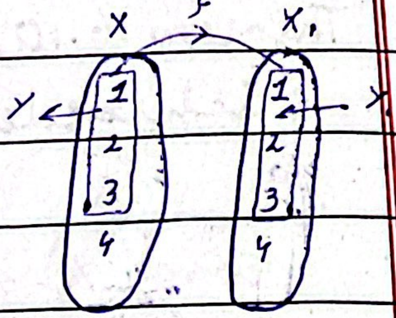
Since $X \approx X_1$, there exist a bijective mapping

$$f : X \longrightarrow X_1$$

Since $Y \subseteq X$, the restriction of $f : Y \longrightarrow X_1$ is one-to-one.

Let

$f(Y) = Y_1$



$\Rightarrow Y \approx Y_1$ & $Y_1 \subseteq X_1 \subseteq Y \subseteq X$

For similar reason,

$f(X_1) = X_2$

& $X_1 \approx X_2$ where $X_2 \subseteq Y_1 \subseteq X_1 \subseteq Y \subseteq X$

$\Rightarrow f : X_1 \to X_2$ is bijective

Accordingly, there exist equipotent sets $X, X_2, X_2, \ldots$ and equipotent set $Y, Y_1, Y_2, \ldots$

such that

$\ldots Y_3 \subseteq X_3 \subset Y_2 \subseteq X_2 \subseteq Y_1 \subseteq X_1 \subseteq Y \subseteq X$

and

$f : X_k \to X_{k+1}$ & $f : Y_k \to Y_{k+1}$

are bijective

Let $B = X \cap Y \cap X_1 \cap Y_1 \cap X_2 \cap Y_2 \cap \ldots$

$A/B = A \cap B^c$

Mon Tue Wed Thu Fri Sat $X/Y = X \cap Y^c$ تاریخ: ________

Then

$$X = (X/Y) \cup (Y/X_1) \cup (X_1/Y_1) \cup \cdots \cup B$$

$$Y = (Y/X_1) \cup (X_1/Y_1) \cup (Y_2/X_2) \cup \cdots \cup B$$

Furthermore,

$X/Y,\ X_1/Y_1,\ X_2/Y_2, \ldots$ are equipotent and

$$f : (X_K \setminus Y_K) \longrightarrow (X_{K+1} \setminus Y_{K+1})$$

is bijective

If we define a function

$$g : X \longrightarrow Y$$

as

$$g(x) = \begin{cases} f(x) & \text{if } x \in X_K/Y_K \text{ or } x \in X/Y \\ x & \text{if } x \in Y_K/X_K \text{ or } x \in B \end{cases}$$

Then g is bijective so $X \approx Y$.

Cardinal Arithmetic:

Let $\alpha = \text{cord}(A$ & $\beta = \text{cord}(B)$

Then addition and multiplication are defined as

i) $\alpha + \beta = \text{cord}(A) + \text{cord}(B)$

$= \text{cord}(A \cup B)$ where $A \cap B = \phi$

ii) $\alpha\beta = \text{cord}(A) \cdot \text{cord}(B)$

$= \text{cord}(A \times B)$

Example:

i) $m = \text{cord}\{1, 3, 5, \ldots, m\}$

$n = \text{cord}\{2, 4, 6, \ldots, n\}$

$\Rightarrow m + n = \text{cord}\{1, 3, 5, \ldots, m\} + \text{cord}\{2, 4, 6, \ldots, n\}$

$= \text{cord}\{1, 3, 5, \ldots, m\} \cup \{2, 4, 6, \ldots, n\}$

$= \text{cord}\{1, 2, 3, 4, 5, 6, \ldots m, n\}$

ii) If n is a cordinality of finite set.

Then $n + \aleph_0 = \aleph_0$

cosider

$n + \aleph_0 = \text{cord}\{1, 2, \ldots, n\} + \text{cord}\{n+1, n+2, \ldots\}$

Clearly $\{1, 2, \ldots n\} \cap \{n+1, n+2, \ldots\} = \phi$

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تاریخ:

so,

$n + N_0 = \text{cord}\{1, 2, \ldots, n\} \cup \{n+1, n+2, \ldots\}$

$= \text{cord}\{1, 2, 3, \ldots, n, n+1, n+2, \ldots\}$

$= \text{cord}(N)$

$n + N_0 = N_0$

iii) prove that $N_0 + N_0 = N_0$

consider

$N_0 + N_0 = \text{cord}\{1, 3, 5, \ldots\} + \text{cord}\{2, 4, 6, \ldots\}$

clearly $\{1, 3, 5, \ldots\} \cap \{2, 4, 6, \ldots\} = \phi$

so

$N_0 + N_0 = \text{cord}\{1, 3, 5, \ldots\} \cup \{2, 4, 6, \ldots\}$

$= \text{cord}\{1, 2, 3, 4, 5, \ldots\}$

$= \text{cord}(N)$

$= N_0$

$N_0 + N_0 = N_0$

iv) Prove that $c + c = c$

cosider

$c + c = \text{cord}\left[0, \frac{1}{2}\right] + \text{cord}\left(\frac{1}{2}, 1\right]$

clearly $\left[0, \frac{1}{2}\right] \cap \left[\frac{1}{2}, 1\right] = \phi$

So

$C+C = \text{card}\left[0, \frac{1}{2}\right] \cup \left(\frac{1}{2}, 1\right]$

$= \text{card}[0, 1]$

$C + C = C$

Ex 6.6

b) prove $\aleph_0 \aleph_0 = \aleph_0$

$\aleph_0 \aleph_0 = \text{card}(\mathbb{N}) . \text{card}(\mathbb{N})$

$\aleph_0 \aleph_0 = \text{card}(\mathbb{N} \times \mathbb{N})$ — ①

we know that

$\mathbb{N} \times \mathbb{N} \simeq \mathbb{N}$

$\text{card}(\mathbb{N} \times \mathbb{N}) = \text{card}(\mathbb{N}) = \aleph_0$

eq① become

$\aleph_0 \aleph_0 = \aleph_0$

c) Prove that $CC = C$

we know that

$C = 2^{\aleph_0}$ — ①

Taking multiplication ① with itself

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تاریخ:

$$c \cdot c = 2^{\aleph_0} \cdot 2^{\aleph_0}$$

$$c \cdot c = 2^{\aleph_0 + \aleph_0}$$

$$c \cdot c = 2^{\aleph_0}$$

$$\Rightarrow \quad c \cdot c = c \qquad \because \aleph_0 + \aleph_0 = \aleph_0$$

d) show that cordinality of $\mathbb{R}^n$ is $2^{\aleph_0}$.

Proof:

we know that

$$\mathbb{R}^n = \mathbb{R} \times \mathbb{R} \times \dots \mathbb{R}$$

$$\text{cord}(\mathbb{R}^n) = \text{cord}(\mathbb{R} \times \mathbb{R} \times \dots \times \mathbb{R})$$

$$= \text{cord}(\mathbb{R}) \dots \text{cord}(\mathbb{R}) \dots \text{cord}(\mathbb{R})$$

$$= c \cdot c \dots c \quad (n\text{-times})$$

$$\text{cord}(\mathbb{R}^n) = c^n$$

$$\text{cord}(\mathbb{R}^n) = (2^{\aleph_0})^n \qquad \because c = 2^{\aleph_0}$$

$$= 2^{\aleph_0 n}$$

$$\text{cord}(\mathbb{R}^n) = 2^{\aleph_0} \qquad \because \aleph_0 n = \aleph_0$$

e) show that cordinality of set of complex number is $2^{\aleph_0}$.

Proof: we know that

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$\mathbb{C} \approx \mathbb{R}^2$

$$\text{cord}(\mathbb{C}) = \text{cord}(\mathbb{R}^2)$$
$$= \text{cord}(\mathbb{R} \times \mathbb{R})$$
$$= \text{cord}(\mathbb{R}) \cdot \text{cord}(\mathbb{R})$$
$$= c \cdot c$$
$$= c$$

$\text{cord}\ \mathbb{C} = 2^{\aleph_0}$

6.12 (solved Problem)

show that $R \approx R^+$

To prove $\mathbb{R} \approx \mathbb{R}^+$

we defined a mapping

$$g : \mathbb{R} \to \mathbb{R}^+$$

$$g(x) = \begin{cases} \frac{x}{1+|x|} + 1 & \text{if } x < 0 \\ x+1 & \text{if } x \geq 0 \end{cases}$$

Now we have to prove

g is

i) **one-to-one**

For $x_1, x_2, \in \mathbb{R}$ we have

$$g(x_1) = g(x_2)$$

case #1

if $x_1, x_2 < 0$

case #2

if $x_1, x_2 \geq 0$

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تاریخ: ____

$$\Rightarrow \frac{x_1}{1+|x_1|} + 1 = \frac{x_2}{1+|x_2|} + 1$$

$$\Rightarrow \frac{x_1}{1+|x_1|} = \frac{x_2}{1+|x_2|}$$

$$\Rightarrow x_1 + x_1|x_2| = x_2 + x_2|x_1|$$

$$\Rightarrow x_1 - x_1 x_2 = x_2 - x_1 x_2$$

$$\Rightarrow x_1 = x_2$$

① $\Rightarrow x_1 + 1 = x_2 + 1$

$\Rightarrow x_1 = x_2$

$$\boxed{|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}}$$

$\Rightarrow g$ is one to one

ii) onto:

clearly ~~eachuma~~

$\frac{x}{1+|x|} + 1 \in \mathbb{R}^+ \ \forall x < 0$

& $x + 1 \in \mathbb{R}^+ \ \forall x \geq 0$

so, g is onto

$\Rightarrow$ Hence $g : \mathbb{R} \to \mathbb{R}^+$ is bijective.

$\mathbb{R} \approx \mathbb{R}^+$