CALCULUS-III

(BY SIR AHMAD)

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Stoke's Theorem:

Statement

According to this theorem line integral of vector function around the closed curv of a surface is equal to the surface integral of curl of a vector function over that ~~closed~~ closed surface. ~~closed~~.

$$\oint \vec{V}\cdot\vec{dl} = \oint_{s} \text{curl}\,\vec{V}\cdot\vec{ds}$$

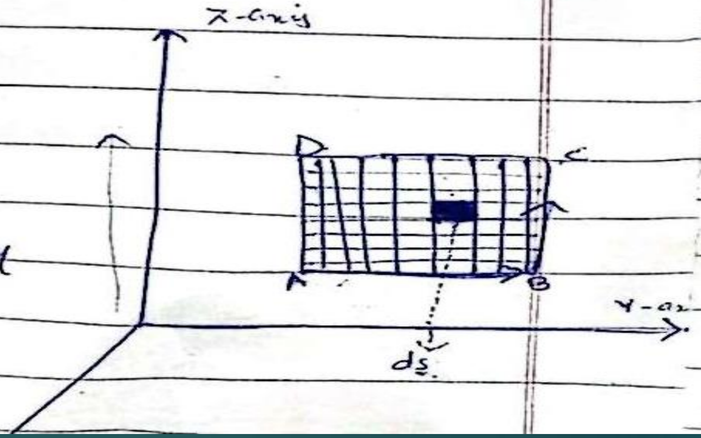
If $\vec{V}$ is the vector function then Stoke's law can be written as.

Explanation:

Consider of surface an enclosed by a curv ABCD. We divided into Large number of Small meshes - Let the Area of each mesh is $\vec{ds}$ as curl $\vec{V}$ is the line integral per unit area. So line integral is around the boundry of element of area.

$$\vec{d}s = \text{Curl}\,\vec{v}\cdot\vec{ds}$$

Suppose that we take the line integral of all the meshes within the curv ABCD will be Takes to twice in



opposite direction.

So it will be cancelled with each other. We take line integral along only curv ABCD.

let $\vec{V}$ is the vector function which can be written as

$$\vec{V} = V_x\hat{i} + V_y\hat{j} + V_z\hat{k}$$

Line integral of this vector function can be written as

$$\oint \vec{V}\cdot\vec{dl} = \oint_{x\text{-axis}} \vec{V}\cdot\vec{dl} + \oint_{y\text{-axis}} \vec{V}\cdot\vec{dl} + \oint_{z\text{-axis}} \vec{V}\cdot\vec{dl} \Longrightarrow \underline{\text{Main Equation}}$$

$$\oint_{x\text{-axis}} \vec{V}\cdot\vec{dl} = \int_A^B \vec{V}\cdot\vec{dl} + \int_B^C \vec{V}\cdot\vec{dl} + \int_C^D \vec{V}\cdot\vec{dl} + \int_D^A \vec{V}\cdot\vec{dl}$$

$$= \int_A^B v\,dl\cos\theta + \int_B^C v\,dl\cos\theta + \int_C^D v\,dl\cos\theta + \int_D^A v\,dl\cos\theta$$

$$\oint_{x\text{-axis}} \vec{v}\cdot\vec{dl} = \int_A^B v\,dl\cos(0^\circ) + \int_B^C v\,dl\cos(0^\circ) + \int_C^D v\,dl\cos(180^\circ) + \int_D^A v\,dl\cos(180^\circ)$$

$$\oint_{x\text{-axis}} \vec{v}\cdot\vec{dl} = \int_A^B v\,dl + \int_B^C v\,dl - \int_C^D v\,dl - \int_D^A v\,dl$$

$\Rightarrow$ The velocity along line $\overline{AB} = v_y - \frac{dv_y}{dz}\frac{dz}{2}$

$\Rightarrow$ The velocity along line $\overline{BC} = v_z + \frac{dv_z}{dy}\frac{dy}{2}$

$\Rightarrow$ The velocity along line $\overline{CD} = v_y + \frac{dv_y}{dz}\frac{dz}{2}$

$\Rightarrow$ The velocity along line $\overline{DA} = v_z - \frac{dv_z}{dy}\frac{dy}{2}$

$$\oint_{x\text{-axis}} \vec{v}\cdot\vec{dl} = \int_A^B \left(v_y - \frac{dv_y}{dz}\frac{dz}{2}\right)dl + \int_B^C \left(v_z + \frac{dv_z}{dy}\frac{dy}{2}\right)dl - \int_C^D \left(v_y + \frac{dv_y}{dz}\frac{dz}{2}\right)dl$$

$$- \int_D^A \left(v_z - \frac{dv_z}{dy}\frac{dy}{2}\right)dl$$

$$\therefore \int_A^B dl = dy$$

$$\therefore \int_B^A dl = dz$$

$$\oint_{x\text{-axis}} \vec{V}\cdot\vec{dl} = \left(V_y - \frac{dV_y}{dz}\frac{dz}{2}\right)dy + \left(V_z + \frac{dV_z}{dy}\,dy/2\right)dz - \left(V_y + \frac{dV_y}{dz}\frac{dz}{2}\right)dy - \left(V_z - \frac{dV_z}{dy}\frac{dy}{2}\right)dZ$$

$$= \cancel{V_y dy} - \frac{dV_y}{dz}\frac{dz}{2}dy + \cancel{V_z dz} + \frac{dV_z}{dy}\frac{dy}{2}dz$$

$$- \cancel{V_y dy} - \frac{dV_y}{dz}\frac{dz}{2}dy - \cancel{V_z dz} + \frac{dV_z}{dy}\frac{dy}{2}dz$$

$$= -\cancel{2}\frac{dV_y}{dz}\frac{dz}{\cancel{2}}dy + \cancel{2}\frac{\partial V_z}{\partial y}\frac{dy}{\cancel{2}}dz$$

$$= dy\,dz\left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right)$$

$$\oint_{x\text{-axis}} \vec{V}\cdot\vec{dl} = \left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right)ds_x$$

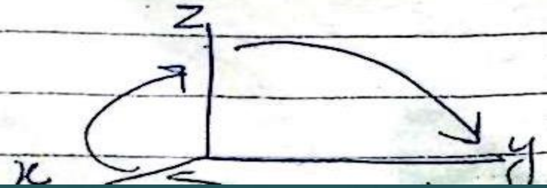
$\int_A^B dl = dy$

$\int_B^C dl = dz$

$\int_C^D dl = dy$

$\int_D^A dl = dz$

$\Rightarrow$ Rotation is clockwise.



Similarly line integral of $\vec{v}$ along y-axis & z-axis is respectively

$$\oint_{y\text{-axis}} \vec{v}\cdot\vec{dl} = \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right) ds_y$$

$$\oint_{z\text{-axis}} \vec{v}\cdot\vec{dl} = \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right) ds_z.$$

Main Equation Becomes

$$\oint \vec{v}\cdot\vec{dl} = \oint_{x\text{-axis}} \vec{v}\cdot\vec{dl} + \oint_{y\text{-axis}} \vec{v}\cdot\vec{dl} + \oint_{z\text{-axis}} \vec{v}\cdot\vec{dl}$$

$$= \oint_s \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right) ds_x + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right) ds_y$$

$$+ \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_y}{\partial y}\right) ds_z$$

Answer

$$= \oint_S \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix} \cdot (ds_x \hat{i} + ds_y \hat{j} + ds_z \hat{k})$$

$$\oint \vec{V} \cdot \vec{dl} = \oint_S (\vec{\nabla} \times \vec{V}) \vec{ds}$$

$$\oint \vec{V} \cdot \vec{dl} = \oint_S \text{curl } \vec{V} \cdot \vec{ds}$$

Hence Stoke's theorem is proved.

Example No:01

Stoke's Theorem

Q. No. (01)

Use Stokes Theorem

where

$F = F(x,y,z) = xz\,\hat{i} + yz\,\hat{j} + xy\,\hat{k}$

S is the part of the Sphere $x^2+y^2+z^2 = 4$ that lies inside the cylinder x^2+y^2-1 and above the xy-plane.

Sol:

$x^2+y^2+z^2 = 4$ and $x^2+y^2 = 1$

So $z^2 = 4-1$

$z^2 = 3$

$z = \sqrt{3}$

So $r(t) = \cos t\,\hat{i} + \sin t\,\hat{j} + \sqrt{3}\,k \qquad 0 \le t \le 2\pi$

$r'(t) = -\sin t\,\hat{i} + \cos t\,\hat{j}$

Also we have

$f(r(t)) = \sqrt{3}\cos t\,\hat{i} + \sqrt{3}\sin t\,\hat{j} + \cos t \sin t\,\hat{k}$

Apply Stoke's Theorem

$$\iint_S \operatorname{curl} f \cdot ds = \int_C F \cdot dr$$

$$= \int_0^{2\pi} F(r(t)) \cdot r'(t)\, dt$$

$$= \int_0^{2\pi} (-\sqrt{3}\cos t \sin t + \sqrt{3}\sin t \cos t)\, dt$$

$$= \sqrt{3} \int_0^{2\pi} 0\, dt$$

$= \underline{0 \text{ Ans}}$

Gauss's Divergence Theorem:

Statement:

The surface integral of a vector point function around a closed surface is equal to volume integral of divergence of that vector point function over that surface enclosing a particular volume.

Mathematically form:

$$\oint_S \vec{v}.\vec{ds} = \oint_V \vec{div}.\vec{dv}$$

let v is the vector point function which should be well-defined, continuous and differentiable.

Explanation:

Let $\vec{V}$ is the vector point function which can be written as

$$\vec{V} = V_x\hat{i} + V_y\hat{j} + V_z\hat{k}$$

As we know

$$\vec{\nabla} = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$

So Divergence of vector point function can be written as

$$div\ \vec{V} = \vec{\nabla} \cdot \vec{V}$$

$$= \left(\frac{d}{dx}\hat{i} + \frac{d}{dy}\hat{j} + \frac{d}{dz}\hat{k}\right) \cdot \left(v_x\hat{i} + v_y\hat{j} + v_z\hat{k}\right)$$

$$= \frac{dv_x}{dx} + \frac{dv_y}{dy} + \frac{dv_z}{dz}$$

The volume integral of div $\vec{V}$ is

$$\int_v div\ \vec{V}\, dv = \int\left(\frac{dv_x}{dx} + \frac{dv_y}{dy} + \frac{dv_z}{dz}\right) dv$$

$$\int_v \vec{div}\,\vec{v}\,dv = \iiint\left(\frac{dv_x}{dx} + \frac{dv_y}{dy} + \frac{dv_z}{dz}\right) dx\,dy\,dz$$

$$\int_v \vec{div}\,\vec{V}\,dv = \iiint \frac{dv_x}{dx}\,dx\,dy\,dz + \iiint \frac{dv_y}{dy}\,dx\,dy\,dz + \iiint \frac{dv_z}{dz}\,dx\,dy\,dz$$

Since for single variable.

$$d \longrightarrow d$$

$$= \iiint \frac{dv_x}{dx} \, dx\,dy\,dz + \iiint \frac{dv_y}{dy} \, dx\,dy\,dz + \iiint \frac{dv_z}{dz} \, dx\,dy\,dz$$

$$= \int dv_x \iint dy\,dz + \int dv_y \iint dx\,dz + \int dv_z \iint dx\,dy$$

= (Derivative is cut the integration)

$$= V_x \iint dy\,dz + V_y \iint dx\,dz + V_z \iint dx\,dy$$

$$= \iint V_x \, dy\,dz + \iint V_y \, dx\,dz + \iint V_z \, dx\,dy$$

$$\int_v \operatorname{div} \vec{V} \cdot dv = \int_s V_x \, ds_x + \int_s V_y \, ds_y + \int_s V_z \, ds_z$$

$$= \int_s \left(V_x \, ds_x + V_y \, ds_y + V_z \, ds_z \right)$$

$$= \int_s \left(V_x \hat{i} + V_y \hat{j} + V_z \hat{k} \right) \cdot \left(ds_x \hat{i} + ds_y \hat{j} + ds_z \hat{k} \right)$$

$$\int_V \operatorname{div} \vec{v} \cdot \vec{dv} = \int_S \vec{v} \cdot \vec{ds}$$

Hence Gauss's divergence theorem is proved.

Example No:01

Divergence Theorem

If $\vec{A} = xz^3\hat{i} - 2x^2yz\hat{j} + 2yz^4\hat{k}$. Find divergence of A at point $(1, -1, 1)$.

Sol:

$$\vec{A} = xz^3\hat{i} - 2x^2yz\hat{j} + 2yz^4\hat{k}$$

$$\vec{\nabla}\cdot\vec{A} = ?$$

$$\vec{\nabla}\cdot\vec{A} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\cdot\left(xz^3\hat{i} - 2x^2yz\hat{j} + 2yz^4\hat{k}\right)$$

$$= \frac{\partial}{\partial x}(xz^3) + \frac{\partial}{\partial y}(-2x^2yz + \frac{\partial}{\partial z}(2yz^4)$$

$$\vec{\nabla}\cdot\vec{A} = z^3 - 2x^2z + 8yz^3$$

Value of $\vec{\nabla}\cdot\vec{A}$ at $(1, -1, 1)$

$$\vec{\nabla}\cdot\vec{A} = (1)^3 - 2(1)^2 + 8(-1)(1)^3$$

$$= -9$$

Thanks For Watching