Date:

Linear Transformation:

Let U and V be two vector spaces over the same field F and let $T: U \to V$ be the function then T is called linear Transformation if the following conditions are satisfied

(i) $T(U_1 + U_2) = T(U_1) + T(U_2)$

(ii) $T(\alpha U) = \alpha T(U)$

Example 1: Check which of the following defined linear transformations from $\mathbb{R}^3$ to $\mathbb{R}^2$?

(i) $T(x_1, x_2, x_3) = (x_1 - x_2, x_1 - x_3)$

Sol: Given transformation is

$T(x_1, x_2, x_3) = (x_1 - x_2, x_1 - x_3)$

Let $U_1 = (x_1, x_2, x_3)$

and $U_2 = (y_1, y_2, y_3) \in \mathbb{R}^3$

then we prove that

(i) $T(U_1 + U_2) = T(U_1) + T(U_2)$

Date:

Now,

$$T(u_1+u_2) = T((x_1,x_2,x_3)+(y_1,y_2,y_3))$$
$$= T(x_1+y_1, x_2+y_2, x_3+y_3)$$
$$= ((x_1+y_1)-(x_2+y_2), (x_1+y_1)-(x_3+y_3))$$
$$= (x_1+y_1-x_2-y_2, x_1+y_1-x_3-y_3)$$
$$= (x_1-x_2+y_1-y_2, x_1-x_3+y_1-y_3)$$
$$= (x_1-x_2, x_1-x_3)+(y_1-y_2, y_1-y_3)$$
$$= T(x_1,x_2,x_3)+T(y_1,y_2,y_3)$$
$$= T(u_1)+T(u_2)$$

(ii) Let $\alpha \in \mathbb{R}$ and $u_1 = (x_1, x_2, x_3) \in \mathbb{R}^3$ then we prove that $T(\alpha u_1) = \alpha T(u_1)$

Now,

$$T(\alpha u_1) = T(\alpha(x_1,x_2,x_3))$$
$$= T(\alpha x_1, \alpha x_2, \alpha x_3)$$
$$= (\alpha x_1 - \alpha x_2, \alpha x_1 - \alpha x_3)$$
$$= (\alpha(x_1-x_2), \alpha(x_1-x_3))$$
$$= \alpha(x_1-x_2, x_1-x_3)$$
$$= \alpha T(x_1, x_2, x_3)$$
$$= \alpha T(u_1)$$

Hence T is a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^2$.

Date: ____________

(ii) $T(x_1, x_2, x_3) = (|x_1|, x_2 - x_3)$

Sol:

Given transformation is

$T(x_1, x_2, x_3) = (|x_1|, x_2 - x_3)$

Let $U_1 = (x_1, x_2, x_3)$

and $U_2 = (y_1, y_2, y_3) \in \mathbb{R}^3$

then we prove

(i) $T(U_1 + U_2) = T(U_1) + T(U_2)$

Now,

$T(U_1 + U_2) = T((x_1, x_2, x_3) + (y_1, y_2, y_3))$

$= T(x_1 + y_1, x_2 + y_2, x_3 + y_3)$

$= (|x_1 + y_1|, (x_2 + y_2) - (x_3 + y_3))$

So $T(U_1 + U_2) = (|x_1 + y_1|, x_2 + y_2 - x_3 - y_3)$ — ①

Now,

$T(U_1) + T(U_2) = T(x_1, x_2, x_3) + T(y_1, y_2, y_3)$

$= (|x_1|, x_2 - x_3) + (|y_1|, y_2 - y_3)$

$= (|x_1| + |y_1|, x_2 - x_3 + y_2 - y_3)$

$= (|x_1| + |y_1|, x_2 + y_2 - x_3 - y_3)$ — ②

from ① and ②

$T(U_1 + U_2) \neq T(U_1) + T(U_2)$

Hence T is not a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^2$.

(iii) $T(x_1, x_2, x_3) = (x_1+1, x_2+x_3)$

Sol:

Given transformation is

$T(x_1, x_2, x_3) = (x_1+1, x_2+x_3)$

Let $u_1 = (x_1, x_2, x_3)$

and $u_2 = (y_1, y_2, y_3) \in \mathbb{R}^3$

then we prove

(i) $T(u_1+u_2) = T(u_1) + T(u_2)$

Now,

$$T(u_1+u_2) = T((x_1, x_2, x_3) + (y_1, y_2, y_3))$$
$$= T(x_1+y_1, x_2+y_2, x_3+y_3)$$
$$= (x_1+y_1+1, x_2+y_2+x_3+y_3)$$
$$T(u_1+u_2) = (x_1+y_1+1, x_2+x_3+y_2+y_3) \quad \text{—} ①$$

Now,

$$T(u_1)+T(u_2) = T(x_1, x_2, x_3) + T(y_1, y_2, y_3)$$
$$= (x_1+1, x_2+x_3) + (y_1+1, y_2+y_3)$$
$$= (x_1+1+y_1+1, x_2+x_3+y_2+y_3)$$
$$T(u_1)+T(u_2) = (x_1+y_1+2, x_2+x_3+y_2+y_3) \quad \text{—} ②$$

from ① and ②

$$T(u_1+u_2) \neq T(u_1) + T(u_2)$$

Hence T is not a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^2$.

Date:

(iv) $T(x_1, x_2, x_3) = (0, x_3)$

Sol:

Do Yourself

T is a L.T from $\mathbb{R}^3$ to $\mathbb{R}^2$.

(v) $T(x_1, x_2, x_3) = \left(\frac{x_1 + x_2}{x_3}, x_3\right)$

Sol:

Do Yourself

T is not a L.T from $\mathbb{R}^3$ to $\mathbb{R}^2$.

Example 2: Show that each of the following defines Linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$.

(i) $T(x_1, x_2, x_3) = (x_1 - x_2, x_2 - x_3, x_1)$

Sol:

Given transformation is

$T(x_1, x_2, x_3) = (x_1 - x_2, x_2 - x_3, x_1)$

let $u_1 = (x_1, x_2, x_3)$

and $u_2 = (y_1, y_2, y_3) \in \mathbb{R}^3$

GO GREEN

Date:

then we prove that:

(i) $T(U_1+U_2)=T(U_1)+T(U_2)$

Now,

$$T(U_1+U_2)=T((x_1,x_2,x_3)+(y_1,y_2,y_3))$$
$$=T(x_1+y_1,x_2+y_2,x_3+y_3)$$
$$=((x_1+y_1)-(x_2+y_2),(x_2+y_2)-(x_3+y_3),x_1+y_1)$$
$$=(x_1+y_1-x_2-y_2,x_2+y_2-x_3-y_3,x_1+y_1)$$
$$=(x_1-x_2+y_1-y_2,x_2-x_3+y_2-y_3,x_1+y_1)$$
$$=(x_1-x_2,x_2-x_3,x_1)+(y_1-y_2,y_2-y_3,y_1)$$
$$=T(x_1,x_2,x_3)+T(y_1,y_2,y_3)$$
$$\Rightarrow T(U_1+U_2)=T(U_1)+T(U_2)$$

Now we prove

(ii) $T(\alpha U_1)=\alpha T(U_1)$

$$T(\alpha U_1)=T(\alpha(x_1,x_2,x_3))$$
$$=T(\alpha x_1,\alpha x_2,\alpha x_3)$$
$$=(\alpha x_1-\alpha x_2,\alpha x_2-\alpha x_3,\alpha x_1)$$
$$=\alpha(x_1-x_2,x_2-x_3,x_1)$$
$$=\alpha T(x_1,x_2,x_3)$$
$$T(\alpha U_1)=\alpha T(U_1)$$

Hence T is a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$.

Date:

(ii) $T(x_1, x_2, x_3) = (x_2, -x_1, -x_3)$

Sol:

Do Yourself

T is a L.T from $\mathbb{R}^3$ to $\mathbb{R}^3$.

(iii) $T(x_1, x_2, x_3) = (x_1 - 3x_2 - 2x_3, x_2 - 4x_3, x_3)$

Sol:

Given transformation is

$T(x_1, x_2, x_3) = (x_1 - 3x_2 - 2x_3, x_2 - 4x_3, x_3)$

Let $u_1 = (x_1, x_2, x_3)$

and $u_2 = (y_1, y_2, y_3) \in \mathbb{R}^3$

then we prove

(i) $T(u_1 + u_2) = T(u_1) + T(u_2)$

Now,

$T(u_1 + u_2) = T((x_1, x_2, x_3) + (y_1, y_2, y_3))$

$= T(x_1 + y_1, x_2 + y_2, x_3 + y_3)$

$= ((x_1+y_1) - 3(x_2+y_2) - 2(x_3+y_3), (x_2+y_2) - 4(x_3+y_3), x_3+y_3)$

$= (x_1 + y_1 - 3x_2 - 3y_2 - 2x_3 - 2y_3, x_2 + y_2 - 4x_3 - 4y_3, x_3 + y_3)$

$= (x_1 - 3x_2 - 2x_3 + y_1 - 3y_2 - 2y_3, x_2 - 4x_3 + y_2 - 4y_3, x_3 + y_3)$

$= (x_1 - 3x_2 - 2x_3, x_2 - 4x_3, x_3) + (y_1 - 3y_2 - 2y_3, y_2 - 4y_3, y_3)$

$= T(x_1, x_2, x_3) + T(y_1, y_2, y_3)$

$\Rightarrow T(u_1 + u_2) = T(u_1) + T(u_2)$

(ii) Let $\alpha \in \mathbb{R}$ and $v_1 = (x_1, x_2, x_3) \in \mathbb{R}^3$

then we prove

$$T(\alpha v_1) = \alpha T(v_1) + \alpha T(v_2)$$

Now,

$$T(\alpha v_1) = T(\alpha(x_1, x_2, x_3))$$

$$= T(\alpha x_1, \alpha x_2, \alpha x_3)$$

$$= (8\alpha x_1 - 3\alpha x_2 - 2\alpha x_3, \alpha x_2 - 4\alpha x_3, \alpha x_3)$$

$$= \alpha(8x_1 - 3x_2 - 2x_3, x_2 - 4x_3, x_3)$$

$$= \alpha T(x_1, x_2, x_3)$$

$$\Rightarrow T(\alpha v_1) = \alpha T(v_1)$$

Hence T is a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$.

(iv) $T(x_1, x_2, x_3) = (x_1 + x_3, x_1 - x_3, x_2)$

Sol:

Do Yourself

Hence T is L.T from $\mathbb{R}^3$ to $\mathbb{R}^3$.