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تاریخ: ____

Chapter 7

Ex 7.1

d) $(\mathbb{Z}, R)$

$a, b \in \mathbb{Z}$ and relation R on $\mathbb{Z}$ is defined as

"$aRb \Leftrightarrow b = a^r$ where r is positive integer"

[O₁] : Reflexive:

we know that

$a = a \Rightarrow a = a^1$

$\Rightarrow aRa$

$\Rightarrow R$ is reflexive

[O₂] : Anti-symmetric

Let aRb & bRa

then there exist two positive integer r & s such that

$b = a^r$ & $a = b^s$

$\Rightarrow b = (b^s)^r$

$\Rightarrow b^1 = b^{sr}$

$\Rightarrow sr = 1$

$\Rightarrow s = 1$ & $r = 1$

$b = a$ & $a = b$

$\Rightarrow a = b$

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[O_3] transitive:

Let aRb & bRc then there exist two positive integer r & s such that

$$b = a^r \quad \& \quad c = b^s$$

$$\Rightarrow c = (a^r)^s$$

$$\Rightarrow c = a^{rs}$$

$$\Rightarrow c = a^t \qquad rs = t \text{ positive integer}$$

$$\Rightarrow aRa.$$

$$\Rightarrow R \text{ is transitive}$$

since R is a partial ordering of $\mathbb{Z}$.

Ordered subset:

Let S be an order set and $A \subseteq S$. Then the order in S induces an order in A in the following Natural way

$a \precsim b \ \forall a, b \in A$ whenever $a \precsim b$ for $a, b \in S$

then set A is called ordered subset.

Comparability

Let S be a partial ordered set and a, b ∈ S.

we say a and b are comparable if

$a \preceq b$ or $b \preceq a$

$\boxed{\text{Note that}}$:

if a & b are caparable the we can write as $a \parallel b$ otherwise it is called non-comparability.

Exp: $(N, <)$ $(N, \mid)$

$1, 2 \in N$ $2 \nmid 3$ $3 \nmid 2$

$1 < 2$ so their no comparable

$1 \parallel 2$ b/w their no.

$2 \npreceq 3$ $3 \npreceq 2$

$2 \nparallel 3$

linearly order set / totally order set:

Let S be a partial ordered set, If every pair of element of set S are comparable then S is called linear order set (totally order set)

ex: $(N, <)$

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otherwise it is non linear order set.

set construction and ordered:

Product order:

Suppose A and B are order sets. Then a relation "$\precsim$" defined on $A \times B$ as "$(a, b) \precsim (a', b')$ if $a \leq a'$ and $b \leq b'$" is called product order relation.

Lexicographical order:

suppose A and B are order sets. Then a relation '$\precsim$' defined on $A \times B$ as

$$(a, b) \precsim (a', b') \begin{cases} \text{if } a < a' \\ \text{or if } a = a' \text{ and } b < b' \end{cases}$$

is called lexicographical order relation.

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$S_1 \times S_2 \times \cdots \times S_n$ as

"$(a_1, a_2, \ldots, a_n) < (a_1', a_2', \ldots a_n')$

$$\begin{cases} \text{if} & a_1 < a_1' \\ \text{or if} & a_1 = a_1' \; ; \; a_2 = a_2', \ldots, a_{n-1} = a_{n-1}' \\ & \text{but } a_n < a_n' \end{cases}$$

Immediate succesior:

Let S be a partial ordered set and $a, b \in S$ such that $a < b$.

Then 'b' is called imediate succesior of 'a' if there is no other element ~~a & b as a < c < b~~ $c \in S$ exist b/w a & b as $a < c < b$.

limit element:

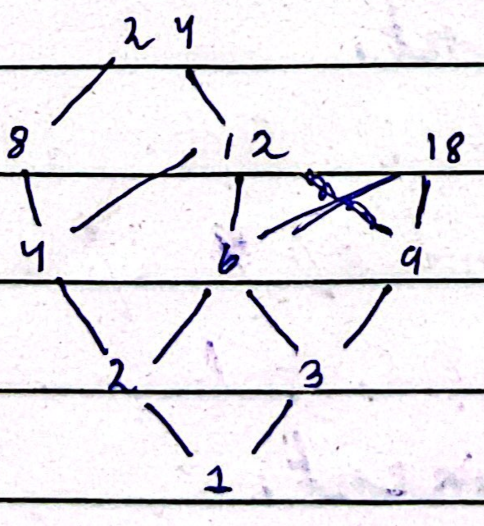
Let S be a partial ordered set and $x, c \in S$ with $x < c$. Then 'c' is called limit element of set S if there exist $y \in S$ such that $x < y < c$.

Hasse Diagrams:

7.3 (a)

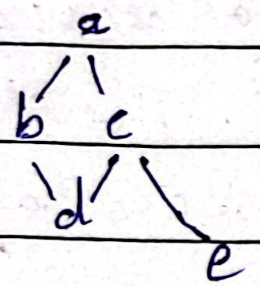
$A = \{1, 2, 3, 4, 6, 8, 9, 12, 18, 24\}$

$R = \{(x, y) \mid x \text{ divides } y\}$



we have to check ka kon kes ko divide karta ha uska accor humna lines lagani hai

$$R = \{(x, y) \mid x \leq y\}$$



Let S be a partial order set

1st element:

An element $a \in S$ is called 1st element of S if

$$a \leq x \quad \forall \; x \in S$$

last element:

An element $b \in S$ is called last element of S if

$$x \leq b, \; \forall \; x \in S.$$

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Maximal element:

An element $l \in S$ is called maximal element if

$$l \leq x \Rightarrow x = l$$

Example:

$$S = \{1, 2, 3, 4\}$$

Suprimum (least upper bounds):

And Let $A \subseteq S$ ~~Then an element $\beta \in S$ is called suprimum~~

An element $\beta \in S$ is called an upper bound of A if

$$x \leq \beta \ \forall \ x \in A$$

If 'β' precedes all upper bound of A then β is called suprimum.

$$\sup(A) = \beta = l.u.b(A)$$

Infimum (Greatest lower bounds)

And Let $A \subseteq S$. An element $\alpha \in S$ is called an lower bound

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of A if

that real line ma chalana hoga

$\alpha \leq x \; \forall \; x \in A$

If 'α' succeeds all lower bound of A then α is called infimum

$Inf(A) = \alpha = g.l.b(A)$

Example:

$A \subseteq \mathbb{R}$

$A = \{1, 2, \ldots, 10\}$

$\forall \; \alpha \leq 1$ lower bounds

$g.l.b = (A) = 1 = inf(A)$

$\forall \beta \geq 10$ upper bounds

$l.u.b(A) = 10 = sup(A)$

Example 7.7

c) $B = \{x \in Q / x > 0 \text{ and } 2 < x^2 < 3\} \subseteq Q$
$\subseteq \mathbb{R}$

$\Rightarrow 2 < x^2 < 3$

$\Rightarrow \sqrt{2} < x < \sqrt{3}$

$\alpha = \sqrt{2} \notin Q \quad \beta = \sqrt{3} \notin Q$

$\alpha = \sqrt{2} \in \mathbb{R} \quad \beta = \sqrt{3} \in \mathbb{R}$

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Example 7.8:

Q a) **Greatest common Divisor of a and b:**

Let $a, b \in \mathbb{Z}$. Then a number $d \in \mathbb{Z}$ is called g.c.d of a and b if

$d > 0$

$d | a$ and $d | b$

For $c \in \mathbb{Z}$ such that $c | a$ and $c | b$ we have $c | d$.

Symbalicaly,

$g.c.d(a, b) = d$

Least common divisor of a & b:

Let $a, b \in \mathbb{Z}$. Then a number $m \in \mathbb{Z}$ is called l.c.m of a & b if

Q $m > 0$

$a | m$ and $b | m$

For $c \in \mathbb{Z}$ such that $a | c$ and $b | c$

we have $m | c$.

Symbalicaly,

$l.c.m(a, b) = m$

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Note that:

Suppose N is ordered set by divisibility '\'

Then

$g.c.d(a,b) = \inf(a,b)$ and

$lcm = (a,b)$

$= \sup(a,b)$

where $a, b \in N$

Q construct D_n set D_{36} and also draw its Hasse Diagram and find sup & inf

b) **Defination of D_N:**

For any positive integer m, set D_m denote the set of positive divisors of m ordered by Divisibility "\".

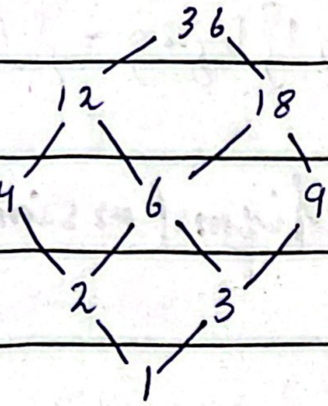
Example:

$D_{36} = \{ d \mid d \text{ divides } m = 36 \}$

$= \{1, 2, 3, 4, 6, 9, 12, 18, 36\}$

$(D_{36}, \backslash) \leftarrow$ order by Divisibility

Hasse Diagram:



$\sup(D_{36}) = \text{lcm}(a,b)$ ← pair wise
$\inf(D_{36}) = \gcd(a,b)$

$\sup(D_{36}) = \text{lcm}(1,2,\ldots,36) = 36$
$\inf(D_{36}) = \gcd(1,2,\ldots,36) = 1$

c) $S \neq \phi$

$P(S)$ = power set of S

Let $A, B \in P(S)$ then

$\sup(A,B) = A \cup B$

$\inf(A,B) = A \cap B$

Example:

$S = \{1, 2\}$

$P(S) = \{\phi, \{1\}, \{2\}, S\}$

$\{1\}, \{2\} \in P(S)$

$\inf(\{1\},\{2\}) = \{1\} \cap \{2\} = \phi$

$sup(\{1\},\{2\}) = \{1\} \cup \{2\} = \{1,2\} = S$

Imp:

Order isomorphism (or similarity mapping):

Suppose X and Y are partial ordered sets.

Then a bijective mapping $f: X \to Y$ is called order isomorphism (or similarity mapping). If f preserves the order as

$$x_1 \leq_x x_2 \iff f(x_1) \leq_y f(x_2)$$

for $x_1, x_2 \in X$.

Note:

If $f: X \to Y$ is order isomorphism then we can say X is order isomorphic to Y and write it as

$$X \simeq Y$$

Ex 7.9

b) **Show that:**

$$\mathbb{N} \simeq 2\mathbb{N}$$

sol:

we define a maping

$f : \mathbb{N} \to 2\mathbb{N}$

$f(x) = 2x$

Now we have to show f is

i) one-to-one:

$f(x_1) = f(x_2)$

$2x_1 = 2x_2$

$\Rightarrow \quad x_1 = x_2$

$\Rightarrow$ f is one-to-one.

ii) onto:

Clearly for each element $2x \in 2\mathbb{N}$ we have an element $x \in \mathbb{N}$ under f

such that

$f(x) = 2x$

$\Rightarrow$ f is onto

iii) order preserving:

$x_1 \lesssim x_2 \Leftrightarrow 2x_1 \leq 2x_2$

$x_1 \leq x_2 \Leftrightarrow f(x_1) \leq f(x_2)$

$\Rightarrow$ f is order preserving

Hence proved

$\mathbb{N} \simeq 2\mathbb{N}$

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c) Prove that

$P \not\sim A$ $\qquad P = \{1, 2, \dots\}$

$A = \{-1, -2, \dots\}$

sol: we define a map

$f : P \longrightarrow A$

$f(x) = -x$

Now we have to ~~show~~ check f is

one-to-one:

$f(x_1) = f(x_2)$

$-x_1 = -x_2$

$\Rightarrow x_1 = x_2$

$\Rightarrow$ f is one-to-one

onto:

clearly, for each element $-x \in A$ we have an element $x \in P$ under f

such that

$f(x) = -x$

f is onto

Order preserving:

$x_1 \leq x_2 \Leftrightarrow -x_1 \geq -x_2$

$x_1 \leq x_2 \Longleftrightarrow f(x_1) \leq f(x_2)$

$\Rightarrow (a \wedge b) \wedge (b \wedge c) = a \wedge b$

$\Rightarrow \cancel{\{(a \wedge b) \wedge b\} \wedge \{(a \wedge b) \wedge c\} = a}$

$\Rightarrow \cancel{[a \wedge}$

$\Rightarrow a \wedge (b \wedge b) \wedge c = a$

$\Rightarrow (a \wedge b) \wedge c = a$

$\Rightarrow a \wedge c = a$

$\Rightarrow a \leq c$

Sublattices, Isomorphic Lattices:

Let L and L' be lattices.
Then L and L' are said to be isomorphic if their exist bijective mapping $f : L \to L'$ such that

$$f(a \wedge b) = f(a) \wedge f(b)$$

and $f(a \vee b) = f(a) \vee f(b)$