CALCULUS-III

BS-MATHEMATICS

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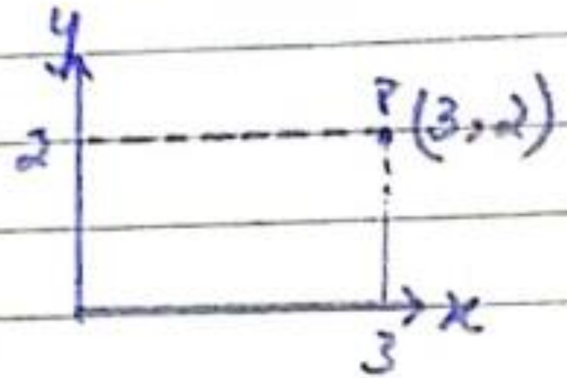
TOPICS

1) **ORTHOGONAL CURVILINEAR COORDINATES**
2) **ARC LENGTH**
3) **CYLINDRICAL COORDINATES**

ORTHOGONAL CURVILINEAR COORDINATES

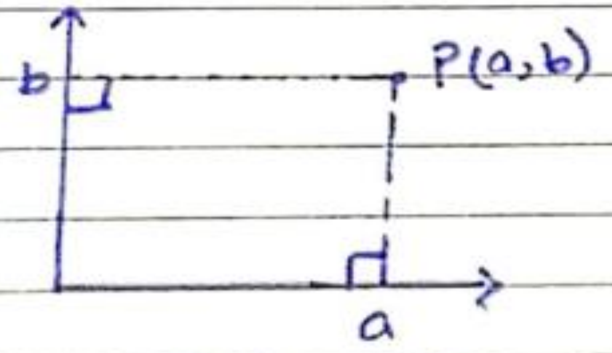
COORDINATES:-

The set of values that shows exact distance.

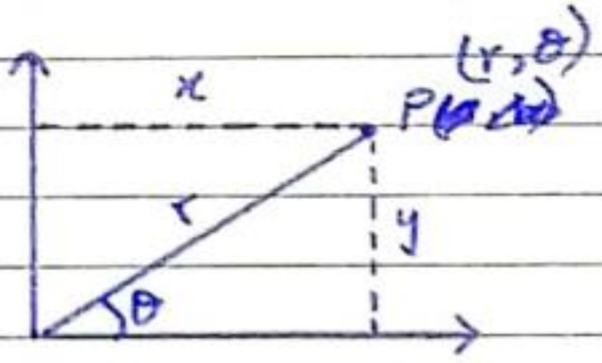


Types:-

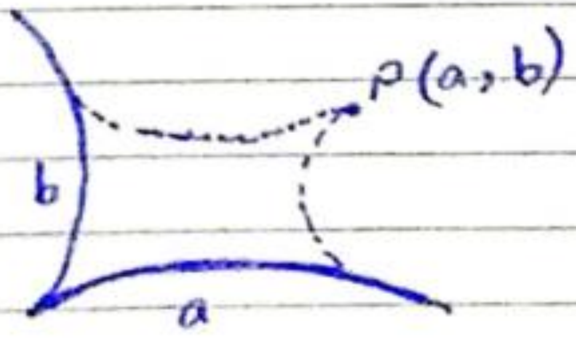
1) Cartesian (x, y)
2) Curvilinear (U_1, U_2)
3) Polar (r, θ)



Cartesian(Rectangular) coordinates

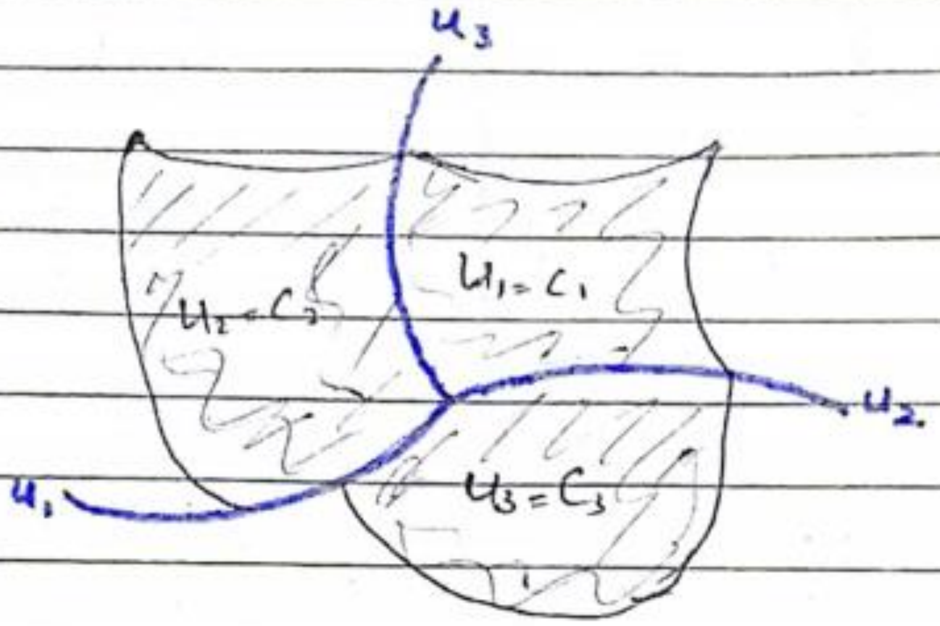


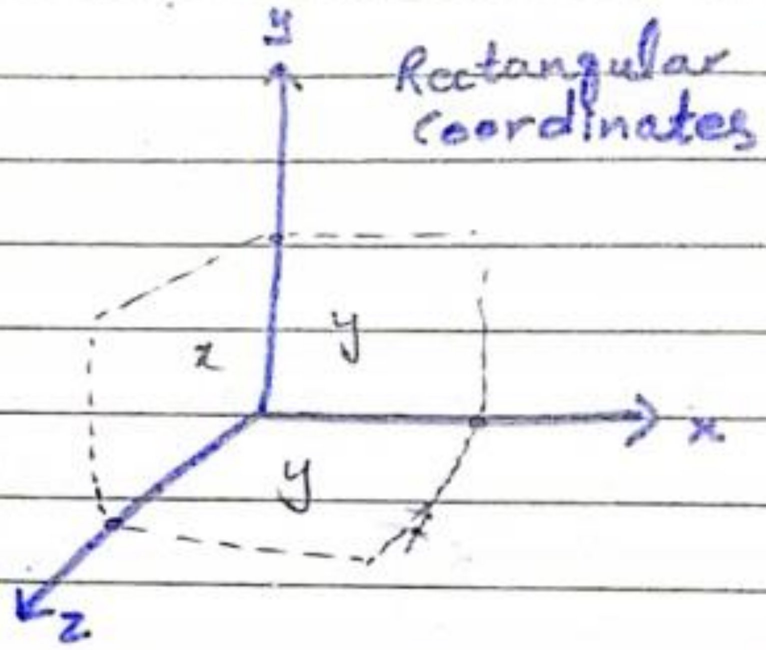
Polar Coordinates



Curvilinear Coordinates

Coordinates Surfaces and Curves





$\Rightarrow u_1 = c_1,\ u_2 = c_2,\ u_3 = c_3$

are the surfaces

⇒ where c_1, c_2 and c_3 are constants.
These are the constants by which surface varies or change when we change there values

e.g.:- $U_1 = 2$, $U_1 = 3$

⇒ because these surfaces are generelly curved.

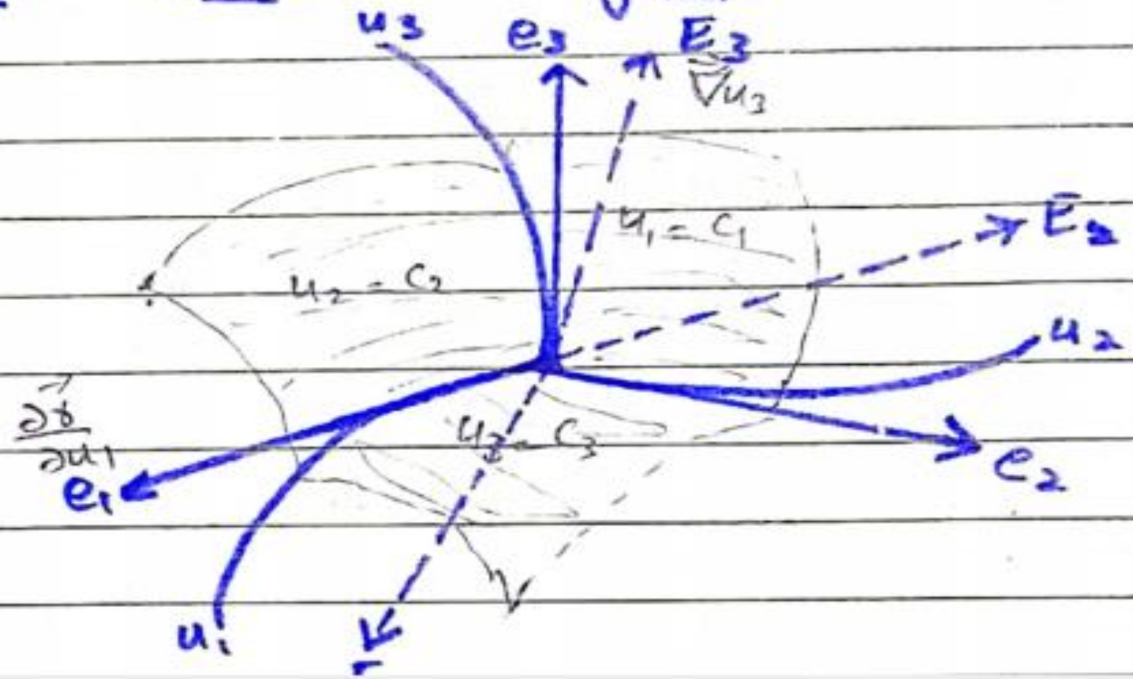
⇒ Each pair of these surfaces intersection on curves called coordinate curves.

⇒ If the coordinates surfaces intersect at right angles, the curvilinear coordinate system is called **orthogonal.**

Unit Vectors in Curvilinear System.

Let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$x = x(u_1, u_2, u_3)$
$y = y(u_1, u_2, u_3)$
$z = z(u_1, u_2, u_3)$



$\vec{r} = \vec{r}(u_1, u_2, u_3)$

e_1

unit tangent vetor

$\Rightarrow \dfrac{\partial \vec{r}}{\partial u_1}$ is a tangent vector to the

$u_1 = c_1$

$\hat{e}_1 = \dfrac{\dfrac{\partial \vec{r}}{\partial u_1}}{\left|\dfrac{\partial \vec{r}}{\partial u_1}\right|}$ is unit tangent vector at $u_1 = c_1$ of u_1 curve

$\left|\dfrac{\partial \vec{r}}{\partial u_1}\right| = h_1 \quad \Rightarrow \hat{e}_1 = \dfrac{\partial \vec{r}}{\partial u_1} \Big/ h_1 \quad \Rightarrow h_1 \hat{e}_1 = \dfrac{\partial \vec{r}}{\partial u_1}$

Similarly

$$h_2 \hat{e}_2 = \frac{\partial \vec{r}}{\partial u_2} \quad \text{at} \quad u_2\text{- curve}$$

$$\& \quad h_3 \hat{e}_3 = \frac{\partial \vec{r}}{\partial u_3} \quad \text{at} \quad u_3\text{- curve}$$

where h_1, h_2, h_3 are scale factors

Unit Normal Vector:

$\Rightarrow$ let $u_1 = c_1$, $u_2 = c_2$ & $u_3 = c_3$ be the three surfaces, we use $\vec{\nabla} u_1$, $\vec{\nabla} u_2$, $\vec{\nabla} u_3$ as normal to the surfaces

$\Rightarrow$ Then unit normals to given surfaces are defined as

$$\hat{E}_1 = \frac{\vec{\nabla} u_1}{|\nabla u_1|} \quad \text{for} \quad u_1 = c_1$$

Similarly

$$\hat{E}_2 = \frac{\vec{\nabla} u_2}{|\nabla u_2|} \quad \text{for} \quad u_2 = c_2$$

$$\vec{E}_3 = \frac{\vec{\nabla} u_3}{|\nabla u_3|} \quad \text{for} \quad u_3 = c_3$$

Result: In curvilinear coordinates at each point generally there exists two unit vectors

$$\underbrace{\hat{e}_1, \hat{e}_2, \hat{e}_3}$$

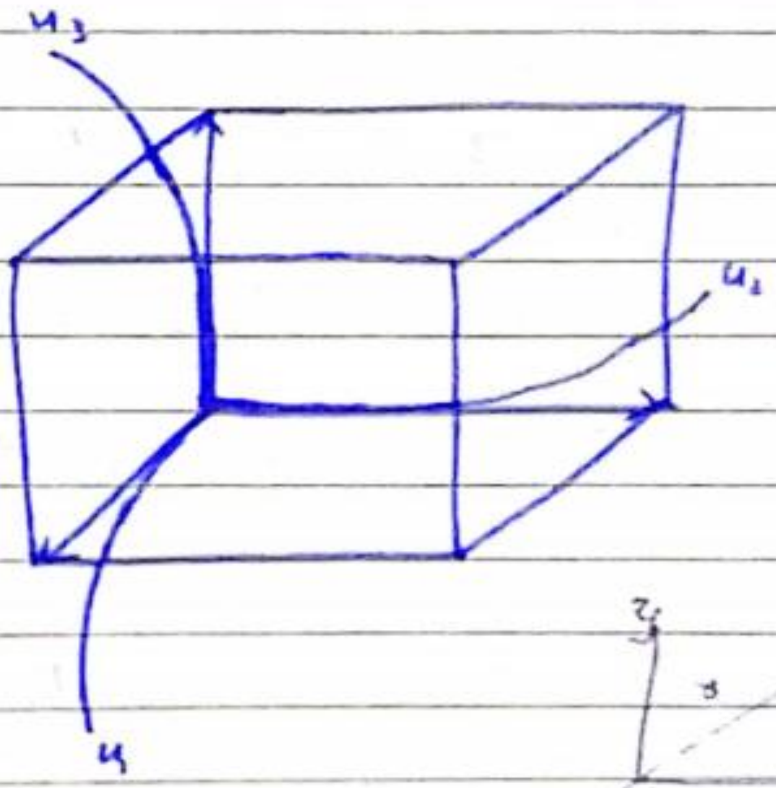
tangent to the coordinate curves

$$\underbrace{\hat{E}_1, \hat{E}_2, \hat{E}_3}$$

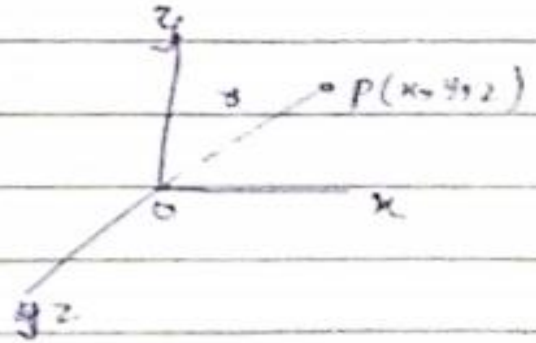
~~tangent~~ Normal to the coordinate curves

ARC LENGTH

Arc Length:



$$(ds)^2 = d\vec{r}\cdot d\vec{r}$$
$$= |d\vec{r}|^2$$



$\vec{r} = \vec{r}\,(u_1, u_2, u_3)$

Taking differential on both sides

$OP = \vec{r} = \vec{r}(x, y, z)$

$OP = \vec{r} = \vec{r}(u_1, u_2, u_3)$

$$d\vec{r} = \frac{\partial \vec{r}}{\partial u_1} du_1 + \frac{\partial \vec{r}}{\partial u_2} du_2 + \frac{\partial \vec{r}}{\partial u_3} du_3$$

$$\because \frac{\partial \vec{r}}{\partial u_1} = h_1 \hat{e}_1 , \frac{\partial \vec{r}}{\partial u_2} = h_2 \hat{e}_2 , \frac{\partial \vec{r}}{\partial u_3} = h_3 \hat{e}_3$$

$$d\vec{r} = h_1 du_1 \hat{e}_1 + h_2 du_2 \hat{e}_2 + h_3 du_3 \hat{e}_3$$

We have to find

$$d\vec{r} \cdot d\vec{r} \quad \text{or} \quad |d\vec{r}|^2$$

$$dr = h_1 du_1 \hat{e}_1 + h_2 du_2 \hat{e}_2 + h_3 du_3 \hat{e}_3$$

$$|dr| = \sqrt{(h_1 du_1 \hat{e}_1)^2 + (h_2 du_2 \hat{e}_2)^2 + h_3 du_3 \hat{e}_3}$$

Taking square on both sides

$$|dr|^2 = (h_1 du_1 \hat{e}_1)^2 + (h_2 du_2 \hat{e}_2)^2 + (h_3 du_3 \hat{e}_3)^2$$

$$\therefore \quad |dr|^2 = (ds)^2$$

$$(ds)^2 \quad h_1^2\, du_1^2\, \hat{e}_1^2 + h_2^2\, du_2^2\, \hat{e}_2^2 + h_3^2\, du_3^2\, \hat{e}_3^2 \quad \text{— (2)}$$

This is called the arc length of curvilinear coordinates.

7.7. Find the square of the element of arc length in cylindrical coordinates and determine the corresponding scale factors.

Solution

First Method.

$$x = \rho \cos\phi, \quad y = \rho \sin\phi, \quad z = z$$
$$dx = -\rho \sin\phi \, d\phi + \cos\phi \, d\rho, \quad dy = \rho \cos\phi \, d\phi + \sin\phi \, d\rho, \quad dz = dz$$

Then

$$\begin{aligned} ds^2 &= dx^2 + dy^2 + dz^2 = (-\rho \sin\phi \, d\phi + \cos\phi \, d\rho)^2 + (\rho \cos\phi \, d\phi + \sin\phi \, d\rho)^2 + (dz)^2 \\ &= (d\rho)^2 + \rho^2 (d\phi)^2 + (dz)^2 = h_1^2 (d\rho)^2 + h_2^2 (d\phi)^2 + h_3^2 (dz)^2 \end{aligned}$$

and $h_1 = h_\rho = 1$, $h_2 = h_\phi = \rho$, $h_3 = h_z = 1$ are the scale factors.

Second Method. The position vector is $\mathbf{r} = \rho \cos\phi \mathbf{i} + \rho \sin\phi \mathbf{j} + z\mathbf{k}$. Then

$$\begin{aligned} d\mathbf{r} &= \frac{\partial \mathbf{r}}{\partial \rho} d\rho + \frac{\partial \mathbf{r}}{\partial \phi} d\phi + \frac{\partial \mathbf{r}}{\partial z} dz \\ &= (\cos\phi \mathbf{i} + \sin\phi \mathbf{j}) d\rho + (-\rho \sin\phi \mathbf{i} + \rho \cos\phi \mathbf{j}) d\phi + \mathbf{k} \, dz \\ &= (\cos\phi \, d\rho - \rho \sin\phi \, d\phi)\mathbf{i} + (\sin\phi \, d\rho + \rho \cos\phi \, d\phi)\mathbf{j} + \mathbf{k} \, dz \end{aligned}$$

Thus

$$\begin{aligned} ds^2 &= d\mathbf{r} \cdot d\mathbf{r} = (\cos\phi \, d\rho - \rho \sin\phi \, d\phi)^2 + (\sin\phi \, d\rho + \rho \cos\phi \, d\phi)^2 + (dz)^2 \\ &= (d\rho)^2 + \rho^2 (d\phi)^2 + (dz)^2 \end{aligned}$$

7.13. Suppose u_1, u_2, u_3 are orthogonal curvilinear coordinates. Show that the Jacobian of x, y, z with respect to u_1, u_2, u_3 is

$$J\left(\frac{x, y, z}{u_1, u_2, u_3}\right) = \frac{\partial(x, y, z)}{\partial(u_1, u_2, u_3)} = \begin{vmatrix} \frac{\partial x}{\partial u_1} & \frac{\partial y}{\partial u_1} & \frac{\partial z}{\partial u_1} \\ \frac{\partial x}{\partial u_2} & \frac{\partial y}{\partial u_2} & \frac{\partial z}{\partial u_2} \\ \frac{\partial x}{\partial u_3} & \frac{\partial y}{\partial u_3} & \frac{\partial z}{\partial u_3} \end{vmatrix} = h_1 h_2 h_3$$

Solution

By Problem 2.38, the given determinant equals

$$\left(\frac{\partial x}{\partial u_1}\mathbf{i} + \frac{\partial y}{\partial u_1}\mathbf{j} + \frac{\partial z}{\partial u_1}\mathbf{k}\right) \cdot \left(\frac{\partial x}{\partial u_2}\mathbf{i} + \frac{\partial y}{\partial u_2}\mathbf{j} + \frac{\partial z}{\partial u_2}\mathbf{k}\right) \times \left(\frac{\partial x}{\partial u_3}\mathbf{i} + \frac{\partial y}{\partial u_3}\mathbf{j} + \frac{\partial z}{\partial u_3}\mathbf{k}\right)$$

$$= \frac{\partial \mathbf{r}}{\partial u_1} \cdot \frac{\partial \mathbf{r}}{\partial u_2} \times \frac{\partial \mathbf{r}}{\partial u_3} = h_1\mathbf{e}_1 \cdot h_2\mathbf{e}_2 \times h_3\mathbf{e}_3$$

$$= h_1 h_2 h_3 \mathbf{e}_1 \cdot \mathbf{e}_2 \times \mathbf{e}_3 = h_1 h_2 h_3$$

If the Jacobian equals zero identically, then $\partial\mathbf{r}/\partial u_1$, $\partial\mathbf{r}/\partial u_2$, $\partial\mathbf{r}/\partial u_3$ are coplanar vectors and the curvilinear coordinate transformation breaks down, that is, there is a relation between x, y, z having the form $F(x, y, z) = 0$. We shall therefore require the Jacobian to be different from zero.

7.3. Prove that a cylindrical coordinate system is orthogonal.

Solution

The position vector of any point in cylindrical coordinates is

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = \rho\cos\phi\mathbf{i} + \rho\sin\phi\mathbf{j} + z\mathbf{k}$$

The tangent vectors to the ρ, ϕ, and z curves are given respectively by $\partial\mathbf{r}/\partial\rho$, $\partial\mathbf{r}/\partial\phi$, and $\partial\mathbf{r}/\partial z$ where

$$\frac{\partial\mathbf{r}}{\partial\rho} = \cos\phi\mathbf{i} + \sin\phi\mathbf{j}, \qquad \frac{\partial\mathbf{r}}{\partial\phi} = -\rho\sin\phi\mathbf{i} + \rho\cos\phi\mathbf{j}, \qquad \frac{\partial\mathbf{r}}{\partial z} = \mathbf{k}$$

The unit vectors in these directions are

$$\mathbf{e}_1 = \mathbf{e}_\rho = \frac{\partial\mathbf{r}/\partial\rho}{|\partial\mathbf{r}/\partial\rho|} = \frac{\cos\phi\mathbf{i} + \sin\phi\mathbf{j}}{\sqrt{\cos^2\phi + \sin^2\phi}} = \cos\phi\mathbf{i} + \sin\phi\mathbf{j}$$

$$\mathbf{e}_2 = \mathbf{e}_\phi = \frac{\partial\mathbf{r}/\partial\phi}{|\partial\mathbf{r}/\partial\phi|} = \frac{-\rho\sin\phi\mathbf{i} + \rho\cos\phi\mathbf{j}}{\sqrt{\rho^2\sin^2\phi + \rho^2\cos^2\phi}} = -\sin\phi\mathbf{i} + \cos\phi\mathbf{j}$$

$$\mathbf{e}_3 = \mathbf{e}_z = \frac{\partial\mathbf{r}/\partial z}{|\partial\mathbf{r}/\partial z|} = \mathbf{k}$$

Then

$$\mathbf{e}_1 \cdot \mathbf{e}_2 = (\cos\phi\mathbf{i} + \sin\phi\mathbf{j}) \cdot (-\sin\phi\mathbf{i} + \cos\phi\mathbf{j}) = 0$$
$$\mathbf{e}_1 \cdot \mathbf{e}_3 = (\cos\phi\mathbf{i} + \sin\phi\mathbf{j}) \cdot (\mathbf{k}) = 0$$
$$\mathbf{e}_2 \cdot \mathbf{e}_3 = (-\sin\phi\mathbf{i} + \cos\phi\mathbf{j}) \cdot (\mathbf{k}) = 0$$

CYLINDRICAL COORDINATES

Cylindrical coordinate system :-

Polar coordinate $\longrightarrow$ Spherical or
cylindrical coordinates.

Polar coordinates $\longrightarrow$ 2D.

Cylindrical coordinates $\longrightarrow$ 3D.

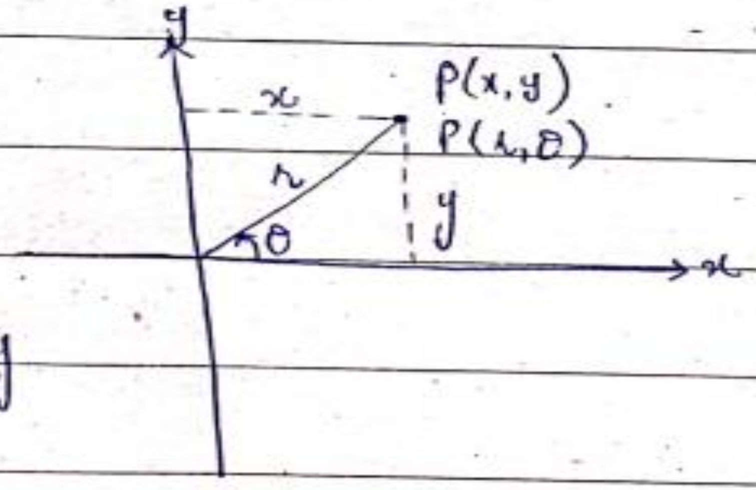
. Cylindrical or spherical $\xrightarrow{\text{extension}}$ polar coordinates.

Polar coordinates:

$x = r\cos\theta$

$y = r\sin\theta$

we find 'r' & 'θ' by this:



$r^2 = x^2+y^2$ (By pathagoras theorem)

$$\boxed{r = \sqrt{x^2+y^2}}$$

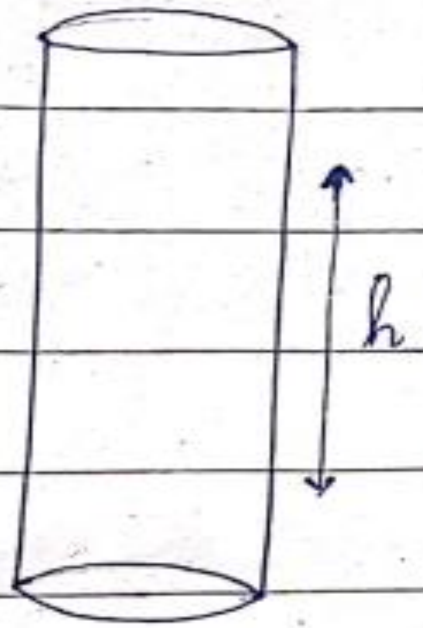
$$\tan\theta = \frac{\text{Prep}}{\text{Base}} = \frac{y}{x}$$

$$\boxed{\theta = \tan^{-1}(y/x)}$$

Now we will discuss about cylindrical coordinates system :

Cylindrical coordinates system :-

In this we need height 'h' and circular region (polar coordinates) that lie in 3D.

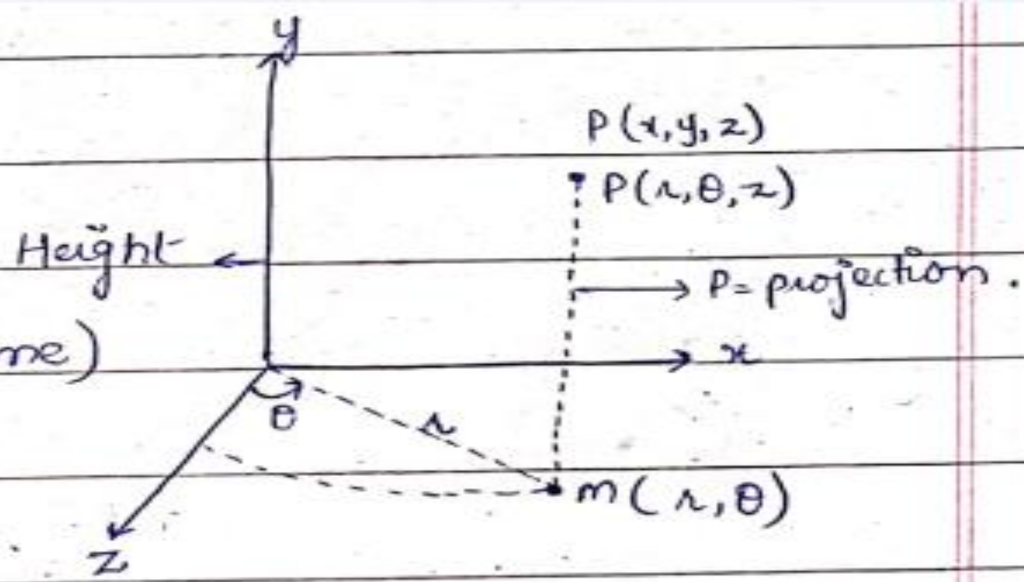


$x = r\cos\theta$

$y = r\sin\theta$

$\theta = \tan^{-1}(y/x)$

$z = z$ (remain same)



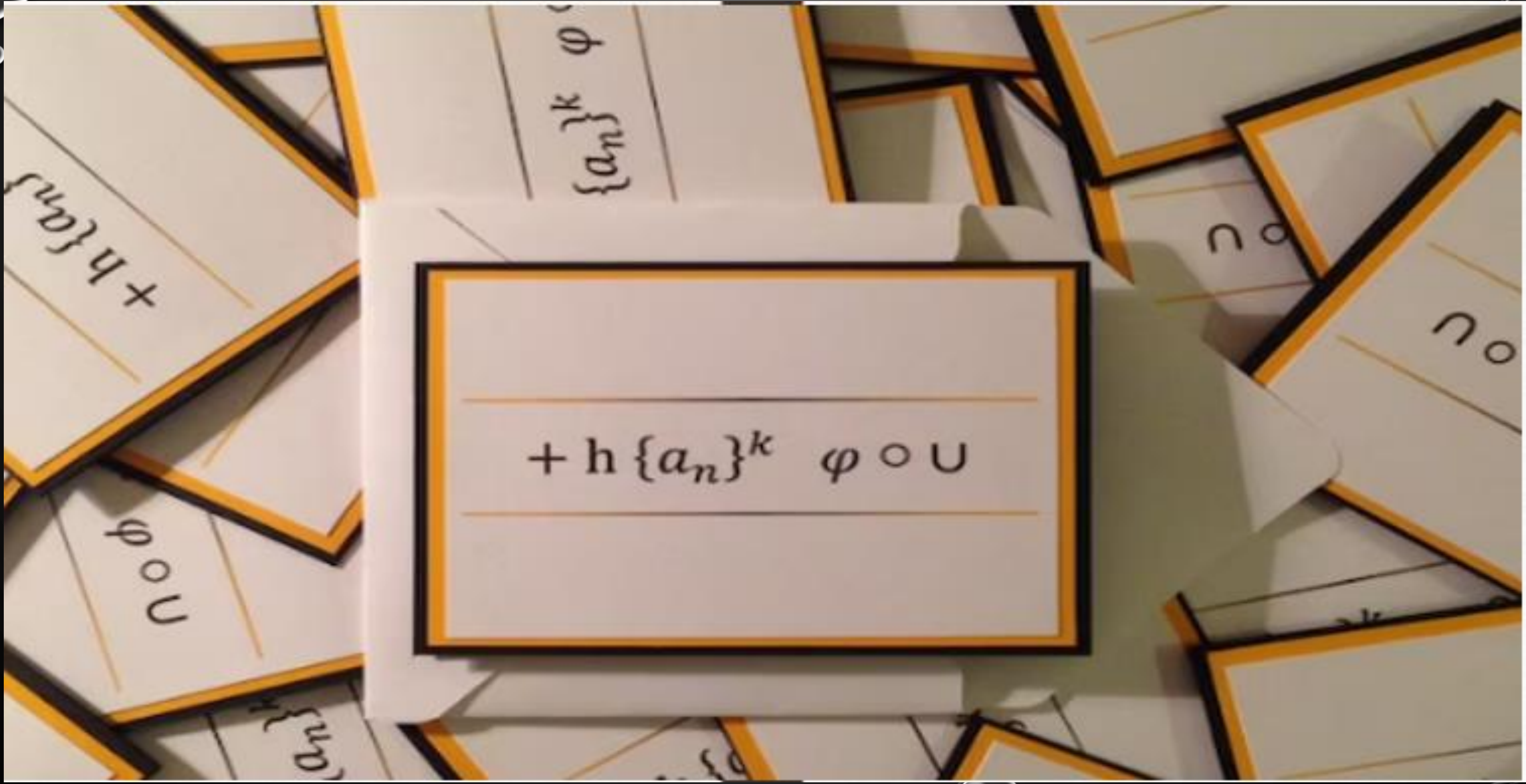
Example :

Express $x^2+y^2+z^2=16$ in cylindrical coordinates.

Since $x^2+y^2=r^2$

So, $\boxed{r^2+z^2=16}$.

Because 'x' & 'y' become neglected.



$+\,\mathrm{h}\,\{a_n\}^k \quad \varphi \circ \cup$