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Calculus III

Unit Vector:-

Let $\vec{u}$ be any vector

i.e. $\vec{u} = \vec{u}(x_1, x_2, x_3, \dots x_n)$

$$\hat{u} = \frac{\vec{u}}{\|\vec{u}\|} = \frac{(x_1, x_2, x_3, \dots x_n)}{\sqrt{x_1^2 + x_2^2 + \dots + x_n^2}}$$

$$= \left(\frac{x_1}{\sqrt{x_1^2 + x_2^2 + \dots x_n^2}}, \frac{x_2}{\sqrt{x_1^2 + x_2^2 + \dots x_n^2}}, \dots, \frac{x_n}{\sqrt{x_1^2 + x_2^2 + \dots x_n^2}} \right)$$

$$= \left(\frac{x_1}{\|\vec{u}\|}, \frac{x_2}{\|\vec{u}\|}, \dots, \frac{x_n}{\|\vec{u}\|} \right)$$

Example:-

$$\vec{u} = (1, -3, 2, 4)$$

$$\|\vec{u}\| = \sqrt{(1)^2 + (-3)^2 + (2)^2 + (4)^2}$$

$$= \sqrt{1 + 9 + 4 + 16} = \sqrt{30}$$

$$\hat{u} = \frac{\vec{u}}{\|\vec{u}\|} = \frac{(1, -3, 2, 4)}{\sqrt{30}}$$

$$= \left(\frac{1}{\sqrt{30}}, \frac{-3}{\sqrt{30}}, \frac{2}{\sqrt{30}}, \frac{4}{\sqrt{30}} \right)$$

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ii) $\vec{v} = (1, -3, 5, 6, \sin\theta, \cos\theta)$

$\|\vec{v}\| = \sqrt{(1)^2 + (-3)^2 + (5)^2 + (6)^2 + \sin^2\theta + \cos^2\theta}$

$= \sqrt{1 + 9 + 25 + 36 + \sin^2\theta + \cos^2\theta}$

$= \sqrt{72}$

$\hat{v} = \dfrac{\vec{v}}{\|\vec{v}\|} = \dfrac{1}{\sqrt{72}}, \dfrac{-3}{\sqrt{72}}, \dfrac{5}{\sqrt{72}}, \dfrac{6}{\sqrt{72}}, \dfrac{\sin\theta}{\sqrt{72}}, \dfrac{\cos\theta}{\sqrt{72}}$

$\vec{u} \cdot \vec{v} = \|u\| \|v\| \cos\theta$ $\qquad \vec{F} \cdot \vec{d} = Fd\cos\theta$

$\cos\theta = \dfrac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|}$

$\theta = 90^\circ$

$\cos 90^\circ = 0$

$\boxed{\vec{u} \cdot \vec{v} = 0}$

$\theta = 0^\circ$

$\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\|$

Proof:-

$\left| \dfrac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|} \right| = |\cos\theta|$

$\dfrac{|\vec{u} \cdot \vec{v}|}{\|\vec{u}\| \|\vec{v}\|} \le 1$

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$$\Rightarrow |\vec{u}\cdot\vec{v}| \le \|\vec{u}\|\,\|\vec{V}\|$$

Schwawz Inequality.

Projection :-

Let $\vec{u}$ and $\vec{V}$ be two vectors then projection of $\vec{u}$ on $\vec{V}$ will be

$$(\vec{u} \text{ on } \vec{v}) = \frac{\vec{u}\cdot\vec{v}}{\|V\|^2}(\vec{V})$$

Example :-

$\vec{u} = (1, -2, 3)$, $\vec{V} = (2, 4, 5)$

$$\vec{u} \text{ on } \vec{v} = \frac{\vec{u}\cdot\vec{V}}{\|V\|^2}\vec{V}$$

$$\vec{u}\cdot\vec{V} = (2 + (-8) + 15)$$

$$\vec{u}\cdot\vec{v} = 9$$

$$\|\vec{v}\| = \sqrt{(2)^2 + (4)^2 + (5)^2}$$

$$= \sqrt{4 + 16 + 25}$$

$$= (\sqrt{45})^2 = 45$$

$$(\vec{u} \text{ on } \vec{v}) = \frac{\cancel{9}}{\cancel{45}\,5}(2, 4, 5)$$

$$= \frac{1}{5}(2, 4, 5)$$

$$= \left(\frac{2}{5}, \frac{4}{5}, \frac{\cancel{5}}{\cancel{5}}\right) = \left(\frac{2}{5}, \frac{4}{5}, 1\right)$$

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Example:

$$f(t) = (\sin t, \cos t, t)$$

$$\vec{v}(t) = (\cos t, -\sin t, 1)$$

Find tangent on $\vec{v}(t)$

$\hat{v}(t)$?

$$\|\vec{v}\| = \sqrt{\cos^2 t + \sin^2 t + 1}$$

$$= \sqrt{1+1} = \sqrt{2}$$

$$\hat{v}(t) = \left(\frac{\cos t}{\sqrt{2}}, \frac{-\sin t}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right)$$

Cross - Product :-

Let $\vec{u}$ and $\vec{v}$ be two vectors $\Rightarrow$ $\vec{u} = (a_1, b_1, c_1)$

$\vec{v} = (a_2, b_2, c_2)$

Then Their cross product is

$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}$$

$$= \hat{i}(b_1c_2 - c_1b_2) - \hat{j}(a_1c_2 - c_1a_2) + \hat{k}(a_1b_2 - a_2b_1)$$

$$= \hat{i}(b_1c_2 - b_2c_1) + \hat{j}(a_2c_1 - a_1c_2) + \hat{k}(a_1b_2 - a_2b_1)$$

in general:

$$\vec{u} \times \vec{v} = \|\vec{u}\| \, \|\vec{v}\| \sin\theta \, \hat{n}$$

Problem:-

Let $\vec{u} = 2\hat{i} - 3\hat{j} + 4\hat{k}$, $\vec{v} = 3\hat{i} + \hat{j} - 2\hat{k}$

$\vec{w} = \hat{i} + 5\hat{j} + 3\hat{k}$

$$\vec{u}\times\vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -3 & 4 \\ 3 & 1 & -2 \end{vmatrix}$$

$= \hat{i}(6-4) - \hat{j}(-4-12) + \hat{k}(2+9)$

$= 2\hat{i} + 16\hat{j} + 11\hat{k}$

(i) $\hat{i}\times\hat{j} = \hat{k}$, $\hat{j}\times\hat{k} = \hat{i}$
$\hat{k}\times\hat{i} = \hat{j}$

(ii) $\hat{j}\times\hat{i} = -\hat{k}$, $\hat{k}\times\hat{j} = -\hat{i}$
$\hat{i}\times\hat{k} = -\hat{j}$

(iii) $\hat{i}\times\hat{i} = 0$, $\hat{j}\cdot\hat{j} = 0$
$\hat{k}\cdot\hat{k} = 0$

$$\vec{v}\times\vec{u} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 1 & -2 \\ 2 & -3 & 4 \end{vmatrix}$$

$= \hat{i}(4-6) - \hat{j}(12+4) + \hat{k}(-9-2)$

$= -2\hat{i} - 16\hat{j} - 11\hat{k}$

$$\vec{u}\times\vec{w} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -3 & 4 \\ 1 & 5 & 3 \end{vmatrix}$$

$= \hat{i}(-9-20) - \hat{j}(6-4) + \hat{k}(10+3)$

$= -29\hat{i} + 2\hat{j} + 13\hat{k}$

Problem: 2 Find $\vec{u}\times\vec{v}$ and $\vec{v}\times\vec{u}$ if

a) $u = (1,2,3)$, $v = (4,5,6)$

b) $u = (-4,7,3)$, $v = (6,-5,2)$

Problem: 3

Find $\vec{v}\times\vec{w}$ also check $\vec{v}\times\vec{w}\cdot v = 0$, $\vec{v}\times\vec{w}\cdot\vec{w} = 0$

$v = (1,3,4)$, $w = (2,-6,-5)$

$\vec{v}\times\vec{w}\cdot\vec{v} = 0$

$\vec{v} \times \vec{w}$

$\hat{i}$	$\hat{j}$	$\hat{k}$
1	3	4
2	-6	-5

$= \hat{i}(-15+24) - \hat{j}(-5-8) + \hat{k}(-6-6)$

$= 9\hat{i} + 13\hat{j} - 12\hat{k}$

$\vec{v} \times \vec{w} \cdot \vec{v} = 0$

$= (9\hat{i} + 13\hat{j} - 12\hat{k}) \cdot (1, 3, 4) = 0$

$= 9 + 39 - 48 = 0$

$48 - 48 = 0$

$0 = 0$

$\vec{v} \times \vec{w} \cdot \vec{w} = 0$

$(9\hat{i} + 13\hat{j} - 12\hat{k}) \cdot (2, -6, -5)$

$= 18 - 78 + 60 = 0$

$-60 + 60 = 0$

$0 = 0$

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Find two unit vectors perpendicular to $(2, 0, -3)$ and $(-1, 4, 2)$

$\vec{u} = (2, 0, -3)$, $\vec{v} = (-1, 4, 2)$

$$\vec{u}\times\vec{v} = \begin{vmatrix} i & j & k \\ 2 & 0 & -3 \\ -1 & 4 & 2 \end{vmatrix}$$

$= \hat{i}(0+12) - \hat{j}(4-3) + \hat{k}(8+0)$

$= 12\hat{i} - \hat{j} + 8\hat{k} = \vec{w}$

$$\vec{v}\times\vec{u} = \begin{vmatrix} i & j & k \\ -1 & 4 & 2 \\ 2 & 0 & -3 \end{vmatrix}$$

$= \hat{i}(-12-0) - \hat{j}(3-4) + \hat{k}(0-8)$

$= -12\hat{i} + \hat{j} - 8\hat{k} = \vec{z}$

$\vec{u}\cdot\vec{w} = (2\hat{i} - 0\hat{j} - 3\hat{k})\cdot(12\hat{i} - \hat{j} + 8\hat{k})$

$= (2\cdot 12)\,\hat{i}\cdot\hat{i} + (0) - 24 = 0$

$\cancel{24} - \cancel{24} = 0$

$0 = 0$

$\vec{v}\cdot\vec{w} = (-1, 4, 2)\cdot(12, -1, 8)$

$= -12 - 4 + 16 = 0$

$-\cancel{16} + \cancel{16} = 0$

$0 = 0$

$\cancel{\vec{u}\cdot\vec{z} = (2, 0, -3)\cdot(-1, 4, 2)}$

$\cancel{= -2 + 0 - 6 = 0}$

$\hat{u}\cdot\vec{z} = (2, 0, -3)\cdot(-12, 1, -8)$

$= -24 + 0 + 24 = 0 \Rightarrow 0 = 0$

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$\vec{V}\cdot\vec{R} = (-1,4,2)\cdot(-12,1,-8)$

$12+4-16 = 0$

$0 = 0$

$\vec{u}\times\vec{v} = 12\hat{i}-\hat{j}+8\hat{k}$

$\|\vec{u}\times\vec{v}\| = \sqrt{144+1+64}$

$= \sqrt{209}$

$$\frac{\vec{u}\times\vec{v}}{\|\vec{u}\times\vec{v}\|} = \frac{12}{\sqrt{209}}\hat{i} - \frac{1}{\sqrt{209}}\hat{j} + \frac{8}{\sqrt{209}}\hat{k}$$

$$\frac{\vec{v}\times\vec{u}}{\|\vec{v}\times\vec{u}\|} = \frac{-12}{\sqrt{209}}\hat{i} + \frac{1}{\sqrt{209}}\hat{j} - \frac{8}{\sqrt{209}}\hat{k}$$

Problem:-

if $\vec{u}\cdot\vec{v} = 0$

Then what will $\vec{u} = ?$ $\vec{v} = ?$

And if $\vec{u}\times\vec{v} = 0$

Then $\vec{u} = ?$ $\vec{v} = ?$

$\vec{u} = 0\hat{i}, 0\hat{j}, 0\hat{k}$

$\vec{v} = 3\hat{i}, 5\hat{j}, 9\hat{k}$

$\vec{u}\cdot\vec{v} = 0+0+0 = 0$

$0 = 0$

$$\vec{u}\times\vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & 0 \\ 3 & 5 & 9 \end{vmatrix} = 0$$

$\vec{u}\times\vec{v} = 0i, 0j, 0\hat{k} = 0$

$0 = 0$

Possibilities:-

If $\vec{u} = 0$ or $\vec{v} = 9\hat{i}$, Some Values

If $\vec{v} = 0$ $\vec{u} =$ Some Values

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If $\vec{u} = 0$ $\vec{v} = 0$

$\vec{u} \cdot \vec{v} = 0 \Rightarrow a_1 b_1 + b_2 b_2 + c_1 c_2$

if $a_1 = b_1 = 0$

$b_2 b_2 = -c_1 c_2$

$\Rightarrow 0 + c_1 c_2 + c_1 c_2 = 0$

$0 = 0$

$\vec{u} = \vec{v}$

$a_1 = a_2$, $b_1 = b_2$, $c_1 = c_2$

Analytic Geometry:-

The branch of mathematics in which the position of any point can be (observed/ located/ determined) by ordered pair. This also called Cartesian geometry.

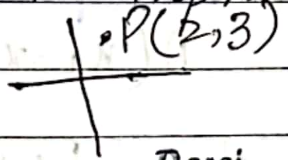
Plane:- A flate surface goes on infinitely in each direction.

Co-ordinates are the two ordered pair which defines the location of point on plane.

Types of co-ordinate System:-

Cartesian Co-ordinates System:-

In which the position of any point represented by (x, y)



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Polar co-ordinates Systems:-

In which the location or position of a point is describe by distance 'r' and angle 'θ'

$\therefore \quad x = r\cos\theta \quad y = r\sin\theta$

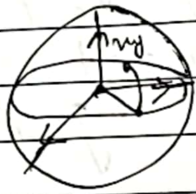
Cylinderical co-ordinate Systems:-

In which all the points are represented by height 'h' distance 'r' and angle θ.

$\therefore \quad P(h, r, \theta) \quad , \quad P(r, \theta, h)$

Spherical co-ordinate Systems:-

In which the point in space is represented by distance 'r' and angle 'θ' with xy-plane and another angle φ making with z-axis



Three Planes with the help of (x, y, z)

1. xy-plane
2. yz-plane
3. xz-plane

Distance Formula:-

The distance b/w two points $A(x_1, y_1)$ and $B(x_2, y_2)$. The distance formula can be written as:-

$$d(A,B) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Mid point:- In general

$A(x_1, x_2, x_3, \ldots x_n)$, $B(y_1, y_2, \ldots y_n)$

$$d = \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2 + \ldots + (y_n - x_n)^2}$$

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Mid-point:

$A(x_1, x_2, \ldots x_n)$, $B(y_1, y_2, \ldots y_n)$

$$M(z_1, z_2, \ldots z_n) = \left(\frac{x_1+y_1}{2}, \frac{x_2+y_2}{2}, \ldots \frac{x_n+y_n}{2}\right)$$

$$y = m_1x + c$$
$$y = m_2x + c$$

$$\tan\theta = \frac{m_1 - m_2}{1+m_1m_2}$$

If $\theta = 0^\circ$ (parallel)

$$0 = \frac{m_1 - m_2}{1+m_1m_2}$$

$$m_1 - m_2 = 0$$

$$\boxed{m_1 = m_2}$$

If $\theta = 90^\circ$ ($\perp$)

$$\tan 90^\circ = \frac{m_1 - m_2}{1+m_1m_2}$$

$$\frac{\sin 90^\circ}{\cos 90^\circ} = \frac{m_1 - m_2}{1+m_1m_2}$$

$$\frac{1}{0} = \frac{m_1 - m_2}{1+m_1m_2}$$

$$\infty = \frac{m_1 - m_2}{1+m_1m_2}$$

$$1+m_1m_2 = \frac{m_1 - m_2}{\infty}$$

$$1+m_1m_2 = 0 \Rightarrow \boxed{m_1m_2 = -1}$$

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Example:-

Find the distance for the following points i) A(3, 2, 11), B(6, 9, 17)

ii) C(5, 9, 11), D(8, 7, 13)

iii) E($\cos\theta$, $3\sin\theta$), F($\sin\theta$, $3\cos\theta$)

Also find their mid points.

Example:-

Find the slope b/w (5, -3) and y-intercept. (0, 1)

Slope:-

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Vector Identities:-

Differential Operator

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

$$= \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$

In general for n variables

$$\vec{\nabla} = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_n}\right)$$

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$f = f(x, y, z)$

$$\vec{\nabla} f = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) f(x, y, z)$$

$$= \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$$

If $f = f(x_1, x_2, x_3, \ldots x_n)$

$$\vec{\nabla} = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_1}, \ldots \frac{\partial}{\partial x_n}\right)$$

$$\vec{\nabla} f = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots \frac{\partial f}{\partial x_n}\right)$$

$$\vec{\nabla} f = (f_{x_1}, f_{x_2}, \ldots f_{x_n})$$

Let $\vec{F}$ be vector field function.

$$\vec{F} = (f_x, f_y, f_z)$$

and

$$\vec{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

$$\vec{\nabla} \cdot \vec{F} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot (f_x, f_y, f_z)$$

$$= \frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} + \frac{\partial f_z}{\partial z}$$

In general

$$\vec{\nabla} \cdot f = \left(\frac{\partial f_1}{\partial x_1}, \frac{\partial f_2}{\partial x_2}, \ldots \frac{\partial f_n}{\partial x_n}\right)$$

If $\vec{G} = \vec{f}(f_1, f_2, f_3, \ldots f_n)$

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$$\vec{\nabla} = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots \frac{\partial}{\partial x_n}\right)$$

Example:-

$$\vec{F} = \left((2+y), (x-4)\right)$$

$$\vec{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$$

$$\vec{\nabla}\cdot\vec{F} = \frac{\partial}{\partial x}(2+y) + \frac{\partial}{\partial y}(x-4)$$

$$= 0 + 0 = 0$$

Example:-

$$\vec{F} = \left(x^2+3y+z,\ \sin^2 y,\ \tan x + \sin y \cos z\right)$$

$$\vec{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

$$\vec{\nabla}\cdot\vec{F} = \frac{\partial}{\partial x}(x^2+3y+z) + \frac{\partial}{\partial y}(\sin^2 y) + \frac{\partial}{\partial z}(\tan x + \sin y \cos z)$$

$$= 2x + 2\sin y \cos y + \sin y(-\sin z)$$

$$= 2x + \sin 2y - \sin y \sin z$$

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial F_z}{\partial y} - \frac{\partial}{\partial z}(F_y)\right) - \hat{j}\left(\frac{\partial F_z}{\partial x} - \frac{\partial}{\partial z}F_x\right) + \hat{k}\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)$$

Example:-

let $\vec{F} = \left(x^2+3y+z,\ \sin^2 y,\ \tan x + \sin y \cos z\right)$

$$\vec{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2+3y+z & \sin^2 y & \tan x+\sin y\cos z \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial}{\partial y}(\tan x+\sin y\cos z) - \frac{\partial}{\partial z}(\sin^2 y)\right) - \hat{j}\left(\frac{\partial}{\partial z}(x^2+3y+z) - \frac{\partial}{\partial x}\tan x + \sin y\cos z\right)$$

$$+\hat{k}\left(\frac{\partial}{\partial x}(\sin^2 y) - \frac{\partial}{\partial y}(x^2+3y+z)\right)$$

$$= \hat{i}(\cos y\cos z - 0) - \hat{j}(\sec^2 x - 1) + \hat{k}(0-3)$$

$$= (\cos y\cos z)\hat{i} - (\sec^2 x - 1)\hat{j} + 3\hat{k}$$

$$= \cos y\cos z - \sec^2 x + 1 - 3$$

$$= \cos y\cos z - \sec^2 x - 2$$

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①

$$\nabla^2 f = (\vec{\nabla}\cdot\vec{\nabla})f$$

$$= \vec{\nabla}\cdot\vec{\nabla}f$$

$$= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\cdot\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$$

$$= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

$$= \vec{\nabla}^2 f$$

②

$$\vec{\nabla}(f+g) = \vec{\nabla}f + \vec{\nabla}g$$

L.H.S

$$= \vec{\nabla}(f+g)$$

$$= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)(f+g)$$

$$= \left(\frac{\partial}{\partial x}(f+g), \frac{\partial}{\partial y}(f+g), \frac{\partial}{\partial z}(f+g)\right)$$

$$= \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial x}, \frac{\partial f}{\partial y} + \frac{\partial g}{\partial y}, \frac{\partial f}{\partial z} + \frac{\partial g}{\partial z}\right)$$

$$= \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) + \left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z}\right)$$

$$= \vec{\nabla} f + \vec{\nabla} g$$

2) $\nabla(cf) = c\nabla f$

L.H.S

$\nabla(cf)$ $\quad$ $\because c$ any constant

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

$$\nabla f = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) f$$

Similarly $\nabla(cf) = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)(cf)$

$$= \left(\frac{\partial}{\partial x}(cf), \frac{\partial}{\partial y}(cf), \frac{\partial}{\partial z}(cf)\right)$$

$$= \left(c\frac{\partial f}{\partial x}, c\frac{\partial f}{\partial y}, c\frac{\partial f}{\partial z}\right) = c\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$$

$$= c\nabla f$$

(3) $\nabla(fg) = g\nabla f + f\nabla g$

L.H.S $\quad \nabla(fg) = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)(fg)$

$$=\left(\frac{\partial}{\partial x}(fg), \frac{\partial}{\partial y}(fg), \frac{\partial}{\partial z}(fg)\right)$$

$$=\left(f\frac{\partial}{\partial x}g+g\frac{\partial f}{\partial x}, f\frac{\partial}{\partial y}g+g\frac{\partial f}{\partial y}, f\frac{\partial}{\partial z}g+g\frac{\partial f}{\partial z}\right)$$

$$=\left(f\frac{\partial g}{\partial x}, f\frac{\partial g}{\partial y}, f\frac{\partial g}{\partial z}\right)+\left(g\frac{\partial f}{\partial x}, g\frac{\partial f}{\partial y}, g\frac{\partial f}{\partial z}\right)$$

$$=f\left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z}\right)+g\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$$

$$=f\nabla g+g\nabla f$$

4) $\nabla(f/g) = \dfrac{g\nabla f - f\nabla g}{g^2}$

L.H.S

$$=\nabla(f/g)$$

$$=\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)(f/g)$$

$$=\left(\frac{\partial}{\partial x}(f/g), \frac{\partial}{\partial y}(f/g), \frac{\partial}{\partial z}(f/g)\right)$$

$$=\left(\frac{g\,\partial/\partial x(f)-f\frac{\partial g}{\partial x}}{g^2}, \frac{g\,\partial/\partial y(f)-f\frac{\partial g}{\partial y}}{g^2}, \frac{g\,\partial/\partial z(f)-f\frac{\partial g}{\partial z}}{g^2}\right)$$

$$=\frac{1}{g^2}\left(g\frac{\partial f}{\partial x}-f\frac{\partial g}{\partial x}, g\frac{\partial f}{\partial y}-f\frac{\partial g}{\partial y}, g\frac{\partial f}{\partial z}-f\frac{\partial g}{\partial z}\right)$$

$$=\frac{1}{g^2}\left(g\frac{\partial f}{\partial x}, g\frac{\partial f}{\partial y}, g\frac{\partial f}{\partial z}\right)-\left(f\frac{\partial g}{\partial x}, f\frac{\partial g}{\partial y}, f\frac{\partial g}{\partial z}\right)$$

$$=\frac{1}{g^2}\left[g\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)-f\left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z}\right)\right]$$

$$=\frac{1}{g^2}\left[g\vec{\nabla}f-f\vec{\nabla}g\right]=\frac{g\vec{\nabla}f-f\vec{\nabla}g}{g^2}$$

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5) Let $\vec{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$

$\vec{F} = (f_1, f_2, f_3)$

$\vec{G} = (g_1, g_2, g_3)$

Prove that

$$\nabla \cdot (\vec{F} + \vec{G}) = \vec{\nabla} \cdot \vec{F} + \vec{\nabla} \cdot \vec{G}$$

$$L.H.S = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)(f_1+g_1, f_2+g_2, f_3+g_3)$$

$$= \frac{\partial}{\partial x}(f_1+g_1) + \frac{\partial}{\partial y}(f_2+g_2) + \frac{\partial}{\partial z}(f_3+g_3)$$

$$= \frac{\partial f_1}{\partial x} + \frac{\partial g_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial g_2}{\partial y} + \frac{\partial f_3}{\partial z} + \frac{\partial g_3}{\partial z}$$

$$= \left(\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}\right) + \left(\frac{\partial g_1}{\partial x} + \frac{\partial g_2}{\partial y} + \frac{\partial g_3}{\partial z}\right)$$

$$= \nabla \cdot \vec{F} + \vec{\nabla} \cdot \vec{G}$$

6) $\vec{\nabla} \times (\vec{F} + \vec{G}) = \vec{\nabla} \times \vec{F} + \vec{\nabla} * \vec{G}$

Let $\vec{F} = (f_1, f_2, f_3)$, $\vec{G} = (g_1, g_2, g_3)$

$$\vec{\nabla} = \left(\frac{\partial}{\partial x}; \frac{\partial}{\partial y}; \frac{\partial}{\partial z}\right)$$

$$\vec{F} + \vec{G} = (f_1+g_1, f_2+g_2, f_3+g_3)$$

$$\vec{\nabla} \times (\vec{F} * \vec{G}) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1+g_1 & f_2+g_2 & f_3+g_3 \end{vmatrix}$$

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$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix} + \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ g_1 & g_2 & g_3 \end{vmatrix}$$

$$= \vec{\nabla} \times \vec{F} + \vec{\nabla} \times \vec{G}$$

Curve: is continuous and smoth flowing line without any sharp turn.

Continuity:-

f is defined

∇' Limit should be exist

$$\lim_{x \to a} f(x) = f(a)$$

Let $\lim_{\delta x \to 0} f(x + \delta x) = f(x)$

if $|f(x+\delta x) - f(x)| < \varepsilon$ for $|\delta x| < \delta$

A scalar function $\phi(u)$ is called continuous a u if

$$\lim_{\Delta u \to 0} \phi(u + \Delta u) = \phi(u)$$

Differentiability:- A vector function of 'u' is called differentiable of order 'n' if its n^{th} derivative exist.

A function whose range is one-dimensional is called scalar function.

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$f(x,y,z) = xyz^2$

Vector function:-
A vector expression of the form $(f(t), g(t), h(t))$ is called a vector function. "A vector whose co-ordinates are itself are functions."

Partial derivatives:-

let $F = F(x,y,z)$

$$\frac{\partial F}{\partial x} = \lim_{\delta x \to 0} \frac{F(x+\delta x, y, z) - F(x,y,z)}{\delta x}$$

Similarly

$$\frac{\partial F}{\partial y} = \lim_{\delta y \to 0} \frac{F(x, y+\delta y, z) - F(x,y,z)}{\delta y}$$

$$\frac{\partial F}{\partial z} = \lim_{\delta z \to 0} \frac{F(x,y, z+\delta z) - F(x,y,z)}{\delta z}$$

let $F = F(x_1, x_2, x_3, \dots, x_n)$

$$\frac{\partial F}{\partial x_1} = \lim_{\delta x_1 \to 0} \frac{F(x_1+\delta x_1, \dots x_n) + F(\dots)}{\partial x_1}$$

Partial derivative of order 2

$$\frac{\partial^2 F}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial x}\right)$$

$$\frac{\partial^2 F}{\partial x \partial y} = \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial x}\right)$$

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$$\frac{\partial^2 F}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial y}\right)$$

$$= \lim_{\delta y \to 0} \frac{F_y(x, y+\delta y, z) - F_y(x, y, z)}{\delta y}$$

Example :-

$$\phi(x, y, z) = xy^2 z$$

$$\vec{A} = x\hat{i} + y\hat{j} + z\hat{k} = (x, y, z)$$

Find

$$\frac{\partial^3}{\partial x^2 \partial z}(\phi \vec{A}) \quad \text{at } P(1, 2, 2)$$

$$P(2, 2, 3)$$

Solution:-

$$\phi\vec{A} = xy^2 z\,(x, y, z)$$

$$= (x^2y^2z^2,\ xy^3z,\ xy^2z^2)$$

$$\frac{\partial}{\partial x}(\phi\vec{A}) = \left(\frac{\partial}{\partial x}(x^2y^2z),\ \frac{\partial}{\partial x}(xy^3z),\ \frac{\partial}{\partial x}(xy^2z^2)\right)$$

$$= 2xy^2z,\ y^3z,\ y^2z^2$$

$$\frac{\partial^2(\phi\vec{A})}{\partial x^2} = (2y^2z, 0, 0)$$

$$\frac{\partial^3(\phi\vec{A})}{\partial x^2 \partial z} = (2y^2, 0, 0)$$

$$= (8, 0, 0) \cdot \text{at } P(1, 2, 2)$$

Let $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$

let $\phi = \phi(x, y, z)$

the gradient is

$$\nabla\phi = \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right)$$

and curl of $\phi = \nabla \times \phi$

if $\vec{\phi} = (\phi_1, \phi_2, \phi_3)$

and Divergence of $\phi = \vec{\nabla} \cdot \vec{\phi}$

Problem:-

$$\nabla\phi = \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right)$$

Since ϕ is scalar function.

Here $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$

$\nabla \times (\nabla\phi)$

Curl of gradient $= \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \partial\phi/\partial x & \partial\phi/\partial y & \partial\phi/\partial z \end{vmatrix}$

$= \hat{i}\left(\partial/\partial y \times \partial\phi/\partial z - \partial/\partial z \times \partial\phi/\partial y\right) - \hat{j}\left(\partial/\partial x \times \partial\phi/\partial z - \partial/\partial z \times \partial\phi/\partial x\right)$
$+ \hat{k}\left(\partial/\partial x \times \partial\phi/\partial y - \partial/\partial y \times \partial\phi/\partial x\right)$

$= \hat{i}\left(\frac{\partial^2\phi}{\partial y\partial z} - \frac{\partial^2\phi}{\partial x\partial y}\right) - \hat{j}\left(\frac{\partial^2\phi}{\partial x\partial z} - \frac{\partial^2\phi}{\partial z\partial x}\right) +$
$\hat{k}\left(\frac{\partial^2\phi}{\partial x\partial y} - \frac{\partial^2\phi}{\partial y\partial x}\right)$

$= 0\hat{i} - 0\hat{j} + 0\hat{k} = \vec{0}$

$\nabla\cdot(\nabla\phi) = ?$ divergence of gradient

$$= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\cdot\left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right)$$

$$= \frac{\partial^2\phi}{\partial x^2}, \frac{\partial\phi^2}{\partial y^2}, \frac{\partial^2\phi}{\partial z^2}$$

$(\vec{\nabla}\times(\vec{\nabla}\times A)) = ?$ curl of curl

$$\nabla\times A = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_1 & A_2 & A_3 \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial}{\partial y}A_3 - \frac{\partial}{\partial z}A_2\right) - \hat{j}\left(\frac{\partial}{\partial x}A_3 - \frac{\partial}{\partial z}A_1\right) + \hat{k}\left(\frac{\partial}{\partial x}A_2 - \frac{\partial}{\partial y}A_1\right)$$

$$\nabla\times(\nabla\times A) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \partial A_3/\partial y - \partial A_2/\partial z & -\partial A_3/\partial x + \frac{\partial A_1}{\partial z} & \partial A_2/\partial x - \partial A_1/\partial y \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial}{\partial y}\times\left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right) - \frac{\partial}{\partial z}\times\left(-\frac{\partial A_3}{\partial x} + \frac{\partial A_1}{\partial z}\right)\right) - \hat{j}\left(\frac{\partial}{\partial x}\times\left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right) - \frac{\partial}{\partial z}\times\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right)\right) + \hat{k}\left(\partial/\partial x\times\left(\frac{-\partial A_3}{\partial x} + \frac{\partial A_1}{\partial z}\right) - \frac{\partial}{\partial y}\times\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right)\right)$$

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if $\vec{A} = (A_1, A_2, A_3)$

1) $d\vec{A} = (dA_1, dA_2, dA_3)$, $d\vec{B} = (dB_1, dB_2, dB_3)$

2) $d(A \cdot B) = A \cdot dB + B \cdot dA$

3) $d(\vec{A} \times \vec{B}) = ?$

$= (A_1, A_2, A_3) \cdot (dB_1, dB_2, dB_3) + (B_1, B_2, B_3) \cdot (dA_1, dA_2, dA_3)$

$= A_1 dB_1 + A_2 dB_2 + A_3 dB_3 + B_1 dA_1 + B_2 dA_2 + B_3 dA_3$

Let

$df = f(x, y, z)$

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz$$

$A_1 = A_1(x, y, z)$

$$dA_1 = \frac{\partial A_1}{\partial x} dx + \frac{\partial A_1}{\partial y} dy + \frac{\partial A_1}{\partial z} dz$$

Problems:-

$\vec{A} = (x^2y^2z, xyz, x^2y^2z)$

$\vec{B} = (x^2, x^2 - y^2, e^{xyz})$

Find $d(\vec{A} \cdot \vec{B}) = ?$

$d(A \cdot B) = A \cdot dB + B \cdot dA$

$= (A_1, A_2, A_3) \cdot (dB_1, dB_2, dB_3) + (B_1, B_2, B_3)(dA_1, \ldots$

$= A_1 dB_1 + A_2 dB_2 + A_3 dB_3 + B_1 dA_1 + B_2 dA_2 + B_3 dA_3 \rightarrow (1)$

$$dA_1 = \frac{\partial}{\partial x}(x^2y^2z)dx + \frac{\partial}{\partial y}(x^2y^2z)dy + \frac{\partial}{\partial z}(x^2y^2z)dz$$

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$= 2xy^2z\,dx + 2x^2yz\,dy + x^2y^2dz$

$dA_2 = yz\,dx + xz\,dy + xy\,dz$

$dA_3 = 2xy^3z\,dx + 3x^2y^2z\,dy + x^2y^3dz$

$dB_1 = 2x\,dx$

$dB_2 = 2x\,dx - 2y\,dy$

$dB_3 = yz\,e^{xyz}dx + xz\,e^{xyz}dy + xy\,e^{xyz}dz$

Put in eq (1)

$= (2x\,dx)(x^2y^2z) + xyz(2x\,dx - 2y\,dy) + x^2y^2z\,($
$(yz\,e^{xyz}dx + xz\,e^{xyz}dy + xy\,e^{xyz}dz) +$
$x^2(2xy^2dx + 2x^2yz\,dy + x^2y^2dz) +$
$(x^2-y^2)(yz\,dx + xz\,dy + xy\,dz) +$
e^{xyz}
$\cancel{(2xy^3z\,dx + 3x^2y^2z\,dy + x^3y^3dz)}$

$= \cancel{2x^3y^2z\,dx} + \cancel{2x^2yz\,dx} - 2xy^2z\,dy + \cancel{x^2y^3z^2e^{xyz}}dx$
$+ \cancel{x^3y^3z^2e^{xyz}dy} + \cancel{x^3y^4z\,e^{xyz}dz} +$

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