Chapter # 5

Further Theory of sets and functions

Operations on Collections of Sets:

Let $\mathcal{A}$ is the collection of set. The **union of $\mathcal{A}$** is doneted by

$\cup\{A : A \in \mathcal{A}\}$ or $\bigcup_{A \in \mathcal{A}} A$ or simply $\cup\mathcal{A}$

Consist of all elements of x such that x belongs to at least one set in $\mathcal{A}$ that is

$\cup\{A : A \in \mathcal{A}\} = \{x : x \in A \text{ for some } A \text{ in } \mathcal{A}\}$

The **intesection of $\mathcal{A}$** is doneted by

$\cap\{A : A \in \mathcal{A}\}$ or $\bigcap_{A \in \mathcal{A}} A$ or simply $\cap\mathcal{A}$

Consist of all elements of x such that x belong to all the sets in $\mathcal{A}$, that is

$\cap\{A : A \in \mathcal{A}\} = \{x : x \in A \text{ for every } A \text{ in } \mathcal{A}\}$

Indexed Collection of sets

Let $I \neq \phi$ and $\mathcal{A}$ be a collection of set. An indexing function

$$f : I \to \mathcal{A}$$

as

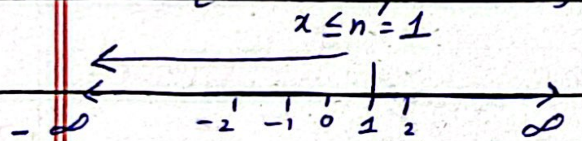
$$f(i) = A_i \in \mathcal{A} \text{ for } i \in I$$

then indexed colection of sets is defined as

$$\{A_i : i \in I\} \text{ or } [A_i]_{i \in I} \text{ or } [A_i]$$

Ex 5.2 **$I = \mathbb{Z}$ is subset of R:**

$A_n = \{x \in \mathbb{R} \,/\, x \leq n\}$ where $n \in \mathbb{Z}$



$$(-\infty, n) = A_n$$

$$\bigcup_{n \in \mathbb{Z}} A_n = \cdots \cup (-\infty, -2] \cup (-\infty, -1] \cup (-\infty, 0] \cup (-\infty, 1] \cup \{\cdots$$

$$= R$$

$$\bigcap_{n \in \mathbb{Z}} A_n = \phi$$

Sequences:

Sequence is function whose domain

is the set of natural number N.

as $f : N \longrightarrow \mathbb{R}$

$f(n) = X_n$

$(x_n) = (x_1, x_2, x_3, \ldots, x_n, \ldots)$

Exp: $f(n) = x_n = n^2$

$f(1) = x_1 = 1^2$

$f(2) = x_2 = 2^2$

$(n^2) = (1^2, 2^2, 3^2, \ldots)$

Summation symbol, sums:

Consider $(x_n) = (x_1, x_2, \ldots, x_n, \ldots)$

is a sequence. Then

$\sum_{n \in N} x_n = x_1 + x_2 + x_3 + \ldots + x_n + \ldots$ is

summation of the sequence

$\Rightarrow \sum_{n=1}^{\infty} x_n = x_1 + x_2 + \ldots + \ldots$

infinite series

$\Rightarrow \sum_{j=1}^{n} a_j = a_1 + a_2 + \ldots + a_n$

~~in~~finite series

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Restriction:

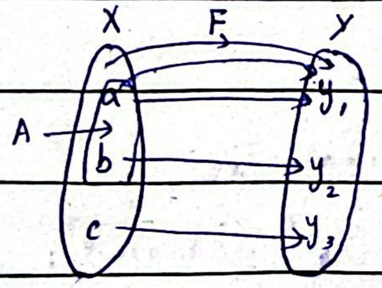
Consider a function $f : X \to Y$.

Let $A \subseteq X$ Then

a function $g = f\downarrow_A : A \to Y$ such that

$g(x) = f(x) \ \forall x \in A$.

is called restriction



Extension:

Consider a function $f : A \to Y$.

Let X be a supper set of A.

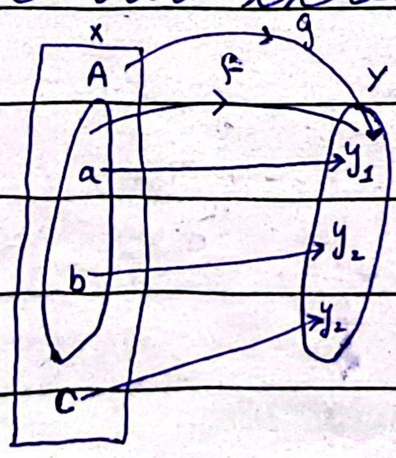
Then a function

$$g = f\uparrow^{X} : X \to y$$

such that

$$g(x) = f(x) \ \forall x \in Y$$

is called extension.



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Mon Tue Wed Thu Fri Sat

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Inclusion Map:

Let $A \subseteq S$. Then a function $i : A \to S$ defined by $i(x) = x$ for $\forall x \in A$ is inclusion map.

Characteristic function:

Let U be a universal set and $A \subseteq U$. Then a function.

$$\chi_A : U \to \{0, 1\}$$

defined by

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$

is called characteristic function.

$U = \{a, b, c, d, e\}$

$A = \{a, d, e\}$

Then

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \in A \end{cases}$$

$x = 0$

$\chi_A(a) = 1$

$\chi_B(b) = 0$

$\frac{A}{R} = \{[a] / a \in A\}$

$f(A) = \{ f(a) / a \in A\}$

image

Mon Tue Wed Thu Fri Sat

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Canonical map:

Let R be an equivalence relation on $S \neq \phi$ and Quatient set of S by R is defined by

$$\frac{S}{R} = \{[a] / a \in S\}$$

Then a function

$$\eta : S \to \frac{S}{R}$$

such that

$$\eta(a) = [a]$$

is called conionical map. It is also called natural map.

Fundamental Factorization of a function:

Consider any function $f : A \to B$ with relation R on A as

"$xRy \Longleftrightarrow f(x) = f(y)$"

which equivalence.

Then a function

$$f^* : \frac{A}{R} \to f(A)$$

such that

$f^*([a]) = f(a)$ this function

Mon Tue Wed Thu Fri Sat

تاریخ:

Show that the fundamental factorization of a function is well defined and bijection.

Proof:

fundamental factorization of a function.

$$f^*: \frac{A}{R} \to f(A)$$

as

$$f^*([a]) = f(a)$$

A fuction is well-defined if $a = b \Rightarrow f(a) = f(b)$

Now we have to show:

i) f^* is well defined:

Suppose

$$[a] = [b]$$

$\Rightarrow (a, b) \in R$

where R is a relation i.e $aRb \Longleftrightarrow f(a) = f(b)$

$\Rightarrow aRb$

$\Rightarrow f(a) = f(b)$

$\Rightarrow f^*([a]) = f^*([b])$

$\Rightarrow f^*$ is well defined

ii) f^* is one - one:

$f^*([a]) = f^*([b])$

$\Rightarrow f(a) = f(b)$

$\Rightarrow a R b$

$\Rightarrow (a, b) \in R$

$\Rightarrow [a] = [b]$

$\Rightarrow f^*$ is one - one

iii) f^* is onto:

clearly for each image

$f(a) \in f(A)$ we have an element

$[a] \in \frac{A}{R}$ under f^* such that $f^*([a]) = f(a)$

$\Rightarrow f^*$ is onto

Hence proved

f^* is well defined and bijective

Associated set function:

Consider a function $f : X \to Y$

If $f[A] = \{f(a) / a \in A\}$ where $A \subseteq X$

$f^{-1}[B] = \{f^{-1}(b) / b \in B\}$ where $B \subseteq Y$

then f & f^{-1} are called

Associated set function.

Choise function:

Consider a collection $\{A_i : i \in I\}$ of subsets of subsets of a set B. Then a function $f : \{A_i : i \in I\} \to B$

as $f(A_i) = a_i \in A_i$

is called choice function.

Defination of G.C.D of a & b.

Let $a, b \in \mathbb{Z}$. Then an integer 'd' of a & b if

i) $d > 0$

ii) $d \mid a$ & $d \mid b$

iii) for any common divisor c of a & b we have $c \mid d$.

symbolically it is written as

$g.c.d(a, b) = d$

Example:

$a = 4$ & $b = 8$

$c_1 = 2$ $c_2 = 4 = d$

i) $d = 4 > 0$

ii) $4 \mid 4$ & $4 \mid 8$ $g.c.d(a,b) = g.c.d(8,4)$

$= 4.$

iii) $c_1 = 2 \mid 4$ & $c_2 = 4 \mid 4$.

CH : 6

Equivalent sets:

Two sets X and Y are said to be equivalent if there exist a bijective map

$$f : X \to Y$$

Notation of Relation between X & Y denoted by '$\approx$'

Mathematically:

$$X \approx Y.$$

Theorem:

Show that the equivalent relation is an equivalence relation.

Proof:

Equivalent relation R is defined as

$$X R Y \iff X \approx Y$$

Now we have to show R is

i) **Reflexive:**

we know that

Identity map $I : X \to X$

defined by $I(x) = x$ exist

which is bijective.

$\Rightarrow X \approx X$

$\Rightarrow X R X$

$\Rightarrow R$ is reflexive

Symmetric:

suppose $X R Y \Longleftrightarrow X \approx Y$

then there exist a bijective

map $f : X \to Y$

we know that

"Inverse of bijective map is bijective"

So

$f^{-1} : Y \to X$ is bijective

$\Rightarrow Y \approx X$

$\Rightarrow Y R X$

$\Rightarrow R$ is symmetric

Transitive:

suppose,

$X R Y \Leftrightarrow X \approx Y$ —— ①

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Mon Tue Wed Thu Fri Sat

$Y R Z \Leftrightarrow Y \simeq Z$ —— ②

From ① & ② there exist bijective maps $f: X \to Y$ and $g: Y \to Z$.

we know that

"Composition of two bijective maps is bijective"

so,

composition $g \circ f : X \to Z$

is bijective

$\Rightarrow X \simeq Z$

$\Rightarrow X R Z$

$\Rightarrow R$ is transitive

Hence prove that equipotent relation is an equivalence relation.

Finite set:

A set X is said to be finite if

$X \simeq \{1, 2, 3, \ldots, n\}$ for $n \in N$.

Example:

$X = \{a, b, c\} \simeq \{1, 2, 3\}$

Infinite set:

A set X is said to be infinite set if it is equivalent to a proper subset of itself.

Example:

$\{2, 4, 6, \dots\} \subset N$

$N \approx \{2, 4, 6, \dots\}$

we defined a map

$f : N \to \{2, 4, 6, \dots\}$

as $f(n) = 2n$

one-one:

$f(n_1) = f(n_2)$

$2n_1 = 2n_2$

$\Rightarrow n_1 = n_2$

$\Rightarrow f$ is 1-1

onto:

Clearly for each element $2n \in \{2, 4, 6, \dots\}$ there exist an element $n \in N$ under f such that

$f(n) = 2n.$

$\Rightarrow f$ is onto

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$N \approx \{2, 4, 6, \dots\}$

$\Rightarrow N$ is infinite

(ii) <u>set of integers $\mathbb{Z}$ is an infinite set.</u>

Proof

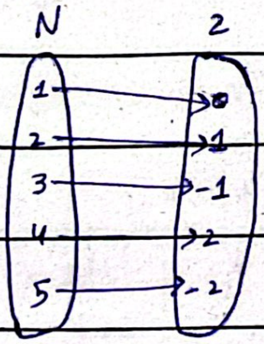
since,

$\mathbb{N} \subseteq \mathbb{Z}$

$\phi : \mathbb{N} \to \mathbb{Z}$

as

$$\phi(x) = \begin{cases} x/2 & \text{if } x \text{ is even} \\ \frac{-x+1}{2} & \text{if } x \text{ is odd} \end{cases}$$



$\Rightarrow$ (Now we have to show that this function is one-one & onto)

i) one-one

consider

$\phi(x_1) = \phi(x_2)$

case # I :

if x_1, x_2 are even

$\Rightarrow \frac{x_1}{2} = \frac{x_2}{2}$

$\Rightarrow x_1 = x_2$

Case # II : if x_1, x_2 is odd

$\Rightarrow \frac{-x_1+1}{2} = \frac{-x_2+1}{2}$

$\Rightarrow -x_1+1 = -x_2+1$

$\Rightarrow -x_1 = -x_2$

$\Rightarrow x_1 = x_2$

Hence f is one-one

Onto:

For clearly each

$\frac{x}{2}, \frac{-x+1}{2} \in \mathbb{Z}$

Hence is an element $x \in N$ under f set.

$\Rightarrow f$ is onto

$K \; N \simeq \mathbb{Z}$

Hence proved $\mathbb{Z}$ is infinite set

Mon Tue Wed Thu Fri Sat

set of integer $\mathbb{R}$ is an infinite set.

Since

$$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \subset \mathbb{R}$$

we define a mapping

$$\phi = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R}$$

as

$$\phi(x) = \tan x$$

Ex 6.2

$$A' = A \times \{1\}$$

$$B' = B \times \{2\}$$

we have to Prove

$$A \approx A' \text{ and } B \approx B'$$

For this we defined two mappings

$f : A \to A'$ & $g : B \to B'$

$f(x) = (x, 1)$ | $g(x) = (x, 2)$

one-one.

onto

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Denumerable set:

An infinite ~~A~~ set X is said to be denumberable if $X \approx N$ (agar ya condition sahi ati ho) otherwise X is called non-denumerable

~~Example~~:

Countable set:

A set X is said to be countable if

Either X is finite or $X \approx N$.

otherwise X is uncountable

Remark:

every Denumerable is countable but every countable set may or may not be Denumerable.

CH # 05

Solved Problems

Generalized operations, Index sets

5.1 Let $\mathcal{A} = \{1,2,3,4\}, \{2,3,4,5\}, \{3,4,5,6\}$
$\{3,4,7,8,9\}$

Find (a) $\cup\mathcal{A}$ and (b) $\cap\mathcal{A}$

a) $\cup\mathcal{A} = \{1,2,3,4,5,6,7,8,9\}$

b) $\cap\mathcal{A} = \{3,4\}$

5.2 Let $A_m = \{m, 2m, 3m, \ldots\}$ where $m \in P$. Find (a) $A_3 \cap A_5$ (b) $A_4 \cap A_6$
(c) $A_5 \cup A_{15}$ (d) $\cup(A_m : m \in S)$

(a) $A_3 \cap A_{15}$

A_3 is positive multiples of 3

A_5 is positive multiples of 5

Multiples of 15 are common in both A_3 and A_5.

as $5 \times 3 = 15$

so $A_3 \cap A_5 = A_{15}$

b) $A_4 \cap A_6$

A_4 means positive multiples of

Mon Tue Wed Thu Fri Sat

$4 = \{4, 8, 12, 16, 20, 24, 28, \ldots\}$

A_6 consists of postive multiples of $6 = \{6, 12, 18, 24, 30, \ldots\}$

L.C.M of 4 and 6 is 12

so $A_4 \cap A_6 = A_{12}$

c) $A_5 \cup A_{15}$

Multiples of 15 is A_{15} and multiples of 5 is A_5. All multiples of 15 are also included in A_5 (As 15 is multiple of 5)

$A_{15} \subseteq A_5$

Hence $A_5 \cup A_{15} = A_5$

d) $\cup(A_m : m \in S)$

$\cup(A_m : m \in S) = \{2, 3, 4, 5, \ldots\}$

$P \setminus \{1\}$

5.3 Let $B_n = \{n, n+1\}$ where $n \in \mathbb{Z}$. Find

(a) $B_1 \cup B_2$ (b) $B_3 \cap B_4$

(c) $\cup_{i=7}^{18} B_i = \cup\{B_i : i \in \{7, 8, \ldots, 18\}\}$:

d) $\cup(B_i : i \in \mathbb{Z})$

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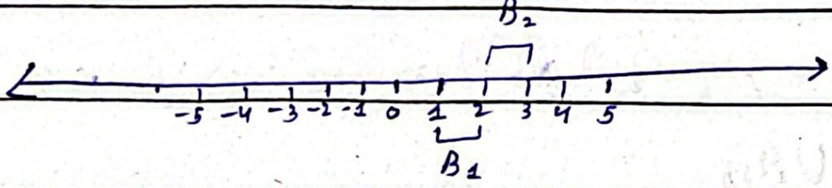
(a) $B_1 \cup B_2$ As it is interval so

$B_1 = \{1, 2\}$, $B_2 = \{2, 3\}$

Plot it on number line to understand.



$B_1 \cup B_2 = \{1, 3\}$ (union)

b) $B_3 \cap B_4$

$B_3 = \{3, 4]$ (means from 3 to 4)

$B_4 = [4, 5]$ (means all points from 4 to 5)

only one point is common in both intervals which is 4.

Hence $B_3 \cap B_4 = \{4\}$

c) $\bigcup_{i=7}^{18} B_i = \bigcup(B_i : i \in \{7, 8, \ldots 18\})$

which is union from B_7 to $_{18}$

$\{7, 8\}, \{8, 9\}, \ldots \{18, 19\}$.

Hence

$\bigcup_{i=7}^{18} B_i = \{7, 19\}$

Mon Tue Wed Thu. Fri Sat تاریخ:

d) $\cup(B_i : i \in Z)$

Z is set of integers like $\{1,2,3,\ldots\}$
where $\mathbb{R}$ is set of real number

$$\mathbb{R} = \{1, 1.1, 1.2, 1.3, \ldots\}$$

When we take union of all B_i
such that i belongs to Z we
get a set which also contain
values with decimal point
so $\cup(B_i : i \in Z) = \mathbb{R}$

5.4 Prove theorem

a) $B \cap (\cup_i A_i) = \cup_i (B \cap A_i)$

b) $B \cap (\cap_i A_i) = \cap_i (B \cap A_i)$

a) $B \cap (\cup_i A_i) = \cup_i (B \cap A_i)$

L.H.S

$B \cap (\cup_i A_i) = \{x : x \in B, x \in \cup_i A_i\}$

$= \{x : x \in B, x \in A_i \text{ for some } i \in I\}$

$= \{x : x \in B \cap A_i \text{ for some } i \in I\}$

$= \{x : x \in \cup_i (B \cap A_i)\}$

$= \cup_i (B \cap A_i) = \text{R.H.S}$

b) $B \cup (\cap_i A_i) = \cap_i (B \cup A_i)$

L.H.S

$B \cup (\cap_i A_i) = \{x : x \in B \text{ or } x \in \cap_i A_i\}$

$= \{x : x \in B \text{ or } x \in A_i \ \forall \ i \in I\}$

$= \{x : x \in B \cup A_i \ \forall \ i \in I\}$

$= \{x : x \in \cap_i (B \cup A_i)\}$

$= \cap_i (B \cup A_i) = R.H.S$

sequences summation symbol

5.7 Write out the first six terms of each sequence:

a) $a_n = (-1)^{n+1} n^2$

$a_1 = (-1)^2 (1)^2 = 1$

$a_2 = (-1)^3 (2)^2 = -4$

$a_3 = (-1)^4 (3)^2 = 9$

$a_4 = (-1)^5 (4)^2 = -16$

$a_5 = (-1)^6 (5)^2 = 25$

$a_6 = (-1)^7 (6)^2 = -36$

b) $b_n = \frac{n}{n+1}$

$b = \frac{1}{2}$, $b_2 = \frac{2}{3}$, $b_3 = \frac{3}{4}$, $b_4 = \frac{4}{5}$

Mon Tue Wed Thu Fri Sat

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$b_5 = \frac{5}{6}$, $b_6 = \frac{6}{7}$

c) $C_n = \begin{cases} 3n & \text{if } n \text{ is odd} \\ 5 & \text{if } n \text{ is even} \end{cases}$

$3 \cdot 4$
$3(1) = 3$
$3(2) = 5$
$3(3) = 9$
$3(4) =$

$c_1 = 3$, $c_2 = 5$, $c_3 = 6$, $c_4 = 5$

$c_5 = 9$ ~~5~~, $c_6 = 5$

5.8 write out the first six terms of each sequence.

(a) $a_1 = 1$, $a_n = n + a_{n-1}$ for $n \geq 1$

$a_2 = 2 + 1 = 3$

$a_3 = 3 + 3 = 6$

$a_4 = 4 + 6 = 10$

$a_5 = 5 + 10 = 15$

$a_6 = 6 + 15 = 21$

b) $b_1 = 1$, $b_2 = 2$, $b_n = 3b_{n-3}$ for $n \geq 2$

$b_3 = 3(1) + 2(2) = 3 + 4 = 7$

$b_4 = 3(2) + 2(7) = 6 + 14 = 20$

$b_5 = 3(7) + 2(20) = 21 + 40 = 61$

$b_5 = 3(20) + 2(61) = 60 + 122 = 182$

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5.9 Find:

a) $\sum_{K=1}^{4} K^3$

$= 1^3 + 2^3 + 3^3 + 4^3$

$= 1 + 8 + 27 + 64 = 100$

b) $\sum_{i=1}^{5} x_i$

$= x_1 + x_2 + x_3 + x_4 + x_5$

c) $\sum_{j=1}^{3} (j^4 - j^2)$

$= (1^4 - 1^2) + (2^4 - 2^2) + (3^4 - 3^2)$

$= (1-1) + (16-4) + (81-9)$

$= 0 + 12 + 72 = 84$

Prove:

5.10 $\sum_{K=1}^{n} [f(K) + g(K)] = \sum_{K=1}^{n} f(K) + \sum_{K=1}^{n} g(K).$

Put $n = 1$

$\sum_{n=1}^{1} [f(K) - g(K)] = f(1) + g(1) = \sum_{K=1}^{1} f(K) + \sum_{K=1}^{1} g(K)$

Hence it is true for $n = 1$

suppose it is true for $n = 1$

$\cancel{\sum_{K=1}^{n^1} [f(K) + g(K)] = f(1) + g(1) = \sum_{K=1}^{1} f(K) + \sum_{K=1}^{1} g(K)}$

then

$$\sum_{K=1}^{n}[f(K)+g(K)]=\sum_{K=1}^{n-1}[f(K)+g(K)]+[f(n)+g(n)]$$

$$=\sum_{K=1}^{n-1}f(K)+\sum_{K=1}^{n-1}g(K)+[f(n)+g(n)]$$

$$=\sum_{K=1}^{n-1}f(K)+f(n)+\sum_{K=1}^{n-1}g(K)+g(n)$$

$$=\sum_{K=1}^{n}f(K)+\sum_{K=1}^{n}g(K)$$

Hence praned

SPEcial function: Extension, choice, characteristic

5.17 Consider the function $f(x)=x$ and $D=[0,\infty)$ is domain. For $x\geq 0$ state wether or not each of the fallowing function is an extension of f.

a) $g_1(x)=x$ where $x\geq -2$

Yes, it is extension of f as $g_1(x)=x$ for $[0,\infty)$ and also has larger Domain.

Mon Tue Wed Thu Fri Sat b) $g_2(x) = (x + |x|)$ تاريخ:

if $x > 0$ then $g_2(x) = \frac{x + x}{2} = x$

which mean $g_2(x) = x$ for $(0, \infty)$ which is domain of f.

Hence $g_2(x)$ is extension.

c) $g_3(x) = x$ where $x \in [-1, 1]$

Domain of $g_3(x)$ is not a super set of Domain of f, it doesn't contain all element which D_f contain. ~~all~~ Hence, it can't be an extension.

5.18 Consider the following subsets of $B = \{1, 2, 3, 4, 5\}$, $A_1 = \{1, 2, 3\}$, $A_2 = \{1, 5\}$, $A_3 = \{2, 4, 5\}$

$A_4 = \{3, 4\}$

State whether or not each of the following functions from $\{A_1, A_2, A_3, A_4\}$ into B is a choice function.

Mon Tue Wed Thu Fri Sat

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(a) $f_1 = \{(A_1, 1), (A_2, 2), (A_3, 3), (A_4, 4)\}$
Since $f_1(A_2) = 2$ is not an element in A_2, f_1 is not a choice function.

b) $f_2 = \{(A_1, 1), (A_2, 1), (A_3, 4), (A_4, 4)\}$
Hence $f_2(A_i)$ belongs to A_i, for each i; Hence f_2 is a choice function.

c) $f_3 = \{(A_1, 2), (A_2, 1), (A_3, 4), (A_4, 3)\}$
Also, $f_3(A_i)$ belongs to A_i, for each i, hence f_3 is a choice function.

d) $f_4 = \{(A_1, 3), (A_2, 5), (A_3, 1) (A_4, 3)\}$
Note that $f_4(A_3) = 1$ doesnot belong to A_3, hence f_4 is not a choice function.

Supplementary Problems

5.28 Let $A = [\{1, 2, 3, 4\}, \{1, 3, 5, 7, 9\}, \{1, 2, 3, 6, 8\}, \{1, 3, 7, 8, 9\}]$. Find

(a) $\cup A$; (b) $\cap A$

(a) $\cup A = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$

(b) $\cap A = \{1, 3\}$

5. For each $m \in P$, let A_m be the following subset of P:

$$A_m = \{m, 2m, 3m, \dots\} = \{\text{multiples of } m\}$$

(a) Find: (1) $A_2 \cap A_7$; (2) $A_6 \cap A_8$; (3) $A_3 \cup A_2$; (4) $A_3 \cap A_{12}$.

(b) Prove $\cap(A_i : i \in J) = \phi$, when J is an infinite subset of P.

(a) Find:

(1) $A_2 \cap A_7$

$A_2 \cap A_7 = A_{14}$ A_2 LCM of 2 and 7 is 14 - Both contain all multiples of 14 which is "A_{14}".