Chapter #8
Ordinal numbers

Find successors and limit elements of Q under usual order "$<$".

Sol:

Suppose $q \in Q$. For any $x \in Q$ such that $x < q$

then there exist $y = \frac{x+q}{2} \in Q$ such that

$x < y < q$

$\Rightarrow$ q is a limit element

Since q was arbitrary so all elemets of set Q are limits elements

we know that "limit elements can not be a successor".

So successers does not exist in Q.

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تاریخ:

Find Successors and limit elements of $\mathbb{Z}$ under usual order "$<$"?

Sol:

suppose $n \in \mathbb{Z}$ clearly $n+1 \in \mathbb{Z}$ is next consective element of set $\mathbb{Z}$ such that

$$n < n+1$$

n replace $n-1$

$$n-1 < n$$

$\Rightarrow$ There is no other integer exist between $n-1$ and n then n is successor.

Since n was orbitrary so all elemets of set $\mathbb{Z}$ are succesors.

we know that "successor can not be a limit element" so limit element does not exist in $\mathbb{Z}$.

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Initial segment:

Let A be well ordered set and $a \in A$. Then initial segment of A, denoted by $S(a)$, is defined as

$$S(a) = \{x \in A / x < a\}$$

Example:

$$\mathbb{N} = \{1, 2, 3, \dots\}$$

$$5 \in \mathbb{N}$$

$$S(5) = \{x \in \mathbb{N} / x < 5\}$$
$$= \{1, 2, 3, 4\}$$

$$1 \in \mathbb{N}$$

$$S(1) = \{x \in \mathbb{N} / x < 1\}$$
$$S(1) = \phi$$

Note that:

$$S(a) \subseteq A$$

Theorem 8.6:

Let $S(A)$ denote the collection of all initial segments of elements in a well-ordered set A and let $S(A)$ be ordered by set inclusion. Then A is similar to $S(A)$.

Proof:

Given

$$S(A) = \{S(x) / x \in A\}$$

is a collection of all initial segements in A.

To Prove:

$$A \simeq S(A)$$

For this we define a mapping

$$f : A \longrightarrow S(A)$$

as

$$f(x) = S(x) \qquad \forall x \in A.$$

Now we have to show f is

i) one-to-one

Let $x, y \in A$ such that $x \neq y$

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$\Rightarrow$ either $x < y$ or $y < x$

suppose $x < y$

$\Rightarrow x \in S(y)$

but $x \in S(x)$

so

$$S(y) \neq S(x)$$

$\Rightarrow S(x) \neq S(y)$

$\Rightarrow f(x) \neq f(y)$

$\Rightarrow f$ is one-to-one

ii) onto:

Clearly, for each element $S(x) \in S(A)$ we have an element $x \in A$ under f such that

$$f(x) = S(x)$$

$\Rightarrow f$ is onto

iii) Order preserving:

Let $x, y \in A$ such that

$$x \lesssim y \quad \text{—} \quad (1)$$

<u>show that</u>

$$f(x) \lesssim f(y)$$

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For this we take

$a \in S(x)$

$\Rightarrow a < x$

$\Rightarrow a < x \leq y$ (from ①)

$\Rightarrow a < y$

$\Rightarrow a \in S(y)$

Thus

$S(x) \subseteq S(y)$

$\Rightarrow f(x) \leq f(y)$

$\Rightarrow f$ is order-preserving

Hence proved

$A \simeq S(A)$

Theorem 8.7:

Let A be a well-ordered set and $B \subseteq A$. If $f: A \to B$ is similarity then show that $a \leq f(a)$ for all $a \in A$.

Proof: To prove above statement consider a set

$D = \{ a \in A \mid f(a) < a \}$ —— ①

if $D = \emptyset$ there is nothing left to prove.

if $D \neq \emptyset$ so ~~Set~~ we can suppose $a_0 \in D$ is a minimal element such that

$$f(a_0) < a_0$$

Let $f(a_0) = a_1$

since $f: A \to B$ is similarity map therefor

$$f(a_1) < f(a_0)$$

$$f(a_1) < a_1$$

$$\Rightarrow a_1 \in D$$

which is contradiction to fact a_0 is minimal element of D.

Hence prove

$$a \leq f(a).$$

Theorem 8.8:

Let A and B be simil

sets. Then there exi

$\Rightarrow g^{-1}\circ f(x) = I(x)$ — (3)

From (a) & (b) we get

$g^{-1}\circ f(x) = I(x)$

$\Rightarrow g[g^{-1}\circ f(x)] = g[I(x)]$

$\Rightarrow (g\circ g^{-1})\circ f(x) = g\circ I(x)$

$\Rightarrow (I\circ f)(x) = (g\circ I)(x)$

$\Rightarrow f(x) = g(x)$

$\Rightarrow f = g$

which is condriction to our supposition.

So $f: A \to B$ is unique

Theorem 8.9.

A well-orderedset can not be a similar to one of its initial segment.

Proof:

Let A be well-ordered set and $a \in A$. If $S(a) = \{x \in A / x < a\}$ is one initial segment of A then we have to prove.

$$A \not\simeq S(a)$$

Suppose

$$A \simeq S(a)$$

then there exist a similarity mapping

$$f : A \to S(a)$$

Since $a \in A \Rightarrow f(a) \in S(a)$

$$\Rightarrow f(a) < a$$

which contradition to the fact

$$a \leq f(a)$$

Hence proved well-ordered set cannot be similarity one of its initial segment.

Chapter # 9

Axiom of choice, Zorn's lemma, well-ordering Theorem.

Choice function:

Let $\{A_K : K \in I\}$ be a family of non-empty subsets of a set X. Then a function $f : \{A_K : K \in I\} \to X$ is called choice function if $f(A_K) = a_K \in A_K$ for every $K \in I$.

Axiom of choice:

Let $X \neq \phi$ and $\mathcal{A} = P(X) - \phi$. Then there exist a choice function $f : \mathcal{A} \to X$ such that $A \in \mathcal{A} \Rightarrow f(A) \in X$.

This is called axiom of choice.

Zermelo's Postulate:

Let $\{A_K : K \in I\}$ be a family

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non-empty sets. Then there exist a set A such that

$A \subseteq \bigcup_{K \in I} A_K$ and $A \cap A_K = \{a\}$

where 'a' is arbitrary.

Example: of choice function

$\{A_1 = \{1, 2\}, A_2 = \{2, 3\}, A_3 = \{3, 4, 5\}\}$

$X = \{1, 2, 3, 4, 5\}$

$f : A \longrightarrow X$ as $f(A_1) = 2 \in A_1$

$f(A_2) = 3 \in A_2$

$X = \{1, 2\}$

$P(X) = \{\phi, \{1\}, \{2\}, X\}$

$\Rightarrow A = \{\{1\}, \{2\}, \{1, 2\}\}$

Example of Zermelo's Postulate:

$A_1 = \{1, 2\}$

$A_2 = \{3, 4\}$

$A_3 = \{5, 6, 7\}$

$\bigcup_{K=1}^{3} A_K = \{1, 2\} \cup \{3, 4\} \cup \{5, 6, 7\}$

$= \{1, 2, 3, 4, 5, 6, 7\}$

There exist a set

$A = \{1, 3, 5\}$

such that

$A \leq \bigcup_{k=1}^{3} A_k$

$A \cap A_1 = \{1\}$

$A \cap A_3 = \{3\}$

$A \cap A_3 = \{5\}$

Zorn's Lemma:

Let X be a non empty partial ordered set in which every linear ordered subset has an upper bound in X. Then X contain at least one maximal element.

$(X = \{1, \ldots, 100\}, \backslash)$

$(A = \{1, 2, 4, 8, 16, 32, 64\}, \backslash)$

$x \in A$, $x \leq 64$

$B = \{1, 25, 50, 100\}$

Basic logical operations:

There are three basic logical operator.

i) conjunction $\rightarrow$ and $\rightarrow \wedge$ (x)

ii) disjunction → or → $\vee$ +

iii) Neggtine → not → $\neg$

Truth table:

Truth table is a chart use to elastrate and determin the truth value of proposition and the validity of their resulting argument.

p, q $\sim(p \wedge \sim q)$

p	q	$\sim q$	$p \wedge \sim q$	$\sim(p \wedge \sim q)$
T	T	F	F	T
T	F	T	T	F
F	T	F	F	T
F	F	T	F	T

state and prove idempotent law

i) $P \vee P = P$ (b) $P \wedge P = P$

P	P	$P \vee P$
T	T	T
F	F	F

P	P	$P \wedge P$
T	T	T
F	F	F

Theorem 10.3

$P_1, P_2, , P_n \vdash Q$

$\underline{P}; \underline{P \to q} \vdash q$

$P_1 . \quad P_2 \vdash Q$

P_1		P_2	$P_1 \wedge P_1$	$P_1 \cup P_2 \to Q$
P	q	$P \to q$	$P \wedge P \to q$	$P \wedge P \to q \to q$
T	T	T	T	T
F	T	T	F	T
T	F	F	F	T
F	F	T	F	T

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$\neg(p \vee q) \vee (\neg p \wedge q) \equiv \neg p$

statement

1) $\neg(p \vee q) \vee (\neg p \wedge q) = (\neg p \wedge \neg q) \vee (\neg p \wedge q)$ ← Reason Demorgin law

$= \neg p \wedge (\neg q \vee q)$ Distributive

$= \neg p \wedge T$ compliment law

$= \neg p$ Identity law

Imp long

To prove $p \to q, \sim q \vdash \sim p$ is valid we have to prove $p \to q \wedge \sim q \to \sim p$ is tatoulgy

$\sim p$: p ka ulat
$\sim q = q$ k ulat
jah T ha g
$\sim p$ mq F aya g

p	q	$p \to q$	$\sim q$	$p \to q \wedge \sim q$	$(p \to q \wedge \sim q) \to \sim p$
T	T	T	F	F	T
T	F	F	T	F	T
F	T	T	F	F	T
F	F	T	T	T	T

$\sim p$
F
F
T
T

11.3

Consider the Boolean algebra D_{210}. List the element and its diagram. Also find its two sub-algebras with eight elements.

sol:

we know that

$D_m = \{d : d|m\} ; m \in \mathbb{N}$

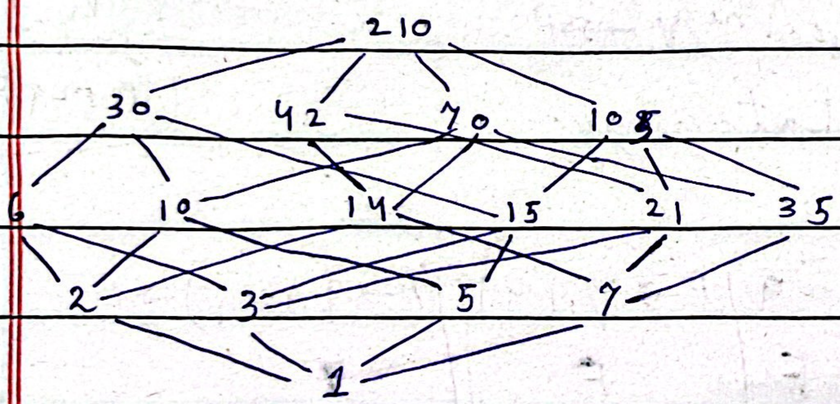
we have to write no jo 210 ko divide karta ha

List of elements : $D_{210} = \{d : d/210\}$

$= \{1, 2, 3, 5, 6, 7, 10, 14, 15, 21, 30,$

$35, 42, 70, 105, 210\}$

$$\begin{array}{r|l} 2 & 210 \\ \hline 3 & 105 \\ \hline 7 & 35 \\ \hline 5 & 5 \\ \hline & 1 \end{array}$$



To prove mapping $f : B \to P(A)$ defined by

$f(x) = \{a_1, a_2, \ldots, a_n\}$

is isomorphism

For this suppose $x, y \in B$

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such that

$$x = a_1 + \dots + a_r + b_1 + \dots + b_s$$

$$y = b_1 + \dots + b_s + c_1 + \dots + c_t$$

where

$$A = \{a_1, \dots, a_r, b_1, \dots, b_s, c_1, \dots, c_t, d_1, \dots, d_k\}$$

Then

$$x + y = a_1 + \dots + a_r + b_1 + \dots + b_s + c_1 + \dots + c_t$$

$$xy = b_1 + \dots + b_s$$

$$\begin{cases} a_i^2 = a_i \\ a_i a_j = 0 \end{cases}$$

$a \wedge a = a$

$b_1 + b_1 = b_1 \wedge b_1 = b_1$

$$f(x+y) = \{a_1, a_2, \dots, a_r, b_1, b_2, \dots, b_s, c_1, c_2, \dots, c_t\}$$

$$= \{a_1, a_2, \dots, b_1, b_2, \dots b_s\} \cup \{b_1, b_2, \dots, c_1, c_2, \dots, c_t\}$$

$$= f(x) \cup f(y)$$

$$x' = c_1 + \dots + c_t + d_1 + \dots + d_k$$

Then

$$x + x' = 1$$

$$x \cdot x' = 0$$

$$f(x') = \{c_1, \dots, c_t, d_1, \dots, d_k\}$$

$$= \{a_1, \dots, a_r, b_1, \dots, b_s\}^c$$