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[illegible]

## Group:

A set which satisfies the following properties is called a group.

- → Closure property.
- → Associative property.
- → Identity
- → Inverse

Along with these properties if a set satisfies commutative property then it is called Abelian Group.

## Field:

A set F which is abelian group under addition, abelian group under multiplication and also satisfy the distributive property is called a Field. For example $\mathbb{R}, \mathbb{Q}, \mathbb{C}$ etc.

OR

A field is a set F with two operations addition and multiplication and satisfy the following axioms (A) (M) and (D)

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## (A) Axioms for addition:

($A_1$) If $x, y \in F$ then $x+y \in F$

($A_2$) Addition is commutative i-e
$$x+y = y+x \quad \forall \; x, y \in F$$

($A_3$) Addition is associative i-e
$$(x+y)+z = x+(y+z) \quad \forall \; x, y, z \in F$$

($A_4$) F contains an element '0' such that $x+0 = x \quad \forall \; x \in F$.

($A_5$) For any element $x \in F \; \exists \; -x \in F$ such that $x+(-x) = 0$

## (M) Axioms for multiplication:

($M_1$) If $x, y \in F$ then $x \cdot y \in F$

($M_2$) Multiplication is commutative i-e
$$x \cdot y = y \cdot x \quad \forall \; x, y \in F$$

($M_3$) Multiplication is associative i-e
$$(xy)z = x(yz) \quad \forall \; x, y, z \in F$$

($M_4$) F contains an element 1 such that
$$x \cdot 1 = x \quad \forall \; x \in F$$

($M_5$) For any non-zero element $x \in F$, $\exists$ an element $1/x$ such that
$$x \cdot \frac{1}{x} = 1$$

(D) The Distributive law:

$$x\cdot(y+z) = xy + xz \quad \forall \; x,y,z \in F$$
$$(x+y)\cdot z = xz + yz \quad \forall \; x,y,z \in F$$

## Vector Space:

Let $F$ be a field and $V$ be a non-empty set on whose elements an operation of addition is defined. Suppose that for every $a \in F$ and every $v \in V$, $av$ is an element of $V$. Then $V$ is called a vector space over $F$ if following axioms hold:

(i) $V$ is an abelian group under addition.

(ii) $a(v+w) = av + aw \quad \forall \; a \in F, \; v,w \in V$

(iii) $(a+b)\cdot v = av + bv \quad \forall \; a,b \in F, \; v \in V$

(iv) $a(bv) = (ab)v \quad \forall \; a,b \in F, \; v \in V$

(v) $1\cdot v = v$, $1$ being M.I of $F$.

The elements of $F$ are called scalars and elements of $V$ are called vectors and $av$, $a \in F$ and $v \in V$ is called scalar multiplication of $v$ by $a$.

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Example 1: The set $\mathbb{R}^3 = \{(x,y,z): x,y,z \in \mathbb{R}\}$ is vector space over field $\mathbb{R}$, under addition and scalar multiplication defined by:

(i) $u+u' = (x,y,z)+(x',y',z')$
$= (x+x', y+y', z+z')$
for $u=(x,y,z)$, $u'=(x',y',z') \in \mathbb{R}^3$

(ii) $au = (ax, ay, az) \in \mathbb{R}^3$, $a \in \mathbb{R}$

Proof:
We check axioms for vector space.

(i) Abelian group under Addition.

(a) Closure Property:
Let $u, v \in \mathbb{R}^3$, such that $u = (x_1, y_1, z_1)$, $v = (x_2, y_2, z_2)$ then

$u+v = (x_1, y_1, z_1) + (x_2, y_2, z_2)$
$= (x_1+x_2, y_1+y_2, z_1+z_2) \in \mathbb{R}^3$

So, closure property holds.

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(b) Commutative property:

Let $u, v \in \mathbb{R}^3$, such that $u = (x_1, y_1, z_1)$, $v = (x_2, y_2, z_2)$ then,

$$u+v = (x_1, y_1, z_1) + (x_2, y_2, z_2)$$
$$= (x_1+x_2, y_1+y_2, z_1+z_2)$$
$$= (x_2+x_1, y_2+y_1, z_2+z_1) \quad \because \mathbb{R} \text{ is field.}$$
$$= (x_2, y_2, z_2) + (x_1, y_1, z_1)$$
$$= v+u$$

So, commutative property holds.

(c) Associative property:

Let $u, v, w \in \mathbb{R}^3$ such that $u = (x_1, y_1, z_1)$, $v = (x_2, y_2, z_2)$, $w = (x_3, y_3, z_3)$ then

$$u+(v+w) = (x_1, y_1, z_1) + ((x_2, y_2, z_2) + (x_3, y_3, z_3))$$
$$= (x_1, y_1, z_1) + (x_2+x_3, y_2+y_3, z_2+z_3)$$
$$= (x_1+x_2+x_3, y_1+y_2+y_3, z_1+z_2+z_3)$$
$$= (x_1+x_2, y_1+y_2, z_1+z_2) + (x_3, y_3, z_3)$$
$$= ((x_1, y_1, z_1) + (x_2, y_2, z_2)) + (x_3, y_3, z_3)$$
$$= (u+v)+w$$

So, associative property holds.

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(d) Identity element:

As $0=(0,0,0)\in\mathbb{R}^3$ such that $\forall\ u\in\mathbb{R}^3$, $0+u=u=u+0$.

(e) Inverse of each element:

$\forall\ u\in\mathbb{R}^3$, $\exists\ -u\in\mathbb{R}^3$ such that

$u+(-u)=0$

$u=(x_1,y_1,z_1)$, $-u=(-x_1,-y_1,-z_1)$

$$u+(-u)=(x_1,y_1,z_1)+(-x_1,-y_1,-z_1)$$
$$=(x_1-x_1,\ y_1-y_1,\ z_1-z_1)$$
$$=(0,0,0)=0$$

So, the given set is Abelian group under addition.

(ii) Let $a\in\mathbb{R}$, and $u,v\in\mathbb{R}^3$ such that $u=(x_1,y_1,z_1)$, $v=(x_2,y_2,z_2)$ then

$$a(u+v)=a((x_1,y_1,z_1)+(x_2,y_2,z_2))$$
$$=a(x_1+x_2,\ y_1+y_2,\ z_1+z_2)$$
$$=(a(x_1+x_2),\ a(y_1+y_2),\ a(z_1+z_2))$$
$$=(ax_1+ax_2,\ ay_1+ay_2,\ az_1+az_2)$$
$$=(ax_1,ay_1,az_1)+(ax_2,ay_2,az_2)$$
$$=a(x_1,y_1,z_1)+a(x_2,y_2,z_2)$$
$$=au+av$$

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(iii) Let $a,b \in \mathbb{R}$, $u \in \mathbb{R}^3$, $u=(x_1,y_1,z_1)$ then

$$(a+b)\cdot u = (a+b)(x_1,y_1,z_1)$$
$$= ((a+b)x_1, (a+b)y_1, (a+b)z_1)$$
$$= (ax_1+bx_1, ay_1+by_1, az_1+bz_1)$$
$$= (ax_1, ay_1, az_1)+(bx_1, by_1, bz_1)$$
$$= a(x_1,y_1,z_1)+b(x_1,y_1,z_1)$$
$$= au+bu$$

(iv) Let $a,b \in \mathbb{R}$ and $u \in \mathbb{R}^3$, $u=(x_1,y_1,z)$ then

$$a(bv) = a(b(x_1,y_1,z_1))$$
$$= a(bx_1, by_1, bz_1)$$
$$= (abx_1, aby_1, abz_1)$$
$$= (ab)(x_1,y_1,z_1)$$
$$= (ab)u$$

(V) As $1 \in \mathbb{R}$, let $u \in \mathbb{R}^3$, $u=(x_1,y_1,z_1)$ then

$$1.u = 1(x_1,y_1,z_1)$$
$$= (1.x_1, 1.y_1, 1z_1) \quad \because \mathbb{R} \text{ is field}$$
$$= (x_1,y_1,z_1) = u$$

As all the conditions of vector space are satisfied. Given set $\mathbb{R}^3$ is vector space over $\mathbb{R}$.

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**Example 2:** The field $(F, +, \cdot)$ is vector space over itself.

**Solution:**

We use axioms of vector space.

(i) As by definition of field $F$. $F$ is the abelian group under addition

(ii) Let $a \in F$ and $u, v \in F$, then
$a(u+v) = au + av$ $\because F$ is distributive

(iii) Let $a, b \in F$ and $u \in F$, then
$(a+b)u = au + bu$ $\because F$ is distributive

(iv) Let $a, b \in F$ and $u \in F$, then
$a(bv) = (ab)v$ $\because F$ has asso. property

(v) As $1 \in F$ and let $u \in F$, then
$1 \cdot u = u$ $\because F$ has identity element

As all the properties of vector space are satisfied. Then every field is vector space over itself.