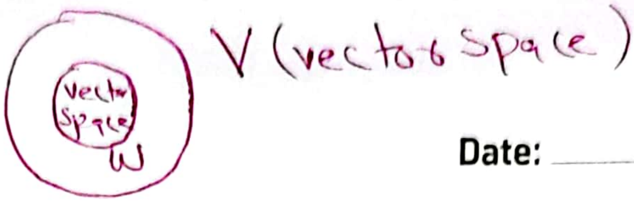

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Subspace of Vector Space:

A non-empty subset W of a vector space V over a field F is said to be a subspace of V if W is itself a vector space over F under the same operations as defined in V.

Every vector space has two subspaces: V itself and the set $\{0\}$, which consists of zero vector only.

Theorem: Let V be a vector space over a field F and let W be a non-empty subset of V. Then W is a subspace of V if and only if

(i) $w_1, w_2 \in W \Rightarrow w_1 + w_2 \in W$

(ii) $w \in W, a \in F \Rightarrow aw \in W$

Proof:

Suppose W is non-empty subset of V satisfying (i) and (ii), then we have to show that W is a vector space.

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let $w_1, w_2 \in W$, then by (ii)

$(-1)w_2 \in W$ i.e $-w_2 \in W$.

Now $w_1, -w_2 \in W$ and so by (i)

$w_1 + (-w_2) = w_1 - w_2 \in W$

Therefore, W is a subgroup of a group V under addition. Since V is abelian group. Also, remaining properties of the vector space are satisfied because all the properties are satisfied in the bigger set V.

Conversely, suppose that W is subspace of V.

$\Rightarrow$ W is itself a vector space

So, by definition of vector space

(i) $w_1, w_2 \in W \Rightarrow w_1 + w_2 \in W$

(ii) $w \in W, a \in F \Rightarrow aw \in W$

Corollary: A non-empty subset W of a vector space V is a subspace of V if for

$w_1, w_2 \in W$ and $a, b \in F$

$\Rightarrow aw_1 + bw_2 \in W$

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Example: Which of the following are subspace of $\mathbb{R}^3$.

(i) $W = \{(x, y, z) : x + y + z = 0\}$

Sol:

let $w_1, w_2 \in W$

$w_1 = \{(x_1, y_1, z_1) : x_1 + y_1 + z_1 = 0\}$

$w_2 = \{(x_2, y_2, z_2) : x_2 + y_2 + z_2 = 0\}$

Now let $a, b \in \mathbb{R}$, then

$aw_1 + bw_2 = a(x_1, y_1, z_1) + b(x_2, y_2, z_2)$

$= (ax_1, ay_1, az_1) + (bx_2, by_2, bz_2)$

$aw_1 + bw_2 = (ax_1 + bx_1, ay_1 + by_2, az_1 + bz_2)$

$aw_1 + bw_2 \in W$ if $ax_1 + bx_2 + ay_1 + by_2$

$+ az_1 + bz_2 = 0$

$L.H.S = ax_1 + bx_2 + ay_1 + by_2 + az_1 + bz_2$

$= a(x_1 + y_1 + z_1) + b(x_2 + y_2 + z_2)$

$= a(0) + b(0)$

$= 0$

So, $aw_1 + bw_2 \in W$

Hence W is a subspace of $\mathbb{R}^3$.

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(ii) $W = \{(x, y, z) : x \geq 0\}$

Sol:

Let $w_1, w_2 \in W$

$w_1 = \{(x_1, y_1, z_1) : x_1 \geq 0\}$

$w_2 = \{(x_2, y_2, z_2) : x_2 \geq 0\}$

Now let $a, b \in \mathbb{R}$, then

$$aw_1 + bw_2 = a(x_1, y_1, z_1) + b(x_2, y_2, z_2)$$
$$= (ax_1, ay_1, az_1) + (bx_2, by_2, bz_2)$$
$$= (ax_1 + bx_2, ay_1 + by_2, az_1 + bz_2)$$

Now $aw_1 + bw_2 \in W$ if $ax_1 + bx_2 \geq 0$

As $a, b \in \mathbb{R}$ and $x_1, x_2 \geq 0$

So $ax_1 + bx_2$ may not be ≥ 0

Hence $aw_1 + bw_2 \notin W \quad \forall a, b \in \mathbb{R}$

So W is not a subspace of $\mathbb{R}^3$.

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(iii) $W = \{(x, 0, y) : x, z \in \mathbb{R}\}$

Sol:

Let $w_1, w_2 \in W$ then

$w_1 = \{(x_1, 0, z_1) : x_1, z_1 \in \mathbb{R}\}$

$w_2 = \{(x_2, 0, z_2) : x_2, z_2 \in \mathbb{R}\}$

Let $a, b \in \mathbb{R}$, then

$aw_1 + bw_1 = a(x_1, 0, z_1) + b(x_2, 0, z_2)$

$= (ax_1, 0, az_1) + (bx_2, 0, bz_2)$

$= (ax_1 + bx_2 + 0, az_1 + bz_2)$

Now $aw_1 + bw_2 \in W$ if $ax_1 + bx_2, az_1 + bz_2 \in \mathbb{R}$

But as $a, b, x_1, z_1, x_2, z_2 \in \mathbb{R}$

So $ax_1 + bx_2, az_1 + bz_2 \in \mathbb{R}$

Hence $aw_1 + bw_2 \in W$

So W is a subspace of $\mathbb{R}^3$

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Linear Combination:

Let V be a vector space over F and let $v_1, v_2, v_3, \cdots, v_n \in V$ then any vector in V of the form $a_1 v_1 + a_2 v_2 + a_3 v_3 + \cdots + a_n v_n$ is called linear combination of $v_1, v_2, \cdots, v_n$.

Example: Express the vector $(2,-5,3)$ in $\mathbb{R}^3$ as a linear combination of the vectors $(1,-3,2), (2,-4,-1), (1,-5,7)$

Sol:

Let $(2,-5,3) \in \mathbb{R}^3$

and let $V = \{\overset{v_1}{(1,-3,2)}, \overset{v_2}{(2,-4,-1)}, \overset{v_3}{(1,-5,7)}\}$ is a set of V.

Let $a, b, c \in F$, then

$$(2,-5,3) = a(1,-3,2) + b(2,-4,-1) + c(1,-5,7)$$

$$(2,-5,3) = (a,-3a,2a) + (2b,-4b,-b) + (c,-5c,7c)$$

$$(2,-5,3) = (a+2b+c, -3a-4b-5c, 2a-b+7c)$$

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$$\Rightarrow \quad a + 2b + c = 2 \quad \text{—} \ (1)$$
$$-3a - 4b - 5c = -5 \quad \text{—} \ (2)$$
$$2a - b + 7c = 3 \quad \text{—} \ (3)$$

multiply eq. (1) by 3 and add in (2)

$$\begin{array}{r} \cancel{3a} + 6b + 3c = 6 \\ -\cancel{3a} - 4b - 5c = -5 \\ \hline 2b - 2c = 1 \end{array}$$

$$b - c = \frac{1}{2} \quad \text{—} \ (4)$$

Now multiply (1) by 2 and subtract (3) from (1)

$$\begin{array}{r} \cancel{2a} + 4b + 2c = 4 \\ \pm\cancel{2a} \mp b \pm 7c = \pm 3 \\ \hline 5b - 5c = 1 \end{array}$$

$$b - c = \frac{1}{5} \quad \text{—} \ (5)$$

from (4) and (5), we cannot find value of b and c. Thus (2, -5, 3) cannot be expressed as a linear combination of (1, -3, 2), (2, -4, -1), (1, -5, 7).

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Example: For what value of k will the vector $(1,-2,k)$ in $\mathbb{R}^3$ be a linear combination of vectors $(3,0,-2)$ and $(2,-1,-5)$?

Sol:

Let $(1,-2,k) = a(3,0,-2) + b(2,-1,-5)$

$= (3a, 0, -2a) + (2b, -b, -5b)$

$(1,-2,k) = (3a+2b, -b, -2a-5b)$

$\Rightarrow \quad 3a + 2b = 1 \quad - \; (1)$

$-b = -2 \quad - \; (2)$

$-2a - 5b = k \quad - \; (3)$

put $\boxed{b = +2}$ in (1)

$3a + 4 = 1 \Rightarrow 3a = -3 \Rightarrow \boxed{a = -1}$

put values of a and b in (3)

$-2(-1) - 5(2) = k$

$2 - 10 = k$

$\Rightarrow \boxed{k = -8}$

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$(1,-2,k)$

So for $k=-8$, the vector is a linear combination of $(3,0,2)$ and $(2,-1,-5)$.

Example: Show that each of the following sets of vectors generate $\mathbb{R}^3$.

(i) $\{(1,2,3),(0,1,2),(0,0,1)\}$

Sol:

Given set is $\{(1,2,3),(0,1,2),(0,0,1)\}$

Let $(x,y,z)\in\mathbb{R}^3$ and suppose

$$(x,y,z)=a(1,2,3)+b(0,1,2)+c(0,0,1)$$
$$=(a,2a,3a)+(0,b,2b)+(0,0,c)$$
$$(x,y,z)=(a,2a+b,3a+2b+c)$$

$\Rightarrow$ $a=x$ — 1

$2a+b=y$ — 2

$3a+2b+c=z$ — 3

from 1 $\boxed{a=x}$

put $a=x$ in 2

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$2x + b = y$

$\boxed{b = y - 2x}$

put values of a and b in ③

$3x + 2(y - 2x) + c = z$

$3x + 2y - 4x + c = z$

$-x + 2y + c = z$

$\boxed{c = x - 2y + z}$

So,

$(x, y, z) = x(1, 2, 3) + (y - 2x)(0, 1, 2) + (x - 2y + z)(0, 0, 1)$

Hence given vectors generate $\mathbb{R}^3$.

(ii) $\{(1, 1, 1), (0, 1, 1), (0, 1, -1)\}$

Sol: Given set is $\{(1, 1, 1), (0, 1, 1), (0, 1, -1)\}$

Let $(x, y, z) \in \mathbb{R}^3$ and suppose

$(x, y, z) = a(1, 1, 1) + b(0, 1, 1) + c(0, 1, -1)$

$= (a, a, a) + (0, b, b) + (0, c, -c)$

$(x, y, z) = (a, a + b + c, a + b - c)$

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$\Rightarrow \quad a = x \quad — \quad ①$

$a+b+c = y \quad — \quad ②$

$a+b-c = z \quad — \quad ③$

from ① $\boxed{a = x}$

Add ② and ③

$$\begin{aligned} a+b+\cancel{c} &= y \\ a+b\cancel{-c} &= z \\ \hline 2a+2b &= y+z \end{aligned}$$

$2x+2b = y+z$ using $a=x$

$2b = y+z-2x$

$$\boxed{b = \frac{y+z-2x}{2}}$$

put values of a and b in ②

$$x + \frac{y+z-2x}{2} + c = y$$

$$c = y - x - \frac{y+z-2x}{2}$$

$$= \frac{2y-\cancel{2x}-y-z+\cancel{2x}}{2}$$

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$$\boxed{c = \frac{y-z}{2}}$$

So,

$$(x,y,z) = x(1,1,1) + \left(\frac{y+z-2x}{2}\right)(0,1,1) + \left(\frac{y-z}{2}\right)(0,1,-1)$$

Hence given vector generate $\mathbb{R}^3$.

Linear Span:

Let S be a non-empty subset of vector space V, then the set of all linear combinations of finite no. of elements of S is called the linear span of S and is denoted by $\langle S \rangle$.

Example: Determine whether the set $S = \{(1,1,2),(1,0,1),(2,1,3)\}$ spans $\mathbb{R}^3$.

Sol:

Given set is $S = \{(1,1,2),(1,0,1),(2,1,3)\}$

Let $(x,y,z) \in \mathbb{R}^3$ and suppose

$$(x,y,z) = a(1,1,2) + b(1,0,1) + c(2,1,3)$$

$$= (a,a,2a) + (b,0,b) + (2c,c,3c)$$

$$(x,y,z) = (a+b+2c,\ a+c,\ 2a+b+3c)$$

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$\Rightarrow \quad a+b+2c = x \quad — (1)$

$a+c = y \quad — (2)$

$2a+b+3c = z \quad — (3)$

Subtract (3) from (1)

$$\begin{array}{r} a+b+2c = x \\ \pm 2a \pm b \pm 3c = \pm z \\ \hline -a-c = x-z \end{array}$$

or $\quad a+c = z-x \quad — (4)$

and $\quad a+c = y \quad — (2)$

The equation (2) and (4) cannot be solved for a and c. Hence we cannot find values of a, b and c. So the set $S = \{(1,1,2),(1,0,1),(2,1,3)\}$ does not span $\mathbb{R}^3$.

Example: Show that the yz-plane $W = \{(0,y,z) : y,z \in \mathbb{R}\}$ is spanned by

(i) $(0,1,1)$ and $(0,2,-1)$

Sol:

Let $W = \{(0,y,z) : y,z \in \mathbb{R}\}$

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Suppose $(0,y,z) \in W$ and

$$(0,y,z) = a(0,1,1) + b(0,2,-1)$$
$$= (0,a,a) + (0,2b,-b)$$
$$(0,y,z) = (0, a+2b, a-b)$$

$$\Rightarrow \quad a + 2b = y \quad — \ (1)$$
$$a - b = z \quad — \ (2)$$

Subtract (2) from (1)

$$\begin{array}{r} \cancel{a} + 2b = y \\ \pm \cancel{a} \mp b = z \\ \hline 3b = y - z \end{array}$$

$$\boxed{b = \frac{y-z}{3}}$$

put value of b in (2)

$$a - \frac{y-z}{3} = z \Rightarrow a = \frac{y-z}{3} + z$$

$$a = \frac{y - z + 3z}{3}$$

$$\boxed{a = \frac{y+2z}{3}}$$

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So

$$(0,y,z) = \left(\frac{y+2z}{3}\right)(0,1,1) + \left(\frac{y-z}{3}\right)(0,2,-1)$$

Hence yz-plane is spanned by $(0,1,1)$ and $(0,2,-1)$.

(ii) $(0,1,2),(0,2,3)$, and $(0,3,1)$

Sol:

Given vectors are $(0,1,2),(0,2,3)$ and $(0,3,1)$

Let $(0,y,z) \in W$ and suppose

$$(0,y,z) = a(0,1,2)+b(0,2,3)+c(0,3,1)$$
$$= (0,a,2a)+(0,2b,3b)+(0,3c,c)$$
$$(0,y,z) = (0,\ a+2b+3c,\ 2a+3b+c)$$

$\Rightarrow$ $a+2b+3c = y$ — ①

$2a+3b+c = z$ — ②

put $a=0$

So

$2b+3c = y$ — ③

$3b+c = z$ — ④

multiply ④ by 3 and subtract from ③

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$$2b + \cancel{3c} = y$$
$$\pm 9b \pm \cancel{3c} = \pm 3z$$
$$-7b = y - 3z$$

$$\boxed{b = \frac{3z - y}{7}}$$

put in 3

$$2\left(\frac{3z - y}{7}\right) + 3c = y$$

$$\frac{6z - 2y}{7} + 3c = y$$

$$3c = y - \frac{6z - 2y}{7}$$

$$= \frac{7y - 6z - 2y}{7}$$

$$3c = \frac{9y - 6z}{7}$$

$$\cancel{3}c = \frac{\cancel{3}(3y - 2z)}{7}$$

$$\boxed{c = \frac{3y - 2z}{7}}$$

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Hence,

$$(0,y,z) = 0(0,1,2) + \left(\frac{3z-y}{7}\right)(0,2,3) + \left(\frac{3y-3z}{7}\right)(0,3,1)$$

So yz-plane is spanned by $(0,1,2)$, $(0,2,3)$ and $(0,3,1)$.

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