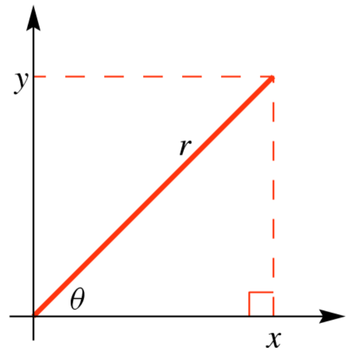
Digital Signal Processing
Computer Science Department
Spring 2024

- Mathematical representation of Periodic Signal
- Complex Numbers
- Euler's Relation

Digital Signal Processing

Sinusoids

SINE and COSINE functions



$$\sin\theta = \frac{y}{r}$$
$$\Longrightarrow \quad y = r\sin\theta$$

$$\cos\theta = \frac{x}{r}$$
$$\Longrightarrow \quad x = r\cos\theta$$

McClellan, Schafer and Yoder, *Signal Processing First,* ISBN 0-13-065562-7.
Pearson Prentice Hall, Inc. Upper Saddle River, NJ 07458. © 2003

SINES and COSINES

- Always use the COSINE FORM

$$\boxed{A\cos(2\pi(440)t + \varphi)}$$

- Sine is a special case:

$$\boxed{\sin(\omega t) = \cos(\omega t - \tfrac{\pi}{2})}$$

SINUSOIDAL SIGNAL

$$\boxed{A\cos(\omega t + \varphi)}$$

- **FREQUENCY** $\boxed{\omega}$
  - **Radians/sec**
  - or, Hertz (cycles/sec)

$$\boxed{\omega = (2\pi) f}$$

- **PERIOD** (in sec)

$$\boxed{T = \frac{1}{f} = \frac{2\pi}{\omega}}$$

- **AMPLITUDE** $\boxed{A}$
  - Magnitude
- **PHASE** $\boxed{\varphi}$

PHASE <--> TIME-SHIFT

- Equate the formulas:

$$\boxed{A\cos(\omega(t - t_m)) = A\cos(\omega t + \varphi)}$$

- and we obtain:

$$\boxed{-\omega t_m = \varphi}$$

- or,

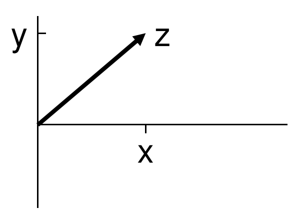
$$\boxed{t_m = -\frac{\varphi}{\omega}}$$

TIME-SHIFT

- Whenever a signal can be expressed in the form $x_1(t)=s(t-t_1)$, we say that $x_1(t)$ is time shifted version of $s(t)$
  - If t_1 is a **+** number, then the shift is to the right, and we say that the signal $s(t)$ has been *delayed* in time.
  - If t_1 is a **-** number, then the shift is to the left, and we say that the signal $s(t)$ was *advanced* in time.

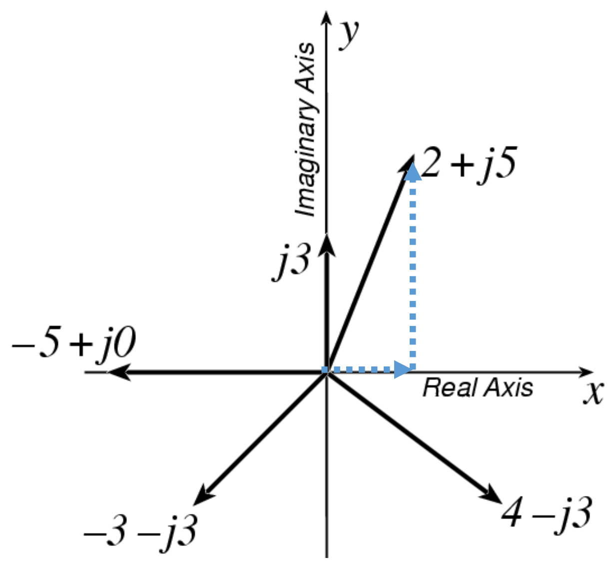
COMPLEX NUMBERS

- To solve: $z^2 = -1$
  - $z = \boldsymbol{j}$
  - Math and Physics use $z = \boldsymbol{i}$
- Complex number: $z = x + \boldsymbol{j}\, y$

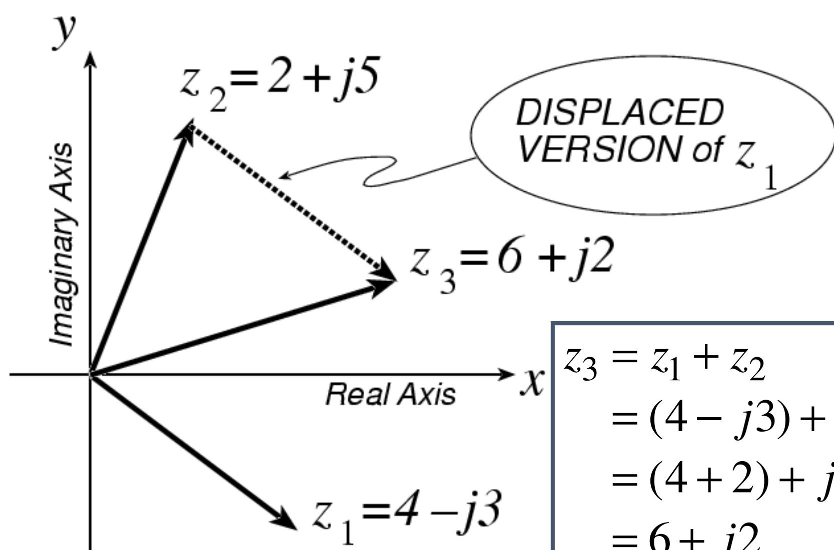


Cartesian coordinate system

PLOT COMPLEX NUMBERS (RECTANGULAR Form)



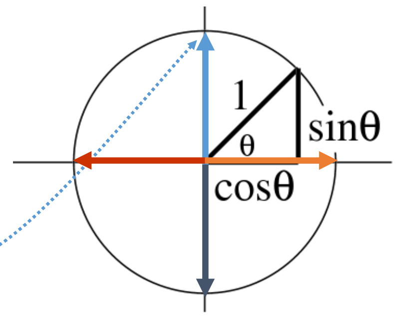
COMPLEX ADDITION = VECTOR Addition



$$
\begin{aligned}
z_3 &= z_1 + z_2 \\
&= (4 - j3) + (2 + j5) \\
&= (4 + 2) + j(-3 + 5) \\
&= 6 + j2
\end{aligned}
$$

*** POLAR FORM ***

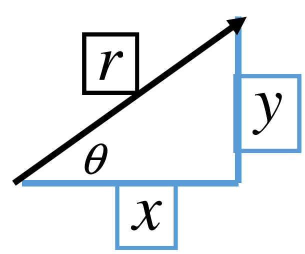
- Vector Form
  - **Length** =1
  - **Angle** = θ
- Common Values



1 has angle of 0
j has angle of 0.5π
–1 has angle of π
– j has angle of 1.5π
also, angle of **–j could** be –0.5π = 1.5π –2π
because the PHASE is **AMBIGUOUS**

POLAR <--> RECTANGULAR

- Relate (x,y) to (r,θ)



$$r^2 = x^2 + y^2$$
$$\theta = \mathrm{Tan}^{-1}\left(\frac{y}{x}\right)$$

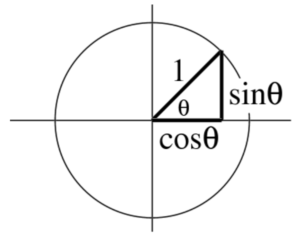
Most calculators do Polar-Rectangular

$$x = r\cos\theta$$
$$y = r\sin\theta$$

Need a notation for POLAR FORM

Euler's FORMULA

- Complex Exponential
  - Real part is cosine
  - Imaginary part is sine
  - Magnitude is one



$$\boxed{e^{j\theta} = \cos(\theta) + j\sin(\theta)}$$

$$\boxed{re^{j\theta} = r\cos(\theta) + jr\sin(\theta)}$$

$$\boxed{e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)}$$

INVERSE Euler's Formula

- Solve for cosine (or sine)

$$\cos(\omega t) = \tfrac{1}{2}(e^{j\omega t} + e^{-j\omega t})$$

$$\sin(\omega t) = \tfrac{1}{2j}(e^{j\omega t} - e^{-j\omega t})$$

Euler's Formula Reversed

- Solve for cosine (or sine)

$$e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)$$

$$e^{-j\omega t} = \cos(-\omega t) + j\sin(-\omega t)$$

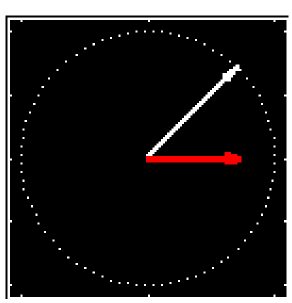
$$e^{-j\omega t} = \cos(\omega t) - j\sin(\omega t)$$

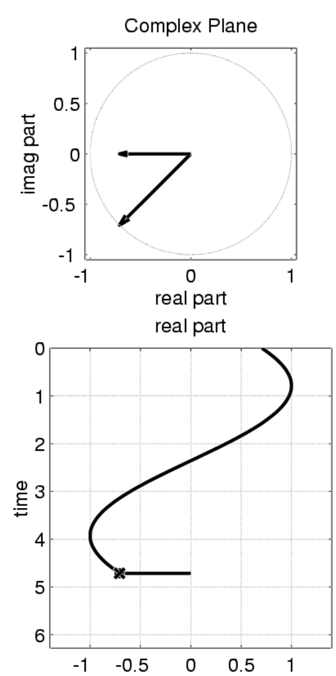
$$e^{j\omega t} + e^{-j\omega t} = 2\cos(\omega t)$$

$$\boxed{\cos(\omega t) = \tfrac{1}{2}(e^{j\omega t} + e^{-j\omega t})}$$

Rotating Phasor

See Demo on CD-ROM
Chapter 2





Phase change over time with complex numbers

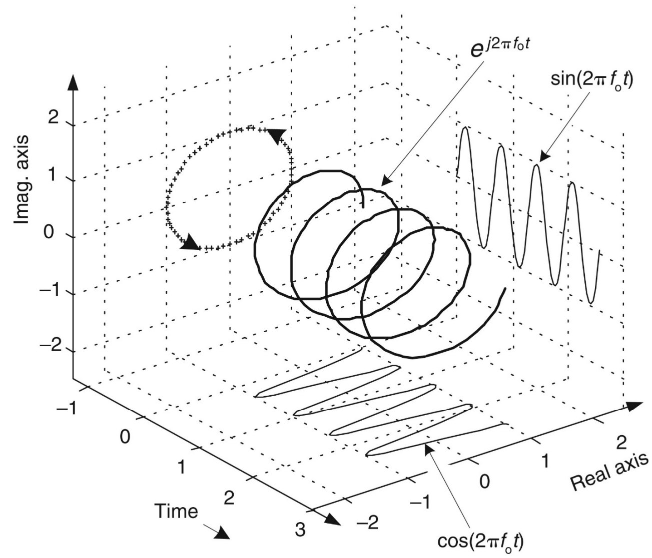


Figure 8-6 The motion of the $e^{j2\pi f_o t}$ complex signal as time increases.

cos = REAL PART

Real Part of Euler's

$$\boxed{\cos(\omega t) = \Re e\{ e^{j\omega t} \}}$$

General Sinusoid

$$\boxed{x(t) = A\cos(\omega t + \varphi)}$$

So,

$$\begin{aligned} A\cos(\omega t + \varphi) &= \Re e\{ A e^{j(\omega t + \varphi)} \} \\ &= \Re e\{ A e^{j\varphi} e^{j\omega t} \} \end{aligned}$$