History of Quadratic Convergence of the Jacobi Algorithm

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1 Overview

1.1 Classical Jacobi Algorithm

1958, Henrici [2, 1958] provided qualitative analysis for quadratic convergence of classical Jacobi algorithm applied on symmetric matrices with *simple eigenvalues*

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1961, Schönhage [3, 1961] gives a shape bound for the quadratic convergence of the classical Jacobi algorithm applied on symmetric matrices with eigenvalues that are *at most double*.

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1964, claimed by van Kempen, Schönhage [4, 1964] and he [5, 1966] proved the ultimate quadratic convergence of the classical Jacobi algorithm for matrices with *multiple eigenvalues* at the same time.

1.2 Cyclic Jacobi Algorithm

1958, Henrici [2, 1958] provided qualitative analysis for quadratic convergence of the cyclic Jacobi algorithm applied on symmetric matrices with *simple eigenvalues*. Notice that the choice of indices for this variant of Jacobi algorithm is not trivial. See [2, 1958, Eq. 1.26] for clarification.

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1961, Schönhage [3, 1961] gives a shape bound for the quadratic convergence of the cyclic Jacobi algorithm applied on symmetric matrices with eigenvalues that are *at most double*.

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1962, Wilkinson [7, 1962] provided sharper bounds for the "general" cyclic Jacobi algorithm and the special cyclic Jacobi algorithm applied on symmetric matrices with *distinct eigenvalues*.

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1966 van Kempen [6, 1966] proved the ultimate quadratic convergence of the cyclic Jacobi algorithm applied on symmetric matrices with *multiple eigenvalues*.

2 Result by Henrici, 1958

2.1 Classical Jacobi Algorithm

The first paper discuss the quadratic convergence of the Jacobi algorithm is by Henrici [2, 1958]. Let us define

$$\mathsf{off}(A) = \sqrt{\sum_{i\neq j}^{n} a_{ij}^2}, \qquad \mu_m = \max_{p\neq q} |\, a_{pq}^{(k)} \,|, \qquad d = \min_{i\neq j} |\, \lambda_i(A) - \lambda_j(A) \,|.$$

The classical Jacobi algorithm has been proven to converge linearly,

$$\mathsf{off}(A^{(k+1)}) \leq (1 - N^{-1})\, \mathsf{off}(A^{(k)}), \tag{2.1}$$

where

$$N = \frac{1}{2} n(n-1).$$

Theorem 2.1 (convergence of classical Jacobi algorithm)**.** *Let A have n distinct eigenvalues λ_i. Let the sequence of matrices $\{A^{(k)}\}$ be generated by the classical Jacobi algorithm. Then for any m such that*

$$4n\mu_m \leq d,$$

there exist an integer ν such that $0 \leq \nu \leq N$ and

$$\mu_{m+\nu} \leq n^2 \mathrm{e}^{0.854n^2} d^{-1} \mu_m^2. \tag{2.2}$$

This result provides an improvement over (2.1). By (2.2), we have

$$\mathsf{off}(A^{(m+N)}) \leq \frac{1}{4} n^6 \mathrm{e}^{1.708n^2} d^{-2}\, \mathsf{off}(A^{(m)})^2.$$

Thus, this conclude that for matrices with distinct eigenvalues, the classical Jacobi method converges *quadratically*.

2.2 Cyclic Jacobi Algorithm

Motivated by the considerations of computational saving, Gregory [1, 1953] proposed to select the pair π_k in some fixed cyclic order.

$$\pi_{k+1} = \begin{cases} (i_k+1, j_k) & (i_k < j_k - 1, j_k \leq n) \\ (1, j_k+1) & (i_k = j_k - 1, j_k < n) \\ (1,2) & (i_k = n-1, j_k = n) \end{cases}. \tag{2.3}$$

Henrici refered this method as the *special cyclic Jacobi method*. More generally, a *cyclic Jacobi method* is a method where in every segment of $N = n(n-1)/2$ consecutive elements of the sequence $\{\pi_k\}$ every pair (p,q) $(1 \leq p < q \leq n)$ occurs exactly once.

Theorem 2.2 (convergence of the cyclic Jacobi algorithm)**.** *Let A have n distinct eigenvalues. Let the sequence $\{A^{(k)}\}$ be generated by an arbitrary cyclic Jacobi method, where each rotation angle $\phi_k \in [-\pi/4, \pi/4]$. If for some m,*

$$4n\mu_m \leq d, \tag{2.4}$$

then

$$\mu_{m+N} \leq \sqrt{2}n(n-2)\mathrm{e}^{0.972n^4 d^{-1}\mu_m} d^{-1}\mu_m^2. \tag{2.5}$$

Theorem 2.2 shows that for matrices with n distinct eigenvalues, the special cyclic methods ultimately converge quadratically. However, the theorem does not establish convergence for arbitrary cyclic methods, since it is not asserted that condition (2.4) is ever satisfied.

3 Result by Schönhage, 1961

Henrici only proves that in the case of simple eigenvalues, quadratic convergence of Jacobi algorithm will occur some point onward. However, the bounds are so rough such that these can only be viewed as qualitative results. In [3, 1961], Schönhage reviewed the qudratic convergence, and show that the convergence still holds for matrix with repeated eigenvalues, provided the algebraic multiplicity for them are at most 2.

3.1 Classical Jacobi Algorithm

Let us first focus on matrix $A \in \mathbb{R}^{n\times n}$ whose eigenvalues are simple and satisfies

$$|\,\lambda_\mu - \lambda_\nu\,| \geq \delta > 0, \qquad (\nu \neq \mu). \tag{3.1}$$

Due to the linear convergence of the classical Jacobi algorithm, the condition $2\,\mathsf{off}(A^{(r)}) < \delta$ will holds after r_0, and the rotation angle φ will become very small. For simplicity, let us assume $r_0 = 0$, and we have

Theorem 3.1 (convergence of cyclic Jacobi algorithm)**.** *From $2\,\mathsf{off}(A^{(0)}) < \delta$ follows with $r_1 = n(n-1)/2$,*

$$\mathsf{off}(A^{(r_1)}) \leq \sqrt{\frac{n}{2}-1}\,\frac{\mathsf{off}(A^{(0)})^2}{\delta - 2\,\mathsf{off}(A^{(0)})}.$$

Theorem 3.2 (convergence for double eigenvalue)**.** *Let the eigenvalues of A be $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ such that $\lambda_{\nu+1} - \lambda_\nu \geq \delta > 0$ except for p pairs $\lambda_{\nu_1+1} - \lambda_{\nu_1} < \delta, \ldots, \lambda_{\nu_p+1} - \lambda_{\nu_p} < \delta$, where $|\,\nu_k - \nu_{k'}\,| \geq 2$ for $k \neq k'$, i.e. they are separated. Then with $2\,\mathsf{off}(A^{(0)}) < \delta$, for $r' \leq r_2 = n(n-1)/2 + p(n-2)$,*

$$\mathsf{off}(A^{(r')}) \leq \frac{n}{2}\cdot\frac{\mathsf{off}(A^{(0)})^2}{\delta - 2\,\mathsf{off}(A^{(0)})}.$$

3.2 Cyclic Jacobi Algorithm

Henrici [2, 1958] demonstrates the quadratic convergence for the cyclic Jacobi algorithm in the case of simple eigenvalues and his estimate is (2.5). Instead, Schönhage [3, 1961, Sec. 4], under his notation and analysis, gives

Theorem 3.3. *Let $2\,\mathsf{off}(A^{(0)}) < \delta \leq |\,\lambda_\nu - \lambda_\mu\,|$; with $r_1 = n(n-1)/2$, for any cyclic choice of i_r, k_r,*

$$\mathsf{off}(A^{(r_1)}) \leq \frac{n}{2}\sqrt{n-2} \cdot \frac{\mathsf{off}(A^{(0)})^2}{\delta - 2\,\mathsf{off}(A^{(0)})}.$$

Notice that, this convergence is still for specific cyclic Jacobi method discussed in [2, 1958]. For other general cases, there is no general convergence proof yet available in case we need $2\,\mathsf{off}(A^{(0)}) \leq \delta$). Also, we are not able to obtain a result analogous to Theorem 3.2 for matrices with double eigenvalues.

4 Result by Wilkinson, 1962

Wilkinson [7, 1962] provided sharper bounds for the general cyclic Jacobi algorithm and the special cyclic Jacobi algorithm applied on symmetric matrix with distinct eigenvalues.

4.1 General Cyclic Jacobi Algorithm

Let us use the same notation, $|\,\lambda_\nu - \lambda_\mu\,| \geq \delta$, and $N = n(n-1)/2$. Then, suppose we reach the point $\mathsf{off}(A^{(0)}) \leq \delta/4$, and notice the difference the assumption Schönhage made. Then, we have, in the same notation,

$$\mathsf{off}(A^{(N)}) \leq \sqrt{\frac{n(n-1)}{2}} \cdot \frac{\mathsf{off}(A^{(0)})^2}{\delta}.$$

4.2 Special Cyclic Jacobi Algorithm

For the special cyclic Jacobi algorithm with choices of indices discussed in (2.3). Under the same condition, we have

$$\mathsf{off}(A^{(N)}) \leq \frac{\mathsf{off}(A^{(0)})^2}{\delta}.$$

5 Result by Schönhage, 1964

Later in 1964, based on previous work, Schönhage [4, 1964] proved that for quadratic convergence of *classical Jacobi algorithm*, the condition that all eigenvalues are simple or at most double is unnecessary. However, the quadratic convergence for *cyclic Jacobi algorithm* applied on the matrix with multiple eigenvalue remains as an open problem.

6 Result by van Kempen, 1966

van Kempen published two papers regarding the quadratic convergence of the classical [5, 1966] and cyclic [6, 1966] Jacobi algorithm. As claimed by van Kempen,

> While this paper was in the press, Schönhage gave a different proof of the ultimate quadratic convergence of the classical Jacobi method.

6.1 Cyclic Jacobi Algorithm

The improvement by van Kempen is made by redefine δ as

$$\delta \geq \min_{\lambda_i \neq \lambda_j} |\, \lambda_i - \lambda_j \,|.$$

Suppose that after r rotations, the matrix $A^{(r)}$ satiesfies the condition,

$$\mathsf{off}(A^{(k)}) \leq \frac{\delta}{4}.$$

Then, the convergence result, in our notation, is

$$\mathsf{off}(A^{(r+N)}) \leq \sqrt{\frac{68}{9}} \cdot \frac{\mathsf{off}(A^{(r)})^2}{\delta}.$$

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