# Collaborative Optimization for Collective Decision-making in Continuous Spaces


Nikhil Garg
Stanford University
nkgarg@stanford.edu

Vijay Kamble
Stanford University
vjkamble@stanford.edu

Ashish Goel
Stanford University
ashishg@stanford.edu

David Marn
Univ. of California, Berkeley
marn@berkeley.edu

Kamesh Munagala
Duke University
kamesh@cs.duke.edu



## ABSTRACT

Many societal decision problems lie in high-dimensional continuous spaces not amenable to the voting techniques common for their discrete or single-dimensional counterparts. These problems are typically discretized before running an election or decided upon through negotiation by representatives. We propose a meta-algorithm called *Iterative Local Voting* for collective decision-making in this setting, in which voters are sequentially sampled and asked to modify a candidate solution within some local neighborhood of its current value, as defined by a ball in some chosen norm. In general, such schemes do not converge, or, when they do, the resulting solution does not have a natural description.

We first prove the convergence of this algorithm under appropriate choices of neighborhoods to plausible solutions in certain natural settings: when the voters' utilities can be expressed in terms of some form of distance from their ideal solution, and when these utilities are additively decomposable across dimensions. In many of these cases, we obtain convergence to the societal welfare maximizing solution.

We then describe an experiment in which we test our algorithm for the decision of the U.S. Federal Budget on Mechanical Turk with over 4,000 workers, employing neighborhoods defined by $\mathcal{L}^1, \mathcal{L}^2$ and $\mathcal{L}^\infty$ balls. We make several observations that inform future implementations of such a procedure.



## Keywords

Crowdsourced Democracy; Participatory Budgeting; Societal Decision-making; Crowdsourcing; Collective intelligence


## 1. INTRODUCTION

Methods and experiments to increase large-scale, direct citizen participation in policy-making have recently become commonplace as an attempt to revitalize democracy. Computational and crowdsourcing techniques involving human-algorithm interaction have been a key driver of this trend [3, 15, 14, 19]. Some of the most important collective decisions, whether in government or in business, lie in high-dimensional, continuous spaces – e.g. budgeting, taxation brackets and rates, collectively bargained wages and benefits, urban planning etc. Direct voting methods originally designed for categorical decisions are typically infeasible for collective decision-making in such spaces. Although there has been some theoretical progress on designing mechanisms for continuous decision-making [18, 4, 16], in practice these problems are either discretized before running an election, or are decided upon through negotiation by committee, such as in a standard representative democracy [3, 21, 22, 8, 23, 9].

We claim that the central challenge in designing good collective decision-making mechanisms in continuous space voting, and the reason for the current gap between theory and practice, is *behavioral.* It is unclear whether existing algorithms can simultaneously be simple enough to explain and use in practice, be robust to the inevitable deviations from ideal models of user behavior and preferences, and result in meaningful solutions with desirable properties. To address this challenge, a social planner must make practically reasonable assumptions on the nature and complexity of feedback that can be elicited from people and then design simple algorithms that operate effectively under these conditions.

We first tackle the question of what type of feedback voters can give. In general, for the types of problems we wish to solve, a voter cannot fully articulate her utility function. Even if voters had the patience to state preferences for a reasonably large number of points, there is no reason to believe that they could do so in any consistent manner.

On the other hand, we posit that it is relatively easy for people to choose their favorite amongst a reasonably small set of options, or articulate how they would like to locally modify a candidate solution to better match their preferences. Note that with such feedback and without any additional assumptions on voter preferences, no algorithm has any hope of finding a desirable solution that depends on the cardinal values of voters' utilities, e.g., the social welfare maximizing solution (i.e., one that maximizes the sum of agent utilities). An algorithm that uses only ordinal information about voter preferences is insensitive to any scaling or even monotonic transformations of those preferences.

In this paper, we study and experimentally test a type of meta-algorithm for large-scale preference aggregation that effectively leverages the possibility of asking voters such easy questions. In this algorithm that we call *Iterative Local Vot-*



*WWW 2017,* April 3–7, 2017, Perth, Australia.
ACM 978-1-4503-4913-0/17/04.
http://dx.doi.org/10.1145/3038912.3052690

**Algorithm 1:** Iterative Local Voting (ILV)

*Inputs:* Initial solution $x_0 \in \mathcal{X}$, tolerance $\epsilon > 0$, a function $N(t)$ that maps time $t$ to an integer in $[0, t-1]$, initial radius $r_0 > 0$, termination time $T$.

*Output:* Solution $x$.

- For $t \geq 1$, sample a voter $v_t \in \mathcal{V}$ at random from the population; set $r_t = r_0/t$ and elicit
$$x_t = [\arg\max_{x \in \{s: \|s - x_{t-1}\|_q \leq r_t\}} f_{v_t}(x)]_{\mathcal{X}}, \quad (1)$$
i.e. ask the voter to move to her favorite point within the constraints, and $[\cdot]_{\mathcal{X}}$ is a projection onto space $\mathcal{X}$.
- Stop when either $t = T$, in which case return $x_T$, or when $\max_{l,\, m \in \{N(t), \ldots, t\}} |x_l - x_m| \leq \epsilon$, in which case return $x = x_t$.

*ing* (ILV), voters are sequentially sampled and asked to modify a candidate solution to their favorite point within some local neighborhood, until a stable solution is obtained (if at all). One has flexibility in deciding how these local neighborhoods are defined – in this paper we focus on neighborhoods that are balls in the $\mathcal{L}^q$ norm, and in particular on the cases where $q = 1$, 2 or $\infty$[1].

More formally, consider a $M$-dimensional societal decision problem in $\mathcal{X} \subseteq \mathbb{R}^M$ and a population of voters $\mathcal{V}$, where each voter $v \in \mathcal{V}$ has bounded utility $f_v(x) \in \mathbb{R}, \forall\, x \in \mathcal{X}$. Then we consider the class of algorithms described in Algorithm 1.

ILV is directly inspired by the stochastic approximation approach to solve optimization problems [20], especially stochastic gradient descent (SGD) and stochastic subgradient descent (SSGD). The idea is that if (a) voter preferences are drawn from some probability distribution and (b) the response of a voter to the query (1) moves the solution approximately in the direction of her utility gradient, then this procedure *almost* implements stochastic gradient descent for minimizing negative expected utility.

The caveat is that although the procedure can potentially obtain the direction of the gradient of the voter utilities, it cannot obtain any information about its magnitude since the movement norm is chosen by the procedure itself. However, we show that for certain plausible utility and voter response models, the algorithm does indeed converge to a unique point with desirable properties. Our main theoretical contributions are as follows:

- **Convergence for $\mathcal{L}^p$ normed utilities:** We show that if the agents cost functions can be expressed as the $\mathcal{L}^p$ distance from their ideal solution, and if agents correctly respond to query (1), then an interesting duality emerges: for $p = 1$, 2 or $\infty$, using $\mathcal{L}^q$ neighborhoods, where $q = \infty$, 2 and 1 respectively, results in the algorithm converging to the unique social welfare optimizing solution. Whether such a result holds for general $(p, q)$, where $q$ is the dual norm to $p$ (i.e., 1/p + 1/q = 1), is an open question. However, we show that such a general result holds if, in response to query (1), the voter instead moves the current solution in the direction of the gradient of her utility function to the neighborhood boundary.
- **Convergence for other utilities:** Next, we show convergence to a unique solution in two cases: (a) when the voter cost can be expressed as a weighted sum of $\mathcal{L}^2$ distances over sub-spaces of the solution space, under $\mathcal{L}^2$ neighborhoods – in which case the solution is also Pareto efficient, and (b) when the voter utility can be additively decomposed across dimensions, under $\mathcal{L}^\infty$ neighborhoods – in which case the algorithm converges to the median of the ideal solutions of the voters on each dimension.

We then build a platform and run the first large-scale experiment in voting in multi-dimensional continuous spaces, in a budget allocation setting. We test three variants of ILV: with $\mathcal{L}^1$, $\mathcal{L}^2$ and $\mathcal{L}^\infty$ neighborhoods. Our main findings are as follows:

- We observe that the algorithm with $\mathcal{L}^\infty$ neighborhoods is the only alternative that satisfies the first-order concern for real-world deployability: consistent convergence to a unique stable solution. Both $\mathcal{L}^1$ and $\mathcal{L}^2$ neighborhoods result in convergence to multiple solutions.
- The consistent convergence under $\mathcal{L}^\infty$ neighborhoods in experiments strongly suggests the decomposability of voter utilities for the budgeting problem. Motivated by this observation, we propose a general class of decomposable utility functions to model user behavior for the budget allocation setting.
- We make several qualitative observations about user behavior and preferences. For instance, voters have large indifference regions in their utilities, with potentially larger regions in dimensions about which they care about less. Further, we show that asking voters for their ideal budget allocations and how much they care about a given item is fraught with UI biases and should be carefully designed.

We remark that an additional attractive feature of such a local update algorithm in a large population setting is that strategic behavior from the voters is less of a concern since the effect of a single voter's decision on the outcome is negligible.

The structure of the paper is as follows. After discussing related work in Section 2, we present convergence results for our algorithm under different settings in Section 3. In Section 4, we introduce the budget allocation problem and describe our experimental platform. In Section 5, we analyze the experiment results, and then we conclude the paper in Section 6. The proofs of our results have been omitted due to space constraints and are available in an online appendix.

## 2. RELATED WORK

**Stochastic Gradient Descent:** As discussed in the introduction, we draw motivation from SGD and SSGD, and our main proof technique is mapping our algorithm to SSGD. Beginning with the original stochastic approximation algorithm by Robbins and Monro [20], a rich literature surrounds SSGD, for instance see [2, 17, 12].

**Iterative local voting:** A version of our algorithm, with $\mathcal{L}^2$ norm neighborhoods, has been proposed several times [10, 5, 1]. Instead of query 1, the movement $\frac{\nabla f_v(x_{t-1})}{\|\nabla f_v(x_{t-1})\|_2}$ is elicited to try to compute the fixed point $\mathbb{E}\left[\frac{\nabla f_v(x)}{\|\nabla f_v(x)\|_2}\right] =$

[1]The $\mathcal{L}^q$ norm $\|x\|_q \triangleq \sqrt[q]{\sum_i |x_i|^q}$. $q = 1$, 2 and $\infty$ neighborhoods correspond to bounds on the sum of absolute values of the changes, the sum of the square of the changes, and the maximum change, respectively.

| Utility Function (p) | Neighborhood (q) |
|---|---|
| 2 | 2 |
| 1 | $\infty$ |
| $\infty$ | 1 |

Table 1: Summary of results of Theorem 3.1. If the voter utilities are $\mathcal{L}^p$, then restricting local movements to a $\mathcal{L}^q$ norm ball leads to convergence to the societal optimum in the above cases.

0. Similarly, a single step of our algorithm with $\mathcal{L}^2$ neighborhoods is similar to quadratic voting [13, 24]. For further information on the relation between our work and these concepts, see Remark 3.1. To the best of our knowledge, no work studies such an algorithm with other neighborhoods and under ordinal feedback, or implements such an algorithm.

**Optimization without gradients:** Because we are concerned with optimization without access to voters' utility functions or its gradients, this work seems to be in the same vein as recent literature on convex optimization without gradients – such as with comparisons or with pairs of function evaluations [7, 11, 6]. However, in the social choice or human optimization setting, we cannot estimate each voter's gradients exactly rather than up to a scaling term, and so cannot directly utilize strategies from such work.

# 3. CONVERGENCE ANALYSIS

In this section, we discuss the convergence properties of ILV under various utility and behavior models. For the rest of the technical analysis, we make the following assumptions on our model.

1. The solution space $\mathcal{X} \subseteq \mathbb{R}^M$ is non-empty, bounded, closed, and convex.
2. Each voter $v$ has a unique ideal solution $x_v \in \mathcal{X}$.
3. The ideal point $x_v$ of each voter is drawn independently from a probability distribution with a bounded density function $h_{\mathcal{X}}$.

Furthermore, "convergence" of ILV refers to the convergence of the sequence of random variables $\{x_t\}_{t\geq 1}$ to some $x \in \mathcal{X}$ with probability 1, assuming that the algorithm is allowed to run indefinitely (this implies the termination of the algorithm with probability 1 for any $N(t) \leq t-1$ and $\epsilon > 0$ as $t \to \infty$). Under this model, for a solution $x \in \mathcal{X}$, the societal utility is given by $\mathrm{E}_v[f_v(x)]$. and the societal welfare optimizing solution is any $x^* \in \arg\max_{x\in\mathcal{X}} \mathrm{E}_v[f_v(x)]$.

## 3.1 Spatial Utilities

Here we consider *spatial* utility functions, where the utilities of each voters can be expressed in the form of some kind of spatial distance from their ideal solutions. First, we consider the following kind of utilities.

**Definition 3.1. $\mathcal{L}^p$ normed utilities:** *The voter utility function is $\mathcal{L}^p$ normed if $f_v(x) = -\|x - x_v\|_p, \forall x \in \mathcal{X}$.*

Under such utilities, for $p = 1$, 2 and $\infty$, restricting voters to a ball in the dual norm leads to convergence to the societal optimum. Our main result is the following theorem, summarized in Table 1:

**Theorem 3.1.** *Suppose that the voter utilities are $\mathcal{L}^p$ normed, and voters correctly respond to query* (1)*. Then, ILV with $\mathcal{L}^q$ neighborhoods converges to the societal optimal point w.p. 1 when $(p, q) = (2, 2)$, $(1, \infty)$, or $(\infty, 1)$.*

A sketch of the proof is as follows. For the given pairs $(p, q)$, we show that, except in certain 'bad' regions, the update rule $x_{t+1} = \arg\min_x[\|x - x_{v_t}\|_p : \|x - x_t\|_q \leq r_t]$ is equivalent to the stochastic subgradient descent (SSGD) update rule $x_{t+1} = x_t - r_t g_t$, for some $g_t \in \partial\mathrm{E}_v[\|x - x_{v_t}\|_p]$, and that the probability of being in a 'bad' region decreases fast enough as a function of $r_t$. We then leverage a standard SSGD convergence result to finish the proof. One natural question is whether the result extends holds for general dual norms $p, q$, where $1/p + 1/q = 1$. Unfortunately, the update rule is not equivalent to SSGD in general, and we leave the convergence to the societal optimum for general $(p, q)$ as an open question.

However, the result does hold for general dual norms $(p,\ q)$ if one assumes an alternative behavior model.

**Theorem 3.2.** *Suppose that the voter utilities are $\mathcal{L}^p$ normed, and in response to query* (1)*, each voter moves in the direction of the gradient of her utility function (i.e. in the direction of greatest increase of utility) to the boundary of the given neighborhood. Then, ILV with $\mathcal{L}^q$ neighborhoods converges to the societal optimal point w.p. 1 for any $p > 0$ and $q > 0$ such that $1/p + 1/q = 1$.*

This behavior model is reasonable when a voter vaguely knows how she would like to change a decision but does not know her exact ideal value within the neighborhood. The proof uses the following property of $\mathcal{L}^p$ normed utilities: the $\mathcal{L}^q$ norm of the gradient of these utilities at any point other than the ideal point is constant. This fact, along with the voter behavior model, allows the algorithm to implicitly capture the magnitude of the gradient of the utilities, and thus a direct mapping to SSGD is obtained. Note that the above result holds even if we assume that a voter moves to her ideal point $x_v$ in case it falls within the neighborhood (since, as explained earlier, the probability of sampling such a voter decreases fast enough).

Next, we introduce another general class of utility functions, which we call *Weighted Euclidean utilities*, for which one can obtain convergence to a unique solution.

**Definition 3.2. Weighted Euclidean utilities:** *Let the solution space $\mathcal{X}$ be decomposable into $K$ different sub-spaces, so that $x = (x^1, \ldots, x^K)$ for each $x \in \mathcal{X}$ (where $\sum_{i=1}^K \dim(i) = M$). Suppose that the utility function of the voter $v$ is*

$$f_v(x) = -\sum_{k=1}^{K} w_v^k \|x^k - x_v^k\|_2.$$

*where $w_v$ is a voter-specific weight vector, then the function is a Weighted Euclidean utility function. We further assume that $w_v \in \mathcal{W} \subset \mathbb{R}_+^K$ and $x_v$ are independently drawn for each voter $v$ from a joint probability distribution with a bounded and measurable density function, with $\mathcal{W}$ nonempty, bounded, closed, and convex.*

This utility function can be interpreted as follows: the decision-making problem is decomposable into $K$ sub-problems, and each voter $v$ has an ideal point $x_v^k$ and a weight $w_v^k$ for each sub-problem $k$, so that the voter's disutility for a solution is the weighted sum of the Euclidean distances to the ideal points in each sub-problems. In this case, we show the following:

**Theorem 3.3.** *Suppose that the voter utilities are Weighted Euclidean, and voters correctly respond to query* (1). *Then, ILV with $\mathcal{L}^2$ neighborhoods converges with probability* 1 *to the unique societal optimal point in the world where the voter utilities are $f_v(x) = -\sum_{k=1}^{K} w_v^k \|x^k - x_v^k\|_2/\|w_v\|_2$.*

This means that the algorithm converges to the societal optimal solution in the world where the voter utilities are scaled down by $\|w_v\|_2$. Note that this implies that the resulting solution is Pareto efficient. The intuition for the result is as follows: as long as the neighborhood does not contain the ideal point of the sampled voter, the correct response to query (1) under weighted Euclidean preferences is to move the solution in the direction of the ideal point to the neighborhood boundary, which, as it turns out, is the same as the direction of the gradient. Thus with radius $r_t$, the effective movement is $\frac{\nabla f_v(x_t)}{||\nabla f_v(x_t)||_2}$. With weighted Euclidean utilities, $\|\nabla f_v(x_t)\|_2 = \|w_v\|_2$ everywhere. Hence, it is as if the algorithm is obtaining the gradient of the function $f_v/\|w_v\|_2$.

**Remark 3.1.** *We conjecture that under the choice of $r_t = 1/t$, under $\mathcal{L}^2$ neighborhoods, and under general concave utilities $f_v$, if the realized sequence of solutions $\{x_t\}$ converges to some $x$, then it must satisfy*[2] $\mathrm{E}_v\left[\frac{\nabla f_v(x)}{||\nabla f_v(x)||_2}\right] = 0$. *Such a point has been called Directional Equilibrium (DE) in recent literature [5]. Several properties of this solution have been studied (e.g. it is Pareto efficient), starting from [10] to more recently, [5] and [1]. Computing such a point has been challenging: these works each have proposed an iterative algorithm similar to ours to compute this point, but except for special cases (see [1]), convergence is an open question.*

## 3.2 Decomposable utilities

Next consider the general class of decomposable utilities, motivated by the fact that the algorithm with $\mathcal{L}^\infty$ neighborhoods is of special interest since they are easy for humans to understand: one can change each dimension up to a certain amount, independent of the others.

**Definition 3.3. Decomposable utilities:** *A voter utility function is decomposable if there exists concave functions $f_i$ for $i \in \{1 \dots M\}$ such that $f_v(x) = \sum_{i=1}^{M} f_v^i(x^i)$.*

If the utility functions for the voters are decomposable, then we can show that our algorithm under $\mathcal{L}^\infty$ neighborhoods converges to the vector of medians of voters' ideal points on each dimension. Suppose that $h_{\mathcal{X}}^i$ is the marginal density function of the random variable $x_v^i$, and let $\bar{x}^i$ be the unique median of $x_v^i$.

**Theorem 3.4.** *Suppose that the voter utilities are decomposable, and voters correctly respond to query* (1). *Then, ILV with $\mathcal{L}^\infty$ neighborhoods converges with probability* 1 *to the vector of medians $\bar{x}$.*

[2]Note that as $r_t \to 0$, $f_v(y)$ can be linearly approximated by the first term of the Taylor series expansion around $x$, for $y \in \{s : ||s - x||_2 \le r_t\}$. Then, to maximize $f_v(y)$ in the region, if the region does not contain $x_v$ voter $v$ chooses $y^*$ *s.t.* $y^* - x \approx r_t \frac{\nabla f(x)}{||\nabla f(x)||_2}$, i.e., the voter moves the solution approximately in the direction of her gradient to the neighborhood boundary. A similar observation is central to the idea of Quadratic Voting in discrete solution spaces [13, 24]

Although simply eliciting each agent's optimal solution and computing the vector of median allocations on each dimension is a viable approach in the case of decomposable utilities, deciding an optimal allocation across multiple dimensions is a more challenging cognitive task than deciding whether one wants to increase or decrease each dimension relative to the current solution (see Section 5.2.3 for experimental evidence). In fact, in this case, the algorithm can be run separately for each dimension, so that each voter expresses her preferences on only one dimension, drastically reducing the cognitive burden of decision-making on the voter, especially in high dimensional settings like budgeting.

# 4. EXPERIMENTS WITH BUDGETS

We built a voting platform and ran a large scale experiment, along with several extensive pilots, on Amazon Mechanical Turk[3] (MTurk), with over 4,000 workers participating in total. The design challenges we faced and voter feedback we received provide important lessons for deploying such systems in a real-world setting.

First we present a theoretical model for our setting. We consider a budget allocation problem on $M$ items. One possibility is to define $\mathcal{X}$ as the space of feasible allocations, such as those below a spending limit, and to run the algorithm as defined, with projections. However, in such cases, it may be difficult to theorize about how voters behave; e.g. if voters knew their answers would be projected onto a budget balanced set, they may respond differently. Rather, we consider an unconstrained budget allocation problem, one in which a voter's utility includes a term for the budget deficit. Let $\mathcal{E} \subseteq \{1 \dots M\}$, $\mathcal{I} = \{1 \dots M\} \backslash \mathcal{E}$ be the expenditure and income items, respectively. Then the general *budget utility function* is $f_v(x) = g_v(x) - d(\sum_{e \in \mathcal{E}} x^e - \sum_{i \in \mathcal{I}} x^i)$, where $d$ is an increasing function on the deficit. In general, nothing is known about convergence of Algorithm 1 with such utilities, as the deficit term may add complex dependencies between the dimensions. However, if the voter utility functions are decomposable across the dimensions and $\mathcal{L}^\infty$ neighborhoods used, then the results of Section 3.2 can be applied. We propose the following class of decomposable utility functions for the budgeting problem, achieved by assuming that the cost for the deficit is linear, and call the class "decomposable with a linear cost for deficit," or DLCD.

**Definition 4.1.** *Let $f_v(x)$ be DLCD if*

$$f_v(x) = \sum_{m=1}^{M} f_v^m(x^m) - w_v \left( \sum_{e \in \mathcal{E}} x^e - \sum_{i \in \mathcal{I}} x^i \right),$$

*where $f_v^m$ is a concave function for each $m$ and $w_v \in \mathbb{R}_+$.*

In the experiments discussed below in the budget setting, ILV consistently and robustly converges with $\mathcal{L}^\infty$ norm neighborhoods. Further, it approximately converges to the medians of the optimal solutions (which are elicited independently), as theorized in Section 3.2. Such a convergence pattern suggests the validity of the DLCD model, though we do not formally analyze this claim.

## 4.1 Experimental Setup

Though we make no normative claims about running a vote in this setting in reality, we asked voters to vote on the

[3]`https://www.mturk.com`; Turkprime (`https://www.turkprime.com`) was used to manage postings and payment.

U.S. Federal Budget across several of its major categories: National Defense; Healthcare; Transportation, Science, & Education; and Individual Income Tax[4]. This setting was deemed the most likely to be meaningful to the largest cross-section of workers and to yield a diversity of opinion, and we consider budgets a prime application area in general. The specific categories were chosen because they make up a substantial portion of the budget and are among the most-discussed items in American politics.

One major concern was that with no way to validate that a worker actually performed the task (since no or little movement is a valid response if the solution presented to the worker was near her ideal budget), we may not receive high-quality responses. This issue is especially important in our setting because a worker's actions influence the initial solution future workers see. We thus restricted the experiment to workers with a high approval rate and who have completed over 500 experiments. Further, we offered a bonus to workers for justifying their movements well, and more than 80% of workers qualified, suggesting that we also received high-quality movements. The experiment was restricted to Americans to best ensure familiarity with the setting.

## 4.2 Experimental Parameters

Our large scale experiment included 2,000 workers[5] and ran over a week in real-time. We tested the $\mathcal{L}^1, \mathcal{L}^2$, and $\mathcal{L}^\infty$ mechanisms, along with a "full elicitation" mechanism in which workers reported their ideal values for each item, and a "weight" in $[0, 10]$ indicating how much they cared about the actual spending in that item being close to their stated value. To test repeatability of convergence, each of the constrained mechanisms had three copies, given to three separate groups of people. Each group consisted of two sets with different starting points, with each worker being asked to vote in each set in her assigned group. We used a total of three different sets of starting points across the three groups, such that each group shared one set of starting points with each of the other two groups. This setup allowed testing for repeatability across different starting points and observing each worker's behavior at two points. Workers in one group in each constrained mechanism type were also asked to do the full elicitation after submitting their movements for the constrained mechanism, and such workers were paid extra. These copies, along with the full elicitation, resulted in 10 different mechanism instances to which workers could be allocated, each completed by about 200 workers.

To update the current point, we waited for 10 submissions and then updated the point to their average. This averaging explains the step-like structure in the convergence plots in the next section. The radius was decreased approximately every 60 submissions, i.e. $r_t \approxeq \frac{r_0}{\lceil t/60 \rceil}$. The averaging and slow radius decay rate were implemented in response to observing in the pilots that the initial few voters with a large radius had a disproportionately high impact, as there were not enough subsequent voters to recover from large initial movements away from an eventual fixed point (though in theory this would not be a problem given enough voters).

[4]The US Federal Government cannot just decide to set tax receipts to some value. We asked workers to assume tax rates would be increased/decreased at proportional rates in hopes of increasing/decreasing receipts.

[5]Participants of any of the pilots were excluded.

## 4.3 User Experience

As workers arrived, they were randomly assigned to a mechanism instance. They had a roughly equal probability of being assigned to each instance, with slight deviations in case an instance was "busy" (another user was currently doing the potential $10^{\text{th}}$ submission before an update of the instance's current point) and to keep the number of workers in each instance balanced. Upon starting, workers were shown mechanism instructions. We showed the instructions on a separate page so as to be able to separately measure the time it takes to read & understand a given mechanism, and the time it takes to do it, but we repeated the instructions on the actual mechanism page as well for reference.

On the mechanism page, workers were shown the current allocation for each of the two sets in their group. They could then move, through sliders, to their favorite allocation under the movement constraint. We explained the movement constraints in text and also automatically calculated for them the number of "credits" their current movements were using, and how many they had left. Next to each budget item, we displayed the percentage difference of the current value from the 2016 baseline federal budget[6], providing important context to workers. We also provided short descriptions of what goes into each budget item as scroll-over text. The resulting budget deficit and its percent change were displayed above the sliders[7]. For the full elicitation mechanism, workers were asked to move the sliders to their favorite points with no constraints (the sliders went from $0 to twice the 2016 value in that category), and then were asked for their "weights" on each budget item, including the deficit. Figure 1 shows part of the interface for the $\mathcal{L}^2$ mechanism, not including instructions, with similar interfaces for the other constrained mechanisms. The full elicitation mechanism additionally included sliders for items' weights. On the final page, workers were asked for feedback on the experiment[8].

# 5. RESULTS AND ANALYSIS

We now discuss the results of our experiments.

## 5.1 Convergence

One basic test of a voting mechanism is that whether it produces a consistent and unique solution, given a voting population and their behaviors. If an election process can produce multiple, distinct solutions purely by chance, opponents can assail any particular solution as a fluke and call for a re-vote. The question of whether the mechanisms consistently converge to the same point thus must be answered before analyzing properties of the equilibrium point itself. In this section, we show that the $\mathcal{L}^2$ and $\mathcal{L}^1$ algorithms do

[6]The 2016 budget estimate was obtained from `http://federal-budget.insidegov.com/l/119/2016-Estimate` and `http://atlas.newamerica.org/education-federal-budget`

[7]Assuming other budget items are held constant.

[8]We plan on posting the data, including feedback. In general, workers seemed to like the experiment, though some complained about the constraints, and others were generally confused. Some expressed excitement about being asked their views in an innovative manner and suggested that everyone could benefit from participating as, at the least, a thought exercise. The feedback and explanations provided by workers were much longer than we anticipated, and they convince us of the procedure's civic engagement benefits.

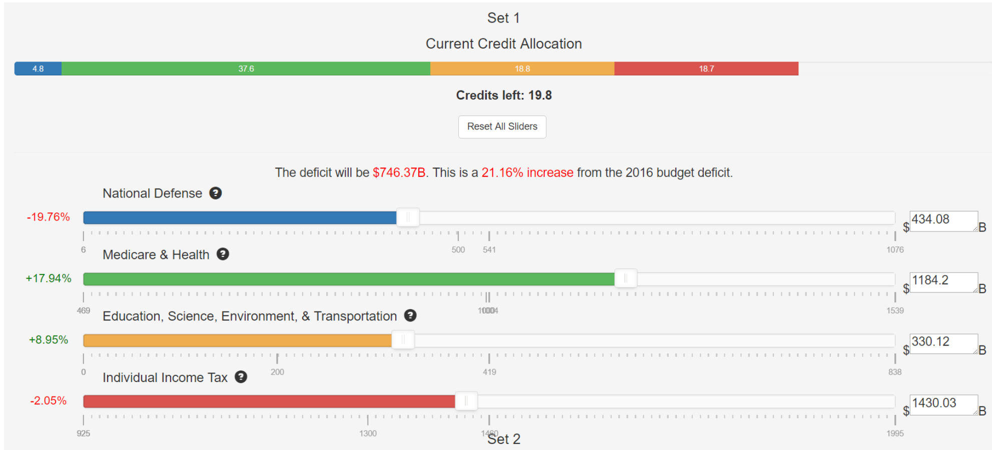



Figure 1: UI Screenshot for 1 set of the $\mathcal{L}^2$ Mechanism

not appear to converge to a unique point, while the $\mathcal{L}^\infty$ mechanism converges to a unique point across several initial points and with distinct worker populations.

The solutions after each voter for each set of starting points, across the 3 separate groups of people for each constrained mechanism are shown in Figure 2. Each plot shows all the trajectories with the given mechanism type, along with the median of the ideal points elicited from the separate voters who only performed the full elicitation mechanism. Observe that the three mechanisms have remarkably different convergence patterns. In the $\mathcal{L}^1$ mechanism, not even the sets done by the same group of voters (in the same order) converged in all cases. In some cases, they converged for some budget items but then diverged again. In the $\mathcal{L}^2$ mechanism, sets done by the same voters starting from separate starting points appear to converge, but the three groups of voters seem to have settled at two separate equilibria in each dimension. Under the $\mathcal{L}^\infty$ neighborhood, on the other hand, all six trajectories, performed by three groups of people, converged to the same allocation very quickly and remained together throughout the course of the experiment. Furthermore, the final points, in all dimensions except Healthcare, correspond almost exactly to the median of values elicited from the separate set of voters who did only the full elicitation mechanism. For Healthcare, the discrepancy could result from biases in full elicitation (see Section 5.2.4), though we make no definitive claims. These patterns shed initial insight on how the use of $\mathcal{L}^2$ constraints may differ from theory in prior literature and offer justification for the use of DLCD utility models and the $\mathcal{L}^\infty$ constrained mechanism.

One natural question is whether these mechanisms really have converged, or whether if we let the experiment continue, the results would change. This question is especially salient for the $\mathcal{L}^2$ trajectories, where trajectories within a group of people converged to the same point, but trajectories between groups did not. Such a pattern could suggest that our results are a consequence of the radius decreasing too quickly over time, or that the groups had, by chance, different distributions of voters which would have been corrected with more voters. However, we argue that such does not seem to be the case, and that the mechanism truly found different equilibria. We can test whether the final points for each trajectory are stable by checking the net movement in a window, normalized by each voter's radius, i.e. $\frac{1}{N}\sum_{s=t-N}^{t}\frac{x_s - x_{s-1}}{r_s}$, for some $N$. If voters in a window are canceling each other's movements, then this value goes to 0, and the algorithm would be stable even if the radius does not decrease. The notion is thus robust to apparent convergence just due to decreasing radii. The net movement normalized in a sliding window of 30 voters, for each dimension and mechanism, is shown in Figure 3. It seems to die down for almost all mechanisms and budget items, except for a few cases which do not change the result. We conclude it likely that the mechanisms have settled into equilibria which are unlikely to change given more voters.

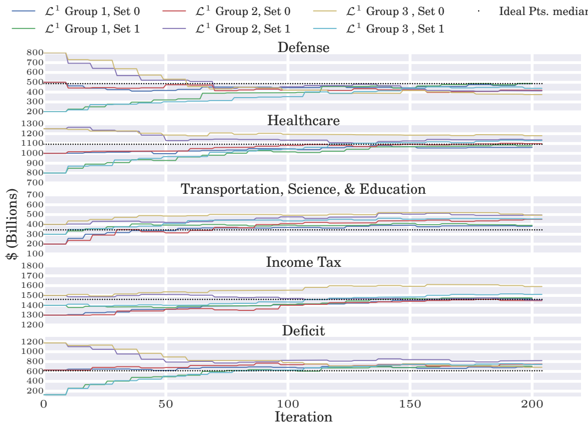
## 5.2 Understanding Voter Behavior

A mechanism's practical impact depends on more than whether it consistently converges, however. We now turn our attention to understanding how voters behave under each mechanism and whether we can learn anything about their utility functions from that behavior. We find that voters understood the mechanisms but that their behaviors suggest large indifference regions, and that the full elicitation scheme is susceptible to biases that can skew the results.

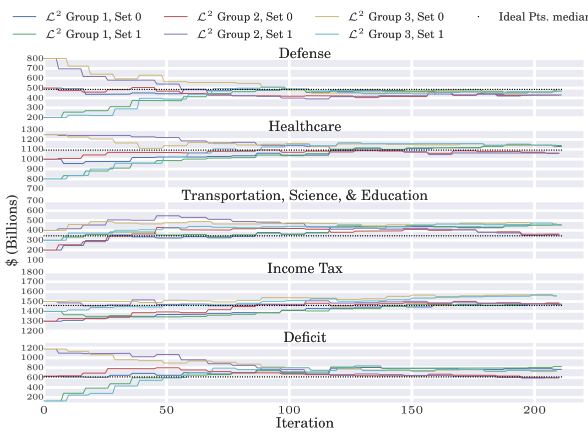
### 5.2.1 *Voter Understanding of Mechanisms*

One important question is whether, given very little instruction on how to behave, voters understand the mechanisms and act approximately optimally under their (unknown to us) utility function. This section shows that the voters behaved as expected for each mechanism.

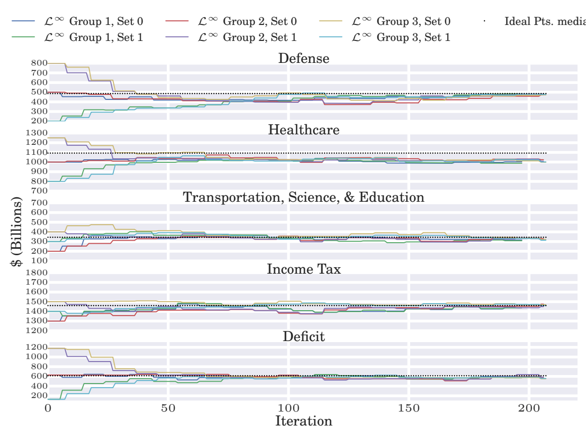
Regardless of the exact form of the utility function, one would expect that, in the $\mathcal{L}^1$ constrained mechanism, a voter would use most of her movement credits in the dimension about which she cares most. In fact, in either the Weighted Euclidean preferences case (and with 'sub-space' being a single dimension) or with a small radius, a voter would move only on one dimension. With $\mathcal{L}^2$ constraints, one would expect a voter to apportion her movement more equally because she pays an increasing marginal cost to move more in one dimension (people were explicitly informed of this consequence in the instructions). Under the Weighted Euclidean preferences model, a voter would move in each dimension proportional to her weight in that dimension. Finally, with $\mathcal{L}^\infty$ constraints, a voter would move, in all dimensions in




(a) $\mathcal{L}^1$




(b) $\mathcal{L}^2$




(c) $\mathcal{L}^\infty$

Figure 2: Solution over time for each mechanism type

which she is not indifferent, to her favorite point in the neighborhood for that dimension (most likely an endpoint), independently of other dimensions. One would thus expect a more equal distribution of movements.


(a) $\mathcal{L}^1$


(b) $\mathcal{L}^2$


(c) $\mathcal{L}^\infty$

Figure 3: Net normalized movement in window of $N = 30$

Figure 4 shows the average movement (as a fraction of the voter's total movement) by each voter for the dimension she moved most, second, third, and fourth, respectively, for each constrained mechanism. We reserve discussion of the full elicitation weights for Section 5.2.4. The movement patterns indicate that voters understood the constraints and moved accordingly – with more equal movements across dimensions in $\mathcal{L}^2$ than in $\mathcal{L}^1$, and more equal movements still in $\mathcal{L}^\infty$. We dig deeper into user utility functions next, but can conclude that, regardless of their exact utility functions, voters responded to the constraint sets appropriately.


Figure 4: Average movement in dimension over total movement for each voter, with dimensions sorted

### 5.2.2 Large Indifference Regions

Although it is difficult to extract a voter's full utility function from their movements, the separability of dimensions (except through the deficit term) under the $\mathcal{L}^\infty$ constraint allows us to test whether voters behave according to some given utility model in that dimension, without worrying about the dependency on other dimensions.

Figure 5 shows, for the $\mathcal{L}^\infty$ mechanism, a histogram of the movement on a dimension as a fraction of the radius (we find no difference between dimensions here). Note that a large percentage of voters moved very little on a dimension, even in cases where their ideal point in that dimension was far away (defined as being unreachable under the current radius). This result cannot be explained away by workers clicking through without performing the task: almost all workers moved at least one dimension, and, given that a worker moved in a given dimension, it would not explain smaller movements being more common than larger movements. That this pattern occurs in the $\mathcal{L}^\infty$ mechanism is key – if a voter feels *any* marginal disutility in a dimension, she can move the allocation without paying a cost of more limited movement in other dimensions. We conclude that, though voters may share a single ideal point for a dimension when asked for it, they are in fact relatively indifferent over a potentially large region – and their actions reflect so.

Furthermore, this lack of movement is correlated with a voter's weights when she was also asked to do the full elicitation mechanism. Conditioned on being far from her ideal point, when a voter ranked an item as one of her top two important items (not counting the deficit term), she moved an average of 74% of her allowed movement in that dimension; when she ranked an item as one of least two important items, she moved an average of 61%, and the difference is significant through a two sample t-test with $p = .013$. We find no significant difference in movement within the top two ranked items or within the bottom two ranked items. This connection suggests that one can determine which dimensions a voter cares about by observing these indifference regions and movements, even in the $\mathcal{L}^\infty$ constrained case.

Furthermore, we note that while such indifference regions conflict with the utility models under which the $\mathcal{L}^2$ constraint mechanism converges in theory, it fits within the DLCD framework introduced in Section 4.

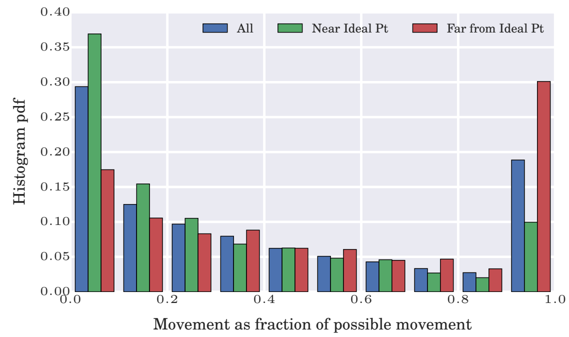



Figure 5: Fraction of possible movement in each dimension in $\mathcal{L}^\infty$, conditioned on distance to ideal pt. The 'All' condition contains data from all three $\mathcal{L}^\infty$ instances, whereas the others only from the instance that also did full elicitation.

### 5.2.3 Mechanism Time

In this section, we note one potential problem with schemes that explicitly elicit voter's optimal solutions – for instance, to find the component-wise median – as compared to the constrained elicitation used in ILV: it seems to be cognitively difficult for voters. In Figure 6, the median time per page, aggregated across each mechanism type, is shown. The "Mechanism" time includes a single user completing both sets in each of the constrained mechanism types, but not does include the time to also do the extra full elicitation task in cases where a voter was asked to do both a constrained mechanism and the full elicitation. The full elicitation bars include only voters who did only the full elicitation mechanism, and so the bars are completely independent. On average, it took longer to do the full elicitation mechanism than it took to do *two* sets of any of the constrained mechanisms, suggesting some level of cognitive difficulty in articulating one's ideal points and weights on each dimension – even though understanding what the instructions are asking was simple, as demonstrated by the shorter instruction reading time for the full elicitation mechanism. The $\mathcal{L}^\infty$ mechanism took the least time to both understand and do, while the $\mathcal{L}^2$ mechanism took the longest to do, among the constrained mechanisms. This result is intuitive: it is easier to move each budget item independently when the maximum movement is bounded than it is to move the items when the sum or the sum of the changes squared is bounded (even when these values are calculated for the voter). In practice, with potentially tens of items on which constituents are voting, these relative time differences would grow even larger, potentially rendering full elicitation or $\mathcal{L}^2$ constraints unpalatable to voters.

### 5.2.4 UI Biases

We now turn our attention to the question of how workers behaved under the full elicitation mechanism and highlight some potential problems that may affect results in real deployments. Figures 7 and 8 show the histogram of values and weights, respectively, elicited from all workers who did the full elicitation mechanism. Note that in the histogram of values, in every dimension, the largest peak is at the slider's default value (at the 2016 estimated budget), and the histograms seem to undergo a phase shift at that peak, suggesting that voters are strongly anchored at the slider's starting value. This anchoring could systematically bias the medians of the elicited values. A similar effect occurs in eliciting voter weights on each dimension. Observe that in Figure 4 the full elicitation weights appear far more balanced than the weights implied by any of the mechanisms (for the full elicitation mechanism, the plot shows the average weight over the sum of the weights for each voter). From the histogram of full elicitation weights, however, we see that this result is a consequence of voters rarely moving a dimension's weight down from the default of 5, but rather moving others up. This pattern demonstrates the difficulty in eliciting utilities from voters directly; even asking voters how much they care about a particular budget item is extremely susceptible to the user interface design. Though such anchoring to the slider default undoubtedly also occurs in the $\mathcal{L}^\infty$ constrained mechanism, it would only slow the rate of convergence, assuming the anchoring affects different voters similarly. These biases can potentially be overcome by changing the UI design, such as by providing no default value through sliders. Such design choices must be carefully thought through before deploying real systems, as they can have serious consequences.

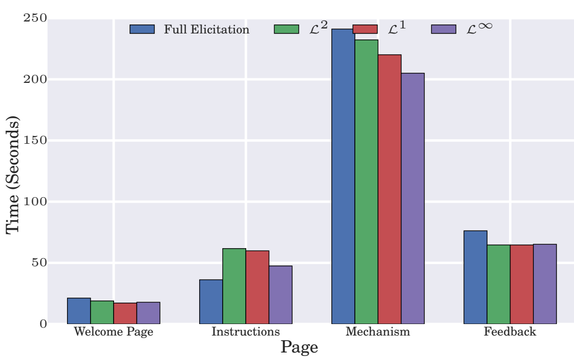



Figure 6: Median time per page

# 6. CONCLUSION

We evaluate a natural class of iterative algorithms for collective decision-making in continuous spaces that makes practically reasonable assumptions on the nature of human feedback. We first introduce several cases in which the algorithm converges to the societal optimum point, and others in which the algorithm converges to other interesting solutions. We then experimentally test such algorithms in the first work to deploy such a scheme. Our findings are significant: even with theoretical backing, two variants fail the basic test of being able to give a consistent decision across multiple trials with the same set of voters. On the other hand, a variant that uses $\mathcal{L}^\infty$ neighborhoods consistently leads to convergence to the same solution, which has attractive properties under a likely model for voter preferences suggested by this convergence. We also make certain observations about other properties of user preferences – most

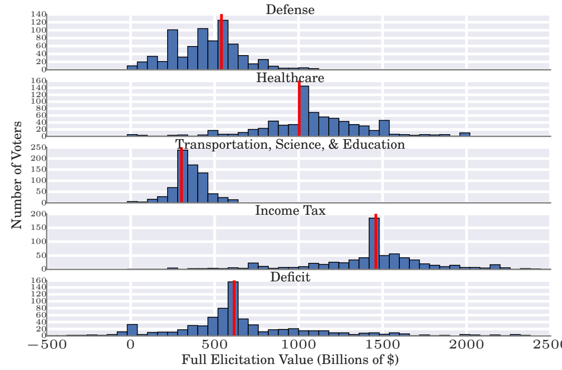



Figure 7: Histogram of values from all full elicitation data. The red vertical lines indicate each slider's default value (at the 2016 estimated budget).

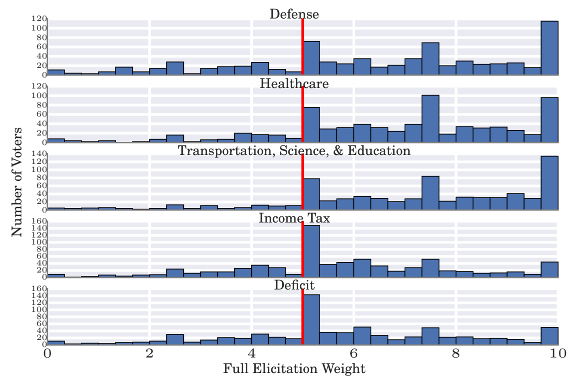



Figure 8: Histogram of weights from all full elicitation data. The red vertical lines indicate the sliders' default value of 5.

saliently, that they have large indifferences on dimensions about which they care less.

In general, this work takes a significant step within the broad research agenda of understanding the fundamental limitations on the quality of societal outcomes posed by the constraints of human feedback, and in designing innovative mechanisms that leverage this feedback optimally to obtain the best achievable outcomes.


# 7. ACKNOWLEDGMENTS

Supported by NSF grant nos. CCF-1408784, CCF-1637397, CCF-1637418, and IIS-1447554, ONR grant no. N00014-15-1-2786, ARO grant no. W911NF-14-1-0526, and the NSF Graduate Research Fellowship under grant no. DGE-114747. This work benefited from many helpful discussions with Oliver Hinder.


# References


[1] D. J. Benjamin, G. Carroll, O. Heffetz, and M. S. Kimball. Aggregating Local Preferences To Guide Policy. Oct. 2014.

[2] S. Boyd and A. Mutapcic. Subgradient methods. *Lecture notes of EE364b, Stanford University, Winter*

*Quarter*, 2007, 2006.

[3] Y. Cabannes. Participatory budgeting: a significant contribution to participatory democracy. *Environment and Urbanization*, 16(1):27–46, Apr. 2004.

[4] Y. Cheng and S. Zhou. A Survey on Approximation Mechanism Design Without Money for Facility Games. In D. Gao, N. Ruan, and W. Xing, editors, *Advances in Global Optimization*, number 95 in Springer Proceedings in Mathematics & Statistics, pages 117–128. Springer International Publishing, 2015. DOI: 10.1007/978-3-319-08377-3_13.

[5] H. Chung and J. Duggan. Directional Equilibria. Feb. 2014.

[6] J. C. Duchi, M. I. Jordan, M. J. Wainwright, and A. Wibisono. Optimal Rates for Zero-Order Convex Optimization: The Power of Two Function Evaluations. *IEEE Transactions on Information Theory*, 61(5):2788–2806, May 2015.

[7] A. D. Flaxman, A. T. Kalai, and H. B. McMahan. Online Convex Optimization in the Bandit Setting: Gradient Descent Without a Gradient. In *Proceedings of the Sixteenth Annual ACM-SIAM Symposium on Discrete Algorithms*, SODA '05, pages 385–394, Philadelphia, PA, USA, 2005. Society for Industrial and Applied Mathematics.

[8] H. R. Gilman. Transformative Deliberations: Participatory Budgeting in the United States. *Journal of Public Deliberation*, 8(2), 2012.

[9] A. Goel, A. K. Krishnaswamy, S. Sakshuwong, and T. Aitamurto. Knapsack voting. *Proceedings of Collective Intelligence*, 2015.

[10] A. Hylland and R. Zeckhauser. *A mechanism for selecting public goods when preferences must be elicited.* 1980.

[11] K. G. Jamieson, R. Nowak, and B. Recht. Query complexity of derivative-free optimization. In *Advances in Neural Information Processing Systems*, pages 2672–2680, 2012.

[12] L. Jiang and J. Walrand. Scheduling and Congestion Control for Wireless and Processing Networks. *Synthesis Lectures on Communication Networks*, 3(1):1–156, Jan. 2010.

[13] S. P. Lalley and E. G. Weyl. Quadratic Voting. SSRN Scholarly Paper ID 2003531, Social Science Research Network, Rochester, NY, Dec. 2015.

[14] D. T. Lee, A. Goel, T. Aitamurto, and H. Landemore. Crowdsourcing for Participatory Democracies: Efficient Elicitation of Social Choice Functions. In *Second AAAI Conference on Human Computation and Crowdsourcing*, Sept. 2014.

[15] P. McDermott. Building open government. *Government Information Quarterly*, 27(4):401–413, Oct. 2010.

[16] H. Moulin. On Strategy-Proofness and Single Peakedness. *Public Choice*, 35(4):437–455, 1980.

[17] A. Nemirovski, A. Juditsky, G. Lan, and A. Shapiro. Robust stochastic approximation approach to stochastic programming. *SIAM Journal on optimization*, 19(4):1574–1609, 2009.

[18] A. D. Procaccia and M. Tennenholtz. Approximate Mechanism Design Without Money. In *Proceedings of the 10th ACM Conference on Electronic Commerce*, EC '09, pages 177–186, New York, NY, USA, 2009. ACM.

[19] D. Quarfoot, D. Von Kohorn, K. Slavin, R. Sutherland, and E. Konar. Quadratic Voting in the Wild: Real People, Real Votes. 2016.

[20] H. Robbins and S. Monro. A Stochastic Approximation Method. *The Annals of Mathematical Statistics*, 22(3):400–407, Sept. 1951.

[21] A. Shah, editor. *Participatory budgeting*. Public sector governance and accountability series. World Bank, Washington, D.C, 2007. OCLC: ocm71947871.

[22] Y. Sintomer, C. Herzberg, and A. Röcke. From Porto Alegre to Europe: potentials and limitations of participatory budgeting. *International Journal of Urban and Regional Research*, 32(1):164–178, 2008.

[23] P. Spada and H. R. Gilman. Budgets for the People. *Foreign Affairs*, Aug. 2015.

[24] T. N. Tideman and F. Plassmann. Efficient Collective Decision-Making, Marginal Cost Pricing, And Quadratic Voting. SSRN Scholarly Paper ID 2836610, Social Science Research Network, Rochester, NY, Aug. 2016.