

Materi 6 Aljabar Linear

Dot Product



Kegunaan

1. Mencari proyeksi orthogonal
2. Hitung jarak dari titik ke garis
3. Hitunglah $r \cdot t$ jika diketahui $r + t$ dan $r - t$



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Operasi Dot Product Sederhana

$u \cdot v = \text{skalar}$

$$u \cdot v = u_1v_1 + u_2v_2 + u_3v_3 + \dots + u_nv_n$$

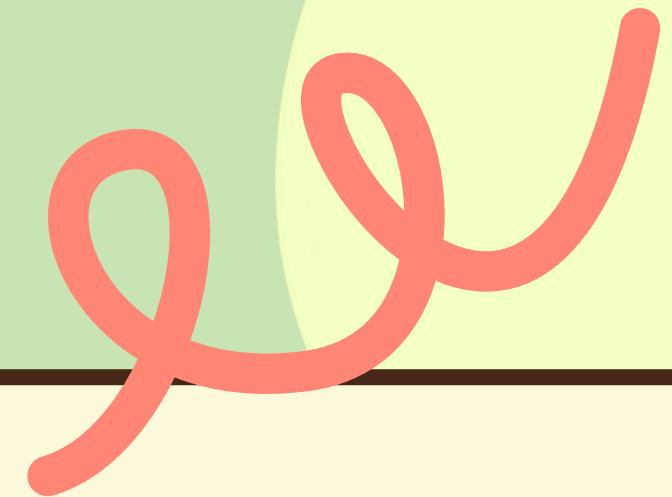
$u \cdot v = 0$ jika u dan v ortogonal

Vektor u dan v di Ruang-2 atau di Ruang-3, dengan θ sudut apit antara u dan v

$$u \cdot v = \begin{cases} \|u\| \|v\| \cos \theta & \text{jika } u \neq 0 \text{ dan } v \neq 0 \\ 0 & \text{jika } u = 0 \text{ atau } v = 0 \end{cases}$$

Catatan: u dan v saling tegak lurus ($\theta = 90^\circ$ & $\cos \theta = 0^\circ$) $\Rightarrow u \cdot v = 0$

Vektor-vektor yang saling tegak lurus disebut vektor-vektor ortogonal



**Vektor u dan v di Ruang-2 atau di Ruang-3, dengan θ sudut
apit antara u dan v**

Catatan: $u, v \in$ Ruang-2 $\rightarrow u = (u_1, u_2), v = (v_1, v_2)$

$u, v \in$ Ruang-3 $\rightarrow u = (u_1, u_2, u_3), v = (v_1, v_2, v_3)$

Formula lain untuk $u \cdot v$:

Ruang-2: $u \cdot v = u_1v_1 + u_2v_2$

Ruang-3: $u \cdot v = u_1v_1 + u_2v_2 + u_3v_3$

Contoh :

1. Misal $\mathbf{u} = (1, 2, 3)$ dan $\mathbf{v} = (-2, 1, 3)$

Maka $\mathbf{u} \cdot \mathbf{v} = -2 + 2 + 9 = 9$

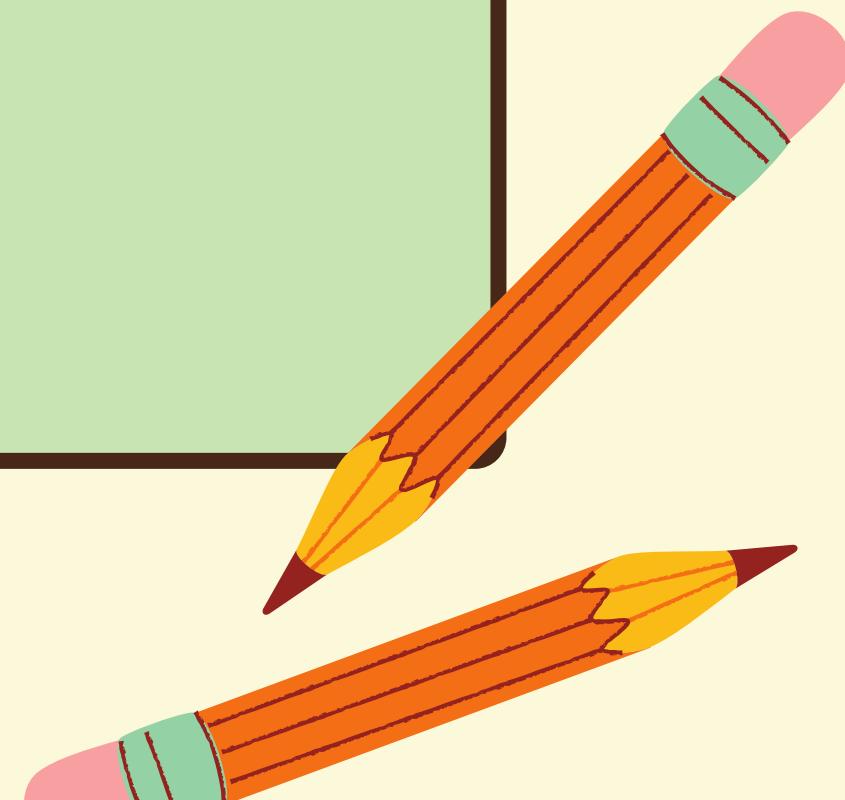
2. Dari soal nomor 1, hitunglah sudut antara \mathbf{u} dan \mathbf{v}

$$\|\mathbf{u}\| = \sqrt{14} \text{ dan } \|\mathbf{v}\| = \sqrt{14}$$

$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| \cos \alpha = 9$ di mana α adalah sudut antara \mathbf{u} dan \mathbf{v}

$$\cos \alpha = 9 / 14$$

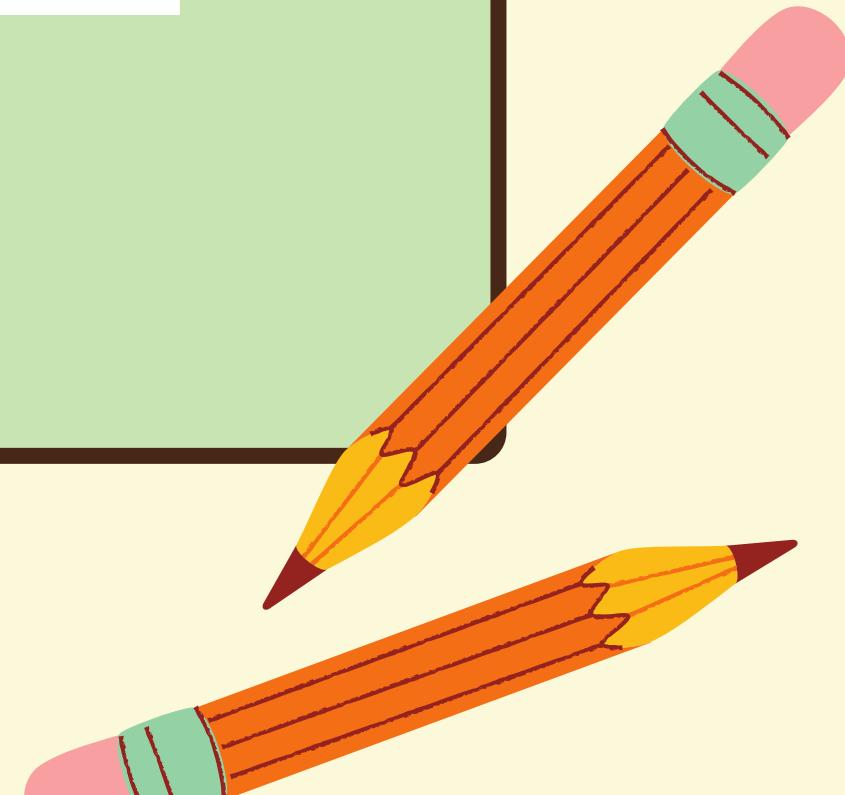
$$\alpha = \arccos(9/14)$$



Contoh :

Example I As shown in Figure 2, the angle between the vectors $u = (0, 0, 1)$ and $v = (0, 2, 2)$ is 45° . Thus,

$$u \cdot v = \|u\| \|v\| \cos \theta = \left(\sqrt{0^2 + 0^2 + 1^2} \right) \left(\sqrt{0^2 + 2^2 + 2^2} \right) \left(\frac{1}{\sqrt{2}} \right) = 2$$



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Contoh :

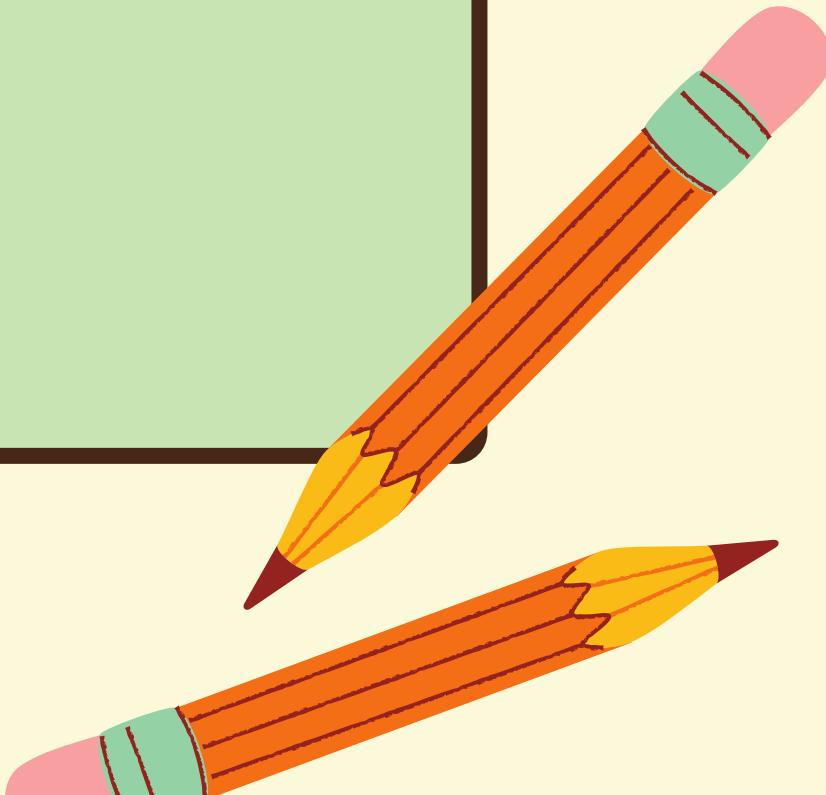
Jika $u = (1, -2, 3)$, $v = (-3, 4, 2)$, $w = (3, 6, 3)$

Maka

- $u \cdot v = -3 - 8 + 6 = -5$
- $v \cdot w = -9 + 24 + 6 = 21$
- $u \cdot w = 3 - 12 + 9 = 0$

Oleh karena itu, maka

- u dan v membentuk suatu sudut tumpul
- w dan v membentuk suatu sudut lancip
- u dan w saling tegak lurus



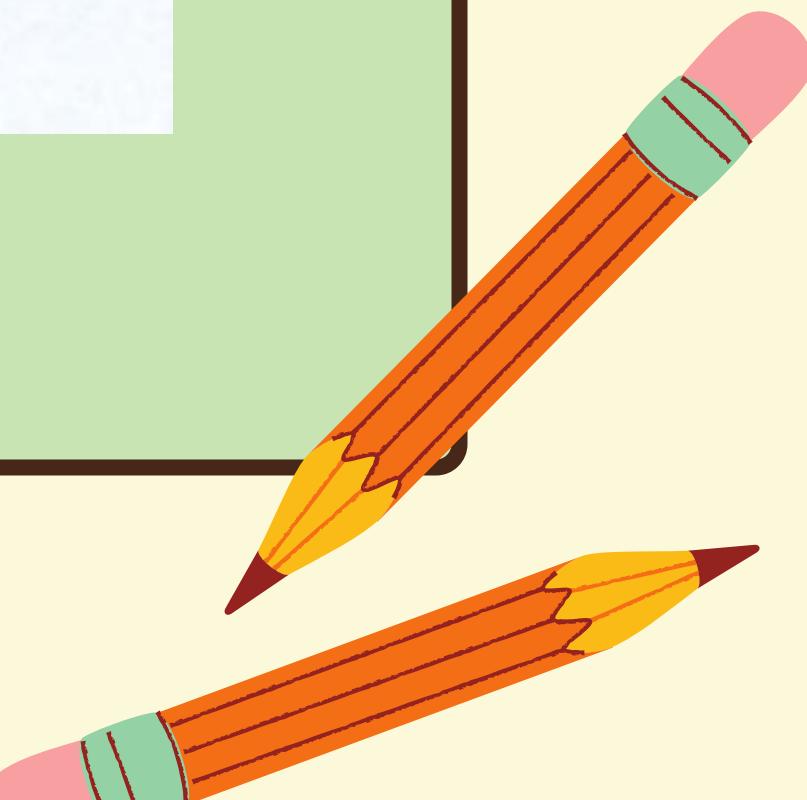
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Contoh soal No. 1

Diketahui $P = \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}$ dan $R = \begin{pmatrix} -5 \\ 7 \\ 1 \end{pmatrix}$

Ditanyakan $P \cdot R$ dan tentukan sudut antara P & R



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Jawaban dari soal No. 1

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jawab :

$$\textcircled{2} \quad P.R = 2 \cdot (-5) + (3 \cdot 7) + (5 \cdot 1)$$
$$= -10 + 21 + 5$$

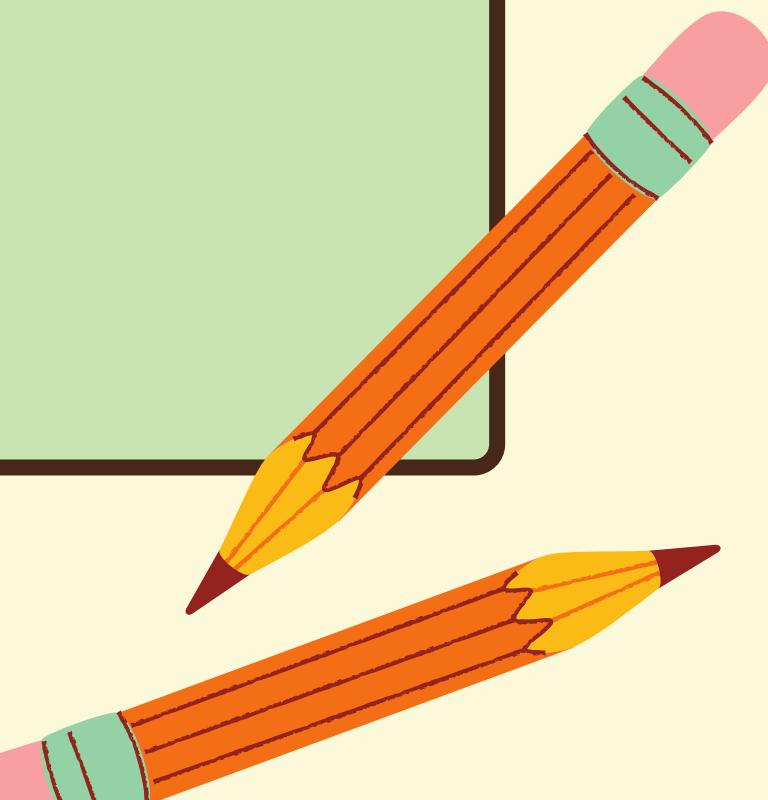
$$\textcircled{2} \quad = 16$$

$$P.R = |P| \cdot |R| \cdot \cos \theta$$
$$\textcircled{2} \quad 16 = \sqrt{2^2 + 3^2 + 5^2} \cdot \sqrt{(-5)^2 + 7^2 + 1^2} \cdot \cos \theta$$
$$16 = \sqrt{38} \cdot \sqrt{75} \cdot \cos \theta$$

$$\textcircled{3} \quad 16 = \sqrt{2850} \cdot \cos \theta$$

$$\textcircled{3} \quad \cos \theta = \frac{16}{\sqrt{2850}}$$

$$\textcircled{3} \quad \theta = \arccos \frac{16}{\sqrt{2850}}$$



Teorema 3.3.1 - 3.3.2:
Vektor-vektor u, v, w di Ruang-2 atau di Ruang-3;
 k adalah skalar

- $\underline{v \cdot v = \|v\|^2}$, atau $\|v\| = (\underline{v \cdot v})^{1/2}$

Bukti: $v \cdot v = \|v\| \|v\| \cos 0^\circ$
 $= \|v\| \|v\| (1) = \|v\|^2$
 $= \|v\|^2$

$$\begin{aligned} v \cdot v &= v_1 v_1 + v_2 v_2 \\ &= v_1^2 + v_2^2 \\ &= \|v\|^2 \end{aligned}$$

- $\underline{u \cdot v = v \cdot u}$

Bukti: $u \cdot v = \|u\| \|v\| \cos \theta$
 $= \|v\| \|u\| \cos \theta$
 $= v \cdot u$

Teorema 3.3.1 - 3.3.2: Vektor-vektor u, v, w di Ruang-2 atau di Ruang-3; k adalah skalar

- $u \cdot (v + w) = u \cdot v + u \cdot w$

Bukti: $u \cdot (v + w) = (u_1, u_2, u_3) \cdot (v_1+w_1, v_2+w_2, v_3+w_3)$

$$\begin{aligned} &= u_1(v_1+w_1) + u_2(v_2+w_2) + u_3(v_3+w_3) \\ &= (u_1v_1+u_1w_1) + (u_2v_2+u_2w_2) + (u_3v_3+u_3w_3) \\ &= (u_1v_1+u_2v_2+u_3v_3) + (u_1w_1+u_2w_2+u_3w_3) \\ &= u \cdot v + u \cdot w \end{aligned}$$

$$k(u \cdot v) = (\cancel{k}u) \cdot v = u \cdot (\cancel{k}v)$$

$$k(u \cdot v) = k(u_1v_1 + u_2v_2 + u_3v_3) \quad \dots \dots \dots$$

$$= (\cancel{k}u_1)v_1 + \cancel{k}u_2v_2 + \cancel{k}u_3v_3 = (u_1\cancel{k}v_1 + u_2\cancel{k}v_2 + u_3\cancel{k}v_3)$$

$$= (\cancel{k}u_1)v_1 + (\cancel{k}u_2)v_2 + (\cancel{k}u_3)v_3 = u_1(\cancel{k}v_1) + u_2(\cancel{k}v_2) + u_3(\cancel{k}v_3)$$

$$= (\cancel{k}u) \cdot v = u \cdot (\cancel{k}v)$$

Dot Product

Teorema 3.3.1 - 3.3.2:

**Vektor-vektor u, v, w di Ruang-2 atau di Ruang-3;
k adalah skalar**

- * $v \cdot v > 0$ jika $v \neq 0$ dan $v \cdot v = 0$ (skalar) jika $v = 0$ (vektor)

Teorema 3.3.1 - 3.3.2:

**Vektor-vektor u, v, w di Ruang-2 atau di Ruang-3;
k adalah skalar**

- $v \cdot v = \|v\|^2$, atau $\|v\| = (\mathbf{v} \cdot \mathbf{v})^{1/2}$
- jika $u \neq 0, v \neq 0$ dan mengapit sudut θ , maka
 - θ lancip $\Leftrightarrow u \cdot v > 0$
 - θ tumpul $\Leftrightarrow u \cdot v < 0$
 - $\theta = 90^\circ \Leftrightarrow u \cdot v = 0$
- $u \cdot v = v \cdot u$
- $u \cdot (v + w) = u \cdot v + u \cdot w$
- $k(u \cdot v) = (ku) \cdot v = u \cdot (kv)$
- $v \cdot v > 0$ jika $v \neq 0$ dan $v \cdot v = 0$ jika $v = 0$

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Contoh :

Jika $u = (1, -2, 3)$, $v = (-3, 4, 2)$, $w = (3, 6, 3)$

$$\text{Maka } u \cdot v = -3 - 8 + 6 = -5$$

$$v \cdot w = -9 + 24 + 6 = 21$$

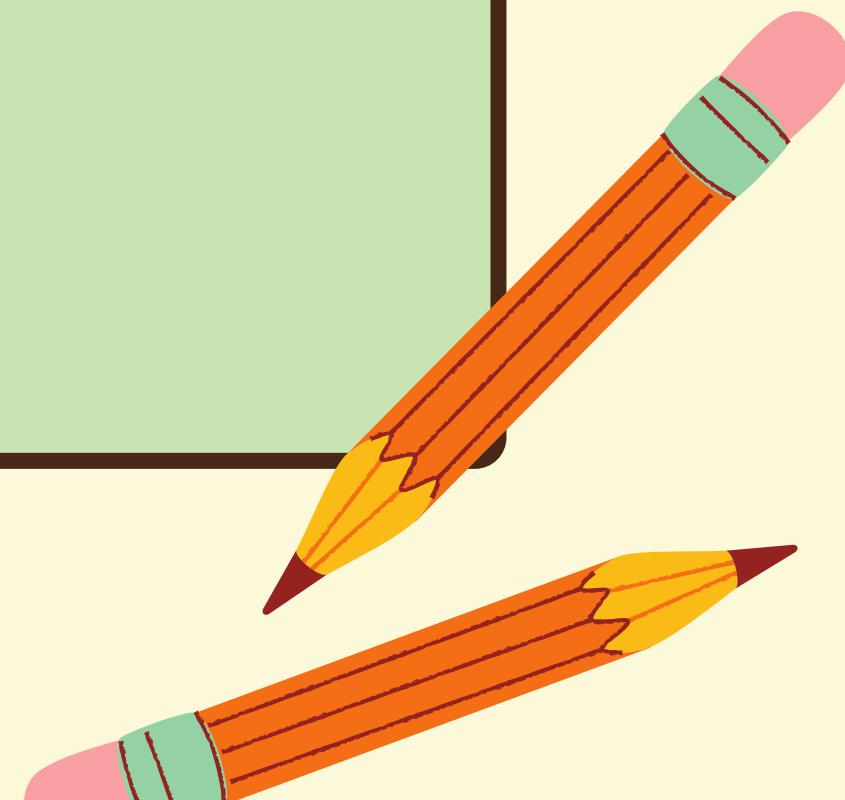
$$u \cdot w = 3 - 12 + 9 = 0$$

Oleh karena itu, maka:

u dan v membentuk suatu sudut tumpul

w dan v membentuk suatu sudut lancip

u dan w saling tegak lurus



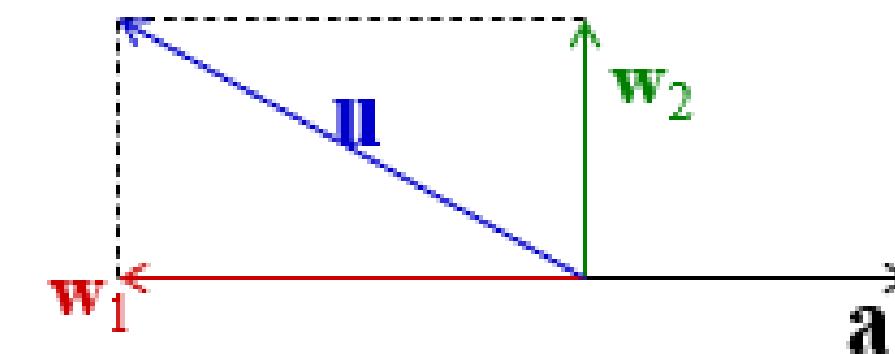
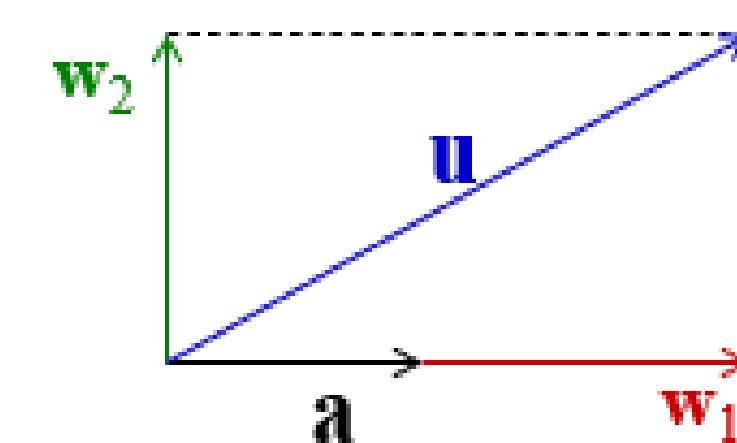
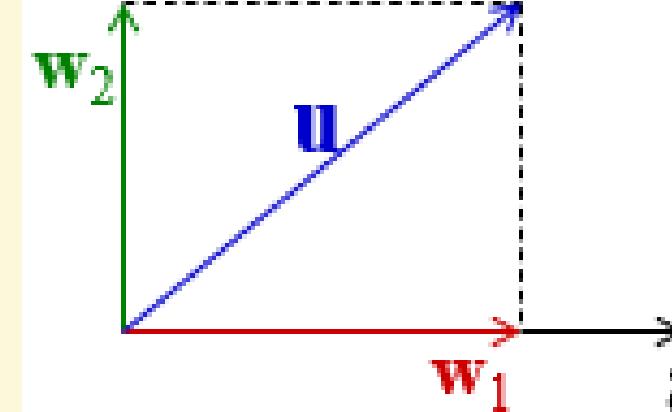
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Proyeksi Ortogonal

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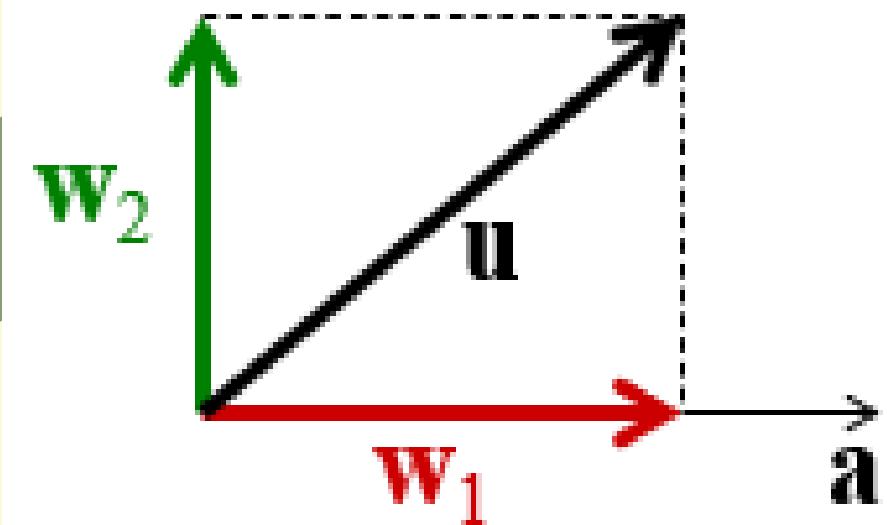
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w_1 = proyeksi ortogonal dari vektor u pada vektor a

= komponen vektor u di sepanjang vektor a

w_2 = komponen vektor u ortogonal terhadap vektor a



w_1 = proyeksi ortogonal dari vektor \vec{u} pada vektor \vec{a}

w_2 = komponen vektor \vec{u} ortogonal terhadap vektor \vec{a}

$$w_1 = (\vec{u} \cdot \vec{a} / \|\vec{a}\|^2) \vec{a}$$

$$w_2 = \vec{u} - (\vec{u} \cdot \vec{a} / \|\vec{a}\|^2) \vec{a}$$

Bukti: $w_1 = (k) \mathbf{a} \rightarrow k = (\mathbf{u} \cdot \mathbf{a} / \|\mathbf{a}\|^2) ?$

$$\mathbf{u} = w_1 + w_2 = k \mathbf{a} + w_2$$

$$\mathbf{u} \cdot \mathbf{a} = (k \mathbf{a} + w_2) \cdot \mathbf{a}$$

$$= k \mathbf{a} \cdot \mathbf{a} + w_2 \cdot \mathbf{a}$$

$$= k \|\mathbf{a}\|^2 + 0 = k \|\mathbf{a}\|^2$$

$$k = (\mathbf{u} \cdot \mathbf{a}) / \|\mathbf{a}\|^2$$

Norm vektor w_1 : $\|w_1\| = |\mathbf{u} \cdot \mathbf{a}| \|\mathbf{a}\| / \|\mathbf{a}\|^2 = |\mathbf{u} \cdot \mathbf{a}| / \|\mathbf{a}\|$

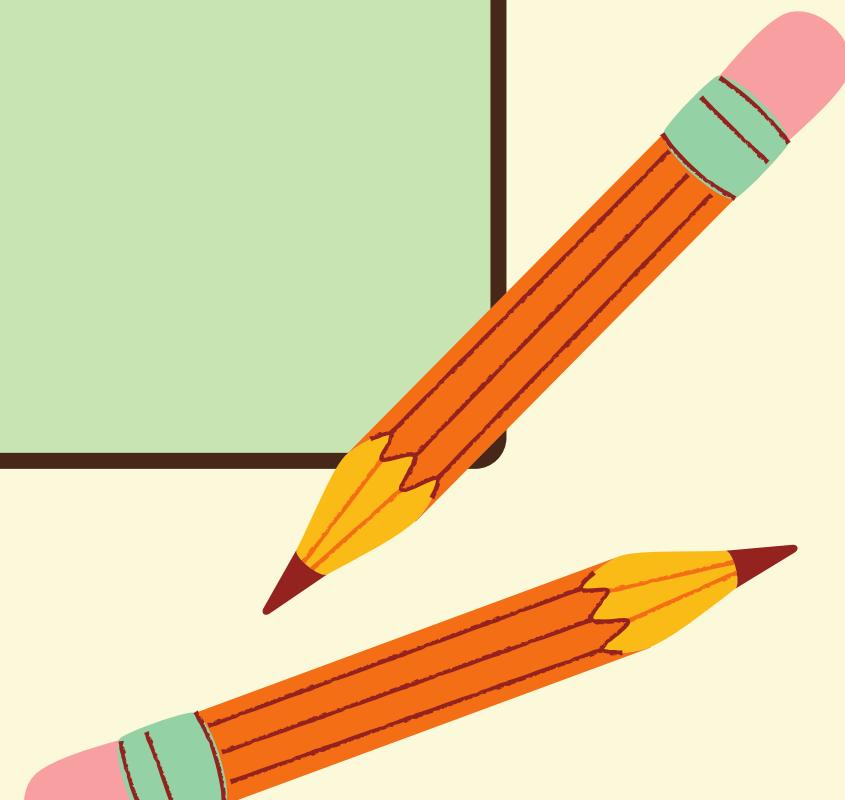
Contoh soal No. 2

Diketahui $d (-3,5,12)$ dan $e (5,7,-2)$

Carilah proyeksi orthogonal vector e pada vector d

Carikah komponen vektor e yang orthogonal terhadap d

Ketelitian 2 angka dibelakang koma (titik)



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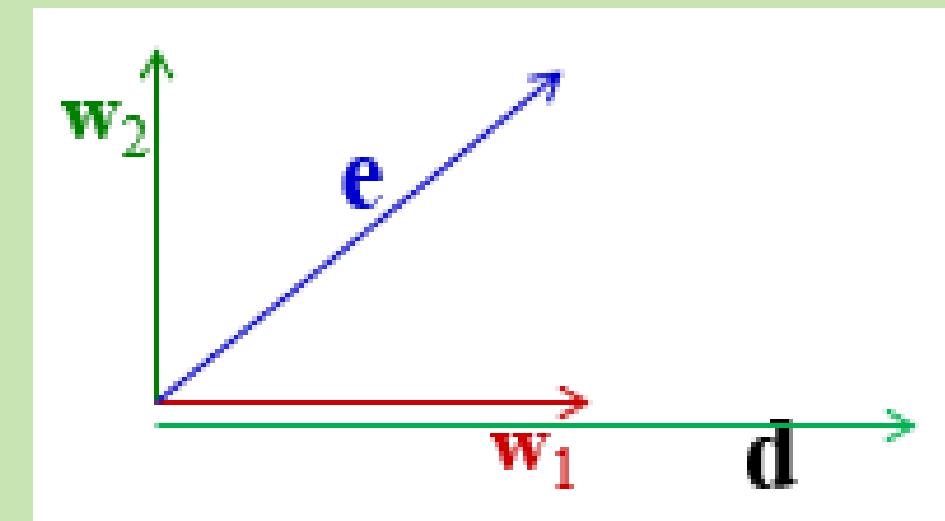
Jawaban soal No. 2

$$w_1 = (e \cdot d / |d|^2) \cdot d$$

$$w_2 = e - w_1$$

$$e \cdot d = -4$$

$$|d|^2 = 178$$



$$w_1 = \left(\begin{array}{ccc} -4 & / & 178 \end{array} \right) \cdot \left(\begin{array}{ccc} -3 & 5 & 12 \end{array} \right)$$

$$w_1 = \begin{pmatrix} 0.07 & -0.11 & -0.27 \end{pmatrix}$$

$$w_2 = e - w_1$$

$$w_2 = \begin{pmatrix} 4.93 & 7.11 & -1.73 \end{pmatrix}$$

Contoh proyeksi

Diketahui $d (-3,5,12), e (5,7,-2)$

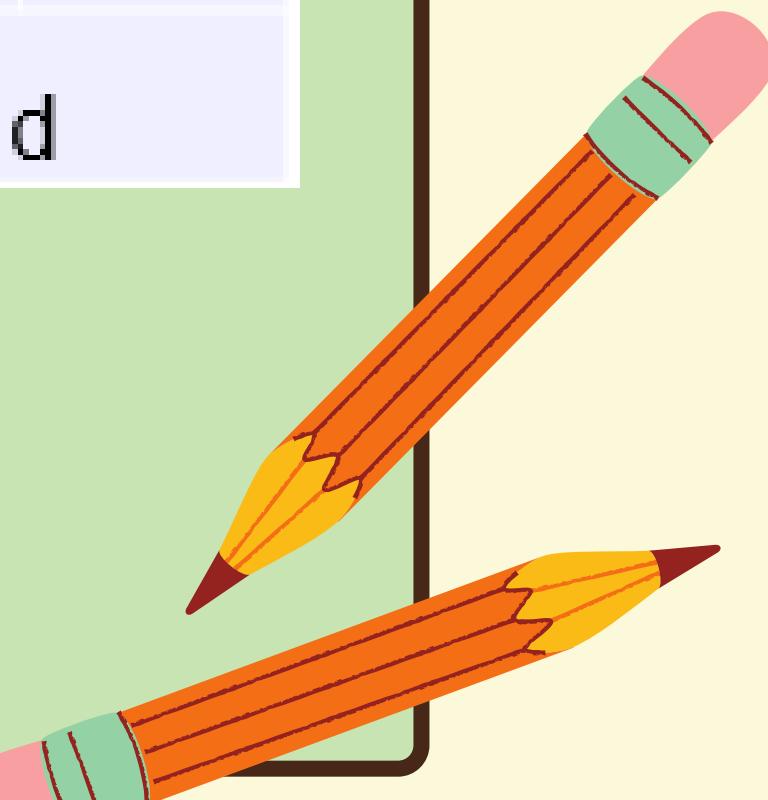
Carilah proyeksi orthogonal vector e pada vector d

Cariakah komponen vektor e yang orthogonal terhadap d

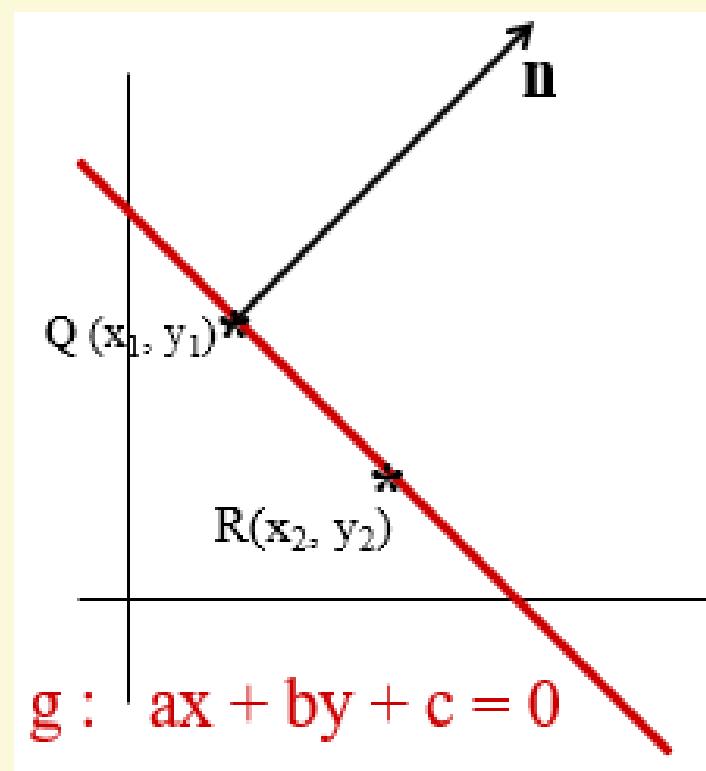
Ketelitian 2 angka dibelakang koma (titik)

Q16	nilai variabel x pada W_1 adalah	0.07
Q17	nilai variabel y pada W_1 adalah	-0.11
Q18	nilai variabel z pada W_1 adalah	-0.27
Q19	nilai variabel x pada W_2 adalah	4.93
Q20	nilai variabel y pada W_2 adalah	7.11
Q21	nilai variabel z pada W_2 adalah	-1.73

$$\begin{array}{l} e = \begin{pmatrix} 5 \\ 7 \\ -2 \end{pmatrix} \\ d = \begin{pmatrix} -3 \\ 5 \\ 12 \end{pmatrix} \\ w_1 = \frac{(e \cdot d)}{|d|^2} \cdot d \end{array}$$



Jarak titik $P_0(x_0, y_0)$ ke garis lurus $g : ax + by + c = 0$



Vektor $\mathbf{n} = (a, b)$ ortogonal garis g

Bukti bahwa $\mathbf{n} = (a, b)$ ortogonal garis g

Vektor $QR = (x_2 - x_1, y_2 - y_1)$

Dengan perkalian titik: $\mathbf{n} \cdot QR = a(x_2 - x_1) + b(y_2 - y_1)$

R terletak pada garis g , maka: $ax_2 + by_2 + c = 0$

Q terletak pada garis g , maka: $ax_1 + by_1 + c = 0$

$$\underline{a(x_2 - x_1) + b(y_2 - y_1) + 0 = 0}$$

Jadi, $\mathbf{n} \cdot QR = a(x_2 - x_1) + b(y_2 - y_1) = 0$

artinya vektor \mathbf{n} ortogonal QR , sehingga vektor \mathbf{n} ortogonal garis g (terbukti)

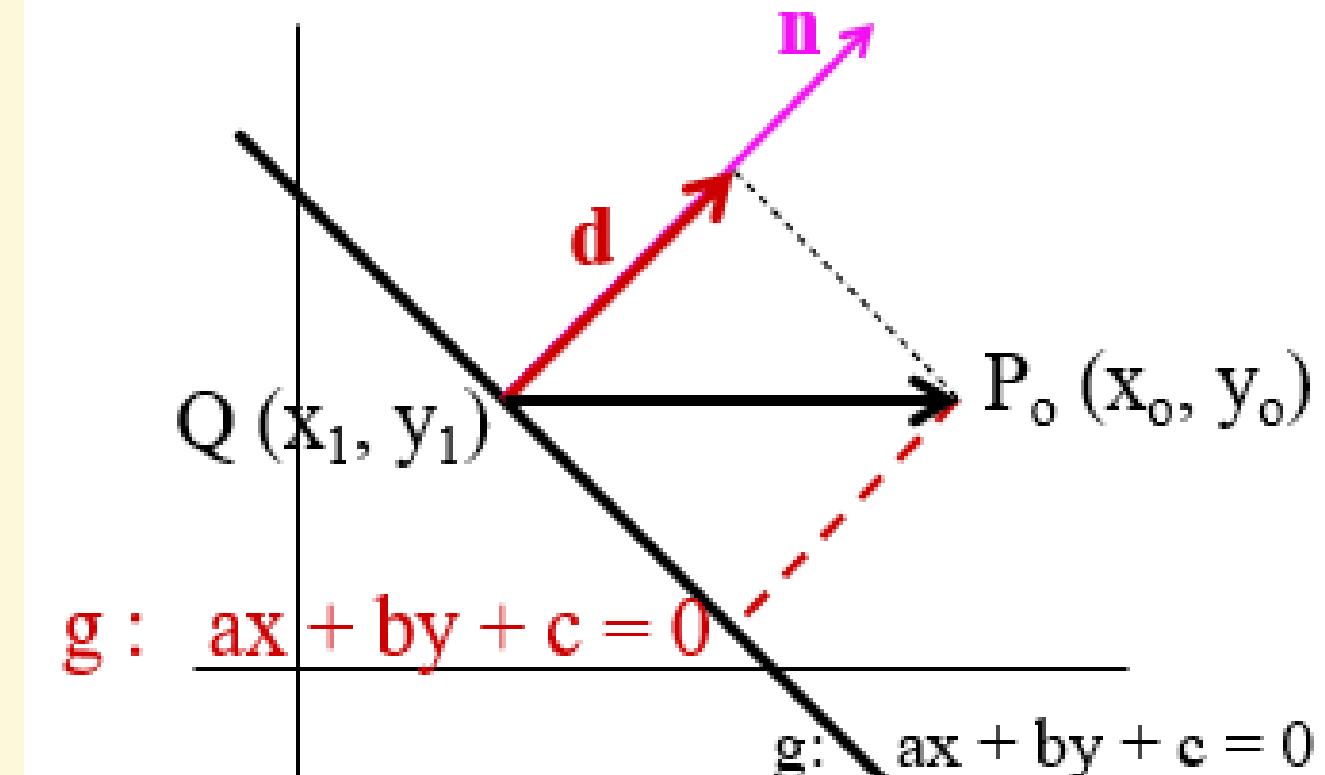
Jarak titik $P_0(x_0, y_0)$ ke garis lurus $g : ax + by + c = 0$

$$\begin{aligned}\|\mathbf{d}\| &= \|\overrightarrow{QP_0} \cdot \mathbf{n}\| / \|\mathbf{n}\| = |(x_0 - x_1, y_0 - y_1) \cdot (a, b)| / \sqrt{a^2 + b^2} \\ &= |(x_0 - x_1)a + (y_0 - y_1)b| / \sqrt{a^2 + b^2} = |x_0a - x_1a + y_0b - y_1b| / \sqrt{a^2 + b^2}\end{aligned}$$

tetapi Q terletak di g , maka $ax_1 + by_1 + c = 0$ atau $c = -ax_1 - by_1$

Maka $\|\mathbf{d}\| = |ax_0 + by_0 - ax_1 - by_1| / \sqrt{a^2 + b^2}$

$$\|\mathbf{d}\| = |ax_0 + by_0 + c| / \sqrt{a^2 + b^2}$$



Vektor $\overrightarrow{QP_0} = (x_0 - x_1, y_0 - y_1)$

(vektor $\overrightarrow{QP_0}$ seperti vektor \mathbf{u} ;
vektor \mathbf{n} seperti vektor \mathbf{a}

vektor \mathbf{d} seperti vektor \mathbf{w}_1)

jarak dari titik P_0 ke garis $g = \|\mathbf{d}\|$

$$\|\mathbf{w}_1\| = |\mathbf{u} \cdot \mathbf{a}| / \|\mathbf{a}\|$$

soal No. 3

Contoh (1) :

Hitunglah jarak antara titik (1,-2) ke garis $3x + 4y - 6 = 0$

Penyelesaian :

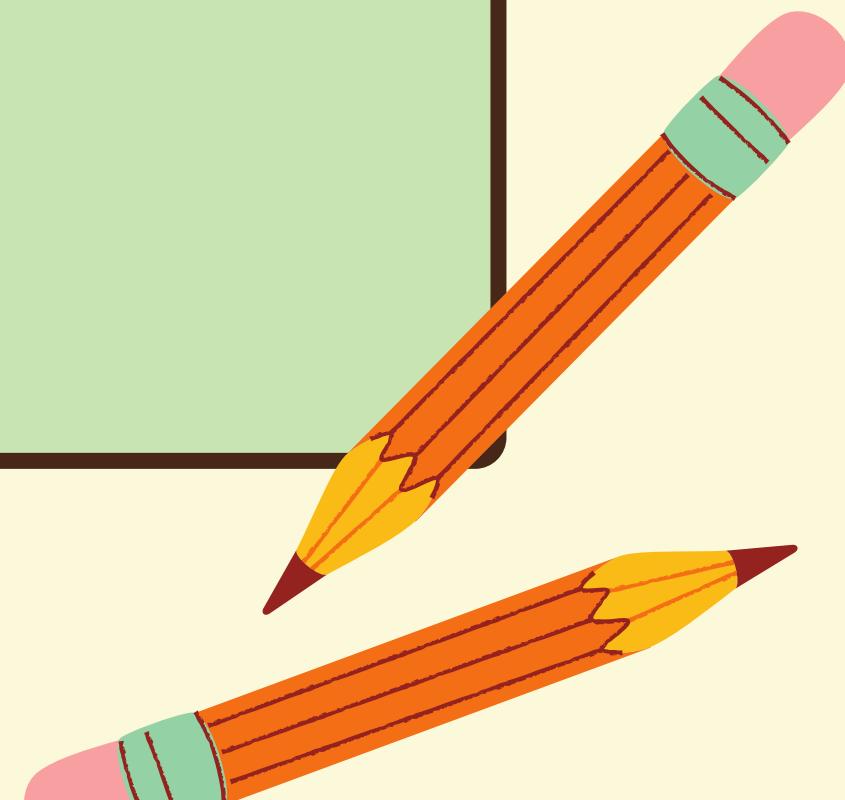
$$D = \frac{|3 \cdot 1 + (4 \cdot -2) - 6|}{\sqrt{3^2 + 4^2}} = \frac{|-11|}{\sqrt{25}} = \frac{11}{5} = 2,2$$

Contoh (2) :

Hitunglah jarak antara titik (1,-2) ke garis $2 = 4y - 2x$

Penyelesaian : garis diubah menjadi $-2x + 4y - 2 = 0$

$$D = \frac{|-2 \cdot 1 + (4 \cdot -2) - 2|}{\sqrt{-2^2 + 4^2}} = \frac{|-12|}{\sqrt{20}} = \frac{12}{\sqrt{20}} = 2,68$$



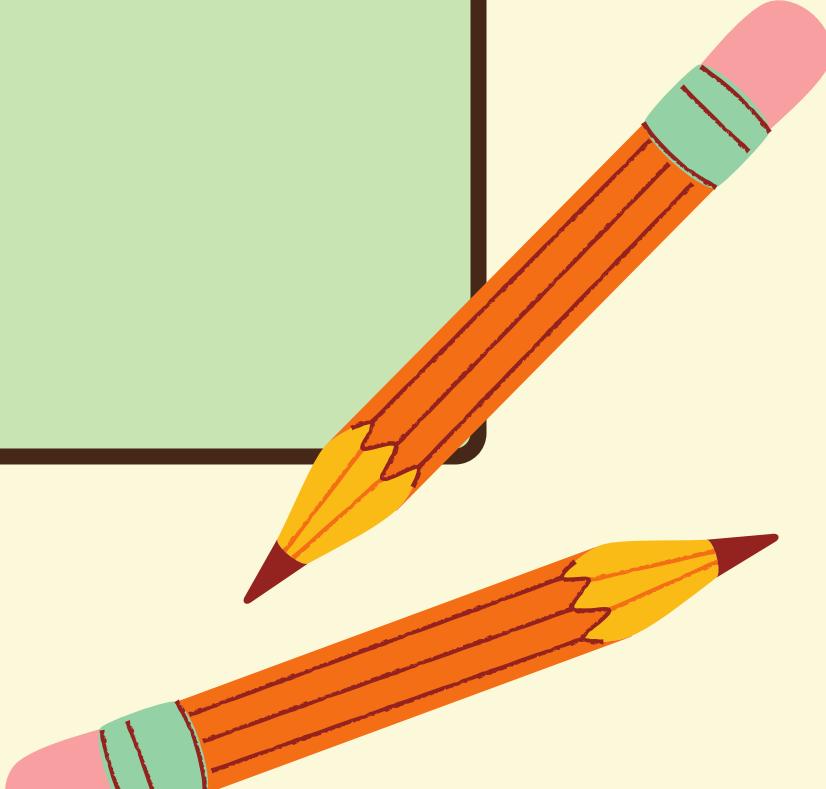
Contoh soal

hitunglah jarak antara titik (-5, 8) ke garis $9 - 4y = -5x$

Ketelitian 2 angka dibelakang koma (titik)

titik	-5	8
persamaan garis		
$5x - 4y + 9 = 0$		

Jarak antara titik dan garis adalah 7.5



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Teorema 4.1.6 – 4.1.7

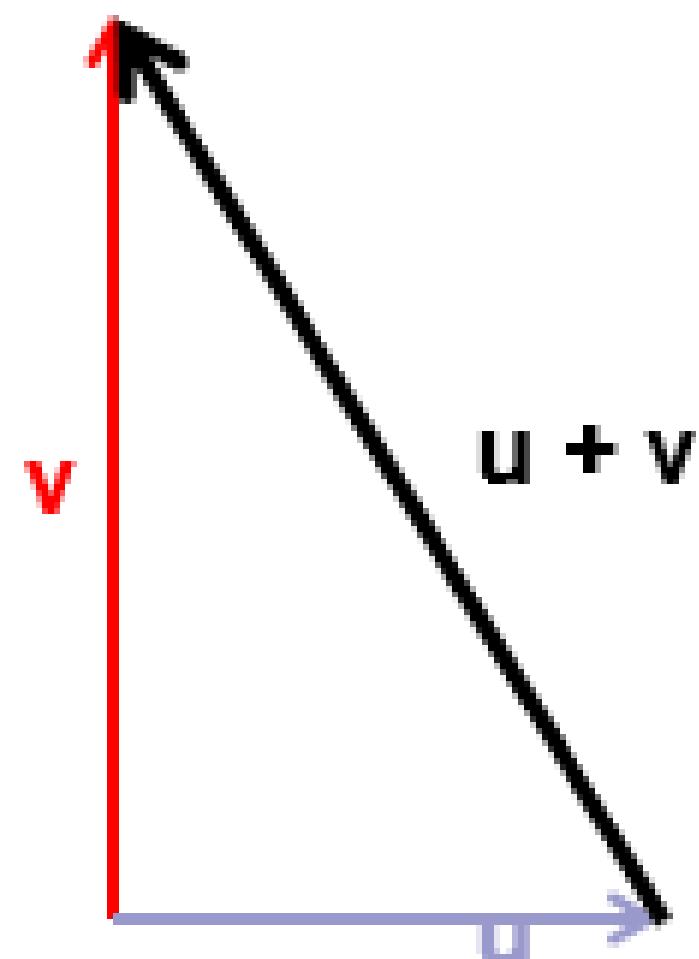
Teorema 4.1.6 – 4.1.7:

$$\mathbf{u} \cdot \mathbf{v} = \frac{1}{4} \|\mathbf{u} + \mathbf{v}\|^2 - \frac{1}{4} \|\mathbf{u} - \mathbf{v}\|^2$$

Teorema Pythagoras

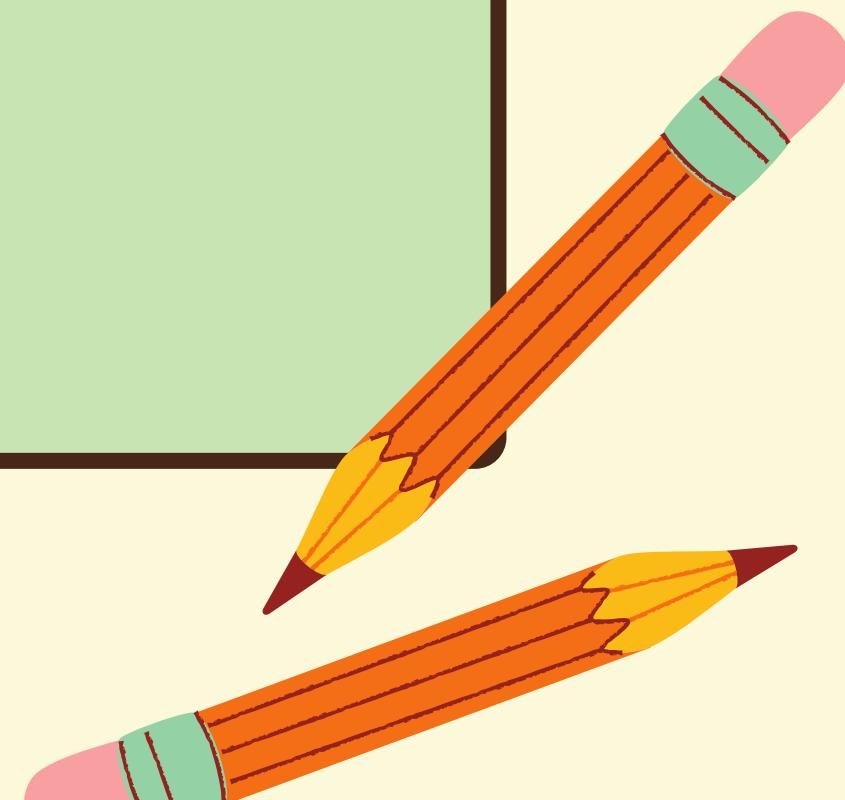
→ jika \mathbf{u} ortogonal \mathbf{v}

$$\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$$



soal No. 4

Hitunglah $u \cdot v$ jika diketahui $u+v = (2, 2, 3)$ dan
 $u-v = (-4, 2, 7)$



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Jawaban No. 4

jawab:

$$u \cdot v = \frac{1}{4} |u+v|^2 - \frac{1}{4} |u-v|^2$$

$$|u+v|^2 = \left(\sqrt{2^2 + 2^2 + 3^2} \right)^2 \quad (3)$$

$$= 4 + 4 + 9$$

$$= 17 \quad (3)$$

$$|u-v|^2 = \left(\sqrt{-4^2 + 2^2 + 7^2} \right)^2 \quad (3)$$

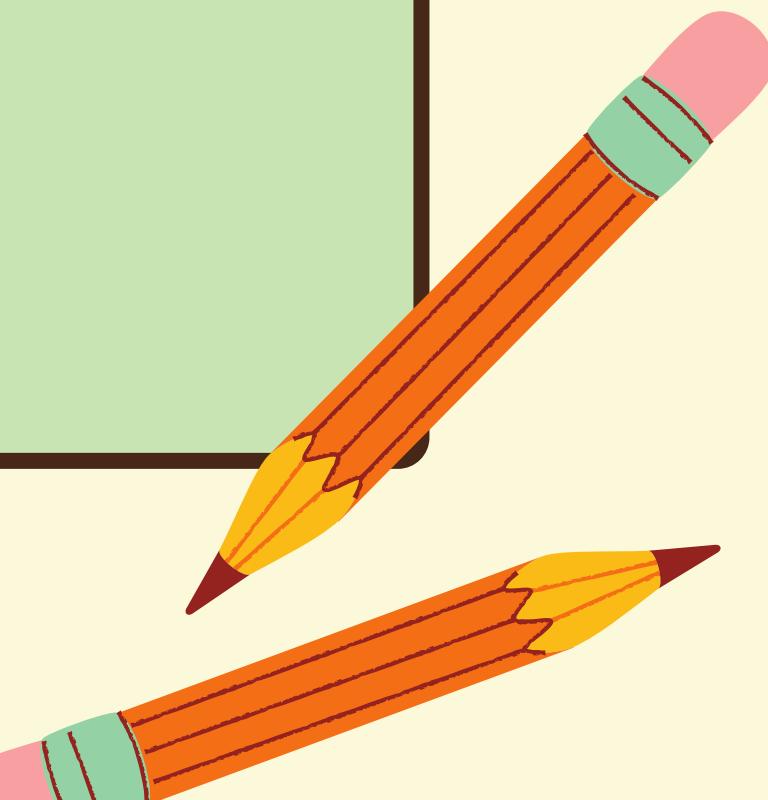
$$= 16 + 4 + 49$$

$$= 69$$

$$u \cdot v = \frac{1}{4} \cdot 17 - \frac{1}{4} \cdot 69 \quad (3)$$

$$= -\frac{52}{4}$$

$$= -13 \quad (3)$$



Contoh soal

Hitunglah $r \cdot t$ jika diketahui $r + t = (-4, 7, 9)$ dan $r - t = (8, -3, 6)$

Ketelitian 2 angka dibelakang koma (titik)

nilai $(r+t)^2$ adalah

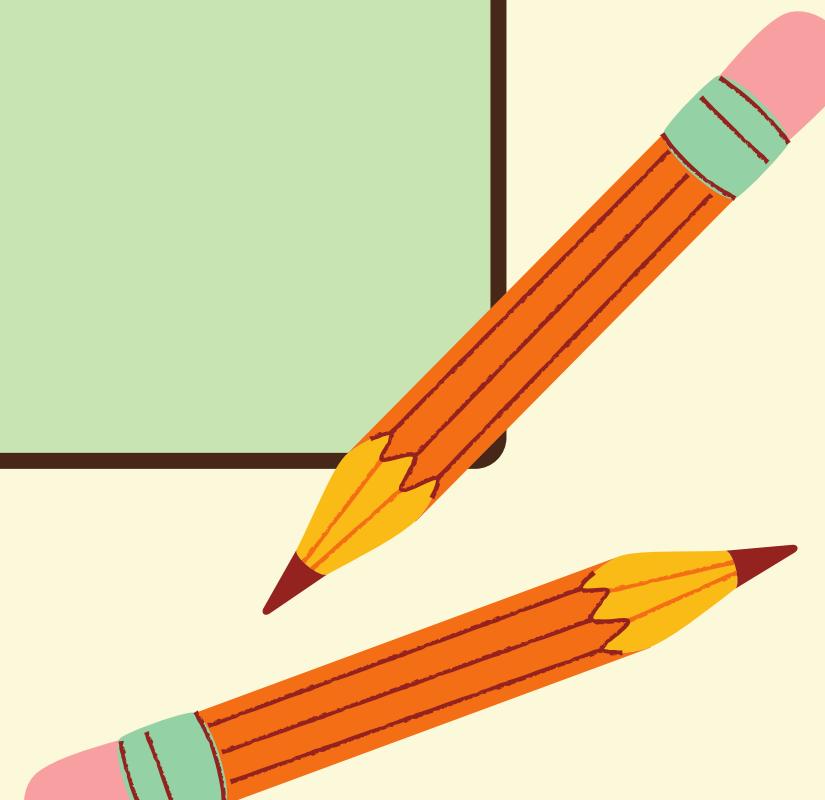
146

nilai $(r-t)^2$ adalah

109

nilai $r \cdot t$ adalah

9.25



Thank You

