→ Recurrence formula for the Bionomial Distribution.

$$P(X=x) = \frac{n-x+1}{x} \frac{p}{q} (X = x-1)$$

→ PROOF:-

$$^nC_x \Rightarrow \frac{n!}{x!} \Rightarrow \frac{n!}{x(n-x)!}$$

$$P(X=x) = \binom{n}{x} p^x q^{n-x} \quad \text{——— (1)}$$

$$P(X=x) = \frac{n!}{x(n-x)!} p^x q^{n-x} \quad \text{——— (2)}$$

$$P(X=x\ 1) = \binom{n}{x} p^x q^{n-x} \quad \text{——— (3)}$$

Put $(X=x-1)$ in eq (iii)

$$P = (X = x-1) = \binom{n}{x-1} p^{x-1} q^{n-x+1}$$

$$P = (X=x-1) = \frac{n!}{(x-1)(n-x+1)!} p^{x-1} q^{n-x+1} \quad \text{—— (4)}$$

Dividing eq (2) by eq (4)

$$\frac{P(X=x)}{P(X=x-1)} = \frac{\frac{\cancel{n!}}{x(n-x)!} p^x q^{\cancel{n-x}}}{\frac{\cancel{n!}}{(x-1)(n-x+1)!} p^{x-1} q^{\cancel{n-x}+1}}$$

$$\frac{P(X=x)}{P(X=x-1)} = \frac{(x-1)!\,(n-x+1)!}{x!\,(n-x)!}\frac{p}{q}$$

$$= \frac{\cancel{(x-1)!}\,\cancel{(n-x)!}\,(n-x+1)}{x\,\cancel{(x-1)!}\,\cancel{(n-x)!}}\frac{p}{q}$$

$$\frac{P(X=x)}{P(X=x-1)} = \frac{n-x+1}{x}\frac{p}{q}$$

$$P(X=x) = \frac{n-x+1}{x}\frac{p}{q}\,P(X=x-1)$$

Mean Of Binomial Distribution

$$\text{Mean} = \mu = E(X) = np$$

$$E(X) = \sum_{x=0}^{n} x \binom{n}{x} p^x q^{n-x} \qquad x = 0,1,2 \ldots n$$

$$E(X) = 0\binom{n}{0}p^0 q^{n-0} + 1\binom{n}{1}p^1 q^{n-1} + 2\binom{n}{2}p^2 q^{n-2}$$

$$\ldots + n\binom{n}{n}p^n q^{n-n}$$

$$E(X) = 0 + npq^{n-1} + 2\binom{n}{2}pq^{n-2} \ldots$$

$$+ n(1)p^n q^0$$

$$E(X) = np\{q^{n-1} + 2\,{}^nC_2\,q^{n-2} \ldots$$

$$+ p^{n-1})$$

$$= np(q^{n-1} + 2\,{}^{n-1}C_2\,q^{n-2} \ldots + p^{n-1})$$

$$E(X) = np[q+p]^{n-1}$$

$$\boxed{\because q = 1-p \rightarrow q+p=1}$$

$$E(X) = np(1)^{n-1}$$

$$\mu = E(X) = np$$

→ Variance of Binomial Probability Distribution :-

$$\sigma^2 = npq$$

$$\sigma^2 = E(X^2) - [E(X)]^2$$

where $E(X^2) = [E(X(x-1)+x)]$

$$E(x^2 - x + x) = E(x^2)$$

$$E(x^2) = [E(x(x-1)) + x]$$

$$E(X^2) = E(x(x-1)) + E(X)$$

first we solve $E(x(x-1))$

discrete

$$E(x(x-1)) = \sum_{x=0}^{n} x(x-1)\, {}^nC_x\, p^n q^{n-x}$$

$$= \sum_{x=0}^{n} x(x-1) \frac{n!}{x!} p^n q^{n-x}$$

$$= \sum_{x=0}^{n} \cancel{x(x-1)} \frac{n(n-1)(n-2)!\ p^2 p^{-2+x} q^{n-x}}{\cancel{x(x-1)}(x-2)!(n-x)!}$$

$$= \sum_{x=0}^{n} n(n-1)p^2 \frac{(n-2)!\ p^2 p^{-2+x} q^{n-x}}{(x-2)!(n-x)!}$$

$$= n(n-1)p^2 \underbrace{\sum_{x=0}^{n} \left({}^{n-2}C_{x-2}\right) p^{x-2} q^{n-x}}_{(q+p)^{n-2}}$$

$$E[X(X-1)] = n(n-1)p^2 (q+p)^{n-2}$$

$$\because q+p=1$$

$$E(X(X-1)) = n(n-1)p^2 (1)^{n-2}$$

$$= n(n-1)p^2$$

$$E(X(X-1)) = n^2p^2 - np^2$$

$$E(X^2) = [E(X(X-1)) + E(X)]$$

$$E(X^2) = n^2p^2 - np^2 + np$$

$$\sigma^2 = E(X^2) - [E(X)]^2$$

$$= \cancel{n^2p^2} - np^2 + np - \cancel{n^2p^2}$$

$$\sigma^2 = np - np^2$$

$$= np(1-p) \qquad \because q+p=1$$

$$\sigma^2 = npq \qquad q = 1-p$$

$S.D = \sqrt{npq}$

MgF of Binomial Probability

$M_0 t = E(e^{tx})$

$\boxed{\because Mgf = (q+Pe^t)^n}$

$$M_0 t = \sum_{x=0}^{n} e^{tx} \cdot {}^nC_x \, P^x q^{n-x}$$

$$M_0 t = \sum_{x=0}^{n} {}^nC_x (pe^t)^x q^{n-x} \quad \text{——— (i)}$$

~~pply~~ Put summation in eq (1)

$$M_0 t = {}^nC_0 (Pe^t)^0 q^{n-0} + {}^nC_1 (Pe^t)^1 q^{n-1} + {}^nC_2 (Pe^t)^2 q^{n-0} \text{———} {}^nC_n (Pe^t)^n q^{n-n}$$

${}^nC_0 = 1$

$$= q^n + {}^nC_1 (Pe^t)^1 q^{n-1} + \text{———} + (1)(Pe^t)^n (1)$$

$\boxed{Mgf = (q+Pe^t)^n}$

Differentiate :-

$$\mu_1' = E(X) = \left[\frac{d}{dt}(q+Pe^t)^n\right]\Bigg|_{t=0}$$

$$\mu_1' = E(X) = npe^t (q + pe^t)^{n-1} \Big|_{t=0}$$

$$= npe^0 (q + pe^0)^{n-1}$$

$$= np (q+p)^{n-1}$$

$$= np(1)$$

$$\mu_1' = E(X) = np$$

$$\mu_2' = E(X^2) = \frac{d^2}{dt^2} [q + pe^t]^n \Big|_{t=0}$$

$$\mu_2' = E(X^2) = npe^t [q+pe^t]^{n-1} + n(n-1)p^2 e^{2t} (q+pe^t)^{n-2} \Big|_{t=0}$$

$$\mu_2' = npe^0 [q+pe^0]^{n-1} + n(n-1)p^2 e^{20} (q+pe^0)^{n-2}$$

$$= np[q+p]^{n-1} + n(n-1)p^2 (q+p)^{n-2}$$

$$= np(1)^{n-1} + n(n-1)p^2 (1)^{n-2}$$

$$= np + n(n-1)p^2$$

$$E(X^2) = \mu_2' = np + n^2p^2 - np^2$$

$$\mu_2 = E(X^2) - [E(X)]^2$$

$$= np + n^2p^2 - np^2 - n^2p^2$$

$$= np - np^2$$

$$= np(1-p)$$

$$\mu_2 = npq$$

Cumulant generating function.

$k(t) = \log_e M_0(t)$

$= \log_e [q + pe^t]^n$

$= n \log_e q + pe^t$

$\therefore e^t = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots \frac{t^n}{n!}$.

$k(t) = n \log_e [q + p(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} \cdots)]$

Expanding

$k_1 = \mu_1 = np$

$k_2 = \mu_2 = npq$

Higher Moments :-

- $\boxed{\mu_1' = E(X) = np}$
- $\boxed{\mu_2' = E(X^2) = n(\ldots)p^2 + np}$

- $\mu_3' = E(X^3) = E[X(X-1)(X-2)] + 3E[X(X-1)] + E(X)$ —— (1)

where:-

$\boxed{E(X) = np \quad \text{and} \quad E(X(X-1)) = n(n-1)p^2}$

let

$E[X(X-1)(X-2)] = x(x-1)(x-2)\, {}^nC_x\, p^x q^{n-x}$ (2)

The series $x(x-1)(x-2)\, {}^nC_x = \frac{n(n-1)(n-2)(n-3)!}{x(x-1)(x-2)(x-3)!\,(n-x)!}$

$$x(x-1)(x-2)\,{}^{n}C_{x} = \frac{n(n-1)(n-2)(n-3)!}{(x-3)!\,(n-x)!}$$

$$= n(n-1)(n-2)\,{}^{n-3}C_{x-3} \quad \text{—— (3)}$$

Put eq (3) in eq (2)

$$E[X(x-1)(x-2)] = n(n-1)(n-2)\sum_{x=3}^{n} {}^{n-3}C_{x-3}\, p\, p^{x-3} p^{3} q^{n-x}$$

$$= n(n-1)(n-2)\,p^{3} \underbrace{\sum_{x=3}^{n} {}^{n-3}C_{x-3}\, q^{n-x} p^{x-3}}$$

$$= n(n-1)(n-2)\,p^{3}\,[q+p]^{n-3}$$

$$= n(n-1)(n-2)\,p^{3}\,(1)^{n-3}$$

$$E(X(X-1)(X-2)) = n(n-1)(n-2)p^{3} \quad \text{—— (4)}$$

put eq (4) in eq (1)

$$u_3 = E(X^3) = n(n-1)(n-P)P^3 + 3E(X(X-1)) + E(X)$$

$$u_3' = E(X^3) = n(n-1)(n-2)P^3 + 3[\quad] + np$$

Derivation OF Hypergeometric Probability Distribution :-

$$P(X=x) = \frac{\binom{k}{x}\binom{N-k}{n-x}}{\binom{N}{n}} ,$$ for x such that $x = 0,1,2 \text{---} n$ & $x = 0,1,2 \text{---} k$

$$P(X=x) = \frac{{}^kC_x \; {}^{N-k}C_{n-x}}{{}^NC_n}$$

$$= \frac{{}^NC_n}{\cancel{{}^NC_n}} \Rightarrow 1$$

Mean Of hypergeometric Probability Distribution :-

Mean = $E(X) = np$

$$E(X) = x \frac{({}^kC_x)({}^{N-k}C_{n-x})}{{}^NC_n}$$

$$E(X) = \frac{\cancel{x} \dfrac{k(k-1)!}{\cancel{x}(x-1)!(k-x)!} \left({}^{N-k}C_{n-x}\right)}{{}^NC_n}$$

$$= \frac{k}{{}^NC_n} \left({}^{k-1}C_{x-1}\right)\left({}^{N-k}C_{n-x}\right)$$

Put $y = x-1$ & $x = y+1$

$$E(X) = \frac{k}{{}^NC_n} \left({}^{k-1}C_y\right)\left({}^{N-k}C_{n-y-1}\right)$$

$$E(X) = \frac{k}{{}^{N}C_n} \left({}^{k-1}C_y\right)\left({}^{N-k}C_{n-y-1}\right)$$

$$\frac{k}{{}^{N}C_n} \sum_{x=0}^{n} \left({}^{N-1}C_{n-1}\right)$$

$$E(X) = \frac{\frac{k}{N(N-1)!}}{n(n-1)!\,(N-n)!} \quad \frac{(N-1)!}{(n-1)!\,(N-n)!}$$

$$E(X) = \frac{k\,[n(n-1)!\,\cancel{(N-n)!}]}{N\,\cancel{(N-1)!}} \cdot \frac{\cancel{(N-1)!}}{\cancel{(n-1)!}\,\cancel{(N-n)!}}$$

$$E(X) = \frac{nk}{N}$$

$$\boxed{\text{where} \quad p = \frac{k}{N}}$$

$$\boxed{E(X) = np}$$

Variance and S.D of HGD :-

$$\sigma^2 = E(X^2) - [E(X)]^2$$

$$\boxed{\begin{aligned} \sigma^2 &= npq \, \frac{N-n}{N-1} \\ \sigma &= \sqrt{npq}\,\sqrt{\frac{N-n}{N-1}} \end{aligned}}$$

$$E(X^2) = E(X(X-1)+X)$$

$$E(X^2) = E(X(X-1)) + E(X)$$

$$E(X^2) =$$

$$E(X^2) = E(X(X-1)) + E(X)$$

$$E(X^2) = \frac{x(x-1)\,{}^kC_x \, {}^{N-k}C_{n-x}}{{}^NC_n} + \frac{nk}{N}$$

$$= \frac{\cancel{x(x-1)}\,\dfrac{k(k-1)(k-2)!}{\cancel{x(x-1)}(x-2)!\,(k-x)!}\left({}^{N-k}C_{n-k}\right)}{{}^NC_n}$$

$$= \frac{k(k-1)(k-2)!}{(x-2)!\,(k-x)!}\,{}^{N-k}C_{n-x} \Big/ {}^NC_n$$

$$E(X^2) = \frac{k(k-1)\sum_{x=2}^{n}\left({}^{k-2}C_{x-2}\right)\cdot\left({}^{N-k}C_{n-x}\right)}{{}^NC_n} + \frac{nk}{N}$$

put $x-2 = y \rightarrow x = y+2$

$$E(X^2) = \frac{k(k-1)\sum_{y=2}^{n}\left({}^{k-2}C_{y}\right)\left({}^{N-k}C_{n-y-2}\right)}{{}^NC_n} + \frac{nk}{N}$$

$$= \frac{k(k-1)\left({}^{N-2}C_{n-2}\right)}{{}^NC_n} + \frac{nk}{N}$$

$$E(X^2) = \frac{k(k-1)\cdot\dfrac{(N-2)!}{(n-2)!\,(N-n)!}}{\dfrac{N(N-1)(N-2)!}{n(n-1)(n-2)(N-n)!}} + \frac{nk}{N}$$

$$= \frac{k(k-1)\,n(n-1)\cancel{(n-2)}\cancel{(N-n)!}\cdot\cancel{(N-2)!}}{N(N-1)\cancel{(N-2)!}\cdot\cancel{(n-2)!}\cancel{(N-n)!}} + \frac{nk}{n}$$

$$E(X^2) = \frac{nk(k-1)(n-1)}{N(N-1)} + \frac{nk}{N}$$

$$\sigma^2 = E(X^2) - [E(X)]^2$$

$$\sigma^2 = \frac{nk(k-1)(n-1)}{N(N-1)} + \frac{nk}{N} - \left(\frac{nk}{N}\right)^2$$

$$= \frac{nk}{N}\left[\frac{(k-1)(n-1)}{(N-1)} + 1 - \frac{nk}{N}\right]$$

$$= \frac{nk}{N}\left[\frac{N(k-1)(n-1) + (N^2-N) - nk(N-1)}{N(N-1)}\right]$$

$$= \frac{nk}{N}\left[\frac{N(kn-k-n+1) + (N^2-N) - nkN + nk}{N(N-1)}\right]$$

$$= \frac{nk}{N}\left[\frac{\cancel{Nkn} - Nk - nN + \cancel{N} + N^2 - \cancel{N} - \cancel{Nkn} + nk}{N(N-1)}\right]$$

$$= \frac{nk}{N}\left[\frac{N^2 - nN - Nk + nk}{N(N-1)}\right]$$

$$\sigma^2 = \frac{nk}{N}\left[\frac{N[N-n] - k(n-N)}{N(N-1)}\right]$$

$$= \frac{nk}{N}\left[\frac{(N-k)(N-n)}{N(N-1)}\right]$$

$$= \frac{nk(N-k)}{N^2}\left[\frac{N-n}{N-1}\right]$$

$\boxed{\text{wher } P = \frac{k}{N} \text{ and } q = \frac{N-k}{N}}$

$$\sigma^2 = \frac{nk}{N} \frac{N-k}{N} \left[\frac{N-n}{N-1}\right]$$

$$\boxed{\sigma^2 = npq\left[\frac{N-n}{N-1}\right]}$$

$$S.D = \sqrt{npq\left(\frac{N-n}{N-1}\right)}$$

Poisson Distribution

$f(x) = P[X = x] \geq 0$

$$\sum f(x) = \sum_{0}^{\infty} \frac{e^{-\mu}\mu^{x}}{x!}$$

$$\sum f(x) = e^{-\mu} \sum_{x=0}^{\infty} \frac{\mu^{x}}{x!} \qquad x = 0, 1, 2 \ldots$$

$$= e^{-\mu}\left[\frac{\mu^0}{0!} + \frac{\mu^1}{1!} + \frac{\mu^2}{2!} + \frac{\mu^3}{3!} \ldots\right]$$

$$\sum f(x) = e^{-\mu}\left[1 + \frac{\mu}{1} + \frac{\mu^2}{2} + \frac{\mu^3}{6} \ldots\right]$$

$$= e^{-\mu} \cdot e^{\mu}$$

$$= e^{-\mu+\mu}$$

$$\sum f(x) = e^{0}$$

$$\sum f(x) = 1$$

Mean

$$E(X) = \sum_{x=0}^{\infty} x\, f(x)$$

$$f(X) = \sum_{x=0}^{\infty} x\, P(x; \mu)$$

where $P(x; \mu) = \dfrac{e^{-\mu}\mu^{x}}{x!}$

$$E(X) = \sum_{x=0}^{\infty} x \frac{e^{-\mu}\mu^{x}}{x!}$$

$$= \frac{0 e^{-\mu} e^{x}}{0!} + \frac{1 e^{-\mu}\mu^{1}}{1!} + \frac{2e^{-\mu}\mu^{2}}{2!} + \frac{3 e^{-\mu}\mu^{3}}{3!} -$$

$$E(X) = e^{-\mu}\mu\left[0 + 1 + \frac{2\mu}{2} + \frac{3\mu^2}{6} \text{------}\right]$$

$$= e^{-\mu}\mu \cdot e^{\mu}$$

$$= e^{-\mu+\mu} \cdot \mu$$

$$E(X) = 1 \cdot \mu$$

$$\boxed{E(X) = \mu}$$

Variance

$$\sigma^2 = E(X^2) - [E(X)]^2$$

$$E(X^2) = [E(X(X-1)) + X]$$

$$E(X^2) = E(X(X-1)) + E(X)$$

$$E(X(X-1)) = x(x-1)\sum_{x=0}^{\infty}\frac{e^{-\mu}\mu^x}{x!}$$

$$= \sum_{x=0}^{\infty}\frac{\cancel{x(x-1)} \cdot e^{-\mu}\mu^x}{\cancel{x(x-1)}(x-2)!}$$

$$= e^{-\mu}\mu^2\sum_{x=2}^{\infty}\frac{\mu^{x-2}}{(x-2)!}$$

let $x - 2 = y \implies x = y + 2$

$$= e^{-\mu}\mu^2\left[\sum_{y=0}^{\infty}\frac{\mu^y}{y!}\right]$$

$$E(x(x-1)) = e^{-\mu}\mu^2\left[\frac{\mu^0}{0!} + \frac{\mu^1}{1!} + \frac{\mu^2}{2!} + \frac{\mu^3}{3!} \cdots\right]$$

$$= e^{-u}\mu^2\left[0 + \frac{u^1}{1} + \frac{u^2}{2} + \frac{u^3}{6} \cdots\right]$$

$$= \cancel{e^{-u}} u^2 \cancel{e^{u}}$$

$$\boxed{E(x(x-1) = u^2}$$

$$E(x^2) = E(x(x-1)) + E(x)$$

$$E(x^2) = u^2 + u$$

$$\sigma^2 = E(x^2) - [E(x)]^2$$

$$= u^2 + u - (\mu)^2$$

$$= \cancel{\mu^2} + u - \cancel{\mu^2}$$

$$\boxed{\sigma^2 = \mu}$$

$$\boxed{S.D = \sqrt{\mu}}$$

Moment Generating function.

$$M_0(t) = E(e^{tx}) = \sum_{x=0}^{\infty} e^{tx}\frac{\mu^x e^{-u}}{x!}$$

$$M_0(t) = \sum_{x=0}^{\infty}\frac{e^{-u}(e^t\mu)^x}{x!}$$

$$= e^{-u}\sum_{x=0}^{\infty}\frac{(e^t u)^x}{x!}$$

$$M_0(t) = e^{-u} \sum_{x=0}^{\infty} \frac{(e^t\mu)^x}{x!}$$

$$= e^{-u}\left[\frac{(e^t\mu)^0}{0!} + \frac{(e^tu)^1}{1!} + \frac{(e^tu)^2}{2!} + \cdots\right]$$

$$= e^{-u}\left(e^{e^t u}\right)$$

$$M_0(t) = e^{-u} e^{ue^t} \Rightarrow \boxed{e^{u(e^t-1)}}$$

$$\mu = E(X) = \left[\frac{d}{dt} e^{u(e^t-1)}\right]\Big|_{t=0}$$

$$= ue^t\left(e^{u(e^t-1)}\right)\Big|_{t=0}$$

$$= \mu e^0\left(e^{u(e^0-1)}\right)$$

$$= \mu\left(e^{\mu(1-1)}\right)$$

$$= \mu(e^0) = u(1)$$

$$\boxed{\mu' = \mu}$$

$$\sigma^2 = \frac{d^2}{dt^2}\left\{e^{\mu(e^t-1)}\right\}\Big|_{t=0}$$

$$\sigma^2 = \mu e^t\left(e^{u(e^t-1)}\right) + \mu e^t(\mu e^t)\left(e^{2u(e^t-1)}\right)\Big|_{t=0}$$

$$= \mu e^t\left(e^{u(e^t-1)}\right) + \mu^2 e^{2t}\left(e^{2u(e^t-1)}\right)\Big|_{t=0}$$

$$= \mu e^0\left(e^{\mu(1-1)}\right) + \mu^2 e^{2(0)}\left(e^{2\mu(0)}\right)$$

$$= \mu + \mu^2$$

$\sigma^2 = E(x^2) - [E(x)]^2$

$= u + \mu^2 - \mu^2$

$\boxed{\sigma^2 = \mu}$

Recurrence formula :-

$P(X=x) = \dfrac{e^{-u}\mu^x}{x!}$ — ①

put $X = x-1$ in eq ②

$P(X = x-1) = \dfrac{e^{-u}\mu^{x-1}}{(x-1)!}$

divide eq (1) by eq ②.

$\dfrac{P(X=x)}{P(X=x-1)} = \dfrac{\cancel{e^{-u}}\mu^x}{x!} \times \dfrac{(x-1)!}{\cancel{e^{-u}}\mu^{x-1}}$

$= \dfrac{\cancel{(x-1)!}}{x\cancel{(x-1)}\,u^{-1}}$

$\dfrac{P(X=x)}{P(X=x-1)} = \dfrac{\mu}{x}$

$P(X=x) = \dfrac{\mu}{x}\, P(X=x-1)$ proved

Reproductive Property :-

if two independent r.v X & Y have a poisson distribution with parmeter μ, λ, then their sum $X+Y$ has also a poisson distribution with parmeter $\mu + \lambda$

Here X is $P(x:\mu)$
Y is $P(y:\gamma)$

$P = (X+Y = k)$ for $k = 0, 1, 2 \ldots$

$k=0$, $P(X+Y=0)$, $P(X=0), P(Y=0)$

$\because$ (X and Y are independ.)

$= e^{-u} \cdot e^{-v} = e^{-(\mu-\gamma)}$

$k=1$, $P(x+y=1) = P(x=0).P(y=1)$, $P(y=0).P(x=1)$

$= e^{-\mu} \lambda e^{-\lambda} + \mu e^{-\mu} \cdot e^{-\lambda}$

$= (\mu+\gamma) e^{-(\mu+v)}$

Derivation of Negative Binomial (ae't)' :-

$$P(X=x) = \binom{x-1}{k-1} p^k q^{x-k} \qquad x = k, k+1, \ldots$$

$$\Sigma b(x; k.p) = \sum_{x=k}^{\infty} {}^{x-1}C_{k-1}\, p^k q^{x-k}$$

let $x-k = y \longrightarrow x = y+k$

$$b(x:k.p) = \Sigma f(y) = \sum_{y=0}^{\infty} \left({}^{y+k-1}C_{k-1}\right) p^k q^y$$

$$= \left[\left({}^{0+k-1}C_{k-1}\right) p^k q^0 + \left({}^{1+k-1}C_{k-1}\right) p^k q^1 + {}^{2+k-1}C_{k-1}\, p^k q^2 \ldots\right]$$

$$= \left[\left({}^{k-1}C_{k-1}\right) p^k + \left({}^{k}C_{k-1}\right) p^k q + \left({}^{k+1}C_{k-1}\right) p^k q^2 \ldots\right]$$

$$= \left[p^k + {}^{k}C_{k-1}\, p^k q + {}^{k+1}C_{k-1}\, p^k q^2 \ldots\right]$$

$$= p^k + \frac{k!}{(k-1)!(\cancel{k}-\cancel{k}+1)!} p^k q + \frac{(k+1)!}{(k-1)!(\cancel{k}+1-\cancel{k}+1)} p^k q^2 \ldots$$

$$= p^k + \frac{k}{(k-1)!\,1!} p^k q^1 + \frac{(k+1)!}{(k-1)!\,2!} p^k q^2 \ldots$$

$$= p^k + \frac{k\cancel{(k-1)!}}{\cancel{(k-1)!}\,1} p^k q + \frac{(k+1)k\cancel{(k-1)!}}{\cancel{(k-1)!}\,2!} p^k q^2 \ldots$$

$$\boxed{\because (1-x)^{-n} = 1 + nx + \frac{n(n+1)}{2!} x^2 \ldots}$$

$$\left[1 + kq + \frac{k(k+1)x^2}{2!} ---- \right] \Rightarrow (1-q)^{-k}$$

$$\Sigma f(y) = p^k (1-q)^{-k}$$

$$\boxed{\because \; 1-q = p}$$

$$\Sigma f(y) = p^k p^{-k}$$

$$\Sigma f(y) = 1$$

mgf $\leftarrow$

$$mgf = m_x(t) = E(e^{tx})$$

$$m_x(t) = \sum_{x=0}^{\infty} e^{tx} \cdot {}^{x-1}C_{k-1} \; p^k q^{x-k}$$

let $x-k = y \rightarrow x = y+k$

$$= \sum_{y=0}^{\infty} e^{tx} \left({}^{y+k-1}C_{k-1}\right) p^k q^y$$

Replace y by x

$$= \sum_{x=0}^{\infty} e^{tx} \left({}^{x+k-1}C_{k-1}\right) p^k q^x$$

$$= p^k \left[e^{t(0)} \left({}^{0+k-1}C_{k-1}\right) q^0 + e^{t(1)} \left({}^{1+k-1}C_{k-1}\right) q^1 + \right.$$

$$\left. e^{t(2)} \left({}^{2+k-1}C_{k-1}\right) q^2 ---- \right]$$

$$M_x(t) = p^k\left({}^{k-1}C_{k-1}\right)(qe^t)^0 + p^k\left({}^{k}C_{k-1}\right)(qe^t)^1 + p^k\left({}^{k+1}C_{k-1}\right)(qe^t)^2 + \text{——}$$

$$M_x(t) = \left[p^k + \frac{p^k(k)!}{(k-1)(k-k+1)!} qe^t + \frac{p^k(k+1)!}{(k-1)!(k+1-k+1)!}(qe^t)^2 + \text{——}\right]$$

$$M_x(t) = p^k\left[1 + \frac{(k)!\cancel{(k-1)!}}{\cancel{(k-1)!}\,1!}qe^t + \frac{k(k+1)\cancel{(k-1)}}{\cancel{(k-1)!}\,2!} \text{——}\right]$$

$$= p^k\left[1 + k^1(qe^t) + \frac{k(k+1)(e^tq)^2}{2!}\right]$$

$$\boxed{\because (1-x)^{-n} = 1 + nx + \frac{n(n+1)x^2}{2!} \text{——} \quad \boxed{n = k},\ \boxed{x = qe^t}}$$

$$\boxed{M_x(t) = p^k(1-qe^t)^{-k}}$$

Negative Binomial

Mean :-

$$\text{Mean} = E(X) = \frac{d}{dt}\left[M_x(t)\right]$$

$$E(X) = \frac{d}{dt}\left[p^k(1-qe^t)^{-k}\right|_{t=0}$$

$$E(X) = p^k k qe^t(1-qe^t)^{-k-1}\Big|_{t=0}$$

$$E(X) = p^k k qe^0(1-qe^0)^{-k-1}$$

$$E(X) = p^k \, kq \, (1-q)^{-k-1}$$

$$E(X) = \cancel{p^k} \, kq \, p^{-k-1}$$

$$\boxed{E(X) = \frac{kq}{p}}$$

Variance

$$\sigma^2 = Var(X) = E(X^2) - [E(X)]^2$$

$$E(X^2) = \frac{d^2}{dt^2} M_x(t) \Big|_{t=0}$$

$$E(X^2) = \frac{d^2}{dt^2} \left[p^k (1-qe^t)^{-k} \right] \Big|_{t=0}$$

$$= p^k kqe^t (1-qe^t)^{-k-1} + p^k k(k+1) q^2 e^{2t} (1-qe^t)^{-k-2} \Big|_{t=0}$$

$$= p^k kqe^0 (1-qe^0)^{-k-1} + p^k k(k+1) q^2 e^{2(0)} (1-qe^0)^{-k-2}$$

$$= p^k kq(1-q)^{-k-1} + p^k k(k+1) q^2 (1-q)^{k-2}$$

$$= \cancel{p^k} kq \, p^{-k-1} + \cancel{p^k} k(k+1) q^2 (p)^{-k-2}$$

$$= \frac{kq}{p} + \frac{k(k+1)q^2}{p^2}$$

Put value in σ^2