Chapter 9

CONTINUOUS PROBABILITY DISTRIBUTIONS

9.1. (b) The given rectangular distribution is

$$f(x) = 1, \quad -\frac{1}{2} \leq x \leq \frac{1}{2},$$

$$= 0, \quad \text{elsewhere.}$$

The *m.g.f.* is

$$M_0(t) = E(e^{Xt})$$

$$= \int_{-1/2}^{1/2} e^{xt}\, dx = \left[\frac{1}{t} \cdot e^{xt}\right]_{-1/2}^{1/2}$$

$$= \frac{1}{t}\left[e^{t/2} - e^{-t/2}\right]$$

$$= \frac{1}{t}\left[\left(1 + \frac{t}{2} + \frac{1}{2!} \cdot \frac{t^2}{4} + \frac{1}{3!} \cdot \frac{t^3}{8} + \frac{1}{4!} \cdot \frac{t^4}{16} + \ldots\right)\right.$$

$$\left. - \left(1 - \frac{t}{2} + \frac{1}{2!} \cdot \frac{t^2}{4} - \frac{1}{3!} \cdot \frac{t^3}{8} + \frac{1}{4!} \cdot \frac{t^4}{16} - \ldots\right)\right]$$

$$= \frac{2}{t}\left[\frac{t}{2} + \frac{1}{3!} \cdot \frac{t^3}{8} + \frac{1}{5!} \cdot \frac{t^5}{32} + \ldots\right]$$

$$= 1 + \frac{1}{3!} \cdot \frac{t^2}{4} + \frac{1}{5!} \cdot \frac{t^4}{16} + \ldots$$

Now μ'_r = co-efficient of $\frac{t^r}{r!}$ in the expansion of $M_0(t)$

$\therefore \quad \mu'_1 = 0, \quad \mu'_2 = \frac{1}{12}, \quad \mu'_3 = 0, \quad \mu'_4 = \frac{1}{80}.$

Hence the mean moments are,

$\mu_1 = 0, \quad \mu_2 = \frac{1}{12}, \quad \mu_3 = 0$ and $\mu_4 = \frac{1}{80}$

9.2. (b) Here the mean, i.e. $\frac{1}{\lambda} = 12$, so that $\lambda = \frac{1}{12}$.

Then the probability that the length of life is equal to or greater than 18 months is given by

$$P(X \geq 18) = \int_{18}^{\infty} \left(\frac{1}{12}\right) e^{-x/12}\, dx$$

$$= \left[-e^{-x/12}\right]_{18}^{\infty} = e^{-1.5} = 0.2231$$

9.3. (a) The exponential distribution is given by

$$f(x) = \frac{1}{2} e^{-x/2}, \quad 0 \leq x < \infty.$$

Now $\quad \mu = E(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx = \int_{0}^{\infty} x \cdot \frac{1}{2} \cdot e^{-x/2}\, dx$

$$= \left[-xe^{-x/2}\right]_0^{\infty} + \int_0^{\infty} e^{-x/2}\, dx \text{ (Integrating by parts)}$$

$$= 0 + \left[\frac{-e^{-x/2}}{1/2}\right]_0^{\infty} = 2$$

Again, $E(X^2) = \int_0^{\infty} x^2 \cdot \frac{1}{2} e^{-x/2}\, dx$

$$= \left[-x^2 e^{-x/2}\right]_0^{\infty} + \int_0^{\infty} 2x \cdot e^{-x/2}\, dx$$

(Integrating by parts)

$$= 0 + \frac{2E(X)}{1/2} = 4(2) = 8.$$

$\therefore \text{Var}(X) = E(X^2) - [E(X)]^2 = 8 - (2)^2 = 4.$

The $m.g.f.$ is $M_0(t) = \int_0^{\infty} e^{tx} \left(\frac{1}{2}\right) e^{-x/2}\, dx$

$$= \frac{1}{2}\int_0^\infty e^{-(1/2-t)x}\,dx$$

$$= \frac{1}{2}\left[\frac{-e^{-(1/2-t)x}}{\frac{1}{2}-t}\right]_0^\infty = \frac{\frac{1}{2}}{\frac{1}{2}-t} = \frac{1}{1-2t}$$

$$P(X>3) = \int_3^\infty \frac{1}{2}e^{-x/2}\,dx$$

$$= \left[-e^{-x/2}\right]_3^\infty = e^{-1.5} = 0.2231.$$

Now $P(X>5 \mid X>2) = \dfrac{P(X>5)}{P(X>2)}$, where

$$P(X>5) = \int_5^\infty \left(\frac{1}{2}\right)e^{-x/2}\,dx$$

$$= \left[-e^{-x/2}\right]_5^\infty = e^{-2.5} = 0.0821, \text{ and}$$

$$P(X>2) = \int_2^\infty \left(\frac{1}{2}\right)e^{-x/2}\,dx$$

$$= \left[-e^{-x/2}\right]_2^\infty = e^{-1} = 0.3679.$$

Hence $P(X>5 \mid X>2) = \dfrac{P(X>5)}{P(X>2)} = \dfrac{0.0821}{0.3679} = 0.2232.$

(b) Let the r.v.X denote the distance in kilometers travelled by customers. Then

$$P(X\le 1) = \frac{1}{5}\int_0^1 e^{-x/5}\,dx = \left[-e^{-x/5}\right]_0^1$$

$$= \left[-e^{-1/5}+1\right] = 1 - 0.8187 \quad (\because e^{-0.2} = 0.8187)$$

$$= 0.1813.$$

$$P(X\ge 15) = \frac{1}{5}\int_{15}^\infty e^{-x/5}\,dx = \left[-e^{-x/5}\right]_{15}^\infty$$

$$= \left[0 + e^{-15/5}\right] = e^{-3} = 0.0498$$

Thus the required proportions of customers are 18.13% and 4.98% respectively.

9.4. (a) The expected value of X is

$$E(X) = \int_0^\infty x \,.\, \alpha e^{-\alpha x}\, dx$$

Let $\alpha e^{-\alpha x}\, dx = dv$ and $x = u$ so that $v = -e^{-\alpha x}$ and $du = dx$. Then integrating by parts, we have

$$E(X) = \Big[-xe^{-\alpha x}\Big]_0^\infty + \int_0^\infty e^{-\alpha x}\, dx = \left[\frac{-e^{-\alpha x}}{\alpha}\right]_0^\infty = \frac{1}{\alpha}.$$

The expected value of X^2 is

$$E(X^2) = \int_0^\infty x^2 \,.\, \alpha e^{-\alpha x}\, dx$$

$$= \Big[-x^2 e^{-\alpha x}\Big]_0^\infty + 2\int_0^\infty xe^{-\alpha x}\, dx \text{ (Integrating by parts)}$$

$$= \frac{2}{\alpha}\Big[-xe^{-\alpha x}\Big]_0^\infty + \frac{2}{\alpha}\int_0^\infty e^{-\alpha x}\, dx = \frac{2}{\alpha^2}$$

$$\therefore \ \operatorname{Var}(X) = E(X^2) - [E(X)]^2 = \frac{2}{\alpha^2} - \left(\frac{1}{\alpha}\right)^2 = \frac{1}{\alpha^2}$$

Hence $\sigma = \dfrac{1}{\alpha}$.

(b) The m.g.f. of the distribution given is obtained as

$$M_0(t) = \frac{1}{\alpha}\int_0^\infty e^{tx} \,.\, e^{-x/\alpha}\, dx$$

$$= \frac{1}{\alpha}\int_0^\infty e^{-x(1/\alpha - t)}\, dx = \frac{1}{\alpha}\left[\frac{-e^{-x(1/\alpha - t)}}{\frac{1}{\alpha} - t}\right]_0^\infty$$

$$= \frac{1}{\alpha} \cdot \frac{\alpha}{1-\alpha t} = \frac{1}{1-\alpha t} = (1-\alpha t)^{-1}$$

$$= 1 + \alpha t + \alpha^2 t^2 + \alpha^3 t^3 + \alpha^4 t^4 + \ldots$$

Now μ'_r is the co-efficient of $\frac{t^r}{r!}$, therfore we get

$$\mu'_1 = \alpha, \mu'_2 = 2\alpha^2, \mu'_3 = 6\alpha^3 \text{ and } \mu'_4 = 24\alpha^4.$$

Hence the moments about the mean are obtained below:

$$\mu_1 = 0;$$

$$\mu_2 = \mu'_2 - \mu'^2_1 = 2\alpha^2 - (\alpha)^2 = \alpha^2;$$

$$\mu_3 = \mu'_3 - 3\mu'_1\mu'_2 + 2\mu'^3_1 = 6\alpha^3 - 3\alpha(2\alpha^2) + 2\alpha^3 = 2\alpha^3, \text{ and}$$

$$\mu_4 = \mu'_4 - 4\mu'_1\mu'_3 + 6\mu'^2_1\mu'_2 - 3\mu'^4_1$$

$$= 24\alpha^4 - 4\alpha(6\alpha^3) + 6\alpha^2(2\alpha^2) - 3\alpha^4 = 9\alpha^4.$$

9.5. The m.g.f. is obtained first as below:

$$M_0(t) = \int_0^\infty e^{tx} \cdot xe^{-x}\, dx = \int_0^\infty x \cdot e^{-x(1-t)}\, dx$$

Let $u = x(1-t)$ so that $dx = \frac{du}{1-t}$. Then

$$M_0(t) = \int_0^\infty \frac{u}{1-t} e^{-u} \frac{du}{1-t} = \frac{1}{(1-t)^2} \int_0^\infty u \cdot e^{-u}\, du$$

$$= \frac{1}{(1-t)^2}\Gamma(2) \qquad \text{(by Gamma function)}$$

$$= (1-t)^{-2}, \qquad t < 1.$$

Thus $M_0(t) = 1 + 2t + 3t^2 + 4t^3 + 5t^4 + \ldots$

$$= 1 + 2(t) + 3 \times 2! \frac{t^2}{2!} + 4 \times 3! \frac{t^3}{3!} + 5 \times 4! \frac{t^4}{4!} + \ldots$$

Now μ'_r is the co-efficient of $\frac{t^r}{r!}$ in the expansion of $M_0(t)$.

Thus $\mu'_1 = 2, \mu'_2 = 6, \mu'_3 = 24, \mu'_4 = 120$

Hence the moments about the mean are

$$\mu_1 = 0; \mu_2 = \mu'_2 - \mu'^2_1 = 2;$$

$$\mu_3 = \mu'_3 - 3\mu'_1\mu'_2 + 2\mu'^3_1 = 4; \text{ and}$$

$$\mu_4 = \mu'_4 - 4\mu'_1\mu'_3 + 6\mu'^2_1\mu'_2 - 3\mu'^4_1 = 24.$$

9.6. First of all, we find the value of y_0 which should be such as to make

$$\int_0^\infty y_0 e^{-x/\sigma} dx = 1 \text{ or } y_0 \left[-\sigma e^{-x/\sigma}\right]_0^\infty = 1 \text{ or } y_0\sigma = 1$$

$$\therefore \quad y_0 = \frac{1}{\sigma}$$

Moments about the origin are obtained below;

$$\mu'_r = \frac{1}{\sigma}\int_0^\infty x^r e^{-x/\sigma} dx$$

Let $\quad u = \frac{x}{\sigma}$ so that $dx = \sigma du$. Then

$$\mu'_r = \sigma^r \int_0^\infty u^r e^{-u} du$$

$$= \sigma^r \Gamma(r+1) \qquad \text{(by Gamma function)}$$

$$= r!\,\sigma^r$$

$$\therefore \quad \mu'_1 = \sigma, \; \mu'_2 = 2\sigma^2, \; \mu'_3 = 6\sigma^3 \text{ and } \mu'_4 = 24\sigma^4,$$

Now mean $= \mu'_1 = \sigma$, and

$$S.D. = \sqrt{\mu'_2 - \mu'^2_1} = \sqrt{2\sigma^2 - \sigma^2} = \sigma.$$

Moments about the mean are obtained as

$$\mu_1 = 0, \; \mu_2 = \sigma^2, \; \mu_3 = 2\sigma^3 \text{ and } \mu_4 = 9\sigma^4.$$

Hence $\quad \beta_1 = \dfrac{\mu_3^2}{\mu_2^2} = \dfrac{(2\sigma^3)^2}{(\sigma^2)^3} = \dfrac{4\sigma^6}{\sigma^6} = 4,$ and

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{9\sigma^4}{(\sigma^2)^2} = 9.$$

The first quartile, Q_1, is given by the relation

$$\int_0^{Q_1} \frac{1}{\sigma} e^{-x/\sigma}\, dx = \frac{1}{4} \text{ or } \frac{1}{\sigma}\Big[-\sigma\, e^{-x/\sigma}\Big]_0^{Q_1} = \frac{1}{4}$$

Or $\quad -e^{-Q_1/\sigma} + 1 = \dfrac{1}{4}$ or $e^{-Q_1/\sigma} = \dfrac{3}{4} \qquad \ldots \text{(A)}$

Also $\quad \displaystyle\int_0^{Q_3} \frac{1}{\sigma} e^{-x/\sigma}\, dx = \frac{3}{4}$, giving

$$e^{-Q_3/\sigma} = \frac{1}{4} \qquad \ldots \text{(B)}$$

Dividing (A) by (B), we get $e^{(Q_3 - Q_1)/\sigma} = 3$

Hence $\quad Q_3 - Q_1 = \sigma \log_e 3.$

9.7. (a) $\quad E(X) = \displaystyle\int_{-\infty}^{\infty} x f(x)\, dx = \frac{1}{2}\int_{-\infty}^{\infty} x \,.\, e^{-|x|}\, dx.$

$$= \frac{1}{2}\int_{-\infty}^{0} x e^{x}\, dx + \frac{1}{2}\int_{0}^{\infty} x e^{-x}\, dx$$

$$= -\frac{1}{2}\int_{0}^{\infty} x e^{-x}\, dx + \frac{1}{2}\int_{0}^{\infty} x e^{-x}\, dx$$

(changing x into $-x$)

$$= 0$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\, dx = \frac{1}{2}\int_{-\infty}^{\infty} x^2 e^{-|x|}\, dx$$

$$= \frac{1}{2}\int_{-\infty}^{0} x^2 e^{x}\, dx + \frac{1}{2}\int_{0}^{\infty} x^2 e^{-x}\, dx$$

$$= \frac{1}{2}\int_{0}^{\infty} x^2 e^{-x}\, dx + \frac{1}{2}\int_{0}^{\infty} x^2 . e^{-x}\, dx$$

$$= \int_{0}^{\infty} x^2 e^{-x}\, dx = \Gamma(3) = 2$$

$\therefore$ $\quad \text{Var}(X) = E(X^2) - [E(X)]^2 = 2.$

(b) The total probability shall be unity, i.e.

$$\int_{-\infty}^{\infty} \frac{a\, dx}{e^{-x} + e^{x}} = 1$$

Or $\quad \int_{-\infty}^{\infty} \frac{a\, e^{x}\, dx}{1 + e^{2x}} = 1 \quad$ (multiplying L.H.S. by $\frac{e^x}{e^x}$)

Now let $t = e^x$, then $dt = e^x\, dx$.

$\therefore$ $\quad \int_{0}^{\infty} \frac{a dt}{1 + t^2} = 1$ Or $a\left[\tan^{-1} t\right]_{0}^{\infty} = 1$

Or $\quad a\frac{\pi}{2} = 1$, or $a = \frac{2}{\pi}$

(ii) Now $P(X<1) = \frac{2}{\pi}\int_{-\infty}^{1} \frac{dx}{e^{-x} + e^{x}}$

$$= \frac{2}{\pi}\int_{-\infty}^{1} \frac{e^x}{1 + e^{2x}}\, dx \text{ [multiplying by } \frac{e^x}{e^x}]$$

$$= \frac{2}{\pi}\int_{0}^{e} \frac{dt}{1 + t^2} \quad \text{(on putting } e^x = 1)$$

$$= \frac{2}{\pi}\left[\tan^{-1} t\right]_{0}^{e}$$

$$= \frac{2}{\pi}\tan^{-1} e = \frac{2}{\pi}\tan^{-1}(2.7183)$$

$$= \frac{2}{\pi}\left(\frac{\pi}{180} \times \frac{349}{5}\right) = 0.7755.$$

Hence the probability that two independent observations will take on values less than 1 = (0.7755) (0.7755) = 0.6014.

9.8. (b) The mode is that value of x for which

(i) $f'(x) = 0$ and (ii) $f''(x) < 0$.

Here $$f(x) = \frac{1}{\Gamma(m)} e^{-x} x^{m-1}.$$

Differentiating, we get

$$f'(x) = \frac{1}{\Gamma(m)}[(-e^{-x})\, x^{m-1} + (m-1)\, x^{m-2}\, e^{-x}]$$

$$= \frac{1}{\Gamma(m)} . e^{-x} [(m-1)\, x^{m-2} - x^{m-1}]$$

Now $f'(x) = 0$ gives

$$e^{-x} [(m-1)\, x^{m-2} - x^{m-1}] = 0$$

or $$e^{-m} . x^{m-2} [(m-1) - x] = 0 \quad \text{or} \quad x = m - 1.$$

Differentiating $f'(x)$, we get

$$f''(x) = \frac{1}{\Gamma(m)}\Big[\{-e^{-x}(m-1)\, x^{m-2} - x^{m-1}\} + e^{-x}\{(m-1) \times (m-2)\, x^{m-3} - (m-1)\, x^{m-2}\}\Big]$$

$$= \frac{1}{\Gamma(m)} e^{-x}\Big[(m-1)(m-2)\, x^{m-3} - (m-1)\, x^{m-2} - (m-1)\, x^{m-2} + x^{m-1}\Big]$$

$$= \frac{1}{\Gamma(m)} e^{-x}\, x^{m-3}\, [x^2 - 2(m-1)x + (m-1)(m-2)]$$

Putting $x = m - 1$ in $f''(x)$, we get

$$f'(m-1) = \frac{1}{\Gamma(m)} e^{1-m}(m-1)^{m-3}[(m-1)^2-2(m-1)^2+(m-1)(m-2)]$$

$$= \frac{1}{\Gamma(m)} e^{1-m}(m-1)^{m-2}[-(m-1)+(m-2)]$$

$$= \frac{1}{\Gamma(m)} e^{1-m}(m-1)^{m-2}(-1)$$

which is negative for an integral value of m.

Hence mode = $m-1$.

9.9. (a) (i) The rth moment about origin of $\gamma(m)$ variate is given by

$$\mu_r' = E(X^r) = \int_0^\infty x^r f(x)\,dx = \int_0^\infty x^r . \frac{1}{\Gamma(m)} x^{m-1} . e^{-x} dx$$

$$= \frac{1}{\Gamma(m)} \int_0^\infty x^{m+r-1} . e^{-x}\,dx = \frac{1}{\Gamma(m)} \Gamma(m+r)$$

$$= \frac{(m+r-1)(m+r-2)\ldots(m+1)\,m\Gamma(m)}{\Gamma(m)}$$

$$= m(m+1)\ldots(m+r-2)(m+r-1).$$

(ii) The rth moment about origin using mgf is obtained as below:

The $m.g.f.$ of $\gamma(m)$ variate is given by

$$M_0(t) = \int_0^\infty e^{tx} . \frac{1}{\Gamma(m)} x^{m-1} e^{-x} dx = \frac{1}{\Gamma(m)} \int_0^\infty x^{m-1} e^{-x(1-t)}\,dx$$

Let $u = x(1-t)$. Then $dx = \dfrac{du}{1-t}$

$$\therefore \quad M_0(t) = \frac{1}{\Gamma(m)} \int_0^\infty \left(\frac{u}{1-t}\right)^{m-1} e^{-u} \frac{du}{1-t}$$

$$= \frac{1}{(1-t)^m} \cdot \frac{1}{\Gamma(m)} \int_0^\infty u^{m-1} e^{-u}\, du$$

$$= (1-t)^{-m}, \text{ provided that } |t| < 1.$$

Differentiating $M_0(t)$ r times $w.r.$ to t and putting $t=0$, we get

$$\mu'_r = m(m+1) \ldots (m+r-1)$$

(b) The m.g.f. of a $\gamma(m)$ variate with respect to origin is given by

$$M_0(t) = \int_0^\infty \frac{e^{tx} e^{-x} \cdot x^{m-1}}{\Gamma(m)} dx = \frac{1}{\Gamma(m)} \int_0^\infty x^{m-1} \cdot e^{-x(1-t)}\, dx$$

$$= \frac{1}{(1-t)^m} = (1-t)^{-m}, \text{ provided that } |t| < 1.$$

Similarly, the $m.g.f.$ of a $\gamma(n)$ variate is

$$M_0(t) = (1-t)^{-n}$$

Now the $m.g.f.$ of the sum is equal to the product of their $m.g.f$'s. That is

$$M_{X+Y}(t) = E[e^{t(X+Y)}], \text{ where}$$

$$X \sim \frac{1}{\Gamma(m)} \int_0^\infty x^{m-1} e^{-x}\, dx \text{ and } Y \sim \frac{1}{\Gamma(n)} \int_0^\infty y^{n-1} \cdot e^{-y}\, dy$$

Thus $M_{X+Y}(t) = M_X(t) \cdot M_Y(t)$

$$= (1-t)^{-m} (1-t)^{-n} = (1-t)^{-(m+n)}$$

But this is the $m.g.f.$ of a $\gamma(m+n)$ variate.

Hence the result.

9.10. (b) The harmonic mean, H, of the Beta distribution is given by

$$\frac{1}{H} = \int_0^1 \frac{1}{B(m,n)} \cdot x^{m-1} (1-x)^{n-1} \cdot \frac{1}{x}\, dx$$

$$= \frac{1}{B(m,n)} \int_0^1 x^{m-2} (1-x)^{n-1}\, dx$$

$$= \frac{B(m-1, n)}{B(m, n)} = \frac{\Gamma(m-1)\,\Gamma(n)}{\Gamma(m+n-1)} \div \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$$

$$= \frac{\Gamma(m-1)\,\Gamma(n)}{\Gamma(m+n-1)} \times \frac{(m+n-1)\,\Gamma(m+n-1)}{(m-1)\,\Gamma(m-1)\,\Gamma(n)} = \frac{m+n-1}{m-1}$$

Hence $H = \dfrac{m-1}{m+n-1}$.

9.11. (a) The joint probability density of X and Y is

$$f(x, y) = \frac{1}{\Gamma(m)\,\Gamma(n)} e^{-(x+y)} . x^{m-1} . y^{n-1}, \quad \begin{matrix} 0 \le x \le \infty \\ 0 \le y \le \infty \end{matrix}$$

Let $u = x + y$ and $v = \dfrac{x}{y}$, so that $x = \dfrac{uv}{1+v}$ and $y = \dfrac{u}{1+v}$.

Then the Jacobian of the transformation is

$$\frac{1}{J} = \frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ \dfrac{1}{y} & \dfrac{-x}{y^2} \end{vmatrix} = -\frac{x+y}{y^2} = -\frac{(1+v)^2}{u}$$

$$\therefore \quad |J| = \frac{u}{(1+v)^2} \text{ and } dxdy = \frac{u}{(1+v)^2}\,dudv.$$

After substitution, the joint probability density for u and v is obtained as

$$g(u,v) = \frac{1}{\Gamma(m)\,\Gamma(n)} e^{-u} \left(\frac{uv}{1+v}\right)^{m-1} . \left(\frac{u}{1+v}\right)^{n-1} . \frac{u}{(1+v)^2}$$

$$= \frac{1}{\Gamma(m)\,\Gamma(n)} . e^{-u}\, u^{m+n-1} \frac{v^{m-1}}{(1+v)^{m+n}}$$

As x and y range from 0 to ∞, the range of u is from 0 to ∞ and of v is from 0 to ∞.

By integrating out $w.r.t.u.$, we get the probability density for v as follows:

$$h(v) = \int_0^\infty g(u, v)\, du = \frac{1}{\beta(m, n)} \cdot \frac{v^{m-1}}{(1+v)^{m+n}}$$

Thus the distribution of v is the Beta distribution of the second kind.

(b) Let X be a $\gamma(l)$ variate. Then

$$f(x) = \frac{1}{\Gamma(l)} x^{l-1} \cdot e^{-x}, \qquad x > 0$$

Put $u = \sqrt{x}$, so that $du = \dfrac{dx}{2\sqrt{x}}$ or $dx = 2udu$, $\qquad 0 < u < \infty$

Then the distribution of u is

$$g(u) = \frac{1}{\Gamma(l)} (u^2)^{l-1} \cdot e^{-u^2} \cdot 2udu$$

$$= \frac{2}{\Gamma(l)} u^{2l-1} \cdot e^{-u^2} du$$

which is another form for Γ-function.

The mean value of u, the positive square root of x, is given by

$$E(u) = \int_0^\infty u\, g(u)\, du$$

$$= \frac{2}{\Gamma(l)} \int_0^\infty u.u^{2l-1}.e^{-u^2} du = \frac{2}{\Gamma(l)} \int_0^\infty e^{2l-1+1/2}.e^{-u^2} du$$

$$= \frac{\Gamma(l + 1/2)}{\Gamma(l)}$$

9.15. (a) The equation of the normal curve is

$$f(x) = ke^{-(x^2-6x+9)/24} = ke^{-(x-3)^2/24}$$

Since the area under the normal curve is unity, therefore

$$1 = k \int_{-\infty}^{\infty} e^{-(x-3)^2/24}\, dx$$

Let $\quad z = \dfrac{x-3}{\sqrt{24}}$ so that $dx = \sqrt{24}\ dz$. Then

$$1 = \sqrt{24}\ k \int_{-\infty}^{\infty} e^{-z^2}\, dz = \sqrt{24}\ k.2 \int_{0}^{\infty} e^{-z^2}\, dz$$

Again let $z^2 = v$. Then $2z\, dz = dv$ or $dz = (1/2)\, v^{-1/2}\, dv$

Thus $\quad 1 = \sqrt{24}\ k \displaystyle\int_{0}^{\infty} e^{-v}\, e^{-1/2}\, dv = \sqrt{24}\ k \sqrt{\pi}$

$$\left(\because \int_{0}^{\infty} e^{-v}\, v^{-1/2}\, dv = \frac{\Gamma(1)}{(2)} = \sqrt{\pi}\right)$$

Hence $\quad k = \dfrac{1}{\sqrt{24\pi}}$.

The equation of the normal curve may be written as

$$f(x) = \frac{1}{\sqrt{12}\ \sqrt{2\pi}}\, e^{-(x-3)^2/2(\sqrt{12})^2}$$

Comparing with the general form of the normal curve, we find that

$$\mu = 3 \text{ and } \sigma = \sqrt{12} = 3.464.$$

9.16. The Quartile Deviation, Q, is found as

$$\frac{1}{\sigma\sqrt{2\pi}} \int_{\mu-Q}^{\mu+Q} e^{-(x-\mu)^2/2\sigma^2}\, dx = \frac{1}{2}$$

Or $\quad \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{0}^{Q/\sigma} e^{-z^2/2}\, dz = \dfrac{1}{4}$, where $z = \dfrac{x-\mu}{\sigma}$

From area tables, we find that $\dfrac{Q}{\sigma} = 0.6745$

Or $\quad Q = \dfrac{2}{3}\sigma$ (approximately)

$$\therefore \qquad \frac{Q}{10} = \frac{\sigma}{15} \qquad \ldots (1)$$

The mean deviation from the mean, μ, is given by

$$M.D. = \int_{-\infty}^{\infty} |x-\mu| \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2} dx$$

$$= \frac{\sigma}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |z| \, e^{-z^2/2} dz, \text{ where } z = \frac{x-\mu}{\sigma}$$

$$= \frac{2\sigma}{\sqrt{2\pi}} \int_{0}^{\infty} z e^{-z^2/2} dz = \sigma\sqrt{\frac{2}{\pi}}$$

$$= 0.7979\,\sigma = \frac{4}{5}\sigma, \text{ approximately.}$$

$$\therefore \qquad \frac{M.D.}{12} = \frac{\sigma}{15} \qquad \ldots (2)$$

Hence from (1) and (2), we find that

$$\frac{Q}{10} = \frac{M.D.}{12} = \frac{\sigma}{15}$$

Thus the Quartile Deviation, the Mean Deviation and the Standard Deviation are approximately in the ratio of 10:12:15.

9.18. (b) The Standardized Normal Variable is

$$Z = \frac{X-\mu}{\sigma}.$$

Now (i) $\quad P(\mu \le X \le \mu + 1.54\sigma) = P\left(\frac{\mu-\mu}{\sigma} \le Z \le \frac{\mu + 1.54\sigma - \mu}{\sigma}\right)$

$$= P(0 \le Z \le 1.54)$$

$$= 0.4382 \quad \text{(From area tables)}$$

(ii) $\quad P(\mu - 1.73\sigma \le X \le \mu + 0.56\sigma)$

$$= P\left(\frac{\mu - 1.73\sigma - \mu}{\sigma} \le Z \le \frac{\mu + 0.56\sigma - \mu}{\sigma}\right)$$

$$= P(-1.73 \le Z \le 0.56)$$

$$= P(-1.73 \le Z \le 0) + P(0 \le Z \le 0.56)$$

$$= 0.4582 + 0.2123 = 0.6705$$

Hence the required percentages of area between the given intervals are 43.82% and 67.05% respectively.

9.19 (a) The $r.v. X$ is $N[0,(0.6)^2]$, so that $S.N.V.$ is

$$Z = \frac{X-0}{0.6}$$

Now (i) $P(X>0) = P\left(\frac{X-0}{0.6} > \frac{0-0}{0.6}\right)$

$$= P(Z>0) = 0.5$$

(ii) $P(0.2<X<1.8) = P\left(\frac{0.2-0}{0.6} < \frac{X-0}{0.6} < \frac{1.8-0}{0.6}\right)$

$$= P(0.33 < Z < 3.00)$$

$$= P(0 < Z < 3.00) - P(0<Z<0.33)$$

$$= 0.49865 - 0.1293$$

$$= 0.36935 \quad \text{(From area tables)}$$

(b) The S.N.V. is $\mathbf{Z = \frac{X-1}{3}}$

(i) $P(3.43 \le X \le 6.19) = P\left(\frac{3.43-1}{3} \le \frac{X-1}{3} \le \frac{6.19-1}{3}\right)$

$$= P(0.81 \le Z \le 1.73)$$

$$= P(0 \le Z \le 1.73) - P(0 \le Z \le 0.81)$$

$$= 0.4582 - 0.2910 \quad \text{(From area tables)}$$

$$= 0.1672$$

(ii) $P(-1.43 \le X \le 6.19) = P\left(\frac{-1.43-1}{3} \le \frac{X-1}{3} \le \frac{6.19-1}{3}\right)$

$$= P(-0.81 \le Z \le 1.73)$$

$$= P(-0.81 \le Z \le 0) + P(0 \le Z \le 1.73)$$

$$= 0.2910 + 0.4582 \quad \text{(From area tables)}$$

$$= 0.7492$$

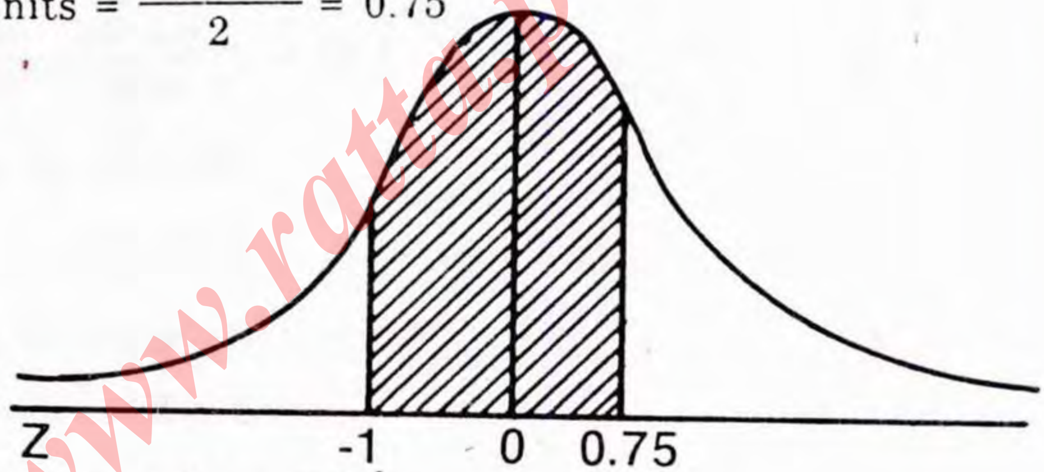
9.20. The normal distribution with $\mu = 12$ and $\sigma = 2$, is

$$f(x) = \frac{1}{2\sqrt{2\pi}} e^{-[(x-12)/2]^2/2}$$

Thus the standardized normal variate is $Z = \dfrac{X-12}{2}$.

(a) 10 in standard units $= \dfrac{10-12}{2} = -1$,

13.5 in standard units $= \dfrac{13.5-12}{2} = 0.75$



Required area

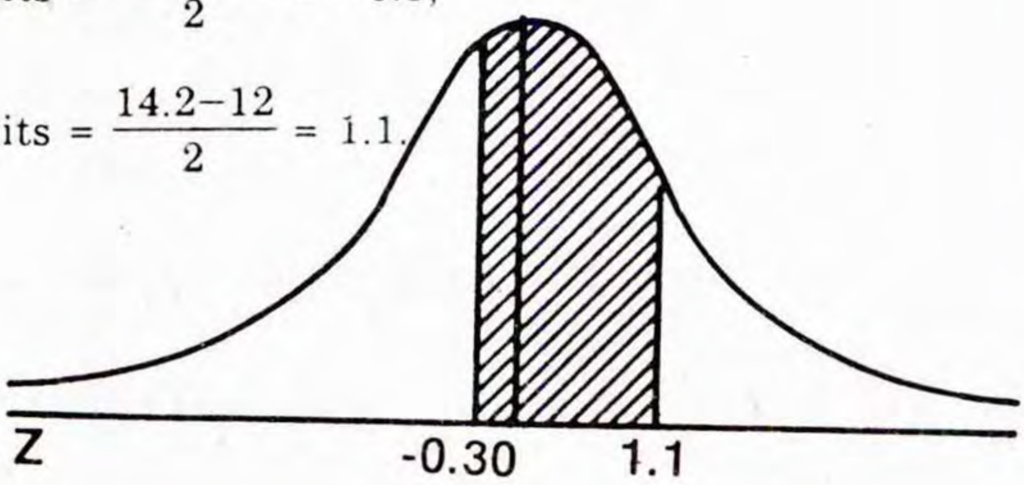
= (area between $z = -1$ and $z = 0.75$)

= (area between $z = -1$ and $z = 0$) + (area between $z = 0$ and $z = 0.75$)

= 0.3413 + 0.2734 = 0.6147 (From area tables)

(b) 11.4 in standard units $= \dfrac{11.4-12}{2} = -0.3$,

14.2 in standard units $= \dfrac{14.2-12}{2} = 1.1$,



$\therefore$ Required area

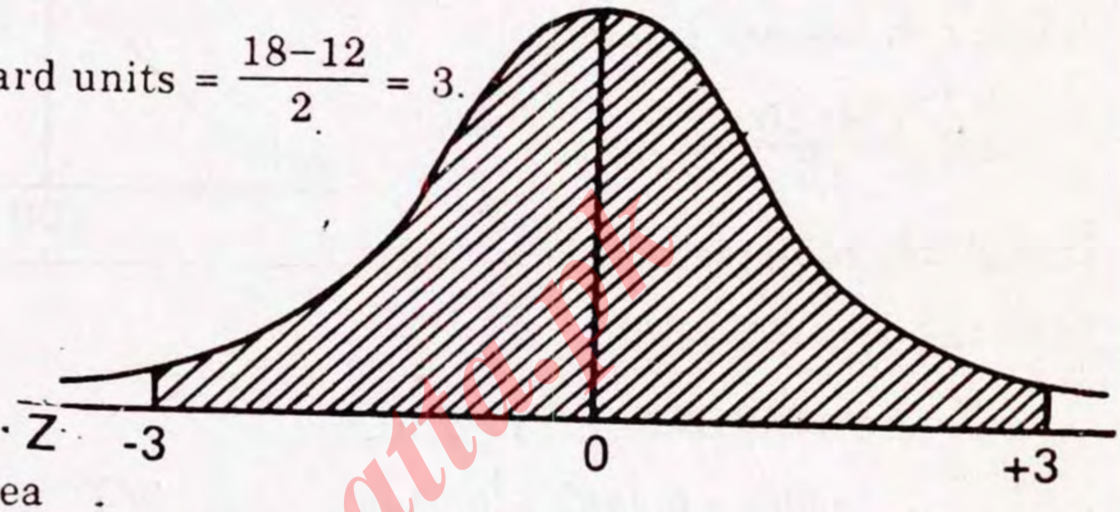
$= (\text{area between } z = -0.3 \text{ and } z = 1.1)$

$= (\text{area between } z = -0.3 \text{ and } z = 0) + (\text{area between } z = 0 \text{ and } z = 1.1)$

$= 0.1179 + 0.3643 = 0.4822$ (From area tables)

(c) 6 in standard units $= \dfrac{6-12}{2} = -3,$

18 in standard units $= \dfrac{18-12}{2} = 3.$



$\therefore$ Required area

$= (\text{area between } z = -3 \text{ and } z = 3)$

$= (\text{area between } z = -3 \text{ and } z = 0) + (\text{area between } z = 0 \text{ and } z = 3)$

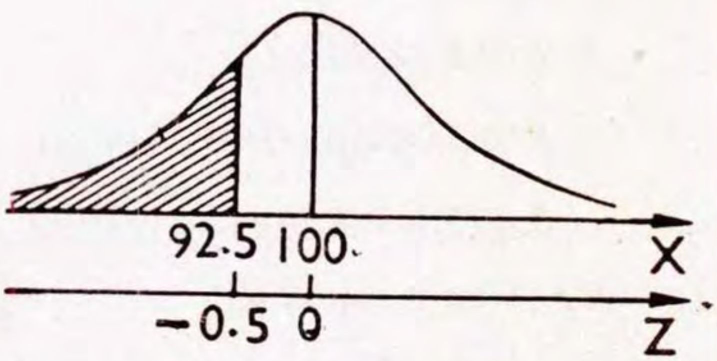
$= 0.49865 + 0.49865 = 0.9973$ (From area tables)

9.21. We draw the normal curve sketch showing x and z values, and the desired area for each part. With $\mu = 100$ and $\sigma = \sqrt{225} = 15$, we have

$$z = \frac{x-100}{15}$$

(a) At $x = 92.5$, we calculate

$$z = \frac{92.5-100}{15} = -0.5.$$



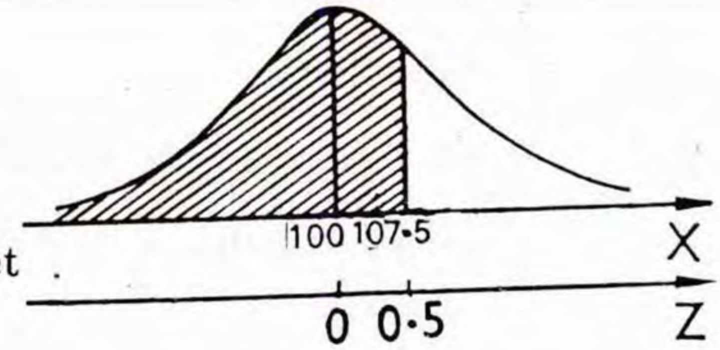
$P(X \leq 92.5) = P(Z \leq -0.5)$

$= 0.5 - P(-0.5 \leq Z \leq 0)$

$= 0.5 - 0.1915 = 0.3085$ (From area tables)

(b) At $x = 107.5$, we compute

$$z = \frac{107.5-100}{15} = 0.5.$$

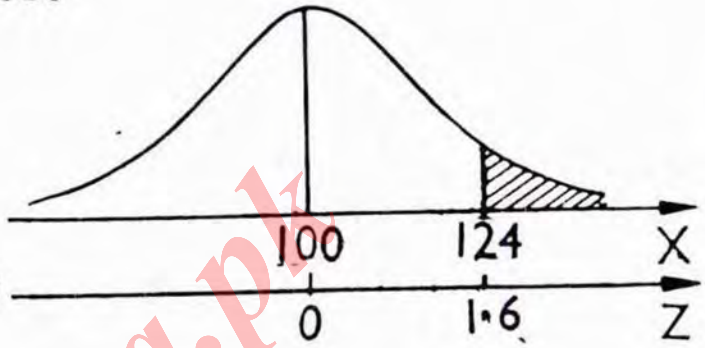


Using area table, we therefore get

$$\begin{aligned} P(X \le 107.5) &= P(Z \le 0.5) \\ &= P(-\infty \le Z \le 0) + P(0 \le Z \le 0.5) \\ &= 0.5 + 0.1915 = 0.6915 \end{aligned}$$

(c) At $x = 124$, we compute

$$z = \frac{124-100}{15} = 1.6.$$

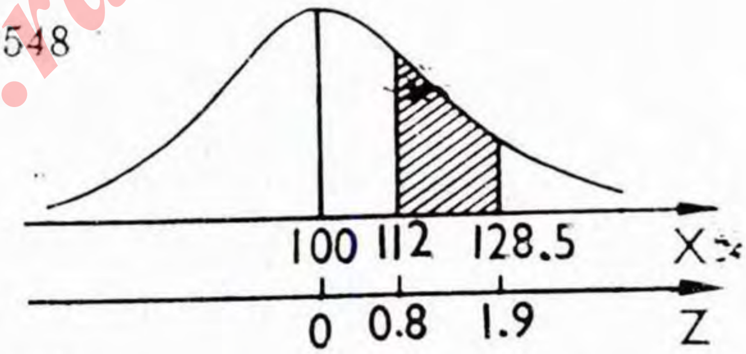


Using area table, we therefore find

$$\begin{aligned} P(X \ge 124) &= P(Z \ge 1.6) \\ &= P(0 \le Z \le \infty) - P(0 \le Z \le 1.6) \\ &= 0.5 - 0.4452 = 0.0548 \end{aligned}$$

(d) We have for $x = 112$,

$$z = \frac{112-100}{15} = 0.8, \text{ and}$$



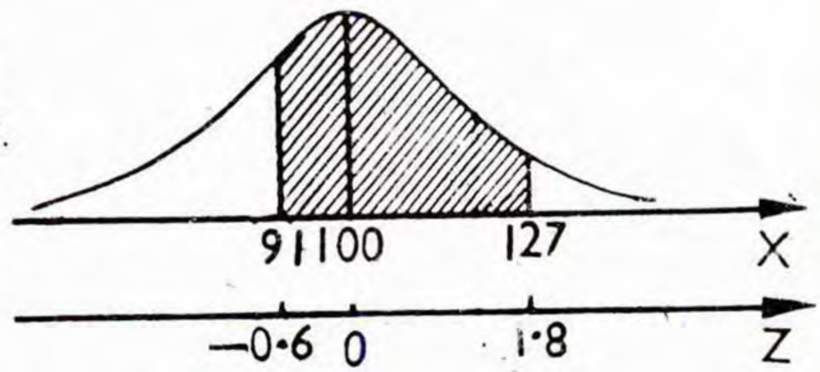
for $x = 128.5$, $z = \dfrac{128.5-100}{15} = 1.9.$

Therefore using area table, we get

$$\begin{aligned} &P(112 \le X \le 128.5) \\ &= P(0.8 \le Z \le 1.9) \\ &= P(0 \le Z \le 1.9) - P(0 \le Z \le 0.8) \\ &= 0.4713 - 0.2881 = 0.1832. \end{aligned}$$

(e) We have for $x = 91$,

$$z = \frac{91-100}{15} = -0.6, \text{ and}$$



for $x = 127$, $z = \dfrac{127-100}{15} = 1.8$.

Using area table, we therefore get

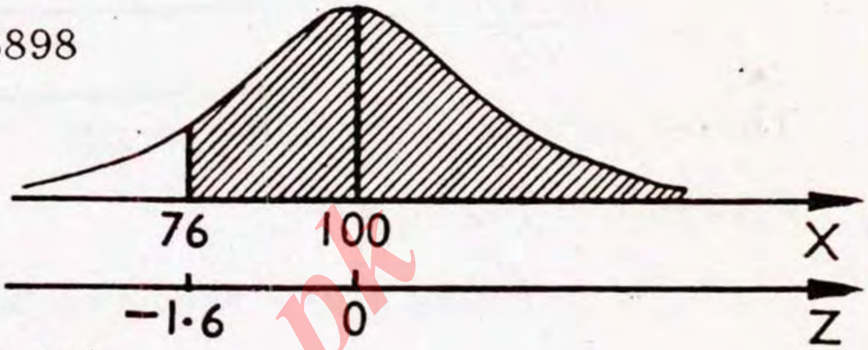
$$P(91 \le X \le 127)$$

$$= P(-0.6 \le Z \le 1.8)$$

$$= P(-0.6 \le Z \le 0) + P(0 \le Z \le 1.8)$$

$$= 0.2257 + 0.4641 = 0.6898$$

(f) At $x = 76$, we calculate



$$z = \frac{76-100}{15} = -1.6.$$

Using area table, we therefore get

$$P(X \ge 76) = P(Z \ge -1.6)$$

$$= P(-0.6 \le Z \le 0) + P(0 \le Z \le \infty)$$

$$= 0.4452 + 0.5 = 0.9452.$$

9.22. (b) Here X is normally distributed with a mean 10 and variance 25. Further $Y = 5X + 10$, therefore

$$E(Y) = 5E(X) + 10 = 5(10) + 10 = 60, \text{ and}$$

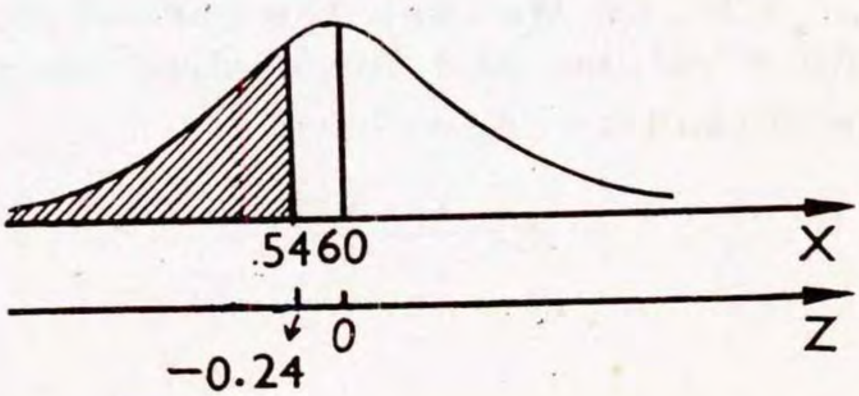
$$\text{Var}(Y) = 25\ \text{Var}(X) = 25(25) = 625.$$

Hence Y is $N\ (60, 625)$.

We draw the normal curve sketch showing x and z values and the desired area for each part. Wtih $\mu = 60$ and $\sigma = \sqrt{625} = 25$, we have

$$z = \frac{y-60}{25}$$

(i) At $y = 54$, we compute



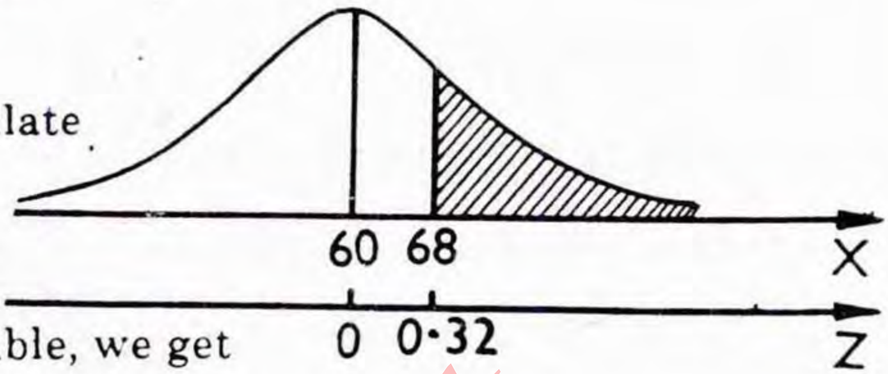
$$z = \frac{54-60}{25} = -0.24$$

Therefore using area table, we get

$$P(Y \le 54) = P(Z \le -0.24)$$
$$= 0.5 - P(-0.24 \le Z \le 0)$$
$$= 0.5 - 0.0948$$
$$= 0.4052.$$

(ii) At $y = 68$, we calculate

$$z = \frac{68-60}{25} = 0.32$$



Therefore using area table, we get

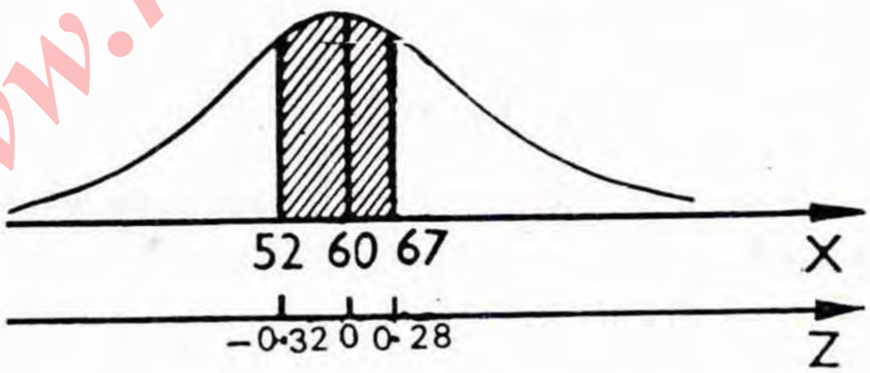
$$P(Y \ge 68) = P(Z \ge 0.32)$$
$$= P(0 \le Z < \infty) - P(0 \le Z < 0.32)$$
$$= 0.5 - 0.1255$$
$$= 0.3745.$$

(iii) We have for $y = 52$

$$z = \frac{52-60}{25} = -0.32,$$

and for $y = 67$.

$$z = \frac{67-60}{26} = 0.28.$$



Using area table, we therefore get

$$P(52 \le Y \le 67) = P(-0.32 \le Z \le 0.28))$$
$$= P(-0.32 \le Z \le 0) + P(0 \le Z \le 0.28)$$
$$= 0.1255 + 0.1103 = 0.2358.$$

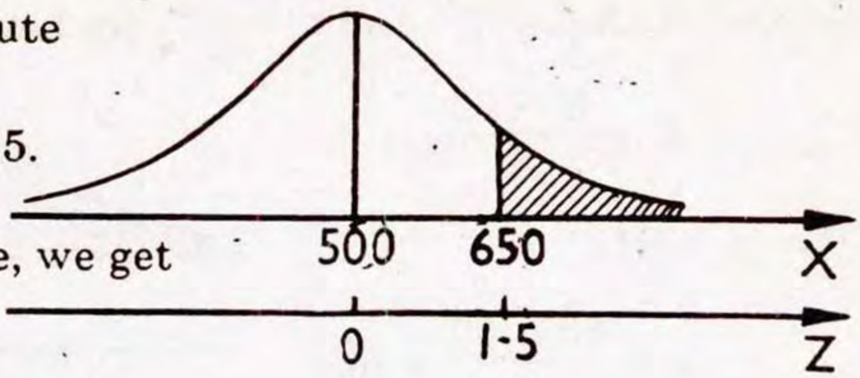
9.23. (a) We draw the normal curve sketch showing x and z values, and the desired area for each part. With $\mu = 500$ and $\sigma = 100$, we have

$$Z = \frac{X-500}{100}.$$

(i) We need the probability that a student will score over 650, *i.e.* $P(X > 650)$.

At $x = 650$, we compute

$$z = \frac{650 - 500}{100} = 1.5.$$



Hence using the area table, we get

$$P(X > 650) = P(Z > 1.5)$$

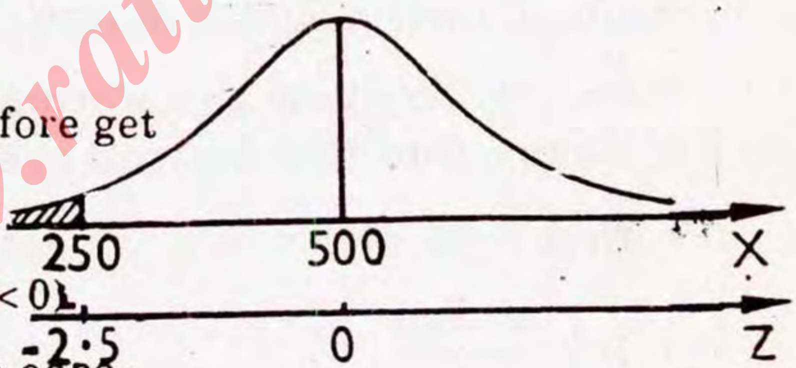
$$= 0.5 - P(0 \leq Z \leq 1.5)$$

$$= 0.5 - 0.4337 = 0.0663.$$

(ii) We need the probability that a student will score less than 250, *i.e.* $P(X < 250)$.

At $x = 250$, we compute

$$z = \frac{250 - 500}{100} = -2.5.$$



Using the area table, we therefore get

$$P(X < 250) = P(Z < -2.5)$$

$$= 0.5 - P(-2.5 < Z < 0)$$

$$= 0.5 - 0.4938 = 0.0062.$$

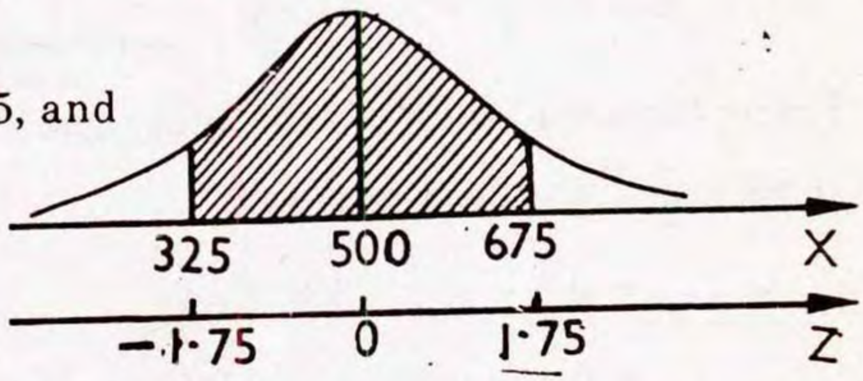
(iii) We need the probability that a student will score between 325 and 675, *i.e.* $P(325 < X < 675)$.

We have for $x = 325$,

$$z = \frac{325 - 500}{100} = -1.75, \text{ and}$$

for $x = 675$,

$$z = \frac{675 - 500}{100} = 1.75.$$



Hence using the area table, we get

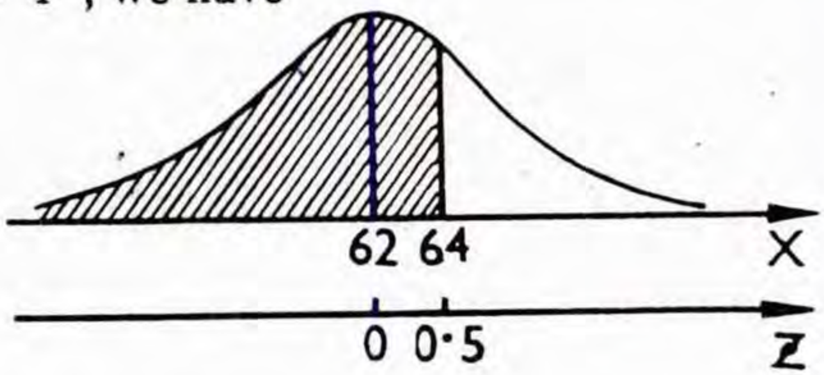
$$P(325 < X < 675) = P(-1.75 < Z < 1.75)$$

$$= P(-1.75 < Z < 0) + P(0 < Z < 1.75)$$

$$= 0.4599 + 0.4599 = 0.9198.$$

(b) The percentage of boys who would be rejected on account of their height is equal to the area under the normal curve for $X < 64$. With $\mu = 62''$ and $\sigma = 4''$, we have

$$Z = \frac{X - 62}{4}$$



For $x = 64$,

$$z = \frac{64 - 62}{4} = 0.5.$$

Therefore using the area table, we obtain

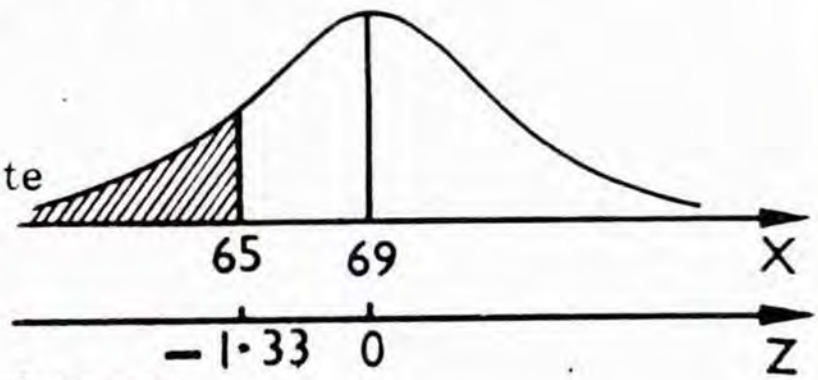
$$P(X < 64) = P(Z < 0.5)$$
$$= 0.5 + P(0 < Z < 0.5)$$
$$= 0.5 + 0.1915 = 0.6915.$$

Hence the desired percentage of boys who would be rejected on account of their height is 69.15%.

9.24. (a) We draw the normal curve sketch showing x and z vlaues, and the desired area for each part.

With $\mu = 69$ and $\sigma = 3$, we have

$$Z = \frac{X - 69}{3}.$$



(i) At $x = 65$, we compute

$$z = \frac{65 - 69}{3} = -1.33.$$

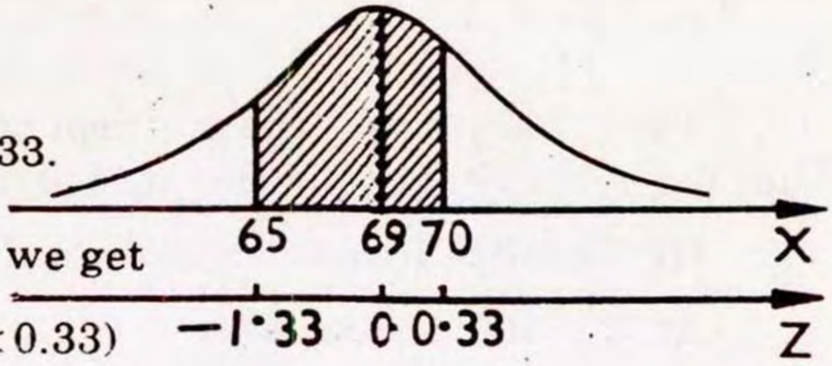
Using the area table, we therefore get

$$P(X < 65) = P(Z < -1.33)$$
$$= 0.5 - P(-1.33 < Z < 0)$$
$$= 0.5 - 0.4082 = 0.0918.$$

(ii) For $x = 65$,

$$z = \frac{65 - 69}{3} = -1.33, \text{ and}$$

for $x = 70$, $z = \frac{70-69}{3} = 0.33$.



Hence, using the area table, we get

$$P(65 \leq X \leq 70) = P(-1.33 < Z < 0.33)$$
$$= P(-1.33 < Z < 0) + P(0 < Z < 0.33)$$
$$= 0.4082 + 0.1293 = 0.5375$$

(b) Comparing $M_0(t) = e^{-6t + 32t^2}$ with the *m.g.f.* of $N(\mu, \sigma^2)$, we find that $\mu = -6$ and $\sigma^2 = 64$ or $\sigma = 8$.

To find the desired probabilities, we transform x values to z values, using $z = \frac{x-(-6)}{8}$. Therefore

at $x = -4$, we get $z = \frac{-4+6}{8} = 0.25$, and

for $x = 16$, $z = \frac{16+6}{8} = 2.75$.

Hence using the area table, we get

$$P(-4 \leq X < 16) = P(0.25 \leq Z < 2.75)$$
$$= P(0 \leq Z \leq 2.75) - P(0 \leq Z \leq 0.25)$$
$$= 0.4970 - 0.0987 = 0.3983.$$

Again, for $x = -10$, we have $z = \frac{-10+6}{8} = -0.5$, and

for $x = 0$, $z = \frac{0+6}{8} = 0.75$.

Using the area table, we therefore get

$$P(-10 < X \leq 0) = P(-0.5 < Z \leq 0.75)$$
$$= P(-0.5 < Z \leq 0) + P(0 < Z \leq 0.75)$$
$$= 0.1915 + 0.2734 = 0.4649.$$

9.25. With mean = 155 and standard deviation = 20, we have

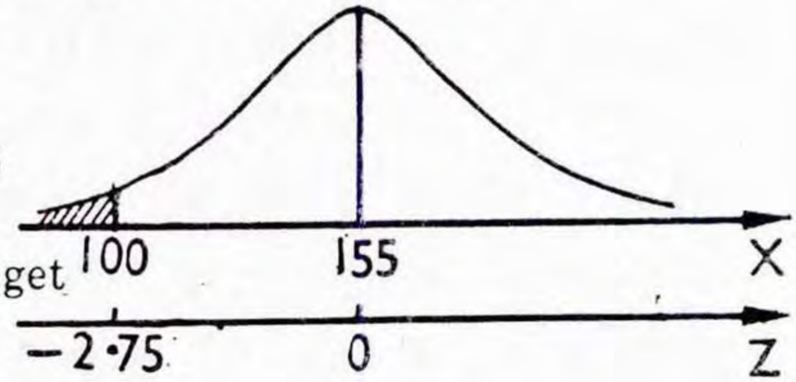
$$Z = \frac{X - 155}{20}$$

The x values and the corresponding z values are shown in normal curve sketch and the needed areas are shaded.

(i) Weights less than or equal to 100 pounds.

At $x = 100$, we compute

$$z = \frac{100 - 155}{20} = -2.75$$



Thus using the area table, we get

$$P(X \le 100) = P(Z \le -2.75)$$
$$= 0.5 - P(-2.75 \le Z \le 0)$$
$$= 0.5 - 0.4970 = 0.0030$$

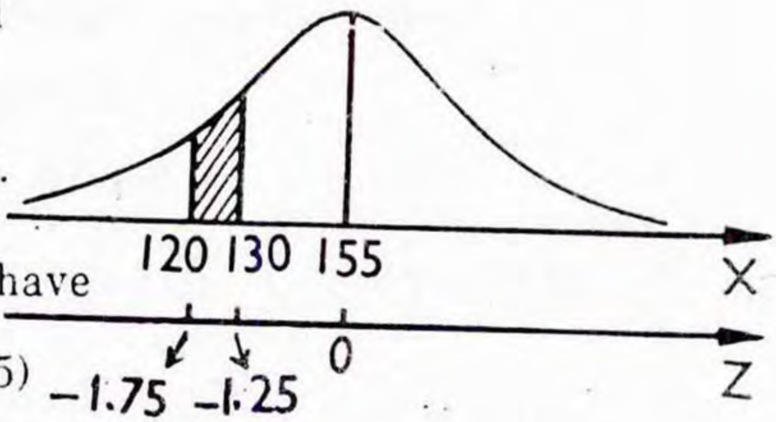
Therefore the number of students with weights less than or equal to 100 pounds is $2{,}000 \times 0.0030 = 6$.

(ii) Weights between 120 and 130 pounds.

At $x = 120$, we have

$$z = \frac{120 - 155}{20} = -1.75, \text{ and}$$

for $x = 130$, $z = \dfrac{130 - 155}{20} = -1.25$.



Using the area table, we therefore have

$$P(120 \le X \le 130) = P(-1.75 \le Z \le -1.25)$$
$$= P(-1.75 \le Z \le 0) - P(-1.25 \le Z \le 0)$$
$$= 0.4599 - 0.3944 = 0.0655.$$

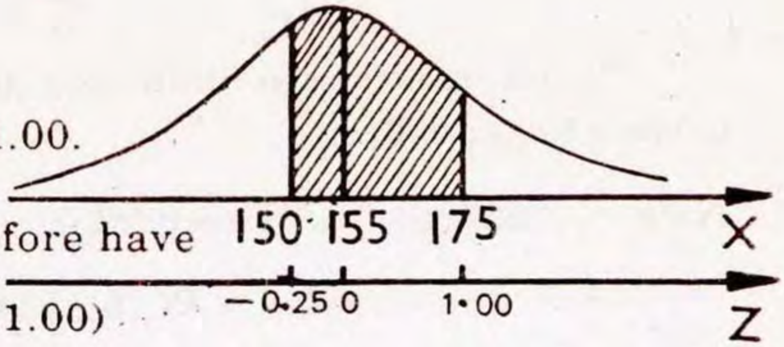
Thus the number of students with weights between 120 and 130 pounds is $2{,}000 \times 0.0655 = 131$.

(iii) Weights between 150 and 175 pounds.

At $x = 15$, we compute

$$z = \frac{150 - 155}{20} = -0.25, \text{ and}$$

for $x = 175$, $z = \dfrac{175-155}{20} = 1.00.$



Using the area table, we therefore have

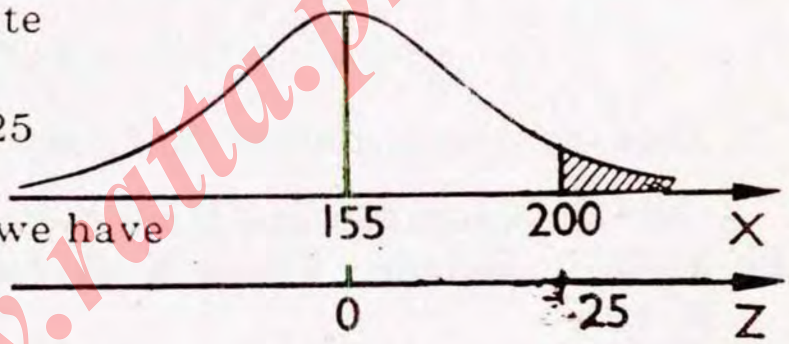
$$P(150 \le X \le 175) = P(-0.25 \le Z \le 1.00)$$
$$= P(-0.25 \le Z < 0) + P(0 \le Z \le 1.00)$$
$$= 0.0987 + 0.3413 = 0.4400$$

Thus the number of students with weights between 150 and 175 pounds is $2{,}000 \times 0.4400 = 880$.

(iv) Weights greater than or equal to 200 pounds.

At $x = 200$, we compute

$$z = \frac{200-155}{20} = 2.25$$



Thus using the area table, we have

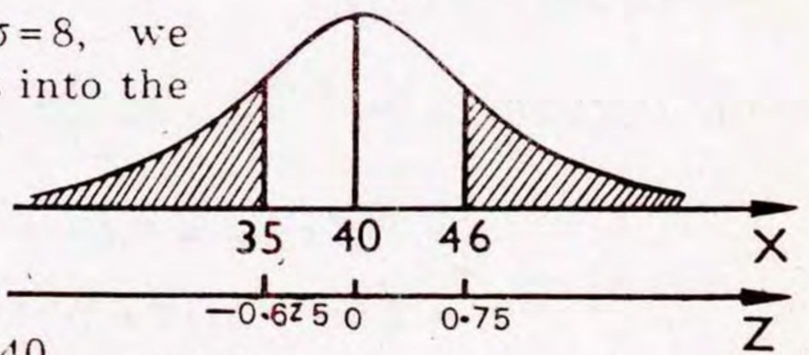
$$P(X \ge 200) = P(Z \ge 2.25)$$
$$= 0.5 - P(0 \le Z \le 2.25)$$
$$= 0.5 - 0.4878 = 0.0122.$$

Hence the number of students with weights greater than or equal to 200 pounds is $2{,}000 \times 0.0122 = 24$.

9.26. (a) The number of pairs of stockings that would need replacement is equal to the area under the normal curve for X≤35 and X≥46 days.

With $\mu = 40$ and $\sigma = 8$, we therefore change x values into the corresponding z values by

$$Z = \frac{X-40}{8}$$



Thus at $x = 35$, $z = \dfrac{35-40}{8} = -0.625.$

and at $x = 46$, $z = \dfrac{46-40}{8} = 0.75.$

These values are indicated in the figure and the needed areas are shaded.

Now $P(X \le 35) = P(Z \le -0.625)$

$$= 0.5 - P(-0.625 \le Z \le 0)$$

$$= 0.5 - 0.2340 = 0.2660.$$

This represents $100{,}000 \times 0.2660 = 26{,}600$ pairs of stockings.

Again $P(X \ge 46) = P(Z \ge 0.75)$

$$= 0.5 - P(0 \le Z \le 0.75)$$

$$= 0.5 - 0.2734 = 0.2266.$$

This represents $100{,}00 \times 0.2266 = 22{,}660$ pairs of stockings.

(b) Let X denote the time taken to deliver the milk to the G.O.R. Estate. Then X is N(12, 2^2). The S.N.V. is $Z = \dfrac{X-12}{2}$.

(i) $P(X > 17) = P\left(\dfrac{X-12}{2} > \dfrac{17-12}{2}\right)$

$$= P(Z > 2.5) = 0.5 - P(0 < Z < 2.5)$$

$$= 0.5 - 0.4938 = 0.0062$$

$\therefore$ The number of days when he takes longer than 17 minutes $= 0.0062 \times 365 = 2$ days.

(ii) $P(X < 10) = P\left(\dfrac{X-12}{2} < \dfrac{10-12}{2}\right)$

$$= P(Z < -1) = 0.5 - P(-1 < Z < 0)$$

$$= 0.5 - 0.3413 = 0.1587$$

$\therefore$ The number of days when he takes less than 10 minutes $= 0.1587 \times 365 = 58$ days.

(iii) $P(9<X<13) = P\left(\frac{9-12}{2} < \frac{X-12}{2} < \frac{13-12}{2}\right)$

$= P(-1.5 < Z < 0.5)$

$= P(-1.5 < Z < 0) + P(0<Z<0.5)$

$= 0.4332 + 0.1915 = 0.6247$

Thus the number of days when he takes between 9 and 13 minutes = $0.6247 \times 365 = 228$ days.

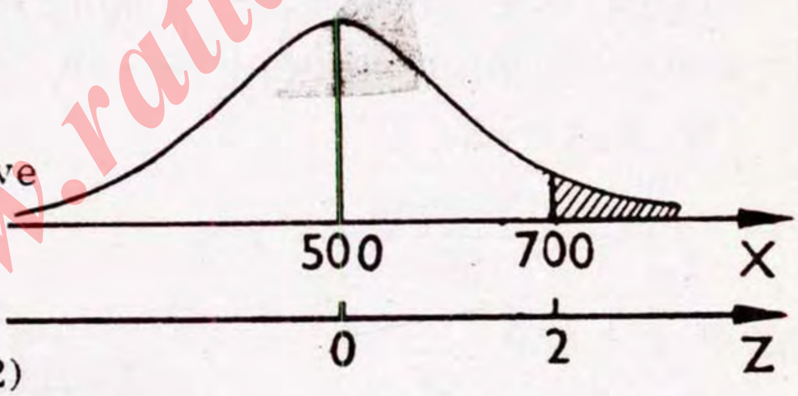
9.27. We draw the normal curve sketch showing x and z values, and the desired area for each part. With $\mu = 500$ and $\sigma = 100$, we have

$$Z = \frac{X-500}{100}$$

(a) (i) *Candidates receiving scores greater than 700.*

At $x = 700$, we compute

$$z = \frac{700-500}{100} = 2.$$



Using the area table, we therefore get

$P(X \geq 700) = P(Z>2)$

$= 0.5 - P(0 \leq Z \leq 2)$

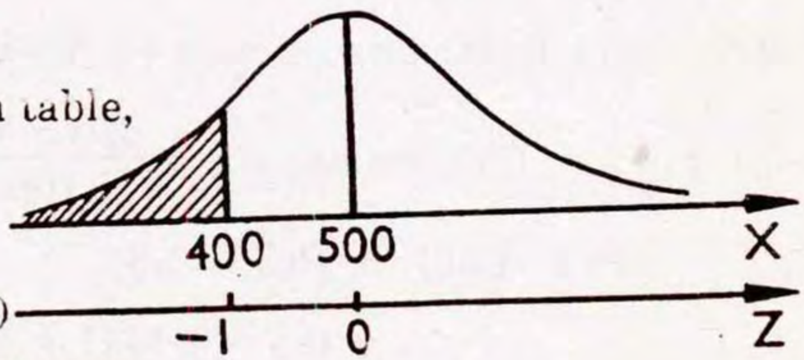
$= 0.5 - 0.4772 = 0.0228$

Hence the required %age = 2.28.

(ii) *Candidates receiving scores less than 400.*

At $x = 400$, we have

$$z = \frac{400-500}{100} = -1.$$



Therefore using the area table, we get

$P(X<400) = P(Z \leq -1)$

$= 0.5 - P(-1 \leq Z \leq 0)$

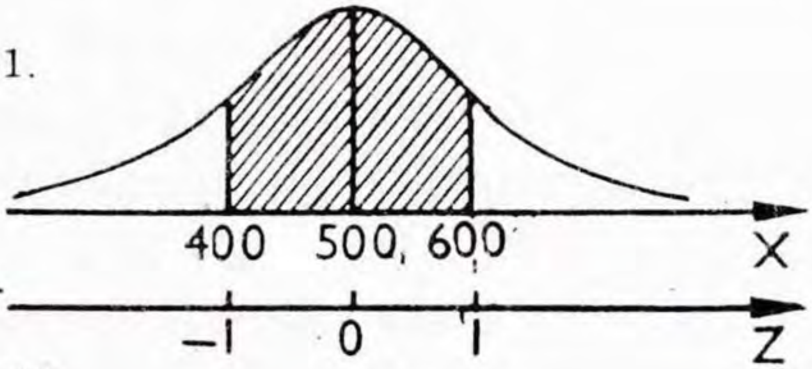
$= 0.5 - 0.3413 = 0.1587$

Hence the desired %age = 15.87.

(iii) *Candidates receiving scores between 400 and 600.*

At $x = 400$, we have

$$z = \frac{400 - 500}{100} = -1.$$



and at $x = 600$,

$$z = \frac{600 - 500}{100} = 1.$$

Using the area table, we therefore get

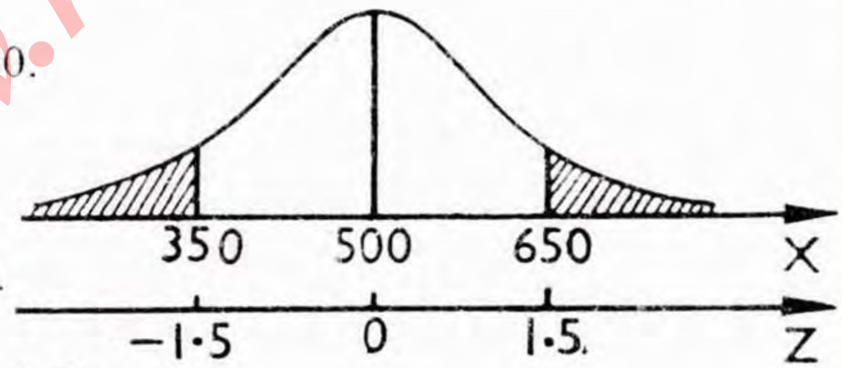
$$P(400 \le X \le 600) = P(-1 \le Z \le 1)$$
$$= 2P(0 \le Z \le 1) = 2(0.3413) = 0.6826$$

Hence the desired %age = 68.26.

(iv) *Which differ from mean by more than 150.* This implies that the scores should be either less than 350 or more than 650.

At $x = 350$,

$$z = \frac{350 - 500}{100} = -1.50.$$



and at $x = 650$,

$$z = \frac{650 - 500}{100} = 1.50.$$

Now $P(X \le 350) = P(Z \le -1.5)$

$= 0.5 - 0.4332 = 0.0668$, and

$P(X \ge 650) = P(Z \le 1.5) = 0.5 - 0.4332 = 0.0668.$

Hence the desired proportion = 0.0668 + 0.0668 = 0.1336 or 13.36%.

(b) For $x = 680$, we have $z = \frac{680 - 500}{100} = 1.8$. Therefore

$P(X > 680) = P(Z > 1.8)$

$= 0.5 - 0.4641 = 0.0359.$

Hence the per cent of candiates having higher scores than 680 is 3.59%.

9.28. The better route would be one which gives the smaller probability of the man's being late for the appointment.

Let X denote the time taken for the journey by the *route through the city centre.* Then with $\mu = 27$ and $\sigma = 5$, we have

$$Z = \frac{X - 27}{5}. \text{ Thus}$$

(i) at $x = 28$, we compute $z = \frac{28 - 27}{5} = 0.2$, and

(ii) at $x = 32$, we calculate $z = \frac{32 - 27}{5} = 1.0$.

Therefore the probabilities of being late are

(i) $P(X \geq 28) = P(Z > 0.2)$

$= 0.5 - P(0 \leq Z \leq 0.2) = 0.5 - 0.0793 = 0.4207$, and

(ii) $P(X \geq 32) = P(Z > 1.0)$

$= 0.5 - P(0 \leq Z \leq 1.0) = 0.5 - 0.3413 = 0.1583$.

Again, t Y denote the time taken for the journey by the *new route*. T en with $\mu = 29$ and $\sigma = 2$, we have $Z = \frac{Y - 29}{2}$. Thus

(i) at $y = 28$, we compute $z = \frac{28 - 29}{2} = -0.5$, and

(ii) at $y = 32$, we have $z = \frac{32 - 29}{2} = 1.5$.

Therefore the probabilities of being late are

(i) $P(Y \geq 28) = P(Z \geq -0.5)$

$= 0.5 + P(-0.5 \leq Z \leq 0)$

$= 0.5 + 0.1915 = 0.6915$, and

(ii) $P(Y \geq 32) = P(Z \geq 1.5)$

$= 0.5 - P(0 \leq Z \leq 1.5)$

$= 0.5 - 0.4332 = 0.0668$.

Hence comparing the probabilities of being late, we find that the route through the city centre is better if the man has 28 minutes, but if he has 32 minutes, the new route is better.

9.29. (a) Given P(X>a) = 0.5636, indicating that a must be less than the mean, 100.

$$\text{Now} \qquad P\left(\frac{X-100}{4} > \frac{a-100}{4}\right) = 0.5636$$

$$\text{i.e.} \qquad P\left(Z > \frac{a-100}{4}\right) = 0.5636$$

Since a is less than 100, so $\frac{a-100}{4}$ is negative and can be expressed as $-\left(\frac{100-a}{4}\right)$.

$$\text{Thus} \qquad P\left(Z > -\frac{100-a}{4}\right) = 0.5636 = 0.0636 + 0.5$$

$$= P(-z<0) + P(z>0)$$

Corresponding to an area of 0.0636, from area tables (under normal curve) we find that $z = 0.16$, so

$$\frac{100-a}{4} = 0.16 \text{ or } 100 - a = 0.64$$

which gives $a = 99.36$.

(b) For a normal distribution with mean μ and variance σ^2, $Q.D. = 0.6745\sigma$ (see property 10 one page 383 of the text).

Let x be the point P_{10} where P_{10} is a point at or below which 10% of the area lies. Then the area to the left of x is 0.1 and the area between μ and x is $0.5-0.1 = 0.4$.

Now $(z/P = 0.4) = 1.28$, where $z = (x-\mu)/\sigma$

So $\frac{x-\mu}{\sigma} = -1.28$, or $x = \mu - 1.28\sigma$ (x lies to the left of μ)

Thus $P_{10} = \mu - 1.28\sigma$

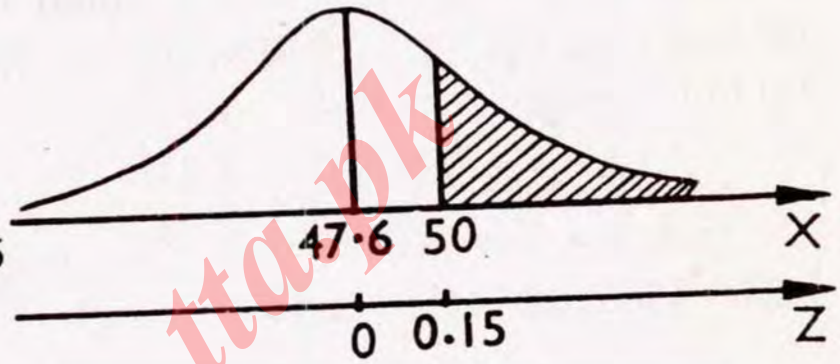
Similarly, we find that $P_{90} = \mu + 1.28\sigma$

$$\text{Hence} \quad k = \frac{Q.D.}{P_{90} - P_{10}} = \frac{0.6745\sigma}{(\mu + 1.28\sigma) - (\mu - 1.28\sigma)}$$

$$= \frac{0.6745\sigma}{2.56\sigma} = 0.263.$$

9.30. We draw the normal curve sketch showing x and z values, and the desired area for each part. With $\mu = 47.6$ and $\sigma = 16.2$, we have

$$Z = \frac{X - 47.6}{16.2}$$



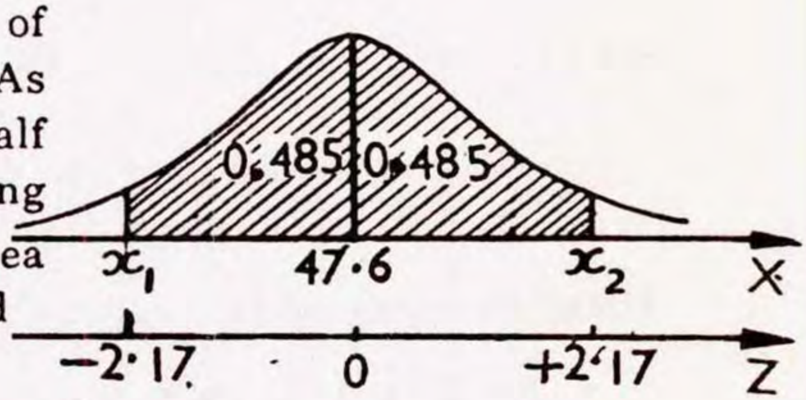
(i) At $x = 50$, we compute

$$z = \frac{50 - 47.6}{16.2} = 0.15$$

Using the area table, we therefore get

$$P(X > 50) = P(Z > 0.15)$$
$$= 0.5 - P(0 < Z < 0.15)$$
$$= 0.5 - 0.0596 = 0.4404.$$

(ii) Let x_1 and x_2 be the two points between which the probability of an observation falling is 0.97. As the curve is symmetrical, so half of 0.97, *i.e.* 0.485 is the area lying on either side of μ. Using area table inversely, we therefore find



$$(|z|\ P = 0.485) = 2.17.$$

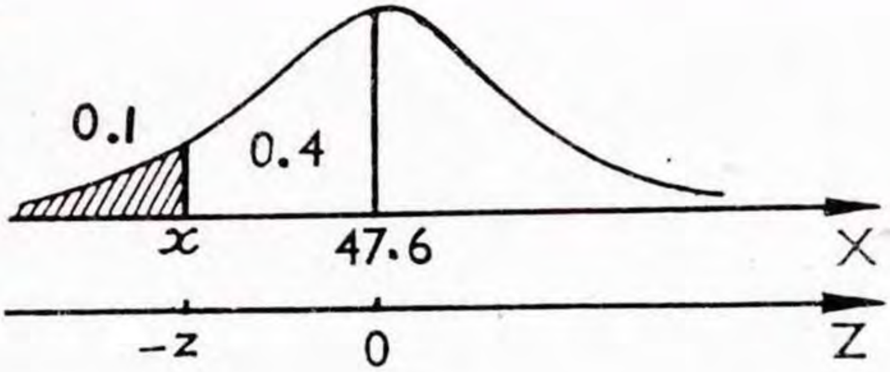
Since x_1 lies to the left of μ, therefore z is negative at this point. Hence $x_1 = \mu + z\sigma$

$$= 47.6 + (-2.17)(16.2) = 12.4.$$

Again, as x_2 lies to the right of μ, therefore z is positive at this point, and $x_2 = \mu + z\sigma$

$$= 47.6 + (2.17)(16.2) = 82.8.$$

(iii) Let x be the point P_{10}, where P_{10} is a point at or below which 10% of the area lies. Then the area to the left of x is 0.1 and area between μ and x is $0.5-0.1=0.4$



Looking at the area table, we find that a probability of 0.4 does not appear in the table, so we take the closest probability to 0.4, which is 0.3997. Thus, using the area table inversely, we find

$$(z \mid P = 0.3997) = 1.28$$

Since x lies to the left of μ, therefore z is negative at this point. Hence $x = \mu + z\sigma$

$$= 47.6 + (-1.28)(16.2) = 26.86.$$

Similarly, we find that $P_{30} = 39.11$ and $P_{99} = 85.3$.

9.31. (a) We first transform x values to z values, using

$$Z = \frac{X - 14.40}{2.50}$$

(i) At $x = 12$, we have $z = \frac{12 - 14.40}{2.50} = -0.96$, and

at $x = 16$, $z = \frac{16 - 14.40}{2.50} = 0.64$.

Using the area table, we therefore get

$$P(12 \le X \le 16) = P(-0.96 \le Z \le 0.64)$$

$$= P(-0.96 \le Z \le 0) + P(0 \le Z \le 0.64)$$

$$= 0.3315 + 0.2389 = 0.5704.$$

Hence the number of rods between 12 and 16 metres is

$$1{,}000 \times 0.5704 = 570 \text{ rods.}$$

(ii) At $x = 15$, we compute $z = \dfrac{15 - 14.40}{2.50} = 0.24.$

Therefore using the area table, we get

$$\begin{aligned} P(X > 15) &= P(Z \geq 0.24) \\ &= 0.5 - P(0 \leq Z \leq 0.24) \\ &= 0.5 - 0.0948 = 0.4052. \end{aligned}$$

(b) With mena = 34.4 and s.d. = 16.6, we have

$$Z = \frac{X - 34.4}{16.6}$$

At $x = 30$, we compute $z = \dfrac{30 - 34.4}{16.6} = -0.27$, and

at $x = 50$, we have $z = \dfrac{60 - 34.4}{16.6} = 1.54$

Therefore using the area table, we obtain

$$\begin{aligned} P(30 \leq X \leq 60) &= P(-0.27 \leq Z \leq 1.54) \\ &= P(-0.27 \leq Z \leq 0) + P(0 \leq Z \leq 1.54) \\ &= 0.1064 + 0.4384 = 0.5446 \end{aligned}$$

Hence the number of students expected to obtain marks between 30 and 60 is $1{,}000 \times 0.5446 = 545$.

The central 70% of the candidates implies that 35% of the candidates will lie on either side of the mean, *i.e.* the area lying on either side of μ is 0.35.

Using area table inversely, we therefore find

$$(z \mid P = 0.35) = 1.04$$

$$\begin{aligned} \text{Thus the limits of marks } &= \mu \pm z\sigma \\ &= 34.4 \pm (1.04)(16.6) \\ &= 34.4 \pm 17.26 = 17.14,\ 51.68 \end{aligned}$$

Hence the desired limits of marks of the central 70% of the candidates are 17 and 52.

9.32. Let X be the r.v. the amount of drink in milliliters. Then X is N[200, (15)^2].

$$\text{The S.N.V. is } Z = \frac{X-\mu}{\sigma} = \frac{X-200}{15}$$

(a) Now $P(X > 240) = P\left(\frac{X-200}{15} > \frac{240-200}{15}\right)$

$$= P(Z > 2.67) = 0.5 - P(0 < Z < 2.67)$$

$$= 0.5 - 0.4962 = 0.0038$$

The required fraction is therefore 0.38%.

(b) $P(191 < X < 209) = P\left(\frac{191-200}{15} < \frac{X-200}{15} < \frac{209-200}{15}\right)$

$$= P(-0.6 < Z < 0.6)$$

$$= P(-0.6 < Z < 0) + P(0 < Z < 0.6)$$

$$= 0.2257 + 0.2257 = 0.4514$$

(b) A cup will likely overflow if it contains 230 or more than 230 milliliters.

$$P(X \geq 230) = P\left(\frac{X-200}{15} \geq \frac{230-200}{15}\right)$$

$$= P(Z \geq 2) = 0.5 - 0.4772 = 0.0228$$

Thus the required number of cups is

$$1000 \times 0.0228 = 23$$

(d) Let x be the value below which the smallest 25% of the drinks lie. Then the area to the left of x is 0.25. Looking the area tables inversely, we find that

$$(z / P = 0.25) = 0.675$$

$$x = \mu + z\sigma = 200 + (-0.675)(15)$$

$$= 200 - 10.125 = 189.875$$

9.33. (a) Let X be the r.v. the height of applicant to police force. Then X is normally distributed with $\mu = 170$ cm and $\sigma = 3.8$ cm.

The *S.N.V.* is $Z = \dfrac{X - 170}{3.8}$

Let x denote the minimum acceptance height for the police force. Then the area to the left of x is 0.30 and the area between μ and x is $0.50 - 0.30 = 0.20$

Looking the are table inversely, we find that

$(z/P = 0.20) = 0.527$

As x lies to the left of μ, so z is negative at this point.

$\therefore \quad x = \mu - z(\sigma) = 170 - (0.527)(3.8) = 168$ cm

(b) Let X be the r.v. the life of motors in years. Then X is $N(10,(2)^2)$. The S.N.V. is $Z = \dfrac{X - 10}{2}$.

Let x be the number of years, the manufacturer should offer as guaranteed period. Then the area between μ and x is $0.5-0.03 = 0.47$. Looking the area table inversely, we find that

$(z/P = 0.47) = 1.88$

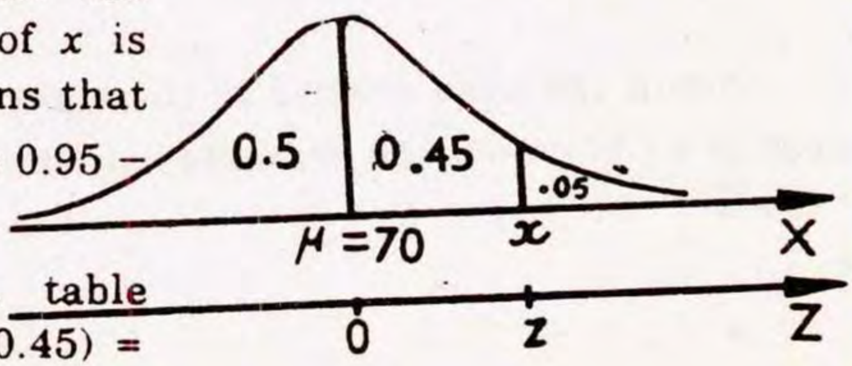
Since x lies to the left of μ, so z is negative at this point.

$\therefore \quad x = \mu - z\sigma = 10 - (1.88)(2)$

$= 10 - 3.76 = 6.24$ years.

9.34. Let x be the point corresponding to the height needed by 95% of men using the door.

Then the area under the normal curve to the left of x is 0.95 (see figure). This means that the area between μ and x is $0.95 - 0.50 = 0.45$.



Looking the area table inversely, we find $(z \mid P = 0.45) = 1.645$.

With $\mu = 70$ and $\sigma = 3$, we have $z = \dfrac{x - 70}{3}$ or $x = 70 + 3z$

As x lies to the right of μ, so z is positive at this point.

Thus $x = 70 + 3\,(1.645) = 70 + 4.935 = 75$ inches.

Since the architect wants to have at least a one-foot clearance, therefore he must make the doors $75 + 12 = 87$ inches high.

9.35. (a) Let μ be the mean and σ the standard deviation of the normal distribution. Then the two quartiles are given by

$$Q_1 = \mu - 0.6745\sigma, \text{ and } Q_3 = \mu + 0.6745\sigma$$

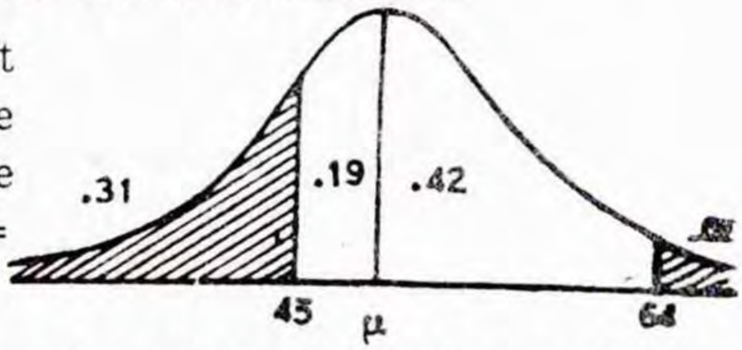
Substituting the values, we get

$$\mu - 0.6745\sigma = 8,$$

$$\mu + 0.6745\sigma = 17$$

Solving, we get $\mu = 12.5$ and $\sigma = 6.67$.

(b) Let μ and σ be the mean and the standard deviation respectively of the normal distribution.



The area shaded in the left hand tail is 0.31 (given), therefore the area lying between the ordinates at 45 and the mean = $0.5 - 0.31 = 0.19$.

From area tables, we find that the value of z corresponding to an area of 0.19 is 0.4958. Therefore

$$\frac{\mu - 45}{\sigma} = 0.4958 \text{ or } \mu - 45 = 0.4958\sigma \qquad \ldots (A)$$

Again the area shaded in the right tail is 0.08 (given) so the area lying between the ordinates at 64 and the mean = $0.5 - 0.08$ $= 0.42$.

The value of z corresponding to an area of 0.42 is 1.4053, implying that $\dfrac{64-\mu}{\sigma} = 1.4053$ or $64 - \mu = 1.4053\sigma$ ... (B)

Solving the equations (A) and (B), we get

$$\mu = 50 \text{ approx. and } \sigma = 10 \text{ approx.}$$

(c) Now $P(X<89) = 0.90$ implies that the probability between μ and 89 is $0.90 - 0.50 = 0.40$.

From area tables, we find that the value of Z corresponding to an area 0.40 is 1.28. Therefore

$$\frac{89-\mu}{\sigma} = 1.28 \quad \text{or} \quad 89 - \mu = 1.28\sigma \qquad \text{... (1)}$$

Again $P(X>94) = 0.05$ implies that the probability between μ and 94 is $0.50 - 0.05 = 0.45$.

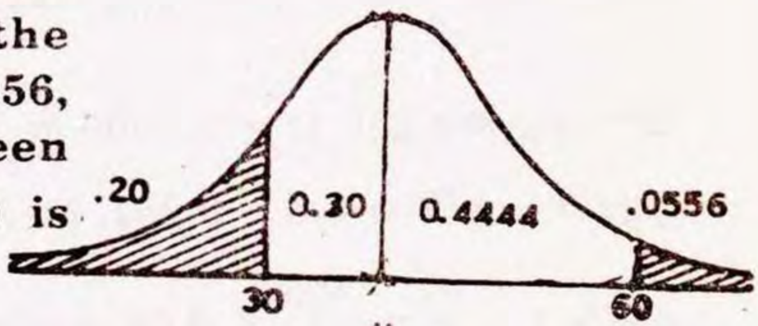
From area tables, we find that the value of Z corresponding to an area 0.45 is 1.645. Therefore

$$\frac{94-\mu}{\sigma} = 1.645 \quad \text{or} \quad 94 - \mu = 1.645\sigma \qquad \text{... (2)}$$

Solving the equations (1) and (2), we get

$$\mu = 71.464 \text{ and } \sigma = 13.70.$$

9.36. Let the total marks be 100, and let μ and σ be the mean and the standard deviation. The number of students getting 60 or more marks is 25, that is $\dfrac{25}{450}$ = 0.0556 of the total number of students. In other words, the area to the right of the ordinate at x = 60 is 0.0556, therefore the area between the ordinates at μ and 60 is 0.5000–0.0556 = 0.4444.



Corresponding to an area of 0.4444, we find from area tables that the value of z is 1.59.

$$\therefore \qquad \frac{60-\mu}{\sigma} = 1.59 \text{ or } 60-\mu = 1.59\sigma \qquad \ldots (A)$$

The number of students who failed (got less than 30 marks) is 90, that is $\frac{90}{450} = 0.20$ of the total number of students. This means that the area to the left of the ordinate at $x=30$ is 0.20. that is the area between the ordinates at μ and 30 is $0.50-0.20=0.30$. For this value of area, the value of z from area tables is 0.84.

$$\therefore \qquad \frac{\mu-30}{\sigma} = 0.84 \text{ or } \mu-30 = 0.84\sigma \qquad \ldots (B)$$

Solving the equatoins (A) and (B), we get

$\mu = 40.37$ and $\sigma = 12.35$.

9.37. Let X denote the height the boy can reach in various attempts. Then we are given

$P(X \geq 1.85) = \frac{1}{5} = 0.2$ and

$P(X \geq 1.70) = \frac{9}{10} = 0.9$, which implies $P(X \leq 1.70) = 0.1$.

Let μ and σ be the mean and standard deviation of the normal distribution of heights. Then $Z = \frac{X-\mu}{\sigma}$.

Looking the area table inversely, we find that $(z_1/P=0.3) = 0.84$ and $(z_2/P=0.4) = -1.28$ as it lies to the left of μ.

Thus $\frac{1.85-\mu}{\sigma} = 0.84$ and $\frac{1.70-\mu}{\sigma} = -1.28$.

Solving, we get $\mu = 1.7905\, m$ and $\sigma = 0.0708$.

Again, $(z/P = 0.5 - 0.001$, i.e. $0.499) = 3.08$,

so that $\frac{x-1.7905}{0.0708} = 3.08$, which gives

$x = 1.7905 + 0.2181 = 2.009$ m.

9.38. (b) Le X denote the number of deaths. Then the *p.d.* **of X is**

$$f(x) = \binom{500}{x}\left(\frac{20}{100}\right)^x\left(\frac{80}{100}\right)^{500-x}$$

and we need $P(70 \le X \le 80)$.

To use the normal curve area, the given values of X are to be adjusted for continuity. Therefore the interval of discrete values $70 \le X \le 80$ is replaced by the interval $69.5 \le X \le 80.5$.

With $\mu = np = 500 \times \frac{20}{100} = 100$, and

$$\sigma = \sqrt{npq} = \sqrt{500 \times \frac{20}{100} \times \frac{80}{100}} = 8.944,$$

we compute the z values.

At $x = 69.5$, we compute $z = \frac{69.5 - 100}{8.944} = -3.41$, and

at $x = 80.5$, we find that $z = \frac{80.5 - 100}{8.944} = -2.18$.

Hence, using the area table, we get

$$\begin{aligned} P(70 \le X \le 80) &= P(-3.41 \le Z \le -2.18) \\ &= P(-3.41 \le Z \le 0) - P(-2.18 \le Z \le 0) \\ &= 0.4997 - 0.4854 = 0.0143. \end{aligned}$$

9.39. (b) Let X denote the number of heads when a fair coin is tossed. Then the *p.d.* of X is

$$f(x) = \binom{200}{x}\left(\frac{1}{2}\right)^x\left(\frac{1}{2}\right)^{200-x}$$

To use the normal curve area, the given values of X are to be adjusted for continuity. Therefore (i) the interval of discrete values $80 \le X \le 120$ becomes $79.5 \le X \le 120.5$, (ii) the value 90 starts at 89.5, and (iii) the discrete value 100 becomes the interval 99.5 to 100.5.

With $\mu = np = 200 \times \frac{1}{2} = 100$, and $\sigma = \sqrt{npq} = \sqrt{200 \times \frac{1}{2} \times \frac{1}{2}} = 7.07$, we compute the z values.

(i) At $x = 79.5$, we find $z = \dfrac{79.5 - 100}{7.07} = -2.90$, and

at $x = 120.5$, we find $z = \dfrac{120.5 - 100}{7.07} = 2.90$.

Using the area table, we therefore get

$$\begin{aligned} P(80 \le X \le 120) &= P(-2.90 \le Z \le 2.90) \\ &= P(-2.90 \le Z \le 0) + P(0 \le Z \le 2.90) \\ &= 0.4981 + 0.4981 = 0.9962. \end{aligned}$$

(ii) At $x = 89.5$, we compute $z = \dfrac{89.5 - 100}{7.07} = -1.49$.

Therefore using the area table, we find

$$\begin{aligned} P(X < 90) &= P(Z \le -1.49) \\ &= 0.5 - P(-1.49 \le Z \le 0) = 0.5 - 0.4319 = 0.0681. \end{aligned}$$

(iii) At $x = 99.5$, we compute $z = \dfrac{99.5 - 100}{7.07} = -0.07$, and

at $x = 100.5$, we find $z = \dfrac{100.5 - 100}{7.07} = 0.07$.

Using the area table, we therefore get

$$\begin{aligned} P(X = 100) &= P(-0.07 \le Z \le 0.07) \\ &= P(-0.07 \le Z \le 0) + P(0 \le Z \le 0.07) \\ &= 0.0279 + 0.0279 = 0.0558. \end{aligned}$$

9.40. (a) Let X denote the number of heads appearing. Then the *p.d.* of X is

$$f(x) = \binom{200}{x} \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{200-x}$$

In order to use the normal curve area table, we adjust the given vlaues of X. Consequently (i) the interval of discrete values

$105 \le X \le 110$ is replaced by the interval $104.5 \le X \le 110.5$, and (ii) the discrete value 95 starts at 94.5.

With $\mu = np = 200 \times \frac{1}{2} = 100$, and $\sigma = \sqrt{200 \times \frac{1}{2} \times \frac{1}{2}} = 7.07$, we compute z values.

(i) At $x = 104.5$, $z = \dfrac{104.5 - 100}{7.07} = 0.64$, and

at $x = 110.5$, $z = \dfrac{110.5 - 100}{7.07} = 1.49$.

Therefore using the area table, we get

$$\begin{aligned} P(105 \le X \le 110) &= P(0.64 \le Z \le 1.49) \\ &= P(0 < Z \le 1.49) - P(0 \le Z \le 0.64) \\ &= 0.4319 - 0.2382 = 0.1925. \end{aligned}$$

(ii) At $x = 94.5$, $z = \dfrac{94.5 - 100}{7.07} = -0.78$.

Hence using the area table, we get

$$\begin{aligned} P(X < 95) &= P(Z < -0.78) = 0.5 - P(-0.78 < Z < 0) \\ &= 0.5 - 0.2823 = 0.2177. \end{aligned}$$

(b) **Let X denote the number of heads appearing**. Then the *p.d.* of X is

$$f(x) = \binom{15}{x}\left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{15-x}$$

(i) We need $P(6 \le X \le 9)$ by applying the binomial distribution. Thus

$$\begin{aligned} P(6 \le X \le 9) &= \sum_{x=6}^{9} \binom{15}{x}\left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{15-x} \\ &= \left(\frac{1}{2}\right)^{15} \left[\binom{15}{6} + \binom{15}{7} + \binom{15}{8} + \binom{15}{9}\right] \\ &= \frac{1}{32768}\left[5005 + 6435 + 6435 + 5005\right] \\ &= \frac{22880}{32768} = 0.6982 \end{aligned}$$

(ii) To use the normal curve area table, the given values of X are to be adjusted for continuity. Accordingly, the interval of discrete values $6 \le X \le 9$ is replaced by the interval $5.5 \le X \le 9.5$.

With $\mu = np = 15 \times \frac{1}{2} = 7.5$, and $\sigma = \sqrt{npq} = \sqrt{15 \times \frac{1}{2} \times \frac{1}{2}} = 1.94$, we compute z values. Thus

$$\text{at } x = 5.5,\ z = \frac{5.5 - 7.5}{1.94} = -1.03, \text{ and}$$

$$\text{at } x = 9.5,\ z = \frac{9.5 - 7.5}{1.94} = 1.03$$

Hence using the area table, we get

$$\begin{aligned} P(6 \le X \le 9) &= P(-1.03 \le Z \le 1.03) \\ &= P(-1.03 \le Z \le 0) + P(0 \le Z \le 1.03) \\ &= 0.3485 + 0.3485 = 0.6970. \end{aligned}$$

9.41. (a) Let the r.v.X be the number of incoming calls in one minute. Then, assuming that the number of incoming calls in one minute to have the Poisson distribution, we have

$$P(X = x) = \frac{e^{-5} \cdot (5)^x}{x!} \text{ for } x = 0, 1, 2, \ldots$$

If Y be the number of incoming calls in 20 minutes, then Y has the Poisson distribution with mean $= 5 \times 20 = 100$. We need $P(Y \le 102)$. To find the required probability, we use a normal approximatioin as the mean of the distribution is large. So Y is $N(100, 100)$ as the mean and variance of the Poisson distribution are equal.

$$\begin{aligned} \text{Hence } P(Y \le 102) &= P\left(\frac{(Y + 1/2) - 100}{10} \le \frac{(102 + 1/2) - 100}{10}\right) \\ &= P(Z \le 0.25) = 0.5 + P(0 \le Z \le 0.25) \\ &= 0.5 + 0.0987 = 0.5987 \end{aligned}$$

(b) Given X is $b(x; 20, 0.4)$.

Now $$P(6 \le X \le 10) = \sum_{x=6}^{10} b(x; 20, 0.4)$$

$$= \sum_{x=0}^{10} b(x; 20, 0.4) - \sum_{x=0}^{5} b(x; 20, 0.4)$$

$$= 0.8724 - 0.1255 = 0.7469$$

Approximations using

(i) The Poisson distribution: We have $\mu = np = 20 \times 0.4 = 8$

So $$P(X=x) = \frac{e^{-8} \cdot (8)^x}{x!}$$

Now $$P(X=6) = \frac{e^{-8} \cdot (8)^6}{6!} = 0.122138$$

Using the recurrence formula $P(X=x) = \frac{\mu}{x} \cdot P(X=x-1)$, we get

$$P(X=7) = \frac{8}{7} P(X=6) = \frac{8}{7} \times 0.122138 = 0.139586$$

$$P(X=8) = \frac{8}{8}(0.139586) = 0.139586$$

$$P(X=9) = \frac{8}{9}(0.139586) = 0.124076$$

$$P(X=10) = \frac{8}{10} P(X=9) = 0.099261$$

$\therefore$ $P(6 \le X \le 10) = P(X=6) + P(X=7) + P(X=8) + P(X=9) + P(X=10) = 0.6246.$

This answer shows a poor approximation.

(ii) The normal distribution:

Here X is $N(np, npq)$, where

$$np = (20)(0.4) = 8 \text{ and } npq = (20)(0.4)(0.6) = 4.8$$

Adjusting the values of X for continuity, the interval of discrete values $6 \leq X \leq 10$ is replaced by the interval $5.5 \leq X \leq 10.5$. Therefore

$$P(5.5 \leq X \leq 10.5) = P\left(\frac{5.5-8}{2.19} \leq \frac{X-8}{2.19} \leq \frac{10.5-8}{2.19}\right)$$

$$= P(-1.14 \leq Z \leq 1.14)$$

$$= 0.3729 + 0.3729 = 0.7458.$$

9.42. (a) From ordinates table, we find that

(i) Ordinate at $z = 0.064$ is 0.3981;

(ii) Ordinate at $z = 1.27$ is 0.1781;

(iii) Ordinate at $z = -0.84 =$ Ordinate at $z = 0.84 = 0.2803$,

and (iv) Ordinate at $z = -2.08 = 0.0459$

(b) **As the bulk of the normal distribution lies between** $\mu - 3\sigma$ and $\mu + 3\sigma$, so the range of classes would be $27.0 \pm 3(2.2)$, *i.e.* 20.4 to 33.6.

As range = 13.2, we use 7 classes with a common width $h = 2$. We then construct the classes as 20.0–22.0, 22.0–24.0, ..., 32.0–34.0. The necessary computations are shown below:

Classes	Upper class boundary	$z = \frac{ucb - \mu}{\sigma}$	$P(Z < z)$ $\Phi(z)$	Probab. $(\hat{p})$	Expected freq.
Upto 22.0	22.0	−2.27	0.0116	0.0116	2
22.0-24.0	24.0	−1.36	0.0869	0.0753	16
24.0-26.0	26.0	−0.45	0.3264	0.2395	50
26.0-28.0	28.0	+0.45	0.6736	0.3472	73
28.0-30.0	30.0	1.36	0.9131	0.2395	50
30.0-32.0	32.0	2.27	0.9884	0.0753	16
Over 32.0	∞	∞	1.0000	0.0116	2

9.43. (a) We first calculate the mean and the standard deviation of the given distribution.

Stature (x)	f	$D(=x-68)$	fD	fD^2
61	2	−7	−14	98
62	10	−6	−60	360
63	11	−5	−55	275
64	38	−4	−152	608
65	57	−3	−171	513
66	93	−2	−186	372
67	106	−1	−106	106
68	126	0	−744	0
69	109	1	109	109
70	87	2	174	348
71	75	3	225	675
72	23	4	92	368
73	9	5	45	225
74	4	6	24	144
$\sum$	750	--	+669 −75	4201

$$\bar{x} = a + \frac{\sum fD}{n} = 68 + \frac{(-75)}{750} = 67.9 \text{ inches, and}$$

$$s = \sqrt{\frac{\sum fD^2}{n} - \left(\frac{\sum fD}{n}\right)^2} = \sqrt{\frac{4201}{750} - \left(\frac{-75}{750}\right)^2} = \sqrt{5.59} = 2.36,$$

Now $\bar{x} + s = 67.9 \pm 2.36 = 65.54,\ 70.26.$

The number of students having stature in this range

$= 93 + 106 + 126 + 109 + 87 = 521$

$\therefore$ Proportion $= \dfrac{521}{750} \times 100 = 69\%$

$\bar{x} + 2s = 67.9 \pm 2(2.36) = 63.18,\ 72.62$

The number of students having stature in this range

$= 38 + 57 + 93 + 106 + 126 + 109 + 87 + 75 + 23 = 714$

$\therefore$ Proportion $= \dfrac{714}{750} \times 100 = 95\%$

$$\bar{x} + 3s = 67.9 \pm 3(2.36) = 60.82,\ \ 74.98$$

The number of students having stature in this range = 750

$\therefore$ Proportion $= \dfrac{750}{750} \times 100 = 100\%$

In a normal distribution, the proportions (from the area tables) lying between these ranges are 68%, 95% and 99.7% respectively. Hence the given distribution can be considered to be reasonably normal.

(b) **Here the mean = 67.9, and s = 2.36. The necessary computations of the expected frequencies are given below:**

(x_i)	f	ucb	$z = \dfrac{ucb - \bar{x}}{s}$	$P(Z<z)$ $\Phi(z_i)$	Probab. (p)	Expected $f_i(np)$
61	2	Upto 61.5	−2.71	0.0034	0.0034	2.55
62	10	62.5	−2.29	0.0110	0.0076	5.70
63	11	63.5	−1.86	0.0314	0.0204	15.30
64	38	64.5	−1.44	0.0749	0.0435	32.62
65	57	65.5	−1.02	0.1539	0.0790	59.25
66	93	66.5	−0.59	0.2776	0.1237	92.78
67	106	67.5	−0.17	0.4325	0.1549	116.18
68	126	68.5	0.25	0.5987	0.1662	124.65
69	109	69.5	0.68	0.7518	0.1531	114.82
70	87	70.5	1.10	0.8643	0.1125	84.38
71	75	71.5	1.53	0.9370	0.0727	54.52
72	23	72.5	1.95	0.9744	0.0374	28.05
73	9	73.5	2.37	0.9911	0.0167	12.52
74	4	$+\infty$	∞	1.0000	0.0089	6.68
Σ	750	--	--	--	--	750

9.44. We first calculate the mean and standard deviation of this distribution.

Computation of mean and standard deviation

Classes	x_i	f	$u=(x-67)/5$	fu	fu^2
40–44	42	7	−5	−35	175
45–49	47	8	−4	−32	128
50–54	52	25	−3	−75	225
55–59	57	38	−2	−76	152
60–64	62	51	−1	−51	51
65–69	67	60	0	0	0
70–74	72	45	1	45	45
75–79	77	32	2	64	128
80–84	82	10	3	30	90
85–89	87	4	4	16	64
Total		280		−114	1058

Now, $\bar{x} = a + \frac{\sum f_i u_i}{\sum f} \times h$

$$= 67 + \frac{(-114)}{280} \times 5 = 67 - 2.04 = 64.96, \text{ and}$$

$$s = h \times \sqrt{\frac{\sum fu^2}{\sum f} - \left(\frac{\sum fu}{\sum f}\right)^2}$$

$$= 5 \times \sqrt{\frac{1058}{280} - \left(\frac{-114}{280}\right)^2} = 5 \times \sqrt{3.7786 - 0.1681}$$

$$= 5 \times \sqrt{3.6105} = 5 \times (1.90) = 9.50$$

The necessary calculations of the expected frequencies of the fitted normal distribution are shown as follows:

Upper class boundary	$z = \dfrac{ucb - \bar{x}}{s}$	$P(Z<z)$ $\Phi(z_i)$	Probability $(\hat{p})$	Expected f $(n\hat{p})$
Upto 39.5	−2.68	0.0037	0.0037	
44.5	−2.15	0.0158	0.0121	4.42
49.5	−1.63	0.0515	0.0357	10.00
54.5	−1.10	0.1357	0.842	23.58
59.5	−0.57	0.2843	0.1486	41.61
64.5	−0.05	0.4801	0.1958	54.82
69.5	+0.48	0.6844	0.2043	5[illegible].[illegible]0
74.5	1.00	0.8413	0.1567	43.80
79.5	1.53	0.9370	0.0957	26.80
84.5	2.06	0.9803	0.0433	12.12
Over 84.5	+∞	1.0000	0.0197	5.52

And the necessary calculations for the heights of the ordinates appear below:

x_i	$z_i = \dfrac{x_i - \bar{x}}{s}$	$\phi(z_i)$	Ordinates $(nh/s)\,\phi(z_i)$
42	−2.42	0.0213	3.14
47	−1.89	0.0669	9.86
52	−1.36	0.1582	23.31
57	−0.84	0.2803	41.31
62	−0.31	0.3802	56.03
67	+0.21	0.3902	57.50
72	0.74	0.3034	44.71
77	1.27	0.1781	26.25
82	1.79	0.0804	11.85
87	2.32	0.0270	3.98

9.45. Let x denote the mid points. Then taking $u = (x-19)/2$, we find that $\sum fu = 70$ and $\sum fu^2 = 2092$.

Hence, in terms of the units of original measurements, $\bar{x} = 19.14$ and $s = 2.89$.

Computation of the Expected Frequencies

ucb	$z_i = \frac{ucb - \bar{x}}{s}$	$P(Z<z)$ $\phi(z_i)$	$\hat{p}$	Expected f $(n\hat{p})$
Upto 10	−3.16	0.0008	0.0008	6.8
12	−2.47	0.0068	0.0060	
14	−1.78	0.0375	0.0307	30.7
16	−1.09	0.1379	0.1004	100.4
18	−0.39	0.3483	0.2104	210.4
20	+0.30	0.6179	0.2696	269.6
22	0.99	0.8389	0.2210	221.0
24	1.68	0.9535	0.1146	114.6
26	2.37	0.9911	0.0376	37.6
∞	∞	1.0000	0.0089	8.9
Total	--	--	--	1000

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