Q#1: L'Hôpital's Rule

Suppose

(i) $f(a) = g(a) = 0$

(ii) f and g are differentiable on $(-\delta+a, a+\delta)$ and $g'(x) \neq 0$ if $x \neq a$. Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

assuming that limit on R.H.S exists.

Q#2: Extreme Value Theorem

If f is continuous on $[a,b]$ then f attains both an absolute maximum value M and an absolute minimum m in $[a,b]$.

Q#3: Give $(\epsilon-\delta)$ definition of Continuous function at a point.

A function f is said to be continuous at point "c" if for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$|f(x) - f(c)| < \epsilon \quad \text{whenever} \quad |x - c| < \delta$$

Q#4 Differential equation of exponential decay

A differential equation

$$\frac{dx}{dt} = kx$$

represents exponential growth if k is a positive constant and it represents exponential decay if k is a negative constant.

Q#5 Point of discontinuity of

$$f(x) = \frac{\ln x + \tan^{-1}x}{x^2 - 1}$$

(i) $\ln x$ is not defined on $(-\infty, 0]$ therefore this function is not continuous on $(-\infty, 0]$

Moreover

(ii) denominator $x^2 - 1 = 0$ iff $x = \pm 1$

Therefore point of discontinuity of this function are

$(-\infty, 0] \cup \{1\}$ $\left(\begin{array}{l}\underline{\text{Note}} \\ -1 \in (-\infty, 0]\end{array}\right)$

Q#6 State intermediate value Theorem.

If f is a continuous function on $[a, b]$ and if y_0 is any value between $f(a)$ and $f(b)$ then $y_0 = f(c)$ for some $c \in [a, b]$.

Q#7 Chain Rule for differentiation.

If $y = f(u)$ is a differentiable at the point $u = g(x)$ and $g(x)$ is differentiable at x then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Q#8 Where is the function $h(x) = \sin(x^2)$ continuous?

This function is continuous for every real number.

In particular all trigonometric functions ($\sin x$, $\cos x$) are continuous on every real number.

③

$\tan x = \frac{\sin x}{\cos x}$ is discontinuous on the point wherever $\cos x = 0$ i.e $x = (2n+1)\frac{\pi}{2}$; $n \in \mathbb{Z}$

Q# 9 Evaluate $\lim_{x \to 0^-} e^{\frac{1}{x}}$

$$\left\{\begin{array}{l} \lim_{x \to 0^+} \frac{1}{x} = \infty \\ \lim_{x \to 0^-} \frac{1}{x} = -\infty \\ \lim_{x \to \infty} \frac{1}{x} = 0 \\ \lim_{x \to -\infty} \frac{1}{x} = 0 \end{array}\right.$$

$$\lim_{x \to 0^-} e^{\frac{1}{x}} = e^{-\infty}$$

$$= \frac{1}{e^{\infty}}$$

$$= \frac{1}{\infty}$$

$$= 0$$

Q#10 $\frac{dy}{dx} = ?$ $y = e^{x^2 f(x)}$

$$\frac{dy}{dx} = \frac{d}{dx}\left(e^{x^2 f(x)}\right)$$

$$= e^{x^2 f(x)} \frac{d}{dx}\left(x^2 f(x)\right)$$

$$= e^{x^2 f(x)} \left[x^2 \frac{d}{dx} f(x) + f(x) \frac{d}{dx}(x^2)\right]$$

$$= e^{x^2 f(x)} \left[x^2 f'(x) + 2x f(x)\right]$$

Q#11 Give an example of increasing function

- A function is increasing if $f(x_2) > f(x_1)$ if $x_2 > x_1$ where $x_1, x_2 \in D(f)$
- A function is decreasing if $f(x_2) < f(x_1)$ if $x_2 > x_1$ where $x_1, x_2 \in D(f)$

4

First derivative Test

Let f be differentiable function on interval I

(i) If $f'(x) > 0$ on I then f is increasing on I

(ii) If $f'(x) < 0$ on I then f is decreasing on I.

Example.

→ $f(x) = x^2$ is increasing on $(0, 4)$

→ $f(x) = x^2$ is decreasing on $(-4, 0)$

(* can be verified using above test).

Q#12 Define Critical point

A number $c \in D(f)$ is called critical point of $y = f(x)$ if

$f'(c) = 0$ or $f'(c)$ does not exist.

Long Questions

Q #2(a): $y = (\sin^{-1}x)^{x^{\frac{1}{x}}}$; $\frac{dy}{dx} = ?$

(Application of logarithmic differentiation)

$$y = (\sin^{-1}x)^{x^{\frac{1}{x}}}$$

$$\Rightarrow \ln y = \ln (\sin^{-1}x)^{x^{\frac{1}{x}}}$$

$$\Rightarrow \ln y = x^{\frac{1}{x}} \ln(\sin^{-1}x)$$

Taking $\frac{d}{dx}$ on both sides

$$\frac{1}{y}\frac{dy}{dx} = x^{\frac{1}{x}} \cdot \frac{d}{dx} \ln(\sin^{-1}x) + \ln(\sin^{-1}x) \frac{d}{dx}\left(x^{\frac{1}{x}}\right)$$

$$\Rightarrow \frac{dy}{dx} = y\left[\frac{x^{\frac{1}{x}}}{(\sin^{-1}x)\sqrt{1-x^2}} + x^{\frac{1}{x}} \ln \sin^{-1}x\left(\frac{1-\ln x}{x^2}\right)\right]$$ ⑤

$$\frac{dy}{dx} = (\sin^{-1}x)^{x^{\frac{1}{x}}}\left[\frac{x^{\frac{1}{x}}}{(\sin^{-1}x)\sqrt{1-x^2}} + \frac{x^{\frac{1}{x}}\ln(\sin^{-1}x)(1-\ln x)}{x^2}\right] \text{ Ans}$$

Q#2(b) Evaluate $\lim_{x\to 2} \frac{\sqrt{6-x}-2}{\sqrt{3-x}-1}$

$$\lim_{x\to 2} \frac{\sqrt{6-x}-2}{\sqrt{3-x}-1} \quad \left(\frac{0}{0}\right) \text{form}$$

$$= \lim_{x\to 2} \frac{\sqrt{6-x}-2}{\sqrt{3-x}-1} \times \frac{\sqrt{3-x}+1}{\sqrt{3-x}+1}$$

$$= \lim_{x\to 2} \frac{(\sqrt{6-x}-2)(\sqrt{3-x}+1)}{(\sqrt{3-x})^2-(1)^2}$$

$$= \lim_{x\to 2} \frac{\sqrt{(6-x)(3-x)}+\sqrt{6-x}-2\sqrt{3-x}-2}{3-x-1}$$

$$= \lim_{x\to 2} \frac{\sqrt{6-x}(\sqrt{3-x}+1)-2(\sqrt{3-x}+1)}{2-x}$$

Rationalization doesn't help here!

$\rightarrow$ Solving by L'Hôpital Rule.

$$= \lim_{x\to 2} \frac{\frac{1}{2\sqrt{6-x}}\frac{d}{dx}(6-x)-0}{\frac{1}{2\sqrt{3-x}}\frac{d}{dx}(3-x)-0}$$

$$\frac{1}{y}\frac{dy}{dx} = x^{\frac{1}{x}} \cdot \frac{1}{\sin^{-1}x} \frac{d}{dx}(\sin^{-1}x) + \ln(\sin^{-1}x)\frac{d}{dx}(x^{\frac{1}{x}})$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{x^{\frac{1}{x}}}{\sin^{-1}x} \cdot \frac{1}{\sqrt{1-x^2}} + \ln(\sin^{-1}x)\frac{d}{dx}(x^{\frac{1}{x}}) \quad \text{--- } (1)$$

let $u = x^{\frac{1}{x}}$

$$\Rightarrow \ln u = \ln x^{\frac{1}{x}}$$

$$\Rightarrow \ln u = \frac{1}{x}\ln x$$

$$\Rightarrow \frac{d}{dx}(\ln u) = \frac{d}{dx}\left(\frac{1}{x} \cdot \ln x\right)$$

$$\Rightarrow \frac{1}{u}\frac{du}{dx} = \frac{1}{x}\frac{d}{dx}(\ln x) + \ln x \frac{d}{dx}\left(\frac{1}{x}\right)$$

$$\Rightarrow \frac{1}{u}\frac{du}{dx} = \frac{1}{x} \cdot \frac{1}{x} + \ln x\left(-\frac{1}{x^2}\right)$$

$$\Rightarrow \frac{1}{u}\frac{du}{dx} = \frac{1}{x^2} - \frac{\ln x}{x^2}$$

$$\Rightarrow \frac{1}{u}\frac{du}{dx} = \frac{1-\ln x}{x^2}$$

$$\Rightarrow \frac{du}{dx} = u\left(\frac{1-\ln x}{x^2}\right)$$

$$\Rightarrow \frac{du}{dx} = x^{\frac{1}{x}}\left(\frac{1-\ln x}{x^2}\right) \quad \text{--- } (2)$$

using (2) in (1)

$$\frac{1}{y}\frac{dy}{dx} = \frac{x^{\frac{1}{x}}}{(\sin^{-1}x)\sqrt{1-x^2}} + \ln \sin^{-1}x \cdot x^{\frac{1}{x}}\left(\frac{1-\ln x}{x^2}\right)$$

$$= \lim_{x\to 2} \frac{-\frac{1}{2\sqrt{6-x}}}{-\frac{1}{2\sqrt{3-x}}}$$

$$= \lim_{x\to 2} \frac{\sqrt{3-x}}{\sqrt{6-x}} = \frac{\sqrt{3-2}}{\sqrt{6-2}} = \frac{\sqrt{1}}{\sqrt{4}} = \frac{1}{2} \text{ Ans}$$

Q#3(a) Prove that every differential function is continuous but converse is not true.

Theorem#1 (Chapter #3) (Differentiability $\Rightarrow$ Continuity)
Chapter #3, Example #4) + (Chapter#2, Example #7)
($|x|$ is continuous at $x=0$ but is not differentiable at $x=0$).

Q#3(b) Evaluate $\lim_{x\to 0} x^{\frac{1}{x-1}}$

Let $y = x^{\frac{1}{x-1}}$

$$\Rightarrow \ln y = \ln\left(x^{\frac{1}{x-1}}\right)$$

$$\Rightarrow \ln y = \frac{1}{x-1} \ln x$$

$$\Rightarrow \ln y = \frac{\ln x}{x-1}$$

$$\Rightarrow \lim_{x\to 0} \ln y = \lim_{x\to 0} \frac{\ln x}{x-1} = \frac{\ln 0}{-1} = \frac{-\infty}{-1} = \infty$$

$$\Rightarrow \lim_{x\to 0} \ln y = \infty$$

$$\Rightarrow \lim_{x\to 0} y = e^{\infty}$$

$$\Rightarrow \lim_{x\to 0} x^{\frac{1}{x-1}} = \infty$$

Q#4(a) If $2-x^2 \le g(x) \le 2\cos x$ for all x, (8)
find $\lim_{x\to 0} g(x)$.

Given that

$$2-x^2 \le g(x) \le 2\cos x$$

$$\Rightarrow \lim_{x\to 0}(2-x^2) \le \lim_{x\to 0} g(x) \le \lim_{x\to 0}(2\cos x)$$

$$\Rightarrow (2-0) \le \lim_{x\to 0} g(x) \le 2\lim_{x\to 0}\cos x$$

$$\Rightarrow 2 \le \lim_{x\to 0} g(x) \le 2\cdot 1$$

$$\Rightarrow 2 \le \lim_{x\to 0} g(x) \le 2$$

By Sandwich theorem

$$\lim_{x\to 0} g(x) = 2$$

Q#4(b) Find Concavity of $f(x) = x^3 - 3x^2 + 2$

$$f(x) = x^3 - 3x^2 + 2$$

$$\Rightarrow f'(x) = 3x^2 - 6x$$

$$\Rightarrow f''(x) = 6x - 6$$

put $f''(x) = 0 \Rightarrow 6x - 6 = 0 \Rightarrow \boxed{x = 1}$

Hence the intervals to be considered for Concavity are $(-\infty, 1)$ and $(1, \infty)$

interval	$(-\infty, 1)$	$(1, \infty)$
Sign of f''	$-ve$	$+ve$
Concavity	Concave down	Concave up

Behaviour at $(-\infty, 1)$

$f(0) = 6(0) - 6 = -6$ (-ve)

Behaviour at $(1, \infty)$

$f(2) = 6(2) - 6$

$= 12 - 6$

$= 6$ (+ve)

Q#5(a) If a ball is thrown vertically with a velocity of 80 ft/s, then its height after t seconds is $s = 80t - 16t^2$. What is maximum height reached by the ball?

$$s = 80t - 16t^2$$

$$\text{velocity} = V(t) = \frac{ds}{dt} = \frac{d}{dt}(80t - 16t^2)$$

$$= 80 - 32t$$

Now maximum height is attained when velocity is zero i.e

$$V = 0$$

$$80 - 32t = 0$$

$$32t = 80$$

$$t = \frac{80}{32}$$

$$\boxed{t = \frac{5}{2} \text{ Seconds}}$$

Q#5(b) $\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}$

let $y = \tan^{-1}x$ —— ① $\Rightarrow \tan y = x$ —— ②

$z = \cot^{-1}x$ —— ③ $\Rightarrow \cot z = x$ —— ④

equating ② & ④

$$\tan y = \cot z$$

$$\Rightarrow \frac{\sin y}{\cos y} = \frac{\cos z}{\sin z}$$

$\Rightarrow \sin y \sin z = \cos y \cos z$

$\Rightarrow \cos y \cos z - \sin y \sin z = 0$

$\Rightarrow \cos(y+z) = 0$

$\Rightarrow y+z = \cos^{-1} 0$

$\Rightarrow y+z = \frac{\pi}{2}$

$\Rightarrow \tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}$ (using ① & ③)

Hence proved.

Q#6(a) Evaluate y' if $y = \cos(e^{\sqrt{\tan 3x}})$

$$\frac{dy}{dx} = \frac{d}{dx}\left[\cos(e^{\sqrt{\tan 3x}})\right]$$

$$= -\sin(e^{\sqrt{\tan 3x}}) \frac{d}{dx}(e^{\sqrt{\tan 3x}})$$

$$= -\sin(e^{\sqrt{\tan 3x}}) . e^{\sqrt{\tan 3x}} . \frac{d}{dx}\sqrt{\tan 3x}$$

$$= -\sin(e^{\sqrt{\tan 3x}}) . e^{\sqrt{\tan 3x}} . \frac{1}{2\sqrt{\tan 3x}} \frac{d}{dx}(\tan 3x)$$

$$= -\sin(e^{\sqrt{\tan 3x}}) e^{\sqrt{\tan 3x}} . \frac{1}{2\sqrt{\tan 3x}} (\sec^2 3x) \frac{d}{dx}(3x)$$

$$= -\sin(e^{\sqrt{\tan 3x}}) e^{\sqrt{\tan 3x}} . \frac{1}{2\sqrt{\tan 3x}} (3\sec^2 3x)$$

$$= \frac{-3e^{\sqrt{\tan 3x}} . \sec^2 3x . \sin(e^{\sqrt{\tan 3x}})}{2\sqrt{\tan 3x}}$$ Ans

Q#6(b) Evaluate

$$\lim_{x\to 1^+}\left(\frac{1}{\ln x}-\frac{1}{x-1}\right)$$

$$\left(= \lim_{x\to 1^+}\frac{1}{\ln x} - \lim_{x\to 1^+}\frac{1}{x-1} \right.$$

$$= \frac{1}{0} - \frac{1}{0^+}$$

$$\left. = (\infty - \infty) \text{ form} \right)$$

$$\lim_{x\to 1^+}\left(\frac{1}{\ln x}-\frac{1}{x-1}\right) = \lim_{x\to 1^+}\left(\frac{x-1-\ln x}{(x-1)(\ln x)}\right) \quad \left(\frac{\infty}{\infty}\right) \text{ form}$$

$$= \lim_{x\to 1^+}\left(\frac{1-0-\frac{1}{x}}{(x-1)(\frac{1}{x})+\ln x\cdot 1}\right)$$

$$= \lim_{x\to 1^+}\left(\frac{1-\frac{1}{x}}{\frac{x-1+x\ln x}{x}}\right)$$

$$= \lim_{x\to 1^+}\frac{\left(\frac{x-1}{x}\right)}{\frac{x-1+x\ln x}{x}}$$

$$= \lim_{x\to 1^+}\frac{x-1}{x-1+x\ln x} \quad \left(\frac{0}{0}\right) \text{form}$$

$$= \lim_{x\to 1^+}\frac{1}{1-0+x\cdot\frac{1}{x}+\ln x\cdot 1}$$

$$= \lim_{x\to 1^+}\frac{1}{1+1+\ln x} = \frac{1}{2+\ln 1}$$

$$= \frac{1}{2+0} = \frac{1}{2} \quad \text{Ans}$$

(11)

CamScanner