What is a relation?

Let A and B be sets. A binary relation from A to B is a subset of $A \times B$.

We use the notation $a\,R\,b$ to denote that $(a, b) \in R$, and $a \not R\, b$ to denote that $(a, b) \notin R$.
When $(a, b) \in R$, a is said to be related to b by R.

Example: A = (1, 2, 3) and B = {x, y, z}, and let

$$R = \{(1, y), (1, z), (3, y)\}.$$

Then R is a relation from A to B ? . **Yes--since R is a subset of A × B**

With respect to this relation,

$$1Ry, 1Rz, 3Ry, \quad \text{but} \quad 1\not{R}x, 2\not{R}x, 2\not{R}y, 2\not{R}z, 3\not{R}x, 3\not{R}z$$

Definition: The **ordered pair** (x , y) is a single element consisting of pair of elements in which

- ✓ x is the first element (coordinate)
- ✓ y is the second element (coordinate).

Note:

- ➢ If {a, b} is a set, {a, b}= {b, a}
- ➢ If (a, b) is an ordered pair, then (a, b) ≠ (b, a)

Definition: Two ordered pair (x , y) and (w , z) will be **equal** if

$$x = w \text{ and } y = z.$$

Sets are unordered, so $\{1, 2, 3\} = \{1, 3, 2\}$. However, sometimes we need to establish an order.

$$A = \{(1, 1), (2, 4), (3, 9), (4, 16), (5, 25), \ldots\}$$

In this example, $(2, 4) \in A$ but $(4, 2) \notin A$.

Definition: The **Cartesian product** of two sets A and B is the set of all ordered pairs (a, b) with $a \in A$ and $b \in B$.

$$A \times B = \{(a, b) | a \in A \text{ and } b \in B\}$$

Example: Let $A = \{x, y\}$ and $B = \{1, 2\}$. Compute $A \times B$.

Note: in general

- $A \times B \neq B \times A$.
- $|A \times B| = |A| \times |B|$.

Example: Consider these relations on the set of integers:

$R_1 = \{(a,b) \mid a \leq b\}$,
$R_2 = \{(a,b) \mid a > b\}$,
$R_3 = \{(a,b) \mid a = b \text{ or } a = -b\}$,
$R_4 = \{(a,b) \mid a = b\}$,
$R_5 = \{(a,b) \mid a = b + 1\}$,
$R_6 = \{(a,b) \mid a + b \leq 3\}$.

Which of these relations contain each of the pairs

(1,1), (1, 2), (2, 1), (1, −1), and (2, 2)?

(1,1) is in R_1, R_3, R_4, and R_6:
(1,2) is in R_1 and R_6:
(2,1) is in R_2, R_5, and R_6:
(1, −1) is in R_2, R_3, and R_6 :
(2,2) is in R_1, R_3, and R_4.

Definition: The **domain** of relation R is the set of all first elements of the ordered pairs which belong to R, denoted by Dom(R).

Definition: The **range** is the set of second elements of the ordered pairs which belong to R, denoted by Ran(R).

Example: $A = (1, 2, 3)$ and $B = \{x, y, z\}$, and consider the relation

$$R = \{(1, y), (1, z), (3, y)\}.$$

Find the domain and range of R.

The domain of R is Dom(R) = $\{1, 3\}$
The range of R is Ran(R) = $\{y, z\}$

Definition: Let R be any relation from set A to B. The inverse of R, denoted by R^{-1}, is the relation from B to A denoted by

$$R^{-1} = \{(b , a) \mid (a , b) \in R\}$$

> *Example:* let $A = \{1, 2, 3\}$ and $B = \{x, y, z\}$. Find the inverse of
> $$R = \{(1, y), (1 , z), (3 , y)\}$$

Solution: $R^{-1} = \{(y , 1), (z , 1), (y , 3)\}$

- If R is any relation, then $(R^{-1})^{-1} = R$.
- The domain and range of R^{-1} are equal to the range and domain of R, respectively.
- If R is a relation on A, then R^{-1} is also a relation on A.

Composition of Relations

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Definition: Suppose A, B and C are sets, and

- ✓ R is a relation from A to B
- ✓ S is a relation from B to C
- ✓ Then the composition of R and S, denoted by $R \circ S$, is a relation from A to C defined by

$$R \circ S = \{(a , c) \mid \exists\, b \in B, \text{ for which } (a , b) \in R \text{ and } (b , c) \in S\}$$

> *Example:* Let $A = \{1, 2, 3, 4\}$, $B = \{a, b, c, d\}$, $C = \{x, y, z\}$ and let
> $R = \{(1, a), (2, d), (3, a), (3, b), (3, d)\}$
> and $S = \{(b, x), (b, z), (c, y), (d, z)\}$
> Compute $R \circ S$.

Using arrow diagram, $R \circ S=\{(2,z), (3,x), (3,z)\}$

Definition: A relation R on a set A is **reflexive** if (a,a) ∈ R for all a ∈ A.
Thus R is **not reflexive** if there exists a ∈ A such that (a, a)∉ R.

- Consider the following relations on {1, 2, 3, 4}:
- *R1 = {(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)},*
- *R2 = {(1, 1), (1, 2), (2, 1)},*
- *R3 = {(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 3), (4, 1), (4, 4)},*
- *R4 = {(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)},*
- *R5 = {(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)},*
- *R6 = {(3, 4)}.*
- Which of these relations are reflexive?
- *Solution: The relations R3 and R5 are reflexive because they both contain all pairs of the form (a, a), namely, (1, 1), (2, 2), (3, 3), and (4, 4). The other relations are not reflexive because* they do not contain all of these ordered pairs.
- In particular, *R1, R2, R4, and R6 are not reflexive* because *(3, 3) is not in any of these relations.*

Reflexive Relation: A relation R on set A is reflexive if (a,a) ∈ R holds for every element of a ∈ A i.e if set A = {a, b} then R = {(a,a), (b,b)} is reflexive.

irreflexive: A relation (R) on set A is called irreflexive if no (a,a) ∈ R holds for every element a ∈ A i.e if set A = {a,b} then R = {(a,b), (b,a)} is irreflexive.

A relation on a set A is a relation from A to A (i.e, a subset of $A \times A$).

A relation R on a set A is reflexive if

$$\forall a \in A, (a, a) \in R.$$

Exercise Are these relations reflexive?

a) The graph of the function $f(x) = x^2$, where $f : \mathbb{Z} \to \mathbb{Z}$.
No, since the relation $R = \{(0,0), (1,1), (2,4), (3,9), \ldots\}$ does not contain all pairs of the form (x, x), $\forall x \in \mathbb{Z}$.

b) The graph of a function $g(x) = x$, where $g : \mathbb{R} \to \mathbb{R}$.
Yes, by the definition itself.

c) The relation "is a subset of" (set inclusion $\subseteq$).
Yes, because it includes the equality.

d) The relation "is greater than" on the set of integers.
No, but it would be the relation "is greater than or equal to".

e) The relation "divides" (divisibility) on the set of all negative integers.
Yes, because any number is a divisor of itself.

Symmetric. A relation R on a set A is called symmetric if $(b,a) \in R$ holds when $(a,b) \in R$

i.e $R = \{(4,5), (5,4), (6,5), (5,6)\}$ on set $A = \{4,5,6\}$ is symmetric

Definition: A relation R on a set A is **antisymmetric** if whenever aRb and bRa then a = b.

- Consider the following relations on {1, 2, 3, 4}:
- $R1 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)\}$,
- $R2 = \{(1, 1), (1, 2), (2, 1)\}$,
- $R3 = \{(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 3), (4, 1), (4, 4)\}$,
- $R4 = \{(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)\}$,
- $R5 = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)\}$,
- $R6 = \{(3, 4)\}$.
- The relations R2 and R3 are symmetric, because in each case (b, a) belongs to the
- relation whenever (a, b) does. For R2, the only thing to check is that both (2, 1) and (1, 2) are
- in the relation. For R3, it is necessary to check that both (1, 2) and (2, 1) belong to the relation, and (1, 4) and (4, 1) belong to the relation. The reader should verify that none of the other relations is symmetric.
- This is done by finding a pair (a, b) such that it is in the relation but (b, a) is not.
- R4, R5, and R6 are all ant symmetric.
- For each of these relations there is no pair of elements a and b with a = b such that both (a, b) and (b, a) belong to the relation.
- The reader should verify that none of the other relations is ant symmetric. This is done by finding a pair (a, b) with a = b such that (a, b) and (b, a) are both in the relation.

The relation "is married to" between any two persons.
Symmetric. Whenever 'a' is married to 'b', then 'b' is also married to 'a'.

The relation $R = \{(0,1), (1,2), (2,1)\}$.
Neither symmetric nor antisymmetric. Not symmetric because $(1,0)$ is missing.
Not antisymmetric because $(1,2) \in R$, $(2,1) \in R$ and $1 \neq 2$.

The relation $S = \{(1,1), (2,2), (3,3)\}$.
Symmetric and antisymmetric. In fact, this is the only way a relation can be both symmetric and antisymmetric.

Definition: A relation R on a set A is **transitive** if whenever aRb and bRc then aRc, that is, if whenever $(a, b) \in R$ and $(b, c) \in R$ then $(a, c) \in R$.

Thus R is **not transitive** if there exist $a, b, c \in R$ such that $(a,b) \in R$ and $(b, c) \in R$ but $(a,c) \notin R$.

Transitive

A relation (R) on set A is called transitive if $(a,b) \in R$ then $(b,c) \in R$ then $(a,c) \in R$ for all $(a,b,c) \in A$ i.e $R = \{(1,2),(2,3),(1,3)\}$ on set $A = \{1,2,3\}$ is Transitive.

- Consider the following relations on $\{1, 2, 3, 4\}$:
- $R1 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)\}$,
- $R2 = \{(1, 1), (1, 2), (2, 1)\}$,
- $R3 = \{(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 3), (4, 1), (4, 4)\}$,
- $R4 = \{(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)\}$,
- $R5 = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)\}$,
- $R6 = \{(3, 4)\}$.

- *R4, R5, and R6 are transitive. For each of these relations, we can show that it is*
- transitive by verifying that if *(a, b) and (b, c) belong to this relation, then (a, c) also does. For*
- instance,
- *R4 is transitive, because (3, 2) and (2, 1), (4, 2) and (2, 1), (4, 3) and (3, 1), and (4, 3) and (3, 2) are the only such sets of pairs, and (3, 1), (4, 1), and (4, 2) belong to R4. The reader should verify that R5 and R6 are transitive.*
- *R1 is not transitive because (3, 4) and (4, 1) belong to R1, but (3, 1) does not. R2 is*
- not transitive because *(2, 1) and (1, 2) belong to R2, but (2, 2) does not. R3 is not transitive*
- because *(4, 1) and (1, 2) belong to R3, but (4, 2) does not.*

Definition: A relation R on a set A is called an **equivalence** relation if R is <u>reflexive, symmetric, and transitive</u>.

- It follows three properties:
  1) For every $a \in A$, aRa.
  2) If aRb then bRa.
  3) If aRb and bRc, then aRc.

Equivalence: A relation is an Equivalence relation if it is reflexive, symmetric, and transitive

i.e $R = \{(1,1),(2,2),(3,3),(1,2),(2,1),(2,3),(3,2),(3,1),(1,3)\}$ is equivalence.

Definition : A relation R on a set S is called a **partial ordering**, or **partial order**, if it is *reflexive, antisymmetric, and transitive.*

Definition: A set A together with a partial ordering R is called a **partially ordered set** or **poset.**

- Suppose R is a relation on A
- If R does not possess a particular relation (reflexive, symmetric, transitive)
- Then we may add as few new pairs as possible until we get a new relation R_1 on A that have that required property.
- If such R_1 exists, we call it the closure of R with respect to that property.
- Example: Reflexive closure, Symmetric closure, Transitive closure.