Chapter No. 9

Numerical Linear Algebra

Roundoff

If the number is greater than 5 we increase the left number at one time. If the number is equal to the 5 so we check number is even or odd if the number is even no increasing if the number is odd we increase

Even → same

Odd → increas

Example

9.82651 → not increas Because 2 < 5

one decimal place = 9.8

Two " " = 9.83

Three " " = 9.826

Check the number is equal to 5 so the left number is even (No change)

Partial

(Gauss EJ

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Exampl

No. 9

Linear Algebra

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-1 => not increas
Because 2 < 5
e = 9.8
9.83
9.826
number
e left
change)

Partial pivoting in Gauss Elimination Method.

In which first we check the first colum and make the greater value on the top by exchange the row. After reach the max element on the top we make element in colum is zero.

After this we reach the second square matrics.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & \boxed{a_{22}} & \boxed{a_{23}} \\ a_{31} & \boxed{a_{32}} & \boxed{a_{33}} \end{bmatrix}$$

→ second square matrics

Example

$$x_1 + x_2 + x_3 = 3$$
$$4x_1 + 3x_2 + 4x_3 = 8$$
$$9x_1 + 3x_2 + 4x_3 = 7$$

$$= \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 4 & 3 & 4 & 8 \\ 9 & 3 & 4 & 7 \end{array}\right] \quad R_1 \leftrightarrow R_3$$

$$= \left[\begin{array}{ccc|c} 9 & 3 & 4 & 7 \\ 4 & 3 & 4 & 8 \\ 1 & 1 & 1 & 3 \end{array}\right]$$

$R_2 \rightarrow R_2 - 4R_1$

$$= \left[\begin{array}{ccc|c} 9 & 3 & 4 & 7 \\ 0 & -1 & 0 & -4 \\ 1 & 1 & 1 & 3 \end{array}\right] \quad R_3 \rightarrow 9R_3 - R_1$$

$$= \left[\begin{array}{ccc|c} 9 & 3 & 4 & 7 \\ 0 & -1 & 0 & -4 \\ 0 & 6 & 5 & 20 \end{array}\right]$$

Exchange Row Because top element is 6

$$= \left[\begin{array}{ccc|c} 9 & 3 & 4 & 7 \\ 0 & 6 & 5 & 20 \\ 0 & -1 & 0 & -4 \end{array}\right]$$

$-x_2 = -4$

$\boxed{x_2 = 4}$

$6x_2 + 5x_3 = 20$

$6(4) + 5x_3 = 20$

$5x_3 = 20 - 24$

$\boxed{x_3 = -4/5}$

$9x_1 + 3(4) + 4(-4/5) = 7$

$9x_1 + 12 - 16/5 = 7$

$9x_1 = \frac{35 + 16}{5} - 12$

$9x_1 = \frac{51}{5} - 12$

$9x_1 = \frac{51 - 60}{5}$

$x_1 = \frac{-9}{9 \times 5} = -1/5$

---

so the

$\boxed{x_1 = -1/5}$,

Band W

in Finite E

The numbe

from either

represents h

taken in

the bandw

$$\begin{bmatrix} 1 & -1 \\ -1 & 2 \\ 0 & -1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

b = h

so the value of the x is

$\boxed{x_1 = -1/5}$, $\boxed{x_2 = 4}$ $\boxed{x_3 = -4/5}$

Band Width matrix in Finite Element

The number of non-zero elements from either side of the diagonal represents half bandwidth and taken in combination at gives the bandwidth

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix} 0$$

b = half bandwidth = 2

$$\begin{array}{c} b \\ \begin{bmatrix} 1 & -1 \\ 2 & -1 \\ 2 & -1 \\ 2 & -1 \\ 1 & 0 \end{bmatrix} \end{array}$$

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Tridiagonal Bidiagonal times bidiagonal.

$$\begin{bmatrix} 1 & -1 & & \\ -1 & 2 & -1 & \\ & -1 & 2 & -1 \\ & & -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & & & \\ -1 & 1 & & \\ & -1 & 1 & \\ & & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & & \\ & 1 & -1 & \\ & & 1 & -1 \\ & & & 1 \end{bmatrix}$$

Sparse Matrix

The spare matrix is a matrix in which almost element are Zero. (OR you say maximum element are zero).

$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 & 0 & 6 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 8 & 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 10 & 0 & 7 & 0 & 0 & 0 \end{bmatrix} \end{array} 7 \times 8$$

Total elemen... occupie size... So in th... element occ... in the ma... are zero. this eleme... this by... in Array...

Row	0	1
Col	4	2
Value	5	

Row...

0	4
2	0
5	7

Total element = 56

occupie size = 56 × 2 = 112 bytes (for int)

so in this case the 56 element occupie the 112 bytes in the matrix many element are zero. Zero mean no need this element so we reduce this by using the spare in Array and LinkedList.

Array

Row	0	1	1	2	4	4	5	6	6
Col	4	2	6	0	1	5	7	2	4
Value	5	4	6	1	8	3	2	10	7

Linked List

Row	Col	Value	next

[0 | 4 | 5 |] → [1 | 2 | 4 |] → [1 | 6 | 6 |]

→ [2 | 0 | 1 |] → [4 | 1 | 8 |] → [4 | 5 | 3 |]

→ [5 | 7 | 2 |] → [6 | 2 | 10 |] → [6 | 9 | 7 |]

Fast orthogonalization

Gram–Schmidt works to find the orthonormal.

Let spose that

V → Vector space.

$V \to v_1, v_2, v_3 \ldots v_n$

$u_1, u_2, u_3 \ldots u_n$ ↑ belong to

↳ orthonormal

or orthomatrix

Theorem:-

Let V be the inner product space with $S = \{u_1, u_2 \ldots u_n\}$ as basis set, there exists an orthonormal basis for $T = \{w_1, w_2 \ldots w_n\}$ corresponding to S.

$$w_i = \frac{v_i}{\|v_i\|}, \quad v_1 = u_1$$

Having the properties

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i) $u_1 = v_1$

ii) $u_2 = v_2 - ($

iii) $u_3 = v_3 -$

Example

$v_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

We know

$v_1 = u_1 =$

$u_2 = v_2 - ($

$\frac{v_2 \cdot u_1}{u_1 \cdot u_1} \times u_1$

$=$

$v_2 - \left(\frac{v_2 \cdot u_1}{u_1 \cdot u_1}\right.$

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U_n

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i) $U_1 = V_1$

ii) $U_2 = V_2 - \left(\frac{V_2 \cdot U_1}{U_1 \cdot U_1}\right)\vec{U_1}$

iii) $U_3 = V_3 - \left(\frac{V_3 \cdot U_1}{U_1 \cdot U_1}\right)U_1 - \left(\frac{V_3 \cdot V_2}{U_2 \cdot U_2}\right)U_2$

Example

$$V_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \quad V_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad V_3 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

We know that

$$V_1 = U_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$U_2 = V_2 - \left(\frac{V_2 \cdot U_1}{U_1 \cdot U_1}\right) \times U_1$$

$$\frac{V_2 \cdot U_1}{U_1 \cdot U_1} \times U_1 \left[\frac{(1)(1) + (0)(-1) + (1)(1)}{(1)(1) + (-1)(-1) + (1)(1)}\right] \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$= \left[\frac{1 - 0 + 1}{1 + 1 + 1}\right] \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$= \frac{2}{3} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2/3 \\ -2/3 \\ 2/3 \end{bmatrix}$$

$$V_2 - \left(\frac{V_2 \cdot U_1}{U_1 \cdot U_1}\right) \times U_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 2/3 \\ -2/3 \\ 2/3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 - 2/3 \\ 0 + 2/3 \\ 1 - 2/3 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 2/3 \\ 1/3 \end{bmatrix}$$

③ $U_3 = V_3 - \left(\frac{V_3\cdot U_1}{U_1\cdot U_1}\right)\times U_1 - \left(\frac{U_3\cdot U_2}{U_2\cdot U_2}\right)\times U_2$

$$= V_3 - \left[\frac{(1)(1)+(1)(-1)+2(1)}{3}\right]\begin{bmatrix}1\\-1\\1\end{bmatrix} - \left\{\frac{(1)(1/3)+1(2/3)+2(1/3)}{(1/3)(1/3)+(2/3)(2/3)+(1/3)(1/3)}\right\}\begin{bmatrix}1/3\\2/3\\1/3\end{bmatrix}$$

$$U_3 = \begin{bmatrix}1\\1\\2\end{bmatrix} - \begin{bmatrix}2/3\\-2/3\\2/3\end{bmatrix} - \begin{bmatrix}5/6\\10/6\\5/6\end{bmatrix} = \begin{bmatrix}-1/2\\0\\1/2\end{bmatrix} \text{ Ans.}$$

if we find the length we ahcive the osthogal normal.

$$U_3 = \sqrt{(-1/2)^2 + (0)^2 + (1/2)^2}$$
$$= \sqrt{\frac{1}{4}+0+\frac{1}{4}} = \sqrt{\frac{2}{4}}$$
$$= \sqrt{\frac{1}{2}} \text{ Ans.}$$

$$U_2 = \sqrt{(1/3)^2 + 2/3 + \left(\frac{1}{3}\right)^2}$$
$$= \sqrt{\frac{1}{9} + 4/9 + \frac{1}{9}}$$
$$= \sqrt{\frac{6}{9}} = \sqrt{\frac{2}{3}} \text{ Ans.}$$

$$U_1 = \sqrt{(1)^2 + (-1)^2 + (1)^2}$$
$$= \sqrt{1+1+1} = \sqrt{3} \text{ Ans.}$$

Exercise
prove th

$\|Ax\|$

We know
norm suc

$\|A\| =$

$\because U =$

So
$\|A\| =$

W

$\|A\| =$

Let a

Exercise 9.2
prove that

$$||A\underline{x}|| \le ||A|| \cdot ||\underline{x}||$$

we know that the natural norm such as

$$||A|| = \max_{||\underline{u}||=1} ||A \cdot \underline{u}||$$

$\longrightarrow$ u is a unit vector

$\because$ $\underline{u} = \dfrac{x}{||x||}$ $\longrightarrow$ Any vector

So

$$||A|| = \max_{x \neq 0} \left|\left| \frac{A \cdot x}{||x||} \right|\right|$$

we know that

$$|| \, ||\underline{x}|| \, || = ||\underline{x}||$$

$$||A|| = \max_{x \neq 0} \frac{||A \cdot \underline{x}||}{||\underline{x}||}$$

Let a set $= \{1, 2, 3\}$

Max = 3

$3 \ge x \;\forall\; x \in S$

So

$$\|A\| \geq \frac{\|Ax\|}{\|x\|}$$

$$\|A\|\cdot\|x\| \geq \|Ax\|$$

Hence prove that

Condition Number

$K(A)$, $cond(A)$

$$K(A) = \|A\|\cdot\|A^{-1}\|$$

Question.

prove that condition number of matrix is always greater then or equal to 1.

$K(A) \geq 1$ $\qquad K(A) = \|A\|\cdot\|A^{-1}\|$

We Know that

$$A\cdot A^{-1} = I$$

Taking norm of b/s

$$\|I\| = \|A\cdot A^{-1}\|$$

$\because \|x\cdot y\| \leq \|x$

so

$\|I\| \leq \|A\|$

$\therefore K($

$\|I\| \leq K$

$\because \|I\| =$

$\|I\| \geq$

$\|I\| \geq \|$

$\|I\| \geq 1$

So we

$K(A) \geq$

Hence pr

mber
A)

n number
greater
.
$K(A) = \|A\| \cdot \|A^{-1}\|$

$\because \|x \cdot y\| \le \|x\| \cdot \|y\|$

so

$$\|I\| \le \|A\| \cdot \|A^{-1}\|$$

$\because K(A) = \|A\| \cdot \|A^{-1}\|$

$\|I\| \le K(A)$ ——— i)

$\because \|I\| = \max_{x \neq 0} \dfrac{\|I \cdot x\|}{\|X\|}$

$\|I\| \ge \dfrac{\|I \cdot X\|}{\|X\|}$ → when any matrix is multiply with the ~~vec~~ Identity so the answer is vector $I \cdot A = A$

$\|I\| \ge \dfrac{\cancel{\|X\|}}{\cancel{\|X\|}}$

$\|I\| \ge 1$ ——— ii)

So we compare the eq i and ii)

$K(A) \ge 1$

Hence prove that

Q. $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$ Find $k(A)$

$k_2(A)$, $k_\infty(A) = ?$

We know that

$k(A) = \|A\| \cdot \|A^{-1}\|$

$A^{-1} = \begin{bmatrix} -1 & 0 & 1 \\ 2/3 & -1/3 & 0 \\ 2/3 & 2/3 & -1 \end{bmatrix}$

sum

$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$ sum

$\|A\| = 1+2+2,\ 2+1+2,\ 1+2+1$

$\|A\| = \text{Max} = 5,\ 5,\ 4$

$\boxed{\text{Max} = 5}$

$\|A\| = 5$

Becom positive

$= \begin{bmatrix} -1 & 0 & 1 \\ 2/3 & -1/3 & 0 \\ 2/3 & 2/3 & -1 \end{bmatrix}$

$\|A^{-1}\| = \max = +1+2/3+2/3,\ 0+1/3+2/3,\ 1+0+1$

$= \dfrac{+3+2+2}{3},\ \dfrac{0+1+2}{3},\ 2$

$\text{Max} = 7/3,\ 3/3,\ 2$

$\text{Max} = 7/3$

$\|A^{-1}\| = 7/3$

اس میں جب $\|A\|$ norm کو find / کرتے ہیں تو کوئی بھی
Elem. کو "−" میں نہیں کریں گے اگر مائنس ہو تو اس کو (+) کر دیں گے

$K(A) = \|A\|\|A^{-1}\|$ ...

$= (5)($ ...

$K(A) =$ ...

$K_2(A) =$ ...

under the ...

$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$ ...

$\|A\|_2 = \sqrt{1+4+4+ \ldots}$

$= \sqrt{2 \ldots}$

$\|A^{-1}_2\| = \sqrt{1+4/ \ldots}$

$=$ ...

$K(A_2) = \|A_2\| \ldots$

$= 5 \ldots$

$\frac{1}{2}\Big]$ Find $k(A)$

$k_{\infty}(A) = ?$

t

$\|A\| \cdot \|A^{-1}\|$

um

$1+2+1$

→ Becom positive

$$\begin{pmatrix} -1 & 0 & 1 \\ 2/3 & -1/3 & 0 \\ 2/3 & 2/3 & -1 \end{pmatrix}$$

$+1/3+2/3, \quad 1+0+1$

$\frac{1+2}{3}, \; 2$

2

[illegible] اس میں

"—" کو Elem.

$K(A) = \|A\| \|A^{-1}\|$

$= (5)(7/3) = 35/3$

$K(A) = 35/3$ Ans.

$K_2(A) =$ اس کے اندر کو squar کرتے گئے

under the square root. ~~under~~

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} \qquad A^{-1} = \begin{bmatrix} -1 & 0 & 1 \\ 2/3 & -1/3 & 0 \\ 2/3 & 2/3 & -1 \end{bmatrix}$$

$$\|A\|_2 = \sqrt{1+4+4+4+1+4+1+4+1}$$
$$= \sqrt{24}$$

$$\|A_2^{-1}\| = \sqrt{1+4/9+4/9+0+\frac{1}{9}+4/9+1+0+1}$$
$$= \sqrt{\frac{9+4+4+1+4+9+9}{9}}$$
$$= \sqrt{\frac{40}{9}}$$

$$K(A_2) = \|A_2\| \|A_2^{-1}\|$$
$$= \sqrt{24} \cdot \sqrt{\frac{20}{9}} \text{ Ans.}$$

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$K_\infty(A) = ?$

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}, \quad A^{-1} \begin{bmatrix} -1 & 0 & 1 \\ 2/3 & -1/3 & 0 \\ 2/3 & 2/3 & 1 \end{bmatrix}$$

$$\|A\| = \max\{1+2+1,\ 2+1+2,\ 2+2+1\}$$

$$\max = \{4, 5, 5\}$$

$$\|A_\infty\| = \max = 5$$

$$\|A_\infty^{-1}\| = +1+0+\phi,\ \frac{2}{3}+\frac{1}{3},\ \frac{2}{3}+\frac{2}{3}+1$$

$$= 2, 1, 7/3$$

$$\|A_\infty^{-1}\| = \max = 7/3$$

~~As A~~ $$K(A)_\infty = \|A_\infty\| \, \|A_\infty^{-1}\|$$

$$= (5)(7/3$$

$$= 35/3 \text{ Ans.}$$