Ch# 9 Continuous P.D

Uniform Distribution:-

The density function of a continous r.v x is called a uniform distribution when b/w end point any two subintervals of the Same length containing x, have the same probability.

$$f(x) = \frac{1}{b-a}, \quad a \leq x \leq b$$
$$= 0 \quad \text{elsewhere}$$

Properties of uniform dis.

1. Let x have the ~~same~~ uniform distribution over $[a,b]$. Then its mean is $\frac{a+b}{2}$ and variance is $\frac{(b-a)^2}{12}$.

Proof:-

$$E(x) = \int_{-\infty}^{\infty} x f(x) dx$$

$$= \int_a^b x \cdot \frac{1}{b-a} dx$$

$$E(x) = \frac{1}{b-a}\int_a^b x\,dx$$

$$= \frac{1}{b-a}\left[\frac{x^2}{2}\right|_a^b$$

$$= \frac{1}{b-a}\left[\frac{b^2}{2} - \frac{a^2}{2}\right]$$

$$= \frac{1}{b-a}\left[\frac{b^2-a^2}{2}\right]$$

$$= \frac{\cancel{(b-a)}(b+a)}{2\cancel{(b-a)}}$$

$$\boxed{E(x) = \frac{a+b}{2}}$$

Now $\quad Var(x) = E(x^2) - [E(x)]^2 \longrightarrow ①$

$\therefore \; E(x^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx$

$$= \int_a^b x^2 \cdot \frac{1}{b-a}\,dx$$

$$= \frac{1}{b-a}\int_a^b x^2\,dx$$

$$= \frac{1}{b-a}\left[\frac{x^3}{3}\right|_a^b$$

$$= \frac{1}{b-a}\left[\frac{b^3}{3} - \frac{a^3}{3}\right]$$

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$a^3 - b^3 = (a-b)(a^2+ab+b^2)$

$$E(x^2) = \frac{1}{b-a}\left[\frac{b^3-a^3}{3}\right]$$

$$= \frac{\cancel{(b-a)}(b^2+ba+a^2)}{3\cancel{(b-a)}}$$

$$E(x^2) = \frac{b^2+ab+a^2}{3} \quad \text{put in } ①$$

$$\text{var}(x) = \frac{b^2+ab+a^2}{3} - \left(\frac{a+b}{2}\right)^2$$

$$= \frac{b^2+ab+a^2}{3} - \frac{(a+b)^2}{4}$$

$$= \frac{4b^2+4ab+4a^2-3a^2-3b^2-6ab}{12}$$

$$= \frac{b^2-2ab+a^2}{12}$$

$\rightarrow ①$

$$\boxed{\text{var}(x) = \frac{(b-a)^2}{12}}$$

2. The shape of the distribution is rectangular.

Moment generating function U.D:-

$$M_0(t) = E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx$$

$$= \int_a^b e^{tx} \cdot \frac{1}{b-a}\,dx$$

$$M_0(t) = \frac{1}{b-a}\int_a^b e^{tx}\,dx$$

$$= \frac{1}{b-a}\left[\frac{e^{tx}}{t}\right]_a^b$$

$$M_0(t) = \frac{1}{b-a}\left[\frac{e^{tb}-e^{ta}}{t}\right]$$

$$\boxed{M_0(t) = \frac{e^{bt}-e^{at}}{(b-a)t}}$$

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2. Exponential distribution :-

A random variable x is said to have an exponential distribution with parameter λ and p.d.f define as.

$$f(x) = \lambda e^{-\lambda x} \quad , x > 0$$

$$= 0$$

$\lambda > 0$ p.d.f written as

$$f(x) = \frac{1}{\beta} e^{-x/\beta} \quad , x > 0$$

$$= 0$$

Properties of Exponential dist.

i. The mean and standard deviation of exponential distribution are equal

Proof:-

$$\text{Mean} = E(x) = \int_{-\infty}^{\infty} x f(x) dx$$

$$= \int_0^{\infty} x \cdot \lambda e^{-\lambda x} dx$$

$$= \int_0^{\infty} u e^{-u} \frac{du}{\lambda}$$

Let $\lambda x = u$

$\lambda dx = du$

$dx = \frac{du}{\lambda}$

$$= \frac{1}{\lambda} \int_0^{\infty} u e^{-u} du$$

$$= \frac{1}{\lambda} \Gamma(2)$$

$$\text{gamma} = \Gamma(n) = \int_0^{\infty} e^{-x} x^{n-1} dx$$

$$\Gamma(2) = \int_0^{\infty} e^{-u} u^{2-1} dx$$

$$\Gamma(n+1) = n!$$

$$\Gamma(2) = \Gamma(1+1) = 1!$$

$$\boxed{E(x) = \frac{1}{\lambda}}$$

$$Var(x) = E(x^2) - [E(x)]^2 \longrightarrow (1)$$

$$E(x^2) = \int_0^{\infty} x^2 f(x) dx$$

$$= \int_0^{\infty} x^2 \lambda e^{-\lambda x} dx$$

$\lambda x = u \Rightarrow x = u/\lambda$

$\lambda dx = du$

$dx = du/\lambda$

$$= \int_0^{\infty} x . x \lambda e^{-\lambda x} dx$$

$$E(x^2) = \int_0^\infty \frac{u}{\lambda}\, u e^{-u} \frac{du}{\lambda}$$

$$= \frac{1}{\lambda^2} \int_0^\infty u^2 e^{-u} du$$

$$= \frac{1}{\lambda^2} \Gamma(3)$$

$$\int_0^\infty e^{-u} u^{3-1} du = \Gamma(3)$$

$$\Gamma(3) = 2! = 2$$

$$E(x^2) = \frac{2}{\lambda^2} \longrightarrow \text{put in } ①$$

$$Var(x) = E(x^2) - [E(x)]^2$$

$$= \frac{2}{\lambda^2} - \frac{1}{\lambda^2}$$

$$= \frac{2-1}{\lambda^2}$$

$$\boxed{Var(x) = \frac{1}{\lambda^2}}$$

$$\boxed{S.D(x) = \frac{1}{\lambda}}$$

3.

ii. The distribution is extremely skewed and does not exist any mode.

Moment generation function E.D:-

$$M_0(t) = E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} f(x) dx$$

$$M_0(t) = \int_0^\infty e^{tx} \cdot \lambda e^{-\lambda x} dx$$

$$M_0(t) = \lambda \int_0^\infty e^{x(t-\lambda)}\, dx$$

$$= \lambda \int_0^\infty e^{-x(\lambda-t)}\, dx$$

$$= \lambda \left[\frac{e^{-x(\lambda-t)}}{-(\lambda-t)} \right]_0^\infty$$

$$\boxed{M_0(t) = \frac{\lambda}{\lambda-t}, \quad t < \lambda}$$

3. GAmma and beta distribution:-

Gamma function:-

The gamma function of any number $n>0$ denoted by $\Gamma(n)$ is define as

$$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x}\, dx$$

$$\therefore \Gamma(n+1) = n!$$

$$\Gamma(n+1) = \int_0^\infty x^n \cdot e^{-x}\, dx \quad , n>0$$

Properties of gamma dist:-

⇒ Gamma distribution :- A continouse r.v x is said to gamma dist with parameter m,

$$f(x) = \frac{1}{\Gamma(m)} x^{m-1} e^{-x}, \quad 0 \le x \le \infty$$

$$= 0$$

i. The mean and variance of gamma distribution are equal to its parameter m.

Proof :-

$$U = E(x) = \int_{-\infty}^{\infty} x f(x)\, dx$$

$$= \int_{0}^{\infty} x \cdot \frac{1}{\Gamma(m)} x^{m-1} e^{-x}\, dx$$

$$= \int_{0}^{\infty} \frac{1}{\Gamma(m)} x^{m-1+1} e^{-x}\, dx$$

$$= \int_{0}^{\infty} \frac{x^m e^{-x}\, dx}{\Gamma(m)}$$

$$= \frac{1}{\Gamma(m)} \int_{0}^{\infty} x^m e^{-x}\, dx$$

By defination of gamma function

$$= \frac{1}{\Gamma(m)} \Gamma(m+1)$$

$$E(x) = \frac{m\,\cancel{\Gamma(m)}}{\cancel{\Gamma(m)}} = m$$

$$\boxed{E(x) = m}$$

$\therefore$ $Var(x) = E(x^2) - [E(x)]^2 \longrightarrow ①$

$$E(x^2) = \int_0^\infty x^2 f(x) dx.$$

$$= \int_0^\infty x^2 \frac{1}{\Gamma(m)} x^{m-1} e^{-x} dx$$

$$= \frac{\Gamma(m+2)}{\Gamma(m)} = \frac{(m+1) m \cancel{\Gamma(m)}}{\cancel{\Gamma(m)}}$$

$E(x^2) = m(m+1) \longrightarrow$ put in ①

$$Var(x) = m(m+1) - m^2$$

$$= \cancel{m^2} + m - \cancel{m^2}$$

$$\boxed{Var(x) = m}$$

Beta Function:-

The beta function for any two postive number m, n denoted by $B(m,n)$ is define, by

$$B(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx \quad , m>0, n>0$$

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Beta distribution:-

A continouse r.v x is said to have a beta distribution with two parameters m and n, if its p.d.f is defined by

$$f(x) = \frac{1}{B(m,n)} x^{m-1}(1-x)^{n-1}, \quad 0 \le x \le 1$$
$$m, n > 0$$
$$= 0 \quad \text{elsewhere}$$

Properties of beta dist:-

1. The Mean and Variance of the distribution are $\frac{m}{m+n}$ and $\frac{mn}{(m+n)^2(m+n+1)}$

Proof:-

$$U = E(x) = \int_0^1 x \cdot \frac{1}{B(m,n)} x^{m-1}(1-x)^{n-1} dx$$

$$= \int_0^1 \frac{1}{B(m,n)} x^{m-1+1}(1-x)^{n-1} dx$$

$$= \int_0^1 \frac{1}{B(m,n)} x^{m}(1-x)^{n-1} dx$$

$$= \frac{B(m+1, n)}{B(m, n)}$$

$$u = \frac{m\Gamma(m)\Gamma(n)}{(m+n)\Gamma(m+n)} \cdot \frac{\Gamma(m+n)}{\Gamma(m)\Gamma(n)}$$

$$\boxed{u = \frac{m}{m+n}}$$

$$\therefore Var(x) = E(x^2) - [E(x)]^2 \longrightarrow ①$$

$$E(x^2) = \int_0^1 x^2 f(x)\,dx$$

$$= \int_0^1 x^2 \cdot \frac{1}{B(m,n)} x^{m-1}(1-x)^{n-1}\,dx$$

$$= \int_0^1 \frac{1}{B(m,n)} x^{m-1+2}(1-x)^{n-1}\,dx$$

$$= \int_0^1 \frac{1}{B(m,n)} x^{m+1}(1-x)^{n-1}\,dx$$

$$= \frac{B(m+2,n)}{B(m,n)} = \frac{m(m+1)}{(m+n)(m+n+1)} \longrightarrow \text{put } ①$$

$$Var(x) = \frac{m(m+1)}{(m+n)(m+n+1)} - \left(\frac{m}{m+n}\right)^2$$

$$= \frac{m(m+1)}{(m+n)(m+n+1)} - \frac{m^2}{(m+n)^2}$$

$$= \frac{m(m+1)(m+n) - m^2(m+n+1)}{(m+n)^2(m+n+1)}$$

$$= \frac{m(m^2 + mn + n + m) - m^3 - m^2 n - m^2}{(m+n)^2(m+n+1)}$$

$$Var(x) = \frac{\cancel{m^3} + \cancel{m^2 n} + \cancel{m^2} + mn - \cancel{m^3} - \cancel{m^2 n} - \cancel{m^2}}{(m+n)^2(m+n+1)}$$ 1.

$$\boxed{Var(x) = \frac{mn}{(m+n)^2(m+n+1)}}$$

Normal distribution:-

. It is the limiting form of the binomial distribution when n (no. trial) is very ~~small~~ large neighter p and nor q is very small normal distribution is also called gaussian distribution.

P.d.f is define as

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, \quad -\infty < x < +\infty$$

$e = 2.17828......$

$\pi = 3.14159...$

μ = Mean of the distribution

σ = standard deviation of distribution

Parameters:-

μ and σ^2 are parameter normal distribution

Properties:-

1. Total area under the curve is unity.

Proof:-

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$L.H.S = \int_{-\infty}^{\infty} f(x)\,dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{6\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-u}{6}\right)^2} dx$$

Let $Z = \frac{x-u}{6}$

$$Z6 = x - u$$

Diff both side

$$dZ6 = dx$$

$$dZ = dx/6$$

$$= \int_{-\infty}^{\infty} \frac{1}{\cancel{6}\sqrt{2\pi}} e^{-\frac{1}{2}(Z^2)} dZ\cancel{6}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-Z^2/2} dZ$$

Let $y = Z^2/2$ $\Rightarrow$

Diff. both side

$$dy . \frac{\cancel{2}Z\,dZ}{\cancel{2}}$$

$$dZ = dy/Z \longrightarrow (1)$$

$$\Rightarrow \frac{z^2}{2} = y$$

$$z^2 = 2y \Rightarrow z = \sqrt{2y} \longrightarrow \text{put } ①$$

$$dz = \frac{y}{\sqrt{2y}}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-y} \frac{dy}{\sqrt{2y}}$$

(x by 2 b) change integral value

$$= \frac{2}{\sqrt{2\pi}\sqrt{2}} \int_0^{\infty} \frac{e^{-y}}{\sqrt{y}} dy$$

$$= \frac{2}{\sqrt{2 \times 2\pi}} \int_0^{\infty} e^{-y} y^{-1/2} dy$$

$$= \frac{\cancel{2}}{\cancel{2}\sqrt{\pi}} \Gamma(1/2)$$

$\Gamma \rightarrow$ gamma

$\Gamma(1/2) = \sqrt{\pi}$

$$= \frac{1}{\cancel{\sqrt{\pi}}} \cancel{\sqrt{\pi}}$$

$$\boxed{\int_{-\infty}^{\infty} f(x)\,dx = 1}$$

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2. The Mean and Variance of the normal distribution μ and σ^2.

Proof:-

$$\mu = E(x) = \int_{-\infty}^{\infty} x f(x) dx$$

$$\mu = E(x) = \int_{-\infty}^{\infty} x \cdot \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} x \cdot e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

Let $Z = \frac{x-\mu}{\sigma}$

$$Z\sigma = x - \mu$$

$$x = Z\sigma + \mu$$

$$dx = dZ\sigma$$

$$E(x) = \frac{1}{\cancel{\sigma}\sqrt{2\pi}} \int_{-\infty}^{\infty} (Z\sigma + \mu) e^{-\frac{1}{2}(Z^2)} \cdot \cancel{\sigma} dZ$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mu e^{-Z^2/2} dZ + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} Z\sigma e^{-Z^2/2} dZ$$

$$= \frac{\mu}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-Z^2/2} dZ + \frac{\sigma}{\sqrt{2\pi}} \int_{-\infty}^{\infty} Z e^{-Z^2/2} dZ$$

First integral represent μ and Second integral odd function equals to zero.

$$\boxed{E(x) = \mu}$$

$$Var(x) = E(x-u^2)^2 = \int_{-\infty}^{\infty} (x-u)^2 f(x)dx$$

$$= \int_{-\infty}^{\infty} (x-u)^2 \frac{1}{6\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-u}{6}\right)^2} dx$$

$$= \frac{1}{6\sqrt{2\pi}} \int_{-\infty}^{\infty} (x-u)^2 e^{-\frac{1}{2}\left(\frac{x-u}{6}\right)^2} dx$$

Let $Z = \frac{x-u}{6}$

$Z6 = x-u \Rightarrow dZ6 = dx$

$$= \frac{1}{\cancel{6}\sqrt{2\pi}} \int_{-\infty}^{\infty} Z^2 6^2 e^{-Z^2/2} \cancel{6} dZ$$

$$= \frac{6^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} Z^2 e^{-Z^2/2} dZ$$

using integration by parts

$$\boxed{Var(x) = 6^2}$$ **Ans.**

Example # 9.6, 9.7

Central limit theoram:-

If $x_1, x_2, x_3, \ldots\ldots, x_n$ is a random sample of size n from any population having mean μ and finite variance σ^2, the sample mean $\bar{x}$ has an approximately normal distribution with mean μ and variance $\frac{\sigma^2}{n}$ as n increase.