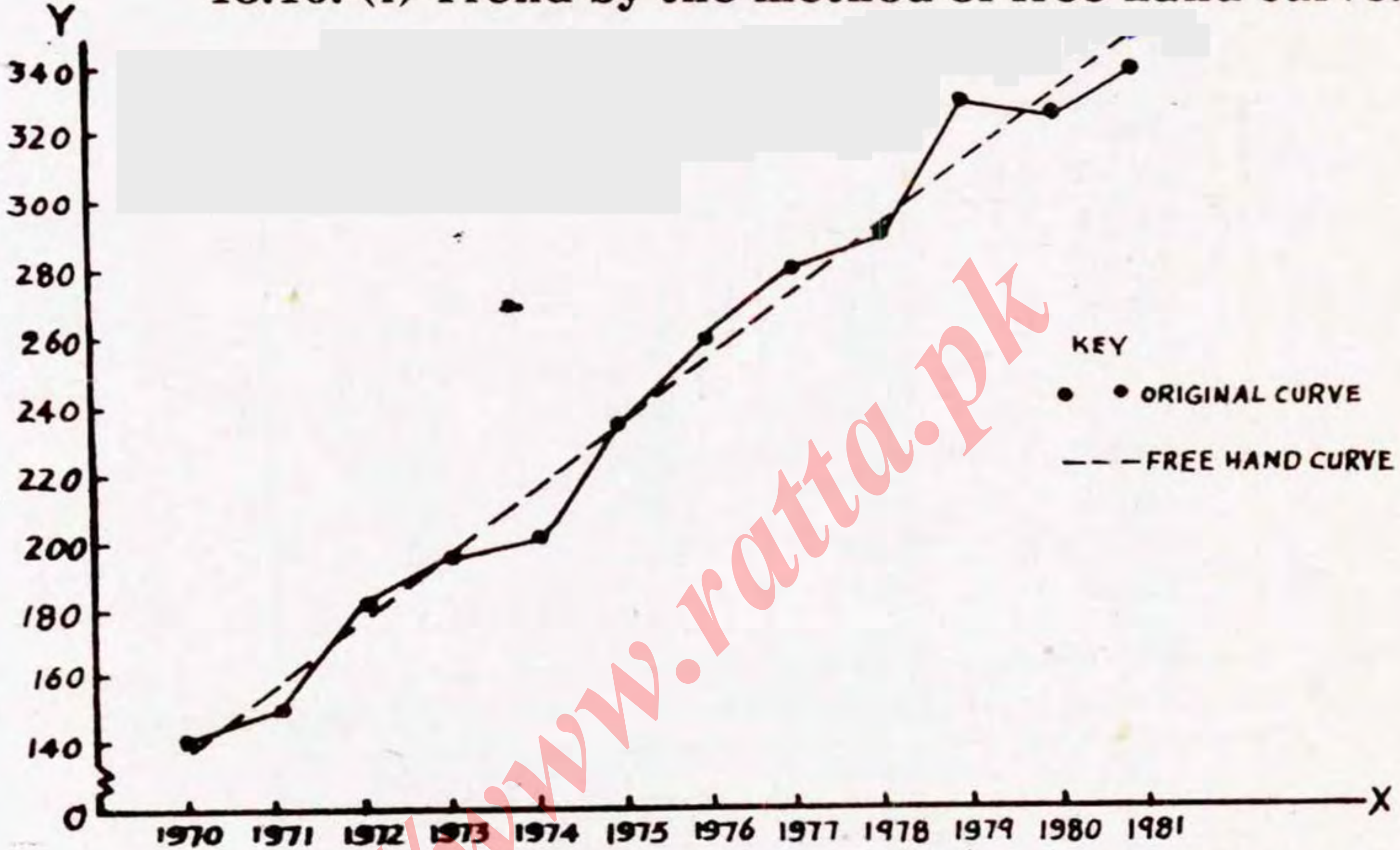
Chapter 13

TIME SERIES ANALYSIS

13.10. (i) Trend by the method of free hand curve.



In this case, the trend values can be read from the graph.

(ii) **Trend by the method of Semi-Averages.**

Year	Wages	Total	Averages	Trend values
1970	140			132.5
1971	148			152.7
1972	180	1098	1098 6 = 183	172.9
1973	195			193.1
1974	200			213.3
1975	235			233.5
1976	260			253.7
1977	280			273.9
1978	290			294.1
1979	330	1825	1825÷6 = 304	314.3
1980	325			334.5
1981	340			354.7

To compute the trend values, we should know the increase or decrease per year. It follows from the two values of the semi-averages that there has been an increase of 304−183 = 121 in 6 years (from the middle of 1972 and 1973 to the middle of 1978 and 1979) or 121 ÷ 6 = 20.2 per year. Thus the trend value for 1973 is 183 + (1/2)(20.2) = 193.1 and for 1974 is 193.1 + 20.2 = 213.3. The trend values obtained by taking into account an increase of 20.2 per year, are shown in the last column of the above table.

13.11. Calculation of the five-year moving averages.

Year	Value	5-Year Moving Total	5-Year Moving Average
1921	102	--	--
1922	108	--	--
1923	130	638	127.6
1924	140	716	143.2
1925	158	804	160.8
1926	180	884	176.8
1927	196	964	192.8
1928	210	1036	207.2
1929	220	--	--
1930	230	--	--

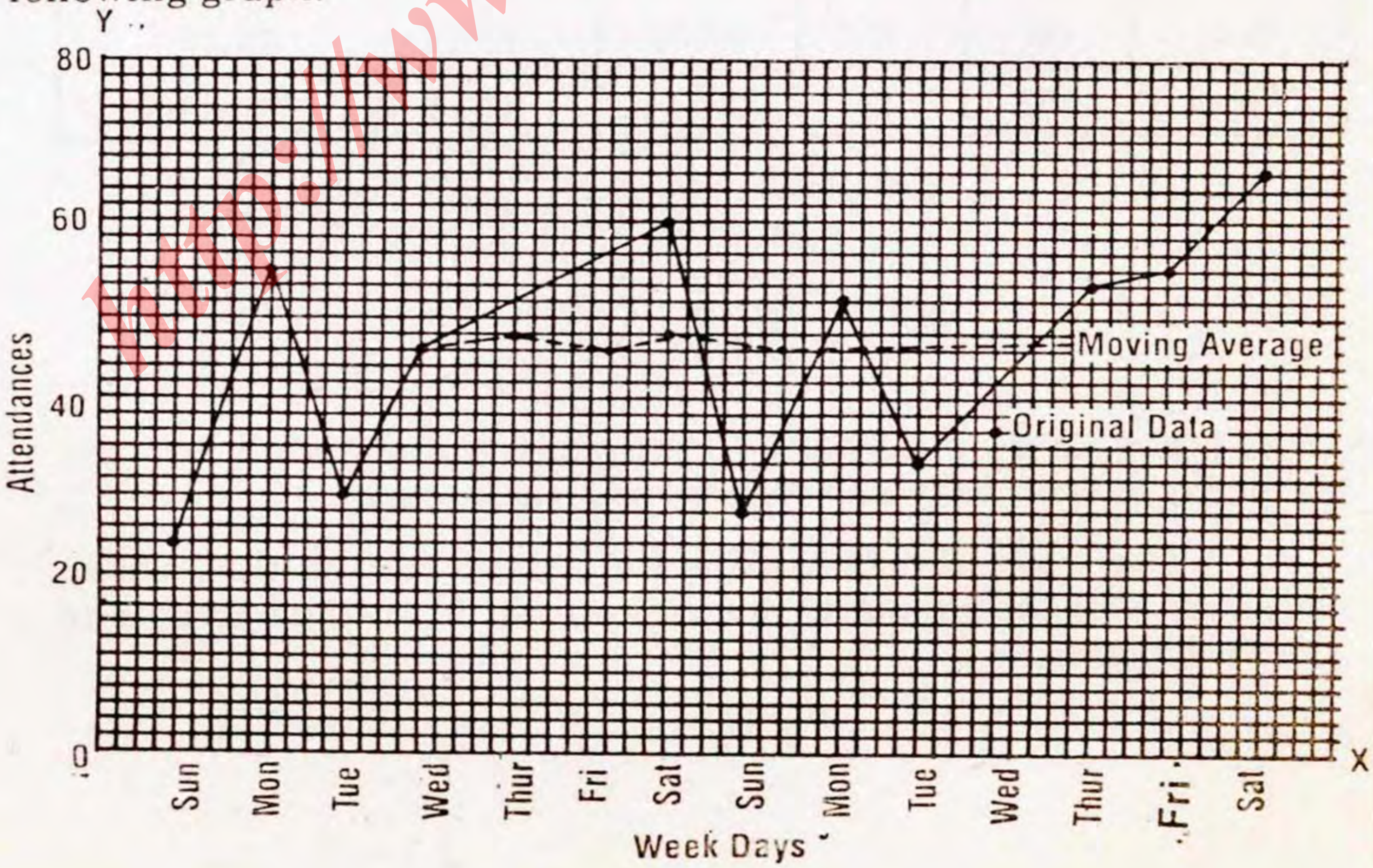
13.12. (ii) Calculation of the 5-year moving averages.

Year	Number	5-Year Moving Total	5-Year Moving Average
1951	170	--	--
1952	210	--	--
1953	188	749	149.8
1954	98	710	142.0
1955	83	705	141.0
1956	131	699	139.8
1957	205	691	138.2
1958	182	700	140.0
1959	90	710	142.0
1960	92	688	137.6
1961	141	641	128.2
1962	183	631	126.2
1963	135	599	119.8
1964	80	565	113.0
1965	60	522	104.4
1966	107	511	102.2
1967	140	---	---
1968	124	---	---

13.3 (a) Calculation of the 7-day moving average for attendances.

Week and Day	Attendances	7-Day Moving Total	7-Day Moving Average
1. Sun.	24	--	--
Mon.	55	--	--
Tues.	29	--	--
Wed.	48	324	46.3
Thur.	52	327	46.7
Fri.	55	324	46.3
Sat.	61	327	46.7
2. Sun.	27	322	46.0
Mon.	52	323	46.1
Tues.	32	324	46.3
Wed.	43	328	46.9
Thur.	53	--	--
Fri.	56	--	--
Sat.	65	--	--

The attendances and moving averages are plotted on the following graph.



(b) **The linear trend to be fitted is $\hat{Y} = a + bX$.**
The arithmetic can be arranged as in the table below:

Week & Day	Y	X	X^2	XY	Trend Values ($\hat{Y} = 46.57 + 1.28X$)
1. Sun.	24	−6.5	42.25	−156.0	38.25
Mon.	55	−5.5	30.25	−302.5	39.53
Tues.	29	−4.5	20.25	−130.5	40.81
Wed.	48	−3.5	12.25	−168.0	42.09
Thur.	52	−2.5	6.25	−130.0	43.37
Fri.	55	−1.5	2.25	−82.5	44.65
Sat.	61	−0.5	0.25	−30.5	45.93
2. Sun.	27	+0.5	0.25	13.5	47.21
Mon.	52	1.5	2.25	78.0	48.49
Tues.	32	2.5	6.25	80.0	49.77
Wed.	43	3.5	12.25	150.5	51.05
Thur.	53	4.5	20.25	238.5	52.33
Fri.	56	5.5	30.25	308.0	53.61
Sat.	65	6.5	42.25	422.5	54.89
Total	652	0	227.50	219.0	651.98

Now $$a = \frac{\sum Y}{n} = \frac{652}{14} = 46.57, \text{ and } \quad (\because \sum X = 0)$$

$$b = \frac{\sum XY}{\sum X^2} = \frac{291.0}{227.5} = 1.28$$

$\therefore \quad \hat{Y} = 46.57 + 1.28X$, with origin at the midpoint between Saturday of first week and Sunday of second week.

The trend values are determined by putting the various values of X in the equation and they appear in the last column of the above table.

13.14. Calculation of the 7-year moving averages.

Year	Index	7-Year Moving Total	7-Year Moving Average
1924	187	--	--
1925	161	--	--
1926	149	--	--
1927	142	1026	146.6
1928	125	966	138.0
1929	129	935	133.6
1930	133	915	130.7
1931	127	902	128.9
1932	130	907	129.6
1933	129	914	130.6
1934	129	933	133.3
1935	130	977	139.6
1936	136	1016	145.1
1937	152	1105	157.9
1938	171	1234	176.3
1939	169	1383	197.6
1940	218	1542	220.3
1941	258	1704	243.4
1942	279	--	--
1943	295	--	--
1944	314	--	--

13.15.(a) Calculation of the Nine-Yer moving average.

Year	Value	9-Year Moving Total	9-Year Moving Average
1	8	--	--
2	7	--	--
3	5	--	--
4	2	--	--
5	4	62	6.9
6	9	60	6.7
7	10	57	6.3
8	9	59	6.6
9	8	68	7.6
10	6	77	8.6
11	4	79	8.8
12	7	78	8.7
13	11	77	8.6
14	13	74	8.2
15	11	78	8.7
16	9	87	9.7
17	8	95	10.6
18	5	96	10.7
19	10	93	10.3
20	13	90	10.0
21	15	87	9.7
22	12	90	10.0
23	10	97	10.8
24	8	103	11.4
25	6	--	--
26	11	--	--
27	12	--	--
28	16	--	--

(b) Computation of 4-month centred moving average.

Month (1)	Value (2)	4-Month moving totals (3)	4-Month *centred* moving Totals (4)	4-Month Centred moving average (Col. 4÷8)
1	23		--	--
2	26		--	--
		107		
3	28		222	27.75
		115		
4	30		239	29.88
		124		
5	31		257	32.12
		133		
6	35		268	33.50
		135		
7	37		273	34.12
		138		
8	32		279	34.88
		141		
9	34		--	--
10	38		--	--

13.16. Calculaton of the centred moving averages of 4-Quarters.

Year and Quarter (1)	Index Number (2)	4-Quarters moving totals (3)	4-Quarters *centred* moving Totals (4)	4-Quarters Centred moving average (Col. 4÷8)
1951-I	86		--	--
II	80		--	--
		333		
III	83		665	83.12
		332		
IV	84		664	83.00
		332		
1952-I	85		661	82.62
		329		
II	80		652	81.50
		323		
III	80		638	79.15
		315		
IV	78		630	78.75
		315		
1953-I	77		631	78.89
		316		
II	80		634	79.25
		318		
III	81		641	80.12
		323		
IV	80		647	80.88
		324		
1954-I	82		650	81.25
		326		
II	81		654	81.75
		328		
III	83		657	82.12
		329		
IV	82		661	82.62
		332		
1955-I	83		666	83.25
		334		
II	84		672	84.00
		338		
III	85		--	--
IV	86		--	--

13.17. Calculation of the 4-quarter centred moving averages.

Year and Quarter (1)	Data (2)	4-Quarters moving totals (3)	4-Quarters centred moving Totals (4)	4-Quarters Centred moving average (Col. 4÷8)
1977-I	102		--	--
II	71		--	--
		318		
III	47		659	82.38
		341		
IV	98		717	89.62
		376		
1978-I	125		778	97.25
		402		
II	106		937	117.12
		535		
III	73		1226	153.25
		691		
IV	231		1505	188.12
		814		
1979-I	281		1764	220.50
		950		
II	229		2157	269.62
		1207		
III	209		2617	327.12
		1410		
IV	488		3038	379.75
		1628		
1980-I	484		3504	438.00
		1876		
II	447		4230	528.75
		2354		
III	457		--	--
IV	966		--	--

13.18. (a) Calculation of the 2-year centred moving averages.

Year (1)	Visitors (2)	2-Year moving totals (3)	2-Year *centred* moving Totals (4)	2-Year Centred moving average (Col. 4÷4) (5)
1971	31		--	--
		80		
1972	49		203	50.75
		123		
1973	74		259	64.75
		136		
1974	62		263	65.75
		127		
1975	65		265	66.25
		138		
1976	73		281	70.25
		143		
1977	70		297	74.25
		154		
1978	84		324	81.00
		170		
1979	86		335	83.75
		165		
1980	79	--	--	--

Calculation of the 3-year weighted moving average with weights 1, 2, 1 respectively.

Year (1)	Visitors (2)	3-Year weighted moving Totals (3)	3-Year weighted moving average (Col. 3÷4) (4)
1971	31	--	--
1972	49	203	50.75
1973	74	259	64.75
1974	62	263	65.75
1975	65	265	66.25
1976	73	281	70.25
1977	70	297	74.25
1978	84	324	81.00
1979	86	335	83.75
1980	79	--	--

The calculations shown above indicate that the 2-year *centred* moving average is equivalent to a 3-year weighted moving average with weights 1, 2, 1 respectively.

(b) Calculation of the 3-year weighted moving average with weights, 1, 4, 1 respectively.

Year (1)	Visitors (2)	3-Year weighted moving Totals (3)	3-Year weighted moving average (Col. 3÷6) (4)
1971	31	--	--
1972	49	301	50.17
1973	74	407	67.83
1974	62	387	64.50
1975	65	395	65.83
1976	73	427	71.17
1977	70	437	72.83
1978	84	492	82.00
1979	86	507	84.50
1980	79	--	--

13.19. Trend by using the method of

(i) Semi-averages **(ii) 3-year moving averages**

Year	Values	Totals	Average	Trend Values	Values	3-Year Moving Total	3-Year Moving Average
1968	2			3.8	2	--	--
1969	4			4.6	4	12	4.0
		20	5				
1970	6			5.4	6	18	6.0
1971	8			6.2	8	21	7.0
1972	7			7.0	7	21	7.0
1973	6			7.8	6	21	7.0
1974	8			8.6	8	24	8.0
		36	9				
1975	10			9.4	10	30	10.0
1976	12			10.2	12	--	--

(iii) Least-squares for fitting a straight line.

Year	Values Y	X	X^2	XY	Trend Values
1968	2	−4	16	−8	3
1969	4	−3	9	−12	4
1970	6	−2	4	−12	5
1971	7	−1	1	−8	6
1972	7	0	0	−40	7
1973	6	1	1	6	8
1974	8	2	4	16	9
1975	10	3	9	30	10
1976	12	4	16	48	11
Total	63	0	60	+100 / + 60	63

Let the equation of the straight line be

$$\hat{Y}_t = a + bX.$$

Then the two normal equations reduce to

$$\sum X = na$$

$$\sum XY = b\sum X^2$$

Substituting the values, we get

$$a = \frac{\sum Y}{n} = \frac{63}{9} = 7, \text{ and}$$

$$b = \frac{\sum XY}{\sum X^2} = \frac{60}{60} = 1$$

Hence the required equation is $\hat{Y}_t = 7 + X$ with origin at 1972. The trend values are obtained by substituting the values of X in the fitted equation. These values appear in the last column of the above table. We prefer the least-squares trend as it is the better fit.

13.20. (b) Determination of the trend by using the method of

(i) 3-year moving averages **(ii) Least-squares for fitting a st. line**

Year	Value	3-year moving Total	3-year moving Average	Value Y	X	X^2	XY	Trend value
1970	12	--	--	12	−4	16	−48	18.09
1971	23	72	24.0	23	−3	9	−69	24.04
1972	37	108	36.0	37	−2	4	−74	29.99
1973	48	126	42.0	48	−1	1	−48	35.94
1974	41	126	42.0	41	0	0	−239	41.89
1975	37	126	42.0	37	1	1	37	47.84
1976	48	146	48.7	48	2	4	96	53.79
1977	61	179	59.7	61	3	9	183	59.74
1978	70	--	---	70	4	16	280	65.69
Total				377	0	60	596 +357	377.01

Let the equation of the straight line be

$$\hat{Y}_t = a + bX$$

Then the two normal equations reduce to

$$\sum Y = na \quad \text{and} \quad \sum XY = b\sum X^2.$$

Substituting the values, we get

$$a = \frac{\sum Y}{n} = \frac{377}{9} = 41.89, \text{ and } b = \frac{\sum XY}{\sum X^2} = \frac{357}{60} = 5.95$$

Hence the equation of the least-squares trend is $\hat{Y}_t = 41.89 + 5.95X$, with origin at 1974 and units of X are 1 year. The trend values are computed by substituting the values of X in the equation and are shown in the last column.

We prefer the *least-squares* trend as it is the better fit.

13.21. (a) Fitting a linear trend.

Let the equation of the linear trend be $\hat{Y}_t = a + bX$. Then the two normal equations for determining a and b are

$$\sum Y = na + b\sum X \text{ and } \sum XY = a\sum X + b\sum X^2$$

Since the number of years in the data is odd, we assign $X=0$ to the middle year 1930, $X=1,2$ to the successive years and $X=-1, -2$ to the preceding years. The normal equatons then reduce to

$$\sum Y = na \text{ and } \sum XY = b\sum X^2$$

The arithmetic can be arranged as in the table below:

Year	X	Index (Y)	XY	X^2	Trend Values
1928	−2	125	−250	4	124.4
1929	−1	114	−114	1	112.0
1930	0	99	0	0	99.6
1931	1	80	80	1	87.1
1932	2	80	160	4	74.7
$\sum$	---	498	−124	10	497.8

Substituting, we get $a = \frac{\sum Y}{n} = \frac{498}{5} = 99.6$, and

$$b = \frac{\sum XY}{\sum X^2} = \frac{-124}{10} = -12.4$$

Hence the equation of the desired linear trend is

$$\hat{Y}_t = 99.6 - 12.4X,$$

with origin at 1930 and units of X are 1 year. The trend values are shown in the last column of the above table.

(b) **The equation of the straight line is** $\hat{Y} = a + bX$

Since $\sum X = 0$ (given), the normal equations are

$$\sum Y = na \text{ and } \sum XY = b\sum X^2.$$

Substituting the values, we get

$$a = \frac{\sum Y}{n} = \frac{438.9}{11} = 39.9, \text{ and} \qquad (\because \; n = 11)$$

$$b = \frac{\sum XY}{\sum X^2} = \frac{-84.4}{110} = -0.7673.$$

Hence the required equation is

$Y = 39.9 - 0.7673X$, with origin at 1953.

Putting $X = -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5$ in the equation, we obtain the trend values as

43.74, 42.97, 42.20, 41.43, 40.67, 39.90, 39.13, 38.37, 37.60, 36.83, 36.06.

13.22. Fitting of a straight line trend and a parabolic trend.

As the number of years in the data is odd, we can assign $X=0$ to the middle year 1980, $X=1,2,3$ to the succeeding years and $X=-1, -2, -3$ to the preceding years.

(i) Let the equation of the straight line be

$$Y_t = a + bX.$$

Then the two normal equations reduce to

$$\sum Y = na \text{ and } \sum XY = b\sum X^2.$$

(ii) Let the equation of the quadratic parabola be $Y_t = a + bX + cX^2$.

Then the three normal equations reduce to

$$\sum Y = na + c\sum X^2$$
$$\sum XY = b\sum X^2$$
$$\sum X^2 Y = a\sum X^2 + c\sum X^4$$

The arithmetic can be arranged as in the table below:

Year	Y	X	X^2	X^4	XY	X^2Y	Y^2
1977	88	−3	9	81	−264	792	7744
1978	101	−2	4	16	−202	404	10201
1979	105	−1	1	1	−105	105	11025
1980	91	0	0	0	0	0	8281
1981	113	1	1	1	113	113	12769
1982	120	2	4	16	240	480	14400
1983	132	3	9	81	396	1188	17424
$\sum$	750	0	28	196	178	3082	81844

Substituting the values in the normal equations of the straight line, we get

$$a = \frac{\sum Y}{n} = \frac{750}{7} = 107.14, \text{ and}$$

$$b = \frac{\sum XY}{\sum X^2} = \frac{178}{28} = 6.38$$

Hence the equation of the straight line is

$$\hat{Y} = 107.14 + 6.38X,$$

with origin at 1980 and units of X are 1 year.

The estimate of the profits in 1985 is obtained by putting $X=5$ in the fitted equation. Thus

$$\hat{Y}_t = 107.14 + 6.38(5) = 139.04$$

Substituting the values in the normal equations of the quadratic parabola, we get

$$b = \frac{\sum XY}{\sum X^2} = \frac{178}{28} = 6.36; \text{ and}$$

$$7a + 28c = 750$$

$$28a + 196c = 3082$$

Solving them simultaneously, we get

$$a = 103.24, \text{ and } c = 0.976$$

Hence the equation of the desired parabola trend is

$$\hat{Y}_t = 103.24 + 6.36X + 0.796X^2,$$

with origin at 1980 and units of X are 1 year.

(iii) In order to determine which is the better fitting trend, we calculate the sum of squares of residuals in both cases. In case of straight line trend, the sum of squares of residuals is

$$\sum(Y-\hat{Y}_t)^2 = \sum Y^2 - a\sum Y - b\sum XY$$

$$= 81844 - (107.14)(350) - (6.38)(178)$$

$$= 81844 - 80355 - 1135.64 = 353.36$$

The sum of squares of residuals in case of parabolic trend is

$$\sum(Y-\hat{Y}_t)^2 = \sum Y^2 - a\sum Y - b\sum XY - c\sum X^2Y$$

$$= 81844 - (103.24)(750) - (6.38)(178) - (0.976)(3.082)$$

$$= 81844 - 77430 - 1135.64 - 3008.03$$

$$= 270.33$$

The parabolic trend is the better fit as it has the smaller sum of squares of residuals.

13.23. Let the equation of the second degree parabola be

$$Y_t = a + bX + cX^2 .$$

Then the normal equations for determining a, b and c are:

$$\sum Y = na + b\sum X + c\sum X^2$$

$$\sum XY = a\sum X + b\sum X^2 + c\sum X^3$$

$$\sum X^2Y = a\sum X^2 + b\sum X^3 + c\sum X^4$$

The arithmetic can be arranged as in the table below:

Year	Production Y	X	X^2	X^3	X^4	XY	X^2Y
1970–71	136	0	0	0	0	0	0
1971–72	162	1	1	1	1	162	162
1972–73	187	2	4	8	16	374	748
1973–74	225	3	9	27	81	675	2025
1974–75	272	4	16	64	256	1088	4352
1975–76	277	5	25	125	625	1385	6925
1976–77	322	6	36	216	1296	1932	11592
$\sum$	1581	21	91	441	2275	5616	25804

Substituting these sums, we get

$$7a + 21b + 91c = 1581$$

$$21a + 91b + 441c = 5616$$

$$91a + 441b + 2275c = 25804$$

Solving these equations simultaneously, we get

$$a = 133.09,\ b = 30.25 \text{ and } c = 0.155$$

Hence the equation of the desired quadratic parabola is

$$Y_t = 133.09 + 30.25X + 0.155X^2$$

Estimated production for 1978–79 is obtained by putting $X = 8$ in the fitted equation. Therefore

$$\hat{Y}_t = 133.09 + 30.25(8) + 0.155(8)^2 = 385$$

13.24. The equation of the parabola of second order is

$$Y_t = a + bX + cX^2.$$

Since the number of years is odd, we therefore assign $X = 0$ to the middle year, *i.e.* 1960 and $X = -1, -2, \ldots$ to the preceding years and $X = 1, 2, \ldots$ to the successive years.

The normal equations then become

$$\sum Y = na + c\sum X^2$$
$$\sum XY = b\sum X^2$$
$$\sum X^2Y = a\sum X^2 + c\sum X^4$$

The arithmetic can be arranged as in the table below:

Year	Production Y	X	X^2	X^4	XY	X^2Y	Trend values
1955	17	−5	25	625	−85	425	18.52
1956	20	−4	16	256	−80	320	18.85
1957	19	−3	9	81	−57	171	20.32
1958	26	−2	4	16	−52	104	22.93
1959	24	−1	1	1	−24	24	26.68
1960	40	0	0	0	0	0	31.57
1961	35	1	1	1	35	35	37.60
1962	35	2	4	16	70	140	44.77
1963	51	3	9	81	153	459	53.08
1964	74	4	16	256	296	1184	62.53
1965	69	5	25	625	345	1725	73.12
$\sum$	410	0	110	1958	601	4587	409.97

Substituting these summations in the normal equations, we get

$$11a + 110c = 410$$

$$110b = 601$$

$$110a + 1958c = 4587$$

Solving them simultaneously, we obtain

$$a = 31.57,\ b = 5.46 \text{ and } c = 0.57$$

Hence the equation of the required parabola is

$$Y_t = 31.57 + 5.46X + 0.57X^2$$

with origin at 1960 and units of X are 1 year.

The trend values are obtained by putting the different values of X in the above equation, and they are given in the last column in the table above.

13.25. The number of years is even and they are equispaced with interval of one year. We therefore assign X=0 to the midpoint of the years 1954 and 1955, X=−1,−3,−5,−7 to the preceding years and X=1,3,5,7 to the successive years (half year units), so that

$$\sum X = 0 = \sum X^3.$$

The equation of the parabola fitting the data is

$$Y_t = a_0 + a_1X + a_2X^2 \quad (t \text{ is coded})$$

The normal equations for determining a_0, a_1 and a_2 then reduce to

$$\sum Y = na_0 + a_2\sum X^2$$

$$\sum XY = a_1\sum X^2$$

$$\sum X^2Y = a_0\sum X^2 + a_2\sum X^4$$

The arithmetic involved in computation is arranged in the following table:

Year (t)	Coded year (X)	Values (Y)	X^2	X^4	XY	X^2Y	Trend values
1951	−7	273.7	49	2401	−1915.9	13411.3	273.96
1952	−5	293.5	25	625	−1467.5	7337.5	292.90
1953	−3	315.0	9	81	−945.0	2835.0	314.32
1954	−1	336.8	1	1	−336.8	336.8	338.22
1955	+1	364.4	1	1	364.4	364.4	364.60
1956	3	394.8	9	81	1184.4	3553.2	393.46
1957	5	424.2	25	625	2121.0	10605.0	424.80
1958	7	458.7	49	2401	3210.9	22476.3	458.62
$\sum$	0	2861.1	168	6216	2215.5	60919.5	2860.88

Substituting these values in the normal equations, we get

$$8a_0 + 168a_2 = 2861.1$$

$$168a_1 = 2215.5$$

$$168a_0 + 6216\,a_2 = 6091.5$$

Solving these equations, we obtain

$$a_0 = 351.1023, \quad a_1 = 13.1875 \quad \text{and} \quad a_2 = 0.3112$$

Hence, the equation of the required parabola is

$$\hat{Y}_t = 351.10 + 13.19X + 0.31X^2,$$

with origin at the midpoint of 1954 and 1955 (i.e. January 1, 1955) and X is measured in units of a half year. The trend values appear in the last column.

13.26. Let the equation of the quadratic parabola be

$$\hat{Y}_t = a + bX + cX^2$$

The normal equations for determining a, b and c are:

$$\sum Y = na + b\sum X + c\sum X^2$$

$$\sum XY = a\sum X + b\sum X^2 + c\sum X^3$$

$$\sum X^2Y = a\sum X^2 + b\sum X^3 + c\sum X^4$$

The arithmetic can be arranged as in the table below:

Year	Y	X	X^2	X^3	X^4	XY	X^2Y
1924	187	0	0	0	0	0	0
1927	142	3	9	27	81	426	1278
1930	133	6	36	216	1296	798	4788
1933	129	9	81	729	6561	1161	10449
1936	136	12	144	1728	20736	1632	19584
1939	169	15	225	3375	50625	2535	38025
1942	279	18	324	5832	104976	5022	90396
Σ	1175	63	819	11907	184275	11574	164520

Substituting these sums, we get

$$7a + 63b + 819c = 1175 \quad \ldots (A)$$

$$63a + 819b + 11907c = 11574 \quad \ldots (B)$$

$$819a + 11907b + 184275c = 164520 \quad \ldots (C)$$

Now 9(A): $63a + 567b + 7371c = 10575 \quad \ldots (D)$

(C)-(D): $252b + 4536c = 999 \quad \ldots (E)$

13(B): $819a + 10647b + 154791c = 150462 \quad \ldots (F)$

(C)-(F): $1260b + 29484c = 14058 \quad \ldots (G)$

5(E): $1260b + 22680c = 4995 \quad \ldots (H)$

(G)-(H): $6804c = 9063$

$\therefore \quad c = 1.332$

Putting $c = 1.332$ in (E), we get

$$252b + 4536\,(1.332) = 999$$

or $\quad 252b = 999 - 6041.952$

or $\quad b = \dfrac{999 - 6041.952}{252} = -20.012$

Putting $b = -20.012$ and $c = 1.332$ in (A), we get $a = 192$

Hence the required equation of the quadratic parabola is

$$\hat{Y}_t = 192 - 20.012X + 1.332\,X^2,$$

with origin at 1924.

To estimate the value of the index for 1935, we put $X = 11$ in the above equation and get

$$\hat{Y} = 192 - 20.012\,(11) + (1.332)\,(11)^2$$

$$= 192 - 220.132 + 161.172 = 133.$$

13.27. The given curve $Y = ab^X$ may be written as

$$\log Y = \log a + X \log b$$

or $\quad Y' = A + BX, \qquad$ (Form of a st. line)

where $\quad Y' = \log Y,\ A = \log a$ and $B = \log b$.

The two normal equations are

$$\sum Y' = nA + B\sum X$$

$$\sum XY' = A\sum X + B\sum X^2$$

The calculations involved are shown in the following table:

Year	X	Y	X^2	$Y' = \log Y$	XY'
1911	0	5.38	0	0.7308	0
1921	1	7.22	1	0.8585	0.8585
1931	2	9.64		0.9841	1.9682
1941	3	12.70	9	1.1038	3.3114
1951	4	17.80	16	1.2504	5.0016
1961	5	24.02	25	1.3806	6.9030
1971	6	31.34	36	1.4961	8.9766
$\sum$	21	--	91	7.8043	27.0193

Substituting these values in the normal equations, we get

$$7A + 21B = 7.8043$$

$$21A + 91B = 27.0193$$

Solving them, we find that $A = 0.7285$ and $B = 0.1288$.

Thus a = antilog of A

= antilog (0.7285) = 5.35, and

b = antilog of B

= antilog (0.1288) = 1.345.

Hence the equation of the required curve is

$\hat{Y} = 5.35\ (1.345)^X$, where origin is at 1911 and X is measured in ten yearly intervals.

In order to forecast the populaton for the year 1991, we put $X=8$ in the fitted equation. Thus $Y_{1991} = 5.35\ (1.345)^8 = 57.30$

13.28. (b) To compute the seasonal indices, we first find the 4-quarter centred moving averages.

Year and Quarter (1)	Y (2)	4-quarter moving totals (3)	4-quarter *centred* moving Totals (4)	4-quarter *centred* moving Averages (Col 4÷8)	$\frac{\text{Data}}{\text{Trend}} \times 100$
	TSCI			*TC*	*SI*
1949 – 1	105		--	--	--
2	77		--	--	--
		345			
3	68		692	86.5	78.6
		347			
4	95		700	87.5	108.6
		353			
1950–1	107		712	89.0	120.2
		359			
2	83		729	91.1	91.1
		370			
3	74		750	93.8	78.9
		380			
4	106		776	97.0	109.3
		396			
1951–1	117		804	100.5	116.4
		408			
2	99		822	102.8	96.3
		414			
3	86		--	--	--
4	112		--	--	--

The percentages are put in the following table in order to obtain the seasonal indices:

Year	Quarters				
	1	2	3	4	
1949	--	--	78.6	108.6	
1950	120.2	91.1	78.9	109.3	
1951	116.4	96.3	--	--	
Total	236.6	187.4	157.5	217.9	Total
Mean (S.I.)	118.30	93.70	78.75	108.95	399.7

The sum of these mean percentages is 399.7 which is close to the desired 400, so that no adjustment is necessary. Hence the desired seasonal indices are 118.30, 93.70, 78.75 and 108.95.

13.29. To compute the seasonal indices, we first find the 4-quarter centred moving averages.

Year and Quarter (1)	Prices Y (2)	4-quarter moving Total (3)	4-quarter *centred* moving Totals (4)	4-quarter *centred* moving Averages (Col 4÷8)	$\frac{\text{Data}}{\text{Trend}} \times 100$
	TSCI			*TC*	*SI*
1961 – I	122		--	--	--
II	125		--	--	--
		482			
III	118		961	120.1	98.3
		479			
IV	117		947	118.4	98.8
		468			
1962–I	119		932	116.5	102.1
		464			
II	114		920	115.0	99.1
		456			
III	114		898	112.2	101.6
		442			
IV	109		869	108.6	100.4
		427			
1963–I	105		833	104.1	100.9
		406			
II	99		792	99.0	100.0
		386			
III	93		753	94.1	98.8
		367			
IV	89		715	89.4	99.6
		348			
1964–I	86		686	85.8	100.2
		338			
II	80		671	83.9	95.4
		333			
III	83	--	--	--	--
IV	84	--	--	--	--

The percentages are put in the following table in order to obtain the seasonal indices:

Year	Quarters				
	I	II	III	IV	
1961	--	--	98.3	98.8	
1962	102.1	99.1	101.6	100.4	
1963	100.9	100.0	98.8	99.6	
1964	120.2	95.4	--	--	
Total	303.2	294.5	298.7	298.8	Total
Mean	101.1	98.2	99.6	99.6	398.5
Seasonal Index	101.5	98.6	100	100	400.1

13.30. (a) To compute the seasonal indices by the percentage of annual average method, we first compute the annual-averages as below:

Year	Quarters				Annual or Yearly	
	I	II	III	IV	Total	Average
1976	118	87	47	83	335	83.75
1977	94	73	41	68	276	69.00
1978	73	61	36	56	226	56.50

We next divide each of the quarterly observations by the corresponding annual-average and express the result as a percentage. The percentages so obtained are given below:

Year	Quarters				Total
	I	II	III	IV	
1976	140.90	103.88	56.12	99.10	
1977	136.23	105.80	59.42	98.55	
1978	129.20	107.96	63.72	99.12	
Total	406.33	317.64	179.26	296.77	
Mean	135.44	105.88	59.75	98.92	399.99

The sum of the mean percentages is 399.99 which is close to the desired total of 400, therefore no adjustment is necessary. Hence the desired seasonal indices are 135.44, 105.88, 59.75 and 98.92.

(b) To compute the indices of seasonal variations using ratio-to-moving average method, we first obtain the 4-quarter centred moving averages.

Year and Quarter (1)	Sales Y (2)	4-quarter moving Total (3)	4-quarter *centred* moving Totals (4)	4-quarter *centred* moving Averages (Col 4÷8)	$\frac{\text{Data}}{\text{Trend}} \times 100$
	TSCI			*TC*	*SI*
1976 – I	118		--	--	--
II	87		--	--	--
		335			
III	47		646	80.8	58.2
		311			
IV	83		608	76.0	109.2
		297			
1977–I	94		588	73.5	127.9
		291			
II	73		567	70.9	103.0
		276			
III	41		531	66.4	61.7
		255			
IV	68		498	62.2	109.3
		243			
1978–I	73		481	60.1	121.5
		238			
II	61		464	58.0	105.2
		226			
III	36	--	--	--	--
IV	56	--	--	--	--

The percentages are put in the following table in order to obtain the seasonal indices:

Year	Quarters				
	I	II	III	IV	
1976	--	--	58.2	109.2	
1977	127.9	103.0	61.7	109.3	
1978	121.5	105.2	--	--	
Total	249.4	208.2	119.9	218.5	Total
Mean	124.7	104.1	60.0	109.2	398.0
Seasonal Index	125.3	104.6	60.3	109.7	399.9

Deseasonalization of Data for 1977.

Quarter	Sales	Seasonal Index	Deseasonalized Data
I	94	125.3	75.0
II	73	104.6	69.8
III	41	60.3	68.0
IV	68	109.7	62.0

13.31. To estimate the seasonal indices, we first compute the trend by means of centred moving averages.

Year and Quarter (1)	Sales Y (2)	4-quarter moving Total (3)	4-quarter *centred* moving Totals (4)	4-quarter *centred* moving Averages (Col 4÷8)	$\frac{\text{Data}}{\text{Trend}} \times 100$
	TSCI			*TC*	*SI*
1974 – I	48		--	--	--
II	52		--	--	--
		151			
III	16		304	38.00	42.1
		153			
IV	35		300	37.50	93.3
		147			
1975–I	50		300	37.50	133.3
		153			
II	46		311	38.88	118.3
		158			
III	22		334	41.75	52.7
		176			
IV	40		340	42.50	94.1
		164			
1976–I	68		332	41.50	163.9
		168			
II	34		331	41.38	82.2
		163			
III	26		351	43.88	59.3
		188			
IV	35		398	49.75	70.4
		210			
1977–I	93		410	51.25	181.5
		200			
II	56		410	51.25	109.3
		210			
III	16		411	51.38	31.1
		201			
IV	45		407	50.88	88.4
		206			
1978–I	84		425	53.12	158.1
		219			
II	61		441	55.12	110.7
		222			
III	29		--	--	--
IV	23		--	--	--

The percentages are put in the following table in order to obtain the seasonal indices:

Year	Quarters				
	I	II	III	IV	
1974	--	--	42.1	93.3	
1975	133.3	118.3	52.7	94.1	
1976	163.9	82.2	59.3	70.4	
1977	181.5	109.3	31.1	88.4	
1978	158.1	110.7	--	--	
Total	636.8	420.5	185.2	346.2	Total
Mean	159.2	105.1	46.3	86.6	397.2
Seasonal Index	160.3	105.8	46.6	87.2	399.9

To forecast sales for each quarter of 1979, first find the projected trend values for these quarters by extending the moving averages (graphically) to thse quarters, then mutliply these projected trend values by the corresponding seasonal indices and divide each product by 100.

13.32. Let the equation of the linear trend be $Y_t = a + bX$. Since the number of quarters in the observed series is even, therefore the middle point of the two middle quarters is taken as X = 0.

Year	Y	X	X^2	XY	Trend $\hat{Y}_t = 103.56 - 1.60X$
1961-I	122	−15	225	−1830	127.56
II	125	−13	169	−1625	124.36
III	118	−11	121	−1298	121.16
IV	117	− 9	81	−1053	117.96
1962-I	119	− 7	49	−833	114.76
II	114	− 5	25	−570	111.56
III	114	− 3	9	−342	108.36
IV	109	− 1	1	− 109	105.16
1963-I	105	1	1	105	101.96
II	99	3	9	297	98.76
III	93	5	25	465	95.56
IV	89	7	49	623	92.36
1964-I	86	9	81	774	89.16
II	80	11	121	880	85.96
III	83	13	169	1079	82.76
IV	84	15	225	1260	79.56
$\sum$	1657	0	1360	−2177	1656.96

Since $\sum X = 0$, the normal equations take the form

$$\sum Y = na \quad \text{and} \quad \sum XY = b\sum X^2$$

Substituting the values, we obtain

$$a = \frac{\sum Y}{n} = \frac{1657}{16} = 103.56, \text{ and}$$

$$b = \frac{\sum XY}{\sum X^2} = \frac{-2177}{1360} = -1.60$$

Thus the equation of the linear trend is

$$\hat{Y}_t = 103.56 - 1.60X,$$

with origin at the middle of IV quarter 1962 and I quarter 1963, and X is measured in half quarter units. The trend values are computed by substituting the values of X in the least-squares equation and they appear in the last column of the above table.

Next, we divide each observation in the original data by the corresponding trend value and multiply it by 100. The percentages so obtained are arranged by quarters as shown in the following table in order to compute the indices of seasonal variation.

Computation of Seasonal Indices

Year	Quarters				
	I	II	III	IV	
1961	95.64	100.51	97.39	99.19	
1962	103.69	102.19	*105.20	103.65	
1963	102.98	100.24	97.32	96.36	
1964	96.46	*93.07	100.29	105.58	
Total	398.77	302.94	295.00	404.78	Total
Mean	99.69	100.98	98.33	101.20	400.20
Seasonal Indices	99.64	100.93	98.28	101.15	400

* Discard these extreme relatives before computing totals

13.33. Let the equation of the linear trend be $Y_t = a + bX$. Since the number of quarters in the observed series is even, therefore the middle point of the two middle quarters is taken as **X = 0.**

Year and quarter	Y	X	XY	X^2	$\hat{Y}_t = 109.42 + 3.81X$
1982-1	70	−11	−770	121	67.51
2	81	−9	−729	81	75.13
3	89	−7	−623	49	82.75
4	115	−5	−575	25	90.37
1983-1	75	−3	−225	9	97.99
2	93	−1	−93	1	105.61
3	108	1	108	1	113.23
4	152	3	456	9	120.85
1984-1	80	5	400	25	128.47
2	105	7	735	49	136.09
3	150	9	1350	81	143.71
4	195	11	2145	121	151.33
Σ	1313	0	2179	572	----

Since $\sum X = 0$, the normal equations take the form

$$\sum Y = na \quad \text{and} \quad \sum XY = b\sum X^2.$$

Substituting the values, we obtain

$$a = \frac{\sum Y}{n} = \frac{1313}{12} = 109.42, \text{ and}$$

$$b = \frac{\sum XY}{\sum X^2} = \frac{2179}{572} = 3.81$$

Thus the trend line is $\hat{Y}_t = 109.42 + 3.81X$, with origin at the middle of 2nd and 3rd quarters of 1983, and X is measured in half quarter units. The trend values are computed by substituting the values of X in the least-squares equation and they appear in the last column of the above table.

Next, we divide each observation in the original data by the corresponding trend value and multiply it by 100. The percentages so obtained are arranged by quarters as shown in the following table in order to compute the indices of seasonal variation.

Computation of Seasonal Indices

Year	Quarters				
	Summer	Autumn	Winter	Spring	
1982	*103.69	*107.81	107.55	127.25	
1983	76.54	88.06	*95.38	125.78	
1984	62.27	77.15	104.38	128.86	
Total	138.81	165.21	211.93	381.89	Total
Mean	69.40	82.60	105.96	127.30	385.26
Seasonal Indices	72.06	85.76	110.02	132.18	400.02

* Discard these extreme relatives before computing totals.

$$\text{Deseasonalized data} = \frac{\text{Period's original value}}{\text{Period's seasonal index}} \times 100$$

$$= \frac{TSCI}{S} \times 100 = TCI \times 100$$

Thus the deseasonalized data for 1984 are shown below:

Deseasonalization of the 1984 values

Quarter	Y	Seasonal Index	Diseasonalized Data
Summer	80	72.06	111.02
Autumn	105	85.76	122.43
Winter	150	110.02	136.34
Spring	195	132.18	147.53

13.34. Let the equation of the linear trend be $Y_t = a + bX$. Since the number of quarters in the observed series in even, therefore the middle of the two middle quarters is taken as X=0.

Year	Y	X	X^2	XY	Trend values $\hat{Y}_t = 114 + 0.247X$
1972-I	107	−15	225	−1605	110.295
II	115	−13	169	−1495	110.789
III	103	−11	121	−1133	111.283
IV	98	− 9	81	−882	111.777
1973-I	118	− 7	49	−826	112.271
II	122	− 5	25	−610	112.765
III	115	− 3	9	−345	113.259
IV	104	− 1	1	− 104	113.753
1974-I	126	+ 1	1	+126	114.247
II	129	3	9	387	114.741
III	118	5	25	590	115.235
IV	107	7	49	749	115.729
1975-I	121	9	81	1089	116.223
II	122	11	121	1342	116.717
III	116	13	169	1508	117.211
IV	103	15	225	1545	117.705
Total	1824	0	1360	336	1824

Since $\sum X = 0$, the normal equations are reduced to

$$\sum Y = na \quad \text{and} \quad \sum XY = b\sum X^2$$

Substituting the values, we get

$$a = \frac{1824}{16} = 114 \quad \text{and} \quad b = \frac{336}{1360} = 0.247$$

Thus the trend line is $Y_t = 114 + 0.247X$, with origin at the middle of IV quarter 1973 and I quarter 1974, and X is measured in half quarter units. The trend values are computed by substituting the values of X in the least squares equation and they are shown in the last column of the table on last page.

Next, we divide each observation in the original data by the corresponding trend value and multiply it by 100. The percentages thus obtained are arranged in the following table in order to compute the indices of seasonal variation.

Computation of Seasonal Indices

Year	Quarters				
	I	II	III	IV	
1972	97.01	103.80	92.56	87.67	
1973	105.10	108.19	101.54	91.43	
1974	110.29	112.43	102.40	92.46	
1975	104.11	104.53	98.97	87.51	
Total	416.51	428.95	395.47	359.07	Total
Mean (S.I.)	104.13	107.24	98.87	89.77	400

13.35. (a) Let the equation of the linear trend be $Y_t = a + bX$. Since the number of quarters in the observed series is even, therefore the middle point of the two middle quarters is taken as X = 0.

Year and quarter	Y	X	XY	X^2	Trend $\hat{Y}_t = 94.08 + 0.83X$
1949-1	105	−11	−1155	121	84.95
2	77	−9	−693	81	86.61
3	68	−7	−476	49	88.27
4	95	−5	−475	25	89.93
1950-1	107	−3	−321	9	91.59
2	83	−1	−83	1	93.25
3	74	1	74	1	94.91
4	106	3	318	9	96.57
1951-1	117	5	585	25	98.23
2	99	7	693	49	99.89
3	86	9	774	81	101.55
4	112	11	1232	121	103.21
Σ	1129	0	473	572	1128.96

Since $\sum X = 0$, the normal equations take the form

$$\sum Y = na \quad \text{and} \quad \sum XY = b\sum X^2.$$

Substituting the values, we obtain

$$a = \frac{\sum Y}{n} = \frac{1129}{12} = 94.08, \text{ and}$$

$$b = \frac{\sum XY}{\sum X^2} = \frac{473}{572} = 0.83.$$

Thus the trend line is $\hat{Y}_t = 94.08 + 0.83X$, with the origin at the middle of 2nd quarter 1950 and 3rd quarter 1950, and X is measured in half quarter units. The trend values are computed by substituting the values of X in the least-squares equation and they appear in the last column of the above table.

(b) (i) Now we divide each observation in the original data by the corresponding trend value and multiply it by 100. The percentages so obtained are arranged by quarters as shown in the following table in order to compute the indices of seasonal variation.

Computation of Seasonal Indices

Year	Quarters				
	1	2	3	4	
1949	123.60	88.90	77.04	105.64	
1950	116.82	89.01	77.97	109.76	
1951	119.11	*99.11	*84.69	108.52	
Total	359.53	177.91	155.01	323.92	Total
Mean	119.84	88.96	77.50	107.97	394.27
Seasonal Indices	121.58	90.25	78.62	109.54	399.99

* Discard these extreme relatives before computing totals.

(ii) Deseasonalization of data

$$\text{Deseasonalized data} = \frac{\text{Period's original value}}{\text{Period's seasonal index}} \times 100$$

$$= \frac{TSCI}{S} \times 100 = TCI \times 100$$

Thus the deseasonalized data are shown below:

Deseasonalization of data

Year	Quarters			
	1	2	3	4
1949	86.36	85.32	86.49	86.73
1950	88.01	91.97	94.12	96.77
1951	96.23	109.70	109.39	102.25

13.36. Expressing the data for each quarter as a percentage of the data for the previous quarter, we get the link relatives as below:

Year	Quarter			
	1	2	3	4
1970	---	111.6	103.2	85.3
1971	108.2	110.3	111.4	78.2
1972	104.3	118.3	105.6	78.7
1973	108.5	118.0	107.3	77.2
Median	108.2	114.8	106.4	78.4

Next, we calculate the chain relatives for the four quarters, taking the value of the first quarter equal to 100%. The chain relatives are:

Quarter	1	2	3	4	1
Chain Relative	100	114.8	122.1	95.7	103.5

Continuing the process, we find that the chain relative for the first quarter works out to be 103.5, which as a matter of fact, ought to have been 100. The increase of (103.5−100), *i.e.* 3.5 is due to secular trend present in the data. Adjusting for the trend, we subtract one-fourth of 3.5 from the second quarter figure, two-fourth from the third quarter figure and three-fourth from the fourth quarter figure. The figures are further adjusted to get a total of 400 and they are given below.

Quarter	1	2	3	4	Total
Adjusted chain relatives	100	113.9	120.3	93.1	427.3
Seasonal Index	93.6	106.6	112.6	87.2	400

13.37. (b) We first fit a straight line $Y_t = a + bX$, by the method of least-squares to the yearly averages, which are assumed to correspond to the midpoint of each year.

Year	Yearly Total	Yearly average (Y)	X	X^2	XY
1981	156	39.00	−1	1	−39.00
1982	180	45.00	0	0	0
1983	279	69.75	+1	1	69.75
Total	--	153.75	0	2	30.75

The two normal equations for determining a and b are

$$\sum Y = na \quad \text{and} \quad \sum XY = b\sum X^2$$

Substituting, we get $a = 51.25$ and $b = 15.38$.

Thus the trend line is $Y_t = 51.25 + 15.38X$.

The value of $b = 15.38$ indicates that Y values increase by 15.38 after every year or 1/4 of (15.38) = 3.84 after every quarter. Assuming that the given quarterly data correspond to the middle of the quarter, we calculate the trend values as below.

When $X = 0$, whcih corresponds to July 1, 1982, $Y = 51.25$. But we need the value of Y, a half quarter later. This is the trend value corresponding to third quarter of 1982.

The quarterly values found by this trend line are shown in the following table.

Year	Quarters			
	1	2	3	4
1981	30.13	33.97	37.81	41.65
1982	45.49	49.33	53.17	57.01
1983	60.85	64.69	68.53	72.37

Dividing each of the actual values by the corresponding trend value and expressing the result as a percentage, we get

Year	Quarters				
	1	2	3	4	
1981	139.40	132.47	87.28	86.43	
1982	65.95	91.22	75.23	114.02	
1983	83.81	106.66	94.85	129.89	Total
Total	289.16	330.35	257.36	330.34	1207.21
Mean	96.39	110.12	85.79	110.11	402.41
Seasonal Indices	95.81	109.46	85.28	109.45	400.03

To forecast the sales for each quarter of 1984, we first get the projected trend values for these quarters, by means of the fitted least-squares equation. The projected values are then multiplied by the corresponding seasonal indices and the product divided by 100. The desired forecasts are given below.

Quarter of 1984	Projected Trend Value	Seasonal Index	Forecast
I	76.21	95.81	73
II	80.05	109.46	88
II	83.89	85.28	72
IV	87.73	109.45	96

13.38. (b) Computations of the cyclical-irregular movements and cyclical relatives are shown in the table below. To remove the irregular variations, a three-quarter moving average has been thought appropriate.

Year & quarter (1)	Y values TSCI (2)	Trend values T (3)	Seasonal Index (S)% (4)	Trend × Seasonal TS ÷ 100 (5)	Cyclical Irregular %s CI (%) (6)	3-quarter moving total (7)	Cyclical relative C (%) (8)
1961-I	122	127.78	101.5	129.70	94.06	--	--
II	125	124.55	98.6	122.81	101.78	293.10	97.70
III	118	121.32	100	121.32	97.26	298.12	99.37
IV	117	118.09	100	118.09	99.08	298.42	99.47
1962-I	119	114.86	101.5	116.58	102.08	304.73	101.58
II	114	111.63	98.6	110.07	103.57	310.82	103.61
III	114	108.40	100	108.40	105.17	312.38	104.13
IV	109	105.17	100	105.17	103.64	310.27	103.42
1963-I	105	101.96	101.5	103.49	101.46	306.82	102.27
II	99	98.71	98.6	97.33	101.72	300.58	100.19
III	93	95.48	100	95.48	97.40	295.60	98.53
IV	89	92.25	100	92.25	96.48	299.05	96.35
1964-I	86	89.02	101.5	90.36	95.17	286.22	95.41
II	80	85.79	98.6	84.59	94.57	290.27	96.76
III	83	82.56	100	82.56	100.53	300.99	100.33
IV	84	79.33	100	79.33	105.89	--	--

13.40. The calculations needed to compute the first serial correlation co-efficient r_1 are shown below:

Y_t	Y_{t+1}	$Y_t-\bar{Y}$	$Y_{t+1}-\bar{Y}$	$(Y_t-\bar{Y})(Y_{t+1}-\bar{Y})$	$(Y_t-\bar{Y})^2$
65	64	1.55	0.55	0.8525	2.4025
64	63	0.55	−0.45	−0.2475	0.3025
63	61	−0.45	−2.45	1.1025	0.2025
61	60	−2.45	−3.45	8.4525	6.0025
60	58	−3.45	−5.45	18.8025	11.9025
58	63	−5.45	−0.45	2.4525	29.7025
63	64	−0.45	0.55	−0.2475	0.2025
64	62	0.55	−1.45	−0.7975	0.3025
62	64	−1.45	0.55	−0.7975	2.1025
64	63	0.55	−0.45	−0.2475	0.3025
63	63	−0.45	−0.45	0.2025	0.2025
63	62	−0.45	−1.45	0.6525	0.2025
62	60	−1.45	−3.45	5.0025	2.1025
60	62	−3.45	−1.45	5.0025	11.9025
62	64	−1.45	0.55	−0.7975	2.1025
64	66	0.55	2.55	1.4025	0.3025
66	68	2.55	4.55	11.6025	6.5025
68	68	4.55	4.55	20.7025	20.7025
68	69	4.55	5.55	25.2525	20.7025
69	--	5.55	--	−3.135 + 75.5275	30.8025
1269	--	0	--	72.3925	148.9500

Now $\bar{Y} = \sum_{t=1}^{n} \frac{Y_t}{n} = \frac{1269}{20} = 63.45$, and

$$r_1 = \frac{\sum_{t=1}^{n-1} (Y_t - \bar{Y})(Y_{t+1} - \bar{Y})}{\sum_{t=1}^{n} (Y_t - \bar{Y})^2} = \frac{72.3925}{148.9500} = 0.49$$

Calculation of the coefficient of auto-correlation of lag 2.

Y_t	Y_{t+2}	$Y_t-\bar{Y}$	$Y_{t+2}-\bar{Y}$	$(Y_t-\bar{Y})(Y_{t+2}-\bar{Y})$
65	63	1.55	−0.45	−0.6975
64	61	0.55	−2.45	−1.3475
63	60	−0.45	−3.45	1.5525
61	58	−2.45	−5.45	13.3525
60	63	−3.45	−0.45	1.5525
58	64	−5.45	0.55	−2.9975
63	62	−0.45	−1.45	0.6525
64	64	0.55	0.55	0.3025
62	63	−1.45	−0.45	0.6525
64	63	0.55	−0.45	−0.2475
63	62	−0.45	−1.45	0.6525
63	60	−0.45	−3.45	1.5525
62	62	−1.45	−1.45	2.1025
60	64	−3.45	0.55	−1.8975
62	66	−1.45	2.55	−3.6975
64	68	0.55	4.55	2.5025
66	68	2.55	4.55	11.6025
68	69	4.55	5.55	25.2525
68	--	4.55	--	+61.7300
69	--	5.55	--	−10.8850
1269	--	0	--	+50.8450

$$r_2 = \frac{\sum_{t=1}^{n-2} (Y_t - \bar{Y})(Y_{t+2} - \bar{Y})}{\sum_{t=1}^{n} (Y_t - \bar{Y})^2} = \frac{50.845}{148.950} = +0.34$$

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