

EXAMPLE 9 Pivot Positions and Columns

Earlier in this section (immediately after Definition 1) we found a row echelon form of

$$A = \begin{bmatrix} 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -10 & 6 & 12 & 28 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$$

to be

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -\frac{7}{2} & -6 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

The leading 1's occur in positions (row 1, column 1), (row 2, column 3), and (row 3, column 5). These are the pivot positions. The pivot columns are columns 1, 3, and 5.

If A is the augmented matrix for a linear system, then the pivot columns identify the leading variables. As an illustration, in Example 5 the pivot columns are 1, 3, and 6, and the leading variables are x_1 , x_3 , and x_6 .

Roundoff Error and Instability

There is often a gap between mathematical theory and its practical implementation—Gauss–Jordan elimination and Gaussian elimination being good examples. The problem is that computers generally approximate numbers, thereby introducing **roundoff errors**, so unless precautions are taken, successive calculations may degrade an answer to a degree that makes it useless. Algorithms (procedures) in which this happens are called **unstable**. There are various techniques for minimizing roundoff error and instability. For example, it can be shown that for large linear systems Gauss–Jordan elimination involves roughly 50% more operations than Gaussian elimination, so most computer algorithms are based on the latter method. Some of these matters will be considered Chapter 9.

Exercise Set 1.2

In Exercises 1–2, determine whether the matrix is in row echelon form, reduced row echelon form, both, or neither.

1. (a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ *B* (b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ *B* (c) $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ *B*

(d) $\begin{bmatrix} 1 & 0 & 3 & 1 \\ 0 & 1 & 2 & 4 \end{bmatrix}$ *B* (e) $\begin{bmatrix} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ *B*

(f) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ *B* (g) $\begin{bmatrix} 1 & -7 & 5 & 5 \\ 0 & 1 & 3 & 2 \end{bmatrix}$ *R*

2. (a) $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ *B* (b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 3 & 4 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 5 & -3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ (e) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(f) $\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 0 & 7 & 1 & 3 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

(g) $\begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}$

In Exercises 3–4, suppose that the augmented matrix for a linear system has been reduced by row operations to the given echelon form. Solve the system.

3. (a) $\begin{bmatrix} 1 & -3 & 4 & 7 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 5 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 & 8 & -5 & 6 \\ 0 & 1 & 4 & -9 & 3 \\ 0 & 0 & 1 & 1 & 2 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 7 & -2 & 0 & -8 & -3 \\ 0 & 0 & 1 & 1 & 6 & 5 \\ 0 & 0 & 0 & 1 & 3 & 9 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & -3 & 7 & 1 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$$4. (a) \begin{bmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 7 \end{bmatrix}$$

$$(b) \begin{bmatrix} 1 & 0 & 0 & -7 & 8 \\ 0 & 1 & 0 & 3 & 2 \\ 0 & 0 & 1 & 1 & -5 \end{bmatrix}$$

$$(c) \begin{bmatrix} 1 & -6 & 0 & 0 & 3 & -2 \\ 0 & 0 & 1 & 0 & 4 & 7 \\ 0 & 0 & 0 & 1 & 5 & 8 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$(d) \begin{bmatrix} 1 & -3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

In Exercises 5–8, solve the linear system by Gaussian elimination.

$$5. \begin{aligned} x_1 + x_2 + 2x_3 &= 8 \\ -x_1 - 2x_2 + 3x_3 &= 1 \\ 3x_1 - 7x_2 + 4x_3 &= 10 \end{aligned} \quad 6. \begin{aligned} 2x_1 + 2x_2 + 2x_3 &= 0 \\ -2x_1 + 5x_2 + 2x_3 &= 1 \\ 8x_1 + x_2 + 4x_3 &= -1 \end{aligned}$$

$$7. \begin{aligned} x - y + 2z - w &= -1 \\ 2x + y - 2z - 2w &= -2 \\ -x + 2y - 4z + w &= 1 \\ 3x &- 3w = -3 \end{aligned}$$

$$8. \begin{aligned} -2b + 3c &= 1 \\ 3a + 6b - 3c &= -2 \\ 6a + 6b + 3c &= 5 \end{aligned}$$

In Exercises 9–12, solve the linear system by Gauss–Jordan elimination.

9. Exercise 5

10. Exercise 6

11. Exercise 7

12. Exercise 8

In Exercises 13–14, determine whether the homogeneous system has nontrivial solutions by inspection (without pencil and paper).

$$13. \begin{aligned} 2x_1 - 3x_2 + 4x_3 - x_4 &= 0 \\ 7x_1 + x_2 - 8x_3 + 9x_4 &= 0 \\ 2x_1 + 8x_2 + x_3 - x_4 &= 0 \end{aligned}$$

$$14. \begin{aligned} x_1 + 3x_2 - x_3 &= 0 \\ x_2 - 8x_3 &= 0 \\ 4x_3 &= 0 \end{aligned}$$

In Exercises 15–22, solve the given linear system by any method.

$$15. \begin{aligned} 2x_1 + x_2 + 3x_3 &= 0 \\ x_1 + 2x_2 &= 0 \\ x_2 + x_3 &= 0 \end{aligned} \quad 16. \begin{aligned} 2x - y - 3z &= 0 \\ -x + 2y - 3z &= 0 \\ x + y + 4z &= 0 \end{aligned}$$

$$17. \begin{aligned} 3x_1 + x_2 + x_3 + x_4 &= 0 \\ 5x_1 - x_2 + x_3 - x_4 &= 0 \end{aligned} \quad 18. \begin{aligned} v + 3w - 2x &= 0 \\ 2u + v - 4w + 3x &= 0 \\ 2u + 3v + 2w - x &= 0 \\ -4u - 3v + 5w - 4x &= 0 \end{aligned}$$

$$19. \begin{aligned} 2x + 2y + 4z &= 0 \\ w - y - 3z &= 0 \\ 2w + 3x + y + z &= 0 \\ -2w + x + 3y - 2z &= 0 \end{aligned}$$

$$20. \begin{aligned} x_1 + 3x_2 + x_4 &= 0 \\ x_1 + 4x_2 + 2x_3 &= 0 \\ -2x_2 - 2x_3 - x_4 &= 0 \\ 2x_1 - 4x_2 + x_3 + x_4 &= 0 \\ x_1 - 2x_2 - x_3 + x_4 &= 0 \end{aligned}$$

$$21. \begin{aligned} 2I_1 - I_2 + 3I_3 + 4I_4 &= 9 \\ I_1 - 2I_3 + 7I_4 &= 11 \\ 3I_1 - 3I_2 + I_3 + 5I_4 &= 8 \\ 2I_1 + I_2 + 4I_3 + 4I_4 &= 10 \end{aligned}$$

$$22. \begin{aligned} Z_3 + Z_4 + Z_5 &= 0 \\ -Z_1 - Z_2 + 2Z_3 - 3Z_4 + Z_5 &= 0 \\ Z_1 + Z_2 - 2Z_3 - Z_5 &= 0 \\ 2Z_1 + 2Z_2 - Z_3 + Z_5 &= 0 \end{aligned}$$

In each part of Exercises 23–24, the augmented matrix for a linear system is given in which the asterisk represents an unspecified real number. Determine whether the system is consistent, and if so whether the solution is unique. Answer “inconclusive” if there is not enough information to make a decision.

$$23. (a) \begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \end{bmatrix} \quad (b) \begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$(c) \begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (d) \begin{bmatrix} 1 & * & * & * \\ 0 & 0 & * & 0 \\ 0 & 0 & 1 & * \end{bmatrix}$$

$$24. (a) \begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & 1 \end{bmatrix} \quad (b) \begin{bmatrix} 1 & 0 & 0 & * \\ * & 1 & 0 & * \\ * & * & 1 & * \end{bmatrix}$$

$$(c) \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & * & * & * \end{bmatrix} \quad (d) \begin{bmatrix} 1 & * & * & * \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

In Exercises 25–26, determine the values of a for which the system has no solutions, exactly one solution, or infinitely many solutions.

$$25. \begin{aligned} x + 2y - 3z &= 4 \\ 3x - y + 5z &= 2 \\ 4x + y + (a^2 - 14)z &= a + 2 \end{aligned}$$

$$\begin{aligned} 26. \quad & x + 2y + z = 2 \\ & 2x - 2y + 3z = 1 \\ & x + 2y - (a^2 - 3)z = a \end{aligned}$$

In Exercises 27–28, what condition, if any, must a , b , and c satisfy for the linear system to be consistent?

$$\begin{aligned} 27. \quad & x + 3y - z = a \\ & x + y + 2z = b \\ & 2y - 3z = c \end{aligned} \quad \begin{aligned} 28. \quad & x + 3y + z = a \\ & -x - 2y + z = b \\ & 3x + 7y - z = c \end{aligned}$$

In Exercises 29–30, solve the following systems, where a , b , and c are constants.

$$\begin{aligned} 29. \quad & 2x + y = a \\ & 3x + 6y = b \end{aligned} \quad \begin{aligned} 30. \quad & x_1 + x_2 + x_3 = a \\ & 2x_1 + 2x_2 = b \\ & 3x_2 + 3x_3 = c \end{aligned}$$

31. Find two different row echelon forms of

$$\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}$$

This exercise shows that a matrix can have multiple row echelon forms.

32. Reduce

$$\begin{bmatrix} 2 & 1 & 3 \\ 0 & -2 & -29 \\ 3 & 4 & 5 \end{bmatrix}$$

to reduced row echelon form without introducing fractions at any intermediate stage.

33. Show that the following nonlinear system has 18 solutions if $0 \leq \alpha \leq 2\pi$, $0 \leq \beta \leq 2\pi$, and $0 \leq \gamma \leq 2\pi$.

$$\begin{aligned} \sin \alpha + 2 \cos \beta + 3 \tan \gamma &= 0 \\ 2 \sin \alpha + 5 \cos \beta + 3 \tan \gamma &= 0 \\ -\sin \alpha - 5 \cos \beta + 5 \tan \gamma &= 0 \end{aligned}$$

[Hint: Begin by making the substitutions $x = \sin \alpha$, $y = \cos \beta$, and $z = \tan \gamma$.]

34. Solve the following system of nonlinear equations for the unknown angles α , β , and γ , where $0 \leq \alpha \leq 2\pi$, $0 \leq \beta \leq 2\pi$, and $0 \leq \gamma < \pi$.

$$\begin{aligned} 2 \sin \alpha - \cos \beta + 3 \tan \gamma &= 3 \\ 4 \sin \alpha + 2 \cos \beta - 2 \tan \gamma &= 2 \\ 6 \sin \alpha - 3 \cos \beta + \tan \gamma &= 9 \end{aligned}$$

35. Solve the following system of nonlinear equations for x , y , and z .

$$\begin{aligned} x^2 + y^2 + z^2 &= 6 \\ x^2 - y^2 + 2z^2 &= 2 \\ 2x^2 + y^2 - z^2 &= 3 \end{aligned}$$

[Hint: Begin by making the substitutions $X = x^2$, $Y = y^2$, $Z = z^2$.]

36. Solve the following system for x , y , and z .

$$\begin{aligned} \frac{1}{x} + \frac{2}{y} - \frac{4}{z} &= 1 \\ \frac{2}{x} + \frac{3}{y} + \frac{8}{z} &= 0 \\ -\frac{1}{x} + \frac{9}{y} + \frac{10}{z} &= 5 \end{aligned}$$

37. Find the coefficients a , b , c , and d so that the curve shown in the accompanying figure is the graph of the equation $y = ax^3 + bx^2 + cx + d$.

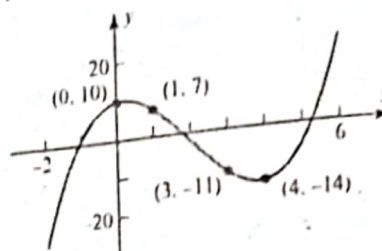


Figure Ex-37

38. Find the coefficients a , b , c , and d so that the circle shown in the accompanying figure is given by the equation $ax^2 + ay^2 + bx + cy + d = 0$.

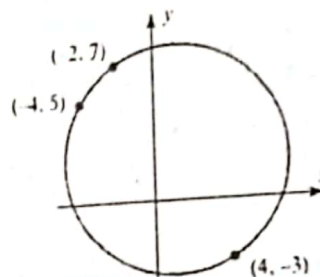


Figure Ex-38

39. If the linear system

$$\begin{aligned} a_1x + b_1y + c_1z &= 0 \\ a_2x + b_2y + c_2z &= 0 \\ a_3x + b_3y + c_3z &= 0 \end{aligned}$$

has only the trivial solution, what can be said about the solutions of the following system?

$$\begin{aligned} a_1x + b_1y + c_1z &= 3 \\ a_2x + b_2y + c_2z &= 7 \\ a_3x + b_3y + c_3z &= 11 \end{aligned}$$

40. (a) If A is a matrix with three rows and five columns, then what is the maximum possible number of leading 1's in its reduced row echelon form?

(b) If B is a matrix with three rows and six columns, then what is the maximum possible number of parameters in the general solution of the linear system with augmented matrix B ?

(c) If C is a matrix with five rows and three columns, then what is the minimum possible number of rows of zeros in any row echelon form of C ?