Chapter 11

MULTIPLE REGRESSION AND CORRELATION

11.2. The estimated multiple linear regression equation is

$$\hat{Y} = a + b_1X_1 + b_2X_2,$$

where a, b_1 and b_2 are the *least-squares estimates* of the parameters α, β_1 and β_2. The three normal equations are

$$\sum Y = na + b_1\sum X_1 + b_2\sum X_2,$$

$$\sum X_1Y = a\sum X_1 + b_1\sum X_1^2 + b_2\sum X_1X_2,$$

$$\sum X_2Y = a\sum X_2 + b_1\sum X_1X_2 + b_2\sum X_2^2.$$

The calculations needed to find a, b_1 and b_2 are shown below:

Y	X_1	X_2	X_1^2	X_2^2	X_1X_2	X_1Y	X_2Y
12	2	1	4	1	2	24	12
10	2	1	4	1	2	20	10
9	3	0	9	0	0	27	0
13	4	0	16	0	0	52	0
20	4	3	16	9	12	80	60
64	15	5	49	11	16	203	82

Substituting the sums in the normal equations, we get

$$5a + 15b_1 + 5b_2 = 64$$

$$15a + 49b_1 + 16b_2 = 203$$

$$5a + 16b_1 + 11b_2 = 82$$

Solving them simultaneously, we obtain

$$a = 3.88,\ b_1 = 2.09 \text{ and } b_2 = 2.65$$

which are the values of the desired least-squares estimates.

11.3. (a) The estimated multiple linear regression equation is

$$\hat{Y} = a + b_1X_1 + b_2X_2,$$

where a, b_1 and b_2 are the least-squares estimates of the parameters α, β_1 and β_2.

The three normal equation are

$$\sum Y = na + b_1\sum X_1 + b_2\sum X_2$$

$$\sum X_1Y = a\sum X_1 + b_1\sum X_1^2 + b_2\sum X_1X_2$$

$$\sum X_2Y = a\sum X_2 + b_1\sum X_1X_2 + b_2\sum X_2^2$$

The calculations needed to find a, b_1 and b_2 are shown below:

Y	X_1	X_2	X_1^2	X_2^2	X_1Y	X_2Y	X_1X_2
2	8	0	64	0	16	0	0
5	8	1	64	1	40	5	8
7	6	1	36	1	42	7	6
8	5	3	25	9	40	24	15
5	3	4	9	16	15	20	12
27	30	9	198	27	153	56	41

Substituting the sums in the normal equations, we get

$$5a + 30b_1 + 9b_2 = 27$$

$$30a + 198b_1 + 41b_2 = 153$$

$$9a + 41b_1 + 27b_2 = 56$$

Solving them simultaneously, we obtain

$$a = 4.49,\ b_1 = -0.04 \text{ and } b_2 = 0.64$$

Hence the desired estimated multiple linear regression is

$$\hat{Y} = 4.49 - 0.04X_1 + 0.64X_2$$

(b) The partial regression co-efficient b_1 measures the average change in Y for a unit change in X_1 while the effect of X_2 is held constant.

Similarly, b_2 measures the average change in Y for a unit change in X_2 while the effect of X_1 is held constant.

11.4. (a) The estimated multiple linear regression equation is

$$\hat{X}_1 = a + b_1X_2 + b_2X_3.$$

The three normal equations are

$$\sum X_1 = na + b_1\sum X_2 + b_2\sum X_3$$
$$\sum X_1X_2 = a\sum X_2 + b_1\sum X_2^2 + b_2\sum X_2X_3$$
$$\sum X_1X_3 = a\sum X_3 + b_1\sum X_2X_3 + b_2\sum X_3^2$$

The calculations needed to find a, b_1 and b_2 are shown below:

X_1	X_2	X_3	X_2^2	X_3^2	X_2X_3	X_1X_2	X_1X_3	X_1^2
1	1	2	1	4	2	1	2	1
4	8	8	64	64	64	32	32	16
1	3	1	9	1	3	3	1	1
3	5	7	25	49	35	15	21	9
2	6	4	36	16	24	12	8	4
4	10	6	100	36	60	40	24	16
16	33	28	235	170	188	103	88	47

Substituting the sums in the normal equations, we get

$$6a + 33b_1 + 28b_2 = 15$$
$$33a + 235b_1 + 188b_2 = 103$$
$$28a + 188b_1 + 170b_2 = 88$$

Solving them simultaneously, we obtain

$$a = 0.04, \quad b_1 = 0.21 \quad \text{and} \quad b_2 = 0.28$$

Hence the desired multiple linear regression is

$$\hat{X}_1 = 0.04 + 0.21X_2 + 0.28X_3$$

(b) The standard error of estimate, $s_{1.23}$ is obtained as below:

$$s_{1.23} = \sqrt{\frac{\sum(X_1 - \hat{X}_1)^2}{n-3}} = \sqrt{\frac{\sum X_1^2 - a\sum X_1 - b_1\sum X_2X_1 - b_2\sum X_3X_1}{n-3}}$$

$$= \sqrt{\frac{47-(0.04)(15)-(0.21)(103)-(0.28)(88)}{6-3}}$$

$$= \sqrt{\frac{0.13}{3}} = \sqrt{0.0433} = 0.2082 = 0.21$$

(c) The co-efficient of multiple determination, $R^2_{1.23}$ is computed as

$$R^2_{1.23} = \frac{\sum(\hat{X}_1 - \bar{X}_1)^2}{\sum(X_1 - \bar{X}_1)^2}$$

$$= \frac{a\sum X_1 + b_1\sum X_1X_2 + b_2\sum X_1X_3 - \frac{(\sum X_1)^2}{n}}{\sum X_1^2 - \frac{(\sum X_1)^2}{n}}$$

$$= \frac{(0.04)(15)+(0.21)(103)+(0.28)(88)-(15)^2/6}{47 - \frac{(15)^2}{6}}$$

$$= \frac{9.37}{9.50} = 0.9863, \text{ and}$$

therefore the multiple correlation co-efficient $R_{1.23}$, is

$$R_{1.23} = \sqrt{0.9863} = 0.99.$$

11.5. (a) The estimated multiple regression of X_3 on X_1 and X_2 is

$$\hat{X}_3 = b_{3.12} + b_{31.2}X_1 + b_{32.1}X_2$$

where b's are the least squares estimates of population parameters.

The three normal equations are

$$\sum X_3 = nb_{3.12} + b_{31.2}\sum X_1 + b_{32.1}\sum X_2,$$

$$\sum X_1X_3 = b_{3.12}\sum X_1 + b_{31.2}\sum X_1^2 + b_{32.1}\sum X_1X_2,$$
$$\sum X_2X_3 = b_{3.12}\sum X_2 + b_{31.2}\sum X_1X_2 + b_{32.1}\sum X_2^2.$$

The calculations needed to find b's are shown in the table below:

X_1	X_2	X_3	X_1^2	X_2^2	X_3^2	X_1X_2	X_2X_3	X_1X_3
3	16	90	9	256	8100	48	1440	270
5	10	72	25	100	5184	50	720	360
6	7	54	36	49	2916	42	378	324
8	4	42	64	16	1764	32	168	336
12	3	30	144	9	900	36	90	360
14	2	12	196	4	144	28	24	168
48	42	300	474	434	19008	236	2820	1818

Substituting the sums in the normal equations, we get

$$6b_{3.12} + 48b_{31.2} + 42b_{32.1} = 300$$
$$48b_{3.12} + 474b_{31.2} + 236b_{32.1} = 1818$$
$$42b_{3.12} + 236b_{31.2} + 434b_{32.1} = 2820$$

Solving them simultaneously, we obtain

$$b_{3.12} = 61.3993, b_{31.2} = -3.6461 \text{ and } b_{32.1} = 2.5385$$

Hence the desired estimated regression equation of X_3 on X_1 and X_2 is

$$\hat{X}_3 = 61.40 - 3.65X_1 + 2.54X_2.$$

(b) When $X_1 = 10$ and $X_2 = 6$, we get

$$X_3 = 61.40 - 3.65\,(10) + 2.54\,(6)$$
$$= 61.40 - 36.5 + 15.24 = 40.$$

(c) The multiple correlation co-efficient is the positive square root of $R^2_{3.12}$, where

$$R^2_{3.12} = \frac{\sum(\hat{X}_3 - \bar{X}_3)^2}{\sum(X_3 - \bar{X}_3)^2}$$

$$= \frac{b_{3.12}\sum X_3 + b_{31.2}\sum X_1X_3 + b_{32.1}\sum X_2X_3 - (\sum X_3)^2/n}{\sum X_3^2 - (\sum X_3)^2/n}$$

$$= \frac{(61.3993)(300) + (-3.6461)(1818) + (2.5385)(2820) - (300)^2/6}{19008 - (300)^2/6}$$

$$= \frac{18419.79 - 6628.6 + 7158.57 - 15000}{19008 - 15000}$$

$$= \frac{3949.75}{4008} = 0.9855$$

Hence $R_{3.12} = \sqrt{0.9855} = 0.99.$

Alternatively:

Now, $\bar{X}_1 = \frac{\sum X_1}{n} = \frac{48}{6} = 8;\ \bar{X}_2 = \frac{\sum X_2}{n} = \frac{42}{6} = 7;$

$$\bar{X}_3 = \frac{\sum X_3}{n} = \frac{300}{6} = 50.$$

$$S_1 = \sqrt{\frac{\sum X_1^2}{n} - \left(\frac{\sum X_1}{n}\right)^2} = \sqrt{\frac{474}{6} - \left(\frac{48}{6}\right)^2}$$

$$= \sqrt{79 - 64} = \sqrt{15} = 3.87;$$

$$S_2 = \sqrt{\frac{\sum X_2^2}{n} - \left(\frac{\sum X_2}{n}\right)^2} = \sqrt{\frac{434}{6} - \left(\frac{42}{6}\right)^2}$$

$$= \sqrt{72.33 - 49} = \sqrt{23.33} = 4.83;$$

$$S_3 = \sqrt{\frac{\sum X_3^2}{n} - \left(\frac{\sum X_3}{n}\right)^2} = \sqrt{\frac{19008}{6} - \left(\frac{300}{6}\right)^2}$$

$$= \sqrt{3168 - 2500} = \sqrt{668} = 25.84;$$

and $$r_{12} = \frac{\sum X_1 X_2 - n\bar{X}_1\bar{X}_2}{\sqrt{[\sum X_1^2 - n\bar{X}_1^2][\sum X_2^2 - n\bar{X}_2^2]}}$$

$$= \frac{236 - 6(8)(7)}{\sqrt{[474 - 6(64)][434 - 6(49)]}} = \frac{236-336}{\sqrt{(90)(140)}}$$

$$= \frac{-100}{112.25} = -0.89;$$

$$r_{23} = \frac{\sum X_2X_3 - n\bar{X}_2\bar{X}_3}{\sqrt{[\sum X_2^2 - n\bar{X}_2^2][\sum X_3^2 - n\bar{X}_3^2]}}$$

$$= \frac{2820 - 6(7)(50)}{\sqrt{[434 - 6(49)][19008 - 6(2500)]}} = \frac{2820-2100}{\sqrt{(140)(4008)}}$$

$$= \frac{720}{749.08} = 0.96;$$

$$r_{31} = \frac{\sum X_1X_3 - n\bar{X}_1\bar{X}_3}{\sqrt{[\sum X_1^2 - n\bar{X}_1^2][\sum X_3^2 - n\bar{X}_3^2]}}$$

$$= \frac{1818 - 2400}{\sqrt{(90)(4008)}} = \frac{-582}{600.6} = -0.97.$$

(a) The linear regression equation of x_3 on x_1 and x_2 is

$$\frac{x_3}{S_3} = \left(\frac{r_{32} - r_{13}r_{12}}{1 - r_{12}^2}\right)\frac{x_2}{S_2} + \left(\frac{r_{31} - r_{23}r_{12}}{1 - r_{12}^2}\right)\frac{x_1}{S_1},$$

where $x_1 = X_1 - \bar{X}_1$, $x_2 = X_2 - \bar{X}_2$ and $x_3 = X_3 - \bar{X}_3$.

Substituting the values, we get

$$\frac{x_3}{25.84} = \left(\frac{(0.96)-(-0.97)(-0.89)}{1-(-0.89)^2}\right)\frac{x_2}{4.83} + \left(\frac{-0.97-(0.96)(-0.89)}{1-(-0.89)^2}\right)\frac{x_1}{3.87}$$

Or $$\frac{x_3}{25.84} = \left(\frac{0.0967}{0.2079}\right)\frac{x_2}{4.83} + \left(\frac{-0.1156}{0.2079}\right)\frac{x_1}{3.87}$$

Or $$x_3 = 2.49x_2 - 3.71x_1$$

Or $$(X_3 - 50) = 2.49(X_2 - 7) - 3.71(X_1 - 8)$$

Or $$X_3 = 62.25 - 3.71X_1 + 2.49X_2.$$

(b) When $X_1 = 10$, and $X_2 = 6$, we get

$$X_3 = 62.25 - 3.71(10) + 2.49(6) = 40$$

$$(c)\ R^2_{3.12} = \frac{r^2_{31} + r^2_{32} - 2r_{12}r_{23}r_{31}}{1 - r^2_{12}}$$

$$= \frac{0.9409 + 0.9216 - 2(-0.89)(0.96)(-0.97)}{1 - 0.7921}$$

$$= \frac{1.8625 - 1.6575}{0.2079} = \frac{0.2050}{0.2079} = 0.9860$$

Hence $R_{3.12} = \sqrt{0.9860} = 0.993.$

11.6. (a) Calculations needed to find b's are shown in the table below:

Y	X_1	X_2	YX_1	YX_2	X_1X_2	X_1^2	X_2^2
57.5	78	2.75	4485.0	158.125	214.50	6084	7.5625
52.8	69	2.15	3643.2	113.520	148.35	4761	4.6225
61.3	77	4.41	4720.1	270.333	339.57	5929	19.4481
67.0	88	5.52	5896.0	369.840	485.76	7744	30.4704
53.5	67	3.21	3584.5	171.735	215.07	4489	10.3041
62.7	80	4.32	5016.0	270.864	345.60	6400	18.6624
56.2	74	2.31	4158.8	129.822	170.94	5476	5.3361
68.5	94	4.30	6439.0	294.550	404.20	8836	18.4900
69.2	102	3.71	7058.4	256.732	378.42	10404	13.7641
548.7	729	32.68	45001.0	2035.521	2702.41	60123	128.6602

Now $\bar{Y} = \frac{\sum Y}{n} = \frac{548.7}{9} = 60.9667,$

$$\bar{X}_1 = \frac{\sum X_1}{n} = \frac{729}{9} = 81,$$

$$\bar{X}_2 = \frac{\sum X_2}{n} = \frac{32.68}{9} = 3.6311,$$

$$\sum x_1^2 = \sum X_1^2 - \frac{(\sum X_1)^2}{n} = 60123 - \frac{(729)^2}{9}$$

$$= 60123 - 59049 = 1074;$$

$$\sum x_2^2 = \sum X_2^2 - \frac{(\sum X_2)^2}{n} = 128.6602 - \frac{(32.68)^2}{9}$$

$$= 128.6602 - 118.6647 = 9.9955;$$

$$\sum x_1 x_2 = \sum X_1 X_2 - \frac{(\sum X_2)(\sum X_1)}{n} = 2702.41 - \frac{(729)\ (32.68)}{9}$$

$$= 2702.41 - 2647.08 = 55.33;$$

$$\sum x_1 y = \sum X_1 Y - \frac{(\sum X_1)(\sum Y)}{n} = 45001.0 - \frac{(729)\ (548.7)}{9}$$

$$= 45001.0 - 44444.7 = 556.3;$$

$$\sum x_2 y = \sum X_2 Y - \frac{(\sum X_2)(\sum Y)}{n} = 2035.521 - \frac{(32.68)(548.7)}{9}$$

$$= 2035.521 - 1992.3907 = 43.1303;$$

$$b_1 = \frac{(\sum x_1 y)\ (\sum x_2^2) - (\sum x_2 y)(\sum x_1 x_2)}{(\sum x_1^2)\ (\sum x_2^2) - (\sum x_1 x_2)^2}$$

$$= \frac{(556.3)\ (9.9955) - (43.1303)\ (55.33)}{(1074)\ (9.9955) - (55.33)^2}$$

$$= \frac{5560.49665 - 2386.39950}{10735.167 - 3061.4089} = \frac{3174.09715}{7673.7581} = 0.4136$$

$$b_2 = \frac{(\sum x_2 y)\ (\sum x_1^2) - (\sum x_1 y)(\sum x_1 x_2)}{(\sum x_1^2)\ (\sum x_2^2) - (\sum x_1 x_2)^2}$$

$$= \frac{(43.1303)\ (1074) - (556.3)\ (55.33)}{(1074)\ (9.9955) - (55.33)^2}$$

$$= \frac{46321.9422 - 30780.079}{7673.7581} = \frac{15541.8632}{7673.7581}$$

$$= 2.0253; \text{ and}$$

$$a = \bar{Y} - b_1 \bar{X}_1 - b_2 \bar{X}_2 = 60.9667 - (0.4136)(81) - (2.0253)(3.6311)$$

$$= 60.9667 - 33.5016 - 7.35406 = 20.111$$

Hence the fitted least squares regression is

$$\hat{Y} = 20.111 + 0.4136X_1 + 2.0253X_2$$

(b) When $X_1 = 75$ days and $X_2 = 3.15$ kg, then the predicted average length of infants is

$$\hat{Y} = 20.111 + (0.4136)(75) + (2.0253)(3.15)$$

$$= 20.111 + 31.02 + 6.3797 = 57.5 \text{ cm}$$

(c) Now $s_{Y.12} = \sqrt{\dfrac{\sum(Y-\hat{Y})^2}{n-3}}$, where

$$\sum(Y-\hat{Y})^2 = \sum Y^2 - a\sum Y - b_1\sum X_1Y - b_2\sum X_2Y$$

$$= 33773.65 - (20.111)(548.7) - (0.4136) \times (45001.0) - (2.0253)(2035.521)$$

$$= 33773.65 - 11034.9057 - 18612.4136 - 4122.54068$$

$$= 33773.65 - 33769.86 = 3.79$$

$$\therefore \quad s_{Y.12} = \sqrt{\frac{3.79}{6}} = \sqrt{0.6317} = 0.79$$

11.7. (b) Calculation of the multiple correlation co-efficient $R_{1.23}$

X_1	X_2	X_3	X_1^2	X_2^2	X_3^2	X_1X_2	X_2X_3	X_1X_3
4	2	8	16	4	64	8	16	32
3	5	10	9	25	100	15	50	30
2	3	13	4	9	169	6	39	26
4	2	15	16	4	225	8	30	60
6	1	17	36	1	289	6	17	102
7	4	16	49	16	256	28	64	112
8	5	20	64	25	400	40	100	160
34	22	99	194	84	1503	111	316	522

Now, $$r_{12} = \frac{n\sum X_1X_2 - (\sum X_1)(\sum X_2)}{\sqrt{n\sum X_1^2 - (\sum X_1)^2}\sqrt{n\sum X_2^2 - (\sum X_2)^2}}$$

$$= \frac{7 \times 111 - (34)(22)}{\sqrt{7 \times 194 - (34)^2}\sqrt{7 \times 84 - (22)^2}}$$

$$= \frac{777 - 748}{\sqrt{(202)(104)}} = \frac{29}{145} = 0.2,$$

$$r_{13} = \frac{n\sum X_1X_3 - (\sum X_1)(\sum X_3)}{\sqrt{n\sum X_1^2 - (\sum X_1)^2}\sqrt{n\sum X_3^2 - (\sum X_3)^2}}$$

$$= \frac{7 \times 522 - (34)(99)}{\sqrt{(202)}\sqrt{7 \times 1503 - (99)^2}}$$

$$= \frac{3654 - 3366}{\sqrt{(202)(72)}} = \frac{288}{381} = 0.76,$$

$$r_{23} = \frac{n\sum X_2X_3 - (\sum X_2)(\sum X_3)}{\sqrt{n\sum X_2^2 - (\sum X_2)^2}\sqrt{n\sum X_3^2 - (\sum X_3)^2}}$$

$$= \frac{7 \times 316 - (27)(99)}{\sqrt{(104)(720)}} = \frac{34}{274} = 0.12.$$

Hence $$R^2_{1.23} = \frac{r^2_{12} + r^2_{13} - 2r_{12}r_{23}r_{13}}{1 - r^2_{23}}$$

$$= \frac{(0.2)^2 + (0.76)^2 - 2(0.2)(0.12)(0.76)}{1 - 0.0144}$$

$$= \frac{0.6176 - 0.03648}{0.9856} = \frac{0.58112}{0.9856} = 0.59, \text{ so that}$$

$$R_{1.23} = \sqrt{0.59} = 0.768$$

11.8. (a) Calculation of the multiple correlations, etc.

X_1	X_2	X_3	X_1^2	X_2^2	X_3^2	X_1X_2	X_2X_3	X_1X_3
32	3	2	1024	9	4	96	6	64
18	2	4	324	4	16	36	8	72
52	5	2	2704	25	4	260	10	104
16	1	5	256	1	25	16	5	80
42	4	3	1764	16	9	168	12	126
48	6	9	2304	36	81	288	54	432
208	21	25	8376	91	139	864	95	878

Now,
$$r_{12} = \frac{n\sum X_1X_2 - (\sum X_1)(\sum X_2)}{\sqrt{n\sum X_1^2 - (\sum X_1)^2}\sqrt{n\sum X_2^2 - (\sum X_2)^2}}$$

$$= \frac{6 \times 684 - (208)(21)}{\sqrt{6 \times 8376 - (208)^2}\sqrt{6 \times 91 - (21)^2}}$$

$$= \frac{5184 - 4368}{\sqrt{(6992)(105)}} = \frac{816}{857} = 0.952;$$

$$r_{13} = \frac{n\sum X_1X_3 - (\sum X_1)(\sum X_3)}{\sqrt{n\sum X_1^2 - (\sum X_1)^2}\sqrt{n\sum X_3^2 - (\sum X_3)^2}}$$

$$= \frac{6 \times 878 - (208)(25)}{\sqrt{6992}\sqrt{6 \times 139 - (25)^2}}$$

$$= \frac{5268 - 5200}{\sqrt{(6992)(209)}} = \frac{68}{1209} = 0.056;$$

$$r_{23} = \frac{n\sum X_2X_3 - (\sum X_2)(\sum X_3)}{\sqrt{n\sum X_2^2 - (\sum X_2)^2}\sqrt{n\sum X_3^2 - (\sum X_3)^2}}$$

$$= \frac{6 \times 95 - (21)(25)}{\sqrt{(105)(209)}} = \frac{45}{148} = 0.304.$$

$$\text{Hence } R^2_{1.23} = \frac{r^2_{12} + r^2_{13} - 2r_{12}r_{23}r_{13}}{1 - r^2_{23}}$$

$$= \frac{(0.952)^2 + (0.056)^2 - 2(0.952)(0.304)(0.056)}{1 - (0.304)^2}$$

$$= \frac{0.877026}{0.907584} = 0.9663; \text{ so that}$$

$$R_{1.23} = \sqrt{0.9663} = 0.98;$$

$$R^2_{2.13} = \frac{r^2_{21} + r^2_{23} - 2r_{12}r_{23}r_{31}}{1 - r^2_{13}}$$

$$= \frac{(0.952)^2 + (0.304)^2 - 2(0.952)(0.304)(0.056)}{1 - (0.056)^2}$$

$$= \frac{0.966306}{0.996864} = 0.9693; \text{ so that}$$

$$R_{2.13} = \sqrt{0.9693} = 0.98; \text{ and}$$

$$R^2_{3.12} = \frac{r^2_{31} + r^2_{32} - 2r_{12}r_{23}r_{13}}{1 - r^2_{12}}$$

$$= \frac{(0.056)^2 + (0.304)^2 - 2(0.952)(0.304)(0.056)}{1 - (0.952)^2}$$

$$= \frac{0.063138}{0.093696} = 0.6739; \text{ so that}$$

$$R_{3.12} = \sqrt{0.6739} = 0.82.$$

(b) The estimated regression equation of x_3 on x_1 and x_2 (in deviation form) in terms of standard deviations and the linear correlation coefficients is

$$\frac{x_3}{S_3} = \left(\frac{r_{13} - r_{12}\,r_{23}}{1 - r^2_{12}}\right)\left(\frac{x_1}{S_1}\right) + \left(\frac{r_{23} - r_{12}\,r_{13}}{1 - r^2_{12}}\right)\left(\frac{x_2}{S_2}\right)$$

Substituting the values, we get

$$\frac{x_3}{1.5} = \left(\frac{0.4-(-0.2)\,(0.5)}{1-(-0.2)^2}\right)\left(\frac{x_1}{1.0}\right) + \left(\frac{0.5-(-0.2)\,(0.4)}{1-(-0.2)^2}\right)\left(\frac{x_2}{2.0}\right)$$

or $$\frac{x_3}{1.5} = \left(\frac{0.5}{0.96}\right)\left(\frac{x_1}{1.0}\right) + \left(\frac{0.58}{0.96}\right)\left(\frac{x_2}{2.0}\right)$$

or $$x_3 = 0.78\,x_1 + 0.45\,x_2$$

To obtain the regression equation in terms of original values, we replace x_1 by $X_1-\bar{X}_1$, x_2 by $X_2-\bar{X}_2$ and x_3 by $X_3-\bar{X}_3$. Thus

$$(X_3-12) = 0.78\,(X_1-20) + 0.45\,(X_2-36)$$

i.e. $$X_3 = 12 + 0.78X_1 - 15.60 + 0.45X_2 - 16.20$$

$$= -19.80 + 0.78X_1 + 0.45X_2$$

11.9. (b) To find the multiple correlation $R_{2.13}$, we first compute the simple correlations.

Now $b_{12} = 0.75$, $b_{13} = 0.58$, $b_{21} = 0.88$,

$b_{23} = 0.53$, $b_{31} = 1.68$ and $b_{32} = 1.30$. Therefore

$$r_{12} = \sqrt{b_{12} \times b_{21}} = \sqrt{(0.75)\,(0.88)} = 0.81,$$

$$r_{13} = \sqrt{b_{13} \times b_{31}} = \sqrt{(0.58)\,(1.68)} = 0.99, \text{ and}$$

$$r_{23} = \sqrt{b_{23} \times b_{32}} = \sqrt{(0.53)\,(1.30)} = 0.83.$$

Now $$R^2_{2.13} = \frac{r^2_{21} + r^2_{23} - 2r_{12}r_{13}r_{23}}{1-r^2_{13}}$$

$$= \frac{(0.81)^2 + (0.83)^2 - 2(0.81)\,(0.99)\,(0.83)}{1-(0.99)^2}$$

$$= \frac{0.013846}{0.0199} = 0.6958$$

Hence $R_{2.13} = \sqrt{0.6958} = 0.83.$

(c) Now $$R^2_{2.13} = \frac{r^2_{21} + r^2_{23} - 2r_{12}r_{23}r_{31}}{1-r^2_{13}}$$

$$= \frac{(0.60)^2 + (0.65)^2 - 2(0.6)(0.7)(0.65)}{1-(0.70)^2}$$

$$= \frac{0.3600 + 0.4225 - 0.546}{1-0.49} = \frac{0.2365}{0.51} = 0.4637$$

Hence $R_{2.13} = \sqrt{0.4637} = 0.68.$

11.10. (a) The multiple correlation co-efficient, $R_{1.23}$ is given by

$$R^2_{1.23} = \frac{r^2_{21} + r^2_{13} - 2r_{12}r_{13}r_{23}}{1-r^2_{23}}$$

$$= \frac{(0.8)^2 + (-0.7)^2 - 2(0.8)(-0.7)(-0.9)}{1-(-0.9)^2}$$

$$= \frac{0.122}{0.19} = 0.64, \text{ so that } R_{1.23} = \sqrt{0.64} = 0.80.$$

(b) To calculate the multiple correlatioin co-efficient and the partial correlation co-efficient, we first calculate the simple correlation co-efficients.

Thus $r_{12} = -\sqrt{b_{12}\times b_{21}} = -\sqrt{(-0.1)(-0.4)} = -0.20.$

$r_{13} = +\sqrt{b_{13}\times b_{31}} = \sqrt{(0.27)(0.6)} = \sqrt{0.162} = 0.40$, and

$r_{23} = +\sqrt{b_{23}\times b_{32}} = \sqrt{(0.67)(0.38)} = \sqrt{0.2546} = 0.50.$

Now $$R^2_{2.13} = \frac{r^2_{21} + r^2_{23} - 2r_{12}r_{13}r_{23}}{1-r^2_{13}}$$

$$= \frac{(-0.2)^2 + (0.5)^2 - 2(-0.2)(0.4)(0.5)}{1-(0.4)^2} = \frac{0.37}{0.84} = 0.44$$

Hence $R_{2.13} = \sqrt{0.44} = 0.66$

And $$r_{23.1} = \frac{r_{23} - r_{21}r_{31}}{\sqrt{1-r^2_{21}}\sqrt{1-r^2_{31}}} = \frac{0.5-(-0.2)(0.4)}{\sqrt{1-(-0.2)^2}\sqrt{1-(0.4)^2}}$$

$$= \frac{0.58}{\sqrt{(0.96)(0.84)}} = \frac{0.58}{0.90} = 0.64$$

11.11. (b) The estimated simple regression equation of X_1 on X_3 is

$$\hat{X}_1 = b_{1.3} + b_{13}X_3,$$

where $b_{13} = \dfrac{n\sum X_1X_3 - (\sum X_1)(\sum X_3)}{n\sum X_3^2 - (\sum X_3)^2}$ and $b_{1.3} = \bar{X}_1 - b_{13}\bar{X}_3$.

The estimated simple regression equation of X_2 on X_3 is

$$\hat{X}_2 = b_{2.3} + b_{23}X_3,$$

where $b_{23} = \dfrac{n\sum X_2X_3 - (\sum X_2)(\sum X_3)}{n\sum X_3^2 - (\sum X_3)^2}$ and $b_{2.3} = \bar{X}_2 - b_{23}\bar{X}_3$.

The computations needed to find the b's are given in the table below:

X_1	X_2	X_3	X_1X_3	X_2X_3	X_3^2
5	10	2	10	20	4
9	12	6	54	72	36
7	8	4	28	32	16
10	9	5	50	45	25
12	11	7	84	77	49
8	7	6	48	42	36
6	5	4	24	20	16
10	8	6	60	48	36
67	70	40	358	356	218

Now $\bar{X}_1 = \dfrac{\sum X_1}{n} = \dfrac{67}{8} = 8.375$, $\bar{X}_2 = \dfrac{\sum X_2}{n} = \dfrac{70}{8} = 8.75$, $\bar{X}_3 = 5$;

$$b_{13} = \frac{(8)(358) - (67)(40)}{(8)(218) - (40)^2} = \frac{184}{144} = 1.28,$$

$$b_{1.3} = 8.375 - (1.28)(5) = 1.975,$$

$$b_{23} = \frac{8(356) - (70)(40)}{8(218) - (40)^2} = \frac{48}{144} = 0.33, \text{ and}$$

$b_{2.3} = 8.75 - (0.33)(5) = 7.10.$

Hence the desired regression equations are

$$X_1 = 1.975 + 1.28X_3 \text{ and } X_2 = 7.10 + 0.33X_3$$

Next, we compute the residuals $X_{1.3} = X_1 - 1.975 - 1.28X_3$ and $X_{2.3} = X_2 - 7.10 - 0.33X_3$, and the simple correlation co-efficient between them. The necessary computations are given in the following table:

X_1	X_2	X_3	$X_{1.3}$	$X_{2.3}$	$X_{1.3}X_{2.3}$	$X_{1.3}^2$	$X_{2.3}^2$
5	10	2	+0.465	2.24	1.04160	0.2162	5.0176
9	12	6	−0.655	2.92	−1.91260	0.4290	8.5264
7	8	4	−0.095	−0.42	0.03990	0.0090	0.1764
10	9	5	1.625	0.25	0.40625	2.6406	0.0625
12	11	7	1.065	1.59	1.69335	1.1342	2.5281
8	7	6	−1.655	−2.08	3.44240	2.7390	4.3264
6	5	4	−1.095	−3.42	3.74490	1.1990	11.6964
10	8	6	0.345	−1.08	−0.37260	0.1190	1.1664
67	70	40	0	0	8.0832	8.4860	33.5002

Hence the co-efficient of correlation between $X_{1.3}$ and $X_{2.3}$, *i.e.* $r_{12.3}$ is obtained as

$$r_{12.3} = \frac{\sum X_{1.3} X_{2.3}}{\sqrt{\sum X_{1.3}^2 \sum X_{2.3}^2}} \qquad (\because \sum X_{1.3} = \sum X_{2.3} = 0)$$

$$= \frac{8.0832}{\sqrt{(8.4860)(33.5002)}} = \frac{8.0832}{16.8607} = 0.48.$$

11.12. $$r_{12.3} = \frac{r_{12} - r_{13} r_{23}}{\sqrt{(1-r_{13}^2)(1-r_{23}^2)}} = \frac{0.80-(-0.40)(-0.56)}{\sqrt{(1-0.16)(1-0.3136)}}$$

$$= \frac{0.80-0.224}{\sqrt{(0.84)(0.6864)}} = \frac{0.576}{0.7593} = 0.759,$$

$$r_{23.1} = \frac{r_{23} - r_{12}r_{13}}{\sqrt{(1-r_{12}^2)(1-r_{13}^2)}} = \frac{-0.56-(0.80)(-0.40)}{\sqrt{(1-0.64)(1-0.16)}}$$

$$= \frac{-0.56+0.32}{\sqrt{(0.36)(0.84)}} = \frac{-0.24}{0.55} = -0.436, \text{ and}$$

$$r_{13.2} = \frac{r_{13} - r_{12}r_{23}}{\sqrt{(1-r_{12}^2)(1-r_{23}^2)}} = \frac{-0.40-(0.80)(-0.56)}{\sqrt{(1-0.64)(1-0.3136)}}$$

$$= \frac{-0.40+0.448}{\sqrt{(0.36)(0.6864)}} = \frac{0.048}{0.497} = 0.097.$$

The linear regression equation of X_1 on X_2 and X_3 is

$$\frac{X_1 - \bar{X}_1}{S_1} = \left(\frac{r_{12}-r_{13}r_{23}}{1-r_{23}^2}\right)\frac{X_2 - \bar{X}_2}{S_2} + \left(\frac{r_{13}-r_{12}r_{23}}{1-r_{23}^2}\right)\frac{X_3 - \bar{X}_3}{S_3}$$

Substituting the values, we get

$$\frac{X_1 - 28.02}{4.42} = \left(\frac{0.576}{0.6864}\right)\frac{X_2 - 4.91}{1.10} + \left(\frac{0.048}{0.6864}\right)\frac{X_3 - 594}{85}$$

Or $\quad X_1 - 28.02 = (3.37)(X_2 - 4.91) + (0.0038)(X_3 - 594)$

Or $\quad X_1 = 28.02 + 3.37X_2 - 16.5467 + 0.0038X_3 - 2.2572$

Or $\quad X_1 = 9.22 + 3.37X_2 + 0.0038X_3,$

which is the required regression equation for hay-crop on spring rainfall and accumulated temperature.

11.13. The regression equation for estimating marks obtained (X_1) is

$$\frac{X_1 - \bar{X}_1}{S_1} = \left(\frac{r_{12}-r_{13}r_{23}}{1-r_{23}^2}\right)\frac{X_2 - \bar{X}_2}{S_2} + \left(\frac{r_{13}-r_{12}r_{23}}{1-r_{23}^2}\right)\frac{X_3 - \bar{X}_3}{S_3}$$

Substituting the values, we get

$$\frac{X_1 - 18.5}{11.2} = \left(\frac{0.60-(0.32)(0.35)}{1-0.1225}\right)\frac{X_2 - 100.6}{15.8} + \left(\frac{0.32-(0.60)\,(0.35)}{1-0.1225}\right)\frac{X_3 - 24}{6.0}$$

Or $$\frac{X_1 - 18.5}{11.2} = (0.5561)\,\frac{X_2 - 100.6}{15.8} + (0.1254)\,.\,\frac{X_3 - 24}{6.0}$$

Or $$X_1 - 18.5 = 0.3942\,(X_2 - 100.6) + 0.2341\,(X_3 - 24)$$

Or $$X_1 = 18.5 + 0.3942X_2 - 39.6565 + 0.2341X_3 - 5.6184$$

$$= 0.39X_2 + 0.23X_3 - 26.77$$

Partial correlaton co-efficients:

$$r_{12.3} = \frac{r_{12} - r_{13}\,r_{23}}{\sqrt{(1-r_{13}^2)\,(1-r_{23}^2)}} = \frac{0.60-(0.32)\,(0.35)}{\sqrt{(1-0.1024)\,(1-0.1225)}}$$

$$= \frac{0.60-0.1120}{\sqrt{(0.8976)\,(0.8775)}} = \frac{0.4880}{0.8875} = 0.55,$$

$$r_{13.2} = \frac{r_{13} - r_{12}\,r_{23}}{\sqrt{(1-r_{12}^2)\,(1-r_{23}^2)}} = \frac{0.32-(0.60)\,(0.35)}{\sqrt{(1-0.36)\,(1-0.1225)}}$$

$$= \frac{0.32-0.21}{\sqrt{(0.64)\,(0.8775)}} = \frac{0.11}{0.75} = 0.15, \text{ and}$$

$$r_{23.1} = \frac{r_{23} - r_{21}\,r_{31}}{\sqrt{(1-r_{21}^2)\,(1-r_{31}^2)}} = \frac{0.35-(0.60)\,(0.32)}{\sqrt{(1-0.36)\,(1-0.1024)}}$$

$$= \frac{0.35-0.192}{\sqrt{(0.64)\,(0.8976)}} = \frac{0.158}{0.758} = 0.21.$$

Interpretation:

(i) $r_{12.3} = 0.55$ represents the correlation co-efficient between the marks obtained and general intelligence scores for college students having the same number of hours of study.

(ii) $r_{13.2} = 0.15$ represents the correlaton co-efficient between marks obtained and hours of studies for college students having the same general intelligence scores.

(iii) $r_{23.1} = 0.21$ represents the correlation co-efficient between general intelligence scores and hours of studies for college students having the same marks obtained.

11.14 (b) The equations of the three regression planes are

$$x_1 = b_{12.3}x_2 + b_{13.2}x_3,$$
$$x_2 = b_{21.3}x_1 + b_{23.1}x_3,$$
$$x_3 = b_{31.2}x_1 + b_{32.1}x_2.$$

Comparing, we find

$$b_{12.3} = 0.41, \quad b_{13.2} = 0.23,$$
$$b_{21.3} = 0.96, \quad b_{23.1} = -0.025,$$
$$b_{31.2} = 1.04, \quad b_{32.1} = -0.05.$$

Now, $r_{12.3} = \sqrt{b_{12.3} \times b_{21.3}} = \sqrt{(0.41)\,(0.96)} = \sqrt{0.3936} = 0.63;$

$r_{13.2} = \sqrt{b_{13.2} \times b_{31.2}} = \sqrt{(0.23)(1.04)} = \sqrt{0.2392} = 0.49$, and

$$r_{23.1} = -\sqrt{b_{23.1} \times b_{32.1}} = -\sqrt{(-0.025)(-0.05)}$$
$$= -\sqrt{0.00125} = -0.035,$$ (both $b_{23.1}$ and $b_{32.1}$ are negative)

Again, $$r_{12} = \frac{r_{12.3} + r_{13.2}\,r_{23.1}}{\sqrt{(1-r^2_{13.2})\,(1-r^2_{23.1})}} = \frac{0.63 + (0.49)\,(-0.035)}{\sqrt{1-(0.49)^2}\,\sqrt{1-(0.035)^2}}$$

$$= \frac{0.63 - 0.01715}{\sqrt{(0.7599)\,(0.998775)}} = \frac{0.61285}{0.8694} = 0.70,$$

$$r_{13} = \frac{r_{13.2} + r_{12.3}\,r_{23.1}}{\sqrt{(1-r^2_{12.3})\,(1-r^2_{23.1})}} = \frac{0.49 + (0.63)\,(-0.035)}{\sqrt{[1-(0.63)^2]\,[1-(-0.035)^2]}}$$

$$= \frac{0.49 - -0.02205}{\sqrt{(0.6031)\,(0.998775)}} = \frac{0.46795}{0.7761} = 0.60,$$

$$r_{23} = \frac{r_{23.1} + r_{12.3}\, r_{13.2}}{\sqrt{(1-r_{12.3}^2)(1-r_{13.2}^2)}} = \frac{-0.035 + (0.63)(0.49)}{\sqrt{[1-(0.63)^2][1-(0.49)^2]}}$$

$$= \frac{-0.035 + 0.3087}{\sqrt{(0.60331)(0.7599)}} = \frac{0.2737}{0.6770} = 0.40.$$

11.15.(a)The regression equation of X_1 on X_2 and X_3 is

$$X_1 = a + b_{12.3} X_2 + b_{13.2} X_3.$$

Let $x_1 = X_1 - \bar{X}_1$, $x_2 = X_2 - \bar{X}_2$ and $x_3 = X_3 - \bar{X}_3$. Then the regression equation can be written more simply as

$$x_1 = b_{12.3}\, x_2 + b_{13.2}\, x_3.$$

The two normal equations are

$$\sum x_1 x_2 = b_{12.3} \sum x_2^2 + b_{13.2} \sum x_2 x_3$$

$$\sum x_1 x_3 = b_{12.3} \sum x_2 x_3 + b_{13.2} \sum x_3^2$$

Expressing the summations in terms of variances and zero order correlation co-efficients, we get

$$nS_1 S_2 r_{12} = b_{12.3}\, nS_2^2 + b_{13.2}\, nS_2 S_3\, r_{23}$$

$$nS_1 S_3 r_{13} = b_{12.3}\, nS_2 S_3\, r_{23} + b_{13.2}\, nS_3^2$$

Simplifying, we obtain

$$S_1 r_{12} = b_{12.3}\, S_2 + b_{13.2}\, S_3\, r_{23},$$

$$S_1 r_{13} = b_{12.3}\, S_2\, r_{23} + b_{13.2}\, S_3$$

Solving them simultaneously, we get

$$b_{12.3} = \frac{S_1}{S_2}\left(\frac{r_{12} - r_{13}\, r_{23}}{1 - r_{23}^2}\right), \quad b_{13.2} = \frac{S_1}{S_3}\left(\frac{r_{13} - r_{12}\, r_{23}}{1 - r_{23}^2}\right)$$

Similarly from the regression equation

$x_3 = b_{32.1}\, x_2 + b_{31.2}\, x_1$, we find that

$$b_{32.1} = \frac{S_3}{S_2}\left(\frac{r_{23} - r_{13}\, r_{12}}{1 - r_{12}^2}\right), \text{ and } b_{31.2} = \frac{S_3}{S_1}\left(\frac{r_{13} - r_{23}\, r_{12}}{1 - r_{12}^2}\right).$$

Hence $b_{13.2} \times b_{31.2} = \dfrac{S_1}{S_3}\left(\dfrac{r_{13} - r_{12}\,r_{23}}{1 - r_{23}^2}\right) \cdot \dfrac{S_3}{S_1}\left(\dfrac{r_{13} - r_{23}\,r_{12}}{1 - r_{12}^2}\right)$

$$= \frac{(r_{13} - r_{12}\,r_{23})^2}{(1 - r_{12}^2)\,(1 - r_{23}^2)} = r_{13.2}^2 .$$

(b) (i) Here $r_{12.3} = \dfrac{r_{12} - r_{13}\,r_{23}}{\sqrt{(1 - r_{13}^2)\,(1 - r_{23}^2)}}$

$$= \frac{0.60 - (0.80)\,(-0.50)}{\sqrt{(1-0.64)\,(1 - 0.25)}} = \frac{0.60 + 0.40}{\sqrt{(0.36)\,(0.75)}}$$

$$= \frac{1.00}{0.52} = 1.92.$$

This is an impossible value of $r_{12.3}$ as it should be numerically less than 1. Hence it is not possible to get these correlations from a set of experimental data.

(ii) Here $r_{12.3} = \dfrac{r_{12} - r_{13}\,r_{23}}{\sqrt{(1 - r_{13}^2)\,(1 - r_{23}^2)}}$

$$= \frac{0.6 - (-0.4)\,(0.7)}{\sqrt{(1-0.16)\,(1 - 0.49)}} = \frac{0.6 + 0.28}{\sqrt{(0.84)\,(0.51)}}$$

$$= \frac{0.88}{0.654} = 1.34.$$

This is an impossible value of $r_{12.3}$ as the partial correlation co-efficient is to be numerically less than 1. Hence there are good reasons to suspect these values by noting that while r_{12} and r_{23} are both large and positive, r_{13} is fairly large and negative, contrary to what would be expected.

(iii) Here $r_{12.3} = \dfrac{r_{12} - r_{13}\,r_{23}}{\sqrt{(1 - r_{13}^2)\,(1 - r_{23}^2)}}$

$$= \frac{0.01 - (0.66)(-0.70)}{\sqrt{(1-(0.66)^2)(1-(-0.7)^2)}} = \frac{0.01 + 0.462}{\sqrt{(0.5644)(0.51)}}$$

$$= \frac{0.472}{0.5365} = 0.88,$$

$$r_{13.2} = \frac{r_{13} - r_{12}\, r_{32}}{\sqrt{(1 - r_{12}^2)(1 - r_{32}^2)}}$$

$$= \frac{0.66 - (0.01)(-0.70)}{\sqrt{(1-(.01)^2)(1-(0.7)^2)}} = \frac{0.66 + 0.007}{\sqrt{(0.9999)(0.51)}}$$

$$= \frac{0.667}{0.7141} = 0.93, \text{ and}$$

$$r_{23.1} = \frac{r_{23} - r_{21}\, r_{31}}{\sqrt{(1 - r_{21}^2)(1 - r_{31}^2)}}$$

$$= \frac{-0.70 - (0.01)(0.66)}{\sqrt{(1-(0.01)^2)(1-(0.66)^2)}}$$

$$= \frac{-0.70 + 0.0066}{\sqrt{(0.9999)(0.5644)}} = \frac{-0.7066}{0.7512} = -0.94.$$

It is possible to obtain the given correlations from a set of experimental data.

11.16. Substituting the given values in the formula for $r_{13.2}$, i.e.

$$r_{13.2} = \frac{r_{13} - r_{21}\, r_{23}}{\sqrt{(1 - r_{12}^2)(1 - r_{23}^2)}}, \text{ we get}$$

$$r_{13.2} = \frac{-0.641 - (0.370)(-0.736)}{\sqrt{(1 - (0.370)^2)(1 - (-0.736)^2)}}$$

$$= \frac{-0.641 + 0.27232}{\sqrt{(0.8631)(0.458304)}} = \frac{-0.36868}{0.62894} = -0.586.$$

Next, by definition, $r_{42} = \dfrac{\text{Cov}(X_4, X_2)}{\sqrt{\text{Var}(X_4)\,\text{Var}(X_2)}}$.

Since $X_4 = X_1 + X_2$, therefore $\bar{X}_4 = \bar{X}_1 + \bar{X}_2$ and

$$X_4 - \bar{X}_4 = (X_1 - \bar{X}_1) + (X_2 - \bar{X}_2).$$

Now
$$\begin{aligned}
\text{Cov}(X_4, X_2) &= \frac{1}{n}\sum(X_4 - \bar{X}_4)(X_2 - \bar{X}_2) \\
&= \frac{1}{n}\sum[(X_1-\bar{X}_1)+(X_2-\bar{X}_2)](X_2-\bar{X}_2) \\
&= \frac{1}{n}\sum[(X_1-\bar{X}_1)(X_2-\bar{X}_2)+(X_2-\bar{X}_2)^2] \\
&= r_{12}S_1S_2 + S_2^2;
\end{aligned}$$

$$\begin{aligned}
\text{Var}(X_4) &= \frac{1}{n}\sum(X_4 - \bar{X}_4)^2 \\
&= \frac{1}{n}\sum[(X_1-\bar{X}_1)+(X_2-\bar{X}_2)]^2 \\
&= \frac{1}{n}\sum[(X_1-\bar{X}_1)^2+(X_2-\bar{X}_2)^2+2(X_1-\bar{X}_1)(X_2-\bar{X}_2)] \\
&= S_1^2 + S_2^2 + 2r_{12}S_1S_2; \text{ and}
\end{aligned}$$

$$\text{Var}(X_2) = \frac{1}{n}\sum(X_2-\bar{X}_2)^2 = S_2^2$$

Substituting these values, we get

$$\begin{aligned}
r_{42} &= \frac{S_1S_2r_{12} + S_2^2}{\sqrt{(S_1^2 + S_2^2 + 2S_1S_2r_{12})(S_2^2)}} \\
&= \frac{(1)(1.3)(0.370) + (1.3)^2}{\sqrt{[(1)^2 + (1.3)^2 + 2(1)(1.3)(0.370)][(1.3)^2]}} \\
&= \frac{0.481 + 1.69}{\sqrt{(3.652)(1.69)}} = \frac{2.171}{2.484} = 0.874.
\end{aligned}$$

Similarly, we find that $r_{43} = -0.836$.

Now $$r_{43.2} = \frac{r_{43} - r_{42} r_{23}}{\sqrt{(1 - r_{42}^2)(1 - r_{23}^2)}}$$

$$= \frac{-0.836 - (0.874)(-0.736)}{\sqrt{1-(0.874)^2}\ \sqrt{1 - (-0.736)^2}}$$

$$= \frac{-0.836 + 0.643264}{\sqrt{(0.236124)(0.458304)}} = \frac{-0.192736}{0.328963} = -0.586.$$

Thus the two partial correlation co-efficients are equal.

11.17. (b) (i) We are given that $R_{1.23} = 1$

Squaring, $R^2_{1.23} = 1$

Or $$\frac{r_{12}^2 + r_{13}^2 - 2r_{12} r_{23} r_{31}}{1 - r_{23}^2} = 1$$

Or $$r_{12}^2 + r_{13}^2 - 2r_{12} r_{23} r_{31} = 1 - r_{23}^2$$

Then $$r_{12}^2 + r_{23}^2 - 2r_{12} r_{23} r_{31} = 1 - r_{13}^2$$

Dividing both sides by $1 - r_{13}^2$, we get

$$\frac{r_{12}^2 + r_{23}^2 - 2r_{12} r_{23} r_{31}}{1 - r_{13}^2} = 1$$

Or $$R^2_{2.13} = 1$$

Or $$R_{2.13} = 1$$ ($\because$ multiple correlation co-efficient is considered non-negative)

(ii) We have shown in (i) that $R_{2.13} = 1$

Squaring, $R^2_{2.13} = 1$

Or $$\frac{r_{12}^2 + r_{23}^2 - 2r_{12} r_{23} r_{31}}{1 - r_{13}^2} = 1$$

Or $\qquad r_{12}^2 + r_{23}^2 - 2r_{12}r_{23}r_{31} = 1 - r_{13}^2$

Or $\qquad r_{13}^2 + r_{23}^2 - 2r_{12}r_{23}r_{31} = 1 - r_{12}^2$

Or $$\frac{r_{13}^2 + r_{23}^2 - 2r_{12}r_{23}r_{31}}{1 - r_{12}^2} = 1$$

Or $\qquad R_{3.12}^2 = 1 \quad \text{or} \quad R_{3.12} = 1$

(c) **Now** $$R_{1.23} = \sqrt{\frac{r_{12}^2 + r_{13}^2 - 2r_{12}r_{13}r_{23}}{1 - r_{23}^2}}$$

$R_{1.23} = 0$, if and only if

$r_{12}^2 + r_{13}^2 - 2r_{12}r_{13}r_{23} = 0,$

i.e. $\qquad$ if $r_{12}^2 + r_{13}^2 = 2r_{12}r_{13}r_{23}$

Again $$R_{2.13} = \sqrt{\frac{r_{12}^2 + r_{23}^2 - 2r_{12}r_{13}r_{23}}{1 - r_{13}^2}}$$

$$= \sqrt{\frac{r_{12}^2 + r_{23}^2 - (r_{12}^2 + r_{13}^2)}{1 - r_{13}^2}} = \sqrt{\frac{r_{23}^2 - r_{13}^2}{1 - r_{23}^2}}$$

which is not necessarily zero.

(d) **Given** $r_{12} = r_{23} = r_{13} = r \neq 1$

Now $$R_{1.23}^2 = \frac{r_{12}^2 + r_{13}^2 - 2r_{12}r_{13}r_{23}}{1 - r_{23}^2}$$

$$= \frac{r^2 + r^2 - 2r.r.r}{1 - r^2} = \frac{2r^2(1-r)}{1 - r^2} = \frac{2r^2}{1+r}$$

$$\therefore \quad R_{1.23} = \frac{r\sqrt{2}}{\sqrt{1 + r}}$$

Similarly, we find that

$$R_{2.13} = \frac{r\sqrt{2}}{\sqrt{1+r}} = R_{3.12}.$$

Hence the result.

11.18. (a) $$r_{12.3} = \frac{r_{12} - r_{13}r_{23}}{\sqrt{(1-r_{13}^2)(1-r_{23}^2)}} = 0, \text{ if and only if}$$

$$r_{12} - r_{13}r_{23} = 0,$$

i.e. if $r_{12} = r_{13}r_{23}$

But $r_{12} = 0,$

$\therefore$ $r_{13}r_{23} = 0$, if either $r_{13} = 0$ or $r_{23} = 0$.

(b) The mean of $aX_1 + bX_2 + cX_3 = 0$ **is** $a\bar{X}_1 + b\bar{X}_2 + c\bar{X}_3 = 0$. **Substracting, we get**

$$a(X_1 - \bar{X}_1) + b(X_2 - \bar{X}_2) + c(X_3 - \bar{X}_3) = 0,$$

i.e. $ax_1 + bx_2 + cx_3 = 0$, whcih can be written as

$x_3 = \left(-\frac{a}{c}\right)x_1 + \left(-\frac{b}{c}\right)x_2$, where x_1 and x_2 are independent and measured from their respective means.

Squaring and summing, we get

$$\Sigma x_3^2 = \frac{a^2}{c^2}\Sigma x_1^2 + \frac{b^2}{c^2}\Sigma x_2^2. \quad \text{(product term vanishes)}$$

$$= \frac{a^2}{c^2}.nS_1^2 + \frac{b^2}{c^2}.nS_2^2$$

Also $$\Sigma x_1x_3 = \left(-\frac{a}{c}\right)\Sigma x_1^2 + \left(-\frac{b}{c}\right)\Sigma x_1x_2$$

$$= \left(-\frac{a}{c}\right)nS_1^2 \qquad (\Sigma x_1x_2 = 0)$$

Thus $$r_{13} = \frac{\Sigma x_1x_3}{\sqrt{\Sigma x_1^2 \Sigma x_3^2}} = \frac{-\frac{a}{c}nS_1^2}{\sqrt{(nS_1^2)\left(\frac{a^2}{c^2}nS_1^2 + \frac{b^2}{c^2}nS_2^2\right)}}$$

$$= \frac{-aS_1}{\sqrt{a^2S_1^2 + b^2S_2^2}}$$

Similarly, we find that $r_{23} = \dfrac{-bS_2}{\sqrt{a^2 S_1^2 + b^2 S_2^2}}$ and $r_{12} = 0$

Now, $$r_{13.2} = \frac{r_{13} - r_{12}\, r_{23}}{\sqrt{(1 - r_{12}^2)\,(1 - r_{23}^2)}}$$

$$= \frac{\dfrac{-aS_1}{\sqrt{a^2 S_1^2 + b^2 S_2^2}} - 0}{\sqrt{(1-0)\left(1 - \dfrac{b^2 S_2^2}{a^2 S_1^2 + b^2 S_2^2}\right)}} = \frac{-aS_1}{+\sqrt{a^2 S_1^2}} = -1;$$

$$r_{12.3} = \frac{r_{12} - r_{13}\, r_{23}}{\sqrt{(1 - r_{13}^2)\,(1 - r_{23}^2)}}$$

$$= \frac{0 - \left(\dfrac{-aS_1}{\sqrt{a^2 S_1^2 + b^2 S_2^2}}\right)\left(\dfrac{-bS_2}{\sqrt{a^2 S_1^2 + b^2 S_2^2}}\right)}{\sqrt{\left(1 - \dfrac{a^2 S_1^2}{a^2 S_1^2 + b^2 S_2^2}\right)\left(1 - \dfrac{b^2 S_2^2}{a^2 S_1^2 + b^2 S_2^2}\right)}}$$

$$= \frac{-ab\, S_1\, S_2}{+\sqrt{a^2 b^2\, S_1^2\, S_2^2}} = -1, \text{ and}$$

$$r_{23.1} = \frac{r_{23} - r_{12}\, r_{13}}{\sqrt{(1 - r_{12}^2)\,(1 - r_{13}^2)}}$$

$$= \frac{\dfrac{-bS_2}{\sqrt{a^2 S_1^2 + b^2 S_2^2}} - 0}{\sqrt{\left(1 - \dfrac{b^2 S_2^2}{a^2 S_1^2 + b^2 S_2^2}\right)(1-0)}} = \frac{-b\, S_2}{+\sqrt{b^2 S_2^2}} = -1.$$

Hence $r_{13.2} = r_{12.3} = r_{23.1} = -1$

11.20. **The estimated least-squares equation of the quadratic regression is**

$$\hat{Y} = a + b_1X + b_2X^2,$$

where a, b_1 and b_2 are the estimates of the parameters.

The three normal equations are

$$\sum Y = na + b_1\sum X + b_2\sum X^2,$$

$$\sum XY = a\sum X + b_1\sum X^2 + b_2\sum X^3,$$

$$\sum X^2Y = a\sum X^2 + b_1\sum X^3 + b_2\sum X^4.$$

The computations needed to find a, b_1 and b_2 are shown below:

X	Y	X^2	X^3	X^4	XY	X^2Y	Y^2
−2	0.4	4	−8	16	−0.8	1.6	0.16
−1	1.3	1	−1	1	−1.3	1.3	1.69
0	2.2	0	0	0	0	0	4.84
1	2.5	1	1	1	2.5	2.5	6.25
2	3.0	4	8	16	6.0	12.0	9.00
0	9.4	10	0	34	6.4	17.4	21.94

Substituting the sums in the normal equations, we get

$$5a + 10b_2 = 9.4$$

$$10b_1 = 6.4$$

$$10a + 34b_2 = 17.4$$

Solving them, we obtain

$$a = 2.08,\ b_1 = 0.64 \text{ and } b_2 = -0.1.$$

Hence the equation of the desired quadratic regression is

$$\hat{Y} = 2.08 + 0.64X - 0.1X^2.$$

The standard error of estimate in this case is

$$s_{y.x,x^2} = \sqrt{\frac{\sum(Y-\hat{Y})^2}{n-3}}$$

$$= \sqrt{\frac{\sum Y^2 - a\sum Y - b_1\sum XY - b_2\sum X^2Y}{n-3}}$$

$$= \sqrt{\frac{21.94-(2.08)(9.4)-(0.64)(6.4)-(-0.7)(17.4)}{5-3}}$$

$$= \sqrt{\frac{0.032}{2}} = \sqrt{0.016} = 0.13$$