University of Sargodha

2562

BS 1st Semester/Term Exam 2021

Subject: I.T/CS/SE **Paper: Calculus & Analytial Geometry (Math-101/2213)**

Time Allowed: 02:30 Hours **Maximum Marks: 60**

Note: i) Objective part is compulsory. Attempt any three questions from subjective part.
ii) The marks of the students, who are repeating the course, will be converted according to 80 marks (for non-practical courses)/60 marks (for practical courses).

Objective Part (Compulsory)

Q.1. Write short answers of the following in 2-3 lines each on your answer sheet. **(2*12)**

(i) Find the slope and y-intercept of the line $3x+4y=12$ (ii) Solve the inequality $\frac{-x}{3}<2x+1$. (iii) Define average rate of change of a function $y=f(x)$ over the interval $[x_1,x_2]$. (iv) Evaluate the limit $\lim\limits_{x\to 1}\frac{x^2+x-2}{x^2-x}$. (v) Evaluate the limit $\lim\limits_{x\to 0}\frac{\sin 2x}{5x}$. (vi) Define continuity of a function at a point. (vii) Evaluate the integral $\int e^{\theta}\sin\theta d\theta$. (viii) Find partial fractions of $\frac{5x-3}{x^2-2x-3}$. (ix) Write chain rule for differentiation. (x) Evaluate the limit $\lim\limits_{x\to 9}\frac{\sqrt{x}-3}{x-9}$. (xi) Let $f(x)=x^3+1$, then find $f^{-1}(x)$. (xii) Find $\frac{dy}{dx}$ if $y=\frac{\ln x}{1+\ln x}$.

Subjective part (3*12)

Q.2. a). Evaluate the limit $\lim\limits_{x\to 4}\frac{4-x}{5-\sqrt{x^2+9}}$.

b). For what value of a is $f(x)=\begin{cases}x^2-1, & \text{if } x<3\\ 2ax, & \text{if } x\geq 3\end{cases}$ continuous at every x ?

Q.3. a). Find $\frac{dy}{dx}$ if $y=\frac{1}{6}\left(1+\cos^2(7t)\right)^3$.

b). Find tangent to the curve $x^2y^2=9$ at the point $(-1,3)$.

Q.4. a). Find the value or values of c that satisfy the equation $\frac{f(b)-f(a)}{b-a}=f'(c)$ in the conclusion of the mean value theorem.

b). Find the absolute maximum and minimum values of the function $f(x)=x^2-1$ in the interval $[-1,2]$.

Q.5. a). Evaluate the integral $\int\frac{\sec x}{\sqrt{\ln(\sec x+\tan x)}}dx$.

b). Find the area of the region enclosed by $y=x^2-2$ and $y=2$.

Q.6. a). Find the volume of the parallelepiped determined by the vectors

$$\vec{u}=i-j+k,\ \vec{v}=2i+j-k,\ \vec{w}=-i+2j-k$$

b). If $\vec{AB}=i+4j-2k$ and B is the point $(5,1,3)$, then find A.

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Q.1 $3x + 4y = 12$

$4y = -3x + 12$

$y = \frac{-3x}{4} + \frac{\cancel{12}3}{\cancel{4}1}$

$y = \frac{-3}{4}x + 3$

$\boxed{y = mx + c}$

Slop = $m = \frac{-3}{4}$; y-intercept $= 3$

ii) $\frac{-x}{3} < 2x + 1$

$-x < 3(2x + 1)$

$-x < 6x + 3$

$-x - 6x < 3$

$-7x < 3$

$x = \frac{3}{-7}$ Ans

iii) $y = f(x)$ $[x_1 , x_2]$

Area $= \frac{y_2 - y_1}{x_2 - x_1}$

$= \frac{f(x_2) - f(x_1)}{x_2 - x_1}$

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(v) Evaluate $\lim_{x \to 0} \frac{\sin 2x}{5x}$

$\times$ by 2

$= \lim_{x \to 0} \frac{2}{2} \frac{\sin 2x}{5x}$

$= \lim_{x \to 0} \frac{2}{5} \frac{\sin 2x}{2x}$

$= \frac{2}{5}(1)$

$= 2/5$ Ans.

(vi) $\text{Limit}_{x \to 1} \frac{x^2 + x - 2}{x^2 - x}$

using L'hospital rule.

$= \lim_{x \to 1} \frac{2x + 1}{2x - 1}$

$= \frac{2(1) + 1}{2(1) - 1}$

$= \frac{2 + 1}{2 - 1} = \frac{3}{1}$

$= 3$ Ans.

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(iii) ~~A limit~~
A function is continuous at a point $x=c$ if:-

- function is define at $x=c$.
- $f(x)$ is $x \to c$ is exists. $= \lim_{x \to c} f(x)$
- $f(c) = \lim_{x \to c} f(x)$

(vii) $\int \underset{I}{e^{\theta}} \underset{II}{\sin\theta}\, d\theta$ $\qquad I = \int e^{\theta} \sin\theta\, d\theta$

Using By Parts.

$= e^{\theta} \int \sin\theta\, d\theta - \int \left[\frac{d}{d\theta}(e^{\theta}) \cdot \int \sin\theta\, d\theta\right] d\theta$

$= -e^{\theta} \cos\theta - \int e^{\theta} \cdot -\cos\theta\, d\theta$

$= -e^{\theta} \cos\theta + \int e^{\theta}\, \cos\theta\, d\theta$

Again By parts

$= -e^{\theta} \cos\theta + \left[e^{\theta} \int \cos\theta\, d\theta - \int \left[\frac{d}{d\theta}(e^{\theta}) \cdot \int \cos\theta\, d\theta\right] d\theta\right]$

$= -e^{\theta} \cos\theta + \left[e^{\theta} \sin\theta - \int e^{\theta} \sin\theta\, d\theta\right]$

$= -e^{\theta} \cos\theta + e^{\theta} \sin\theta - I$

$= -e^{\theta} (\cos\theta + \sin\theta) - I$

Ans

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(viii) $\frac{5x-3}{x^2-2x-3}$ By partial fraction

$$\frac{5x-3}{x^2-2x-3} = \frac{5x-3}{x^2+1x-3x-3} = \frac{5x-3}{x(x-1)-3(x+3)}$$

$$\frac{5x-3}{(x-1)(x+3)} = \frac{A}{(x-1)} + \frac{B}{(x+3)} \quad \text{—— (i)}$$

$\times$ by $(x-1)$ $(x+3)$ on B.S.

$$5x-3 = A(x+3) + B(x-1) \quad \text{—— (ii)}$$

put $x = 1$ in (ii)

$$(5)(1) - 3 = A(1+3) + B(1-1)$$

$$5-3 = 4A + 0$$

$$2 = 4A$$

$$\cancel{2}/\cancel{4}_2 = A$$

$$\boxed{1/2 = A}$$

put $x = -3$ in (ii)

$$5(-3) - 3 = A(-3+3) + B(-3-1)$$

$$-15-3 = 0 - 4B$$

$$\frac{\cancel{-18}\,9}{\cancel{-4}\,2} = B \Rightarrow \boxed{B = 9/2}$$

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Now

$$\frac{5x-3}{x^2-2x-3} = \frac{1}{2(x-1)} + \frac{9}{2(x+3)} \text{ Ans}$$

(ix) Chain rule:-

Differentiate $f(x)$ w.r.t $g(x)$

Let

$$y = f(x) \quad , \quad t = g(x)$$

$$\frac{dy}{dx} = \frac{d}{dx} f(x) \quad ; \quad \frac{dt}{dx} = \frac{d}{dx} g(x)$$

Now :

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

which is

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$

(x)

$$\lim_{x \to 9} \frac{\sqrt{x} - 3}{x - 9}$$

$$= \lim_{x \to 9} \frac{\sqrt{x} - 3}{(\sqrt{x})^2 - (3)^2} = \lim_{x \to 9} \frac{\cancel{\sqrt{x} - 3}}{(\sqrt{x}+3)\cancel{(\sqrt{x}-3)}}$$

$$= \lim_{x \to 9} \frac{1}{\sqrt{x} - 3}$$

$$= \frac{1}{\sqrt{9} - 3} = \frac{1}{3+3} = \frac{1}{6} \text{ Ans}$$

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(XII) $f(x) = x^3+1$ find $f^{-1}(x)$.

$y = x^3+1$

Swap x with y & y with x.

$x = y^3+1$

$x+1 = y^3$

$(x+1)^{1/3} = y$

$f^{-1}(x) = (x+1)^{1/3}$

(XII) find $\frac{dy}{dx}$ if $y = \frac{\ln x}{1+\ln x}$.

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\ln x}{1+\ln x}\right)$$

$$= \frac{(1+\ln x)\frac{d}{dx}(\ln x) - (\ln x)\frac{d}{dx}(1+\ln x)}{(1+\ln x)^2}$$

$$= \frac{1+\ln x\left(\frac{1}{x}\right) - \ln x\frac{1}{x}}{}$$

$$= \frac{\frac{1}{x} + \cancel{\frac{\ln x}{x}} - \cancel{\frac{\ln x}{x}}}{(1+\ln x)^2}$$

$$\frac{dy}{dx} = \frac{1/x}{(1+\ln x)^2} = \frac{1}{x(1+\ln x)^2}$$ ~~Ans~~

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Subjective Part.

Q2 (1) $\lim_{x\to 4} \frac{4-x}{5-\sqrt{x^2+9}}$

$= \lim_{x\to 4} \frac{(4-x)}{5-\sqrt{x^2+9}} \times \frac{5+\sqrt{x^2+9}}{5+\sqrt{x^2+9}}$

$= \lim_{x\to 4} \frac{(4-x)(5+\sqrt{x^2+9})}{(5)^2-(\sqrt{x^2+9})^2}$

$= \lim_{x\to 4} \frac{(4-x)(5+\sqrt{x^2+9})}{25-x^2-9}$

$= \lim_{x\to 4} \frac{(4-x)(5+\sqrt{x^2+9})}{\cancel{25} -x^2+16}$

$= \lim_{x\to 4} \frac{\cancel{(4-x)}(5+\sqrt{x^2+9})}{\cancel{(4-x)}(4+x)}$

$= \lim_{x\to 4} \frac{(5+\sqrt{x^2+9})}{4+x} = \frac{5+\sqrt{(4)^2+9}}{4+4}$

$= \frac{5+\sqrt{16+9}}{8} = \frac{5+\sqrt{25}}{8} = \frac{5+5}{8}$

$= \frac{10}{8} = \frac{5}{4}$ Ans.

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(b) $f(x) = \begin{cases} x^2 - 1 & \text{if } x < 3 \\ 2ax & \text{if } x \geq 3 \end{cases}$ continuous everywhere.

continuous at every x

$\lim_{x \to 3^-} x^2 - 1 = \lim_{x \to 3^+} 2ax$

$(3)^2 - 1 = 2a(3)$

$9 - 1 = 6a$

$8 = 6a$

$\cancel{8}\, 4 = a$

$\cancel{6}\, 3$

$\boxed{\frac{4}{3} = a}$ Ans

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Q.3

a) find $\frac{dy}{du}$ if $y = \frac{1}{6}(1+\cos^2(7t))^3$

$\frac{d}{dt}y = \frac{1}{6}\frac{d}{dt}(1+\cos^2(7t))^3$

$= \frac{1}{6}\left[3\cdot(1+\cos^2(7t))^2 \cdot \frac{d}{du}(1+\cos^2(7t))\right]$

$= \frac{1}{\cancel{6}_2}\left[\cancel{3}(1+\cos^2(7t))^2 \cdot 2\cos(7t)\ \frac{d}{dt}\cos(7t)\right]$

$= \frac{1}{\cancel{2}}\left[(1+\cos^2(7t))^2 \cdot \cancel{2}\cos(7t)\ \{-\sin 7t\}\right] \cdot \frac{d}{dt}(7t)$

$= (1+\cos^2(7t))^2 (\cos 7t)\ -\sin 7t\ (7))$

$\frac{dy}{dt} = -7((1+\cos^2(7t))^2 \cos 7t\ \sin 7t)$

Q.3(b) $x^2y^2 = 9$ $\quad(-1, 3)$

$\frac{d}{dx}(x^2y^2) = \frac{d}{du}(9)$

$x^2\ 2y\frac{dy}{du} + y^2\ 2u\cdot(1) = 0$

$2x^2y\frac{dy}{du} + 2xy^2 = 0$

$$2x^2 y\frac{dy}{dx} = -2xy^2$$

$$\frac{dy}{dx} = \frac{-\cancel{2}x y^{\cancel{2}}}{\cancel{2}x^2 \cancel{y}}$$

$$\frac{dy}{dx} = -\frac{y}{x}$$

$$\frac{dy}{dx} = \frac{-3}{-1} = 3 = m = \text{slop}$$

$$y_2 - y_1 = m(x_2 - x_1)$$

$$y - 3 = 3(x+1)$$

$$y = 3(x+1) + 3$$

$$y = 3x + 3 + 3$$

$$y = 3x + 6 \text{ —— Tangant}$$

$$\cancel{3x - 4.}$$

for normal — if:

$$y - 3 = \frac{-1}{3}(x+1).$$

$$y = \frac{-x}{3} + \frac{1}{3} - 3 \Rightarrow y = \frac{-x}{3} \frac{+1-9}{3}$$

$$y = \frac{-1}{3}x + \frac{8}{3} \quad \text{normal}$$

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Q6 (a)

$$\int \frac{\sec x}{\sqrt{\ln(\sec x + \tan x)}}\,dx$$

$t = \ln(\sec x + \tan x)$

$$dt = \frac{1}{\sec x + \tan x}(\sec x \tan x + \sec^2 x)\,dx$$

$$= \frac{\sec x(\tan x + \sec x)}{\sec x + \tan x}\,dx$$

$dt = \sec x\,dx$

$$= \int \frac{dt}{\sqrt{t}}$$

$$= \int t^{-1/2}\,dt$$

$$= \frac{t^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c = \frac{t^{1/2}}{1/2} + c = 2\sqrt{t} + c$$

$= 2\sqrt{\ln(\sec x + \tan x)} + c$

Ans

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$$\int \frac{(1+\tan x)\,dx}{(x+\log \sec x)}$$

$x + \log \sec x = t$

$1 + \frac{d}{dx}(\log \sec x) = \frac{dt}{dx} \Rightarrow 1 + \frac{1}{\sec x}(\sec x \tan x)\,dx = dt$

$1 + \tan x \, dx = dt$

$$\int \frac{dt}{t}$$

$= \log t + c$

$= \log(x + \log \sec x) + c$ Ans.

(b) Find area —

$y = x^2 - 2$, $y = 2$

$y = 4$

$x^2 - 2 = 2$

$x^2 = 4$

$x = \pm 2$

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$$\int_{-2}^{2} (2-(x^2-2))\,dx$$

$$\int_{-2}^{2} 2x \quad 2-x^2+2\,dx$$

$$= \int_{-2}^{2} 4-x^2\,dx$$

$$= \left[4x - \frac{x^3}{3} + c\right]_{-2}^{2}$$

$$= \left(4(2) - \frac{(2)^3}{3}\right) - \left(4(-2) - \frac{(-2)^3}{3}\right)$$

$$= 8 - \frac{8}{3} - \left(-8 + \frac{8}{3}\right)$$

$$= +8 - \frac{8}{3} + 8 - \frac{8}{3}$$

$$= \cancel{\frac{-8-8}{3}} = \cancel{\frac{8-8}{3}} = \cancel{\frac{-16}{3}}$$

$$= 16 - \frac{16}{3}$$

$$= \frac{48-16}{3} = \frac{36}{3}$$ Ans