Theorem 4.5.3 Plus/Minus Theorem

Let S be a nonempty set of vectors in a vector space V.

(a) If S is a linearly independent set, and if v is a vector in V that is outside of span(S), then the set $S \cup \{v\}$ that results by inserting v into S is still linearly independent.

(b) If v is a vector in S that is expressible as a linear combination of other vectors in S, and if $S-\{v\}$ denotes the set obtained by removing v from S, then S and $S-\{v\}$ span the same space that is

$$\text{span}(S) = \text{span}(S-\{v\})$$

Proof (a)

Let $S = \{v_1, v_2, \ldots, v_r\}$ is a linearly independent set of vectors in V and v is a vector that is outside of span(S).

we need to show $\{v_1, v_2, \ldots, v_r, v\}$ is a linearly independent set.

Let $k_1 v_1 + k_2 v_2 + \cdots + k_r v_r + k_{r+1} v = 0$ —— ①

Note that k_{r+1} must be zero because if $k_{r+1} \neq 0$ then

① $\Rightarrow$ $k_{r+1} v = -k_1 v_1 - k_2 v_2 - \cdots - k_r v_r$

$\Rightarrow$ $v = -\dfrac{k_1}{k_{r+1}} v_1 - \dfrac{k_2}{k_{r+1}} v_2 - \cdots - \dfrac{k_r}{k_{r+1}} v_r$

i.e $v \in$ Span(S) which is not possible.

Hence $\boxed{k_{r+1} = 0}$ — ②

Now using this value in ①

$\Rightarrow k_1 v_1 + k_2 v_2 + \cdots + k_r v_r = 0$

$\Rightarrow k_1 = k_2 = \cdots = k_r = 0$ — ③ ($\because$ S is a linearly independent set)

Combining ② & ③

$k_1 = k_2 = \cdots = k_r = k_{r+1} = 0$

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which shows that $S \cup \{v\}$ is linearly independent.

(b) Assume that $S = \{v_1, v_2, \ldots, v_{k-1}, v_k\}$ is a set of vectors in V and suppose that v_k is a linear combination of $v_1, v_2, \ldots, v_{k-1}$ ~~Also~~ that is;

$$v_k = c_1 v_1 + c_2 v_2 + \cdots + c_{k-1} v_{k-1} \quad \text{——} \; (4)$$

we want to show

$$\text{Span}(S) = \text{Span}(S - \{v_k\})$$

Take $w \in \text{Span}(S)$

$$\Rightarrow w = k_1 v_1 + k_2 v_2 + \cdots + k_{k-1} v_{k-1} + k_k v_k$$

using (4) $\Rightarrow w = k_1 v_1 + k_2 v_2 + \cdots + k_{k-1} v_{k-1} + k_k (c_1 v_1 + \cdots + c_{k-1} v_{k-1})$

i.e $w = (k_1 + k_k c_1) v_1 + (k_2 + k_k c_2) v_2 + \cdots + (k_{k-1} + k_k c_{k-1}) v_{k-1}$

$$\Rightarrow w \in \text{Span}(S - \{v_k\})$$

Hence $\text{Span}(S) = \text{Span}(S - \{v_k\})$

Theorem 4.5.5 Let S be a finite set of vectors in a finite-dimensional vector space V.

(a) If S spans V but is not a basis for V, then S can be reduced to a basis for V by removing appropriate vectors from S.

(b) If S is a linearly independent set that is not already a basis for V, then S can be enlarged to a basis for V by inserting appropriate vectors into S.

Proof: (a) If S is a set of vectors that spans V but is not a basis for V then S is a linearly dependent set.

$\Rightarrow$ Some vector v in S is a linear combination of other vectors in S.

Now Take $S' = S - \{v\}$

then by plus/minus theorem

$$Span(S) = Span(S') = Span(S - \{v\}) = V$$

If S' is linearly independent. Then S' is a basis for V.

If not then some vector v' can be written as linear combination of other vectors in V.

Take $S'' = S' - \{v'\}$

then by plus/minus theorem

$$Span(S'') = Span(S') = V$$

If S'' is linearly independent, we are done. Otherwise we will continue in same manner until we reached a linearly independent set that form a basis for V.

(b) Suppose that $\dim(V) = n$.

If S is a linearly independent set that is not already a basis for V then S fails to span V.

$\Rightarrow$ there exist $v \in V$ which is not present in $Span(S)$.

Take $S' = S \cup \{v\}$

Then by plus/minus theorem S' is still linearly independent.

If S' spans V then we are done. and S' is a basis for V.

If S' does not span V, this means there exist some vector $w \in V$ which is not present in $Span(S')$.

Take $S'' = S' \cup \{w\}$

If S'' spans V, we are done otherwise we will continue in same manner to get a set S^* with n linearly independent vectors in V.

The set S^* will be a basis for V (by Theorem; "Let V be an n-dimensional vector space and let S be a set in V with exactly n vectors. Then S is a basis for V iff S spans V or S is linearly independent.)

Exercise 4.5

Q#10: Find the dimension of the subspace P_3 consisting of all polynomials $a_0 + a_1x + a_2x^2 + a_3x^3$ for which $a_0 = 0$

Solution

An element of above subspace W looks like

$p(x) = a_1x + a_2x^2 + a_3x^3$

$\Rightarrow$ $p(x) \in Span\{x, x^2, x^3\}$

Also $\{x, x^2, x^3\}$ is linearly independent.

$\Rightarrow$ $\{x, x^2, x^3\}$ is a basis.

$\Rightarrow$ $\dim W = 3$.

Q#11 Show that the set W of all polynomials in P_2 such that $p(1) = 0$ is a subspace of P_2. Find its basis and dimension.

Solution To show that W is a subspace of P_2. We need to show

(a) If $f, g \in W \Rightarrow f+g \in W$

(b) If $f \in W$, k is a scalar then $kf \in W$

(a) Let $f, g \in W$

$\Rightarrow f(1) = 0$ — (1), $g(1) = 0$ — (2)

Now

$(f+g)(1) = f(1) + g(1) = 0 + 0 = 0$ (using (1) & (2))

$\Rightarrow f+g \in W$

(b) Let $f \in W \Rightarrow f(1) = 0$

Now

$(kf)(1) = k(f(1)) = k(0) = 0$

$\Rightarrow kf \in W$

Hence W is a subspace of P_2.

If $P(x) \in P_2$

$\Rightarrow P(x) = a_0 + a_1 x + a_2 x^2$ — (3)

Now for $P(x)$ to be a member of W we need $P(1) = 0$

using in (3)

$a_0 + a_1(1) + a_2(1) = 0$

$\Rightarrow a_0 = -a_1 - a_2$

using in (3)

$P(x) = -a_1 - a_2 + a_1 x + a_2 x^2$

$= a_1(x-1) + a_2(x^2-1)$

~~i.e~~ $\cancel{W = Span(1-x, 1-x^2)}$

i.e $\quad W = Span(x-1, x^2-1)$

Let $\quad a_1(x-1) + a_2(x^2-1) = 0$

$\Rightarrow a_1x - a_1 + a_2x^2 - a_2 = 0$

$\Rightarrow -a_1 - a_2 + a_1x + a_2x^2 = 0$

Comparing coefficient

Constant; $\quad -a_1 - a_2 = 0$

x ; $\quad \boxed{a_1 = 0}$

x^2 ; $\quad \boxed{a_2 = 0}$

$\Rightarrow \{x-1, x^2-1\}$ is linearly independent.

$\Rightarrow \{x-1, x^2-1\}$ is a basis for W

$\Rightarrow \dim W = 2.$

Q#14 Let $\{v_1, v_2, v_3\}$ be a basis for a vector space V. Show that $\{u_1, u_2, u_3\}$ is also a basis where $u_1 = v_1$, $u_2 = v_1 + v_2$, $u_3 = v_1 + v_2 + v_3$.

Solution

Use theorem 4.5.4 To prove this

"Let V be an n-dimensional vector space and let S be a set in V with exactly n vectors. Then S is a basis for V iff S spans V or S is linearly independent."

As $\{v_1, v_2, v_3\}$ is a basis for V

$\Rightarrow \dim V = 3$

Now $\{u_1, u_2, u_3\}$ will be a basis if

$\mathrm{Span}\{u_1, u_2, u_3\} = \mathrm{Span}\{v_1, v_2, v_3\} = V$

Take $w \in \mathrm{Span}\{u_1, u_2, u_3\}$

$\Rightarrow w = c_1u_1 + c_2u_2 + c_3u_3$

$\Rightarrow w = c_1(v_1) + c_2(v_1+v_2) + c_3(v_1+v_2+v_3)$

$\Rightarrow w = (c_1+c_2+c_3)v_1 + (c_2+c_3)v_2 + c_3v_3$

$= k_1v_1 + k_2v_2 + k_3v_3$

where $k_1 = c_1+c_2+c_3$

$k_2 = c_2+c_3$

$k_3 = c_3$

$\Rightarrow w \in \mathrm{Span}\{v_1, v_2, v_3\}$

i.e $\mathrm{Span}\{u_1, u_2, u_3\} = \mathrm{Span}\{v_1, v_2, v_3\}$

Hence $\{u_1, u_2, u_3\}$ form a basis for V.