Chapter 8

DISCRETE PROBABILITY DISTRIBUTIONS

8.2. (b) The binomial probability distribution with n = 3 and p = 0.4 is

$$f(x) = \binom{3}{x}(0.4)^x (0.6)^{3-x}, \quad \text{for } x = 0, 1, 2, 3.$$

Now $P\left(X = \frac{3}{2}\right) = f\left(\frac{3}{2}\right) = 0$; because a random variable X with a binomial distribution takes only one of the integer values 0, 1, 2, ..., n.

$$P(X=2) = \binom{3}{2}(0.4)^2 (0.6)^{3-2} = 0.288;$$

$$P(X \le 2) = \sum_{x=0}^{2} \binom{3}{x}(0.4)^x (0.6)^{3-x}$$

$$= \binom{3}{0}(0.4)^0(0.6)^3 + \binom{3}{1}(0.4)^1(0.6)^2 + \binom{3}{2}(0.4)^2(0.6)^1$$

$$= 0.216 + 0.432 + 0.288 = 0.936;$$

$P(X=-2) = f(-2) = 0$; because a random variable X with a binomial distribution takes only one of the non-negative integer values 0, 1, 2, 3, ..., n.

$$P(X \ge 2) = \sum_{x=2}^{3} \binom{3}{x}(0.4)^x (0.6)^{3-x}$$

$$= \binom{3}{2}(0.4)^2 (0.6)^1 + \binom{3}{3}(0.4)^3 (0.6)^0$$

$$= 0.288 + 0.064 = 0.352.$$

8.3. (a) We observe that

(i) there are two possible outcomes, *i.e.* we will get *or* will not get a "5 or 6";

(ii) the probability of getting a "5 or 6" in each trial is $p = \frac{2}{6}$,

(iii) the successive trials are independent, and

(iv) there ore 5 trials.

Therefore the binomial distribution with $n = 5$ and $p = 1/3$ is appropriate.

Let X denote the number of successes. Then

(i) $P(\text{no success}) = P(X=0) = \binom{5}{0}\left(\frac{1}{3}\right)^0\left(\frac{2}{3}\right)^5$

$$= \frac{32}{243} = 0.1317;$$

(ii) $P(\text{at least 2 successes}) = P(X \geq 2) = 1 - P(X < 2)$

$$= 1 - \sum_{x=0}^{1}\binom{5}{x}\left(\frac{1}{3}\right)^x\left(\frac{2}{3}\right)^{5-x}$$

$$= 1 - \left[\binom{5}{0}\left(\frac{1}{3}\right)^0\left(\frac{2}{3}\right)^5 + \binom{5}{1}\left(\frac{1}{3}\right)^1\left(\frac{2}{3}\right)^{5-1}\right]$$

$$= 1 - \left[\frac{32}{243} + \frac{80}{243}\right] = \frac{131}{243} = 0.5391$$

(iii) $P(\text{at least 1 but not more than 3}) = P(1 \leq X \leq 3)$

$$= \sum_{x=1}^{3}\binom{5}{x}\left(\frac{1}{3}\right)^x\left(\frac{2}{3}\right)^{5-x}$$

$$= \binom{5}{1}\left(\frac{1}{3}\right)^1\left(\frac{2}{3}\right)^4 + \binom{5}{2}\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^3 + \binom{5}{3}\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^2$$

$$= \frac{80}{243} + \frac{80}{243} + \frac{40}{243} = \frac{200}{243} = 0.8230.$$

(b) (i) Let X denote the number of successes. Then using the binomial distribution, we get

$$P(X=3) = \binom{8}{3}(0.4)^3(0.6)^{8-3}$$

$$= (56)(0.064)(0.07776) = 0.2787$$

(ii) In the binomial distribution with $n=6$, 2 failures imply four successes, *i.e.* P(2 failures in 6 trials) = P(4 successes in 6 trials). Thus

$$P(X=4) = \binom{6}{4}(0.6)^4(0.4)^2$$

$$= (15)(0.1296)(0.16) = 0.3110$$

(iii) Let X denote the number of successes. Then

$$P(X\le 2) = \sum_{x=0}^{2}\binom{9}{x}(0.4)^x(0.6)^{9-x}$$

$$= \binom{9}{0}(0.4)^0(0.6)^9 + \binom{9}{1}(0.4)(0.6)^8 + \binom{9}{2}(0.4)^2(0.6)^7$$

$$= 0.010078 + 0.060466 + 0.161243 = 0.2318$$

8.4. (a) Let X denote the number of heads when 6 coins are tossed. Then

(i) $P(X = \text{exactly 4 heads}) = \binom{6}{4}\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right)^2 = 15 \cdot \frac{1}{64} = \frac{15}{64}$.

(ii) $P(X = \text{not more than 4 heads}) = 1 - P(X>4)$

$$= 1 - \sum_{x=5}^{6}\binom{6}{x}\left(\frac{1}{2}\right)^6$$

$$= 1 - \left(\frac{6}{64} + \frac{1}{64}\right) = \frac{57}{64}$$

(b) Let X denote the number of heads in a single toss of 6 fair coins. Then

(i) $P(X\ge 2) = 1 - P(X < 2)$

$$= 1 - \sum_{x=0}^{1}\binom{6}{x}\left(\frac{1}{2}\right)^6 = 1 - \left(\frac{1}{64} + \frac{6}{64}\right) = \frac{57}{64}$$

(ii) $P(X<4) = \sum_{x=0}^{3}\binom{6}{x}\left(\frac{1}{2}\right)^6$

$$= \frac{1}{64}+\frac{6}{64}+\frac{15}{64}+\frac{20}{64} = \frac{42}{64} = \frac{21}{32}$$

8.5. (a) Let p denote the probability of getting caught copying in the exam. Then $p=0.2$, $q=0.8$ and $n=3$. Since the attempts are independent, therefore

$$P(\text{not getting caught}) = q^3 = (0.8)^3 = 0.512$$

(b) Let X denote the number of voters who prefer candidate A and p, the probability that a voter prefers A.

Then $X=7$, $p=0.60$ and $n=12$. Hence

$$P(X=7) = \binom{12}{7}(0.60)^7(0.40)^{12-7}$$

$$= (792)(0.0279936)(0.01024) = 0.2270$$

(c) Let X denote the number of patients recovering from a delicate heart operation, and p, the probability that a patient recovers. Then $p=0.9$, $n=7$ and $X=5$.

Thus $P(X=5) = \binom{7}{5}(0.9)^5(0.1)^2$

$$= (21)(0.59049)(0.01) = 0.1240$$

8.6 (a) Let X denote the number of workmen, and p, the probability that a workman catches the disease. Then

$$p = \frac{20}{100} = \frac{1}{5}, \text{ so that } q = 1-\frac{1}{5} = \frac{4}{5}, \text{ and } n = 6.$$

(i) We seek the probability that "not more than 2" will catch the disease, *i.e.* $X \le 2$. Hence

$$P(X\le 2) = \sum_{x=0}^{2}\binom{6}{x}\left(\frac{1}{5}\right)^x\left(\frac{4}{5}\right)^{6-x}$$

$$= \binom{6}{0}\left(\frac{4}{5}\right)^6 + \binom{6}{1}\left(\frac{1}{5}\right)\left(\frac{4}{5}\right)^5 + \binom{6}{2}\left(\frac{1}{5}\right)^2\left(\frac{4}{5}\right)^4$$

$$= \frac{4096}{15625} + \frac{6144}{15625} + \frac{3840}{15625} = \frac{14080}{15625} = \frac{2816}{3125}$$

(ii) We seek the probability that 4 workmen, 5 workmen or 6 workmen will catch the disease. Hence

$$P(X \geq 4) = \sum_{x=4}^{6} \binom{6}{x} \left(\frac{1}{5}\right)^x \left(\frac{4}{5}\right)^{6-x}$$

$$= \binom{6}{4}\left(\frac{1}{5}\right)^4 \left(\frac{4}{5}\right)^2 + \binom{6}{5}\left(\frac{1}{5}\right)^5 \left(\frac{4}{5}\right) + \left(\frac{1}{5}\right)^6$$

$$= \frac{240}{15625} + \frac{24}{15625} + \frac{1}{15625} = \frac{265}{15625} = \frac{53}{3125}$$

(b) Here $p = \frac{12}{30} = 0.4$ so that $q = 0.6$. Thus

(i) P(1st 3 days fine and the remaining 4 days wet)

$$= q^3p^4 = (0.6)^3\,(0.4)^4$$

$$= (0.216)\,(0.0256) = 0.0055.$$

(ii) P(rain falls on just three days) $= \binom{7}{3} p^3q^4$

$$= 35\,(0.4)^3\,(0.6)^4$$

$$= 35\,(0.064)\,(0.1296)$$

$$= 0.2903$$

8.7. We observe that

(i) there are two possible outcomes, *i.e.* a man will be alive *or* will not be alive.

(ii) the probability of being alive for each man is 2/3,

(iii) the men will remain alive independently, and

(iv) there are 5 men.

Therefore the binomial distribution with $n=5$ and $p = \frac{2}{3}$ is appropriate to get the desired probabilities.

Let X denote the number of people who will be alive in 30 years hence. Then we seek (i) $P(X=5)$, (ii) $(P\geq 3)$, (iii) $P(X=2)$ and (iv) $P(X\leq 1)$. Hence

(i) $$P(X=5) = \binom{5}{5}\left(\frac{2}{3}\right)^5\left(\frac{1}{3}\right)^0 = \frac{32}{243} = 0.1317,$$

(ii) $$P(X\geq 3) = \sum_{x=3}^{5}\binom{5}{x}\left(\frac{2}{3}\right)^x\left(\frac{1}{3}\right)^{5-x}$$

$$= \binom{5}{3}\left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right)^2 + \binom{5}{4}\left(\frac{2}{3}\right)^4\left(\frac{1}{3}\right) + \binom{5}{5}\left(\frac{2}{3}\right)^5$$

$$= \frac{80}{243} + \frac{80}{243} + \frac{32}{243} = \frac{192}{243} = 0.7901;$$

(iii) $$P(X=2) = \binom{5}{2}\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^3 = \frac{40}{243} = 0.1646; \text{ and}$$

(iv) $$P(X\leq 1) = \sum_{x=0}^{1}\binom{5}{x}\left(\frac{2}{3}\right)^x\left(\frac{1}{3}\right)^{5-x}$$

$$= \binom{5}{0}\left(\frac{2}{3}\right)^0\left(\frac{1}{3}\right)^5 + \binom{5}{1}\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^4$$

$$= \frac{1}{243} + \frac{10}{243} = \frac{11}{243} = 0.0453.$$

8.8. (a) The probability of a correct answer is $\frac{1}{4}$ = 0.25, and n = 15.

Let X denote the number of correct answers. Then

$$P(5\leq X\leq 10) = \sum_{x=5}^{10}\binom{15}{x}(0.25)^x(0.75)^{15-x} = \sum_{x=5} b(x;\, 15, 0.25)$$

$$= \sum_{x=0}^{10} b(x;\, 15, 0.25) - \sum_{x=0}^{4} b(x;\, 15, 0.25)$$

$$= 0.9999 - 0.6865 \quad \text{(From binomial tables)},$$

$$= 0.3134$$

(b) Let p denote the probability that a spot light is red when the commuter gets to it. Then $p = 0.2$, $q = 1 - 0.2 = 0.8$ and $n = 10$.

Let X be a *r.v.* that denotes the number of times, the commuter must stop for a red light on her way to work. Then X is $b(x; 10, 0.2)$. To evaluate $P(X=0)$ and $P(X \le 5)$, we do the following calculations.

Now $P(X=x) = \binom{10}{x}(0.2)^x (0.8)^{10-x}$, so

$$P(X=0) = \binom{10}{0}(0.2)^0 (0.8)^{10} = 0.107374 \text{ and}$$

$$P(X \le 5) = \sum_{x=0}^{5}\binom{10}{x}(0.2)^x (0.8)^{10-x}$$

$$= (0.8)^{10} + \binom{10}{1}(0.2)(0.8)^9 + \binom{10}{2}(0.2)^2(0.8)^8 + \binom{10}{3}(0.2)^3(0.8)^7 + \binom{10}{4}(0.2)^4(0.8)^6 + \binom{10}{5}(0.2)^5(0.8)^5$$

$$= 0.107374 + 0.268345 + 0.301990 + 0.201327 + 0.088080 + 0.026424 = 0.9936.$$

8.9. (a) The successive terms of the binomial distribution 600 (0.3 + 0.7) ^{6} are calculated below:

X	Probability	$f(x)$	$600\,f(x)$
0	$q^6 = (0.3)^6$	$= 0.000729$	0.4374
1	${}^6C_1\, q^5 p = 6(0.3)^5(0.7)$	$= 0.010206$	6.1236
2	${}^6C_2\, q^4 p^2 = 15(0.3)^4(0.7)^2$	$= 0.059535$	35.7210
3	${}^6C_3\, q^3 p^3 = 20(0.3)^3(0.7)^3$	$= 0.185220$	111.1320
4	${}^6C_4\, q^2 p^4 = 15(0.3)^2(0.7)^4$	$= 0.324135$	194.4810
5	${}^6C_5\, q p^5 = 6(0.3)(0.7)^5$	$= 0.302526$	181.5156
6	$p^6 = (0.7)^6$	$= 0.117649$	70.5894

(b) Let p denote the probability of getting 4, 5 or 6 with one die.

Then $\quad p = \frac{3}{6} = \frac{1}{2}$, so that $q = \frac{1}{2}$

We seek the expected frequencies of getting 0, 1, 2, ..., 5 successes with 5 dice in tossing 96 times, which are the successive terms in the binomial expansion of

$$96\left(\frac{1}{2} + \frac{1}{2}\right)^5$$

Hence the expected frequencies are:

$$96\left(\frac{1}{2}\right)^5; 96\binom{5}{1}\left(\frac{1}{2}\right)^5; 96\binom{5}{2}\left(\frac{1}{2}\right)^5, 96\binom{5}{3}\left(\frac{1}{2}\right)^5,$$

$$96\binom{5}{4}\left(\frac{1}{2}\right)^5 \text{ and } 96 \cdot \left(\frac{1}{2}\right)^5 \text{ or } 3, 15, 30, 30, 15 \text{ and } 3$$

8.10. (a) Let p denote the probability of getting a 5 or a 6 with one die.

Then $\quad p = \frac{2}{6} = \frac{1}{3}$, so that $q = 1 - \frac{1}{3} = \frac{2}{3}$.

As the dice are in sets of 8, the binomial distribution is

$$N\left(\frac{2}{3} + \frac{1}{3}\right)^8$$

Thus the expected number of 3 successs

$$= N \cdot \binom{8}{3}\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^5 = N \cdot \frac{1792}{6561}$$

Hence the proportion of the sets in which 3 successes are expected

$$= N \cdot \frac{1792}{6561} \div N = \frac{1792}{6561} \times 100\% = 27.31\%$$

(b) Let p denote the probability of getting an even number.

Then the expected number of getting 5 even numbers in 10 throws $= N\cdot\binom{10}{5}p^5q^5$, where N is the no. of sets and $q = 1-p$.

Similarly, the expected number of getting 4 even numbers in 10 throws $= N\cdot\binom{10}{4}p^4q^6$.

Now, by the question, we have

$$N\cdot\binom{10}{5}p^5q^5 = 2N\cdot\binom{10}{4}p^4q^6$$

Or $\quad \dfrac{10!}{5!\,5!}p^5q^5 = 2\cdot\dfrac{10!}{4!\,6!}p^4q^6$

Or $\quad \dfrac{p}{5} = \dfrac{q}{3}$ Or $3p = 5(1-p)$

Or $\quad p = \dfrac{5}{8}$ and hence $q = \dfrac{3}{8}$.

Then the binomial distributioin becomes

$$10{,}000\left(\frac{3}{8}+\frac{5}{8}\right)^{10}$$

Hence the expected number of getting no even number

$$= 10{,}000\left(\frac{3}{8}\right)^{10} = \frac{36905625}{67108864} = 1 \text{ approximately.}$$

8.11. Let p be the probability of getting a 6 with one die. Then $p = \dfrac{1}{6}$ and $q = 1-p = \dfrac{5}{6}$.

Let X denote the number of *sixes* when 4 dice are thrown. Then the theoretical frequencies of 0, 1, 2, 3, and 4 sixes are the successive terms of the binomial distribution $108\left(\dfrac{5}{6}+\dfrac{1}{6}\right)^4$, which are calculated as follows:

x	Probability $= f(x)$	$108 \times f(x)$
0	$q^4 = \left(\frac{5}{6}\right)^4 = \frac{625}{1296}$	52.08
1	$\binom{4}{1} q^3p = 4 \cdot \left(\frac{5}{6}\right)^3 \left(\frac{1}{6}\right) = \frac{500}{1296}$	41.67
2	$\binom{4}{2} q^2p^2 = 6 \cdot \left(\frac{5}{6}\right)^2 \left(\frac{1}{6}\right)^2 = \frac{150}{1296}$	12.50
3	$\binom{4}{3} qp^3 = 4 \cdot \left(\frac{5}{6}\right) \left(\frac{1}{6}\right)^3 = \frac{20}{1296}$	1.67
4	$p^4 = \left(\frac{1}{6}\right)^4 = \frac{1}{1296}$	0.08

The mean number of sixes in a single throw $= np = 4 \times \frac{1}{6} = \frac{2}{3}$.

8.12. (a) Calculation of the mean and standard deviation of the binomial $(q+p)^3$.

x	$f(x)$	$x \cdot f(x)$	$x^2 \cdot f(x)$
0	q^3	0	0
1	$3q^2p$	$3q^2p$	$3q^2p$
2	$3qp^2$	$6qp^2$	$12qp^2$
3	p^3	$3p^3$	$9p^3$
$\sum$	1		

Now $\sum x \cdot f(x) = 3q^2p + 6qp^2 + 3p^3$

$= 3p\,[q^2 + 2qp + p^2] = 3p(q+p)^2 = 3p.$

$\sum x^2 \cdot f(x) = 3q^2p + 12qp^2 + 9p^3$

$= 3p\,[q^2 + 4qp + 3p^2]$

$= 3p\,[(q^2 + 2qp + p^2) + (2qp + 2p^2)]$

$= 3p\,[(q+p)^2 + 2p\,(q+p)]$

$= 3p\,[1 + 2p] = 3p + 6p^2$

Hence $\mu = \sum x \cdot f(x) = 3p$, and $\qquad (n = \sum f(x) = 1)$

$$\sigma = \sqrt{\sum x^2 \cdot f(x) - [\sum x \cdot f(x)]^2}$$

$$= \sqrt{3p + 6p^2 - (3p)^2}$$

$$= \sqrt{3p - 3p^2} = \sqrt{3p\,(1-p)} = \sqrt{3pq}$$

(c) As the mean and variance of the binomial $(q+p)^n$ are np and npq, therefore np = 3 and npq = 2.

Dividing, we get

$$\frac{npq}{np} = \frac{2}{3} \text{ or } q = \frac{2}{3}.$$

$$\therefore \quad p = 1 - q = \frac{1}{3}, \text{ and } n\left(\frac{1}{3}\right) = 3, \text{ gives } n = 9.$$

Thus the binomial distribution is $\left(\frac{2}{3} + \frac{1}{3}\right)^9$.

Hence $P(X=7) = \binom{9}{7}\left(\frac{1}{3}\right)^7\left(\frac{2}{3}\right)^{9-7} = \frac{16}{2187} = 0.0073.$

8.13 (a) The mean and the standard deviation of the binomial $(q+p)^n$ are np and $\sqrt{npq}$. Then

$$np = 36 \text{ and } \sqrt{npq} = 4.8 \text{ or } npq = (4.8)^2$$

Dividing, we get

$$\frac{npq}{np} = \frac{(4.8)^2}{36} \text{ or } q = 0.64$$

$$\therefore \quad p = 1 - q = 0.36$$

Putting $p = 0.36$ in $np = 36$, we get

$$n = \frac{36}{0.36} = 100.$$

(b) If X is a binomial r.v. with parameters n and p, then mean = np and variance = npq.

Now Mean = 12.38 so $np = 12.38$, and

Variance = 8.64 so $npq = 8.64$.

Substituting for np in the second equation, we get

$$(12.38)\, q = 8.64 \quad \text{or} \quad q = \frac{8.64}{12.38} = 0.6979$$

so $\quad p = 1 - 0.6979 = 0.3021$, and

$n(0.3021) = 12.38$ givs $n = 41$.

(c) Let the binomial distribution be $(q+p)^n$; $q + p = 1$.

Then $\quad np = 5$ and $\sqrt{npq} = 3$ or $npq = 9$

Dividing, we get

$$\frac{npq}{np} = \frac{9}{5} \text{ or } q = 1.8, \text{ which is greater than unity.}$$

Hence it is not possible to have a binomial distribution with mean = 5 and *s.d.* = 3.

8.17. (a) The binomial distribution of the r.v. X is given by $P(X=x) = \binom{25}{x} (0.2)^x (0.8)^{25-x}$.

Now $\quad \mu = np = 25\,(0.2) = 5$, and

$\sigma = \sqrt{npq} = \sqrt{25\,(0.2)\,(0.8)} = 2.$

Hence $P(X<\mu-2\sigma) = P(X<1) \quad (\because \mu-2\sigma = 5-2(2) = 1)$

$= P(X=0)$

$= \binom{25}{0} (0.2)^0 (0.8)^{25} = 0.00378.$

(b) To find the median and mode of the binomial $\binom{10}{x}\left(\frac{1}{2}\right)^x\left(\frac{1}{2}\right)^{10-x}$, **we expand it. Therefore, expanding the binomial, we get**

x	0	1	2	3	4	5	6	7	8	9	10
$f(x)$	$\frac{1}{1024}$	$\frac{10}{1024}$	$\frac{45}{1024}$	$\frac{120}{1024}$	$\frac{210}{1024}$	$\frac{252}{1024}$	$\frac{210}{1024}$	$\frac{120}{1024}$	$\frac{45}{1024}$	$\frac{10}{1024}$	$\frac{1}{1024}$
F	$\frac{1}{1024}$	$\frac{11}{1024}$	$\frac{56}{1024}$	$\frac{176}{1024}$	$\frac{386}{1024}$	$\frac{638}{1024}$	$\frac{848}{1024}$	$\frac{968}{1024}$	$\frac{1013}{1024}$	$\frac{1023}{1024}$	1

Since the binomial distribution is a *discrete* distribution, therefore

Median = a value at or below which 50% probabilities lie

= 5; and

Mode $=$ a value that corresponds to maximum probability

$= 5.$

8.18. (a) To find the probability of obtaining a head, we need to calculate the mean of the frequency distribution. Thus

$$\bar{x} = \frac{\sum fx}{\sum f} = \frac{(0)(12) + (1)(50) + (2)(151) + (3)(200) + (4)(87)}{500}$$

$$= \frac{1300}{500} = 2.6$$

Let X denote the number of heads obtained in 4 tosses. Then X is a binomial $r.v.$ with $n=4$, and mean $= np = 4p$.

Now $\quad 4p = 2.6$ or $p = 0.65$

Thus the probability that the coin will show heads is 0.65.

(b) Now X is b(x; 4, 0.65) and therefore

$$P(X=x) = \binom{4}{x}(0.65)^x(0.35)^{4-x} \text{ for } x = 0,1,2,3,4.$$

The theoretical binomial probabilities and frequencies are computed as follows:

x	Probability $= f(x)$	Frequency
0	$\binom{4}{0}(0.65)^0(0.35)^4 = 0.015006$	8
1	$\binom{4}{1}(0.65)^1(0.35)^3 = 0.111475$	56
2	$\binom{4}{2}(0.65)^2(0.35)^2 = 0.310538$	155
3	$\binom{4}{3}(0.65)^3(0.35)^1 = 0.384475$	192
4	$\binom{4}{4}(0.65)^4(0.35)^0 = 0.178506$	89

8.19. **We first calculate the mean of the given distribution and equate it to np to find the value of p.**

Now $\bar{x} = \frac{\sum f_i x_i}{\sum f_i} = \frac{0+70+122+75+28+5+0}{200} = \frac{300}{200} = 1.5.$

Thus $6p = 1.5$ or $p = 0.25$ and hence $q = 0.75.$

The fitted binomial distribution is

$$b(x; 6, 0.25) = \binom{6}{x} (0.25)^x (0.75)^{6-x}$$

The theoretical binomial prbabilities and frequencies are computed as below:

x	Probability		Theoretical Frequency
0	$\binom{6}{0} (0.75)^6$	$= 0.177978$	35.60
1	$\binom{6}{1} (0.25) (0.75)^5$	$= 0.355957$	71.19
2	$\binom{6}{2} (0.25)^2 (0.75)^4$	$= 0.296631$	59.33
3	$\binom{6}{3} (0.25)^3 (0.75)^3$	$= 0.131836$	26.37
4	$\binom{6}{4} (0.25)^4 (0.75)^2$	$= 0.032959$	6.59
5	$\binom{6}{5} (0.25)^5 (0.75)$	$= 0.004394$	0.88
6	$\binom{6}{6} (0.25)^6$	$= 0.000244$	0.05
Σ		$= 0.999999$	200.01

8.20. Fitting of the Binomial Distribution.

We first calculate the mean of the given distribution and equate it to np to find the value of p.

Now $\bar{x} = \dfrac{\sum f_i x_i}{\sum f_i} = \dfrac{0 + 62 + 92 + 30 + 8}{150} = \dfrac{192}{150} = 1.28.$

Thus $4p = 1.28$ or $p = 0.32$ and hence $q = 0.68$.

The fitted binomial distribution is

$$b(x; 4, 0.32) = \binom{4}{x} (0.32)^x (0.68)^{4-x}$$

The expected frequencies are computed as below:

x	Probability	Expected frequency
0	$(0.68)^4 = 0.213814$	32
1	$\binom{4}{1}(0.32)(0.68)^3 = 0.402473$	60
2	$\binom{4}{2}(0.32)^2(0.68)^2 = 0.284099$	43
3	$\binom{4}{3}(0.32)^3(0.68) = 0.089129$	13
4	$(0.32)^4 = 0.010486$	2
$\sum$	$= 1.000001$	150

8.21. Calculation of the mean and standard deviation.

x	f	fx	fx^2
0	0	0	0
1	1	1	1
2	3	6	12
3	8	24	72
4	16	64	256
5	28	140	700
6	18	108	648
7	13	91	637
8	9	72	576
9	4	36	324
10	0	0	0
Total	100	542	3226

$$\therefore \quad \bar{x} = \frac{\sum fx}{\sum f} = \frac{542}{100} = 5.42, \text{ and}$$

$$s = \sqrt{\frac{\sum fx^2}{\sum f} - \left(\frac{\sum fx}{\sum f}\right)^2} = \sqrt{\frac{3226}{100} - \left(\frac{542}{100}\right)^2}$$

$$= \sqrt{32.26 - 29.3764} = \sqrt{2.8836} = 1.70.$$

Now putting the mean equal to np, we get

$10p = 5.42$ so that $p = 0.542$ and $q = 0.458$.

Thus the fitted binomial distribution is

$$b(x;\, 10,\, 0.542) = \binom{10}{x} (0.542)^x \cdot (0.458)^{10-x}$$

The expected frequencies are computed as below:

x	Probability		Expected frequency
0	$(0.458)^{10}$	$= 0.000406$	0.0
1	$\binom{10}{1}(0.542)\,(0.458)^9$	$= 0.004808$	0.5
2	$\binom{10}{2}(0.542)^2\,(0.458)^8$	$= 0.0256$	2.6
3	$\binom{10}{3}(0.542)^3\,(0.458)^7$	$= 0.0808$	8.1
4	$\binom{10}{4}(0.542)^4\,(0.458)^6$	$= 0.1673$	16.7
5	$\binom{10}{5}(0.542)^5\,(0.458)^5$	$= 0.2376$	23.8
6	$\binom{10}{6}(0.542)^6\,(0.458)^4$	$= 0.2343$	23.4
7	$\binom{10}{7}(0.542)^7\,(0.458)^3$	$= 0.1584$	15.8
8	$\binom{10}{8}(0.542)^8\,(0.458)^2$	$= 0.0703$	7.0
9	$\binom{10}{9}(0.542)^9\,(0.458)$	$= 0.01849$	1.8
10	$(0.542)^{10}$	$= 0.002188$	0.2
$\sum$		$= 1.000192$	99.9

Mean and variance of the expected distribution are

$$\mu = np = 10\,(0.542) = 5.42$$

$$\sigma = \sqrt{npq} = \sqrt{10(0.542)\,(0.458)}$$

$$= \sqrt{2.48236} = 1.57$$

8.22. For a symmetrical binomial distribution, we have

$$p = q = \frac{1}{2}$$

Then the symmetrical binomial distribution of degree n and observations, N, is $N\left(\frac{1}{2} + \frac{1}{2}\right)^n$.

Let T_r and T_{r+1} denote the rth and $(r+1)$th terms. Then

$$T_r = N \,.\, \binom{n}{r-1} \left(\frac{1}{2}\right)^{r-1} \left(\frac{1}{2}\right)^{n-r+1}$$

$$= N \,.\, \binom{n}{r-1} \left(\frac{1}{2}\right)^{n}, \text{ and}$$

$$T_{r+1} = N \,.\, \binom{n}{r} \left(\frac{1}{2}\right)^{r} \left(\frac{1}{2}\right)^{n-r}$$

$$= N \,.\, \binom{n}{r} \left(\frac{1}{2}\right)^{n}$$

Superposing in such a way that the rth term of one coincides with the $(r+1)$th term of the other, we get

$$T_r + T_{r+1} = N \,.\, \binom{n}{r-1} \left(\frac{1}{2}\right)^{r} + N \,.\, \binom{n}{r} \left(\frac{1}{2}\right)^{n}$$

$$= N \left[\binom{n}{r-1} + \binom{n}{r}\right] \left(\frac{1}{2}\right)^{n}$$

$$= N \binom{n+1}{r} + \left(\frac{1}{2}\right)^{n} = 2N \binom{n+1}{r} \left(\frac{1}{2}\right)^{n+1}$$

$$= 2N \,.\, \binom{n+1}{r} \left(\frac{1}{2}\right)^{r} \left(\frac{1}{2}\right)^{n+1-r}$$

$= (r+1)$th term of the binomial distribution

$2N \cdot \left(\frac{1}{2}+\frac{1}{2}\right)^{n+1}$, which is symmetrical binomial of degree $(n+1)$.

8.23. (b) Given that

$$M_0(t) = \left(\frac{1}{4}+\frac{3}{4}e^t\right)^{12}. \text{ Then}$$

$$E(X) = \left[\frac{d}{dt}\left(\frac{1}{4}+\frac{3}{4}e^t\right)^{12}\right]_{t=0}$$

$$= \left[12\left(\frac{3}{4}e^t\right)\left(\frac{1}{4}+\frac{3}{4}e^t\right)^{11}\right]_{t=0} = 12\left(\frac{3}{4}\right)(1) = 9;$$

$$E(X^2) = \left[\frac{d^2}{dt^2}\left(\frac{1}{4}+\frac{3}{4}e^t\right)^{12}\right]_{t=0}$$

$$= \left[12\left(\frac{3}{4}e^t\right)\left(\frac{1}{4}+\frac{3}{4}e^t\right)^{11}\right]_{t=0} + \left[12\,(11)\left(\frac{3}{4}\right)^2 e^{2t}\left(\frac{1}{4}+\frac{3}{4}e^t\right)^{10}\right]_{t=0}$$

$$= 12\left(\frac{3}{4}\right)(1) + 12(11)\left(\frac{3}{4}\right)^2(1) = 9 + \frac{297}{4} = \frac{333}{4}.$$

$$\therefore \quad \mathrm{Var}(X) = E(X^2) - [E(X)]^2 = \frac{333}{4} - (9)^2 = \frac{333-324}{4} = \frac{9}{4} = 2.25.$$

Alternatively. Comparing $\left(\frac{1}{4}+\frac{3}{4}e^t\right)^{12}$ with $(q+pe^t)^n$, we find that $p = \frac{3}{4}$, $q = \frac{1}{4}$ and $n = 12$.

Therefore $E(X) = np = 12 \times \frac{3}{4} = 9$; and

$$\mathrm{Var}(X) = npq = 12 \times \frac{3}{4} \times \frac{1}{4} = \frac{9}{4} = 2.25.$$

$$\text{Now} \quad P(X \geq 10) = \sum_{x=10}^{12}\binom{12}{x}\left(\frac{3}{4}\right)^x\left(\frac{1}{4}\right)^{12-x}$$

$$= \binom{12}{10}\left(\frac{3}{4}\right)^{10}\left(\frac{1}{4}\right)^2 + \binom{12}{11}\left(\frac{3}{4}\right)^{11}\left(\frac{1}{4}\right) + \left(\frac{3}{4}\right)^{12}$$

$$= (66)\left(\frac{59049}{1048576}\right)\left(\frac{1}{16}\right) + 12\left(\frac{177147}{4194304}\right)\left(\frac{1}{4}\right) + \left(\frac{531441}{16777216}\right)$$

$$= 0.232293 + 0.126705 + 0.031676$$

$$= 0.3907.$$

8.24. The m.g.f. of the binomial distribution $(q+p)^n$ is

$M_0(t) = (q + pe^t)^n$, and

the cumulant generating function is

$\kappa(t) = n \log (q + pe^t)$, and

therefore *rth* order cumulant is given by

$$\kappa_r = n \left[\frac{d^r}{dt^r} \log (q + pe^t)\right]_{t=0}$$

Differentiating with respect to p, we get

$$\frac{d\kappa_r}{dp} = n \left[\frac{d^r}{dt^r}\left(\frac{-1+e^t}{q+pe^t}\right)\right]_{t=0} \qquad \because q+pe^t = 1-p+pe^t,$$

Now $\quad pq \dfrac{d\kappa_r}{dp} = npq \left[\dfrac{d^r}{dt^r}\left(\dfrac{-1+e^t}{q+pe^t}\right)\right]_{t=0} \qquad \therefore \dfrac{d}{dp} = -1 + e^t$

Again $\quad \kappa_{r+1} = n \left[\dfrac{d^{r+1}}{dt^{r+1}} \log (q + pe^t)\right]_{t=0}$

$$= n \left[\frac{d^r}{dt^r} \cdot \frac{d}{dt} \log (q + pe^t)\right]_{t=0} = np \left[\frac{d^r}{dt^r}\left(\frac{e^t}{q+pe^t}\right)\right]_{t=0}$$

$$\therefore \quad \kappa_{r+1} - pq \frac{d\kappa_r}{dp} = np \left[\frac{d^r}{dt^r}\left(\frac{e^t}{q+pe^t} - \frac{q(-1+e^t)}{q+pe^t}\right)\right]_{t=0}$$

$$= np \left[\frac{d^r}{dt^r}\left(\frac{q+pe^t}{q+pe^t}\right)\right]_{t=0} = np \left[\frac{d^r}{dt^r}(1)\right]_{t=0} = 0,$$

so that $\kappa_{r+1} = pq \dfrac{d\kappa_r}{dp}$.

Putting $r = 1$, we get $\kappa_2 = pq \dfrac{d\kappa_1}{dp}$, where $\kappa_1 = np$

$$= npq$$

Putting $r = 2$, we get $\kappa_3 = pq \dfrac{d\kappa_2}{dp} = npq \dfrac{d}{dp}(qp)$

$$= npq(1 - 2p) = npq(q - p)$$

$$= n(p - 3p^2 + 2p^3)$$

Putting $r = 3$, we get $\kappa_4 = pq \dfrac{d\kappa_3}{dp} = npq \dfrac{d}{dp}(p - 3p^2 + 2p^3)$

$$= npq(1 - 6p + 6p^2)$$

$$= npq(1 - 6pq).$$

8.26. (b) Let X denote the number of white beads. Then the probability that X assumes a value x, is

$$P(X=x) = \frac{\binom{k}{x}\binom{N-k}{n-x}}{\binom{N}{n}}, \text{ for } x \text{ such that } 0 \le x \le n \text{ and } 0 \le x \le k.$$

where k = number of white beads, which is 4;

n = number of beads drawn, which is 5,

N = total number of beads in the bowl which is 11, and

x = 0,1,2,3, and 4. (Here possible values of x are equal to k)

Now we calculate the probabilities for various values of x as below:

$$P(X=0) = \binom{4}{0}\binom{11-4}{5-0} \div \binom{11}{5} = \frac{21}{462},$$

$$P(X=1) = \binom{4}{1}\binom{7}{4} \div \binom{11}{5} = \frac{140}{462},$$

$$P(X=2) = \binom{4}{2}\binom{7}{3} \div \binom{11}{5} = \frac{210}{462},$$

$$P(X=3) = \binom{4}{3}\binom{7}{2} \div \binom{11}{5} = \frac{84}{462},$$

$$P(X=4) = \binom{4}{4}\binom{7}{1} \div \binom{11}{5} = \frac{7}{462},$$

Hence the desired probability distribution is

x	0	1	2	3	4	Total
$P(X=x)$	$\frac{21}{462}$	$\frac{140}{462}$	$\frac{210}{462}$	$\frac{84}{462}$	$\frac{7}{462}$	1

Now $$\sum xf(x) = (0)\left(\frac{21}{462}\right) + 1\left(\frac{140}{462}\right) + 2\left(\frac{210}{462}\right) + 3\left(\frac{84}{462}\right) + 4\left(\frac{7}{462}\right)$$

$$= 0 + \frac{140}{462} + \frac{420}{462} + \frac{252}{462} + \frac{28}{462} = \frac{840}{462}, \text{ and}$$

$$\sum x^2 f(x) = 0\left(\frac{21}{462}\right) + 1^2\left(\frac{140}{462}\right) + 2^2\left(\frac{210}{462}\right) + 3^2\left(\frac{84}{462}\right) + 4^2\left(\frac{7}{462}\right)$$

$$= 0 + \frac{140}{462} + \frac{840}{462} + \frac{756}{462} + \frac{112}{462} = \frac{1848}{462}.$$

Hence $$\text{Mean} = \sum x_i f(x_i) = \frac{840}{462} = 1.82, \text{ and}$$

$$\text{Variance} = \sum x_i^2 f(x_i) - \{\sum x_i f(x_i)\}^2 = \frac{1848}{462} - \left(\frac{840}{462}\right)^2$$

$$= 4 - 3.31 = 0.69.$$

The formulas for the mean and variance of the distribution are:

$$\mu = np \text{ and } \sigma^2 = npq\frac{N-n}{N-1}, \text{ where } p = \frac{k}{N} \text{ and } q = \frac{N-k}{N}$$

Here $p = \frac{4}{11}$ and $q = \frac{7}{11}$. Therefore

$$\mu = 5\left(\frac{4}{11}\right) = \frac{20}{11} = 1.82, \text{ and}$$

$$\sigma^2 = 5\left(\frac{4}{11}\right)\left(\frac{7}{11}\right)\frac{11-5}{11-1} = \frac{84}{121} = 0.69.$$

8.27. (a) Here N = 4 men + 2 women = 6 persons,

$n = 3$ persons to be selected,

$k = 4$, and $x = 0, 1, 2, 3$.

Hence the Hypergeometric distribution is

$$h(x;\,6,\,3,\,4) = \frac{\binom{4}{x}\binom{2}{3-x}}{\binom{6}{3}}$$

When $x=0$, $h(0;\,6,\,3,\,4) = 0$, (the event is impossible)

When $x = 1$, the probability is

$$h(1;\,6,\,3,\,4) = \frac{\binom{4}{1}\binom{2}{2}}{\binom{6}{3}} = \frac{4}{20}$$

When $x = 2$, the probability is

$$h(2;\,6,\,3,\,4) = \frac{\binom{4}{2}\binom{2}{1}}{\binom{6}{3}} = \frac{12}{20}$$

When $x = 3$, the probability is

$$h(3;\,6,\,3,\,4) = \frac{\binom{4}{3}\binom{2}{0}}{\binom{6}{3}} = \frac{4}{20}$$

(b) Here 6 bulbs can be selected in $\binom{8}{6}$ ways,

4 tulip bulbs can be selected in $\binom{4}{4}$ ways, and

2 daffodil bulbs can be selected in $\binom{4}{2}$ ways.

$\therefore$ the total number of ways to select 2 daffodil bulbs and 4 tulip bulbs is $\binom{4}{2}\binom{4}{4}$ ways.

Hence the required probability $= \dfrac{\binom{4}{2}\binom{4}{4}}{\binom{8}{6}} = \dfrac{4\times 3}{2} \times \dfrac{2}{8\times 7} = \dfrac{3}{14}$

Or Applying Hypergeometric distribution, we identify

$$N = 8, n = 6, k = 4, x = 2.$$

$$\therefore \quad h(2; 8, 6, 4) = \frac{\binom{4}{2}\binom{8-4}{6-2}}{\binom{8}{6}} = \frac{3}{14}.$$

8.28. Here **N = 10 cans to draw from,**

$n = 5$, the number of cans to be drawn.

$k = 5$ of the 10 cans are tomatoes.

Thus the Hypergeometric distribution is

$$h(x; 10, 5, 5) = \frac{\binom{5}{x}\binom{5}{5-x}}{\binom{10}{5}}.$$

Hence the probability that all contain tomatoes, *i.e.* $x = 5$ is

$$h(5; 10, 5, 5) = \frac{\binom{5}{5}\binom{5}{0}}{\binom{10}{5}} = 0.003968.$$

Let A denote the event that 3 or more cans contain tomatoes. Then

$$P(A) = \sum_{x=3}^{5}\binom{5}{x}\binom{5}{5-x} \div \binom{10}{5}$$

$$= \left[\binom{5}{3}\binom{5}{2} + \binom{5}{4}\binom{5}{1} + \binom{5}{5}\binom{5}{0}\right] \div \binom{10}{5}$$

$$= (100 + 25 + 1) \div 252 = 0.5$$

8.29. (a) Let X denote the number of income tax returns with illegitimate deductions. Then we have

$$N = 20, \; n = 6, \; k = 8 \text{ and } x = 3.$$

Hence using the hypergeometric distribution

$$h(x;\, 20,\, 6,\, 8) = \frac{\binom{8}{x}\binom{12}{6-x}}{\binom{20}{6}},$$ we get the desired probability,

i.e. $P(X=3)$, as

$$h(3;\, 20,\, 6,\, 8) = \frac{\binom{8}{3}\binom{12}{3}}{\binom{20}{6}} = \frac{12320}{38760} = \frac{308}{969} = 0.3179.$$

(b) Let X denote the number of narcotic tablets. Then we have

$$k = 6,\ n = 3 \text{ and } N = 6 + 9 = 15.$$

The hypergeometric distribution is then

$$P(X=x) = h(x;\, 15,\, 3,\, 6) = \frac{\binom{6}{x}\binom{9}{3-x}}{\binom{15}{3}}.$$

The traveller will be arrested for illegal possession of narcotics if the customs official detects *at least* 1 narcotic tablet, *i.e.* if $x = 1, 2, 3$.

Hence the desired probability is

$$\sum_{x=1}^{3} h(x;\, 15,3,6) = \sum_{x=1}^{3} \frac{\binom{6}{x}\binom{9}{3-x}}{\binom{15}{3}}.$$

$$= \frac{\binom{6}{1}\binom{9}{2}}{455} + \frac{\binom{6}{2}\binom{9}{1}}{455} + \frac{\binom{6}{3}\binom{9}{0}}{455}$$

$$= \frac{216}{455} + \frac{135}{455} + \frac{20}{455} = \frac{371}{455} = 0.82.$$

8.31. (a) When a sample is obtained under sampling without replacement, the *hypergeometric* distribution is used to find the desired probability. We are given that

$$N = 150,\ n = 4,\ x = 3 \text{ and } k = 20\% \text{ of } 150 = 30.$$

$$\therefore \quad P(X=3) = h(3;\,150,\,4,\,30) = \frac{\binom{30}{3}\binom{120}{1}}{\binom{150}{4}} = 0.0240$$

(b) When a sample is obtained under sampling with replacement from a finite population or sampling from infinite population, the *binomial* probability is appropriate to calculate the desired probability. We are given that $n = 4$, $x = 3$ and $p = 0.20$.

$$\therefore \quad P(X=3) = b(3;\,4,\,0.20) = \binom{4}{3}(0.20)^3\,(0.80) = 0.0256.$$

(c) There is a slight difference because the sample size is small.

8.32. (c) Since X is a Poisson r.v. with μ = 1.6, therefore

$$P(X=0) = e^{-1.6}\frac{(1.6)^0}{0!} = 0.2019, \qquad (\because\ e^{-1.6} = 0.2019)$$

$$P(X=1) = \frac{e^{-1.6}\,(1.6)}{1!} = 0.3230,$$

$$P(X=2) = \frac{e^{-1.6}\,(1.6)^2}{2!} = 0.2584, \text{ and}$$

$$\begin{aligned} P(X>2) &= 1 - P(X \le 2) \\ &= 1 - [P(X=0) + P(X=1) + P(X=2)] \\ &= 1 - 0.7833 = 0.2167. \end{aligned}$$

8.33. (a) The desired probabilities for x = 0, 1, 2, 3 and 4 are computed as below:

$$P(X=0) = \frac{e^{-3}\,.\,3^0}{0!} = e^{-3} = 0.0498,$$

$$P(X=1) = \frac{e^{-3}\,.\,3}{1!} = (0.0498)\,(3) = 0.1494,$$

$$P(X=2) = \frac{e^{-3}\,.\,3^2}{2!} = (0.1494)\left(\frac{3}{2}\right) = 0.2241,$$

$$P(X=3) = \frac{e^{-3} \cdot 3^3}{3!} = (0.2241)(1) = 0.2241, \text{ and}$$

$$P(X=4) = \frac{e^{-3} \cdot 3^4}{4!} = (0.2241)\left(\frac{3}{4}\right) = 0.1681,$$

Note that $e^{-3} = \dfrac{1}{(2.71828)^3} = 0.0498$

(b) Let X have the Poisson distribution with parameter μ. Then

$$P(X=x) = \frac{e^{-\mu} \cdot \mu^x}{x!}, \quad x = 0, 1, 2, \ldots, \infty$$

It is given that

$$P(X=1) = e^{-\mu} \cdot \mu = 0.3, \text{ and}$$

$$P(X=2) = \frac{e^{-\mu} \cdot \mu^2}{2!} = 0.2$$

Dividing the second equation by the first, we get

$$\frac{\mu}{2!} = \frac{0.2}{0.3}, \quad \text{or} \quad \mu = 1.33$$

$$\therefore \quad P(X=0) = e^{-1.33} = 0.2644, \text{ and}$$

$$P(X=3) = \frac{e^{-1.33}(1.33)^3}{3!}$$

$$= \frac{(0.2644)(2.352637)}{6} = 0.1037$$

To evaluate $e^{-1.33}$, we let $y = e^{-1.33}$

Then $\log y = -1.33 \log e = -1.33\,(0.4343)$

$= -0.5776 = \bar{1}.4223.$

$\therefore \quad y = \text{Antilog}\,(\bar{1}.4223) = 0.2644$

(c) Let the Poisson distribution be

$$P(X=x) = \frac{\mu^x \cdot e^{-\mu}}{x!} \quad \text{for } x = 0, 1, 2, \ldots$$

Given that $P(X=2) = 3P(X=4)$

i.e. $\frac{e^{-\mu} \cdot \mu^2}{2!} = 3 \cdot \frac{e^{-\mu} \cdot \mu^4}{4!}$, i.e. $\frac{\mu^2}{2} = \frac{\mu^4}{8}$ giving $\mu = 2$.

(i) $P(X=0) = e^{-2} = 0.1353 = 0.135$

(ii) $P(X \le 3) = \sum_{x=0}^{3} \frac{e^{-2}(2)^x}{x!}$

$= e^{-2} + \frac{e^{-2} \cdot 2}{1!} + \frac{e^{-2} \cdot 4}{2!} + \frac{e^{-2} \cdot 8}{3!}$

$= e^{-2}\left(1 + 2 + 2 + \frac{4}{3}\right)$

$= 0.1353\ (6.3333) = 0.857$

8.34. (b) Here p = 0.03 and n = 500.

$\therefore$ the mean of the Poisson distribution is

$$\mu = np = 500\ (0.03) = 15.$$

Let X denote the number of defective components.

The Poisson distribution then becomes

$$p(x;\ 15) = \frac{e^{-15}(15)^x}{x!},\quad x = 0, 1, 2, 3, \ldots$$

(i) $P(X \ge 3) = 1 - P(X < 3)$

$= 1 - \left\{e^{-15} + e^{-15}(15) + \frac{e^{-15}(15)^2}{2!}\right\}$

$= 1 - e^{-15}\ \{1 + 15 + 112.5\}$

$= 1 - (0.00000031)\ (128.5)\ (\because e^{-15} = 0.00000031)$

$= 1 - 0.000060 = 0.999940$

(ii) $P(X \ge 6) = 1 - P(X < 6)$

$= 1 - 0.028298 = 0.971702.$

Hence the probability that 2 successive boxes contain 6 or more defective $= (0.971702)^2 = 0.9442.$

8.35. (b) Here p, the probability of a defective tool = 0.1, so that q = 0.9 and n = 10.

(i) $P(2 \text{ defective tools in } 10) = \binom{10}{2}(0.1)^2(0.9)^9 = 0.1937$

(ii) Let μ be the mean of the Poisson distribution.

Then $\mu = np = 10(0.1) = 1.$

$$\therefore \quad P(2 \text{ defective tools in } 10) = \frac{\mu^2 \cdot e^{-\mu}}{2!}$$

$$= \frac{(1)^2 \cdot e^{-1}}{2!} = \frac{1}{2e}$$

$$= \frac{1}{2(2.718)} = 0.1839$$

8.36. (a) Let X denote the number of claims. Then the Poisson distribution with $\mu = 0.05$, is

$$p(x; 0.05) = \frac{e^{-0.05}(0.05)^x}{x!}, \quad x = 0, 1, 2, 3, \ldots$$

Therefore (i) $P(X=0) = e^{-0.05} = 0.9513$

(ii) $P(1 \text{ or fewer claims}) = P(X \le 1) = e^{-0.05} + e^{-0.05}(0.05)$

$$= 0.9513 + (0.9513)(0.05)$$

$$= 0.9513 + 0.0476 = 0.9989.$$

(b) Let p be the chance of being killed in an accident.

Then the mean of the Poisson distribution is

$$\mu = np = 350 \times \frac{1}{1400} = \frac{1}{4} = 0.25$$

Let X denote the number of fatal accidents. Then

$P(\text{At least one fatal accident}) = P(X \ge 1)$

$$= 1 - P(X < 1)$$

$$= 1 - e^{-0.05} \qquad (X=0)$$

$$= 1 - 0.779 = 0.221$$

8.37. (a) Let X be the r.v. the number of errors per page. Then X has a Poisson distribution with parameter $\mu = 2$.

So $P(X=x) = \dfrac{e^{-2}(2)^x}{x!}$.

Now (i) $P(X \geq 4) = 1 - \sum_{x=0}^{3} \dfrac{e^{-2}\,(2)^x}{x!}$, where $e^{-2} = 0.1353$.

$$= 1 - e^{-2}\,[1 + 2 + 2 + 4/3]$$

$$= 1(0.1353)(6.3333) = 1 - 0.8569 = 0.1431, \text{ and}$$

(ii) $P(X=0) = e^{-2} = 0.1353$.

(b) Given that the number of demands for a car is distributed as a Poisson distribution with mean $\mu = 1.5$.

$\therefore\ P(X=x) = \dfrac{e^{-1.5}(1.5)^x}{x!}$, where x denotes the number of demands

Proportion of days on which neither car is used

= probability of there being no demand for the car

$$= \frac{e^{-1.5}\,(1.5)^0}{0!} = e^{-1.5} = 0.2231$$

Proportion of days on which some demand is refused

= probability for the number of demands to be more than two.

$$= 1 - P(X \leq 2), = 1 - \sum_{x=0}^{2} \frac{e^{-1.5}\,(1.5)^x}{x!}$$

$$= 1 - \left[e^{-1.5} + e^{-1.5}(1.5) + \frac{e^{-1.5}\,(1.5)^2}{2!}\right]$$

$$= 1 - (0.2231 + 0.3346 + 0.2510) = 1 - 0.8087 = 0.1913.$$

8.38. Let p be the probability of the product being defective. Then p = 5/100 = 0.05 and n = 100.

The mean of the Poisson distribution, $\mu = np = 0.05 \times 100 = 5$

and the Poisson distribution then becomes

$$p(x; 5) = \frac{e^{-5} \cdot (5)^x}{x!}, \quad x = 0, 1, 2, \ldots \text{ and } e^{-5} = 0.0067$$

We seek the probability that a box will fail to meet the guaranteed quality, *i.e.* the probability of there being more than 4 defective pins in a box of 100 pins. In other words, we are to find $P(X>4)$.

Thus $\quad P(X>4) = 1 - P(X \leq 4) = 1 - \sum_{x=0}^{4} \frac{e^{-5} \cdot (5)^x}{x!}$

$$= 1 - e^{-5}\left[1 + 5 + \frac{(5)^2}{2} + \frac{(5)^3}{6} + \frac{(5)^4}{24}\right]$$

$$= 1 - 0.0067\,[1 + 5 + 12.5 + 20.83 + 26.04]$$

$$= 1 - (0.0067)\,(65.37)$$

$$= 1 - 0.4380 = 0.5620.$$

8.39. (b) As the mean and variance of the Poisson distribution are equal, therefore $\sigma^2 = \mu = 1$; and the Poisson distribution will be

$$P(X=x) = \frac{e^{-1}(1)^x}{x!}.$$

Hence the desired probability is

$$P(X=2) = \frac{e^{-1}(1)^2}{2!} = \frac{e^{-1}}{2!} = \frac{1}{2e} = \frac{1}{2(2.7183)}$$

$$= \frac{1}{5.4366} = 0.1839.$$

8.40. (a) Let μ be the parameter of the Poisson distribution.

Then mean $= \mu$ and $\sigma = \sqrt{\mu}$, *i.e.* $\sigma = \sqrt{\text{mean}}$

In the given statement, mean $= 5$ and $\sigma = 4$

$\therefore \quad \sqrt{\text{mean}}$, *i.e.* $\sqrt{5} \neq \sigma$

Hence the given statement is wrong.

(b) Let μ be the parameter of the Poisson distribution and N be the total number of observations. Then the Poisson distribution is

$$\frac{N \cdot e^{-\mu} \cdot \mu^x}{x!}, \; x = 0, 1, 2, \ldots, \infty$$

It is given that

$$N \cdot e^{-\mu} = 250, \text{ and } N \cdot e^{-\mu} \cdot \mu = 160.$$

Dividing the second equation by the first, we get

$$\mu = \frac{16}{25} = 0.64$$

$$\therefore \quad e^{-0.64} = 1 - (0.64) + \frac{(0.64)^2}{2} - \frac{(0.64)^3}{3!} + \frac{(0.64)^4}{4!} - \ldots$$

$$= 0.5273$$

The frequencies of the next two values are

$$N \cdot e^{-\mu} \cdot \frac{\mu^2}{2!} \text{ and } Ne^{-\mu} \cdot \frac{\mu^3}{3!}$$

Now $\quad N \cdot e^{-\mu} \cdot \dfrac{\mu^2}{2!} = (Ne^{-\mu} \cdot \mu)\left(\dfrac{\mu}{2}\right) = 160 \times \dfrac{0.64}{2} = 51$, and

$$Ne^{-\mu} \cdot \frac{\mu^3}{3!} = \left(Ne^{-\mu} \cdot \frac{\mu^2}{2!}\right)\left(\frac{\mu}{3}\right) = 51 \times \frac{0.64}{3} = 11.$$

8.44. Fitting of the Poisson distribution.

We first calculate the mean of the given distribution and equate it to μ, the parameter of the Poisson distribution $p(x; \mu) = \dfrac{e^{-\mu} \cdot \mu^x}{x!}$, where x denotes the yeast cell counts.

Now $\quad \bar{x} = \dfrac{\sum f_i x_i}{\sum f_i} = \dfrac{0 + 128 + 74 + 54 + 12 + 5}{400} = \dfrac{273}{400} = 0.6825$

Therefore $\mu = 0.6825$ and the fitted Poisson distribution is

$$p(x; 0.6852) = \frac{e^{-.6825}\,(0.6825)^x}{x!}, \; x = 0, 1, 2, 3, \ldots$$

The expected frequencies are obtained by multiplying the probabilities by 400. Thus the expected probabilities and frequencies are calculated as below:

x	Probability $p(x; 0.6825)$		Expected frequencies $400 \times p$
0	$e^{-0.6825}$	$= 0.5054$	202.16
1	$e^{-0.6825}(0.6825)$	$= 0.3449$	137.96
2	$\frac{e^{-0.6825}(0.6825)^2}{2!}$	$= 0.1177$	47.08
3	$\frac{e^{-0.6825}(0.6825)^3}{3!}$	$= 0.0268$	10.72
4	$\frac{e^{-0.6825}(0.6825)^4}{4!}$	$= 0.0046$	1.84
5	$\frac{e^{-0.6825}(0.6825)^5}{5!}$	$= 0.0006$	0.24
Total		1.0000	400

8.45. (a) We first calculate the mean of the given frequency distribution and equate it to (i) μ and (ii) np.

$$\text{Now } \bar{x} = \frac{\sum fx}{\sum f} = \frac{(0)(531)+(1)(354)+(2)(99)+(3)(15)+(4)(1)}{1000}$$

$$= \frac{601}{1000} = 0.60$$

(i) Therefore $\mu = 0.60$ and the fitted Poisson distribution is

$$P(X=x) = \frac{e^{-0.60}(0.60)^x}{x!} \text{ for } x = 0, 1, 2, 3, \ldots$$

(ii) Equating to np, *i.e.* $4p$, we get $p = \frac{0.60}{4} = 0.15$. Thus the fitted binomial distribution is

$$P(X=x) = \binom{4}{x}(0.15)^x(0.85)^{4-x}, \text{ for } x = 0, 1, 2, 3, 4.$$

The expected frequencies are obtained by multiplying the probabilities by 1000. The probabilities and the expected frequencies are given below:

x	(i) Poisson Distribution		(ii) Binomial Distribution	
	$P(X=x)$	Expected f	$P(X=x)$	Expected f
0	0.5488	549	0.5220	522
1	0.3293	329	0.3685	368
2	0.0988	99	0.0975	98
3	0.0198	20	0.0115	12
4	0.0030	3	0.0005	0
>4	0.0003	0	--	--

(b) The mean of the given distribution is

$$\bar{x} = \frac{\sum f_i x_i}{\sum f_i} = \frac{0 + 96 + 68 + 27 + 4 + 0}{440}$$

$$= \frac{195}{440} = 0.443 = \mu$$

Thus the fitted Poisson distribution is

$$p(x; 0.443) = \frac{e^{-0.443}(0.443)^x}{x!}, \quad x = 0, 1, 2, \ldots$$

To find the value of $e^{-0.443}$, we let $y = e^{-0.443}$

Taking logs, we get

$$\log y = -0.443 \log e, \text{ where } e = 2.7183$$

$$= -0.443\,(0.4343) = -0.1924 = \bar{1}.8076$$

$\therefore \quad y = \text{Anti-log}\,(\bar{1}.8076) = 0.6421$

(i) $P(\text{no accident}) = P(X=0) = e^{-0.443} = 0.6421$

(ii) $P(\text{more than one accident}) = P(X>1)$

$$= 1 - \sum_{x=0}^{1} \frac{e^{-0.443}(0.443)^x}{x!}$$

$$= 1 - [0.6421 + 0.2845]$$

$$= 1 - 0.9266 = 0.0734$$

8.46. Computation of the frequencies of the fitted Poisson distribution.

Here $$\bar{x} = \frac{\sum f_i x_i}{\sum f_i} = \frac{0+90+84+36+36+15+6}{300} = \frac{267}{300} = 0.89$$

Thus the fitted Poisson distribution is

$$p(x;\, 0.89) = \frac{e^{-0.89}\,(0.89)^x}{x!}, \quad x = 0, 1, 2, \ldots$$

To find the value of $e^{-0.89}$, we put $y = e^{-0.89}$.

Taking logs, we get

$$\log y = -0.89 \log e = -0.89\,(0.4343) \qquad (\text{where } e = 2.718)$$

$$= -0.3865 = \bar{1}.6135, \quad \therefore\ y = \text{Anti-log}\,(\bar{1}.6135) = 0.4107$$

Hence the expected frequencies are computed as below:

x	Probabilities		Expected f
0	$e^{-0.89}$	$= 0.4107$	123.2
1	$e^{-0.89}\dfrac{(0.89)}{1!} = \dfrac{0.4107\times 0.89}{1}$	$= 0.3655$	109.6
2	$e^{-0.89}\dfrac{(0.89)^2}{2!} = \dfrac{0.3655\times 0.89}{2}$	$= 0.1626$	48.8
3	$e^{-0.89}\dfrac{(0.89)^3}{3!} = \dfrac{0.1626\times 0.89}{3}$	$= 0.0482$	14.5
4	$e^{-0.89}\dfrac{(0.89)^4}{4!} = \dfrac{0.0482\times 0.89}{4}$	$= 0.0107$	3.2
5	$e^{-0.89}\dfrac{(0.89)^5}{5!} = \dfrac{0.0107\times 0.89}{5}$	$= 0.0019$	0.6
6	$e^{-0.89}\dfrac{(0.89)^6}{6!} = \dfrac{0.0019\times 0.89}{6}$	$= 0.0003$	0.1

8.47. The mean number of accidents a day is

$$\bar{x} = \frac{\sum f_i x_i}{\sum f_i} = \frac{0+113+128+63+28+15+6+7}{300} = \frac{360}{300} = 1.2$$

Thus the fitted Poisson distribution is

$$p(x;\,1.2)=\frac{e^{-1.2}\,(1.2)^x}{x!}.$$

To evaluate $e^{-1.2}$, we let $y=e^{-1.2}$

Then $\log y \;=\; -1.2 \log e$

$$= -1.2\,(0.4343) = -0.52116 = \bar{1}.47884$$

$\therefore \quad y \;=\; \text{Anti-log}\,(\bar{1}.47884) = 0.3011.$

The expected frequencies are calculated as below:

Accidents per day (x)	Probabilities $p(x;\,1.2)$	Expected f $300\,f(x)$
0	$e^{-1.2} = 0.3011$	90.3
1	$e^{-1.2}\cdot\frac{(1.2)}{1!} = (0.3011)(1.2) = 0.3613$	108.4
2	$e^{-1.2}\cdot\frac{(1.2)^2}{1!} = (0.3613)(0.6) = 0.2168$	65.0
3	$e^{-1.2}\cdot\frac{(1.2)^3}{3!} = (0.2168)(0.4) = 0.0867$	26.0
4	$e^{-1.2}\cdot\frac{(1.2)^4}{4!} = (0.0867)(0.3) = 0.0260$	7.8
5	$e^{-1.2}\cdot\frac{(1.2)^5}{5!} = (0.0260)(0.24) = 0.0062$	1.9
6	$e^{-1.2}\cdot\frac{(1.2)^6}{6!} = (0.0062)(0.2) = 0.0012$	0.4
7+	$1-\sum_{x=0}^{6}\frac{e^{-1.2}(1.2)^x}{x!} = 0.0007$	0.2
Total	$= 1.0000$	300.0

And $P(4 \text{ or more accidents}) = 1 - P\,(\text{less than } 4 \text{ accidents})$

$$= 1-(0.3011+0.3613+0.2168+0.0867)$$

$$= 1-0.9659 = 0.0341.$$

8.48. (b) Let X denote the number of customers entering the shop in t units of time.

Then we have

$$\lambda = 30 \text{ persons per hour} = 30 \text{ persons per 60 minutes}$$

As 2 minutes is $\frac{2}{60} = \frac{1}{30}$ units of time, so $t = \frac{1}{30}$ and therefore the average number of persons per 2 minutes interval, *i.e.* $\lambda t = 30 \times \frac{1}{30} = 1.$

Hence, using the Poisson process formula

$$p(x; \lambda t) = \frac{e^{-\lambda t}(\lambda t)^x}{x!}, \text{ we get}$$

$$p(0; 1) = \frac{e^{-1} \cdot (1)^0}{0!} = e^{-1} = \frac{1}{2.7183} = 0.3679.$$

8.49. (a) Taking 50 square feet as the unit of area, we have

$$\lambda = 1 \text{ flaw per 50 square feet.}$$

As 32 square feet are $\frac{32}{50} = 0.64$ units of area, so $t = 0.64$, and therefore the average number of flaws per 32 square feet, *i.e.* $\lambda t = 1 \times 0.64 = 0.64$. Assuming the flaws are a Poisson process, we have

$$P(\text{no flaw occurs in 32 square feet}) = \frac{e^{-0.64}(0.64)^0}{0!} = 0.5272$$

$$P(\text{atmost one flaw occurs in 32 square feet}) = \sum_{x=0}^{1} \frac{e^{-0.64}(0.64)^x}{x!}$$

$$= e^{-0.64} + e^{-0.64}(0.64)$$

$$= 0.5272 + 0.3374$$

$$= 0.8646$$

(b) Taking 12 hours from 9 p.m. until 9 a.m. the next morning as the unit of time, we have $\lambda = 3$ calls per 12 hours

As 6 hours from midnight to 6 a.m. are $\frac{6}{12} = \frac{1}{2}$ units of time, so $t = \frac{1}{2}$ and therefore the average number of calls per 6 hours, *i.e.* $\lambda t = 3\times\frac{1}{2} = 1.5$.

As the arrivals of calls are assumed to be Poisson process, therefore we have

$$P(\text{doctor receives no call in 6 hours}) = \frac{e^{-1.5}(1.5)^0}{0!}$$

$$= e^{-1.5} = 0.2231$$

8.50. Taking 25 days as unit of time, we have

$$\lambda = 1 \text{ breakdown in 25 days.}$$

As 10 days is $\frac{10}{25} = 0.4$ unit of time, so $t = 0.4$ and therefore the average number of breakdowns per 10 days, *i.e.* $\lambda t = 1\times 0.4 = 0.4$.

Hence, using the Poisson process formula $p(x; \lambda t) = \frac{e^{-\lambda t}(\lambda t)^x}{x!}$,

we have (i) $P(\text{exactly one breakdown in 10 days}) = e^{-0.4}\frac{(0.4)^1}{1!}$

$$= (0.6704)(0.4)$$

$$= 0.2682$$

(ii) $P(\text{more than one breakdown in 10 days})$

$$= 1 - P(0 \text{ or } 1 \text{ breakdown}) = 1 - \sum_{x=0}^{1} \frac{e^{-0.4}(0.4)^x}{x!}$$

$$= 1 - [0.6704 + 0.2682] = 1 - 0.9386 = 0.0614.$$

8.51. Taking 60 minutes, (10 to 11 A.M.) as unit of time, we have

$$\lambda = 300 \text{ cars passing in 60 minutes.}$$

As 1 minute is $\frac{1}{60}$ unit of time, so $t = \frac{1}{60}$ and therefore the average number of cars passing per minute, *i.e.* $\lambda t = 300 \times \frac{1}{60} = 5$

Hence, using the Poisson process formula $p(x; \lambda t) = \frac{e^{-\lambda t}(\lambda t)^x}{x!}$,

we have (i) P(not more than 4 cars in 1-minute) $= \sum_{x=0}^{4} \frac{e^{-5} \cdot (5)^x}{x!}$

$$= e^{-5}\left[1+5+\frac{(5)^2}{2}+\frac{(5)^3}{6}+\frac{(5)^4}{24}\right]$$

$$= (0.0067)\,[1+5+12.5+20.83+26.04]$$

$$= (0.0067)\,(65.37) \quad (\because e^{-5} = 0.0067)$$

$$= 0.4380$$

(ii) P(5 or more cars pass in 1-minute)

$$= 1 - P(\text{less than 5 cars pass})$$

$$= 1 - \sum_{x=0}^{4} \frac{e^{-5} \cdot (5)^x}{x!} = 1 - 0.4380 = 0.5620.$$

8.53. (b) Let N be equal to number of eggs laid by an insect. Then N is assumed to have a Poisson distribution with mean m.

It is given that the probability of an egg developing is p.

If X denotes the number of eggs that develop, then for given $N=n$, the variate X has a binomial distribution based on n repetitions with parameter p. That is

$$P(X=k \mid N=n) = \binom{n}{k} p^k q^{n-k}, \quad k = 0, 1, 2, \ldots, n$$

Hence according to the theorem of total probability, we have

$$P(X=k) = \sum_{n=k}^{\infty} P(X=k \mid N=n) \,.\, P(N=n)$$

$$= \sum_{n=k}^{\infty} \binom{n}{k} p^k q^{n-k} \cdot \frac{e^{-m} \cdot m^n}{n!}$$

$$= \frac{e^{-m} p^k}{k!} \sum_{n=k}^{\infty} \frac{m^n q^{n-k}}{(n-k!)} = \frac{e^{-m} . p^k m^k}{k!} \sum_{n=k}^{\infty} \frac{(mq)^{n-k}}{(n-k)!}$$

$$= \frac{e^{-m} (mp)^k}{k!} \sum_{j=0}^{\infty} \frac{(mq)^j}{j!} \quad \text{where } j = n-k$$

$$= \frac{e^{-m} (mp)^k}{k!} . e^{mq} = \frac{e^{-m} (mp)^k . e^{m(1-p)}}{k!} = \frac{e^{-mp} . (mp)^k}{k!},$$

which is a Poisson distribution with mean mp.

8.54. (b) We observe that

(i) there are two possible outcomes, *i.e.* the swimmer will *or* will not cross the lake,

(ii) probability of success in swimming across the lake for each swimmer is $p = 0.4$,

(iii) the swimmers will cross the lake independently, and

(iv) the experiment is repeated 10 times to get fourth success.

Therefore the negative binomial distribution with $k = 4$(4th success) and $x = 10$, is appropriate. Hence

$$b^*(10; 4, 0.4) = \binom{10-1}{4-1} (0.4)^4 (0.6)^{10-4}$$

$$= 84 (0.4)^4 (0.6)^6 = 0.10033$$

8.55. (b) Here E(X) = 10, and σ^2 = 9.

Since mean is greater than variance, so X cannot have a negative binomial distribution.

(c) Here p, the probability of getting a head = $\frac{1}{2}$,

$k = 3$ (third success), and $x = 7$ (seventh trial).

$$\therefore \quad b^*(7; 3, \tfrac{1}{2}) = \binom{7-1}{3-1} \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^{7-3}$$

$$= 15 . \left(\frac{1}{2}\right)^7 = \frac{15}{128} = 0.1172.$$

8.56. (b) Here p, the probability of getting installed a black telephone = 0.3.

$k = 5$ (5th success), and $x = 10$ (10th telephone)

$$\therefore \quad b^*(10; 5, 0.3) = \binom{10-1}{5-1}(0.3)^5 (0.7)^{10-5}$$

$$= 126\,(0.3)^5 (0.7)^5 = 0.0515$$

8.57. (b) The Negative Binomial distribution gives the probability that the kth success occurs on the *x*th trial for x=k, k+1, k+2, ... Its probability function is

$$p(x) = \binom{x-1}{k-1} p^k q^{x-k}, \text{ where } x = k, k+1, k+2, \ldots$$

It has two parameters k and $p > 0$. To show that the probabilities add to 1, we proceed as below:

$$\sum p(x) = \sum_{x=k}^{\infty} \binom{x-1}{k-1} p^k q^{x-k}, \quad (x = k, k+1, k+2, \ldots)$$

Let $y = x - k$. Then

$$\text{Sum of prob} = \sum_{y=0}^{\infty} \binom{y+k-1}{k-1} p^k q^y, \qquad (y = 0, 1, 2, \ldots)$$

$$= p^k \sum_{y=0}^{\infty} \binom{y+k-1}{k-1} q^y$$

$$= p^k \left[\binom{k-1}{k-1} q^0 + \binom{k}{k-1} q + \binom{k+1}{k-1} q^2 + \ldots\right]$$

$$= p^k \left[1 + kq + \frac{k(k+1)}{2!} q^2 + \ldots\right]$$

$$= p^k [1-q]^{-k} = p^k p^{-k} = 1.$$

The *m.g.f.* about the origin is

$$M_0(t) = E(e^{tY}) = p^k \sum_{y=0}^{\infty} \binom{y+k-1}{k-1} q^y \cdot e^{ty}$$

$$= p^k \sum_{y=0}^{\infty} \binom{y+k-1}{k-1} (qe^t)^y = p^k(1 - qe^t)^{-k}$$

$$= \left[\frac{p}{1 - qe^t}\right]^k = \left[\frac{p}{p + q - qe^t}\right]^k$$

$$= \left[\frac{1}{1 - \frac{q}{p}(e^t - 1)}\right]^k$$

$$= [1 - \lambda (e^t - 1)]^{-k}, \quad \text{where} \quad \lambda = q/p.$$

Taking logs, we have

$$\kappa(t) = \log M_0(t) = -k \log [1 - \lambda(e^t - 1)]$$

$$= k\left[\lambda(e^t-1) + \tfrac{1}{2}\lambda^2(e^t-1)^2 + \tfrac{1}{3}\lambda^3(e^t-1)^3 + \tfrac{1}{4}\lambda^4(e^t-1)^4 + \ldots\right]$$

$$= k\Big[\lambda\left(t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \ldots\right) + \tfrac{1}{2}\lambda^2 t^2\left(1 + \frac{t}{2!} + \frac{t^2}{3!} + \frac{t^3}{4!} + \ldots\right)^2$$

$$+ \tfrac{1}{3}\lambda^3 t^3\left(1 + \frac{t}{2!} + \ldots\right)^3 + \frac{1}{4}\lambda^4 t^4 + \ldots\Big]$$

$$= k\Big[\lambda t + (\lambda + \lambda^2)\frac{t^2}{2!} + (\lambda + 3\lambda^2 + 2\lambda^3)\frac{t^3}{3!}$$

$$+ (\lambda + 7\lambda^2 + 12\lambda^3 + 6\lambda^4)\frac{t^4}{4!} + \ldots\Big]$$

$$= k\left[\frac{q}{p}.t + \frac{q}{p^2}\cdot\frac{t^2}{2!} + \frac{q(1+q)}{p^3}\cdot\frac{t^3}{3!} + \frac{q(1+4q+q^2)}{p^4}\cdot\frac{t^4}{4!} + \ldots\right]$$

dentifying co-efficients, we get

$$\kappa_1 = \frac{kq}{p}, \quad \kappa_2 = \frac{kq}{p^2}, \kappa_3 = \frac{kq(1 + q)}{p^3}, \quad \text{and}$$

$$\kappa_4 = \frac{kq(1 + 4q + q^2)}{p^4}.$$

8.58. (c) Let X denote the number of trials required o get the occurrence of 1st head.

The head occurs on the third trial, *i.e.* the first success :curs on the 3rd trial, therefore the geometric distribution with $= 1/2$ and $x = 3$ is used.

Hence, using the geometric distribution $P(X=x) = pq^{x-1}$, we get

$$P(X=3) = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^2 = \frac{1}{8}.$$

8.59. (b) Sample space S associated with this experiment contains $\binom{52}{13}$ sample points.

Let A be the event that a bridge hand contains 5 spades, 2 hearts, 3 diamonds and 3 clubs.

Then A contains $\binom{13}{5}\cdot\binom{13}{2}\cdot\binom{13}{3}\cdot\binom{13}{3}$ sample points.

$$\text{Hence } P(A) = \frac{\binom{13}{5}\binom{13}{2}\binom{13}{3}\binom{13}{3}}{\binom{52}{13}} = \frac{(1287)\ (78)\ (286)\ (286)}{635{,}013{,}559{,}600}$$

$$= 0.0129.$$

This question actually relates to the extension of the hypergeometric distribution where the population is partitioned into 4 cells.

8.60. (a) Here $p_1 = P(\text{bulb is red}) = 0.50,$

$p_2 = P(\text{bulb is blue}) = 0.30,$

$p_3 = P(\text{bulb is green}) = 0.20,$ and

$x_1 = 2, x_2 = 1$ and $x_3 = 2.$

$\therefore$ P(2 are red, 1 is green, 2 are blue).

$$= \frac{5!}{2!\ 1!\ 2!}(0.5)^2\ (0.3)\ (0.2)^2$$

$$= \frac{30 \times 3 \times 4}{4 \times 10 \times 100} = \frac{9}{100} = 0.09.$$

(b) Here $p_1 = P(\text{red marble is drawn}) = \frac{5}{10},$

$p_2 = P(\text{white marble is drawn}) = \frac{3}{10},$

$$p_3 = P(\text{blue marble is drawn}) = \frac{2}{10}, \text{ and}$$

(i) $x_1 = 3, x_2 = 2$ and $x_3 = 1$.

Thus $P(3 \text{ are red, } 2 \text{ are white, } 1 \text{ is blue})$.

$$= \frac{6!}{3!\,2!\,1!}\left(\frac{5}{10}\right)^3\left(\frac{3}{10}\right)^2\left(\frac{2}{10}\right)^1$$

$$= \frac{60 \times 1 \times 9 \times 2}{8 \times 100 \times 10} = \frac{27}{200} = 0.135.$$

(ii) $x_1 = 2, x_2 = 3$ and $x_3 = 1$.

Thus $P(2 \text{ are red, } 3 \text{ are white, } 1 \text{ is blue})$.

$$= \frac{6!}{2!\,3!\,1!}\left(\frac{5}{10}\right)^2\left(\frac{3}{10}\right)^3\left(\frac{2}{10}\right)^1$$

$$= \frac{60 \times 27 \times 2}{4 \times 1000 \times 10} = \frac{81}{1000} = 0.081.$$

(iii) $x_1 = 2, x_2 = 2$ and $x_3 = 2$.

Thus $P(2 \text{ of each colour appear})$

$$= \frac{6!}{2!\,2!\,2!}\left(\frac{5}{10}\right)^2\left(\frac{3}{10}\right)^2\left(\frac{2}{10}\right)^2$$

$$= \frac{90 \times 9 \times 4}{4 \times 100 \times 100} = \frac{81}{1000} = 0.081.$$

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