Exercise 4.3

Q#1 (a) $u_1 = -5u_2$

$\Rightarrow u_1$ & u_2 are linearly dependent

(Two vectors are linearly dependent if they are scalar multiple of each other).

Q#1 (b) Let $a_1u_1 + a_2u_2 + a_3u_3 = 0$

$\Rightarrow a_1(3,-1) + a_2(4,5) + a_3(-4,7) = (0,0)$

$$\Rightarrow \begin{aligned} 3a_1 + 4a_2 - 4a_3 &= 0 \\ -a_1 + 5a_2 + 7a_3 &= 0 \end{aligned}$$

i.e. $\begin{bmatrix} 3 & 4 & -4 \\ -1 & 5 & 7 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow$ they are linearly dependent.

($\because$ no. of equations is less than no. of variables involved)

Q#1 (c)

Let $a_1 p_1 + a_2 p_2 = 0$

$a_1(3-2x+x^2) + a_2(6-4x+2x^2) = 0$

$\Rightarrow (3a_1 + 6a_2) + x(-2a_1 - 4a_2) + x^2(a_1 + 2a_2) = 0$

Comparing coefficients of x^2, x and constant term

$$\begin{aligned} 3a_1 + 6a_2 &= 0 \\ -2a_1 - 4a_2 &= 0 \\ a_1 + 2a_2 &= 0 \end{aligned}$$

$$\underset{A}{\begin{bmatrix} 3 & 6 \\ -2 & -4 \\ 1 & 2 \end{bmatrix}} \underset{X}{\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}} = \underset{B}{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}}$$

Consider Augmented matrix $\left[\begin{array}{cc|c} 3 & 6 & 0 \\ -2 & -4 & 0 \\ 1 & 2 & 0 \end{array}\right]$

Reducing into row echelon form

Hint [redacted]

$\mathbb{R}^3$ 0 in same plane iff

(2)

$$\left[\begin{array}{cc|c} 3 & 6 & 0 \\ -2 & -4 & 0 \\ 1 & 2 & 0 \end{array}\right]$$

$$\underset{\sim}{R} \left[\begin{array}{cc|c} 1 & 2 & 0 \\ -2 & -4 & 0 \\ 3 & 6 & 0 \end{array}\right] R_1 \leftrightarrow R_3$$

$$\underset{\sim}{R} \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \begin{array}{l} R_2 + 2R_1 \\ R_3 - 3R_1 \end{array}$$

writing in equation form

$$a_1 + 2a_2 = 0$$

$$\Rightarrow a_1 = -2a_2$$

if $\boxed{a_2 = 1} \Rightarrow \boxed{a_1 = -2}$

which is a non-trivial sol.
Hence vectors are linearly dependent.

OR

$$P_2 = 2P_1$$

Two vectors are linearly dependent if they are scalar multiple of each other.

Q # 1 (d)

$$B = -A$$

i.e B is scalar multiple of A

$\Rightarrow$ A & B are linearly dependent.

Q# 2 (a)

let $a_1u_1 + a_2u_2 + a_3u_3 = 0$

$\Rightarrow a_1(-3, 0, 4) + a_2(5, -1, 2) + a_3(1, 1, 3) = (0, 0, 0)$

$$-3a_1 + 5a_2 + a_3 = 0$$
$$0a_1 - a_2 + a_3 = 0$$
$$4a_1 + 2a_2 + 3a_3 = 0$$

or

$$\begin{bmatrix} -3 & 5 & 1 \\ 0 & -1 & 1 \\ 4 & 2 & 3 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$AX = B$$

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Hint

... $\mathbb{R}^3$ l.i. in same plane iff

Here A is a Square matrix. (3)

This homogeneous system has trivial solution iff $\det A \neq 0$.

i.e vectors are linearly independent iff $\det A \neq 0$

(Check yourself $\det A$).

Q#2(b)

Same like Q#1(b)

more unknowns than no. of equations in system.

Q#3(a) & (b)

Same like Q#2(a)

Consider $\det A$ to verify if vectors are linearly independent or not.

Q#4(a) Same like Q#2(a)

Q#4(b) more unknowns less equations case.

Q#5(a)

$$A=\begin{bmatrix}1 & 0\\ 1 & 2\end{bmatrix},\ B=\begin{bmatrix}1 & 2\\ 2 & 1\end{bmatrix},\ C=\begin{bmatrix}0 & 1\\ 2 & 1\end{bmatrix}$$

let $a_1A+a_2B+a_3C=0$

$$\begin{bmatrix}a_1 & 0\\ a_1 & 2a_1\end{bmatrix}+\begin{bmatrix}a_2 & 2a_2\\ 2a_2 & a_2\end{bmatrix}+\begin{bmatrix}0 & a_3\\ 2a_3 & a_3\end{bmatrix}=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$$

$a_1+a_2=0 \Rightarrow \boxed{a_1+a_2=0}$

$0+2a_2+a_3=0 \Rightarrow \boxed{2a_2+a_3=0}$

$a_1+2a_2+2a_3=0 \Rightarrow a_1+2(a_2+a_3)=0$

$2a_1+a_2+a_3=0 \Rightarrow 2a_1+a_2+a_3=0$

or $\begin{bmatrix}1 & 1 & 0\\ 0 & 2 & 1\\ 1 & 2 & 2\\ 2 & 1 & 1\end{bmatrix}\begin{bmatrix}a_1\\ a_2\\ a_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$

(Hint) Solving by Guass elimination

Q#5 (b) Do yourself
solution like part (a)

Q#6

let $a_1 \begin{bmatrix} 1 & 0 \\ 1 & k \end{bmatrix} + a_2 \begin{bmatrix} -1 & 0 \\ k & 1 \end{bmatrix} + a_3 \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$$\Rightarrow \quad a_1 - a_2 + 2a_3 = 0$$
$$a_1 + ka_2 + a_3 = 0$$
$$ka_1 + a_2 + 3a_3 = 0$$

OR
$$\begin{bmatrix} 1 & -1 & 2 \\ 1 & k & 1 \\ k & 1 & 3 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$AX = 0$$

for vectors to be linearly independent, the system must have a trivial solution only. the system has trivial solution only iff $\det A \neq 0$

$$\begin{vmatrix} 1 & -1 & 2 \\ 1 & k & 1 \\ k & 1 & 3 \end{vmatrix} \neq 0$$

$$1(3k-1) + 1(3-k) + 2(1-k^2) \neq 0$$
$$3k - 1 + 3 - k + 2 - 2k^2 \neq 0$$
$$-2k^2 + 2k + 4 \neq 0$$
$$\Rightarrow \quad k^2 - k - 2 \neq 0$$
$$\Rightarrow \quad (k-2)(k+1) \neq 0$$

i.e $k - 2 \neq 0$ and $k + 1 \neq 0$
$k \neq 2$ and $k \neq -1$

Hint

Q#7 three vectors of $\mathbb{R}^3$ lie in same plane iff they are linearly dependent. (5)
iff $\vec{u}\cdot(\vec{v}\times\vec{w})=0$

Q#8

Two vectors lie on the same line in $\mathbb{R}^3$ if they are scalar multiple of each other.
Three vectors lie on same line in $\mathbb{R}^3$ if two vectors are scalar multiple of one vector.

Q#8 (a) $v_2 = -2v_1$ but v_3 is not scalar multiple of v_1 or v_2.
Therefore they do not lie on same line.

(b) Not lie on same line

(c) $v_1 = 2v_2$

$v_3 = -v_2$

Hence they lie on same line.

Q#9(a) Do yourself

(b) Let $v_1 = k_2 v_2 + k_3 v_3$

$(0,3,1,-1) = k_2(6,0,5,1) + k_3(4,-7,1,3)$

$(0,3,1,-1) = (6k_2, 0, 5k_2, k_2) + (4k_3, -7k_3, k_3, 3k_3)$

$(0,3,1,-1) = (6k_2+4k_3, -7k_3, 5k_2+k_3, k_2+3k_3)$

$6k_2 + 4k_3 = 0 \Rightarrow 6k_2 = -4k_3$

$\Rightarrow \boxed{k_2 = -\frac{2}{3}k_3}$ — (1)

$-7k_3 = 3$

$\boxed{k_3 = -\frac{3}{7}}$ — (2)

using in (1)

$k_2 = \left(-\frac{2}{3}\right)\left(-\frac{3}{7}\right)$

$\boxed{k_2 = \frac{2}{7}}$

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Hence
$$V_1 = \frac{2}{7} V_2 - \frac{3}{7} V_3$$

* with similar calculation V_2 can be expressed as linear combination of V_1 & V_3. V_3 can also be expressed as linear combination of V_1 and V_2.

Q # 10: Same as 9

Q # 11: Same as Q # 6
However for vectors to be linearly dependent in $\mathbb{R}^3$, we must have
$$\det A = 0$$

Q # 12: If $\{u\}$, where $u = 0$ and it is linearly dependent.
If $\{u\}$, where $u \neq 0$ then it is linearly independent.

Q # 13
(a) $T_A(u_1) = Au_1 = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1-2 \\ 0+4 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$

$T_A(u_2) = Au_2 = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1-1 \\ 0+2 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$

$\{T_A(u_1), T_A(u_2)\} = \{(-1,4), (-2,2)\}$ Now check its linear independence by usual techniques.

Q # 14 Same as 13

Q#16(a) Let $u_1 = 6$, $u_2 = 3\sin^2 x$, $u_3 = 2\cos^2 x$

$\because$ $3(2\cos^2 x) + 2(3\sin^2 x) = 6\cos^2 x + 6\sin^2 x$

$= 6(\cos^2 x + \sin^2 x)$

$= 6 \cdot 1$

$= 6$

i.e $3u_3 + 2u_2 = u_1$

or $3u_3 + 2u_2 - u_1 = 0$

Hence u_1, u_2, u_3 are linearly dependent.

Q#16 (d) $u_1 = \cos 2x$, $u_2 = \sin^2 x$, $u_3 = \cos^2 x$

As $\cos 2x = \cos^2 x - \sin^2 x$

$u_1 = u_3 - u_2$

$\Rightarrow u_1 + u_2 - u_3 = 0$

Hence u_1, u_2, u_3 are linearly dependent.

Q#16(e)

$u_1 = (3-x)^2$, $u_2 = (x^2 - 6x)$, $u_3 = 5$

Now

$u_1 = (3-x)^2$

$= x^2 + 9 - 6x$

$= (x^2 - 6x) + 9$

$= (x^2 - 6x) + \frac{9}{5}(5)$

$= u_2 + \frac{9}{5}u_3$

or $u_1 - u_2 - \frac{9}{5}u_3 = 0$

Hence u_1, u_2, u_3 are linearly dependent.

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Q#16(b) $f_1 = x$; $f_2 = \cos x$

Use Wronskian to check linear independence.

$$W = \begin{vmatrix} f_1 & f_2 \\ f_1' & f_2' \end{vmatrix}$$

$$= \begin{vmatrix} x & \cos x \\ +1 & -\sin x \end{vmatrix}$$

$$W = -x\sin x - \cos x \neq 0$$

e.g at $x = \frac{\pi}{2}$

$$W = -\frac{\pi}{2}\sin\frac{\pi}{2} - \cos\frac{\pi}{2}$$

$$= -\frac{\pi}{2}(1) - 0$$

$$= -\frac{\pi}{2}$$

Hence f_1 & f_2 are linearly independent.

Similarly check for (c) & (f) parts.

Q#17–21 Check yourself by Wronskian.

Q#22 Let $u_1 = u - v$
$u_2 = v - w$
$u_3 = w - u$

$\because u_1 + u_2 + u_3 = 0$

i.e $1.u_1 + 1.u_2 + 1.u_3 = 0$

$\Rightarrow u_1, u_2, u_3$ form a linearly dependent set.