CH # 3

Ex = 3.1

In exercise 3-4 find the components of vector $\overrightarrow{P_1P_2}$

(3) (a). $P_1(3, 5)$, $P_2(2, 8)$

Sol:

$\overrightarrow{P_1P_2} = P_2 - P_1$

$= (2, 8) - (3, 5)$

$= (2-3, 8-5)$

$\overrightarrow{P_1P_2} = (-1, 3)$ Ans.

Same (5-6)

5(a): Find terminal point B of vector that is equivalent to $u = (1, 2)$ and whose initial point is $A(1, 1)$.

Sol:

According to vector equation:

$u = \overrightarrow{AB}$

$u = B - A$

$B = u + A$

$B = (1, 2) + (1, 1)$

$B = (2, 3)$

(b): Find initial point A of vector that is equivalent to $u = (1, 1, 3)$ and whose terminal point is $B(-1, -1, 2)$.

Sol: As, $u = \cancel{BA}\ \overrightarrow{AB}$

$u = B - A$

$A = (-1, -1, 2) - (1, 1, 3)$

$A = (-2, -2, -1)$ Ans

Same (9-10)

9): Let $u = (4, -1)$, $v = (0, 5)$ and $w = (-3, -3)$. Find the components of:

(a): $u + w$

Sol:

$u + w = (4, -1) + (-3, -3)$

$= (1, -4)$

(b): $v - 3u$

Sol:

$v - 3u = (0, 5) - (12, -3)$

$= (-12, 8)$

(c): $2(u - 5w)$

(d): $3v - 2(u + 2w)$

Same as (a) and (b)

Same (17-18)

17): Let $u = (1, -1, 3, 5)$ & $v = (2, 1, 0, -3)$. Find scalars a and b so that $au + bv = (1, -4, 9, 18)$.

Sol:

$a(1, -1, 3, 5) + b(2, 1, 0, -3) = (1, -4, 9, 18)$

$(a, -a, 3a, 5a) + (2b, b, 0, -3b) = (1, -4, 9, 18)$

$(a + 2b, -a + b, 3a + 0, 5a - 3b) = (1, -4, 9, 18)$

$a + 2b = 1$; $-a + b = -4$; $3a = 9$; $5a - 3b = 18$

$3 + 2b = 1$; $a = 9/3$

$2b = 1 - 3$; $\boxed{a = 3}$

$2b = -2$

$\boxed{b = -1}$

Ex: 3.2

In exercise 1-2 find the norm of v, and a unit vector that is oppositely directed to v.

1): (a) => $v = (2, 2, 2)$

Sol:

$$\text{Norm} = \frac{\hat{v}}{\|v\|}$$

$$\|v\| = \sqrt{(2)^2 + (2)^2 + (2)^2}$$

$$\|v\| = \sqrt{12}$$

v. $\text{Norm} = \frac{1}{\sqrt{12}}(2, 2, 2)$

$$= \frac{2}{\sqrt{12}}, \frac{2}{\sqrt{12}}, \frac{2}{\sqrt{12}}$$

Now, unit vector opposite to v.

Unit vector: $\frac{-2}{\sqrt{12}}, \frac{-2}{\sqrt{12}}, \frac{-2}{\sqrt{12}}$

In exercise 3-4 evaluate the given expression with $u = (2, -2, 3)$, $v = (1, -3, 4)$ and $w = (3, 6, -4)$

(3) (a): $\|u+v\|$

Sol.

$$\|u+v\| = \sqrt{(2+1)^2 + (-2-3)^2 + (3+4)^2}$$

$$= (3)^2 + (-5)^2 + (7)^2$$

$$= \sqrt{83}$$

(b), (c) and (d) Same as (a).

In Exercise 5-6 evaluate

$u = (-2, -1, 4, 5)$, $v = (3, 1, -5, 7)$, $w = (-6, 2, 1, 1)$

(a): $\|3u - 5v + w\|$

Sol:

$3u - 5v + w = (-6, -3, 12, 15) - (15, 5, -25, 35) + (-6, 2, 1, 1)$

$= (-6 - 15 - 6, -3 - 5 + 2, 12 + 25 + 1, 15 - 35 + 1)$

$= (-27, -6, 38, -19)$

$\|3u - 5v + w\| = \sqrt{(-27)^2 + (-6)^2 + (38)^2 + (-19)^2}$

$= \sqrt{2570}$ Ans.

(b), (c) same as (a)

Same 7-8.

7): let $v = (-2, 3, 0, 6)$. find all scalars k such that $\|kv\| = 5$.

Sol:

$kv = (-2k, 3k, 0, 6k)$

$\|kv\| = \sqrt{4k^2 + 9k^2 + 0 + 36k^2}$

$\|kv\| = \sqrt{49k^2}$

as

$\|kv\| = 5$

$\sqrt{49k^2} = 5$

Squaring on b.s

$49k^2 = 25$

$k^2 = \frac{25}{49}$

$$\boxed{k = \pm \frac{5}{7}}$$

In exersice 9-10, find U·V, U·U, V·V.

(9). (a). U = (3, 1, 4), V = (2, 2, -4)

Sol:

$$U \cdot V = U^T V = \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix} [2, 2, -4] = 6+2-16 = -8.$$

$$U \cdot U = U^T \cdot U = \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix} [3 \quad 1 \quad 4] = 9+1+16 = 26$$

$$V \cdot V = V^T \cdot V = \begin{bmatrix} 2 \\ 2 \\ -4 \end{bmatrix} [2 \quad 2 \quad -4] = 4+4+16 = 24$$

Ex = 3.3

Orthognality

In exercise 3-6, find a point-normal form of equation of line passing through P has a normal

(5) $\Rightarrow P(-1, 3, -2)$; $n = (-2, 1, -1)$

Sol:

As point normal form is

$\Rightarrow -2(x-(-1)) + 1(y-3) - 1(z-(-2))$

$\Rightarrow -2(x+1) + (y-3) - 1(z+2)$ Ans.

In Exercise 13-14 find $\|proj_a\, u\|$.

(13) $\Rightarrow u = (1, -2)$; $a = (-4, -3)$

Sol:

$$\|proj_a\, u\| = \frac{u \cdot a}{\sqrt{\|a\|}}$$

$u \cdot a = (1, -2) \cdot (-4, -3)$

$u \cdot a = (-4, 6)$

$\sqrt{\|a\|} = \sqrt{(-4)^2 + (-3)^2}$

$= \sqrt{16+9}$

$= \sqrt{25}$

$= \pm 5$

$$\|proj_a\, u\| = \pm \frac{2}{5}$$

In Exercise 15-20, find the vector component of u along a and the vector component of u orthognal to a:

(17) $\Rightarrow$ $u = (3, 1, -7)$, $a = (1, 0, 5)$

Sol:

vector component of u along a is,

$$\text{proj}_a u = \frac{u \cdot a}{\|a^2\|} \cdot \hat{a}$$

$$u \cdot a = (3, 1, -7) \cdot (1, 0, 5)$$
$$= (3 + 0 + -35)$$
$$= -32$$

$$\|a^2\| = \left(\sqrt{(1)^2 + (0)^2 + (5)^2}\right)^2$$
$$= (\sqrt{26})^2$$
$$= 26$$

$$\text{proj}_a u = \frac{-32}{26}(1, 0, 5)$$
$$= \frac{-32}{26}, 0, \frac{-160}{26}$$
$$= \left(\frac{-16}{13}, 0, \frac{-80}{13}\right)$$

Now, u orthognal to a:

$$= u - \text{proj}_a u$$

$$= (3, 1, -7) - \left(\frac{-16}{13}, 0, \frac{-80}{13}\right)$$

$$= \left(3 + \frac{16}{13}, 1 - 0, -7 + \frac{80}{13}\right)$$

$$= \left(\frac{55}{13}, 1, \frac{-11}{13} \right)$$

In Exercise 21-24 find the distance b/w the points and the line.

21): $(-3, 1)$; $4x + 3y + 4 = 0$

Sol:

Distance b/w point and line is:

$$= \frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}}$$

So,

$$= \frac{|4(-3) + 3(1) + 4|}{\sqrt{(4)^2 + (3)^2}}$$

$$= \frac{|-12 + 3 + 4|}{\sqrt{25}}$$

$$= \frac{|-5|}{5}$$

$$= 1$$

In Exercise 25-26, find the distance b/w the points and the plane.

25) $(3, 1, -2)$; $x + 2y - 2z = 4$

Sol:

Distance b/w point and the plane is,

$$\frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

Now

$$x + 2y - 2z - 4 = 0$$

and Distance is;

$$D = \frac{|(3) + 2(1) - 2(-2) - 4|}{\sqrt{(1)^2 + (2)^2 + (-2)^2}}$$

$$D = \frac{|3 + 2 + 4 - 4|}{\sqrt{9}}$$

$$D = \frac{5}{3}$$

In Exercise 27-28, find the distance b/w the given parallel planes.

27. $2x-y-z=5$; $-4x+2y+2z=12$

Sol:

To find the distance D, b/w the planes, we can select to arbitrary point in one of the plane and compute its distance to the other plane. By setting $y=z=0$ in the eq, ~~x=2y~~ $2x-y-z=5$, we obtain point $P_0(2,0,0)$ in plane. As distance b/w P_0 and plane $-4x+2y+2z=12$ is

$$D = \frac{|-4(2)+2(0)+2(0)-12|}{\sqrt{(-4)^2+(2)^2+(2)^2}}$$

$$D = \frac{|-8+0+0-12|}{\sqrt{24}}$$

$$D = \frac{|-20|}{\sqrt{24}}$$

$$D = \frac{20}{\sqrt{24}}$$

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29): Find a unit vector that is orthognal to both $u = (1, 0, 1)$ and $v = (0, 1, 1)$.

Sol:

To find unit vector that is orthognal to both u and v we use,

$$= \frac{u \times v}{|u \times v|}$$

~~$u \times v = (1, 0, 1) \cdot (0, 1, 1)$~~

~~$= (1$~~

$$u \times v = \begin{vmatrix} i & j & k \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix}$$

$$u \times v = i(0-1) - j(1-0) + k(1-0)$$

$$u \times v = -i - j + k$$

$$u \times v = (-1, -1, 1)$$

$$|u \times v| = \sqrt{(-1)^2 + (-1)^2 + (1)^2}$$

$$= \sqrt{1+1+1}$$

$$= \sqrt{3}$$

So, unit vector is:= $\frac{1}{\sqrt{3}}(-1, -1, 1)$

$$= \boxed{\left(\frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)}$$

Ex: 3.4

In Exercise 1-4 find vector and parametric equation of the line containing the point and parallel to vector

(1). Point: $(-4, 1)$; vector: $v = (0, -8)$

Sol: ∴ vector eq: $(x, y) = \text{point} + t(\text{vector})$

Vector eq: $(x, y) = (-4, 1) + t(0, -8)$

parametric eq: $x = -4$; $y = 1 - 8t$.

In exercise 9-12 find vector and parametric eq of the plane that contains the given point and is parallel to two vectors.

9). Point: $(-3, 1, 0)$; vectors: $v_1 = (0, -3, 6)$ $v_2(-5, 1, 2)$

Sol:

Vector eq: $(x, y, z) = \text{point} + t_1 v_1 + t_2 v_2$

$(x, y, z) = (-3, 1, 0) + t_1(0, -3, 6) + t_2(-5, 1, 2)$

parametric eq: $x = -3 + 0t_1 - 5t_2$; $y = 1 - 3t_1 + t_2$

$z = 0 + 6t_1 + 2t_2$

In exercise 13-14, find vector and parametric eq. of the line in R^2 that passes through the origin and is orthagonal to v.

(13) ⇒ $v = (-2, 3)$

Sol:

As vector eq. of line is $x = tv$, so let $x = (x, y)$ eq. becomes:

$(x, y) = t(2, 3)$

v_1
v_2
v_3

and parametric eq becomes:

$x = -2t \quad , \quad y = 3t$

Ex: 3.5

In exercise 1-2, let $u = (3, 2, -1)$, $v = (0, 2, -3)$ and $w = (2, 6, 7)$. Compute indicated vectors.

(1) (a): $v \times w$

Sol:

$$v \times w = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 2 & -3 \\ 2 & 6 & 7 \end{vmatrix}$$

$$v \times w = \hat{i}\begin{vmatrix} 2 & -3 \\ 6 & 7 \end{vmatrix} - \hat{j}\begin{vmatrix} 0 & -3 \\ 2 & 7 \end{vmatrix} + \hat{k}\begin{vmatrix} 0 & 2 \\ 2 & 6 \end{vmatrix}$$

$$= \hat{i}(14+18) - \hat{j}(0+6) + \hat{k}(0-4)$$

$$= 32\hat{i} - 6\hat{j} - 4\hat{k}$$

$$= (32, -6, -4)$$

(b), (c), (d), (e) and (f) same as a.

In Exercise 7-8, use the cross product to find a vector that is orthagonal to both U and V.

7). $U = (-6, 4, 2)$, $V = (3, 1, 5)$

Sol=

$$U \times V = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -6 & 4 & 2 \\ 3 & 1 & 5 \end{vmatrix}$$

$$= \hat{i}\begin{vmatrix} 4 & 2 \\ 1 & 5 \end{vmatrix} - \hat{j}\begin{vmatrix} -6 & 2 \\ 3 & 5 \end{vmatrix} + \hat{k}\begin{vmatrix} -6 & 4 \\ 3 & 1 \end{vmatrix}$$

$$= \hat{i}(20-2) - \hat{j}(-30-6) + \hat{k}(-6-12)$$

$$= 18\hat{i} + 36\hat{j} - 18\hat{k}$$

$$= (18, 36, -18)$$

In exercise 9-10 Find area of parallelogram

9). $U = (1, -1, 2)$, $V = (0, 3, 1)$

Sol= $\therefore$ Area of parallelogram $= \frac{U \times V}{\|U \times V\|}$

$$U \times V = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 2 \\ 0 & 3 & 1 \end{vmatrix}$$

$$= \hat{i}\begin{vmatrix} -1 & 2 \\ 3 & 1 \end{vmatrix} - \hat{j}\begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} + \hat{k}\begin{vmatrix} 1 & -1 \\ 0 & 3 \end{vmatrix},$$

$$= \hat{i}(-1-6) - \hat{j}(1-0) + \hat{k}(3+0)$$

$$= -7\hat{i} - \hat{j} + 3\hat{k}$$

$$= (-7, -1, 3)$$

$$\|U \times V\| = \sqrt{(-7)^2 + (1)^2 + (3)^2}$$

$$= \sqrt{49 + 1 + 9}$$

$$= \sqrt{59}$$

In Exercise 15-16: Find the area of the triangle in 3-space that has given vertices.

15): $P_1(2, 6, -1)$, $P_2(1, 1, 1)$, $P_3(4, 6, 2)$

Sol:-

Area of triangle is $= \frac{1}{2} \|\overrightarrow{P_1P_2} \times \overrightarrow{P_2P_3}\|$

$\overrightarrow{P_1P_2} = P_2 - P_1 = (-1, -5, 2)$

$\overrightarrow{P_2P_3} = P_3 - P_2 = (3, 5, 1)$

Now

$$\overrightarrow{P_1P_2} \times \overrightarrow{P_2P_3} = \begin{vmatrix} i & j & k \\ -1 & -5 & 2 \\ 3 & 5 & 1 \end{vmatrix}$$

$$= \hat{i}(-5+10) - \hat{j}(-1+6) + \hat{k}(-5+15)$$

$$= 5\hat{i} - 5\hat{j} + 20k$$

$$= (5, -5, 20)$$

$$\|\overrightarrow{P_1P_2} \times \overrightarrow{P_2P_3}\| = \sqrt{(5)^2 + (5)^2 + (20)^2}$$

$$= \sqrt{374}$$

Area of triangle $= \frac{\sqrt{174}}{2}$

In Exercise 17-18 Find area of parallelepiped.

17) => $U = (2, -6, 2)$, $V = (0, 4, -2)$, $W = (2, 2, 4)$

Sol=

Area of parallelepiped $= |U \cdot (V \times W)|$

$$U \cdot V \times W = \begin{vmatrix} 2 & -6 & 2 \\ 0 & 4 & -2 \\ 2 & 2 & 4 \end{vmatrix}$$

$$U \cdot V \times W = 2\begin{vmatrix} 4 & -2 \\ 2 & 4 \end{vmatrix} - (-6)\begin{vmatrix} 0 & -2 \\ 2 & 4 \end{vmatrix} + 2\begin{vmatrix} 0 & 4 \\ 2 & 2 \end{vmatrix}$$

$$= 2(-16+4) + 6(0+4) + 2(0-8)$$

$$= 2(-12) + 24 + (-16)$$

$$= -\cancel{24} + \cancel{24} - 16$$

$$= -16$$

Now,

$$|U \cdot V \times W| = |-16|$$

$$= 16$$

In Exercise 21-24 compute the scalar triple product $U\cdot(V\times W)$.

21) ⇒ $U=(-2,0,6)$, $V=(1,-3,1)$, $W=(-5,-1,1)$

Sol:=

$$U\cdot(V\times W)=\begin{vmatrix}-2 & 0 & 6\\ 1 & -3 & 1\\ -5 & -1 & 1\end{vmatrix}$$

$$=-2\begin{vmatrix}-3 & 1\\ -1 & 1\end{vmatrix}-0\begin{vmatrix}1 & 1\\ -5 & 1\end{vmatrix}+6\begin{vmatrix}1 & -3\\ -5 & -1\end{vmatrix}$$

$$=-2(-3+1)-0+6(-1-15)$$

$$=4-0-96$$

$$=-92 \quad \text{Ans}$$