Section 4.8

Theorem: The rowspace and column space of a matrix A have the same dimension.

Definition The common dimension of the row space and column space of a matrix A is called rank of A and is denoted by rank(A); the dimension of the null space of A is called the nullity of A and is denoted by nullity (A)

Remark (The no. of non-zero rows in reduced echelon/echelon form is called rank)

Example 1

Find rank and nullity of matrix

$$A = \begin{bmatrix} -1 & 2 & 0 & 4 & 5 & -3 \\ 3 & -7 & 2 & 0 & 1 & 4 \\ 2 & -5 & 2 & 4 & 6 & 1 \\ 4 & -9 & 2 & -4 & -4 & 7 \end{bmatrix}$$

Solution

The reduced row echelon form of A is

$$\begin{bmatrix} 1 & 0 & -4 & -28 & -37 & 13 \\ 0 & 1 & -2 & -12 & -16 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

As there are two rows with leading 1's.

$\Rightarrow$ dim (Row space) = dim (Column space) = rank $A = 2$

For dimension of null space of A, writing in equation form

$$x_1 - 4x_3 - 28x_4 - 37x_5 + 13x_6 = 0$$
$$x_2 - 2x_3 - 12x_4 - 16x_5 + 5x_6 = 0$$

$\rightarrow x_1 = 4x_3 + 28x_4 + 37x_5 - 13x_6$
$x_2 = 2x_3 + 12x_4 + 16x_5 - 5x_6$

put $x_3 = r$
$x_4 = s$
$x_5 = t$
$x_6 = u$

$\rightarrow x_1 = 4r + 28s + 37t - 13u$
$x_2 = 2r + 12s + 16t - 5u$

i.e

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = r\begin{bmatrix} 4 \\ 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} 28 \\ 12 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 37 \\ 16 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + u\begin{bmatrix} -13 \\ -5 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Hence four vector form basis of null space of A. Therefore nullity $(A) = 4$

Remark (Maximum Value for Rank)
Let A be a matrix of order $m \times n$.
$$\text{rank}(A) \le \min(m, n)$$

Theorem (Dimension theorem for Matrices)
If A is a matrix with n columns, then
$$\text{rank}(A) + \text{nullity}(A) = n.$$

In previous example 1

$\text{rank}(A) + \text{nullity } A = 2 + 4 = 6 =$ no. of columns of A.

Theorem If A is an $m \times n$ matrix. then

(a) rank (A) = the number of leading variables in the general solution of $AX = 0$

(* leading variables are those corresponding to leading 1's in echelon form)

(b) nullity (A) = the number of parameters in general solution of $AX = 0$

Example 2

(a) Find the number of parameters in general solution of $Ax = 0$ if A is a 5×7 matrix of rank 3.

(b) Find the rank of a 5×7 matrix A for which $AX = 0$ has a two-dimensional solution space.

Solution

(a) Given that rank $(A) = 3$

no. of colums $= n = 7$

$\because$ rank (A) + nullity $(A) = n$ —— (1)

$\Rightarrow$ 3 + nullity $(A) = 7$

$\Rightarrow$ nullity $(A) = 7 - 3 = 4$

$\Rightarrow$ no. of parameters $= 4$

(b) Given that no. of columns $= n = 7$

nullity $(A) = 2$

rank $(A) = ?$

$\because$ rank(A) + nullity $(A) = n \Rightarrow$ rank$(A) = 7 - 2 = 5$

Theorem

For any matrix A,
$$\operatorname{rank}(A) = \operatorname{rank}(A^T)$$

Definition

If W is a subspace of $\mathbb{R}^n$, then the set of all vectors in $\mathbb{R}^n$ that are orthogonal to every vector in W are called the orthogonal complement of W and is denoted by $W^\perp$.

Theorem

If W is a subspace of $\mathbb{R}^n$. Then

(a) $W^\perp$ is a subspace of $\mathbb{R}^n$.

(b) The only vector common to W and $W^\perp$ is 0.

(c) The orthogonal complement of $W^\perp$ is W.

Theorem Let A be an $m \times n$ matrix.

(a) (Overdetermined case). If $m > n$ then the linear system $Ax = b$ is inconsistent for atleast one vector b in $\mathbb{R}^n$.

(b) If $m < n$, then for each vector b in $\mathbb{R}^m$ the linear system $Ax = b$ is either inconsistent or has infinitely many solutions.

The row space and null space are orthognal compliment of each other.

Exercise set 4.8

Q # 2 (a). Find rank and nullity of matrix A by reducing it to row echelon form.

$$A = \begin{bmatrix} 1 & 0 & -2 & 1 & 0 \\ 0 & -1 & -3 & 1 & 3 \\ -2 & -1 & 1 & -1 & 3 \\ 0 & 1 & 3 & 0 & -4 \end{bmatrix}$$

$$\underset{R}{\sim} \begin{bmatrix} 1 & 0 & -2 & 1 & 0 \\ 0 & -1 & -3 & 1 & 3 \\ 0 & -1 & -3 & 1 & 3 \\ 0 & 1 & 3 & 0 & -4 \end{bmatrix} R_3 + 2R_1$$

$$\underset{R}{\sim} \begin{bmatrix} 1 & 0 & -2 & 1 & 0 \\ 0 & 1 & 3 & -1 & -3 \\ 0 & -1 & -3 & 1 & 3 \\ 0 & 1 & 3 & 0 & -4 \end{bmatrix} -1\,R_2$$

$$\underset{R}{\sim} \begin{bmatrix} 1 & 0 & -2 & 1 & 0 \\ 0 & 1 & 3 & -1 & -3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix} \begin{matrix} \\ R_3 + R_2 \\ R_4 - R_2 \\ \\ \end{matrix}$$

Hence $\quad \text{rank}(A) = 3 \quad$ (no. of rows with leading 1's)

Here $\quad n = 5 \quad$ (no. of columns)

$$\text{nullity}(A) = n - \text{rank}(A) = 5 - 3 = 2$$

<u>Q # 1 (a), (b), Q # 2 (b)</u> Do yourself

(6) Q#6 Given matrix A and its reduced echelon form R, (a) Find rank and nullity of A using R
(b). Confirm the formula rank A + nullity$(A) = n$
(c) Find no. of leading variables and no. of parameters in general solution of $AX=0$ without solving the system.

$$A = \begin{bmatrix} 0 & 2 & 2 & 4 \\ 1 & 0 & -1 & -3 \\ 2 & 3 & 1 & 1 \\ -2 & 1 & 3 & 2 \end{bmatrix}, \quad R = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Solution

(a) using R

$\boxed{\text{rank}(R) = \text{rank}(A) = 3}$

For nullity, by backward substitution

$x_4 = 0$

$x_2 + x_3 = 0 \Rightarrow x_2 = -x_3$

$x_1 - x_3 = 0 \Rightarrow x_1 = x_3$

$\boxed{\text{put } x_3 = t}$

$$\begin{aligned} x_1 &= t \\ x_2 &= -t \\ x_3 &= t \\ x_4 &= 0 \end{aligned} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = t \begin{bmatrix} +1 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$

Thus $\boxed{\text{nullity}(A) = 1}$

(b) Now rank(A) + nullity$(A) = 3+1 = 4$ = no. of columns of A

(c) no. of leading variables in general solution of $Ax=0$
$= \text{rank}(A) = 3$

no. of parameters in general solution of $Ax=0$
$= \text{nullity}(A) = 1$

Q# 3,4,5 (Same as 6) Do yourself.

Q#8: If A is an $m\times n$ matrix what is largest possible value for its rank and smallest possible value for its nullity?

Solution: If A is $m\times n$ matrix

$\because \text{rank}(A) \leq \min(m,n)$

$\Rightarrow \boxed{\text{largest possible value of rank } A = \min(m,n)}$

$\because n = \text{rank}(A) + \text{nullity } A$

$\Rightarrow \text{nullity } A = n - \text{rank}(A)$ $\left(\begin{array}{l} \because \text{rank}(A) \leq \min(m,n) \\ \Rightarrow -\text{rank}(A) \geq -\min(m,n) \end{array}\right.$

$\Rightarrow \text{nullity } A = n - \text{rank}(A) \geq n - \min(m,n)$

$\Rightarrow \boxed{\text{smallest possible value of nullity } A = n - \min(m,n)}$

Q#7 (using Q#8)

Q#7(b) Find largest possible value for rank A and smallest possible value for nullity of A?
A is 3×5.

Largest possible value of rank $A = \min(3,5) = 3$.
smallest possible value of $\text{nullity}(A) = n - \min(m, n)$
$= 5 - 3$
$= 2$

Q#7(a),(c) Do yourself

S#11(b)

Find an equation relating nullity(A) and nullity(A^T) for a general $m \times n$ matrix

Solution

If A is $m \times n$ matrix
$\rightarrow A^T$ is $n \times m$ matrix

Therefore

$\text{rank}\, A + \text{nullity}(A) = n$ —— ①
$\text{rank}(A^T) + \text{nullity}(A^T) = m$ —— ②
using $\text{rank}(A) = \text{rank}(A^T)$ in ②

$\text{rank}(A) + \text{nullity}(A^T) = m$ —— ③
subtracting ① from ③

$$\boxed{\text{nullity}(A^T) - \text{nullity}(A) = m - n}$$

Q#9(a) Given size of $A = 3 \times 3$
rank $(A) = 3$
rank $(A|b) = 3$

(i) find dimension of row space of A, column space of A, null space of A and null space of A^T

(ii) determine whether or not the linear system $Ax = b$ is consistent.

(iii) find number of parameters in general solution of $Ax = b$ if it is consistent.

Solution

(i) * dimension of row space of A = dimension of column space of A
$= \text{rank}(A)$
$= 3$

no. of columns in $A = n = 3$

* dim of null space of A = nullity of A
$= n - \text{rank}(A)$
$= 3 - 3$
$= 0$

* dimension of null space of A^T
$= \text{nullity}(A^T)$
$= m - \text{rank}(A)$
$= 3 - 3 = 0$

(ii) Since rank $A = \text{rank}(A|b) = 3$
$\Rightarrow Ax = b$ is consistent.

(iii) number of parameters in $(Ax = b)$ if consistent
$=$ number of parameters in $(Ax = 0)$
$=$ nullity of A
$= 0$

Q#9 (e)

size of $A = 5 \times 9$

rank $A = 2$

Rank $(A|b) = 3$

(i) * dim Row space of A

= dim Column space of A

= rank (A)

= 2

* nullity $A = n - \text{rank}(A)$

$= 9 - 2$

$= 7$

* nullity $A^T = m - \text{rank}(A)$

$= 5 - 2$

$= 3$

(ii) Since rank$(A) \neq$ rank$(A|b)$

$\Rightarrow Ax = b$ is not consistent.

(iii) $\because Ax = b$ is not consistent

we do not have to find no. of parameters.

Q#10: Do yourself.

Q#11 (Hint))

Let $T: \mathbb{R}^2 \to \mathbb{R}^3$ be linear transformation defined by

$$T(x_1, x_2) = (x_1 + 3x_2, x_1 - x_2, x_1)$$

In matrix form $Tx = y$.

$$\begin{bmatrix} 1 & 3 \\ 1 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + 3x_2 \\ x_1 - x_2 \\ x_1 \end{bmatrix}$$

$$T = \begin{bmatrix} 1 & 3 \\ 1 & -1 \\ 1 & 0 \end{bmatrix}$$ is standard matrix for T

Solve (a) & (b) for this matrix.

Q#13 (Hint)

$$T(x_1, x_2, x_3, x_4, x_5) = (x_1 + x_2, x_2 + x_3 + x_4, x_4 + x_5)$$

$$Tx = y$$

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ x_2 + x_3 + x_4 \\ x_4 + x_5 \end{bmatrix}$$

$$T = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$ is standard matrix.

Q#20: Let A be a 7×6 matrix such that $AX=0$ has only trivial solution. Find rank and nullity of A.

Solution

Here $m=7$
$n=6$

$Ax=0$ has only trivial solution.

$\rightarrow$ $\boxed{\text{nullity } A = 0}$

Now
$$\text{rank } A = n - \text{nullity } A$$
$$= 6-0$$
$$= 6$$

Q#21 Do yourself.

Q#27: Try yourself


Imposter