Theorem: A system of linear equations $Ax = b$ is consistent if and only if b is in the column space of A.

i.e b can be expressed as linear combination of the column vectors of A.

Example 2 page 238: Do yourself

Q#3(a) Determine whether b is in column space of A and if so express b as a linear combination of the column vectors of A. $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{bmatrix}$, $b = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$

Solution

b is in column space of A iff $Ax = b$ has a solution.

Solving

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} \quad \text{——— (1)}$$

using Gauss-elimination method

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & -1 \\ 1 & 0 & 1 & 0 \\ 2 & 1 & 3 & 2 \end{array}\right]$$

$$\underset{\sim}{R} \left[\begin{array}{ccc|c} 1 & 1 & 2 & -1 \\ 0 & -1 & -1 & 1 \\ 0 & -1 & -1 & 4 \end{array}\right] \begin{array}{l} R_2 - R_1 \\ R_3 - 2R_1 \end{array}$$

$$\underset{\sim}{R} \left[\begin{array}{ccc|c} 1 & 1 & 2 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & -1 & -1 & 4 \end{array}\right]$$

②

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 3 \end{array}\right]$$

which has no solution.

Hence b is not in Solution space.

Q#3 (b) $A = \begin{bmatrix} 1 & -1 & 1 \\ 9 & 3 & 1 \\ 1 & 1 & 1 \end{bmatrix}$, $b = \begin{bmatrix} 5 \\ 1 \\ -1 \end{bmatrix}$

Solution

b is in Column space of A iff $Ax = b$ has a Solution.

$$Ax = b$$

$$\begin{bmatrix} 1 & -1 & 1 \\ 9 & 3 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ -1 \end{bmatrix}$$

Solving by Gauss's elimination method

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 5 \\ 9 & 3 & 1 & 1 \\ 1 & 1 & 1 & -1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 1 & 5 \\ 0 & 12 & -8 & -44 \\ 0 & 2 & 0 & -6 \end{array}\right] \begin{array}{l} \\ R_2 - 9R_1 \\ R_3 - R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 1 & 5 \\ 0 & 2 & 0 & -6 \\ 0 & 12 & -8 & -44 \end{array}\right] \quad R_2 \leftrightarrow R_3$$

$$\underset{\sim}{R}\left[\begin{array}{ccc|c} 1 & -1 & 1 & 5 \\ 0 & 1 & 0 & -3 \\ 0 & 12 & -8 & -44 \end{array}\right]$$

$$\underset{\sim}{R}\left[\begin{array}{ccc|c} 1 & -1 & 1 & 5 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & -8 & -8 \end{array}\right] R_3 - 12R_2$$

By backward substitution

$$-8x_3 = -8$$

$$\boxed{x_3 = 1}$$

$$\boxed{x_2 = -3}$$

$$x_1 - x_2 + x_3 = 5$$

$$x_1 - (-3) + 1 = 5$$

$$x_1 + 3 + 1 = 5$$

$$x_1 + 4 = 5$$

$$x_1 = 5 - 4$$

$$\boxed{x_1 = 1}$$

Now b can be written as linear Combination of columns of A as

$$x_1 c_1 + x_2 c_2 + x_3 c_3 = b$$

ie $$1\begin{bmatrix} 1 \\ 9 \\ 1 \end{bmatrix} + (-3)\begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix} + (1)\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ -1 \end{bmatrix}$$

<u>Q#4(a), (b)</u>. Do yourself

Theorem: (Relation b/w Solution of $Ax=b$ and $Ax=0$)

The general solution of a consistent linear system can be expressed as the sum of a particular solution of that system and the general solution of corresponding homogeneous system. i.e

$$x = x_0 + x_h$$

where x is a general solution of $Ax=b$
x_0 is a particular solution of $Ax=b$
x_h is a general solution of $Ax=0$

Example 3: Try Yourself

Q#8(a) Find vector form of general solution of linear system $Ax=b$ and then use that result to find the vector form of general solution of $Ax=0$.

$$x_1 - 2x_2 + x_3 + 2x_4 = -1$$
$$2x_1 - 4x_2 + 2x_3 + 4x_4 = -2$$
$$-x_1 + 2x_2 - x_3 - 2x_4 = 1$$
$$3x_1 - 6x_2 + 3x_3 + 6x_4 = -3$$

$$\Rightarrow \left[\begin{array}{cccc:c} 1 & -2 & 1 & 2 & -1 \\ 2 & -4 & 2 & 4 & -2 \\ -1 & 2 & -1 & -2 & 1 \\ 3 & 6 & 3 & 6 & -3 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 1 & -2 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 12 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 + R_1 \\ R_4 - 3R_1 \end{array}$$

by backward substitution

$$12x_2 = 0$$

$$\Rightarrow \boxed{x_2 = 0}$$

$$x_1 - 2x_2 + x_3 + 2x_4 = -1$$

$$x_1 - 0 + x_3 + 2x_4 = -1$$

$$\Rightarrow x_1 = -1 - x_3 - 2x_4$$

put $x_3 = t$; $x_4 = s$

$$\Rightarrow x_1 = -1 - t - 2s$$

Now general solution of system in column vector form

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -1 - t - 2s \\ 0 \\ t \\ s \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{} + \underbrace{t\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \end{bmatrix}}_{}$$

★ Remark: The column with no multiplication of parameter is particular solution x_0 and remaining sum of columns is x_h.

$$\Rightarrow \quad x = x_0 + x_h$$

Thus the general solution of homogeneous system $Ax = 0$ is

$$x_h = t\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

<u>Verification</u> (No need to do it in paper. Just to show you that x_h is really the solution of $Ax=0$)

Consider $AX=0$

$$x_1-2x_2+x_3+2x_4=0$$
$$2x_1-4x_2+2x_3+4x_4=0$$
$$-x_1+2x_2-x_3-2x_4=0$$
$$3x_1-6x_2+3x_3+6x_4=0$$

$$\left[\begin{array}{cccc|c} 1 & -2 & 1 & 2 & 0 \\ 2 & -4 & 2 & 4 & 0 \\ -1 & 2 & -1 & -2 & 0 \\ 3 & +6 & 3 & 6 & 0 \end{array}\right]$$

$$R_1 \left[\begin{array}{cccc|c} 1 & -2 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 12 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \\ R_2-2R_1 \\ R_3+R_1 \\ R_4-3R_1 \end{array}$$

By backward substitution.

$$12x_2=0$$
$$\Rightarrow \boxed{x_2=0}$$

$$x_1-2x_2+x_3+2x_4=0.$$
$$x_1-2(0)+x_3+2x_4=0.$$
$$\Rightarrow x_1=-x_3-2x_4$$
$$\Rightarrow x_1=-t-2s \quad ; \text{ put } x_3=t;\ x_4=s$$

$$x_h=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}=\begin{bmatrix} -t-2s \\ 0 \\ t \\ s \end{bmatrix}=t\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}+s\begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

As expected.

H.W Q# 7(a), (b), Q#8(b)

→ Q#21, 22 are same like Q#7, 8
You have to Construct System yourself first.
for example.

Q#21(b)

$$AX = b$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 0 \\ 1 & -1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

then solve for general solution

Q#22(b)

$$AX = b$$

$$\Rightarrow \begin{bmatrix} 2 & 0 \\ 0 & 1 \\ 1 & 1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}$$

then solve for general solution.

Exercise set 4.7 Q# 1,2 are same.

Q#2(a) Express the product Ax as a linear combination of column vectors of A.

$$\begin{bmatrix} -3 & 6 & 2 \\ 5 & -4 & 0 \\ 2 & 3 & -1 \\ 1 & 8 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix} = -1 \begin{bmatrix} -3 \\ 5 \\ 2 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 6 \\ -4 \\ 3 \\ 8 \end{bmatrix} + 5 \begin{bmatrix} 2 \\ 0 \\ -1 \\ 3 \end{bmatrix}$$

Q#1(a), (b), Q#2(b) Do yourself

Q#5,6 are Same

(8)

Q# 6: Suppose $x_1=-1, x_2=2, x_3=4; x_4=-3$ is a solution of nonhomogeneous linear system $Ax=b$ and that the solution set of the homogeneous system $Ax=0$ is given by the formulas

$x_1=-3r+4s$; $x_2=r-s$; $x_3=r$; $x_4=s$

(a) Find vector form of general solution of $Ax=0$.
(b) Find a vector form of general solution of $Ax=b$.

Solution

Given that $x_0=\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=\begin{bmatrix}-1\\2\\4\\-3\end{bmatrix}$

$$x_h=\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=\begin{bmatrix}-3r+4s\\r-s\\r\\s\end{bmatrix}=r\begin{bmatrix}-3\\1\\1\\0\end{bmatrix}+s\begin{bmatrix}4\\-1\\0\\1\end{bmatrix}$$

(a) $x=x_h=r\begin{bmatrix}-3\\1\\1\\0\end{bmatrix}+s\begin{bmatrix}4\\-1\\0\\1\end{bmatrix}$

(b) $x=x_0+x_h=\begin{bmatrix}-1\\2\\4\\-3\end{bmatrix}+r\begin{bmatrix}-3\\1\\1\\0\end{bmatrix}+s\begin{bmatrix}4\\-1\\0\\1\end{bmatrix}$

Definition

An inner product on a real vector space V is a function that associates a real number $\langle u, v \rangle$ with each pair of vectors in V in such a way that the following axioms are satisfied for all vectors u, v and w in V and all scalars k.

1. $\langle u, v \rangle = \langle v, u \rangle$
2. $\langle u+v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$
3. $\langle ku, v \rangle = k \langle u, v \rangle$
4. $\langle v, v \rangle \geq 0$ and $\langle v, v \rangle = 0$ iff $v = 0$

A real vector space with an inner product is called real inner product space.

Definition

If V is a real inner product space, then the norm of a vector v in V is denoted by $\|v\|$ and is defined by

$$\|v\| = \sqrt{\langle v, v \rangle}$$

Examples

(i) The standard inner product on M_{nn}

Let $u, v \in M_{nn}$ be square matrices of order $n \times n$

then

$$\langle u, v \rangle = tr(u^T v)$$

Example

Let $U = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $V = \begin{bmatrix} -1 & 0 \\ 3 & 2 \end{bmatrix}$; $\langle u, v \rangle = ?$
$\|u\| = ?$
$\|v\| = ?$

$$\langle u, v \rangle = tr(u^T v)$$
$$= tr\left\{ \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 3 & 2 \end{bmatrix} \right\}$$
$$= tr\left\{ \begin{bmatrix} -1+9 & 0+6 \\ -2+12 & 0+8 \end{bmatrix} \right\}$$
$$= tr \begin{bmatrix} 8 & 6 \\ 10 & 8 \end{bmatrix}$$
$$= 8+8$$
$$\boxed{\langle u, v \rangle = 16}$$

$\|u\| = \sqrt{\langle u, u \rangle}$ —— (1)

Now

$$\langle u, u \rangle = tr(u^T u)$$
$$= tr\left\{ \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \right\}$$
$$= tr\left\{ \begin{bmatrix} 1+9 & 2+12 \\ 2+12 & 4+16 \end{bmatrix} \right\}$$
$$= tr \begin{bmatrix} 10 & 14 \\ 14 & 20 \end{bmatrix}$$
$$= 10+20 = 30$$

using in (1)

$$\boxed{\|u\| = \sqrt{30}}$$

$\|v\| = \sqrt{\langle v, v \rangle}$ —— (2)

Now

$$\langle v, v \rangle = tr(v^T v)$$
$$= tr\left\{ \begin{bmatrix} -1 & 3 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 3 & 2 \end{bmatrix} \right\}$$
$$= tr \begin{bmatrix} 1+9 & 0+6 \\ 0+6 & 0+4 \end{bmatrix}$$
$$= tr \begin{bmatrix} 10 & 6 \\ 6 & 4 \end{bmatrix}$$
$$= 10+4$$
$$= 14$$

using in (2)

$$\boxed{\|v\| = \sqrt{14}}$$

(ii) The standard inner product on P_n

$p = a_0 + a_1 x + \cdots + a_n x^n$

$q = b_0 + b_1 x + \cdots + b_n x^n$

$$\boxed{\langle p, q \rangle = a_0 b_0 + a_1 b_1 + \cdots + a_n b_n}$$

Also $\langle p, p \rangle = a_0 a_0 + a_1 a_1 + \cdots + a_n a_n$

$= a_0^2 + a_1^2 + \cdots + a_n^2$

$$\boxed{\|p\| = \sqrt{\langle p, p \rangle} = \sqrt{a_0^2 + a_1^2 + \cdots + a_n^2}}$$

example let $V = P_2$

$p(x) = x^2$; $q(x) = 1 + x$

Here $a_0 = 0$; $a_1 = 0$; $a_2 = 1$

$b_0 = 1$; $b_1 = 1$; $b_2 = 0$

$\langle p, q \rangle = (0)(1) + (0)(1) + (1)(0) = 0$

$\|p\| = \sqrt{\langle p, p \rangle} = \sqrt{0^2 + 0^2 + 1^2} = \sqrt{1} = 1$

$\|q\| = \sqrt{\langle q, q \rangle} = \sqrt{1^2 + 1^2 + 0^2} = \sqrt{1+1} = \sqrt{2}$

(iii) The standard inner product on R^n

let $u, v \in R^n$

$u = (u_1, u_2, \ldots, u_n)$

$v = (v_1, v_2, \ldots, v_n)$

Now

$\langle u, v \rangle = u \cdot v = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$

$\|u\| = \sqrt{\langle u, u \rangle} = \sqrt{u_1^2 + u_2^2 + \cdots + u_n^2}$

Angle and orthogonality in inner product Spaces

Let V be a vector space. $u, v \in V$ then angle b/w u and v is defined by

$$\theta = \cos^{-1}\left(\frac{\langle u, v\rangle}{\|u\|\,\|v\|}\right)$$

or $\cos\theta = \dfrac{\langle u, v\rangle}{\|u\|\,\|v\|}$

Example

Let M_{22} have standard inner product.

Find the cosine of angle between the vectors

$$U = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad V = \begin{bmatrix} -1 & 0 \\ 3 & 2 \end{bmatrix}$$

Solution As solved earlier

$$\langle u, v\rangle = 16, \quad \|u\| = \sqrt{30}, \quad \|v\| = \sqrt{14}$$

$$\cos\theta = \frac{\langle u, v\rangle}{\|u\|\,\|v\|} = \frac{16}{\sqrt{30}\sqrt{14}} \approx 0.78$$

Definition

Two vectors u and v in an inner product space V called orthogonal if $\langle u, v\rangle = 0$

Example Show that $u = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, $v = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$ are orthogonal.

$$\langle u, v\rangle = tr\{u^T v\}$$

$$= tr\left\{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}\right\}$$

$$= tr\begin{bmatrix} 0+0 & 2+0 \\ 0+0 & 0+0 \end{bmatrix} = tr\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} = 0+0 = 0$$

Hence u, v are orthogonal.

Definition A set of two or more vectors in a real inner product space is said to be orthogonal if all pairs of distinct vectors in the set are orthogonal. An orthogonal set in which each vector has norm 1 is said to be orthonormal.

Example

orthogonal set in $\mathbb{R}^3$

Let $v_1 = (0,1,0)$
$v_2 = (1,0,1)$
$v_3 = (1,0,-1)$

Now $\langle v_1, v_2 \rangle = (0)(1)+(1)(0)+(0)(1) = 0+0+0 = 0$
$\langle v_2, v_3 \rangle = (1)(1)+(0)(0)+(1)(-1) = 1+0-1 = 0$
$\langle v_3, v_1 \rangle = (0)(1)+(1)(0)+(0)(-1) = 0+0+0=0$

$\Rightarrow \{v_1, v_2, v_3\}$ is orthogonal set.

Constructing an orthonormal set

In above example
$\|v_1\| = 1, \|v_2\| = \sqrt{2}, \|v_3\| = \sqrt{2}$

Normalizing u_1, u_2 and u_3, we have

$$u_1 = \frac{v_1}{\|v_1\|} = (0,1,0)$$

$$u_2 = \frac{v_2}{\|v_2\|} = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$$

$$u_3 = \frac{v_3}{\|v_3\|} = \left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right)$$

Note that
$\langle u_1, u_2 \rangle = \langle u_2, u_3 \rangle = \langle u_3, u_1 \rangle = 0$
Also
$\|u_1\| = 1; \|u_2\| = 1; \|u_3\| = 1$
Hence $\{u_1, u_2, u_3\}$ form an orthonormal set.

Theorem

If $S = \{v_1, v_2, \ldots, v_n\}$ is an orthogonal set of non-zero vectors in an inner product space, then S is linearly independent.

Example

$u_1 = (0, 1, 0)$, $u_2 = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$, $u_3 = \left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right)$
form an orthonormal basis of $\mathbb{R}^3$.

Solution $\because$ $\{u_1, u_2, u_3\}$ are orthogonal

$\Rightarrow \{u_1, u_2, u_3\}$ are linearly independent. (by above theorem)

$\because \dim \mathbb{R}^3 = 3$

$\rightarrow$ Hence the three linearly independent vectors $\{u_1, u_2, u_3\}$ form a basis for $\mathbb{R}^3$.

Example 6 page 367 (Do yourself)

The Gram-Schmidt Process

To convert a basis $\{u_1, u_2, \ldots, u_k\}$ into an orthogonal basis $\{v_1, v_2, \ldots, v_k\}$, perform the following computations.

Step I: $v_1 = u_1$

Step 2: $v_2 = u_2 - \dfrac{\langle u_2, v_1 \rangle}{\|v_1\|^2} v_1$

Step 3: $v_3 = u_3 - \dfrac{\langle u_3, v_1 \rangle}{\|v_1\|^2} v_1 - \dfrac{\langle u_3, v_2 \rangle}{\|v_2\|^2} v_2$

$\vdots$

continue for k steps.

* To convert the orthogonal basis into an orthonormal basis $\{q_1, q_2, \dots, q_k\}$, normalize the orthogonal basis vectors.

Example

Assume vector space $\mathbb{R}^3$ has Euclidean inner product. Apply Gram-Schmidt process to transform the basis vectors

$$u_1 = (1,1,1)$$
$$u_2 = (0,1,1)$$
$$u_3 = (0,0,1)$$

into an orthogonal basis $\{v_1, v_2, v_3\}$ and then normalize the orthogonal basis vectors to obtain an orthonormal basis $\{q_1, q_2, q_3\}$.

Sol: **Step 1:** $v_1 = u_1 = (1,1,1)$

Step 2: $$v_2 = u_2 - \cancel{\text{proj}} \frac{\langle u_2, v_1 \rangle}{\|v_1\|^2} v_1$$

$$= (0,1,1) - \frac{1\cdot0 + 1\cdot1 + 1\cdot1}{(\sqrt{1^2+1^2+1^2})^2}(1,1,1)$$

$$= (0,1,1) - \frac{2}{(\sqrt{3})^2}(1,1,1) = (0,1,1) - \left(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}\right)$$

$$= \left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right)$$

Step 3: $$v_3 = u_3 - \frac{\langle u_3, v_1 \rangle}{\|v_1\|^2} v_1 - \frac{\langle u_3, v_2 \rangle}{\|v_2\|^2} v_2$$

$$v_3 = (0,0,1) - \frac{0+0+1}{(\sqrt{1^2+1^2+1^2})^2}(1,1,1) - \frac{0+0+\frac{1}{3}}{(\sqrt{\frac{4}{9}+\frac{1}{9}+\frac{1}{9}})^2}\left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right)$$

$= (0,0,1) - \frac{1}{3}(1,1,1) - \frac{1/3}{2/3}\left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right)$

$\boxed{v_3 = \left(0, -\frac{1}{2}, \frac{1}{2}\right)}$

Now

$v_1 = (1,1,1)$, $v_2 = \left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right)$, $v_3 = \left(0, -\frac{1}{2}, \frac{1}{2}\right)$

form orthogonal basis for $\mathbb{R}^3$.

$\|v_1\| = \sqrt{3}$, $\|v_2\| = \frac{\sqrt{6}}{3}$, $\|v_3\| = \frac{1}{\sqrt{2}}$

so an orthonormal basis for $\mathbb{R}^3$ is

$q_1 = \frac{v_1}{\|v_1\|} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

$q_2 = \frac{v_2}{\|v_2\|} = \left(-\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$

$q_3 = \frac{v_3}{\|v_3\|} = \left(0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

H.W (Exercise 6.3)

Q# 1, 2

Q# 25—31

Orthogonal Matrix

A square matrix is said to be orthogonal if its transpose is the same as its inverse i.e if $A^{-1} = A^T$ or if

$$A^T A = A A^T = I$$

Example

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Now

$$A^T A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta + \sin^2\theta & -\cos\theta\sin\theta + \cos\theta\sin\theta \\ -\cos\theta\sin\theta + \cos\theta\sin\theta & \sin^2\theta + \cos^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Theorem:

(i) The transpose of an orthogonal matrix is orthogonal

(ii) The inverse of an orthogonal matrix is orthogonal.

(iii) The product of orthogonal matrices is orthogonal

(iv) If A is orthogonal, then $\det A = 1$ or $\det A = -1$

Exercise Set 7.1

H.W Q# 1, 2.

<u>Definition</u> If A and B are square matrices then we say that B is orthogonally similar to A if there is an orthogonal matrix P such that $B = P^T A P$.

<u>Definition</u>

If A is orthogonally similar to some diagonal matrix, say

$$P^T A P = D$$

Then we say that A is orthogonally diagonalizable and that P orthogonally diagonalizes A.

<u>Theorem</u>

If A is an $n \times n$ matrix with real entries then following are equivalent.

(i) A is orthogonally diagonalizable

(ii) A has an orthonormal set of n eigenvectors.

(iii) A is symmetric

* Hence A matrix is orthogonally diagonalizable iff A is symmetric.

Orthogonal Diagonalization of Symmetric Matrices

<u>Step 1</u> Find a basis for each eigenspace of A.

<u>Step 2</u> Apply Gram-Schmidt process to find an orthonormal basis corresponding to each eigenbasis obtained in Step 1.

<u>Step 3</u> Form the matrix P whose columns are the vectors constructed in step 2.

Now $D = P^T A P$

where D is diagonal matrix with eigenvalues, having same order as their corresponding eigenvectors in P.

Example Find an orthogonal matrix P that diagonalizes

$$A = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix}$$

Solution

To find eigen values

$$\det(\lambda I - A) = 0$$

$$\Rightarrow \begin{vmatrix} \lambda-4 & -2 & -2 \\ -2 & \lambda-4 & -2 \\ -2 & -2 & \lambda-4 \end{vmatrix} = 0$$

$$\Rightarrow (\lambda-2)^2(\lambda-8) = 0$$

$$\lambda = 2 \; ; \; \lambda = 8$$

The eigenvectors corresponding to $\lambda = 2$ are

$$u_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \text{ and } u_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

form a basis for the eigenspace corresponding to $\lambda = 2$. Applying Gram Schmidt process to $\{u_1, u_2\}$ yields the following orthonormal eigenvectors $\{q_1, q_2\}$

$$q_1 = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ +\frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}, \quad q_2 = \begin{bmatrix} -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{bmatrix}$$

The eigenvector corresponding to $\lambda = 8$ which form basis is

$$u_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

normalizing, we get orthonormal eigenvector as

$$q_3 = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$$

Now $P = [q_1 \quad q_2 \quad q_3]$

$$= \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

and

$$P^T A P = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$

H.W

<u>Exercise 7.2</u>

Q# 7, 8, 10