Singular Value Decomposition

Definition:

If A is an $m\times n$ matrix and if $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of A^TA, then the numbers

$$\sigma_1 = \sqrt{\lambda_1}, \quad \sigma_2 = \sqrt{\lambda_2}, \ldots, \sigma_n = \sqrt{\lambda_n}$$

are called the singular values of A.

Example: Find singular values of matrix

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Solution

Here $A^T = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

Now $A^TA = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$

$$= \begin{bmatrix} 1+0+1 & 1+0+0 \\ 1+0+0 & 1+1+0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

To find eigenvalues of A^TA, put

~~$\det(A^TA - \lambda$~~

$$\det(\lambda I - A^TA) = 0$$

$$\Rightarrow \begin{vmatrix} \lambda-2 & -1 \\ -1 & \lambda-2 \end{vmatrix} = 0$$

$$\Rightarrow (\lambda-2)^2 - 1 = 0$$

$$\Rightarrow \lambda^2 - 4\lambda + 4 - 1 = 0$$

$$\Rightarrow \lambda^2 - 4\lambda + 3 = 0$$

$$\Rightarrow \lambda^2 - 3\lambda - \lambda + 3 = 0$$

$$\Rightarrow \lambda(\lambda-3) - 1(\lambda-3) = 0 \Rightarrow (\lambda-3)(\lambda-1) = 0$$

$$\Rightarrow \lambda - 3 = 0 \; ; \; \lambda - 1 = 0 \Rightarrow \boxed{\lambda = 3} ; \boxed{\lambda = 1}$$

Now Singular values of A are (2)

$\sigma_1 = \sqrt{\lambda_1} = \sqrt{3}$

$\sigma_2 = \sqrt{\lambda_2} = \sqrt{1} = 1$

Exercise 9.4 {Q # 1 – 4}

Singular value Decomposition (Brief form)

If A is an $m \times n$ matrix of rank k, then A can be expressed in form $A = U\Sigma V^T$ where Σ has size $m \times n$ and can be expressed in partitioned form as

$$\Sigma = \left[\begin{array}{c|c} D & O_{k\times(n-k)} \\ \hline O_{(m-k)\times k} & O_{(m-k)\times(n-k)} \end{array}\right]$$

in which D is a diagonal $k \times k$ matrix whose successive entries are the first k singular values of A in nonincreasing order, U is an $m \times m$ orthogonal matrix and V is an $n \times n$ orthogonal matrix.

Singular Value Decomposition (Expanded form)

If A is an $m \times n$ matrix of rank k, then A can be factored as

$$A = U\Sigma V^T = [u_1\ u_2\ \cdots\cdots u_k \mid u_{k+1} \cdots\cdots u_m] \left[\begin{array}{cccc|c} \sigma_1 & 0 & \cdots & 0 & \\ 0 & \sigma_2 & \cdots & 0 & O_{k\times(n-k)} \\ \vdots & \vdots & & & \\ 0 & 0 & \cdots & \sigma_k & \\ \hline & O_{(m-k)\times k} & & & O_{m-k\times(n-k)} \end{array}\right] \left[\begin{array}{c} v_1^T \\ v_2^T \\ \vdots \\ v_k^T \\ \hline v_{k+1}^T \\ \vdots \\ v_n^T \end{array}\right]$$

in which U, Σ and V have sizes $m\times m$, $m \times n$, and $n \times n$ respectively and in which

(a) $V = [v_1 \; v_2 \; \cdots \; v_n]$ orthogonally diagonalizes A^TA.

(b) The non-zero diagonal entries of Σ are $\sigma_1 = \sqrt{\lambda_1}$, $\sigma_2 = \sqrt{\lambda_2}$, ..., $\sigma_k = \sqrt{\lambda_k}$ where $\lambda_1, \lambda_2, \ldots, \lambda_k$ are non-zero eigenvalues of A^TA corresponding to the column vectors of V.

(c) The column vectors of V are ordered so that

$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_k > 0.$

(d) $$u_i = \frac{Av_i}{\|Av_i\|} = \frac{1}{\sigma_i} Av_i \quad (i = 1, 2, \ldots, k)$$

(e) $\{u_1, u_2, \ldots, u_k\}$ is an orthonormal basis for $Col(A)$

(f) $\{u_1, u_2, \ldots, u_k, u_{k+1}, \ldots, u_m\}$ is an extension of $\{u_1, u_2, \ldots, u_k\}$ to an orthonormal basis for $\mathbb{R}^m$.

Example

Find a singular value decomposition of matrix

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$\boxed{\text{Here } m = 3;\ n = 2}$

Sol: we calculated eigenvalues of $A^TA = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ already in previous example. i.e $\lambda_1 = 3$ and $\lambda_2 = 1$ and corresponding singular values of A are $\sigma_1 = \sqrt{3}$ and $\sigma_2 = 1$.

(a) **Calculating v_1 & v_2**

eigen vector of A^TA corresponding to $\lambda_1 = 3$ is

$$(\lambda I - A^TA)x = 0$$

$$\begin{bmatrix} 3-2 & -1 \\ -1 & 3-2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Solving by Guass elimination

$$\left[\begin{array}{cc|c} 1 & -1 & 0 \\ -1 & 1 & 0 \end{array}\right]$$

$$R_2 \quad \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

By backward substitution

$$x_1 - x_2 = 0$$

$$x_1 = x_2 = t$$

$$\Rightarrow V_1' = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

<u>eigen vector of A^TA corresponding to eigen value $\lambda_2 = 1$</u>

put

$$(\lambda I - A^TA)(x) = 0$$

$$\left\{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\right\}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Solving by Guass elimination

$$\left[\begin{array}{cc|c} -1 & -1 & 0 \\ -1 & -1 & 0 \end{array}\right]$$

$$R_2 \quad \left[\begin{array}{cc|c} -1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

by backward substitution

$$-x_1 - x_2 = 0$$

$$\Rightarrow x_1 = -x_2 \quad \text{or} \quad x_2 = -x_1$$

put $x_1 = s \Rightarrow x_2 = -s$

$$\Rightarrow V_2' = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} s \\ -s \end{bmatrix} = s\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$\because$ u_1' & u_2' are orthogonal to each other.

Just normalizing them can produce an orthonormal basis for $\mathbb{R}^2$

i.e

$$v_1 = \frac{v_1'}{\|v_1'\|} \; ; \; v_2 = \frac{v_2'}{\|v_2'\|}$$

$$v_1 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} ; \; v_2 = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$$

Therefore $V^T = \begin{bmatrix} v_1^T \\ v_2^T \end{bmatrix}$

$$V^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

Now by (d)

$$u_1 = \frac{1}{\sigma_1} A v_1 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \frac{1}{\sqrt{3}} \begin{bmatrix} \frac{2}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix}$$

$$= \frac{1}{\sqrt{6}} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

$\boxed{\text{Note } \|u_1\| = 1, \|u_2\| = 1}$

$$u_2 = \frac{1}{\sigma_2} A v_2 = \frac{1}{1} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

as $\underline{m = 3}$; we want to find $U = [u_1 \; u_2 \; u_3]$ an extended basis for $\mathbb{R}^3$. Now to find u_3 which is orthogonal to both u_1 & u_2. Let $u_3 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$$\langle u_1, u_3 \rangle = 0$$

and $\langle u_2, u_3 \rangle = 0$

$\Rightarrow \frac{1}{\sqrt{6}}(2x_1 + x_2 + x_3) = 0$ or $2x_1 + x_2 + x_3 = 0$

$\frac{1}{\sqrt{2}}(-x_2 + x_3) = 0$ or $-x_2 + x_3 = 0$

Now solving this system by Gauss elimination (6)

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 \end{array}\right]$$

$$R_1 \left[\begin{array}{ccc|c} 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & -1 & 1 & 0 \end{array}\right]$$

By backward substitution

$-x_2 + x_3 = 0$

$\boxed{x_3 = x_2 = t}$

$x_1 + \frac{1}{2} x_2 + \frac{1}{2} x_3 = 0$

$\Rightarrow x_1 + \frac{t}{2} + \frac{t}{2} = 0$

$\Rightarrow x_1 + t = 0$

$\Rightarrow x_1 = -t$

Thus

$$u_3' = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -t \\ t \\ t \end{bmatrix} = t\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

Normalizing, we have

$$u_3 = \frac{u_3'}{\|u_3'\|}$$

$$= \frac{1}{\sqrt{3}}(-1, 1, 1)$$

$$\text{or } u_3 = \begin{bmatrix} -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$$

CamScanner

Section 7.3

7

Thus

$$A = U \Sigma V^T$$

$$= [u_1 \ u_2 \ u_3] \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \\ \hline 0 & 0 \end{bmatrix} \begin{bmatrix} v_1^T \\ \hline v_2^T \end{bmatrix}$$

or

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \frac{2}{\sqrt{6}} & 0 & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

As required.

Positive definite matrix

A symmetric matrix A is positive definite iff all eigenvalues of A are positive.

Negative definite matrix

A symmetric matrix A is negative definite iff all eigenvalues of A are negative.

indefinite matrix

A symmetric matrix is indefinite iff A has atleast one positive eigenvalue and atleast one negative value.

★ A symmetric matrix is semi positive definite if each of the eigen values is either positive or zero.

Similarly A symmetric matrix is semi negative definite if each of the eigen value is either negative or zero.

Exercise 17 Determine by inspection whether the matrix is positive definite, negative definite, indefinite positive semidefinite or negative semidefinite.

(a) $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

$\because$ matrix is diagonal. eigen values are diagonal entries ie $\lambda_1 = 1$; $\lambda_2 = 2$

Both are positive. Hence matrix is positive definite

(b) $\begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$

Here eigen values $\lambda_1 = -1$; $\lambda_2 = -2$ are both negative. Therefore matrix is negative definite.

(c) $\begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$

eigen values are $\lambda_1 = -1$; $\lambda_2 = 2$.

One eigen value is negative, other is positive.

Therefore matrix is indefinite.

(d) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

eigen values are $\lambda_1 = 1$; $\lambda_2 = 0$.

$\because$ $\lambda_1 > 0$ and $\lambda_2 = 0$

Therefore it is semi-positive definite

(e) $\begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix}$

eigen values are $\lambda_1 = 0$; $\lambda_2 = -2$

$\because$ $\lambda_1 = 0$ and $\lambda_2 < 0$

Therefore it is semi-negative definite

Q#18 Do yourself

Another method to identify positive definite matrices

We define the k^{th} principal submatrix of an $n \times n$ matrix A to be the $k \times k$ submatrix consisting of the first k rows and columns of A. For example.

the principal submatrices of 3×3 matrix are

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

(First principal submatrix) (Second principal submatrix) (3rd principal submatrix = A)

Theorem If A is a Symmetric matrix. Then

(a) A is positive definite iff the determinant of every principal submatrix is positive.

(b) A is negative definite iff the determinant of principal submatrices alternate between negative and positive values starting with a negative value for the determinant of the first principal submatrix.

(c) A is indefinite iff it is neither positive definite nor negative definite and atleast one principal submatrix has a positive determinant and atleast one has a negative determinant.

Exercise 27(a) Use above theorem to classify whether the matrix is positive definite, negative definite or indefinite.

$$A = \begin{bmatrix} 3 & 1 & 2 \\ 1 & -1 & 3 \\ 2 & 3 & 2 \end{bmatrix}$$

Here $A_1 = [3] \Rightarrow |A_1| = 3$

$$A_2 = \begin{bmatrix} 3 & 1 \\ 1 & -1 \end{bmatrix} \Rightarrow |A_2| = -3-1 = -4$$

$$A_3 = \begin{bmatrix} 3 & 1 & 2 \\ 1 & -1 & 3 \\ 2 & 3 & 2 \end{bmatrix} \Rightarrow |A_3| = 3\begin{vmatrix} -1 & 3 \\ 3 & 2 \end{vmatrix} - 1\begin{vmatrix} 1 & 3 \\ 2 & 2 \end{vmatrix} + 2\begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix}$$

$$= 3[-2-9] - 1[2-6] + 2[3+2]$$

$$= -33 + 4 + 10$$

$$= -19$$

by part (c), A is indefinite.

Q#27(b), Q#28(a), (b) : Do yourself.