Noreen Firdous

Notes

Subject:

Discrete Structure

Written By:

Prof. Shahid Imran

Class:

M. Sc IT Term II

Institute:

Superior Collage Jauharabad

Discrete Mathematics

Logic:- The rules of logic specify the meaning of mathematical statements.

Prepositional logic:- The rules of logic give precise meaning of mathematical statements, these rules are used to distinguish between valid and invalid statements or arguments.

Preposition:- A declarative sentence which declares a fact, that is either true or false, but not both.

Example(1):-

i):- Washington D.C is the capital of USA.

ii):- $1+1=2$

So, the above two examples shows two prepositions.

→ Following are examples which are not prepositions.

i) What time is it?

ii) $x + 1 = 2$

Prepositional variables: Those variables which represent prepositions, and they denote numerical variables.

p, q, r and s ..... are conventional variables which are used for prepositions.

→ Truth value of preposition is denoted by "T" whereas false value is denoted by "F".

Negation: The preposition $\neg p$ is read "not p". The truth value of negation of p is "$\neg p$", It is actually opposite value of "p".

We can also denote negation of p as "$\bar{p}$".

3

Example:

p	q
T	T
T	F
F	T
F	F

→ Negation of p is

p	¬p
T	F
F	T

Conjunction: let p and q are prepositions. The Conjunction of p and q, denoted by $p \wedge q$. The conjunction p and q is true when both p and q are true and false otherwise.

Example:

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Disjunction

Let p and q be prepositions, disjunction of p and q is denoted as "$p \vee q$". The disjunction $p \vee q$ is false when both p and q are false and is true otherwise.

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Exclusive OR

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

Conditional statement

Let p, q be prepositions, Conditional statemen

p → q is the preposition "if p then q". conditional statement p → q is false when p is true and q is false, and true otherwise. In (p → q) "p" is called hypothesis whereas "q" is called conclusion.

→ Truth table of Conditional statement

p	q	p → q
T	T	T
T	F	F.
F	T.	T
F	F	T

Statemen

(6)

Biconditional

Let p and q be prepositions, denoted by "p ↔ q". means (p ↔ q) "p if and only if q". biconditional p ↔ q is true when p and q have the same truth values and it would be false otherwise.

Truth Table

p	q	p ↔ q
T	T	T
T	F	F
F	T	F
F	F	T

Plain English example

"You can take flight if and only if you buy a ticket."

(7)

Precendence of different operators in Discrete Math

Operator	Precendence
$\neg$	First
$\wedge$	second
$\vee$	Third
$\rightarrow$	Fourth
$\leftrightarrow$	Fifth

$\rightarrow$ Truth Table of Compound preposition

p	q	$\neg q$	$p \vee \neg q$	$p \wedge q$	$(p \vee \neg q) \rightarrow (p \wedge q)$
T	T	F	T	T	T
T	F	T	T	F	F
F	T	F	F	F	T
F	F	T	T	F	F

(8)

Tautology :- Such compound preposition which is always true, no matter what the truth values of the preposition that occurs in it.

Contradiction :- A compound statement which is always false is called contradiction.

Contingency :- A compound statement or preposition which is not tautology and not a contradiction is called contingency.

→ Truth Table of Tautology

p	$\neg p$	$p \vee \neg p$
T	F	T
F	T	T

→ Truth Table of Contradition

p	$\neg p$	$p \wedge \neg p$
T	F	F
F	T	F

Logical Equivalences

A compound preposition that has same truth value in all possible cases is said to be logical equivalences.

Method of Showing equivalences

i) $p \equiv q$

ii) $p \Leftrightarrow q$

The above Examples denoting equivalences between p and q prepositions.

Example. Show that $\neg(p \vee q)$ and $\neg p \wedge \neg q$ are logically equivalent.

Example. Show that $p \rightarrow q$ and $\neg p \vee q$ are are logically equivalent.

(11)

Universal Quantifiers

→ Universal quantifiers of $P(x)$ is the preposition "$P(x)$ is true for all values of x in the universe of discourse".

→ Notation $\forall x\, P(x)$.

It can also be expressed as "for all x, $P(x)$ or "for every x, $P(x)$". We call '$\forall$' the universal quantifier.

Example: "Every student in MATH class knows what discrete Math is".

Let $P(x)$ denote the statement "x knows what discrete math is". The statement can be written as $\forall x\, P(x)$.

Existential Quantifiers

→ Existential quantifier of $P(x)$ is the preposition "There exists an element x in the universe of discourse such that $P(x)$ is true".

Example

$$\mathcal{U} = \{x_1, x_2, \cdots x_n\}$$

$$\exists x\, P(x) \equiv P(x_1) \vee P(x_2) \vee \cdots \vee P(x_n)$$

$\rightarrow$ $P(x)$ is true for some x

Universal quantifier

Example

$$\mathcal{U} = \{x_1, x_2, x_3, \cdots x_n\}$$

$$\forall x\, P(x) \equiv P(x_1) \wedge P(x_2) \wedge \cdots \wedge P(x_n)$$

$\rightarrow$ $P(x)$ is true for all x.

Denote ||

Predicate :

A declarative sentence which contains one or more variables

- is not a preposition, but becomes a preposition when variables in it are replaced by allowable choices.

→ Universe of discourse $\mathcal{U}$.
(set of allowable choices).

- Z : integers
- N : Natural Numbers
- Z^+ : Positive integers
- Q : rational Numbers etc.

Predicate Example(1) :

$$\mathcal{U} = N$$

$P(x)$: $x+2$ is an even integer.

$P(5)$: F (false)
$P(8)$: T (True)

$\neg P(x)$: $x+2$ is not an even integer.

Example (2) :

$$\mathcal{U} = N$$

$q(x,y)$: $x+y$ and $x-2y$ are even integers.

$q(11,3)$: F , $q(14,4)$: T

which

Denoted By:

$\exists x\ P(x)$. We call '$\exists$' the existential quantifier.

Example (1):
There is one big bird.

Example (2):
There exists two prime numbers whose sum is prime number.

Example (3):
Every one has one good friend.

Let $P(x, y)$ be the statement "y is a good friend of x". Then statement can be expressed as:

$$\forall x\ \exists y\ P(x, y).$$

Sets

Set is an unordered Collection of " i.e. concept science

→ Objects in a set are called elemen

The Notation used to describe membership is as follows

$$a \in A \longrightarrow ①$$

The above equation shows that lowerca 'a' is element / member of a set 'A'.

→ Members are listed in braces like

$$\{a, b, c, d\}$$

Example (1): The set V of all vowels in English alphabet can be written as $V = \{a, e, i, o, u\}$

Example (2): The set O of ODD positive integers less than 10 can be expressed as

$$O = \{1, 3, 5, 7, 9\}.$$

→ Give some examples of Sets as

$$Z = \{0, \pm 1, \pm 2, \pm 3, \pm 4, \ldots\}$$

$N = \{0, 1, 2, 3, \ldots\}$ set of Natural Numbers.

ote: concept of data type or type in computer science is built upon sets.

→ Following example of set describes the boolean data Type as follows.

Example:

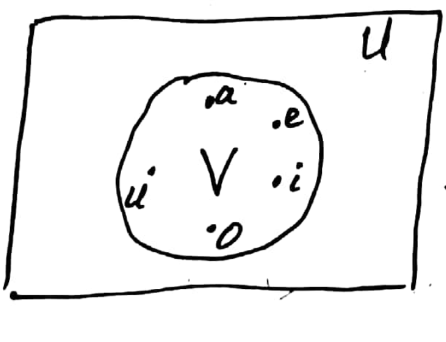
$$Set = \{0, 1\}$$

→ Above set contains two elements/ members (0,1) which indicates the concept of boolean data type which is true or false (or 0,1).

Venn Diagram

Venn diagrams are also used to express the concept of sets. as following.

Example: Draw a Venn diagram that represents V, set of vowels in English alphabet.



→ Rectangular shape shows universal set. Inside this, there is a circular shape representing set of vowel 'V'.

→ A set having no elements is called Empty or null set. It is denoted as $\emptyset$ or $\{ \}$.

Example of Null set: Set of all positive integers that are greater than their square is a null set.

Singleton Set: A set having one element or member is called singleton set.

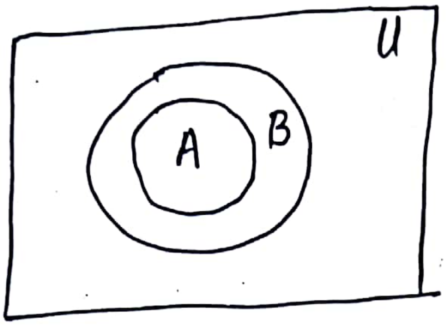
Subset: Set 'A' is said to be a subset of 'B' if and only if every element of A is also an element of B.

Notation: $A \subseteq B$

Quantification:

→ $\forall x\,(x \in A \rightarrow x \in B)$ is true.

nn Diagram to show $A \subseteq B$ is following.



→ Following two sets are Equal.

$A = \{ \emptyset, \{a\}, \{b\}, \{a,b\}\}$ and

$B = \{ x \mid x$ is a subset of the set $\{a,b\}\}$.

→ Sets are widely used in counting so we need to consider the following examples.

(1). Let 'A' be the set of positive integers less than 10 is $|A| = 5$.

(2). Let 'S' be the set of letters in English alphabet. Then $|S| = 26$.

<u>Cartesian Product</u>. $A \times B (A | AB | a \in A \wedge b \in B$ $A \times B (a \in A \wedge B \in B$

$$A \times B = \{ (a,b) \mid a \in A \wedge b \in B \}.$$

The above notation is called set Builder notation.

Union : Let A and B be sets. The union of A and B denoted as $A \cup B$, is the set that contains those elements that are either in A or in B or in Both.

OR

$$A \cup B = \{ x \mid x \in A \vee x \in B \}$$

Intersection : Intersection of set A and B denoted by $A \cap B$, is the set containing those elements in Both A and B.

$$A \cap B = \{ x \mid x \in A \wedge x \in B \}.$$

→ Draw Venn Diagrams for both union and intersection.

Sequence

"A set of things (usually numbers) that are in order."

3, 5, 7, 9, ...

(1st Term) (2nd Term)

Arithmetic Sequence:

In an arithmetic sequence the difference between one term and the next is constant.

Example:

1, 4, 7, 10, 13, 16, 19, 22, ...

→ above sequence has a difference of 3 between each number.

Generally it look like as follows.

$$\{a, a+d, a+2d, a+3d, \ldots\}$$

where 'a' is first Term and 'd' is Common difference.

1, 4, 7, 10, 13, ...

in this example

→ $a = 1$ (first Term)

→ $d = 3$ (common difference)

$$\{1, 1\times3, 1+2\times3, 1+3\times3, \ldots\}$$

$$\{1, 4, 7, 10, \ldots\}$$

Rule of Arithmetic Sequence:

$$X_n = a + d(n-1)$$

Example:

Write Rule, and calculate the 4th Term for

3, 8, 13, 18, 23, 28, 33, 38, ...

Here $a = 3$ (First Term)
$d = 5$ (Common difference)

$$X_n = a + d(n-1)$$
$$= 3 + 5(n-1)$$
$$= 3 + 5n - 5$$
$$= 5n - 2$$

So, the 4th Term is

$$X_4 = 5 \times 4 - 2 = 18$$

→ Addition of first 10 terms of Arithmetic Sequence.

$$\sum_{K=0}^{n-1} (a + Kd) = \frac{n}{2}(2a + (n-1)d)$$

for 1, 4, 7, 10, 13, ...

$$\sum_{K=0}^{10-1} (1 + K.3) = \frac{10}{2}(2 \cdot 1 + (10-1) \cdot 3)$$

$$= 5(2+9\cdot 3) = 5(29)$$
$$= \boxed{145}$$

Geometric Sequence :

In a geometric sequence each term is found by multiplying the previous term by a constant.

Example :

2, 4, 8, 16, 32, 64, 128, ...

This sequence has a factor of 2 between each other.

→ Each term (except the first term) is found by multiplying the previous term by 2.

For Example :

$$\{a, ar, ar^2, ar^3, \ldots\}$$

where

$a \to$ {first Term}

r (common Ratio)

$\{1, 2, 4, 8, \ldots\}$

→ $a = 1$ (first Term)

$r = 2$ (common ratio)

$$= \{a, ar, ar^2, ar^3, \ldots\}$$
$$= \{1, 1\times 2, 1\times 2^2, 1\times 2^3, \ldots\}$$
$$= \{1, 2, 4, 8, \ldots\}$$

Sum

→ Note: 'r' should not be zero, if r is zero then $\{a, 0, 0, \dots\}$ which is not a geometric sequence.

Rule:

$$X_n = ar^{(n-1)}$$

(we use "n−1", because ar^0 is for 1st term.

Example:

10, 30, 90, 270, 810, ...

→ Above sequence has factor of 3 between each other.

→ $a = 10$ (first Term)

→ $r = 3$ (Common Ratio)

$$X_n = 10 \times 3^{(n-1)}$$

So, the 4^{th} Term is:

$$X_4 = 10 \times 3^{(4-1)} = 10 \times 3^3 = 10 \times 27 = \boxed{2}$$

$$\boxed{X_4 = 270}$$

Summing a Geometric Series:

$$a + ar + ar^2 + \dots + ar^{(n-1)}$$

→ Each term is ar^k, where k starts at '0' and goes upto $(n-1)$

Use Formula:

$$\sum_{k=0}^{n-1} (ar^k) = a\left(\frac{1-r^n}{1-r}\right)$$

→ a (First Term)
→ r (Common Ratio)
→ n (Number of Terms)

Example:

$$10, 30, 90, 270, 810, \dots$$

→ $a = 10$, $r = 3$, $n = 4$ (4 Terms)

$$\sum_{k=0}^{4-1} (10 \cdot 3^k) = 10\left(\frac{1-3^4}{1-3}\right) = 400$$

We can check:

$$10 + 30 + 90 + 270 = \boxed{400}$$

20

Relations

SGDM E2013 – *Discrete Mathematics*

Relations

Lorena Ronquillo

lron@itu.dk

24/09/2013 [week 5]

Today's lecture

1. Relations and their properties
   - What is a relation?
   - Functions as relations
   - Relations on numbers
   - Properties of relations
   - Composition of relations
2. n-ary Relations and Their Applications
   - n-ary relations
3. Closures of Relations
4. Equivalence Relations
   - Equivalence relations
   - Equivalence classes
5. Order relations
   - Partial orderings
   - Total orderings

Sets and Cartesian product (reminder)

Sets are unordered, so $\{1, 2, 3\} = \{1, 3, 2\}$. However, sometimes we need to establish an order.

$A = \{(1, 1), (2, 4), (3, 9), (4, 16), (5, 25), \ldots\} = \{(x, y) \mid x, y \in \mathbb{N} \text{ and } y = x^2\}$

In this example, $(2, 4) \in A$ but $(4, 2) \notin A$.

An ordered n-tuple $(a_1, a_2, \ldots, a_n)$ is a collection with an established order, where a_1 is the first element, a_2 is the second element, etc. If $n = 2$ then they are called ordered pairs.

The Cartesian product of two or more sets $A_1, A_2, \ldots, A_n$, denoted by $A_1 \times A_2 \times \ldots \times A_n$, is the set of all ordered n-tuples $(a_1, a_2, \ldots, a_n)$, where $a_i \in A_i$ for $1 \leq i \leq n$.

In general, the Cartesian product is not commutative, so $A \times B \neq B \times A$ unless $A = \emptyset$, $B = \emptyset$ or $A = B$.

Cartesian product (reminder)

Example

Consider the sets *Student*, *Course*, *Grade*, *Textbook*, and *Classroom*:

Student is a set of names of the students in ITU

Course is the set of courses in ITU

Grade is the set of all the possible grades that can be given to a student

Classroom is the set of classrooms at ITU

Then we may have Cartesian products like:

Student $\times$ Course

Course $\times$ Classroom

Student $\times$ Course $\times$ Grade

establish an

What is a relation?

IT University 22

Let A and B be sets. A binary relation from A to B is a subset of $A \times B$.

We use the notation $a\ R\ b$ to denote that $(a, b) \in R$, and $a\ \not R\ b$ to denote that $(a, b) \notin R$.
When $(a, b) \in R$, a is said to be related to b by R.

Example

Let the sets *Course* and *Classroom* be the ones previously defined, and let R be the relation that consists of those pairs (a, b), where a is a course given in classroom b.

$$R \subseteq \mathit{Course} \times \mathit{Classroom}$$

- (Discrete Mathematics, 2A14) $\in R$
- (Discrete Mathematics, 2A12) $\notin R$ (or Discrete Mathematics $\not R$ 2A12)
- If a certain course c is not offered by ITU ($c \notin \mathit{Course}$), then there will be no pairs in R having this course as the first element.

Functions as relations

A function f from a set A (domain) to a set B (codomain) assigns exactly one element of B to each element of A.

Let us consider the set of ordered pairs

$$C = \{(a, b) \mid a \in A \text{ and } b \in B \text{ and } b = f(a)\}.$$

This points form the graph of f.

Since $C \subseteq A \times B$, we have that the graph of any function f from A to B is a relation from A to B.

Relations on numbers

Relations on numbers can be defined by using $\leq, <, =, \geq, >$.

Examples

$$R_1 = \{(a,b) \mid a \leq b\}$$
$$R_2 = \{(a,b) \mid a > b\}$$
$$R_3 = \{(a,b) \mid a = 2b+3\}$$
$$R_4 = \{(a,b) \mid a - b \geq 1\}$$

Properties of relations (I)

A relation on a set A is a relation from A to A (i.e, a subset of $A \times A$).

A relation R on a set A is reflexive if

$$\forall a \in A, (a,a) \in R.$$

Exercise Are these relations reflexive?

a) The graph of the function $f(x) = x^2$, where $f : \mathbb{Z} \to \mathbb{Z}$.
No, since the relation $R = \{(0,0), (1,1), (2,4), (3,9), \ldots\}$ does not contain all pairs of the form (x,x), $\forall x \in \mathbb{Z}$.

b) The graph of a function $g(x) = x$, where $g : \mathbb{R} \to \mathbb{R}$.
Yes, by the definition itself.

c) The relation "is a subset of" (set inclusion $\subseteq$).
Yes, because it includes the equality.

d) The relation "is greater than" on the set of integers.
No, but it would be the relation "is greater than or equal to".

e) The relation "divides" (divisibility) on the set of all negative integers.
Yes, because any number is a divisor of itself.

Properties of relations (II)

24

A relation R on a set A is called symmetric if

$$\forall a, b \in A, (a, b) \in R \rightarrow (b, a) \in R.$$

Similarly, a relation R on a set A is antisymmetric if

$$\forall a, b \in A,\ ((a, b) \in R \wedge (b, a) \in R) \rightarrow a = b.$$

Exercise Are these relations symmetric and/or antisymmetric?

a) The relation "divides" (divisibility) between any two natural numbers.
Antisymmetric. The only way two numbers can be divisible by each other is if the two are the same number.

b) The relation "is married to" between any two persons.
Symmetric. Whenever 'a' is married to 'b', then 'b' is also married to 'a'.

c) The relation $R = \{(0, 1), (1, 2), (2, 1)\}$.
Neither symmetric nor antisymmetric. Not symmetric because $(1, 0)$ is missing. Not antisymmetric because $(1, 2) \in R$, $(2, 1) \in R$ and $1 \neq 2$.

d) The relation $S = \{(1, 1), (2, 2), (3, 3)\}$.
Symmetric and antisymmetric. In fact, this is the only way a relation can be both symmetric and antisymmetric.

Properties of relations (III)

A relation R on a set A is transitive if

$$\forall a, b, c \in A,\ ((a, b) \in R \wedge (b, c) \in R) \rightarrow (a, c) \in R.$$

Example The following are transitive relations.

a) The relation "is greater than" on the set of integers: whenever $a > b$ and $b > c$, then also $a > c$.

b) The relation "is a subset of" (set inclusion).

c) The relation "divides" (divisibility) on the set of natural numbers.

d) The relation "implies" (implication).

The relation "is the mother of" is not transitive. If Alice is the mother of Beth, and Beth is the mother of Charlie, then Alice is NOT the mother of Charlie.

Composition of relations

Given two relations $R \subseteq A \times B$ and $S \subseteq B \times C$, the composition of R and S, denoted by $S \circ R$, is the relation defined by

$$S \circ R = \{(a, c) \in A \times C \mid \exists b \in B : (a, b) \in R \wedge (b, c) \in S\}$$

Note that if R and S are functions, then $S \circ R$ is the usual function composition.

Exercise

a) What is the composition of the relations "is brother of" and "is the parent of"?
The relation "is uncle of": if Alex is the brother of Bob, and Bob is the parent (father) of Charlie, then Bob is the uncle of Charlie.

b) What is the composition of the relations
$R = \{(1, 2), (1, 6), (2, 4), (3, 4), (3, 6), (3, 8)\}$ and
$S = \{(2, u), (4, s), (4, t), (6, t), (8, u)\}$?
The relation $S \circ R = \{(1, u), (1, t), (2, s), (2, t), (3, s), (3, t), (3, u)\}$.

n-ary relations (I)

Let $A_1, A_2, \ldots, A_n$ be sets. An n-ary relation on these sets is a subset of $A_1 \times A_2 \times \ldots \times A_n$.

The sets $A_1, A_2, \ldots, A_n$ are called the domains of the relation, and n is its degree.

Example Given the sets *Student*, *Course*, *Grade*, and *Classroom*, we said one of the possible Cartesian products could be (among others):

$$\text{Student} \times \text{Course} \times \text{Grade}$$

Let us consider the relation $R \subseteq$ Student $\times$ Course $\times$ Grade. This is a ternary relation of degree 3.

n-ary relations (II)

26

Example (cont.)

$$R \subseteq \text{Student} \times \text{Course} \times \text{Grade}$$

where

Student = { Daniel M., Maria S., Mads T., Sarah D., Tobias A.}

Course = { Discr. Math., Global SW dev., SW Eng., Alg. and Data Str., Syst. Arch. and Security }

Grade = $\{-3, 0, 2, 4, 7, 10, 12\}$

$$R = \{(\text{Daniel M.}, \text{SW Eng.}, 7), (\text{Daniel M.}, \text{Syst. Arch. and Security}, 4),$$
$$(\text{Maria S.}, \text{Disc. Math.}, 12), (\text{Mads T.}, \text{SW Eng.}, -3),$$
$$(\text{Sarah D.}, \text{Discr. Math.}, 2), (\text{Tobias A.}, \text{Global SW dev.}, 0), \ldots\}$$

n-ary relations (III)

Example (cont.)

$$R = \{(\text{Daniel M.}, \text{SW Eng.}, 7), (\text{Daniel M.}, \text{Syst. Arch. and Security}, 4),$$
$$(\text{Maria S.}, \text{Disc. Math.}, 12), (\text{Mads T.}, \text{SW Eng.}, -3),$$
$$(\text{Sarah D.}, \text{Discr. Math.}, 2), (\text{Tobias A.}, \text{Global SW dev.}, 0), \ldots\}$$

$S = \{a : (a, b, c) \in R \wedge c \geq 7\}$ is a relation on *Student* being the selection from R of all students registered to some course in ITU, having a grade greater than or equal 7.

The result of the selection would be
$S = \{(\text{Daniel M.}, \text{SW Eng.}, 7), (\text{Maria S.}, \text{Disc. Math.}, 12)\}$

Closures of relations (I)

The closure of a relation R with respect to a certain property p is the relation obtained by adding the minimum number of ordered pairs to R to obtain property p.

- To find the reflexive closure of R: add all the pairs of the form (a, a) (if not already included in R).
- To find the symmetric closure of R: for every pair (a, b), add the pair (b, a) (if not already included in R).
- To find the transitive closure of R: if there is a pair (a, b) and also (b, c), add the pair (a, c) (if not already included in R).

Reflexive and symmetric closures are easy. Transitive closures can be very complicated.

Closures of relations (II)

Exercise Let R be the relation

$$R = \{(1,1), (1,2), (2,1), (2,2), (3,4), (4,1), (4,4)\}.$$

a) Is the relation R reflexive? If not, what is the reflexive closure of R?
   No. We just need to add the pair $(3,3)$ for R to be reflexive.

b) Is it symmetric? If not, what is the symmetric closure of R?
   No. We need to add the pairs $(4,3)$ and $(1,4)$ for R to be symmetric.

c) Is it transitive? If not, what is the transitive closure of R?
   No. We need to add the pair $(3,1)$ (from $(3,4)$ and $(4,1)$), the pair $(4,2)$ (from $(4,1)$ and $(1,2)$), and also add the pair $(3,2)$ (from $(3,4)$ and $(4,2)$).

Equivalence relations

28

A relation R is an equivalence relation if R is reflexive, symmetric, and transitive.

Two elements a and b that are related by an equivalence relation are called equivalent. Often, the notation $a \sim b$ is used.

Example The relation $=$ is an equivalence relation on $\mathbb{N}$.

Exercise The relation $<$ on $\mathbb{N}$ is not an equivalence relation. Why? And what about the relation $\leq$?
The relation $<$ on $\mathbb{N}$ is not an equivalence relation because it is not reflexive nor symmetric. The relation $\leq$ is not an equivalence relation either, because it is reflexive and transitive, but not symmetric. It is, however, a partial order, as we will later see.

Equivalence classes (I)

Let R be an equivalence relation on a set A. The set of elements that are related to an element a of A is called the equivalence class of a.

$$[a]_R = \{x \in A \mid (a, x) \in R\}.$$

It can also be denoted by $[a]$, if there is only one relation in consideration.

If $b \in [a]_R$, then b is called a representative of the equivalence class $[a]_R$. Any element in a class can be used as a representative of this class.

Equivalence classes (II)

Example Let S be the set $S = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$. We can define many equivalence relat... on this set.

For instance,
$\forall x, y \in S, (x, y) \in R$ if and only if $x \equiv y \mod 3$ (i.e. $x - y$ is divisible by 3).

So $R = \{(0,3), (3,0), (3,6), (6,3), (0,6), (6,0),$
$(1,4), (4,1), (4,7), (7,4), (1,7), (7,1), (2,5), (5,2), (5,8), (8,5), (2,8), (8,2)\}$
(for simplicity, the reflexive pairs (x, x) where $0 \leq x \leq 8$ have been omitted)

Then our equivalence classes are:

$$[0] = \{0, 3, 6\} \text{ remainder 0 when divided by 3}$$

$$[1] = \{1, 4, 7\} \text{ remainder 1 when divided by 3}$$

$$[2] = \{2, 5, 8\} \text{ remainder 2 when divided by 3}$$

Note that we could have taken any other element within each class as its representative.

These classes are called congruence classes.

Equivalence classes (III)

Example Let T be the set of all cars in the world:

$T = \{$blueCar1, blueCar2, redCar1, redCar2, redCar3, blackCar1, blackCar2, ...$\}$.

We can now define the following equivalence relation R:

$$\forall x, y \in T, (x, y) \in R \text{ if and only if } x \text{ has the same color as } y.$$

Then our equivalence classes are:

[redCar1] = { redCar1, redCar2, redCar3, ...} all the cars that are red
[blackCar1] = { blackCar1, blackCar2, ...} all the cars that are black
[blueCar1] = { blueCar1, blueCar2, ...} all the cars that are blue
...

Equivalence classes (IV)

80

Theorem Let R be an equivalence relation on a set A. Then, given any two elements $a, b \in A$, the following statements are equivalent:

i) $(a, b) \in R$

ii) $[a] = [b]$

iii) $[a] \cap [b] \neq \emptyset$

Proof (see textbook)

Equivalence classes (V)

Recall that a partition of a set is a collection of mutually disjoint subsets whose union is the original set.

Theorem Let R be an equivalence relation on a set A. Then $\{[a]_R \mid a \in A\}$ is a partition of A.

Example Given the relation of congruence mod 5 on the integers $\mathbb{Z}$, we obtain the partition:

$$\mathbb{Z} = [0] \cup [1] \cup [2] \cup [3] \cup [4]$$

It is easy to check that these congruence classes are disjoint, and every integer is in exactly one of them.

Order relations (I)

Sometimes relations are used to order some or all the elements of a set.

A relation R on a set S is a partial ordering if it is reflexive, antisymmetric, and transitive.

A set S together with a partial ordering R is called a partially ordered set (or poset), denoted by (S, R).

Example Let us consider the relation $\geq$ on the set $\mathbb{Z}$. For any three integers $a, b, c \in \mathbb{Z}$, the relation $\geq$ is

- Reflexive: $a \geq a$.
- Antisymmetric: if $a \geq b$ and $b \geq a$, then $a = b$.
- Transitive: $a \geq b$ and $b \geq c$ imply that $a \geq c$.

Therefore, the relation $\geq$ is a partial ordering on the set $\mathbb{Z}$, and $(\mathbb{Z}, \geq)$ is a poset.

Order relations (II)

Given a partial order R, we use the notation

$$a \preccurlyeq b$$

to indicate that $(a, b) \in R$, and

$$a \prec b$$

when $(a, b) \in R$ but $a \neq b$.

Note: The notation $\preccurlyeq$ is not the same as $\leq$ ("less than or equal to"). Rather, $\preccurlyeq$ is used to denote **any** partial ordering.

Two elements a and b of a poset $(S, \preccurlyeq)$ are comparable if either $a \preccurlyeq b$ or $b \preccurlyeq a$. Otherwise, they are called incomparable.

32

Order relations (III)

If **every pair** of elements in a poset $(S, \preccurlyeq)$ is comparable, then the set S is called a total order set (or a chain), while $\preccurlyeq$ is called a total order.

Example The poset $(\mathbb{Z}, \geq)$ is a total order because for every $a, b \in \mathbb{Z}$ we have that either $a \geq b$ or $b \geq a$.

Example Take the power set $\mathcal{P}(S) = \{\emptyset, \{0\}, \{1,\}, \{0,1\}\}$ of the set $S = \{0, 1\}$. The relation $\subset$ on the set $\mathcal{P}(S)$ is a partial order but NOT a total order.

We can see that, for any three elements $a, b, c \in \mathcal{P}(S)$, the relation $\subseteq$ satisfies the axioms for a partial order:

- $a \subseteq a$
- if $a \subseteq b$ and $b \subseteq a$, then $a = b$.
- $a \subseteq b$ and $b \subseteq c$ imply that $a \subseteq c$.

So $(\mathcal{P}(S), \subseteq)$ is a poset. However, there are pairs of elements in this poset which are not comparable: neither $\{0\} \subseteq \{1\}$ nor $\{1\} \subseteq \{0\}$, then $(\mathcal{P}(S), \subseteq)$ is not a total order.

Order relations (IV)

Lexicographic ordering is the same as any dictionary or phone book. We use alphabetical order starting with the first character in the string, then the next character (if the first was equal), etc. (you can consider "no character" for shorter words to be less than "a").

Formally, lexicographic ordering is defined by combining two other orderings.

Let $(A_1, \preccurlyeq_1)$ and $(A_2, \preccurlyeq_2)$ be two posets. The lexicographic ordering $\preccurlyeq$ on the Cartesian product $A_1 \times A_2$ is defined by

$$(a_1, a_2) \prec (a_1', a_2')$$

if $a_1 \prec_1 a_1'$ or if both $a_1 = a_1'$ and $a_2 \prec_2 a_2'$.

Order relations (V)

The lexicographic order can be extended to Cartesian products of arbitrary length by recursively applying this definition.

Define $\preccurlyeq$ on $A_1 \times A_2 \times \ldots \times A_n$ by

$$(a_1, a_2, \ldots, a_n) \prec (b_1, b_2, \ldots, b_n)$$

if $(a_1 \prec b_1)$ or if there is an integer $i > 0$ such that

$$a_1 = b_1, a_2 = b_2, \ldots, a_i = b_i$$

and $a_{i+1} \prec b_{i+1}$.

Exercise Given the poset $(\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}, \preccurlyeq)$, where $\preccurlyeq$ is the lexicographic ordering constructed from the usual $\leq$ relation on $\mathbb{Z}$, determine whether $(8, 2, 3) \preccurlyeq (8, 0, 5)$. Since, $8 = 8$, and $2 > 0$, we have that $(8, 2, 3) \not\preccurlyeq (8, 0, 5)$.

Order relations (VI)

An element a in a poset $(S, \preccurlyeq)$ is maximal if it is not less than any other element in the poset. If there is only one maximal element, we call it the greatest element.

An element a in a poset $(S, \preccurlyeq)$ is minimal if it is not greater than any other element in the poset. If there is only one minimal element, we call it the least element.

Example Determine whether the following posets have a greatest element and a least element. (For simplicity, posets have been written omitting the pairs coming from reflexivity, transitivity and antisymmetry).

a) $(S, \preccurlyeq) = \{(a, b), (a, c), (b, d), (c, d)\}$.
Least element: a. Greatest element: d.

b) $(S, \preccurlyeq) = \{(a, c), (b, c), (c, d)\}$.
It has no least element (min. elements: a, b). Greatest element: d.

c) $(S, \preccurlyeq) = \{(a, b), (b, c), (b, d), (b, e)\}$.
Least element: a. It has no greatest element (max. elements: c, d, e).

Order relations (VII)

34

Sometimes it is possible to find an element that is greater than or equal (or less than or equal) to all the elements **in a subset** A of a poset $(S, \preccurlyeq)$.

An element $l \in S$ is a lower bound of the subset A if $l \preccurlyeq a$ for all elements $a \in A$. (Similarly, for upper bound)

An element $x \in S$ is called the least upper bound of the subset A if x is an upper bound that is less than every other upper bound of A. (Similarly, for greatest lower bound)

Example Let us consider the poset $(S, |)$ where $S = \{3, 5, 9, 15, 24, 45\}$. We start by finding the elements of the poset.
$(S, |) = \{(3, 9), (3, 15), (3, 24), (5, 15), (5, 45), (9, 45), (15, 45)\}$. (Again, for simplicity, this poset has been written omitting the pairs coming from reflexivity, transitivity and antisymmetry, but you should think they are actually there).

- All upper bounds of $\{3, 5\}$: $15, 45$.
- The least upper bound of $\{3, 5\}$: 15.
- All lower bounds of $\{15, 45\}$: $3, 5, 15$.
- The greatest lower bound of $\{15, 45\}$: 15.

Order relations (VIII)

A lattice is a partially ordered set in which every pair of elements has both a least upper bound and a greatest lower bound.

Example The poset $(S, \preccurlyeq) = \{(a, b), (b, c), (b, d), (c, e), (d, e), (e, f)\}$ is a lattice.

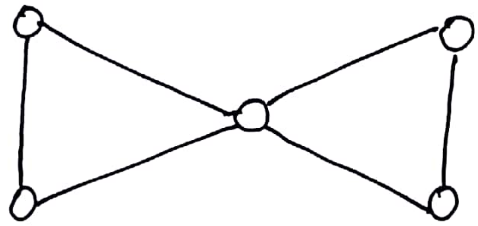
Lattices are used to represent different information flow policies, access control, etc.

Planar Graph:-

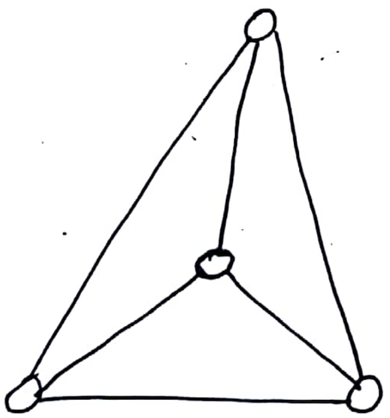
A planar graph is a graph that can be drawn on a plane in such a way that no edges cross each other, such a drawing is called a planar graph.

→ in other words, edges intersect only at their endpoints.

Examples:



(i)



(ii)

Graphs (i) and (ii) are the examples of planar graph.

Matrices

subtraction

" A Matrix is an Array of numbers:

$$\begin{bmatrix} 6 & 4 & 24 \\ 1 & -9 & 8 \end{bmatrix}$$

→ The above matrix has 2 Rows and three columns.

Adding:

$$\begin{bmatrix} 3 & 8 \\ 4 & 6 \end{bmatrix} + \begin{bmatrix} 4 & 0 \\ 1 & -9 \end{bmatrix} = \begin{bmatrix} 7 & 8 \\ 5 & -3 \end{bmatrix}$$

calculations are:

$3+4=7$, $8+0=8$

$4+1=5$, $6-9=-3$

Example: A matrix with 3 rows and 5 columns can be added to another matrix of 3 rows and 5 columns.

But the columns don't match the size as we add 3 rows and 4 columns. (size don't match).

Negative: " Negative of matrix is also simple:

$$-\begin{bmatrix} 2 & -4 \\ 7 & -10 \end{bmatrix} = \begin{bmatrix} -2 & 4 \\ -7 & -10 \end{bmatrix}$$

Subtracting

$$\underbrace{\begin{bmatrix} 3 & 8 \\ 4 & 6 \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ 1 & -9 \end{bmatrix}} = \begin{bmatrix} -1 & 8 \\ 3 & 15 \end{bmatrix}$$

$3-4=-1$, $8-0=8$

$4-1=3$, $6-(-9)=15$

Transposing

"Swap the rows and columns".

$$\begin{bmatrix} 6 & 4 & 24 \\ 1 & -9 & 8 \end{bmatrix}^T = \begin{bmatrix} 6 & 1 \\ 4 & -9 \\ 24 & 8 \end{bmatrix}$$

→ Matrix is usually shown by capital letters (such as A, B)

→ **Matrix Multiplication**

example

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \times \begin{bmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{bmatrix} = \begin{bmatrix} 58 & 64 \\ 139 & 154 \end{bmatrix}$$

Identity Matrix

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

→ A (3×3) Identity Matrix.

→ If we do $\begin{pmatrix} A \times I = A \\ I \times A = A \end{pmatrix}$ return same results.

Inverse of Matrix:

: Reciprocal of Number.

as for (8) is (1/8)

The idea for Matrix is same, but we write inverse of Matrix (A) as (A^{-1}).

Example: Calculate inverse of (2x2) Matrix.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \underbrace{\frac{1}{(ad-bc)}}_{\text{(determinant)}} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

for example:

$$\begin{bmatrix} 4 & 7 \\ 2 & 6 \end{bmatrix}^{-1} = \frac{1}{(4\times6)-(7\times2)} \begin{bmatrix} 6 & -7 \\ -2 & 4 \end{bmatrix}$$

$$= \frac{1}{10} \begin{bmatrix} 6 & -7 \\ -2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 0.6 & -0.7 \\ -0.2 & 0.4 \end{bmatrix}$$

→ check for true Answer:

$$\begin{bmatrix} 4 & 7 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} 0.6 & -0.7 \\ -0.2 & 0.4 \end{bmatrix} = \begin{bmatrix} 4\times0.6+7\times-0.2 & 4\times-0.7+7\times0.4 \\ 2\times0.6+6\times-0.2 & 2\times-0.7+6\times0.4 \end{bmatrix}$$

$$= \begin{bmatrix} 2.4-1.4 & -2.8+2.8 \\ 1.2-1.2 & -1.4+2.4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

So, we end up with identity Matrix.

So, it should also be true.

$$\rightarrow \boxed{A^{-1} \times A = I}$$

check this, example $\quad AA^{-1} = A^{-1}A = I$

Simple Formula For inverse:

In case of 2x2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

- if $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then $A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

→ Note that quantity $(ad-bc)$ is determinant. Furthermore, $\frac{1}{ad-bc}$ is not defined when $(ad-bc = 0)$, because it is not possible to divide by zero so, in this case inverse of (A) does not exist.

- **Example:** Find the inverse of $A = \begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix}$

Solution:

$$A^{-1} = \frac{1}{(3)(2)-(1)(4)}\begin{bmatrix} 2 & -1 \\ -4 & 3 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 2 & -1 \\ -4 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1/2 \\ -2 & 3/2 \end{bmatrix}$$

we can check for true answer by performing $A \times A^{-1}$.
The result should be identity Matrix $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Big O Notation

Big O notation is used in computer science to describe the performance or complexity of an algorithm. Big O specifically describes the worst case scenario, and can be used to describe the execution time required or the space used (e.g in memory or on disk) by an algorithm.

Examples:

O(1): Big O(1) describes an algorithm that will always execute in the same time (or space) regardless of the size of input.

O(N): Big O(N) describes an algorithm whose performance will grow linearly and indirect proportion to the size of input. for example a for loop having iterations till (N). and targeted element can be found by any iteration. In this scenario, we have to iterate till (N). That's why worst time complexity is Big O(N)

Science 2

$O(N^2)$:

Big $O(N^2)$ represents an algorithm whose performance is directly proportional to the square of the size of input.

→ This complexity is common with algorithms that involve nested iterations over the data set.

Note: Deeper (Nested) iterations will result in $O(N^3)$, $O(N^4)$ etc.

$O(2^N)$:

Big $O(2^N)$ denotes an algorithm whose growth with each addition doubles. This type of growth is also called exponential growth.

Example: An Example of $O(2^N)$ is the recursive calculation of Fibonacci Series.

$O(\log N)$:

Big $O(\log N)$ is used in Binary Search technique.

→ Big $O(\log N)$ is oftenly associated with Divide and conquer algorithms e.g Merge Sort, Binary Search etc.

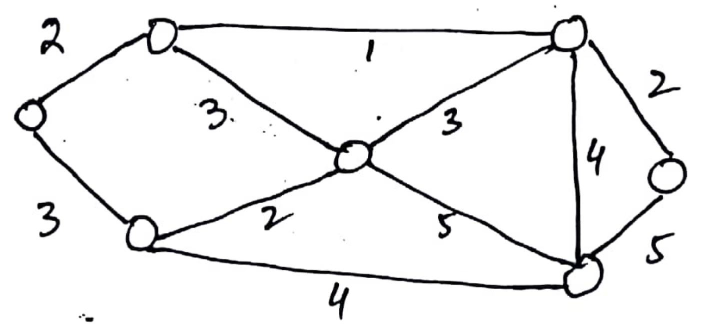
Step (i):- Find the edge with smallest weight incident to a. Add it to T also include in (T), the next vertex b.

Step (2):- Find the edge of smallest weight incident to either a or b. include in (T) and pick the next incident vertex c.

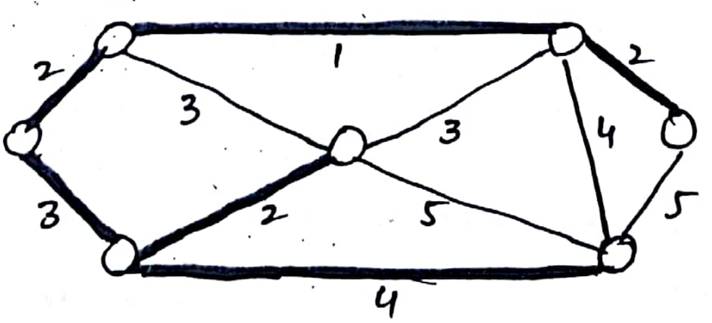
Step (3):- Repeat step (2). choosing the edge of smallest weight that does not form a cycle untill all vertices are in (T).

→ Finally we will get minimum spanning Tree.

Q:

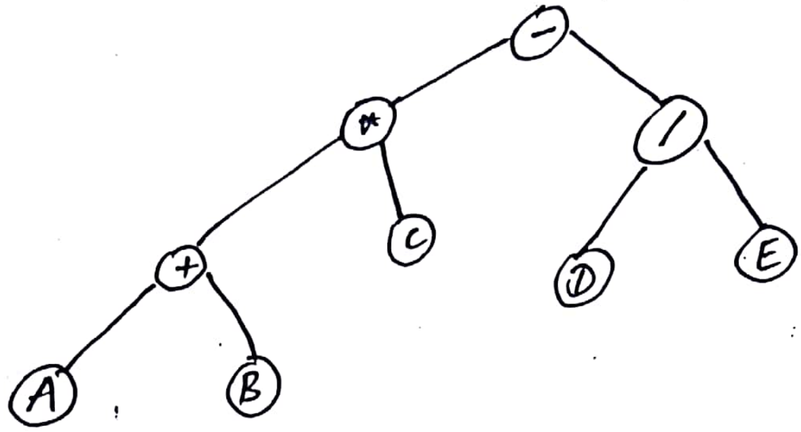


a



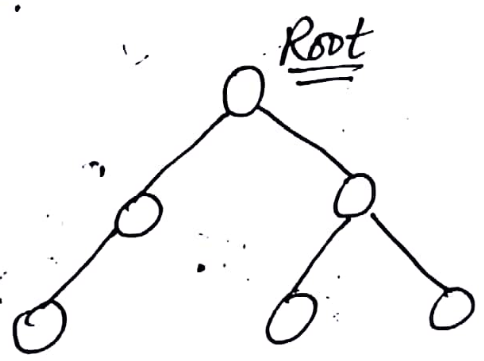
Arithmetic Expression

i) (A + B) * C − D/E

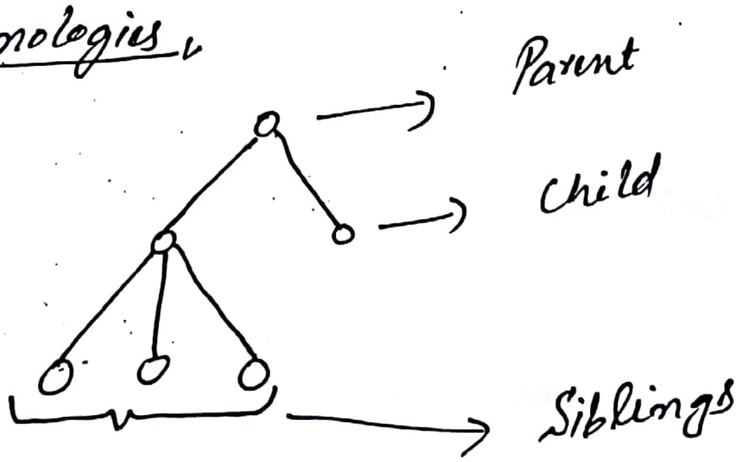


Rooted Tree :- A Tree where one of its vertices is designed as root.

Example :-

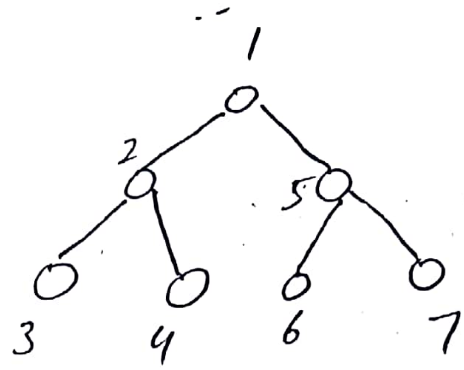


Different Terminologies :-

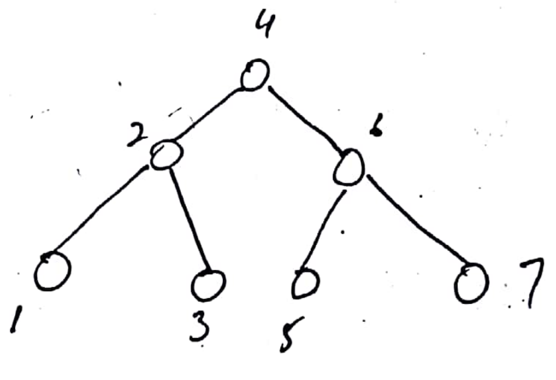


Tree Traversals

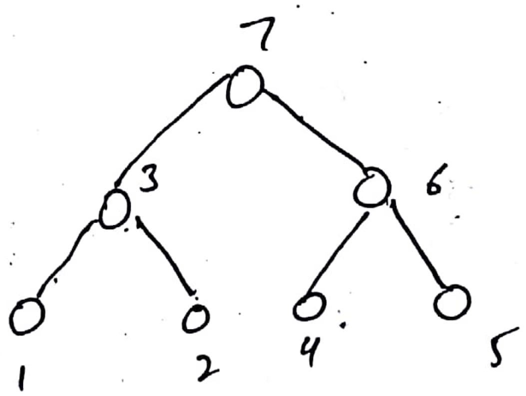
(1). Pre-Order



(2). In-Order



(3). Post Order



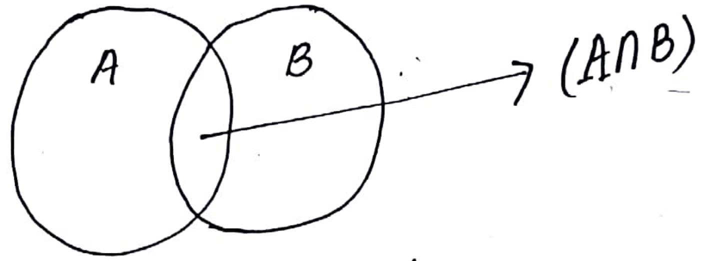
Prim's Algorithm

Step 0. Pick any vertex as a starting vertex (call it a) as $T = \{a\}$.

Inclusion - Exclusion Principle

The inclusion - exclusion principle is a counting technique which generalizes the familiar method of obtaining the number of elements in the union of two finite sets.

Example :



→ Symbolically it can be expressed as

→ $|A \cup B| = |A| + |B| - |A \cap B|$

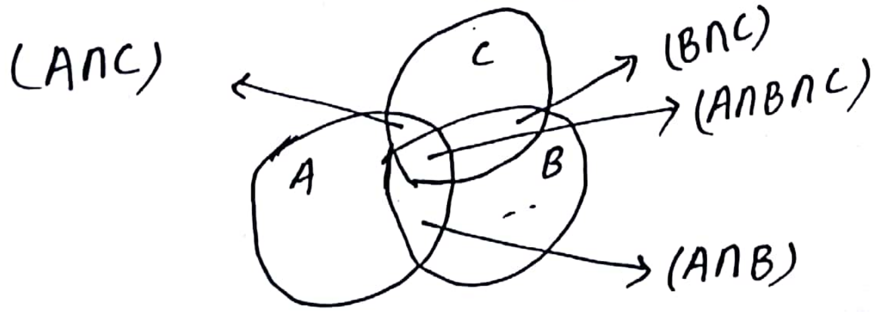
where A and B are two finite sets. The formula expresses the fact that the sum of the sizes of two sets may be too large since some elements may be counted twice.

The double counted elements are those in the intersection of the two sets and the count is corrected by subtracting the size of intersection.

→ The Principle is more clearly seen in case of three sets.

→ $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$

Venn Diagram of the above equation.



Spanning Trees

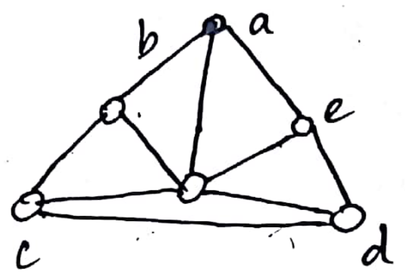
→ Given a graph G, a tree T is a spanning tree of G if:

- T is a subgraph of G

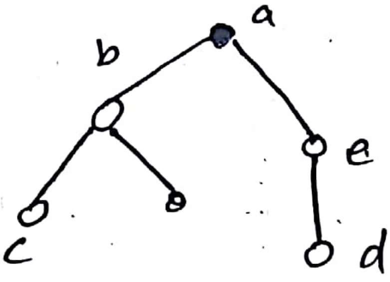
  and

- T contains all the vertices of G.

Example



(i)



(ii)

→ The second Tree (ii) is a spanning Tree because it is a subgraph of (G) as well as it contains all the vertices of (G).

Generalizing the results of these examples give the principle of inclusion - exclusion.

→ To find the cardinality of (n) sets.

(1) include the cardinality of the sets.

(2) Exclude the cardinalities of the pairwise intersections.

(3) include the cardinalities of triple-wise intersections.

(4) Exclude the cardinalities of quadruple-wise intersections.

Permutation

Formula

$$P(n, r) = \frac{n!}{(n-r)!}$$

Task #1: How many ways are there to select first prize winner, second prize and third prize winner from 100 different people.

Sol:

$$P(100, 3) = \frac{100!}{(100-3)!}$$

$$= 970,200$$

or

$$P(100, 3) = 100 \cdot 99 \cdot 98 = \boxed{970,200}$$

Task #2: Suppose there are eight runners in a race. The winner receives gold medal, second place finisher receives silver medal and third place finisher receives bronze medal. How many ways to award these medals.

Sol:

$$P(8, 3) = 8 \cdot 7 \cdot 6 = 336 \text{ ways}$$

or

$$P(8, 3) = \frac{n!}{(n-r)!} = \frac{8!}{(8-3)!} = 336$$

Combination

Formula

$$C(n,r) = \frac{n!}{(n-r)!\,r!}$$

Task #1: How many ways are there to select five players from a 10-member team?

Sol:

$$C(10,5) = \frac{n!}{(n-r)!\,r!}$$

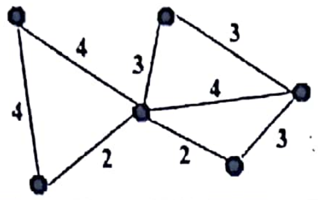
$$= \frac{10!}{(10-5)!\,5!}$$

$$= \frac{10!}{5!\,5!}$$

$$= \boxed{252 \text{ ways}}$$

Prim's Algorithm

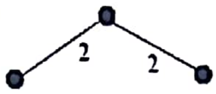
1 *Given a network...........*



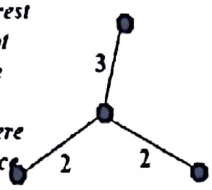
2 *Choose a vertex*

3 *Choose the shortest edge from this vertex.*

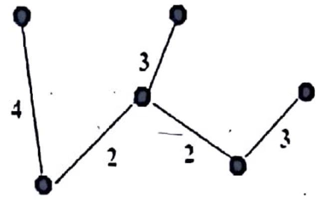
4 *Choose the nearest vertex not yet in the solution.*



5 *Choose the next nearest vertex not yet in the solution, when there is a choice choose either.*

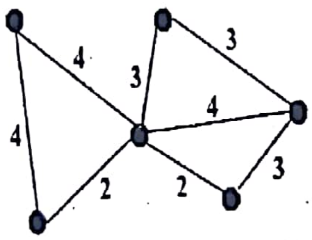


6 *Repeat until you have a minimal spanning tree.*

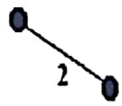


Kruskal's Algorithm

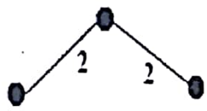
1 *Given a network...........*



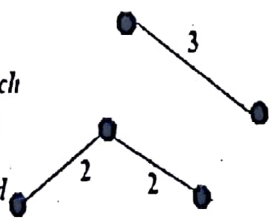
2 *Choose the shortest edge (if there is more than one, choose any of the shortest)........*



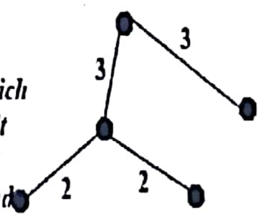
3 *Choose the next shortest edge and add it.........*



4 *Choose the next shortest edge which wouldn't create a cycle and add it.*



5 *Choose the next shortest edge which wouldn't create a cycle and add it.*



6 *Repeat until you have a minimal spanning tree.*