Important Short Questions of LA

Q1. What is Characteristic equation and one of its major advantages?

Ans: The characteristic equation is the equation which is solved to find a matrix's eigenvalues. If A is an **n** x **n** matrix, then λ is an eigenvalue of A if it satisfies the equation.

$$\det (A - d\ I) = 0$$

This is called the characteristic equation of A.

Advantage: For finding eigen values we use characteristic equation.

Q2. Define Eigen Vector and Eigen Value?

Ans:

Eigen Vector	**Eigen Value**
Let A be an n × n matrix. An *eigenvector* of A is a *nonzero* vector v in R^n such that Av = λ v , for some scalar λ .	Let A be an n × n matrix. An *eigenvalue* of A is a scalar λ such that the equation Av = λ v has a *nontrivial* solution.
If Av = λ v for v ≠ 0, we say that λ is the *eigenvalue for* v , and that v is an *eigenvector for* λ .	

Q3. Define similar matrices?

Ans: If A and B are square matrices, we say that B is similar to A if there is an invertible matrix P such that $B = P^{-1}AP$. Where P is a non-singular n × n matrix.

Q4. Define linearly independent vectors with example?

Ans: The vectors $v_1, ..., v_n$ are called **linearly independent** if there are no non-trivial combination of these vectors equal to the zero vector. That is, the vector $V_1, ..., V_n$ are linearly independent if $K_1V_1 + ... + K_nV_n = 0$ if and only if $K_1 = 0, ..., K_n = 0$.

Q5. Define linearly dependent vectors with example?

Ans: The vectors $v_1, ..., v_n$ are called **linearly dependent** if there exists a non-trivial combination of these vectors is equal to the zero vector.

Q6. What is meant by echelon form and reduced echelon form of matrix?

Ans: A matrix is in **row echelon form** when it satisfies the following conditions.

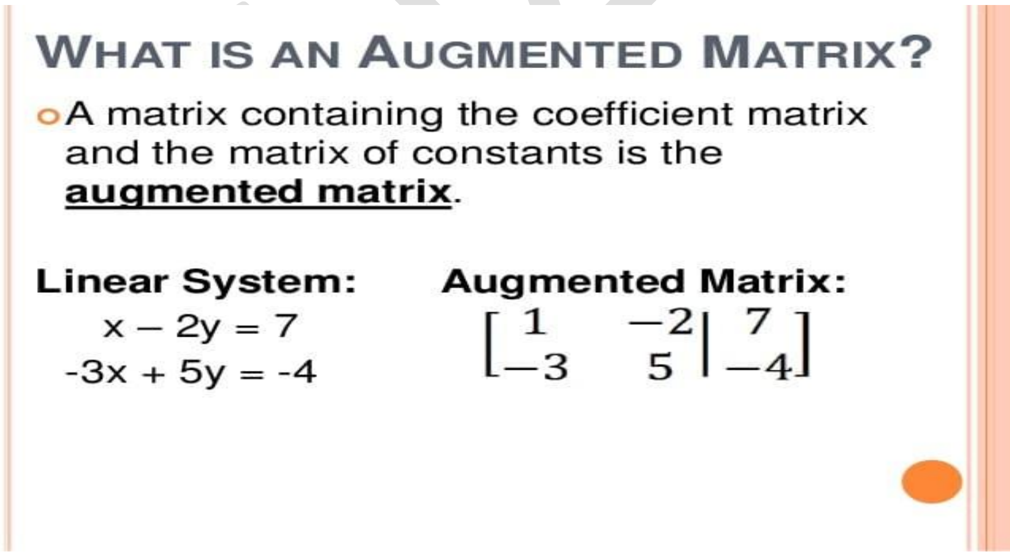
- The first non-zero element in each row, called the **leading entry**, is 1.
- Each leading entry is in a column to the right of the leading entry in the previous row.
- Rows with all zero elements, if any, are below rows having a non-zero element.

A matrix is in **reduced row echelon form** when it satisfies the following conditions.

- The matrix is in row echelon form (i.e., it satisfies the three conditions listed above).
- The leading entry in each row is the only non-zero entry in its column.

Q7. Give example of augmented matrix?

Ans:



Q8. Differentiate between singular matrix and non-singular matrix?

Ans: A square matrix A is called singular, if det(A) = 0.

If det(A) ≠ 0 then it is a non-singular matrix.

Q9. Define Hermitian and skew Hermitian matrix?

Ans:

Hermition matrix

A square matrix A is said to be Hermitian matrix if it is equal to its conjugate transposed matrix

That is A is hermitian iff $A = A^{\theta}$

If $A = \begin{bmatrix} 1 & 4-5i \\ 4+5i & 5 \end{bmatrix}$ then $A = A^{\theta}$

Skew-Hermitian matrix

and it is skew hermitian iff $A = -A^{\theta}$

If $A = \begin{bmatrix} 0 & -4-5i \\ 4-5i & 0 \end{bmatrix}$ then $A = -A^{\theta}$

Q10. What is trace of a matrix?

Ans:

Definition

Let A be a square matrix. The **trace** of A, denoted $tr(A)$ is the sum of the diagonal elements of A. Thus if A is an $n \times n$ matrix.

$$tr(A) = a_{11} + a_{22} + \ldots + a_{nn}$$

Example 4

Determine the trace of the matrix $A = \begin{bmatrix} 4 & 1 & -2 \\ 2 & -5 & 6 \\ 7 & 3 & 0 \end{bmatrix}$.

Solution

We get

$$tr(A) = 4 + (-5) + 0 = -1.$$

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Q11. What is L-U decomposition?

Ans: LU decomposition of a matrix is the factorization of the given square matrix into two triangular matrices, one upper triangular matrix and one lower triangular matrix, such that the product of these two matrices gives the original matrix.

Q12. Define linear span?

Ans:

Definition

Let us start with a formal definition of span.

Definition Let S be a linear space. Let $x_1, \ldots, x_n \in S$ be n vectors. The **linear span** of $x_1, \ldots, x_n$, denoted by

$$\text{span}(x_1, \ldots, x_n)$$

is the set of all the linear combinations

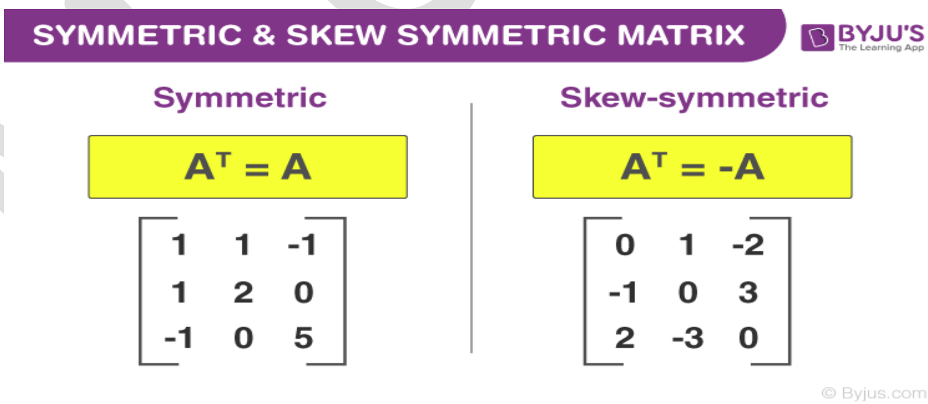
$$x = \alpha_1 x_1 + \ldots + \alpha_n x_n$$

that can be obtained by arbitrarily choosing n scalars $\alpha_1, \ldots, \alpha_n$.

Q13. Define symmetric and skew symmetric matrix?

Ans: A square matrix A is called a symmetric matrix, if $A^T = A$.

A square matrix A is called a skew – symmetric matrix, if $A^T = -A$.



Q14. Write two application areas of Eigen vectors?

Ans: Many applications of matrices in both engineering and science utilize eigenvalues and, sometimes, eigenvectors. Control theory, vibration analysis, electric circuits, advanced dynamics and quantum mechanics are just a few of the application areas.

Q15. Define Row equivalent matrices?

Ans: Matrices A and B are said to be row equivalent if each can be obtained from the other by a sequence of elementary row operations.

Q16. If A and B are invertible matrices with the same size then show

$(AB)^{-1} = B^{-1} A^{-1}$?

Ans:

THEOREM 1.4.6 *If A and B are invertible matrices with the same size, then AB is invertible and*

$$(AB)^{-1} = B^{-1}A^{-1}$$

Proof We can establish the invertibility and obtain the stated formula at the same time by showing that

$$(AB)(B^{-1}A^{-1}) = (B^{-1}A^{-1})(AB) = I$$

But

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I$$

and similarly, $(B^{-1}A^{-1})(AB) = I$.

Although we will not prove it, this result can be extended to three or more factors:

A product of any number of invertible matrices is invertible, and the inverse of the product is the product of the inverses in the reverse order.

Q17. Find vector $P_1 P_2$ when P_1 (2, -1 , 4) and P_2 (7, 5, -8)?

Ans:

▶ EXAMPLE 1 **Finding the Components of a Vector**

The components of the vector $\mathbf{v} = \overrightarrow{P_1P_2}$ with initial point $P_1(2, -1, 4)$ and terminal point $P_2(7, 5, -8)$ are

$$\mathbf{v} = (7 - 2, 5 - (-1), (-8) - 4) = (5, 6, -12)$$ ◀

Q18. Define orthogonal vector? show that u = (-2, 3, 1, 4) and v = (1, 2, 0, -1) are orthogonal vectors?

Ans:

DEFINITION 1 Two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ in R^n are said to be ***orthogonal*** (or ***perpendicular***) if $\mathbf{u} \cdot \mathbf{v} = 0$. We will also agree that the zero vector in R^n is orthogonal to *every* vector in R^n.

Solution (a) The vectors are orthogonal since

$$\mathbf{u} \cdot \mathbf{v} = (-2)(1) + (3)(2) + (1)(0) + (4)(-1) = 0$$

Q19. Define subspace of a vector space?

Ans: A subset **W** of a vector space **V** is called a subspace of **V** if **W** is itself a vector space under the addition and scalar multiplication defined on **V**.

Q20. Define Zero vector space and subspace?

Ans:

▶ EXAMPLE 1 **The Zero Vector Space**

Let V consist of a single object, which we denote by $\mathbf{0}$, and define

$$\mathbf{0} + \mathbf{0} = \mathbf{0} \quad \text{and} \quad k\mathbf{0} = \mathbf{0}$$

for all scalars k. It is easy to check that all the vector space axioms are satisfied. We call this the ***zero vector space***. ◀

▶ EXAMPLE 1 **The Zero Subspace**

If V is any vector space, and if $W = \{\mathbf{0}\}$ is the subset of V that consists of the zero vector only, then W is closed under addition and scalar multiplication since

$$\mathbf{0} + \mathbf{0} = \mathbf{0} \quad \text{and} \quad k\mathbf{0} = \mathbf{0}$$

for any scalar k. We call W the ***zero subspace*** of V.

Q21. Write standard basis of R^n?

Ans:

Standard Basis for R^n

If $\mathbf{e}_1 = (1, 0, 0, \ldots, 0)$, $\mathbf{e}_2 = (0, 1, 0, \ldots, 0)$, …, $\mathbf{e}_n = (0, 0, 0, \ldots, 1)$, then $S = \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ is a linearly independent set in R^n. This set also spans R^n since any vector $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ in R^n can be written as

$$\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + \ldots + v_n\mathbf{e}_n$$

Thus, S is a basis for R^n; it is called the standard basis for R^n.

Q22. What is meant by $X = X_p + X_h$?

Ans: General form of solution sets. If Xp is any particular solution to the linear system AX = b, then any other solutions to AX = b may be written in the form

X = Xp + Xh ,

where Xh is some solution to AX = 0.

Q23. What is meant by diagonalization?

Ans: Diagonalization is the process of transforming a matrix into diagonal form.

Diagonalizable ??

- A square matrix **M** is called diagonalizable if we can find an invertible matrix, say **P**, such that the product $\mathbf{P}^{-1}\,\mathbf{M}\,\mathbf{P}$ is a diagonal matrix.
  - Example $$\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}^{-1}\begin{bmatrix}3 & 1\\ 1 & 3\end{bmatrix}\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix} = \begin{bmatrix}2 & 0\\ 0 & 4\end{bmatrix}$$
- Some matrix cannot be diagonalized.
  - Example $$\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}$$

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Q24. Define the fourier series?

Ans:

What is Fourier Series?

A Fourier series may be defined as an expansion of a function in a series of sines and cosines such as.

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L} \right)$$

Q25. What do you mean by the diagonal matrix?

Ans:

(4) Diagonal Matrix: A square matrix, all of whose non-diagonal elements are zero and atleast one diagonal element is non-zero.

eg. $\begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

Q26. Define basis of a vector space?

Ans:

DEFINITION 1 If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a set of vectors in a finite-dimensional vector space V, then S is called a ***basis*** for V if:

(a) S spans V.

(b) S is linearly independent.

Q27. Define linear equation by writing its standard form?

Ans: An equation between two variables that gives a straight line when plotted on a graph is called linear equation. The standard form of a linear equation in

one variable is represented as **ax + b = 0** where, a ≠ 0 and x is the variable.
The standard form of a linear equation in two variables is represented as

ax + by + c = 0, where, a ≠ 0, b ≠ 0 , x and y are the variables.

Q28. what is meant by positive definite matrix?

Ans: A square matrix is positive definite if pre-multiplying and post-multiplying it by the same vector always gives a positive number as a result, independently of how we choose the vector. Positive definite symmetric matrices have the property that all their eigenvalues are positive.

- A nxn real matrix **M** is positive definite

$$\text{if } \underline{z}^T M \underline{z} > 0 \quad \forall \underline{z} \neq \underline{0}$$

Q29. Write the standard basis for R^3?

Ans: The standard basis for R^3 is following:

$$\begin{aligned} \mathbf{e}_1 &= \mathbf{e}_x = (1, 0, 0) \\ \mathbf{e}_2 &= \mathbf{e}_y = (0, 1, 0) \\ \mathbf{e}_3 &= \mathbf{e}_z = (0, 0, 1). \end{aligned}$$

Q30. Define linear equation in xy-plane?

Ans:

- Any straight line in xy-plane can be represented algebraically by an equation of the form: $a_1x + a_2y = b$

this equation is called linear equation.

Q31. What do you mean by the linear combination of a vector space?

Ans:

DEFINITION 2 If $\mathbf{w}$ is a vector in a vector space V, then $\mathbf{w}$ is said to be a ***linear combination*** of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ in V if $\mathbf{w}$ can be expressed in the form

$$\mathbf{w} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_r\mathbf{v}_r \qquad (2)$$

where $k_1, k_2, \ldots, k_r$ are scalars. These scalars are called the ***coefficients*** of the linear combination.

Q32. Define dimension of vector space and subspace?

Ans:

DEFINITION 1 The ***dimension*** of a finite-dimensional vector space V is denoted by $\dim(V)$ and is defined to be the number of vectors in a basis for V. In addition, the zero vector space is defined to have dimension zero.

Q33. Differentiate between Eigen space and Eigen basis?

Ans:

Eigen space	**Eigen basis**
The set of all solutions to $\mathbf{A}\cdot\mathbf{x} = \lambda\cdot\mathbf{x}$ or equivalently $(\mathbf{A} - \lambda\cdot\mathbf{I})\cdot\mathbf{x} = \mathbf{0}$ is called the eigenspace of "A" corresponding to "l".	A basis for a vector space consisting entirely of eigenvectors is called eigen basis.

Q34. Define Laplace Expansion?

Ans: The Laplace expansion is a formula that allows to express the determinant of a matrix as a linear combination of determinants of smaller matrices, called minors. The Laplace expansion also allows to write the inverse of a matrix in terms of its signed minors, called cofactors.

Technical fact

The Laplace expansion theorem

The determinant of any $n \times n$ matrix can be computed in one of the following two ways:

- By picking any row $\mathbf{r}_i$ and computing:

$$|\mathbf{A}| = \sum_{j=1}^{n} \left(a_{ij} (-1)^{i+j} \left| \mathbf{A}_{(i,j)} \right| \right)$$

- By picking any column $\mathbf{c}_j$ and computing:

$$|\mathbf{A}| = \sum_{i=1}^{n} \left(a_{ij} (-1)^{i+j} \left| \mathbf{A}_{(i,j)} \right| \right)$$

Q35. Differentiate between Hermitian and unitary Matrices?

Ans: A matrix 'A' is called Hermitian matrix if transpose of conjugate of matrix A is equal to A. A matrix 'A' is called Unitary matrix if transpose of conjugate of matrix A is equal to inverse of A.

Hermitian, Skew-Hermitian, and Unitary Matrices

A square matrix $\mathbf{A} = [a_{kj}]$ is called

Hermitian	if $\bar{\mathbf{A}}^{\mathsf{T}} = \mathbf{A}$,	that is,	$\bar{a}_{kj} = a_{jk}$
skew-Hermitian	if $\bar{\mathbf{A}}^{\mathsf{T}} = -\mathbf{A}$,	that is,	$\bar{a}_{kj} = -a_{jk}$
unitary	if $\bar{\mathbf{A}}^{\mathsf{T}} = \mathbf{A}^{-1}$.		

Q36. Define cofactor Expansion of Matrices?

Ans:

DEFINITION 2 If A is an $n \times n$ matrix, then the number obtained by multiplying the entries in any row or column of A by the corresponding cofactors and adding the resulting products is called the ***determinant of A***, and the sums themselves are called ***cofactor expansions of A***. That is,

$$\det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj} \tag{7}$$

[cofactor expansion along the jth column]

and

$$\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in} \tag{8}$$

[cofactor expansion along the ith row]

Q37. Define complex vector space?

Ans: A complex vector space is one in which the scalars are complex numbers. Thus, if v_1, v_2, … , v_m are vectors in a complex vector space, then linear combination is of the form

$$c_1 v_1 + c_2 v_2 + \dots + c_m v_m$$

where the scalars c_1 , c_2 , … , c_m are complex numbers. The complex version of R^n is the complex vector space C^n consisting of ordered n-tuples of complex numbers.

Q38. Define Orthogonal projecton?

Ans: If **u** and **a** are vectors in R^n , and of $\mathbf{a} \neq \mathbf{0}$, then **u** can be expressed in exactly one way in the form $\mathbf{u} = \mathbf{w_1} + \mathbf{w_2}$, where $\mathbf{w_1}$ is a scalar multiple of **a** and $\mathbf{w_2}$ is orhtogonal to **a**. The $\mathbf{w_1}$ is called the orthogonal projection of **u** on **a**, and the vector $\mathbf{w_2}$ is called the vector component of **u** orthogonal to **a**.

$$\text{proj}_{\mathbf{a}}\mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{a}}{\|\mathbf{a}\|^2}\mathbf{a} \quad \textit{(vector component of } \mathbf{u} \textit{ along } \mathbf{a})$$

$$\mathbf{u} - \text{proj}_{\mathbf{a}}\mathbf{u} = \mathbf{u} - \frac{\mathbf{u} \cdot \mathbf{a}}{\|\mathbf{a}\|^2}\mathbf{a} \quad \textit{(vector component of } \mathbf{u} \textit{ orthogonal to } \mathbf{a})$$

Q39. Define upper and lower triangular matrices?

Ans:

(8) Upper Triangular Matrix : A square matrix in which all the elements below the diagonal are zero.

eg $\begin{bmatrix} -2 & 3 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & -3 & 2 \\ 0 & 5 & -4 \\ 0 & 0 & 8 \end{bmatrix}$

(9) Lower Triangular Matrix : A square matrix in which all the elements above the diagonal are zero.

eg. $\begin{bmatrix} -2 & 0 \\ 3 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 \\ -3 & 5 & 0 \\ 2 & -4 & 8 \end{bmatrix}$

Q40. Define the determinant of the transpose?

Ans:

6. The determinant of the transpose of a matrix is equal to the determinant of the original matrix:

$$\det \mathbf{A}^t = \det \mathbf{A}, \quad \text{or} \quad \begin{vmatrix} a & c \\ b & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix}.$$

$$\begin{vmatrix} 3 & 2 \\ 4 & 5 \end{vmatrix} = 3(5) - 2(4) = 7 \quad \text{and} \quad \begin{vmatrix} 3 & 4 \\ 2 & 5 \end{vmatrix} = 3(5) - 4(2) = 7$$

Q41. Define the properties of the Determinants?

Ans:

Basic Properties of Determinants

Some basic properties of determinants are given below:

1. If In is the identity matrix of the order m ×m, then det(I) is equal to1
2. If the matrix X^T is the transpose of matrix X, then det (X^T) = det (X)
3. If matrix X^{-1} is the inverse of matrix X, then det (X^{-1}) = 1/det (x) = $\det(X)^{-1}$
4. If two square matrices x and y are of equal size, then det (XY) = det (X) det (Y)
5. If matrix X retains size a × a and C is a constant, then det (CX) = C^a det (X)

Q42. Define the column space, row space and null space?

Ans:

DEFINITION 2 If A is an $m \times n$ matrix, then the subspace of R^n spanned by the row vectors of A is called the ***row space*** of A, and the subspace of R^m spanned by the column vectors of A is called the ***column space*** of A. The solution space of the homogeneous system of equations $A\mathbf{x} = \mathbf{0}$, which is a subspace of R^n, is called the ***null space*** of A.

Q43. Show that a finite set that contains 0 is linearly dependent?

Ans:

Proof (a) For any vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$, the set $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r, \mathbf{0}\}$ is linearly dependent since the equation

$$0\mathbf{v}_1 + 0\mathbf{v}_2 + \cdots + 0\mathbf{v}_r + 1(\mathbf{0}) = \mathbf{0}$$

expresses $\mathbf{0}$ as a linear combination of the vectors in S with coefficients that are not all zero. •

Q44. Explain why the following form linearly dependent set of vectors.

$u_1 = (-1, 2, 4)$ and $u_2 = (5, -10, -20)$?

Ans:

1- Explain why the following form linearly dependent of Sets.

(a) $u_1 = (-1, 2, 4)$ and $u_2 = (5, -10, -20)$ in R^3

Sol:-

$u_2 = -5(-1, 2, 4)$

$u_2 = -5u_1$

so u_2 is a scalar multiple of u_1.

Q45. What is relation between rank, nullity and dimension of vector space?

Ans:

DEFINITION 1 The common dimension of the row space and column space of a matrix A is called the ***rank*** of A and is denoted by $\text{rank}(A)$; the dimension of the null space of A is called the ***nullity*** of A and is denoted by $\text{nullity}(A)$.

Q46. Find eigen values of following matrix ?

$$A = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$$

Ans:

Solution the eigenvalues of A are the solutions of the equation $\det(\lambda I - A) = 0$, which we can write as

$$\begin{vmatrix} \lambda - 3 & 0 \\ -8 & \lambda + 1 \end{vmatrix} = 0$$

from which we obtain

$$(\lambda - 3)(\lambda + 1) = 0$$

This shows that the eigenvalues of A are $\lambda = 3$ and $\lambda = -1$.

Q47. Define unit vector?

Ans:

Unit Vector

A unit vector is a vector with magnitude 1.

To find a unit vector, **u**, in the same direction of a vector, **v**, we divide the vector by its magnitude.

$$\vec{u} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{1}{\|\vec{v}\|}\vec{v}$$

Q48. If u and v are orthogonal vectors in a real inner product space, then

$$\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$$

Ans:

THEOREM 6.2.3 Generalized Theorem of Pythagoras

If **u** *and* **v** *are orthogonal vectors in a real inner product space, then*

$$\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$$

Proof The orthogonality of **u** and **v** implies that $\langle \mathbf{u}, \mathbf{v}\rangle = 0$, so

$$\|\mathbf{u}+\mathbf{v}\|^2 = \langle \mathbf{u}+\mathbf{v}, \mathbf{u}+\mathbf{v}\rangle = \|\mathbf{u}\|^2 + 2\langle \mathbf{u}, \mathbf{v}\rangle + \|\mathbf{v}\|^2$$
$$= \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$$

Q49. Check whether A is singular or not?

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Ans:

A square matrix A is called singular if det (A) = 0

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$|A| = -1\begin{vmatrix} 2 & 0 \\ 0 & 0 \end{vmatrix} + 0\begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix} - 0\begin{vmatrix} 0 & 2 \\ 0 & 0 \end{vmatrix}$$

$$|A| = -1(0) + 0 - 0 = 0$$

A is singular matrix.

Q50. Find C_{21} , if

$$A = \begin{bmatrix} 2 & 3a & 0 \\ 3a & 1 & b \\ 0 & b & 0 \end{bmatrix}$$

Ans:

$$A = \begin{bmatrix} 2 & 3a & 0 \\ 3a & 1 & b \\ 0 & b & 0 \end{bmatrix}$$

$$M_{21} = \begin{vmatrix} 3a & 0 \\ b & 0 \end{vmatrix} = 0 - 0 = 0$$

$$C_{21} = (-1)^{2+1} \times 0 = 0$$

Q51. Find the angle between u and v, u = (1, -5, 1), v = (0, 0, -1)?

Ans:

$$u = (1, -5, 1) \quad v = (0, 0, -1)$$

$$\cos\theta = \frac{u \cdot v}{\|u\| \|v\|} = \frac{1.0 + (-5).0 + 1.-1}{\sqrt{1^2 + (-5)^2 + 1^2}\ \sqrt{0^2 + 0^2 + (-1)^2}}$$

$$= \frac{0 + 0 - 1}{\sqrt{1 + 25 + 1}\ \sqrt{0 + 0 + 1}}$$

$$\cos\theta = \frac{-1}{\sqrt{27}\ \sqrt{1}} = \frac{-1}{3\sqrt{3}}$$

Q52. If u = (5, -1, 2), find norm of u?

Ans:

$$u = (5, -1, 2)$$

$$\|u\| = \sqrt{5^2 + (-1)^2 + 2^2}$$

$$= \sqrt{25 + 1 + 4}$$

$$= \sqrt{30}$$

Q53. Let A be the matrix

$$A = \begin{bmatrix} 2 & 0 \\ 4 & 1 \end{bmatrix}$$

find P(A) where P(x) = x^3 -2x + 4?

Ans:

$$P(x) = x^3 - 2x + 4 \quad \text{and} \quad A = \begin{bmatrix} 2 & 0 \\ 4 & 1 \end{bmatrix}$$

$$P(A) = A^3 - 2A + 4I$$

$$= \begin{bmatrix} 2 & 0 \\ 4 & 1 \end{bmatrix}^3 - 2\begin{bmatrix} 2 & 0 \\ 4 & 1 \end{bmatrix} + 4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 0 \\ 64 & 1 \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ 8 & 2 \end{bmatrix} + \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 0 \\ 64 & 1 \end{bmatrix} - \begin{bmatrix} 8 & 0 \\ 8 & 6 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 56 & -5 \end{bmatrix}$$

Q54. Is given matrix symmetric?

$$A = \begin{bmatrix} 2 & 3a & 0 \\ 3a & 1 & b \\ 0 & b & 0 \end{bmatrix}$$

Ans:

$$A = \begin{bmatrix} 2 & 3a & 0 \\ 3a & 1 & b \\ 0 & b & 0 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 2 & 3a & 0 \\ 3a & 1 & b \\ 0 & b & 0 \end{bmatrix}$$

$A = A^T$ So this is a symmetric matrix

Q55. Find P(A) for P(x) = x -2x -3 and

$$A = \begin{bmatrix} -1 & 2 \\ 0 & 3 \end{bmatrix}$$

Ans:

▶ EXAMPLE 12 **A Matrix Polynomial**

Find $p(A)$ for

$$p(x) = x^2 - 2x - 3 \quad \text{and} \quad A = \begin{bmatrix} -1 & 2 \\ 0 & 3 \end{bmatrix}$$

Solution

$$p(A) = A^2 - 2A - 3I$$
$$= \begin{bmatrix} -1 & 2 \\ 0 & 3 \end{bmatrix}^2 - 2\begin{bmatrix} -1 & 2 \\ 0 & 3 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 4 \\ 0 & 9 \end{bmatrix} - \begin{bmatrix} -2 & 4 \\ 0 & 6 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

or more briefly, $p(A) = 0$. ◀

Q56. Find inverse of

$$A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

Ans:

$$A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$\text{if } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ then } A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$A^{-1} = \frac{1}{\cos^2\theta + \sin^2\theta}\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$A^{-1} = \frac{1}{1}\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Q57. Find ‘a’ and ‘b’ if

$$\begin{bmatrix} a+3 & 1 \\ -3 & 3b-4 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -3 & 2 \end{bmatrix}$$

Ans:

$$\begin{bmatrix} a+3 & 1 \\ -3 & 3b-4 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -3 & 2 \end{bmatrix}$$

$a+3 = 2$

$a = 2-3$

$a = -1$

$3b-4 = 2$

$3b = 2+4$

$3b = 6$

$b = \frac{6}{3}$

$b = 2$

Q58. Find AB & BA

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 5 \end{bmatrix} \quad B = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$$

Ans:

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 5 \end{bmatrix} \quad B = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$$

Find AB and BA

$$AB = \begin{bmatrix} (2\times 3)+(3\times 1) & (2\times 5)+(3\times 2) \\ (1\times 3)+(5\times 1) & (1\times 5)+(5\times 2) \end{bmatrix} = \begin{bmatrix} 6+3 & 10+6 \\ 3+5 & 5+10 \end{bmatrix}$$

$$AB = \begin{bmatrix} 9 & 16 \\ 8 & 15 \end{bmatrix}$$

$$BA = \begin{bmatrix} (3\times 2)+(5\times 1) & (3\times 3)+(5\times 5) \\ (1\times 2)+(2\times 1) & (1\times 3)+(2\times 5) \end{bmatrix} = \begin{bmatrix} 11 & 34 \\ 4 & 13 \end{bmatrix}$$

Q59. Prove $(AB)^T = B^T A^T$

Ans:

Example 3: If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}$, verify that $(AB)^T = B^T A^T$.

Solution:

The product of A and B is:

$$(AB) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 5 & -2 \\ 11 & -4 \end{bmatrix}$$

And the transpose of (AB) is:

$$(AB)^T = \begin{bmatrix} 5 & 11 \\ -2 & -4 \end{bmatrix}$$

If we take the transpose of A and B separately and multiply A with B, then we have:

$$B^T A^T = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 5 & 11 \\ -2 & -4 \end{bmatrix}$$

Hence $(AB)^T = B^T A^T$.

Q60. What is linear programming?

Ans:

- Linear programming is a method of finding a maximum or minimum value of an objective function that satisfies a given set of conditions called constraints

Q61. Is (5, 11) a linear combination of (1, 0) and (0, 1)

Ans:

$u = (1, 0) \quad v = (0, 1)$

Since linear combination

$\because \quad k_1 u + k_2 v = (a, b)$

$k_1(1, 0) + k_2(0, 1) = (5, 11)$

$(1k_1, 0k_1) + (0k_2, 1k_2) = (5, 11)$

$k_1 + 0k_2 = 5 \quad \text{—— (1)}$

$0k_1 + k_2 = 11 \quad \text{—— (2)}$

$k_1 = 5 \quad k_2 = 11$

So we check for linear combination

$k_1 u + k_2 v = 5(1, 0) + 11(0, 1)$

$= (5, 0) + (0, 11)$

$= (5, 11)$

Since $(5, 11) = 5u + 11v$ is a linear combination of u and v

Q62. Express the matrix equation as a system of linear equation

$$\begin{bmatrix} 5 & 6 & -7 \\ -1 & -2 & 3 \\ 0 & 4 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}$$

Ans:

$$\begin{bmatrix} 5 & 6 & -7 \\ -1 & -2 & 3 \\ 0 & 4 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}$$

$$5x_1 + 6x_2 - 7x_3 = 2$$
$$-1x_1 - 2x_2 + 3x_3 = 0$$
$$4x_2 - 1x_3 = 3$$

Q63. Find M_{12} , M_{22} , M_{32} for the matrix

$$A = \begin{bmatrix} 4 & 4 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

Ans:

$$A = \begin{bmatrix} 4 & 4 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$M_{12} = \begin{vmatrix} -1 & 0 \\ 0 & 5 \end{vmatrix} = -5 - 0 = -5$$

$$M_{22} = \begin{vmatrix} 4 & 0 \\ 0 & 5 \end{vmatrix} = 20 - 0 = 20$$

$$M_{32} = \begin{vmatrix} 4 & 0 \\ -1 & 0 \end{vmatrix} = 0 - 0 = 0$$

Q64. Check given matrix is Hermitian or not

$$A \begin{bmatrix} 1 & i & 1+i \\ -i & -5 & 2-i \\ 1-i & 2+i & 3 \end{bmatrix}$$

Ans:

$$A \begin{bmatrix} 1 & i & 1+i \\ -i & -5 & 2-i \\ 1-i & 2+i & 3 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & -i & 1-i \\ i & -5 & 2+i \\ 1+i & 2-i & 3 \end{bmatrix}$$

$$A^{\theta} = \begin{bmatrix} 1 & i & 1+i \\ -i & -5 & 2-i \\ 1-i & 2+i & 3 \end{bmatrix}$$

$A = A^{\theta}$ so this is a Hermitian matrix

Q65. Define Vector Space?

Ans:

Vector Space Definition

A space comprised of vectors, collectively with the associative and commutative law of addition of vectors and also the associative and distributive process of multiplication of vectors by scalars is called vector space. A

Q66. Compute u + v and ku for u = (-1, 2), v = (3, 4) and k=3?

Ans:

(a) Compute $u+v$ and ku for $u=(-1,2)$ $v=(3,4)$, and $k=3$

Solution

(a) $u+v=(-1,2)+(3,4)$

$=(-1+3,\ 2+4)$

$=(2,6)$

$3u = 3(-1,2) = (0,6)$ $\because ku=(0,ku_2)$

Q67. Find the cosine of the angle between the vectors w.r.t the Euclidean inner product u = (1, -3), v = (2, 4)

Ans:

$u=(1,-3),\ v=(2,4)$

$$\cos\theta = \frac{u\cdot v}{\|u\|\|v\|} = \frac{1\cdot 2+(-3)\cdot 4}{\sqrt{1^2+(-3)^2}\ \sqrt{2^2+4^2}}$$

$$= \frac{2-12}{\sqrt{1+9}\ \sqrt{4+16}} = \frac{-10}{\sqrt{10}\ \sqrt{20}}$$

Q68. Determine whether the vectors are orthogonal w.r.t the Euclidean inner product u = (-1, 3, 2), v = (4, 2, -1)?

Ans:

$u = (-1, 3, 2) \quad V = (4, 2, -1)$

$$u \cdot V = (-1)(4) + (3)(2) + (2)(-1)$$
$$= -4 + 6 - 2$$
$$= 6 - 6 = 0$$

The vectors u and v are orthogonal

Q69. What do you mean by linear algebra?

Ans: Linear algebra is a branch of mathematics concerned with the study of linear combinations. It is the study of vector spaces, lines and planes, and some mappings that are required to perform the linear transformations. It includes vectors, matrices and linear functions. It is the study of linear sets of equations and its transformation properties.

Q70. Differentiate between scalar and vector quantities?

Ans:

Difference between Scalar and Vector Quantities

Scalar Quantities	Vector Quantities
1. Scalar quantities are specified by magnitude only.	1. Vector quantities are specified by both magnitude and direction.
2. Scalar quantities change with change in magnitude only.	2. Vector quantities change either with the change in magnitude or with the change in direction or with the change of both magnitude and direction.
3. Scalar quantities with same units can be added or subtracted according to the ordinary rules of algebra.	3. Vector quantities cannot be added or subtracted algebraically.
4. These are represented by ordinary letters.	4. These are represented by bold-faced letters or letters having arrow over them e.g. $\vec{A}$ is read as vector A.

Q71. Differentiate between dot product and length?

Ans:

DEFINITION 1 If $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ is a vector in R^n, then the ***norm*** of $\mathbf{v}$ (also called the ***length*** of $\mathbf{v}$ or the ***magnitude*** of $\mathbf{v}$) is denoted by $\|\mathbf{v}\|$, and is defined by the formula

$$\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2} \tag{3}$$

DEFINITION 3 If $\mathbf{u}$ and $\mathbf{v}$ are nonzero vectors in R^2 or R^3, and if θ is the angle between $\mathbf{u}$ and $\mathbf{v}$, then the ***dot product*** (also called the ***Euclidean inner product***) of $\mathbf{u}$ and $\mathbf{v}$ is denoted by $\mathbf{u} \cdot \mathbf{v}$ and is defined as

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\|\,\|\mathbf{v}\| \cos\theta \tag{12}$$

If $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}$, then we define $\mathbf{u} \cdot \mathbf{v}$ to be 0.

Q72. What do you mean by Matrix?

Ans: Matrix is an arrangement of numbers into rows and columns.

For example, matrix A has two **rows** and three **columns**.

$$\begin{array}{cc} & \text{3 columns} \\ & \downarrow\ \ \downarrow\ \ \downarrow \\ A = & \begin{bmatrix} -2 & 5 & 6 \\ 5 & 2 & 7 \end{bmatrix} \begin{array}{l} \leftarrow \\ \leftarrow \end{array} \text{2 rows} \end{array}$$

Q73. Give a short description on discrete transformation?

Ans:

DFT
DFT stands for Discrete Fourier Transform.
DFT is the discrete version of the Fourier Transform.
It is the algorithm that transforms the time domain signals to the frequency domain components.
It establishes a relationship between the time domain and the frequency domain representation
Applications of DFT include solving partial differential applications, detection of targets from radar echoes, correlation analysis, computing polynomial multiplication, spectral analysis, etc.

Q74. What do you mean by Markov matrices?

Ans:

A matrix A is a **Markov matrix** if

- Its entries are all ≥ 0
- Each **column**'s entries **sum to 1**

Typicaly, a Markov matrix's entries represent **transition probabilities** from one state to another.

1 The matrix

$$A = \begin{bmatrix} 1/2 & 1/3 \\ 1/2 & 2/3 \end{bmatrix}$$

is a Markov matrix.

Markov matrices are also called **stochastic matrices**.

Q75. What do you mean by a solution?

Ans: A solution to a linear system is an assignment of values to the variables such that all the equations are simultaneously satisfied. For example the system given below

5x + y = 3

2x – y = 4

has the solution x = 1, y = -2.

Q76. Define a complex vector?

Ans: An imaginary vector is defined as a sum of two real vectors of which one has been multiplied by the imaginary scalar or unit imaginary $i = \sqrt{-1}$. An imaginary vector A is then defined as

$$A = \mathbf{a} + i\mathbf{b}.$$

Q78. What do mean by orthonormal and orthogonal bases?

Ans:

Definition

Let V be a subspace in $\mathbb{R}^n$.

1. If a basis B for V is an orthogonal set, then B is called an **orthogonal basis.**
2. If a basis B for V is an orthonormal set, then B is called an **orthonormal basis.**

Q79. What do you mean by the elimination in matrices?

Ans: Gaussian elimination, also known as **row reduction**, is an algorithm for solving systems of linear equations. It consists of a sequence of operations performed on the corresponding matrix of coefficients. This method can also be used to compute the rank of a matrix, the determinant of a square matrix, and the inverse of an invertible matrix.

Q80. Determine whether the given matrix is elementary or not

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Ans:

DEFINITION 2 A matrix E is called an ***elementary matrix*** if it can be obtained from an identity matrix by performing a *single* elementary row operation.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

This is not an elementry matrix because it cannot be obtained from identity matrix by performing a single elementry row operation.

Q81. Find inverse of given matrix if exists.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

Ans:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $A^{-1} = \dfrac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ $\det(A) \neq 0$

$$\det(A) = 4 - 6 = -2$$

$$A^{-1} = \frac{1}{-2} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1/-2 & -1 \\ 3/-2 & -2 \end{bmatrix}$$

Q82. Find the value of k for which the matrix is invertible

$$\begin{bmatrix} k-3 & -2 \\ -2 & k-2 \end{bmatrix}$$

Ans:

15. $\det(A) = (k-3)(k-2) - 4$

$= k^2 - 5k + 6 - 4$

$= k^2 - 5k + 2$

Use the quadratic formula to solve $k^2 - 5k + 2 = 0$.

$$k = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(1)(2)}}{2(1)} = \frac{5 \pm \sqrt{17}}{2}$$

A is invertible for $k \neq \dfrac{5 \pm \sqrt{17}}{2}$.

Q83. Let v = (0 , 5) and w = (-1 , 4) find the components of v+w?

Ans:

$v = (0, 5) \qquad w = (-1, 4)$

$$v + w = (0 + (-1), 5 + 4)$$
$$= (-1, 9)$$

Q84. Let v = (-2, 3, 0, 6), find all scalars k such that ||kv|| = 5?

Ans:

7. $\|k\mathbf{v}\| = |k|\,\|(-2, 3, 0, 6)\|$
$$= |k|\sqrt{(-2)^2 + 3^2 + 0 + 6^2}$$
$$= |k|\sqrt{49}$$
$$= 7|k|$$

$7|k| = 5$ if $|k| = \frac{5}{7}$, so $k = \frac{5}{7}$ or $k = -\frac{5}{7}$.

Q85. Is the vector v = (2, 1, 1) a linear combination of u = (1, 1, 1) and v = (1,0,1).

Ans:

$u = (1, 1, 1) \qquad v = (1, 0, 1)$

Since linear combination

$\because \quad k_1 u + k_2 v = (a, b, c)$

$k_1(1, 1, 1) + k_2(1, 0, 1) = (2, 1, 1)$

$(k_1, k_1, k_1) + (k_2, 0k_2, k_2) = (2, 1, 1)$

$k_1 + k_2 = 2 \quad —— (1)$

$k_1 + 0k_2 = 1 \quad —— (2)$

$k_1 + k_2 = 1 \quad —— (3)$

$\Rightarrow (2)$

$k_1 = 1$

put in (1)

$1 + k_2 = 2 \quad \Rightarrow \quad k_2 = 2 - 1 = 1$

so we check for linear combination

$k_1 u + k_2 v = (1, 1, 1) + (1, 0, 1)$

$= (2, 1, 2)$

is not a linear combination

of $1u + 1v \neq (2, 1, 1)$

Q86. Determine whether u and v are orthogonal vectors or not u = (1, 2, -1) v=(0, 1, 1)?

Ans:

$u = (1, 2, -1) \quad v = (0, 1, 1)$

$u \cdot v = (1)(0) + (2)(1) + (-1)(1)$
$= 0 + 2 - 1 = 1$

u and v are not orthogonal

Q87. State Cauchy Schwarz Inequality?

Ans:

THEOREM 3.2.4 Cauchy–Schwarz Inequality

If $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ *and* $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ *are vectors in* R^n, *then*

$$|\mathbf{u} \cdot \mathbf{v}| \leq \|\mathbf{u}\| \|\mathbf{v}\| \tag{22}$$

or in terms of components

$$|u_1v_1 + u_2v_2 + \cdots + u_nv_n| \leq (u_1^2 + u_2^2 + \cdots + u_n^2)^{1/2}(v_1^2 + v_2^2 + \cdots + v_n^2)^{1/2} \tag{23}$$

Q89. Find $\bar{u}$, Re(u), Im(u) and ||u|| if u = (2 - i, 4i , 1 + i)?

Ans:

1. $\bar{\mathbf{u}} = \left(\overline{2-i}, \overline{4i}, \overline{1+i}\right) = (2+i, -4i, 1-i)$

$\text{Re}(\mathbf{u}) = (\text{Re}(2-i), \text{Re}(4i), \text{Re}(1+i)) = (2, 0, 1)$

$\text{Im}(\mathbf{u}) = (\text{Im}(2-i), \text{Im}(4i), \text{Im}(1+i)) = (-1, 4, 1)$

$$\begin{aligned}\|\mathbf{u}\| &= \sqrt{|2-i|^2 + |4i|^2 + |1+i|^2} \\ &= \sqrt{\left(\sqrt{2^2+1^2}\right)^2 + \left(\sqrt{4^2}\right)^2 + \left(\sqrt{1^2+1^2}\right)^2} \\ &= \sqrt{5+16+2} \\ &= \sqrt{23}\end{aligned}$$

Q90. Show that the given set form a basis for R^2 {(2, 1)(3, 0)}?

Ans:

$u = (2, 1)$, $v = (3, 0)$

$k_1 u + k_2 v = 0$

$k_1(2, 1) + k_2(3, 0)$

$2k_1 + 3k_2 = 0$

$k_1 + 0k_2 = 0$

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix} = 0 - 3 = -3 \neq 0$$

so vectors are linearly independent and forms basis for R^2.

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