Definition

An inner product on a real vector space V is a function that associates a real number $\langle u, v \rangle$ with each pair of vectors in V in such a way that the following axioms are satisfied for all vectors u, v and w in V and all scalars k.

1. $\langle u, v \rangle = \langle v, u \rangle$
2. $\langle u+v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$
3. $\langle ku, v \rangle = k\langle u, v \rangle$
4. $\langle v, v \rangle \geq 0$ and $\langle v, v \rangle = 0$ iff $v = 0$

A real vector space with an inner product is called real inner product space.

Definition

If V is a real inner product space, then the norm of a vector v in V is denoted by $\|v\|$ and is defined by

$$\|v\| = \sqrt{\langle v, v \rangle}$$

Examples

(i) The standard inner product on M_{nn}

Let $u, v \in M_{nn}$ be square matrices of order $n \times n$

then

$$\langle u, v \rangle = tr(u^T v)$$

Example

Let $U = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $V = \begin{bmatrix} -1 & 0 \\ 3 & 2 \end{bmatrix}$; $\langle U, V\rangle = ?$ $\|u\| = ?$ $\|v\| = ?$

$$\langle u, v\rangle = tr(u^T v)$$
$$= tr\left\{\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 3 & 2 \end{bmatrix}\right\}$$
$$= tr\left\{\begin{bmatrix} -1+9 & 0+6 \\ -2+12 & 0+8 \end{bmatrix}\right\}$$
$$= tr\begin{bmatrix} 8 & 6 \\ 10 & 8 \end{bmatrix}$$
$$= 8+8$$
$$\boxed{\langle u, v\rangle = 16}$$

$$\|u\| = \sqrt{\langle u, u\rangle} \quad \text{--- (1)}$$

Now

$$\langle u, u\rangle = tr(u^T u)$$
$$= tr\left\{\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\right\}$$
$$= tr\left\{\begin{bmatrix} 1+9 & 2+12 \\ 2+12 & 4+16 \end{bmatrix}\right\}$$
$$= tr\begin{bmatrix} 10 & 14 \\ 14 & 20 \end{bmatrix}$$
$$= 10+20 = 30$$

using in (1)

$$\boxed{\|u\| = \sqrt{30}}$$

$$\|v\| = \sqrt{\langle v, v\rangle} \quad \text{--- (2)}$$

Now

$$\langle v, v\rangle = tr(v^T v)$$
$$= tr\left\{\begin{bmatrix} -1 & 3 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 3 & 2 \end{bmatrix}\right\}$$
$$= tr\begin{bmatrix} 1+9 & 0+6 \\ 0+6 & 0+4 \end{bmatrix}$$
$$= tr\begin{bmatrix} 10 & 6 \\ 6 & 4 \end{bmatrix}$$
$$= 10+4$$
$$= 14$$

using in (2)

$$\boxed{\|v\| = \sqrt{14}}$$

(ii) The standard inner product on P_n

$p = a_0 + a_1 x + \cdots + a_n x^n$

$q = b_0 + b_1 x + \cdots + b_n x^n$

$$\boxed{\langle p, q \rangle = a_0 b_0 + a_1 b_1 + \cdots + a_n b_n}$$

Also $\langle p, p \rangle = a_0 a_0 + a_1 a_1 + \cdots + a_n a_n$

$= a_0^2 + a_1^2 + \cdots + a_n^2$

$$\boxed{\|p\| = \sqrt{\langle p, p \rangle} = \sqrt{a_0^2 + a_1^2 + \cdots + a_n^2}}$$

example let $V = P_2$

$p(x) = x^2$; $q(x) = 1 + x$

Here $a_0 = 0$; $a_1 = 0$; $a_2 = 1$

$b_0 = 1$; $b_1 = 1$; $b_2 = 0$

$\langle p, q \rangle = (0)(1) + (0)(1) + (1)(0) = 0$

$\|p\| = \sqrt{\langle p, p \rangle} = \sqrt{0^2 + 0^2 + 1^2} = \sqrt{1} = 1$

$\|q\| = \sqrt{\langle q, q \rangle} = \sqrt{1^2 + 1^2 + 0^2} = \sqrt{1+1} = \sqrt{2}$

(iii) The standard inner product on $\mathbb{R}^n$

let $u, v \in \mathbb{R}^n$

$u = (u_1, u_2, \ldots, u_n)$

$v = (v_1, v_2, \ldots, v_n)$

Now

$\langle u, v \rangle = u \cdot v = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$

$\|u\| = \sqrt{\langle u, u \rangle} = \sqrt{u_1^2 + u_2^2 + \cdots + u_n^2}$

Angle and orthogonality in inner product Spaces

Let V be a vector space. $u, v \in V$ then angle b/w u and v is defined by

$$\theta = \cos^{-1}\left(\frac{\langle u, v\rangle}{\|u\|\,\|v\|}\right)$$

or $\cos\theta = \dfrac{\langle u, v\rangle}{\|u\|\,\|v\|}$

Example

Let M_{22} have standard inner product. Find the cosine of angle between the vectors

$$U = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad V = \begin{bmatrix} -1 & 0 \\ 3 & 2 \end{bmatrix}$$

Solution As solved earlier

$$\langle u, v\rangle = 16, \quad \|u\| = \sqrt{30}, \quad \|v\| = \sqrt{14}$$

$$\cos\theta = \frac{\langle u, v\rangle}{\|u\|\,\|v\|} = \frac{16}{\sqrt{30}\sqrt{14}} \approx 0.78$$

Definition

Two vectors u and v in an inner product space V called orthogonal if $\langle u, v\rangle = 0$

Example Show that $u = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, v = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$ are orthogonal.

$$\langle u, v\rangle = tr\{u^T v\}$$
$$= tr\left\{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}\right\}$$
$$= tr\begin{bmatrix} 0+0 & 2+0 \\ 0+0 & 0+0 \end{bmatrix} = tr\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} = 0+0=0$$

Hence u, v are orthogonal.

Definition A set of two or more vectors in a real inner product space is said to be orthogonal if all pairs of distinct vectors in the set are orthogonal. An orthogonal set in which each vector has norm 1 is said to be orthonormal.

Example

orthogonal set in $\mathbb{R}^3$

let $v_1 = (0,1,0)$
$v_2 = (1,0,1)$
$v_3 = (1,0,-1)$

Now $\langle v_1, v_2 \rangle = (0)(1) + (1)(0) + (0)(1) = 0+0+0 = 0$
$\langle v_2, v_3 \rangle = (1)(1) + (0)(0) + (1)(-1) = 1+0-1 = 0$
$\langle v_3, v_1 \rangle = (0)(1) + (1)(0) + (0)(-1) = 0+0+0 = 0$

$\Rightarrow \{v_1, v_2, v_3\}$ is orthogonal set.

Constructing an orthonormal set

In above example
$\|v_1\| = 1$, $\|v_2\| = \sqrt{2}$, $\|v_3\| = \sqrt{2}$

Normalizing u_1, u_2 and u_3, we have

$$u_1 = \frac{v_1}{\|v_1\|} = (0, 1, 0)$$

$$u_2 = \frac{v_2}{\|v_2\|} = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$$

$$u_3 = \frac{v_3}{\|v_3\|} = \left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right)$$

Note that
$\langle u_1, u_2 \rangle = \langle u_2, u_3 \rangle = \langle u_3, u_1 \rangle = 0$

Also
$\|u_1\| = 1$; $\|u_2\| = 1$; $\|u_3\| = 1$

Hence $\{u_1, u_2, u_3\}$ form an orthonormal set.

Theorem

If $S = \{v_1, v_2, \ldots, v_r\}$ is an orthogonal set of non-zero vectors in an inner product space, then S is linearly independent.

Example

$u_1 = (0, 1, 0)$, $u_2 = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$, $u_3 = \left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right)$

form an orthonormal basis of R^3.

Solution $\because$ $\{u_1, u_2, u_3\}$ are orthogonal

$\Rightarrow$ $\{u_1, u_2, u_3\}$ are linearly independent. (by above theorem)

$\because$ $\dim R^3 = 3$

$\Rightarrow$ Hence the three linearly independent vectors $\{u_1, u_2, u_3\}$ form a basis for R^3.

Example 6 page 367 (Do yourself)

The Gram-Schmidt Process

To convert a basis $\{u_1, u_2, \ldots, u_r\}$ into an orthogonal basis $\{v_1, v_2, \ldots, v_r\}$, perform the following computations.

Step I: $v_1 = u_1$

Step 2: $v_2 = u_2 - \frac{\langle u_2, v_1 \rangle}{\|v_1\|^2} v_1$

Step 3: $v_3 = u_3 - \frac{\langle u_3, v_1 \rangle}{\|v_1\|^2} v_1 - \frac{\langle u_3, v_2 \rangle}{\|v_2\|^2} v_2$

$\vdots$

continue for r steps.

* To convert the orthogonal basis into an orthonormal basis $\{q_1, q_2, \ldots, q_s\}$ normalize the orthogonal basis vectors.

<u>Example</u>

Assume vector space $\mathbb{R}^3$ has Euclidean inner product. Apply Gram-Schmidt process to transform the basis vectors

$u_1 = (1,1,1)$
$u_2 = (0,1,1)$
$u_3 = (0,0,1)$

into an orthogonal basis $\{v_1, v_2, v_3\}$ and then normalize the orthogonal basis vectors to obtain an orthonormal basis $\{q_1, q_2, q_3\}$.

<u>Sol:</u> <u>Step 1:</u> $v_1 = u_1 = (1,1,1)$

<u>Step 2:</u> $v_2 = u_2 - \cancel{\text{proj}}\frac{\langle u_2, v_1\rangle}{\|v_1\|^2} v_1$

$$= (0,1,1) - \frac{1.0 + 1.1 + 1.1}{(\sqrt{1^2+1^2+1^2})^2}(1,1,1)$$

$$= (0,1,1) - \frac{2}{(\sqrt{3})^2}(1,1,1) = (0,1,1) - \left(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}\right)$$

$$= \left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right)$$

<u>Step 3</u>: $v_3 = u_3 - \frac{\langle u_3, v_1\rangle}{\|v_1\|^2} v_1 - \frac{\langle u_3, v_2\rangle}{\|v_2\|^2} v_2$

$$v_3 = (0,0,1) - \frac{0+0+1}{(\sqrt{1^2+1^2+1^2})^2}(1,1,1) - \frac{0+0+\frac{1}{3}}{\left(\sqrt{\frac{4}{9}+\frac{1}{9}+\frac{1}{9}}\right)^2}\left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right)$$

$= (0,0,1) - \frac{1}{3}(1,1,1) - \frac{1/3}{2/3}\left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right)$

$$\boxed{v_3 = \left(0, -\frac{1}{2}, \frac{1}{2}\right)}$$

Now

$v_1 = (1,1,1)$, $v_2 = \left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right)$, $v_3 = \left(0, -\frac{1}{2}, \frac{1}{2}\right)$

form orthogonal basis for R^3.

$\|v_1\| = \sqrt{3}$, $\|v_2\| = \frac{\sqrt{6}}{3}$, $\|v_3\| = \frac{1}{\sqrt{2}}$

so an orthonormal basis for R^3 is

$$q_1 = \frac{v_1}{\|v_1\|} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

$$q_2 = \frac{v_2}{\|v_2\|} = \left(\frac{-2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$$

$$q_3 = \frac{v_3}{\|v_3\|} = \left(0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

H.W (Exercise 6.3)

Q#1, 2

Q#25—31