Chapter 7

RANDOM VARIABLES

7.1. (b) The sample space for this experiment is

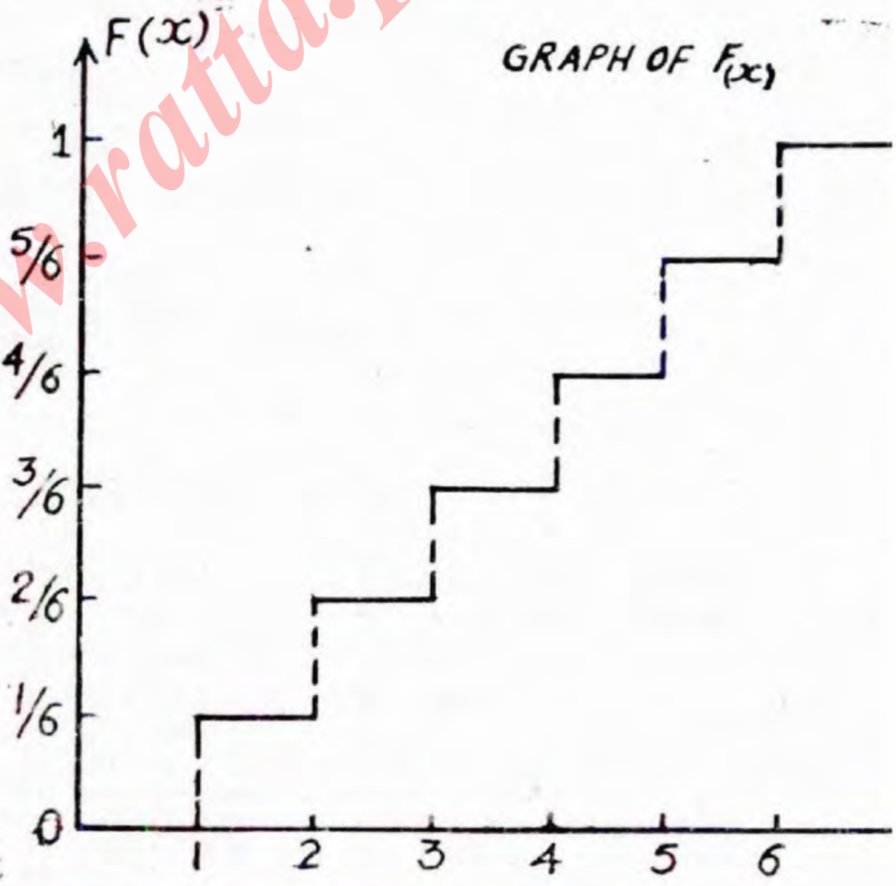
$$S = \{1, 2, 3, 4, 5, 6\}$$

Let X be the r.v. that denote the number of points appearing. Then the values of x are 1, 2, 3, 4, 5 and 6. As all six faces are equally likely to appear, therefore

$P(1) = P(2) = P(3) = P(4) = P(5) = P(6) = 1/6$

Hence the desired distribution function, $F(x)$, is

$$F(x) = \begin{cases} 0, \text{ for } x < 1 \\ \frac{1}{6}, \text{ for } 1 \le x < 2 \\ \frac{2}{6}, \text{ for } 2 \le x < 3 \\ \frac{3}{6}, \text{ for } 3 \le x < 4 \\ \frac{4}{6}, \text{ for } 4 \le x < 5 \\ \frac{5}{6}, \text{ for } 5 \le x < 6 \\ 1, \text{ for } x \ge 1 \end{cases}$$



7.2 (b) Given that

x	-1	0	1	Total
$f(x)$	$3c$	$3c$	$6c$	$12c$

(i) Since the sum of probabilities in a *p.d.* should be equal to 1, therefore $12c = 1$ or $c = \frac{1}{12}$.

(ii) The probability distribution of $Y = 2X + 1$ is found as below:

x_i	-1	0	1
$f(x_i)$	$3c = \frac{3}{12}$	$\frac{3}{12}$	$\frac{6}{12}$
$y_i = 2x_i + 1$	-1	1	3

Hence the desired *p.d.* of $Y = 2X + 1$ is

y_i	-1	1	3
$f(y_i)$	$3/12$	$3/12$	$6/12$

(c) Let X be the random variable that denotes the number of aces in a bridge hand. Then the values that X can take, are 0, 1, 2, 3, and 4. The desired *p.d.* is given below:

x_i	$f(x_i)$
0	$\binom{4}{0}\binom{48}{13} \div \binom{52}{13} = \frac{6327}{20825} = 0.3038$
1	$\binom{4}{1}\binom{48}{12} \div \binom{52}{13} = \frac{9139}{20825} = 0.4389$
2	$\binom{4}{2}\binom{48}{11} \div \binom{52}{13} = \frac{4446}{20825} = 0.2135$
3	$\binom{4}{3}\binom{48}{10} \div \binom{52}{13} = \frac{858}{20825} = 0.0412$
4	$\binom{4}{4}\binom{48}{9} \div \binom{52}{13} = \frac{55}{20825} = 0.0026$
Total	1

7.3. (b) S contains $\binom{10}{4} = 210$ sample points.

Let X be the r.v. that denotes the number of red balls. Then the values of X are 0, 1, 2, 3 and 4, and their probabilities are

$$f(0) = P(X=0) = \frac{\binom{4}{0}\binom{6}{4}}{\binom{10}{4}} = \frac{15}{210}$$

$$f(1) = P(X=1) = \frac{\binom{4}{1}\binom{6}{3}}{\binom{10}{4}} = \frac{80}{210}$$

$$f(2) = P(X=2) = \frac{\binom{4}{2}\binom{6}{2}}{\binom{10}{4}} = \frac{90}{210}$$

$$f(3) = P(X=3) = \frac{\binom{4}{3}\binom{6}{1}}{\binom{10}{4}} = \frac{24}{210}$$

$$f(4) = P(X=4) = \frac{\binom{4}{4}\binom{6}{0}}{\binom{10}{4}} = \frac{1}{210}$$

Putting this information in a tabular form, we obtain the desired probability distribution of X as

No. of red balls: x_i	0	1	2	3	4
Probability: $f(x_i)$	$\frac{15}{210}$	$\frac{80}{210}$	$\frac{90}{210}$	$\frac{24}{210}$	$\frac{1}{210}$

7.4. (a) S consists of $\binom{15}{3}$ = 455 sample points.

Let X be the r.v. that denotes the number of defective clock radios in the sample. Then the values of x are 0, 1, 2, 3 and their probabilities are

$$f(0) = P(X=0) = \binom{5}{0}\binom{10}{3} \div \binom{15}{3} = \frac{120}{455}$$

$$f(1) = P(X=1) = \binom{5}{1}\binom{10}{2} \div \binom{15}{3} = \frac{225}{455}$$

$$f(2) = P(X=2) = \binom{5}{2}\binom{10}{1} \div \binom{15}{3} = \frac{100}{455}$$

$$f(3) = P(X=3) = \binom{5}{3}\binom{10}{0} \div \binom{15}{3} = \frac{10}{455}$$

Hence the desired probability distribution of X is

No. of defectives: x_i	0	1	2	3	Total
Probability: $f(x_i)$	$\frac{120}{455}$	$\frac{225}{455}$	$\frac{100}{455}$	$\frac{10}{455}$	1

(b) S consists of $\binom{8}{3} = 56$ sample points.

Let X be the r.v. that denotes the number of white balls, drawn from the bag. Then the values of x are 0, 1, 2, 3 and their probabilities are:

$$f(0) = P(X=0) = \binom{5}{0}\binom{3}{3} \div \binom{8}{3} = \frac{1}{56}$$

$$f(1) = P(X=1) = \binom{5}{1}\binom{3}{2} \div \binom{8}{3} = \frac{15}{56}$$

$$f(2) = P(X=2) = \binom{5}{2}\binom{3}{1} \div \binom{8}{3} = \frac{30}{56}$$

$$f(3) = P(X=3) = \binom{5}{3}\binom{3}{0} \div \binom{8}{3} = \frac{10}{56}$$

Putting this information in a tabular form, we obtain the desired probability distribution of X as

No. of white balls: x_i	0	1	2	3	Total
Probability: $f(x_i)$	$\frac{1}{56}$	$\frac{15}{56}$	$\frac{30}{56}$	$\frac{10}{56}$	1

7.5. (b) The function f(x) will be a density function, if

(i) $f(x) \geq 0$, and (ii) $\int_{-\infty}^{\infty} f(x)\,dx = 1$

The first condition is satisfied for every x in the given range, and the second coindition will be satisfied if

$$A \int_0^2 (4x - 2x^2)\,dx = 1.$$

$$\text{i.e. if} \quad A\left\{\left[\frac{4x^2}{2}\right]_0^2 - 2\left[\frac{x^3}{3}\right]_0^2\right\} = 1,$$

$$\text{i.e. if} \quad A\left\{8 - 2\left(\frac{8}{3}\right)\right\} = 1,$$

$$\text{i.e. if} \quad A\left\{\frac{24-16}{3}\right\} = 1.$$

This gives $A = \dfrac{3}{8}$.

7.6. (b) Clearly $f(x) \geq 0$, and

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_0^1 \frac{x}{2}\,dx + \frac{1}{4}\int_1^2 (3-x)dx + \frac{1}{4}\int_2^3 dx + \frac{1}{4}\int_3^4 (4-x)\,dx$$

$$= \left[\frac{x^2}{4}\right]_0^1 + \frac{1}{4}\left[3x - \frac{x^2}{2}\right]_1^2 + \frac{1}{4}\left[x\right]_2^3 + \frac{1}{4}\left[4x - \frac{x^2}{2}\right]_3^4$$

$$= \frac{1}{4} + \frac{1}{4}\left[(6-2) - \left(3 - \frac{1}{2}\right)\right] + \frac{1}{4}[3-2] + \frac{1}{4}\left[\left(16 - \frac{16}{2}\right) - \left(12 - \frac{9}{2}\right)\right]$$

$$= \frac{1}{4} + \frac{1}{4}\left(\frac{3}{2}\right) + \frac{1}{4} + \frac{1}{4}\left(\frac{1}{2}\right)$$

$$= \frac{1}{4} + \frac{3}{2} + \frac{1}{4} + \frac{1}{3} = 1.$$

As $f(x)$ is a density function, therefore

$$P(X \geq 3) = \frac{1}{4}\int_3^4 (4-x)\,dx = \frac{1}{4}\left[4x - \frac{x^2}{2}\right]_3^4$$

$$= \frac{1}{4}\left[\left(16 - \frac{16}{2}\right) - \left(12 - \frac{9}{2}\right)\right] = \frac{1}{8};$$

$P(X=2) = 0$, because for a continuous r.v., the probability at any particular value is equal to zero.

$$P(|X| < 1.5) = P(-1.5 < X < 1.5)$$

$$= \int_{-1.5}^{0} 0.\, dx + \int_{0}^{1} \frac{x}{2}\, dx + \frac{1}{4} \int_{1}^{1.5} (3-x)\, dx$$

$$= \left[\frac{x^2}{4}\right]_0^1 + \frac{1}{4}\left[3x - \frac{x^2}{2}\right]_1^{3/2} = \frac{1}{4} + \frac{1}{4}\left[\left(\frac{9}{2} - \frac{9}{8}\right) - \left(3 - \frac{1}{2}\right)\right]$$

$$= \frac{1}{4} + \frac{1}{4}\left[\frac{27}{8} - \frac{5}{2}\right] = \frac{1}{4} + \frac{7}{32} = \frac{15}{32}, \text{ and}$$

$$P(1 < X < 3) = \frac{1}{4}\int_{1}^{2} (3-x)\, dx + \frac{1}{4}\int_{2}^{3} dx$$

$$= \frac{1}{4}\left[3x - \frac{x^2}{2}\right]_1^2 + \frac{1}{4}\left[x\right]_2^3$$

$$= \frac{1}{4}\left[\left(6 - \frac{4}{2}\right) - \left(3 - \frac{1}{2}\right)\right] + \frac{1}{4}[3 - 2]$$

$$= \frac{1}{4}\left(4 - \frac{5}{2}\right) + \frac{1}{4} = \frac{3}{8} + \frac{1}{4} = \frac{5}{8}.$$

7.7. (b) Given $f(x) = A(2-x)(2+x)$, if $0 \le x \le 2$
$= 0$, elsewhere

(i) To find the value of A, we must have that

$$\int_0^2 A(2-x)(2+x)\, dx = 1$$

Solving, $A \int_0^2 (4 - x^2)\, dx = 1$ or $A\left[4x - \frac{x^3}{3}\right]_1^2 = 1$ which gives $A = \frac{3}{16}$.

(ii) $P\left(X = \frac{1}{2}\right) = 0$, as probability at a particular value is zero.

(iii) $P(X \le 1) = \dfrac{3}{16}\int_0^1 (4 - x^2)\,dx = \dfrac{3}{16}\left[4x - \dfrac{x^3}{3}\right]_0^1$

$$= \frac{3}{16}\left[\left(4 - \frac{1}{3}\right) - 0\right] = \frac{11}{16}.$$

(iv) $P(X \ge 2) = 0$, as outside the range 0 to 2, $f(x) = 0$.

(v) $P(1 \le X \le 2) = \dfrac{3}{16}\int_1^2 (4 - x^2)\,dx = \dfrac{3}{16}\left[4x - \dfrac{x^3}{3}\right]_1^2$

$$= \frac{3}{16}\left[\left(8 - \frac{8}{3}\right) - \left(4 - \frac{1}{3}\right)\right]$$

$$= \frac{3}{16}\left[\frac{16}{3} - \frac{11}{3}\right] = \frac{3}{16} \times \frac{5}{3} = \frac{5}{16}.$$

7.8. (i) The function f(x) will be a density function, if

$$f(x) \ge 0 \text{ for every } x, \text{ and } \int_{-\infty}^{\infty} f(x)\,dx = 1.$$

Now the first condition is obvious and the second condition will be satisfied if

$$\int_0^1 6x\,(1 - x)\,dx = 1,$$

i.e. $\quad 6\int_0^1 (x - x^2)\,dx = 1 \quad$ *i.e.* $\quad 6\left[\dfrac{x^2}{2} - \dfrac{x^3}{3}\right]_0^1 = 1$

i.e. $\quad 6\left[\dfrac{1}{2} - \dfrac{1}{3}\right] = 1$, which is true.

Therefore $f(x)$ is a legitimate *p.d.f.*

(ii) The cumulative distribution function (*c.d.f.*) is obtained as

$$F(x) = \int_{-\infty}^{x} f(x)\,dx = 6\int_0^x x(1 - x)\,dx$$

$$= 6\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^x = 3x^2 - 2x^3$$

(iii) Now the probability in the interval $\frac{1}{3}$ to $\frac{2}{3}$ is

$$P\left(\frac{1}{3} < X < \frac{2}{3}\right) = \int_{1/3}^{2/3} f(x)\, dx$$

$$= 6 \int_{1/3}^{2/3} (x - x^2)\, dx = 6 \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_{1/3}^{2/3}$$

$$= 6 \left[\left(\frac{1}{2} \cdot \frac{4}{9} - \frac{1}{3} \cdot \frac{8}{27}\right) - \left(\frac{1}{2} \cdot \frac{1}{9} - \frac{1}{3} \cdot \frac{1}{27}\right)\right]$$

$$= 6 \left[\frac{10}{81} - \frac{7}{162}\right] = 6 \times \frac{13}{162} = \frac{13}{27}; \text{ and}$$

$$P\left(X \le \tfrac{1}{2} / \tfrac{1}{3} \le X \le \tfrac{2}{3}\right) = \frac{P\left(\frac{1}{3} \le X \le \frac{1}{2}\right)}{P\left(\frac{1}{3} \le X \le \frac{2}{3}\right)}, \text{ where}$$

$$P\left(\tfrac{1}{3} \le X \le \tfrac{1}{2}\right) = \int_{1/3}^{1/2} 6(x - x^2)\, dx = 6 \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_{1/3}^{1/2}$$

$$= 6 \left[\left(\frac{1}{2} \cdot \frac{1}{4} - \frac{1}{3} \cdot \frac{1}{8}\right) - \left(\frac{1}{2} \cdot \frac{1}{9} - \frac{1}{3} \cdot \frac{1}{27}\right)\right]$$

$$= 6 \left[\frac{1}{12} - \frac{7}{162}\right] = 6 \times \frac{13}{324} = \frac{13}{54},$$

$$\therefore \quad P\left(X \le \tfrac{1}{2} / \tfrac{1}{3} \le X \le \tfrac{2}{3}\right) = \left(\frac{13}{54}\right) \Big/ \left(\frac{13}{27}\right) = \frac{13}{54} \times \frac{27}{13} = \frac{1}{2}.$$

(iv) Now $\quad P(X < b) = 6 \int_0^b x\,(1 - x)\, dx$

$$= 6 \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^b = 3b^2 - 2b^3; \text{ and}$$

$$P(X > b) = 6 \int_b^1 x(1-x)\,dx$$

$$= 6\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_b^1 = 6\left[\left(\frac{1}{2} - \frac{1}{3}\right) - \left(\frac{b^2}{2} - \frac{b^3}{3}\right)\right]$$

$$= 1 - 3b^2 + 2b^3$$

We are given that $P(X < b) = 2P(X > b)$

i.e. $3b^2 - 2b^3 = 2(1 - 3b^2 + 2b^3)$

i.e. $6b^3 - 9b^2 = -2$

i.e. $b^3 - \frac{3}{2}b^2 + \frac{1}{3} = 0.$

Solving this equation by the Newton-Raphson method, we find that $b = 0.6130$.

Note: The Newton-Raphson method and solution is given below:

Newton-Raphson Formula. Let x_r be an approximation to real root of a function $f(x) = 0$. Then a better approximation, x_{r+1} is given by

$$x_{r+1} = x_r - \frac{f(x_r)}{f'(x_r)}.$$

That is, if x_0 is the initial root, then

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)},$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}, \text{ etc.}$$

are the first approximation, the second approximation, etc.

Here $f(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{3}$, and (replacing b by x)

$f'(x) = 3x^2 - 3x.$

Thus $$x_{r+1} = x_r - \frac{x_r^3 - \frac{3}{2}x_r^2 + \frac{1}{3}}{3x_r^2 - 3x_r} = \frac{2x_r^3 - \frac{3}{2}x_r^2 - \frac{1}{3}}{3x_r^2 - 3x_r}$$

Let the approximate root be 0.5. Then for $r = 0$, and $x_0 = 0.5$, we have

$$x_1 = \frac{2(0.5)^3 - \frac{3}{2}(0.5)^2 - \frac{1}{3}}{3(0.5)^2 - 3(0.5)} = \frac{-0.458}{-0.75} = 0.6107$$

Putting $r = 1$ and $x_1 = 0.6107$, we find

$x_2 = 0.6130$ as a better approximation.

7.9. (i) Let A be the event that a tube will last less than 200 hours, and B be the event that the tube functions after 150 hours of service. We seek P(A/B) which is given as

$$P(A/B) \; \frac{P(A \cap B)}{P(B)}$$

Now $$P(A \cap B) = \int_{150}^{200} \frac{100}{x^2}\,dx$$

$$= \left[\frac{-100}{x}\right]_{150}^{200} = \frac{100}{150} - \frac{100}{200} = \frac{2}{3} - \frac{1}{2} = \frac{1}{6}$$

$$P(B) = \int_{150}^{\infty} \frac{100}{x^2}\,dx = \left[\frac{-100}{x}\right]_{150}^{\infty} = \frac{100}{150} = \frac{2}{3}$$

Hence $$P(A/B) = \frac{1}{6} \div \frac{2}{3} = \frac{1}{6} \times \frac{3}{2} = \frac{1}{4}.$$

(ii) Probability that exactly one out of 3 such tubes will have to be replaced after 150 hours of service (i.e. one not functioning and 2 functioning) is

$$\binom{3}{1}\left(1 - \frac{1}{4}\right)\left(\frac{1}{4}\right)^2 = \frac{9}{64}.$$

(iii) The maximum no. n, of tubes is obtained by solving

$$\left(\frac{2}{3}\right)^n = 0.5, \text{ which gives } n = 2.$$

7.10. (b) The given joint p.d. of two r.v.'s is presented in the following table:

X \ Y	1	2	3	$P(X=x_i)$ $g(x)$
1	6/30	1/30	1/30	8/30
2	4/30	5/30	1/30	10/30
3	2/30	4/30	6/30	12/30
$P(Y=y_i):h(y)$	$\frac{12}{30}$	$\frac{10}{30}$	$\frac{8}{30}$	1

The marginal *p.d.* of X is obtained by adding over the rows and that of Y, by adding over the columns. Thus the two marginal distributions are:

x	1	2	3
$g(x)$	$\frac{8}{30}$	$\frac{10}{30}$	$\frac{12}{30}$

and

y	1	2	3
$h(y)$	$\frac{12}{30}$	$\frac{10}{30}$	$\frac{8}{30}$

By definition the conditional *p.d.* of $X = x_i$ given $Y = y_j$ is

$$f(x_i/y_j) = P(X=x_i/Y=y_j) = \frac{f(x_i, y_j)}{h(y_j)}, \text{ for } i = 1,2,3; j = 1, 2, 3.$$

Thus $$f(1/1) = \frac{f(1, 1)}{h(1)} = \frac{6/30}{12/30} = \frac{1}{2};$$

$$f(2/1) = \frac{f(2, 1)}{h(1)} = \frac{4/30}{12/30} = \frac{1}{3};$$

$$f(3/1) = \frac{f(3, 1)}{h(1)} = \frac{2/30}{12/30} = \frac{1}{6}.$$

Hence the conditional *p.d.* of X given that $Y = 1$ is

x	1	2	3
$f(x/1)$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{6}$

Similarly the conditional distributions of X given that $y = 2, 3$ could be found.

The conditional *p.d.* of Y for given $X = x_i$ is

$$f(y_j/x_i) = P(Y=y_j/X=x_i) = \frac{f(x_i/y_j)}{g(x_i)}.$$

Thus $f(Y/2) = \dfrac{f(2, y)}{g(2)}$ for $y = 1, 2, 3$. That is

$$f(1/2) = \frac{f(2, 1)}{g(2)} = \frac{4/30}{10/30} = \frac{2}{5};$$

$$f(2/2) = \frac{f(2, 2)}{g(2)} = \frac{5/30}{10/30} = \frac{1}{2}; \text{ and}$$

$$f(3/2) = \frac{f(2, 3)}{g(2)} = \frac{1/30}{10/30} = \frac{1}{10}.$$

Hence the conditional *p.d.* of Y given that $X = 2$ is

y	1	2	3
$f(y/2)$	$\frac{2}{5}$	$\frac{1}{2}$	$\frac{1}{10}$

Similarly the conditional distributions of Y given that $X=1$ and 3 could be found.

7.11. The marginal distributions g(x) and h(y) are obtained by adding over the columns and rows respectively in the following table:

Y \ X	1	2	3	$h(y)$
1	1/12	1/6	0	3/12
2	0	1/9	1/5	14/45
3	1/18	1/4	2/15	79/180
$g(x)$	$\frac{5}{36}$	$\frac{19}{36}$	$\frac{5}{15}$	1

Thus the marginal distributions are:

x	1	2	3
$g(x)$	$\frac{5}{36}$	$\frac{19}{36}$	$\frac{5}{15}$

and

y	1	2	3
$h(y)$	$\frac{3}{12}$	$\frac{14}{45}$	$\frac{79}{180}$

By definition, the conditional *p.d.* of $X = x_i$ given $Y = y_j$ is

$$f(x_i/y_j) = P(X=x_i/Y=y_j) = \frac{f(x_i, y_j)}{h(y_j)}$$

For $y = 1$, the conditional distribution of X is

$$f(1/1) = \frac{f(1,1)}{h(1)} = \frac{1/12}{3/12} = \frac{1}{3};$$

$$f(2/1) = \frac{f(2,1)}{h(1)} = \frac{1/6}{3/12} = \frac{2}{3}; \text{ and}$$

$$f(3/1) = \frac{f(3,1)}{h(1)} = \frac{0}{3/12} = 0.$$

Hence the conditional *p.d.* of X given that $Y = 1$ is

x	1	2	3
$f(x/1)$	$\frac{1}{3}$	$\frac{2}{3}$	0

Similarly the conditional distributions of X could be found for given $y = 2, 3$ respectively.

The conditional *p.d.* of $Y = y_j$ for given $X = x_i$ are

$$f(y_j/x_i) = P(Y=y_j/X=x) = \frac{f(x,y)}{g(x)}$$

Thus for x = 3, the conditional distribution of Y is

$$f(1/3) = \frac{f(3,1)}{g(3)} = \frac{0}{5/15} = 0;$$

$$f(2/3) = \frac{f(3,2)}{g(3)} = \frac{1/5}{5/15} = \frac{3}{5}; \text{ and}$$

$$f(3/3) = \frac{f(3,3)}{g(3)} = \frac{2/15}{5/15} = \frac{2}{5}$$

Hence the conditional distribution of Y given that $X = 3$ is

y	1	2	3
$f(y/3)$	0	$\frac{3}{5}$	$\frac{2}{5}$

Similarly for $x=1$ and $x=2$, conditional $p.d.$ of y could be found.

(b) The r.v.'s X and Y will be independent, if

$$f(x, y) = g(x)\ h(y).$$

Now $f(1, 1) = \frac{1}{12}$ and $g(1) \cdot h(1) = \left(\frac{3}{12}\right)\left(\frac{5}{36}\right) = \frac{5}{144}$.

Since $f(1, 1) \neq g(1) \cdot h(1)$, therefore X and Y are not indpendent.

7.12. (i) The marginal probability distribution for X is

$$g(x) = \sum_y f(x, y)$$
$$= \sum_{y=1}^{3} \frac{xy}{66} = \frac{x}{66} + \frac{2x}{66} + \frac{3x}{66} = \frac{x}{11}, \text{ for } x=2,4,5;$$

and the marginal probability distribution for Y is

$$h(y) = \sum_x f(x, y)$$
$$= \sum_x \frac{xy}{66} = \frac{2y}{66} + \frac{4y}{66} + \frac{5y}{66} = \frac{y}{6}, \text{ for } x=1,2,3.$$

Now $\frac{xy}{66} = \frac{x}{11} \times \frac{y}{6}$.

i.e. $f(x,y)=g(x)h(y)$, therefore X and Y are independent.

(ii) The marginal $p.d.$ for X is

$$g(x) = \sum_y f(x, y)$$
$$= \sum_{y=1}^{2} \frac{xy^2}{30} = \frac{x}{30} + \frac{4x}{30} = \frac{x}{6}, \text{ for } x=1,2,3; \text{ and}$$

the marginal $p.d.$ for Y is

$$h(y) = \sum_x f(x, y)$$
$$= \sum_{x=1}^{2} \frac{xy^2}{30} = \frac{y^2}{30} + \frac{2y^2}{30} + \frac{3y^2}{30} = \frac{y^2}{5}, \text{ for } y=1,2.$$

Now $\frac{xy^2}{30} = \frac{x}{6} \times \frac{y^2}{5}$, *i.e.* $f(x, y) = g(x)\ h(y)$

Hence X and Y are independent.

7.13. (b) Given the joint p.d.f. $f(x,y)=3xy(x+y)$, for $0\le x\le 1$, $0\le y\le 1$.

The marginal *p.d.f.* of X is

$$g(x) = 3\int_0^1 xy\,(x+y)\,dy, \qquad 0\le x\le 1,$$

$$= 3\int_0^1 (x^2y + xy^2)\,dy, \qquad 0\le x\le 1,$$

$$= 3\left[\frac{x^2y^2}{2} + \frac{xy^3}{3}\right]_0^1 \qquad 0\le x\le 1,$$

$$= 3\left[\frac{x^2}{2} + \frac{x}{3}\right] \qquad 0\le x\le 1,$$

$$= \frac{3x^2}{2} + x, \qquad 0\le x\le 1.$$

Similarly, the marginal *p.d.f.* of Y is

$$h(y) = 3\int_0^1 (x^2y + xy^2)\,dx, \qquad 0\le y\le 1,$$

$$= 3\left[\frac{x^3y}{3} + \frac{x^2y^2}{2}\right]_0^1 \qquad 0\le y\le 1,$$

$$= 3\left[\frac{y}{2} + \frac{y^2}{2}\right] \qquad 0\le y\le 1,$$

$$= y + \frac{3}{2}y^2. \qquad 0\le y\le 1.$$

The conditional *p.d.f.* of X given $Y=y$ is

$$f(x/y) = \frac{f(x,y)}{h(y)}, \quad \text{where } h(y) > 0$$

$$= \frac{3xy\,(x+y)}{y + \frac{3}{2}y^2} = \frac{3x\,(x+y)}{1+\frac{3}{2}y},$$

and the conditional $p.d.f.$ of Y given $X = x$, is

$$f(y/x) = \frac{f(x, y)}{g(y)} = \frac{3xy\,(x + y)}{x\left(1 + \frac{3}{2}x\right)} = \frac{3y\,(x + y)}{1 + \frac{3}{2}x}.$$

7.14. Given that

$$f(x, y) = x^2 + \frac{1}{3}xy, \qquad 0 \le x \le 1, \qquad 0 \le y \le 2$$

(a) $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)\, dx\, dy = \int_0^1\int_0^2 \left(x^2 + \frac{1}{3}xy\right) dy\, dx$$

$$= \int_0^1 \left[x^2y + \frac{1}{6}xy^2\right]_0^2 dx$$

$$= \int_0^1 \left(2x^2 + \frac{2x}{3}\right) dx$$

$$= \left[\frac{2x^3}{3} + \frac{x^2}{3}\right]_0^1 = \frac{2}{3} + \frac{1}{3} = 1.$$

(b) (i) $$P\left(X > \frac{1}{2}\right) = \int_{1/2}^1\int_0^2 \left(x^2 + \frac{1}{3}xy\right) dy\, dx$$

$$= \int_{1/2}^1 \left[x^2y + \frac{xy^2}{6}\right]_0^2 dx = \int_{1/2}^1 \left(2x^2 + \frac{2x}{3}\right) dx$$

$$= \left[\frac{2x^3}{3} + \frac{x^2}{3}\right]_{1/2}^1 = \left(\frac{2}{3} + \frac{1}{3}\right) - \left(\frac{1}{12} + \frac{1}{12}\right)$$

$$= 1 - \frac{2}{12} = \frac{5}{6}.$$

(ii) $$P(Y < X) = \int_0^1\int_0^x \left(x^2 + \frac{1}{3}xy\right) dy\, dx \qquad (\because\ y < x)$$

$$= \int_0^1 \left[x^2y + \frac{1}{6}xy^2 \right]_0^x dx = \int_0^1 \left(x^3 + \frac{x^3}{6}\right) dx$$

$$= \frac{7}{6}\int_0^1 x^3 dx = \frac{7}{6}\left[\frac{x^4}{4}\right]_0^1 = \frac{7}{6} \times \frac{1}{4} = \frac{7}{24}.$$

(iii) $P(X + Y > 1) = 1 - P(X + Y < 1)$

$$= 1 - \int_0^1 \int_0^{1-x} \left(x^2 + \frac{1}{3}xy\right) dy\, dx$$

$$(\because\ x + y < 1,\ \therefore\ y = 1 - x)$$

$$= 1 - \int_0^1 \left[x^2y + \frac{1}{6}xy^2 \right]_0^{1-x} dx$$

$$= 1 - \int_0^1 \left\{x^2(1 - x) + \frac{1}{6}x(1 - x)^2\right\} dx$$

$$= 1 - \int_0^1 \frac{1}{6}(x + 4x^2 - 5x^3)\, dx$$

$$= 1 - \frac{1}{6}\left[\frac{x^2}{2} + \frac{4x^3}{3} - \frac{5x^4}{4}\right]_0^1$$

$$= 1 - \frac{1}{6}\left[\frac{1}{2} + \frac{4}{3} - \frac{5}{4}\right] = 1 - \frac{1}{6}\left(\frac{7}{12}\right)$$

$$= 1 - \frac{7}{72} = \frac{65}{72}.$$

(iv) Now $P(Y < \frac{1}{2} \mid X < \frac{1}{2}) = \dfrac{P(Y < \frac{1}{2} \text{ and } X < \frac{1}{2})}{P(X < \frac{1}{2})}$, and

$$P(Y < \tfrac{1}{2} \text{ and } X < \tfrac{1}{2}) = \int_0^{1/2}\int_0^{1/2} (x^2 + \tfrac{1}{3}xy)\, dy\, dx$$

$$= \int_0^{1/2} \left[x^2 y + \frac{1}{6}xy^2\right]_0^{1/2} dx$$

$$= \int_0^{1/2} \left(\frac{x^2}{2} + \frac{x}{24}\right) dx$$

$$= \left[\frac{x^3}{6} + \frac{x^2}{48}\right]_0^{1/2} = \frac{1}{48} + \frac{1}{192} = \frac{5}{192}$$

$$\therefore P\left(Y<\tfrac{1}{2} \mid X < \tfrac{1}{2}\right) = \frac{5/192}{1/6} = \frac{5}{32}, \quad (\because P(X<\tfrac{1}{2}) = \tfrac{1}{6}).$$

7.15. (b) The marginal density functions are obtained as below:

$$g(x) = \int_{-\infty}^{\infty} f(x, y)\, dy, \qquad 0 < x < 1,$$

$$= \frac{2}{3}\int_0^1 (x + 2y)\, dy, \qquad 0 < x < 1,$$

$$= \frac{2}{3}\left[xy + \frac{2y^2}{2}\right]_0^1 = \frac{2}{3}[x + 1] \qquad 0 \le x < 1.$$

$$h(y) = \int_{-\infty}^{\infty} f(x, y)\, dx \qquad 0 < y < 1,$$

$$= \frac{2}{3}\int_0^1 (x + 2y)\, dx, \qquad 0 < y < 1,$$

$$= \frac{2}{3}\left[\frac{x^2}{2} + 2xy\right]_0^1, \qquad 0 < y < 1,$$

$$= \frac{2}{3}\left[\frac{1}{2} + 2y\right]_0^1, \qquad 0 < y < 1,$$

$$= \frac{1}{3} + \frac{2}{3}y \qquad 0 < y < 1.$$

The conditional *p.d.f.* of X given $Y = y$ is

$$f(x/y) = \frac{f(x, y)}{h(y)}, \quad \text{where } h(y) > 0.$$

$$= \frac{\frac{2}{3}(x + 2y)}{\frac{2}{3}\left[\frac{1}{2} + 2y\right]} = \frac{x + 2y}{\frac{1}{2} + 2y} = \frac{2x + 4y}{1 + 4y};$$

and the conditional *p.d.f.* of Y given $X = x$ is

$$f(y/x) = \frac{f(x, y)}{g(y)}, \quad \text{where } g(x) > 0$$

$$= \frac{\frac{2}{3}(x + 2y)}{\frac{2}{3}(x + 1)} = \frac{x + 2y}{x + 1}.$$

We know that $f(x/y) = \dfrac{2x + 4y}{1 + 4y}$, therefore

$$P(X < \tfrac{1}{2} / y = \tfrac{1}{2}) = \int_0^{1/2} f(x/\tfrac{1}{2})\, dx = \int_0^{1/2} \frac{2x + 4(\frac{1}{2})}{1 + 4(\frac{1}{2})} dx$$

$$= \int_0^{1/2} \frac{2x + 2}{3} . dx = \frac{2}{3}\left[\frac{x^2}{2} + x\right]_0^{1/2}$$

$$= \frac{2}{3}\left[\frac{1}{8} + \frac{1}{2}\right] = \frac{2}{3} \times \frac{5}{8} = \frac{5}{12}.$$

7.16. The marginal density functions are obtained below:

$$g(x) = \int_{-\infty}^{\infty} f(x, y)\, dy$$

$$= 3\int_0^1 (x^2y + y^2x)\, dy = 3\left[\frac{x^2y^2}{2} + \frac{y^3x}{3}\right]_0^1$$

$$= \frac{3}{2}x^2 + x, \qquad 0 \le x \le 1.$$

$$h(y) = \int_{-\infty}^{\infty} f(x, y)\, dx$$

$$= 3\int_0^1 (x^2y + y^2x)\, dx = 3\left[\frac{x^3y}{3} + \frac{y^2x^2}{2}\right]_0^1$$

$$= y + \frac{3}{2}y^2. \qquad 0 \le y \le 1.$$

The conditional distributions are

$$f(x/y) = \frac{f(x, y)}{h(y)} = \frac{3xy(x + y)}{y + (3/2)\,y^2} = \frac{6x(x + y)}{2 + 3y}, \text{ and}$$

$$f(y/x) = \frac{f(x, y)}{g(x)} = \frac{3xy(x + y)}{x + (3/2)\,x^2} = \frac{6y(x + y)}{2 + 3x}.$$

The conditional probability is obtained below:

$$P\left[\frac{1}{2} \le x \le \frac{3}{4} \,\middle|\, \frac{1}{2} \le y \le \frac{2}{3}\right] = \frac{\int_{1/3}^{3/4}\int_{1/2}^{2/3} (3x^2y + 3y^2x)\, dydx}{\int_{1/2}^{2/3} (y + \frac{3}{2}y^2)\, dy}$$

Now $\int_{1/2}^{3/4}\int_{1/2}^{2/3} (3x^2y + 3y^2x)\, dydx = \int_{1/2}^{3/4} \left[\frac{3x^2y^2}{2} + \frac{3y^2x}{3}\right]_{1/2}^{2/3} dx$

$$= \int_{1/2}^{3/4} \left(\frac{7}{24}x^2 + \frac{37}{216}x\right) dx$$

$$= \left[\frac{7}{24} \cdot \frac{x^2}{3} + \frac{37}{216} \cdot \frac{x^2}{2}\right]_{1/2}^{3/4} = \frac{769}{13824}$$

and $\int_{1/2}^{2/3}\left(y + \frac{3}{2}y^2\right) dy = \left[\frac{y^2}{2} + \frac{3y^3}{6}\right]_{1/2}^{2/3} = \frac{79}{432}$

Hence the required conditional probability is

$$\frac{769}{13824} \times \frac{432}{79} = \frac{769}{2528} = 0.304.$$

7.17. The marginal distribution of X is obtained as

$$g(x) = \int_{-\infty}^{\infty} f(x, y)\, dy$$

$$= 24 \int_0^1 x^2y(1 - x)\, dy = 24 \left[\frac{x^2y^2(1-x)}{2}\right]_0^1$$

$$= 12x^2 (1 - x)$$

The conditional distribution of Y given X is

$$f(y/x) = \frac{f(x, y)}{g(x)} = \frac{24\, x^2y\, (1 - x)}{12x^2\, (1 - x)} = 2y$$

Also the marginal distribution of Y is obtained as

$$h(y) = \int_{-\infty}^{\infty} f(x, y)\, dx$$

$$= 24 \int_0^1 x^2y(1 - x)\, dx = 24 \left[y\left(\frac{x^3}{3} - \frac{x^4}{4}\right)\right]_0^1 = 2y$$

Now obviously $f(x, y) = g(x)\,.\,h(y)$

Hence X and Y are independent.

7.19. (c) Let the r.v. X represent the sum of the spots on the dice. Then X has the following p.d.

x	2	3	4	5	6	7	8	9	10	11	12
$f(x)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

The expected value of the payment is

$$E(X) = 2 \times \frac{1}{36} + 3 \times \frac{2}{36} + 4 \times \frac{3}{36} + \ldots + 12 \times \frac{1}{36} = \frac{252}{36} = 7$$

7.20. (a) Let the r.v. X represent the number of women on the committee. Then X has the following p.d.

x	$f(x)$	$xf(x)$
0	$\binom{3}{0}\binom{5}{5} \div \binom{8}{5} = \frac{1}{56}$	0
1	$\binom{3}{1}\binom{5}{4} \div \binom{8}{5} = \frac{15}{56}$	$\frac{15}{56}$
2	$\binom{3}{2}\binom{5}{3} \div \binom{8}{5} = \frac{30}{56}$	$\frac{60}{56}$
3	$\binom{3}{3}\binom{5}{2} \div \binom{8}{5} = \frac{10}{56}$	$\frac{30}{56}$
Σ	--- 1	$\frac{105}{56}$

Hence the expected number of women on the committee is

$$E(X) = \sum xf(x) = \frac{105}{56}$$

(b) The mathematical expectation of a random variable X, i.e. E(X) is defined as

$$E(X) = \Sigma_i \, x_i \, f(x_i)$$

the summation extending over all possible values of X, provided that $\Sigma_i \mid x_i \mid f(x_i)$ converges.

Now $\quad x_i = 2, -2, \frac{8}{3}, -4, \ldots, (-1)^{j+1} \frac{2j}{j}, \ldots$

$$f(x_i) = \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots, \left(\frac{1}{2}\right)^j, \ldots$$

$$\therefore \quad \Sigma_j \, |x_j| \, f(x_j) = \Sigma_j \frac{2j}{j}\left(\frac{1}{2}\right)^j$$

$$= \sum_{j=1}^{\infty} \frac{1}{j},$$

which is a divergent series, and hence $E(X)$ does not exist. It is however interesting to note that

$$\sum_j x_j f(x_j) = \sum_{j=1}^{\infty} (-1)^{j+1}\frac{1}{j}, \text{ converges.}$$

7.21. (b) As the variables X and Y are independent, therefore their joint distribution is

Y \ X	1	2	3	4	$f(y)$	$yf(y)$
0	1/32	1/32	1/32	1/32	1/8	0
1	3/32	3/32	3/32	3/32	3/8	3/8
2	3/32	3/32	3/32	3/32	3/8	6/8
3	1/32	1/32	1/32	1/32	1/8	3/8
$f(x)$	1/4	1/4	1/4	1/4	1	12/8
$xf(x)$	1/4	2/4	3/4	4/4	10/4	

Now $\quad E(X) = \sum xf(x) = \frac{10}{4} = 2.5$; and

$$E(Y) = \sum yf(y) = \frac{12}{4} = 1.5;$$

Again $\quad E(X+Y) = \sum(x + y)\, f(x, y)$

$$= \left(1\times\frac{1}{32}\right) + \left(2\times\frac{1}{32}\right) + \left(3\times\frac{1}{32}\right) + \left(4\times\frac{1}{32}\right)$$
$$+ \left(2\times\frac{3}{32}\right) + \left(3\times\frac{3}{32}\right) + \left(4\times\frac{3}{32}\right) + \left(5\times\frac{3}{32}\right)$$
$$+ \left(3\times\frac{3}{32}\right) + \left(4\times\frac{3}{32}\right) + \left(5\times\frac{3}{32}\right) + \left(6\times\frac{3}{32}\right)$$
$$+ \left(4\times\frac{1}{32}\right) + \left(5\times\frac{1}{32}\right) + \left(6\times\frac{1}{32}\right) + \left(7\times\frac{1}{32}\right)$$

$$= \frac{128}{32} = 4$$

Also $\quad E(X) + E(Y) = 2.5 + 1.5 = 4$

Hence $\quad E(X) + E(Y) = E(X + Y)$

7.22 (a) Computation of the expected values E(X) and E(X²).

x_i	$f(x_i)$	$x_i f(x_i)$	$x_i^2 f(x_i)$
0	$\binom{3}{0}\left(\frac{1}{4}\right)^0\left(\frac{3}{4}\right)^3 = \frac{27}{64}$	0	0
1	$\binom{3}{1}\left(\frac{1}{4}\right)^1\left(\frac{3}{4}\right)^2 = \frac{27}{64}$	$\frac{27}{64}$	$\frac{27}{64}$
2	$\binom{3}{2}\left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right)^1 = \frac{9}{64}$	$\frac{18}{64}$	$\frac{36}{64}$
3	$\binom{3}{3}\left(\frac{1}{4}\right)^3\left(\frac{3}{4}\right)^0 = \frac{1}{64}$	$\frac{3}{64}$	$\frac{9}{64}$
Σ	1	$\frac{48}{64}$	$\frac{72}{64}$

Hence $E(X) = \sum x_i f(x_i) = \frac{48}{64} = \frac{3}{4}$, and

$$E(X^2) = \sum x_i^2 f(x_i) = \frac{72}{64} = 1\frac{1}{8}.$$

(b) (i) Computation of E(X) and Var(X).

x_i	$f(x_i)$	$x_i f(x_i)$	$x_i^2 f(x)$
−1	0.125	−0.125	0.125
0	0.500	0.000	0.000
1	0.200	0.200	0.200
2	0.050	0.100	0.200
3	0.125	0.375	1.125
Total	1.000	0.550	1.650

Thus $E(X) = \sum xf(x) = 0.55$; and

$$Var(X) = E(X^2) - [E(X)]^2 = 1.650 - (0.55)^2 = 1.650 - 0.3025$$
$$= 1.3475 = 1.35$$

(ii) Computation of the p.d. of $Y = 2X + 1$, $E(Y)$ and $\text{Var}(Y)$.

$y = (2x+1)$	$f(y)$	$yf(y)$	$y^2f(y)$
-1	0.125	-0.125	0.125
1	0.500	0.500	0.500
3	0.200	0.600	1.800
5	0.050	0.250	1.250
7	0.125	0.875	6.125
Total	1.000	2.100	9.800

$\therefore \qquad E(Y) = \sum yf(y) = 2.1$, and

$$\text{Var}(Y) = E(Y^2) - [E(Y)]^2 = 9.80 - (2.1)^2 = 9.80 - 4.41 = 5.39.$$

(iii) For the relationship between $E(X)$ and $E(Y)$ where $Y = 2X + 1$, we have

$$E(Y) = 2.1$$
$$= 2(0.55) + 1 = 2E(X) + 1.$$

For the relationship between $\text{Var}(X)$ and $\text{Var}(Y)$, we have

$$\text{Var}(Y) = 5.39$$
$$= 4(1.3475) = (2)^2 \text{Var}(X).$$

7.23 (a) The function f(x) will be a density function, if

$$A \int_0^1 x^3(1-x)\, dx = 1, \text{ i.e. if } A \int_0^1 (x^3 - x^4)\, dx = 1,$$

$$\text{i.e. if } A \left[\frac{x^4}{4} - \frac{x^5}{5}\right]_0^1 = 1, \text{ i.e. if } A = 20.$$

$\therefore \qquad f(x) = 20x^3(1-x)$ is a proper density function.

(b) The mean and variance are calculated below:

Now $\qquad E(X) = \int_{-\infty}^{\infty} xf(x)\, dx$

$$= 20 \int_0^1 x(x^3-x^4)\,dx = 20 \int_0^1 (x^4 - x^5)\,dx$$

$$= 20 \left[\frac{x^5}{5} - \frac{x^6}{6} \right]_0^1 = 20 \left[\frac{1}{5} - \frac{1}{6} \right] = \frac{2}{3},$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx$$

$$= 20 \int_0^1 x^2(x^3-x^4)\,dx = 20 \int_0^1 (x^5 - x^6)\,dx$$

$$= 20 \left[\frac{x^6}{6} - \frac{x^7}{7} \right]_0^1 = 20 \left[\frac{1}{6} - \frac{1}{7} \right] = \frac{10}{21}$$

$$\therefore \quad \text{Var}(X) = E(X^2) - [E(X)]^2 = \frac{10}{21} - \left(\frac{2}{3}\right)^2 = \frac{2}{63}$$

(c) The distribution function is

$$F(x) = P(X \le x) = 20 \int_0^x x^3(1-x)\,dx = 20 \int_0^x (x^3 - x^4)\,dx$$

$$= 20 \left[\frac{x^4}{4} - \frac{x^5}{5} \right]_0^x = 20 \left[\frac{x^4}{4} - \frac{x^5}{5} \right]$$

$$= 5x^4 - 4x^5$$

Now $P\left(\frac{1}{4} < X < \frac{1}{2}\right) = P\left(X < \frac{1}{2}\right) - P\left(X < \frac{1}{4}\right)$

$$= F\left(\frac{1}{2}\right) - F\left(\frac{1}{4}\right)$$

$$= \left[5\left(\frac{1}{2}\right)^4 - 4\left(\frac{1}{2}\right)^5\right] - \left[5\left(\frac{1}{4}\right)^4 - 4\left(\frac{1}{4}\right)^5\right]$$

$$= \left(\frac{5}{16} - \frac{4}{32}\right) - \left(\frac{5}{256} - \frac{4}{1024}\right)$$

$$= \frac{6}{32} - \frac{16}{1024} = \frac{3}{16} - \frac{1}{64} = \frac{11}{64}$$

7.24. (a) $E[X-E(X)]^2 = E[X^2 - 2XE(X) + \{E(X)\}^2]$

$$= E(X^2) - 2E(X) \,.\, E(X) + [E(X)]^2$$

$$= E(X^2) - 2[E(X)]^2 + [E(X)]^2$$

$$= E(X^2) - [E(X)]^2.$$

(b) Given that $\text{Var}(X_1) = k$, $\text{Var}(X_2) = 2$, and

$\text{Var}(3X_2 - X_1) = 25.$

Since X_1 and X_2 are independent r.v.'s, therefore

$$\text{Var}(3X_2 - X_1) = 9\,\text{Var}(X_2) + \text{Var}(X_1)$$

$$= 9(2) + k$$

Thus $9(2) + k = 25$

or $k = 25 - 18 = 7.$

(c) (i) The probability distribution of r.v. X is

x	1	2	3	4
$f(x)$	1/8	2/8	3/8	2/8

(ii) $E(X) = \sum x f(x)$

$$= \left(1 \times \frac{1}{8}\right) + \left(2 \times \frac{2}{8}\right) + \left(3 \times \frac{3}{8}\right) + \left(4 \times \frac{2}{8}\right)$$

$$= \frac{22}{8} = 2.75.$$

$\text{Var}(X) = E(X^2) - [E(X)]^2$, where

$$E(X^2) = \sum x^2 f(x)$$

$$= \left(1^2 \times \frac{1}{8}\right) + \left(2^2 \times \frac{2}{8}\right) + \left(3^2 \times \frac{3}{8}\right) + \left(4^2 \times \frac{2}{8}\right)$$

$$= \frac{68}{8}$$

$$\therefore \quad \text{Var}(X) = \left(\frac{68}{8}\right) - \left(\frac{22}{8}\right)^2 = \frac{60}{64} = \frac{15}{16}.$$

7.25. (a) Calculation of mean and variance.

x_i	$f(x_i) = \dfrac{6 - \lvert 7-x \rvert}{36}$	$x_i f(x_i)$	$x_i^2 f(x_i)$
2	1/36	2/36	4/36
3	2/36	6/36	18/36
4	3/36	12/36	48/36
5	4/36	20/36	100/36
6	5/36	30/36	180/36
7	6/36	42/36	294/36
8	5/36	40/36	320/36
9	4/36	36/36	324/36
10	3/36	30/36	300/36
11	2/36	22/36	242/36
12	1/36	12/36	144/36
Σ	1	252/36 = 7	1974/36

$\therefore$ Mean $= E(X) = \sum x_i f(x_i) = 7$, and

$$\text{Var}(X) = E(X^2) - [E(X)]^2$$

$$= \sum x_i^2 f(x_i) - [\sum x_i f(x_i)]^2$$

$$= \frac{1974}{36} - (7)^2 = \frac{210}{36} = \frac{35}{6}.$$

(b) Here $x_i = 1, 2, 3, \ldots, n$

$$f(x_i) = \frac{1}{n}, \frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}$$

$$\therefore \quad E(X) = \sum x_i f(x)$$

$$= 1 \cdot \frac{1}{n} + 2 \cdot \frac{1}{n} + 3 \cdot \frac{1}{n} + \ldots + n \cdot \frac{1}{n}$$

$$= \frac{1}{n}[1 + 2 + 3 + \ldots + n]$$

$$= \frac{1}{n}\left[\frac{n(n+1)}{2}\right] = \frac{n+1}{2}.$$

$$E(X^2) = 1^2 \cdot \frac{1}{n} + 2^2 \cdot \frac{1}{n} + 3^2 \cdot \frac{1}{n} + \ldots + n^2 \cdot \frac{1}{n}$$

$$= \frac{1}{n}[1^2 + 2^2 + 3^2 + \ldots + n^2]$$

$$= \frac{1}{n}\left[\frac{n(n+1)(2n+1)}{6}\right] = \frac{(n+1)(2n+1)}{6}$$

$$\therefore \quad \text{Var}(X) = E(X^2) - [E(X)]^2$$

$$= \frac{(n+1)(2n+1)}{6} - \left(\frac{n+1}{2}\right)^2$$

$$= \frac{2n^2 + 3n + 1}{6} - \frac{n^2 + 2n + 1}{4} = \frac{n^2 - 1}{12}$$

(c) Here $x_i = 0, 1, 2, 3, \ldots, n$

$$f(x_i) = k\binom{n}{0},\ k\binom{n}{1},\ k\binom{n}{2}, \ldots, k\binom{n}{n}.$$

where k is the constant of proportionality.

Since $\sum f(x_i) = 1$, therefore we first find the value of k, the constant of proportionality. The sum is

$$k\binom{n}{0} + k\binom{n}{1} + k\binom{n}{2} + \ldots + k\binom{n}{n} = 1$$

or $$k\left[1 + n + \frac{n(n-1)}{2!} + \ldots + 1\right] = 1$$

or $$k\,[1+1]^n = 1 \text{ or } k2^n = 1 \text{ or } k = \frac{1}{2^n}$$

Now $$E(X) = \sum x_i\, f(x_i)$$

$$= \frac{1}{2^n}\left[0.1 + 1.n + 2.\frac{n(n-1)}{2!} + 3.\frac{n(n-1)(n-2)}{3!} + \ldots + n.1\right]$$

$$= \frac{n}{2^n}\left[1 + (n-1) + \frac{(n-1)(n-2)}{2!} + \ldots + 1\right]$$

$$= \frac{n}{2^n}[1+1]^{n-1} = \frac{n}{2^n}(2)^{n-1} = \frac{n}{2}.$$

$$E(X^2) = \frac{1}{2^n}\left[0^2.1 - 1^2.n + 2^2.\frac{n(n-1)}{2!} + 3^2.\frac{n(n-1)(n-2)}{3!} + \ldots + n^2.1\right]$$

$$= \frac{1}{2^n}\left[n + 2\,.\,\frac{n(n-1)}{1!} + 3\,.\,\frac{n(n-1)(n-2)}{2!} + \ldots + n^2\right]$$

$$= \frac{n}{2^n}\left[\left\{1 + (n-1) + \frac{(n-1)(n-2)}{2!} + \ldots + 1\right\} + \left\{(n-1) + \frac{2(n-1)(n-2)}{2!} + \ldots + (n-1)\right\}\right]$$

$$= \frac{n}{2^n}\left[(1+1)^{n-1} + (n-1)(1+1)^{n-2}\right]$$

$$= \frac{n}{2^n}\left[2^{n-1} + (n-1)2^{n-2}\right] = \frac{n}{2} + \frac{n(n-1)}{2^2}$$

Hence $\quad \text{Var}(X) = E(X^2) - [E(X)]^2$

$$= \frac{n}{2} + \frac{n(n-1)}{4} - \left(\frac{n}{2}\right)^2 = \frac{2n + n^2 - n - n^2}{4} = \frac{n}{4}.$$

7.26. (a) The probability of throwing a 6 with one die is $\frac{1}{6}$ and that of not throwing a 6 with one die is $\frac{5}{6}$.

As A has the first throw, therefore he can win in the first, third, fifth, ..., throws.

The probability that A wins in the first throw $= \frac{1}{6}$

The probability that A wins in the 3rd throw $= \left(\frac{5}{6}\right)^2 . \frac{1}{6}$, as both A and B have failed once before A's 3rd throw.

The probability that A wins in the 5th throw $= \left(\frac{5}{6}\right)^4 . \frac{1}{6}$ and so on. Since these cases are mutually exclusive, therefore the probability that A wins $= \frac{1}{6} + \left(\frac{5}{6}\right)^2 . \frac{1}{6} + \left(\frac{5}{6}\right)^4 . \frac{1}{6} + \ldots$

$$= \frac{1/6}{1-\left(\frac{5}{6}\right)^2} = \frac{1}{6} \times \frac{36}{11} = \frac{6}{11}.$$

The probability that B wins $= 1 - P(A \text{ wins})$

$$= 1 - \frac{6}{11} = \frac{5}{11}.$$

Hence $\quad A$'s expectation $= \frac{6}{11} \times$ Rs. 11 = Rs. 6, and

B's expectation $= \frac{5}{11} \times$ Rs. 11 = Rs. 5.

(b) The probability of getting a spade $= \frac{13}{52} = \frac{1}{4}$, and the probability of not getting a spade $= 1 - \frac{1}{4} = \frac{3}{4}$.

Now A can cut a spade in the first, fifth, ninth, ..., drawings, the respective probabilities of which are

$$\frac{1}{4},\ \left(\frac{3}{4}\right)^4 \times \frac{1}{4},\ \left(\frac{3}{4}\right)^8 \times \frac{1}{4}, \ldots$$

Thus the probability that A cuts a spade first

$$= \frac{1}{4} + \left(\frac{3}{4}\right)^4 \cdot \frac{1}{4} + \left(\frac{3}{4}\right)^8 \cdot \frac{1}{4} + \ldots$$

$$= \frac{1}{4}\left[1 + \left(\frac{3}{4}\right)^4 + \left(\frac{3}{4}\right)^8 + \ldots\right]$$

$$= \frac{1/4}{1-\left(\frac{3}{4}\right)^4} = \frac{1}{4} \times \frac{256}{175} = \frac{64}{175}.$$

B can cut a spade in the second, sixth, tenth, ..., drawings, the respective probabilities of which are $\frac{3}{4} \times \frac{1}{4},\ \left(\frac{3}{4}\right)^5 \times \frac{1}{4},\ \left(\frac{3}{4}\right)^9 \times \frac{1}{4}, \ldots$

Therefore $P(B \text{ cuts a spade first}) = \frac{3}{4} \cdot \frac{1}{4} + \left(\frac{3}{4}\right)^5 \cdot \frac{1}{4} + \left(\frac{3}{4}\right)^9 \cdot \frac{1}{4} + \ldots$

$$= \frac{3}{4} \cdot \frac{1}{4} \left[1 + \left(\frac{3}{4}\right)^4 + \left(\frac{3}{4}\right)^8 + \ldots\right]$$

$$= \frac{\frac{3}{4} \times \frac{1}{4}}{1 - \left(\frac{3}{4}\right)^4} = \frac{3}{16} \times \frac{256}{175} = \frac{48}{175}$$

C can cut a spade in the third, seventh, eleventh, ..., drawings, the respective probabilities of which are

$$\left(\frac{3}{4}\right)^2 \cdot \frac{1}{4}, \left(\frac{3}{4}\right)^6 \cdot \frac{1}{4}, \left(\frac{3}{4}\right)^{10} \frac{1}{4}, \ldots \text{ Thus}$$

$$P(C \text{ cuts a spade first}) = \left(\frac{3}{4}\right)^2 \cdot \frac{1}{4} + \left(\frac{3}{4}\right)^6 \cdot \frac{1}{4} + \left(\frac{3}{4}\right)^{10} \frac{1}{4} + \ldots$$

$$= \left(\frac{3}{4}\right)^2 \cdot \frac{1}{4} \left[1 + \left(\frac{3}{4}\right)^4 + \left(\frac{3}{4}\right)^8 + \ldots\right]$$

$$= \frac{\left(\frac{3}{4}\right)^2 \cdot \frac{1}{4}}{1 - \left(\frac{3}{4}\right)^4} = \frac{9}{64} \times \frac{256}{175} = \frac{36}{175}$$

and $P(D \text{ cuts a spade first}) = 1 - \left(\frac{64}{175} + \frac{48}{175} + \frac{36}{175}\right)$

$$= 1 - \frac{148}{175} = \frac{27}{175}.$$

Hence expectation of $A = \frac{64}{175} \times £\,175 = £\,64$:

expectation of $B = \frac{48}{175} \times £\,175 = £\,48$:

expectation of $C = \frac{36}{175} \times £\,175 = £\,36$: and

expectation of $D = \frac{27}{175} \times £\,175 = £\,27$.

(c) The bag contians 2 white and 2 black balls.

Let A denote the event that A draws a white ball first. Then

$$P(A) = \binom{2}{1} \div \binom{4}{1} = \frac{1}{2}$$

If A fails to draw the white ball, then the bag will contain 2 white and 1 black ball, as the ball drawn by A is not replaced.

Let B denote the event that B draws a white ball after A fails to draw it. Then

$P(B) = P(B$ draws white ball/A has failed to draw white ball)

$$= \frac{\binom{2}{1}}{\binom{3}{1}} \times \frac{2}{4} = \frac{1}{3}$$

If both A and B have failed to draw white ball, then the bag will contain 2 white balls as the balls being drawn are not replaced.

Let C denote the event that C draws a white ball given that both A and B have failed to draw it. Then

$$P(C) = \frac{2}{2} \times \frac{2}{4} \times \frac{1}{3} = \frac{1}{6}$$

Hence expectation of $A = \frac{1}{2} \times$ Rs. 18 = Rs. 9.

expectation of $B = \frac{1}{3} \times$ Rs. 18 = Rs. 6, and

expectation of $C = \frac{1}{6} \times$ Rs. 18 = Rs. 3

7.27. (a) The total area under the curve is

$$\int_{-\infty}^{\infty} f(x)\, dx = \int_{0}^{1} x^2(1-x)\, dx$$

$$= \left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12} \text{ } (=N, \text{ say})$$

Hence this is not a density function.

$$\text{Now mean} = \frac{1}{N}\int x \cdot f(x)\,dx$$

$$= 12\int_0^1 (x^3 - x^4)\,dx = 12\left[\frac{x^4}{4} - \frac{x^5}{4}\right]_0^1$$

$$= 12\left[\frac{1}{4} - \frac{1}{5}\right] = 12 \times \frac{1}{20} = \frac{3}{5} = 0.06.$$

$$E(X^2) = 12\left[\int_0^1 x^2 f(x)\,dx\right]$$

$$= 12\int_0^1 (x^4 - x^5)\,dx = 12\left[\frac{x^5}{5} - \frac{x^6}{6}\right]_0^1$$

$$= 12\left[\frac{1}{5} - \frac{1}{6}\right] = \frac{12}{30} = \frac{2}{5} = 0.4.$$

Thus σ^2 or $\mathrm{Var}(X) = E(X^2) - [E(X)]^2 = 0.4 - (0.6)^2 = 0.04$, so that

$$\sigma = \sqrt{0.04} = 0.2.$$

(b) In order that $f(x)$ may be a density function, we should have $f(x) \geq 0$ for every x, and

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

The first condition is clearly satisfied and the second condition will be satisfied if

$$c\int_1^2 x\,dx = 1, \text{ i.e. if } c\left[\frac{x^2}{2}\right]_1^2 = 1,$$

$$\text{i.e. if } c\left(\frac{3}{2}\right) = 1 \text{ i.e. if } c = \frac{2}{3}.$$

(ii) $\quad \text{Mean} = E(X) = \int_{-\infty}^{\infty} x \,.\, f(x)\, dx = \frac{2}{3}\int_{1}^{2} x \,.\, x dx$

$$= \frac{2}{3}\int_{1}^{2} x^2 dx = \frac{2}{3}\left[\frac{x^3}{3}\right]_1^2$$

$$= \frac{2}{3}\left[\frac{8}{3} - \frac{1}{3}\right] = \frac{2}{3} \cdot \frac{7}{3} = \frac{14}{9},$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\, dx = \frac{2}{3}\int_{1}^{2} x^2 x dx = \frac{2}{3}\int_{1}^{2} x^3 dx = \frac{2}{3}\left[\frac{x^4}{4}\right]_1^2$$

$$= \frac{2}{3}\left[\frac{16}{4} - \frac{1}{4}\right] = \frac{2}{3} \times \frac{15}{4} = \frac{5}{2}.$$

$$\text{Var}(X) = E(X^2) - \{E(X)\}^2 = \frac{5}{2} - \left(\frac{14}{9}\right)^2 = \frac{5}{2} - \frac{196}{81} = \frac{13}{162},$$

and $S.D.(X) = \sqrt{\text{Var}(X)} = \sqrt{\frac{13}{162}}$.

(c) $\quad E(X) = \int_{-\infty}^{\infty} x \,.\, f(x)\, dx \quad$ (by definition)

$$= \int_{-\infty}^{0} x \,.\, 0 dx + \frac{3}{8}\int_{0}^{2} x\,(x-2)^2\, dx + \int_{2}^{\infty} x \,.\, 0\, dx$$

$$= \frac{3}{8}\int_{0}^{2} x(x^2 - 4x + 4) dx = \frac{3}{8}\int_{0}^{2} (x^3 - 4x^2 + 4x) dx$$

$$= \frac{3}{8}\left[\frac{x^4}{4} - \frac{4x^3}{3} + \frac{4x^2}{2}\right]_0^2$$

$$= \frac{3}{8}\left[4 - \frac{32}{3} + 8\right] = \frac{3}{8}\left(\frac{4}{3}\right) = \frac{1}{2}, \text{ and}$$

$\sigma = \sqrt{E(X^2) - [E(X)]^2}$, where

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\, dx$$

$$= \frac{3}{8}\int_0^2 x^2(x-2)^2\, dx = \frac{3}{8}\int_0^2 (x^4-4x^3+4x^2)dx$$

$$= \frac{3}{8}\left[\frac{x^5}{5} - \frac{4x^4}{4} + \frac{4x^3}{3}\right]_0^2 = \frac{3}{8}\left[\frac{32}{5} - 16 + \frac{32}{3}\right]$$

$$= \frac{3}{8}\left(\frac{16}{15}\right) = \frac{2}{5}$$

$$\therefore \quad \sigma = \sqrt{\frac{2}{5} - \left(\frac{1}{2}\right)^2} = \sqrt{0.4 - 0.25}$$

$$= \sqrt{0.15} = 0.387$$

7.28. (a) The function f(x) will be a density function, if

$$k\int_0^2 x^3(1-x)\, dx = 1, \text{ i.e. if } k\int_0^1 (x^3 - x^4)\, dx = 1,$$

or $$k\left[\frac{x^4}{4} - \frac{x^5}{5}\right]_0^1 = 1, \text{ or } k\left[\frac{1}{4} - \frac{1}{5}\right] = 1, \text{ or } k = 20$$

$\therefore$ $f(x) = 20x^3(1-x)$ is a proper density function.

The mean, $E(X)$ and the Var(X) are determined as below:

$$E(X) = \int_{-\infty}^{\infty} xf(x)\, dx$$

$$= 20\int_0^1 x \cdot x^3(1-x)\, dx = 20\left[\frac{x^5}{5} - \frac{x^6}{6}\right]_0^1$$

$$= 20\left[\frac{1}{5} - \frac{1}{6}\right] = \frac{20}{30} = \frac{2}{3}$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\, dx$$

$$= 20 \int_0^1 x^2 . x^3(1-x)\, dx = 20 \int_0^1 (x^5 - x^6)\, dx$$

$$= 20 \left[\frac{x^6}{6} - \frac{x^7}{7} \right]_0^1 = 20 \left(\frac{1}{42} \right) = \frac{10}{21}, \text{ and}$$

$$\text{Var}(X) = E(X^2) - [E(X)]^2 = \frac{10}{21} - \left(\frac{2}{3}\right)^2 = \frac{10}{21} - \frac{4}{9} = \frac{2}{63}.$$

(b) The function $f(x)$ will be a density function, if

$$\int_0^3 Ax\,(9 - x^2)\, dx = 1$$

i.e. $$A \left[\frac{9x^2}{2} - \frac{x^4}{4} \right]_0^3 = 1$$

i.e. $$A \left[\frac{81}{2} - \frac{81}{4} \right] = 1, \text{ which gives } A = \frac{4}{81}.$$

$$\therefore \quad f(x) = \frac{4}{81} x\,(9 - x^2), \quad \text{for } 0 \le x \le 3$$

$$= 0, \quad \text{elsewhere}$$

Now Mean $$= E(X) = \frac{4}{81} \int_0^3 x \,.\, x\,(9 - x^2)\, dx$$

$$= \frac{4}{81} \left[\frac{9x^3}{3} - \frac{x^5}{5} \right]_0^3$$

$$= \frac{4}{81} \left[81 - \frac{243}{5} \right] = \frac{4}{81} \times \frac{162}{5} = \frac{8}{5}$$

$$E(X^2) = \frac{4}{81}\int_0^3 x^2(9x - x^3)\,dx$$

$$= \frac{4}{81}\left[\frac{9x^4}{4} - \frac{x^6}{6}\right]_0^3 = \frac{4}{81}\left[\frac{729}{4} - \frac{729}{6}\right]$$

$$= \frac{4}{81} \times \frac{729}{12} = 3.$$

$$\therefore \quad S.D. = \sqrt{\Sigma(X^2) - [E(X)]^2} = \sqrt{3 - \left(\frac{8}{5}\right)^2}$$

$$= \sqrt{3 - 2.56} = 0.66$$

7.29. The total area must be unity.

$$\frac{1}{16}\int_{-3}^{-1}(3+x)^2\,dx + \frac{1}{16}\int_{-1}^{+1}(6-2x^2)\,dx + \frac{1}{16}\int_{1}^{3}(3-x)^2\,dx$$

$$= \frac{1}{16}\left[\frac{(3+x)^3}{3}\right]_{-3}^{-1} + \frac{1}{16}\left[6x - \frac{2x^3}{3}\right]_{-1}^{+1} + \frac{1}{16}\left[\frac{-(3-x)^3}{3}\right]_1^3$$

$$= \frac{1}{16}\left[\frac{8}{3} + \frac{32}{3} + \frac{8}{3}\right] = 1.$$

$$\text{Now } \mu = \int_{-\infty}^{\infty} xf(x)\,dx$$

$$= \frac{1}{16}\int_{-3}^{-1}x(3+x)^2dx + \frac{1}{16}\int_{-1}^{+1}x(6-2x^2)\,dx + \frac{1}{16}\int_1^3 x(3-x)^2dx$$

$$= \frac{1}{16}\left[\int_{-3}^{-1}(9x+6x^2+x^3)dx + \int_{-1}^{1}(6x-2x^3)dx + \int_1^3(9x-6x^2+x^3)\,dx\right]$$

$$= \frac{1}{16}\left[\frac{9x^2}{2} + \frac{6x^3}{3} + \frac{x^4}{4}\right]_{-3}^{-1} + \frac{1}{16}\left[\frac{6x^2}{2} - \frac{2x^4}{4}\right]_{-1}^{+1} +$$

$$\frac{1}{16}\left[\frac{9x^2}{2} - \frac{6x^3}{3} + \frac{x^4}{4}\right]_1^3$$

$$= \frac{1}{16}\left[\left(\frac{9}{2} - 2 + \frac{1}{4}\right) - \left(\frac{81}{2} - 54 + \frac{81}{4}\right)\right] +$$

$$\frac{1}{16}\left[\left(3-\frac{1}{2}\right)-\left(3-\frac{1}{2}\right)\right]+\frac{1}{16}\left[\left(\frac{81}{2}-54+\frac{81}{4}\right)\right.$$
$$\left.-\left(\frac{9}{2}-2+\frac{1}{4}\right)\right]$$

$$=\frac{1}{16}\left[\frac{11}{4}-\frac{189}{4}\right]+0+\frac{1}{16}\left[\frac{189}{4}-\frac{11}{4}\right]=0$$

$$\sigma^2 = \int_{-\infty}^{\infty} x^2 f(x)\,dx-(\mu)^2$$

$$=\frac{1}{16}\int_{-3}^{-1} x^2(3+x)^2\,dx+\frac{1}{16}\int_{-1}^{+1} x^2(6-2x^2)\,dx+$$
$$\frac{1}{16}\int_{1}^{3} x^2(3-x)^2\,dx \qquad (\because\ \mu=0)$$

$$=\frac{1}{16}\int_{-3}^{-1}(9x^2+6x^3+x^4)\,dx+\frac{1}{16}\int_{-1}^{+1}(6x^2-2x^4)\,dx+$$
$$\frac{1}{16}\int_{1}^{3}(9x^2-6x^3+x^4)\,dx$$

$$=\frac{1}{16}\left[3x^3+\frac{3x^4}{2}+\frac{x^5}{5}\right]_{-3}^{-1}+\frac{1}{16}\left[2x^3-\frac{2x^5}{5}\right]_{-1}^{+1}+$$
$$\frac{1}{16}\left[3x^3-\frac{3x^4}{2}+\frac{x^5}{5}\right]_{1}^{3}$$

$$=\frac{1}{16}\left[\left(-3+\frac{3}{2}-\frac{1}{5}\right)-\left(-81+\frac{243}{2}-\frac{243}{5}\right)\right]+$$
$$\frac{1}{16}\left[\left(2-\frac{2}{5}\right)-\left(-2+\frac{2}{5}\right)\right]+$$
$$\frac{1}{16}\left[\left(81-\frac{243}{2}+\frac{243}{5}\right)-\left(3-\frac{3}{2}+\frac{1}{5}\right)\right]$$

$$=\frac{1}{16}\left[\frac{81}{10}-\frac{17}{10}\right]+\frac{1}{16}\left[\frac{8}{5}+\frac{8}{5}\right]+\frac{1}{16}\left[\frac{81}{10}-\frac{17}{10}\right]$$

$$= \frac{2}{5} + \frac{1}{5} + \frac{2}{5} = 1$$

Hence $\sigma = \sqrt{1} = 1$

7.30. The function f(x) will be a proper density function, if

$$k \int_0^3 (6 + x - x^2)\, dx = 1,$$

$$\text{i.e. if } k \left[6x + \frac{x^2}{2} - \frac{x^3}{3} \right]_0^3 = 1, \text{ i.e. if } k \left[18 + \frac{9}{2} - \frac{27}{3}\right] = 1,$$

$$\text{i.e. if } k \left[\frac{27}{2} \right] = 1, \text{ i.e. if } k = \frac{2}{27}.$$

Now $\mu = E(X) = \int_{-\infty}^{\infty} x \,.\, f(x)\, dx$

$$= \frac{2}{27} \int_0^3 x \,.\, (6 + x - x^2)\, dx = \frac{2}{27} \int_0^3 (6x + x^2 - x^3)\, dx$$

$$= \frac{2}{27} \left[\frac{6x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \right]_0^3 = \frac{2}{27} \left[27 + 9 - \frac{81}{4}\right]$$

$$= \frac{2}{27} \left(\frac{63}{4}\right) = \frac{7}{6}.$$

For the mode of the distribution, $f'(x) = 0$ and $f''(x) < 0$.

Now $f(x) = \frac{2}{27}(6 + x - x^2)$

$\therefore \quad f'(x) = \frac{2}{27}(1 - 2x)$, and $f''(x) = \frac{-4}{27} < 0.$

Thus $f'(x) = 0$ gives $x = \frac{1}{2}$

Hence Mode $= \frac{1}{2}$.

Again $\text{Var}(X) = E(X^2) - [E(X)]^2$, where

$$E(X^2) = \frac{2}{27}\int_0^3 x^2(6 + x - x^2)\,dx = \frac{2}{27}\int_0^3 (6x^2 + x^3 - x^4)\,dx$$

$$= \frac{2}{27}\left[\frac{6x^3}{3} + \frac{x^4}{4} - \frac{x^5}{5}\right]_0^3 = \frac{2}{27}\left[54 + \frac{81}{4} - \frac{243}{5}\right]$$

$$= \frac{2}{27}\left[\frac{513}{20}\right] = \frac{19}{10}.$$ Therefore

$$\sigma^2 = \frac{19}{10} - \left(\frac{7}{6}\right)^2 = \frac{19}{10} - \frac{49}{36} = \frac{97}{180}.$$

7.31. Since f(x) is a probability density function, therefore

$$k\int_2^5 (2 - x)(x - 5)\,dx = 1.$$

We find the value of k as follows:

$$k\int_2^5 (2 - x)(x - 5)\,dx = 1$$

or $$k\int_2^5 (-x^2 + 7x - 10)\,dx = 1$$

i.e. $$k\left[-\frac{x^3}{3} + \frac{7x^2}{2} - 10x\right]_2^5 = 1$$

i.e. $$k\left[\left(-\frac{125}{3} + \frac{175}{2} - 50\right) - \left(-\frac{8}{3} + 14 - 20\right)\right] = 1$$

i.e. $$k\left[-\frac{25}{6} + \frac{26}{3}\right] = 1, \text{ i.e. } k\left(\frac{9}{2}\right) = 1 \text{ giving } k = 2/9$$

Now the mean of X is

$$E(X) = \frac{2}{9}\int_2^5 x(2-x)(x-5)\,dx$$

$$= \frac{2}{9}\int_2^5 (-x^3 + 7x^2 - 10x)\,dx$$

$$= \frac{2}{9}\left[-\frac{x^4}{4} + \frac{7x^3}{3} - \frac{10x^2}{2}\right]_2^5$$

$$= \frac{2}{9}\left[\left(-\frac{625}{4} + \frac{875}{3} - \frac{250}{2}\right) - \left(-4 + \frac{56}{3} - 20\right)\right]$$

$$= \frac{2}{9}\left[\frac{125}{12} + \frac{16}{3}\right] = \frac{2}{9}\left(\frac{189}{12}\right) = \frac{7}{2}, \text{ and}$$

$\mathrm{Var}(X) = E(X^2) - [E(X)]^2$, where

$$E(X^2) = \int_2^5 x^2 f(x)\,dx = \frac{2}{9}\int_2^5 x^2(2-x)(x-5)\,dx$$

$$= \frac{2}{9}\int_2^5 (-x^4 + 7x^3 - 10x^2)\,dx$$

$$= \frac{2}{9}\left[-\frac{x^5}{5} + \frac{7x^4}{4} - \frac{10x^3}{3}\right]_2^5 = \frac{2}{9}\left(\frac{1143}{20}\right) = \frac{127}{10}$$

$$\therefore \quad \mathrm{Var}(X) = \frac{127}{10} - \left(\frac{7}{2}\right)^2 = \frac{9}{20}$$

For the <u>mode</u> of the distribution, $f'(x) = 0$ and $f''(x) < 0$.

Now $\quad f(x) = \frac{2}{9}(-x^2 + 7x - 10)$

$\therefore \quad f'(x) = \frac{2}{9}(-2x + 7)$, and

$$f'(x) = \frac{2}{9}(-2) = \frac{-4}{9}, \text{ which is negative.}$$

Thus $\quad f'(x) = 0$ gives $x = \dfrac{7}{2}$

Hence mode of the distribution is at $x = \dfrac{7}{2}$.

The median, a, is given by $\displaystyle\int_{-\infty}^{a} f(x)\,dx = \frac{1}{2}$. Thus

$$\frac{2}{9}\int_{2}^{a}(2-x)(x-5)\,dx = \frac{1}{2}$$

or $$\frac{2}{9}\left[-\frac{x^3}{3} + \frac{7x^2}{2} - 10x\right]_2^a = \frac{1}{2}$$

or $$\frac{2}{9}\left[-\frac{a^3}{3} + \frac{7a^2}{2} - 10a + \frac{26}{3}\right] = \frac{1}{2}$$

This equation reduces to

$$4a^3 - 42a^2 + 120a - 77 = 0.$$

Since the probability density function, which is parabolic, is symmetrical, so the equation has solution $a = 7/2$ and thus we may factorize the equation given above as

$$(2a-7)(2a^2 - 14a + 11) = 0$$

Now $\quad 2a - 7 = 0$ gives $a = \dfrac{7}{2}$ and

$2a^2 - 14a + 11 = 0$ gives $a = 0.902$ and 6.098

But both the values 0.902 and 6.098 are unacceptable since a must lie in the interval (2, 5). The median is therefore given by $a = 7/2$.

7.32. By definition, we have

$$\log G = E[\log X] = \int_{-\infty}^{\infty} \log x\, f(x)\, dx, \text{ if it exists.}$$

$$= 6\int_1^2 (\log x)(2-x)(x-1)\, dx$$

$$= 6\int_1^2 (\log x)(-x^2+3x-2)\, dx$$

$$= -6\int_1^2 x^2 \log x\, dx + 18\int_1^2 x\log x\, dx$$

$$-12\int_1^2 \log x\, dx$$

To evaluate $\int_1^2 x^2 \log x dx$, we put $u = \log x$ and $dv = x^2 dx$ so that $du = \frac{1}{x}$ and $v = \frac{x^3}{3}$. Thus integrating by parts, we have

$$\int_1^2 x^2 \log x\, dx = \left[\frac{x^3}{3}\log x\right]_1^2 - \int_1^2 \frac{x^3}{3}\cdot\frac{1}{x}\, dx$$

$$\left[\because \int_0^t u\,dv = uv\Big]_0^t - \int_0^t v\,du\right]$$

$$= \left[\frac{8}{3}\log 2 - 0\right] - \frac{1}{3}\left[\frac{x^3}{3}\right]_1^2 = \frac{8}{3}\log 2 - \frac{7}{9};$$

$$\int_1^2 x\log x\, dx = \left[\frac{x^2}{2}\log x\right]_1^2 - \int_1^2 \frac{x^2}{2}\cdot\frac{1}{x}\, dx$$

(Integrating by parts)

$$= 2\log 2 - \frac{1}{2}\left[\frac{x^2}{2}\right]_1^2 = 2\log 2 - \frac{3}{4}$$

$$\int_1^2 x \log x \, dx = \left[x \log x\right]_1^2 - \int_1^2 x \cdot \frac{1}{x} \, dx$$

(Integrating by parts)

$$= 2 \log 2 - \left[\, x \,\right]_1^2 = 2 \log 2 - 1.$$

Thus $\log G = -6\left[\frac{8}{3}\log 2 - \frac{7}{9}\right] + 18\left[2 \log 2 - \frac{3}{4}\right] - 12\,[2 \log 2 - 1]$

$$= -4 \log 2 + \frac{19}{6} = -\log 2^4 + \frac{19}{6}$$

or $\quad \log G + \log 16 = \frac{19}{6}$ or $\log(16\,G) = \frac{19}{6}$

or $\quad 6 \log(16\,G) = 19.$

7.33. A function f(x) is a proper density function, if

$\int_{-\infty}^{\infty} f(x)\, dx = 1$. Therefore

$$k \int_0^1 x^2(1-x)^3 \, dx = 1$$

Using the given relationship $\int_0^1 x^m(1-x)^n \, dx = \frac{m!\,n!}{(m+n+1)!}$, we get

$$k \left[\frac{2!\;3!}{(2+3+1)!}\right] = 1, \text{ or } k = 60.$$

Hence $f(x) = 60x^2\,(1-x)^3$, where $0 \le x \le 1$, is a proper $p.d.f.$

The skewness, in this case, is measured by μ_3, which in symmetrical distribution is zero.

Now $\quad \mu_1' = E(X) = \int_{-\infty}^{\infty} x \cdot f(x)\, dx$

$$= 60 \int_0^1 x \cdot x^2(1-x)^3 \, dx = 60 \int_0^1 x^3(1-x)^3 \, dx$$

Using the relationship $\int_0^1 x^m(1-x)^n\,dx = \frac{m!\,n!}{(m+n+1)!}$, we get

$$\mu_1' = 60 \cdot \frac{3!\,3!}{(3+3+1)!} = 60\left(\frac{1}{140}\right) = \frac{3}{7}.$$

$$\mu_2' = E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx = 60\int_0^1 x^2 \cdot x^2(1-x)^3\,dx$$

$$= 60\int_0^1 x^4(1-x)^3\,dx = 60 \cdot \frac{4!\,3!}{(4+3+1)!} = \frac{3}{14}, \text{ and}$$

$$\mu_3' = E(X^3) = \int_{-\infty}^{\infty} x^3 f(x)\,dx = 60\int_0^1 x^5(1-x)^3\,dx$$

$$= 60\,\frac{5!\,3!}{(5+3+1)!} = \frac{5}{42}.$$

Hence $\mu_3 = \mu_3' - 3\mu_1'\mu_2' + 2\mu_1'^3$

$$= \frac{5}{42} - 3\left(\frac{3}{7}\right)\left(\frac{3}{14}\right) + 2\left(\frac{3}{7}\right)^3$$

$$= \frac{5}{42} - \frac{27}{98} + \frac{54}{343} = \frac{2}{2058} = 0.001.$$

7.34 (a) First of all, we find the value of k, which should be such as to make

$$\int_{-a}^{a} k\,dx = 1$$

or $$k\left[x\right]_{-a}^{a} = 1 \text{ or } 2ak = 1 \text{ or } k = \frac{1}{2a}$$

Now, the mean, μ, is given by

$$\mu = \int_{-a}^{a} x \cdot \frac{1}{2a}\,dx = \frac{1}{2a}\left[\frac{x^2}{2}\right]_{-a}^{a} = \frac{1}{2a}(0) = 0.$$

The variance, σ^2 or μ_2, is given by

$$\sigma^2 = \frac{1}{2a}\int_{-a}^{a} x^2\,dx - (\mu)^2 = \frac{1}{2a}\left[\frac{x^3}{3}\right]_{-a}^{a} \qquad (\because\ \mu = 0)$$

$$= \frac{1}{2a}\left[\frac{a^3}{3} + \frac{a^3}{3}\right] = \frac{a^2}{3}.$$

Again Mean Deviation about the mean is

$$M.D. = \int_{-\infty}^{\infty} |x - \mu|\, f(x)\,dx = \frac{1}{2a}\int_{-a}^{a} |x - \mu|\,dx$$

$$= \frac{1}{2a}\left\{\int_{-a}^{0} -x\,dx + \int_{0}^{a} x\,dx\right\} \qquad (\because\ \mu = 0)$$

$$= \frac{2}{2a}\int_{0}^{a} x\,dx = \frac{1}{a}\left[\frac{x^2}{2}\right]_{0}^{a} = \frac{a}{2}.$$

(b) Calculation of the mean moments and the mean deviation.

$$\mu = \int_{-\infty}^{\infty} x\,f(x)\,dx = \int_{0}^{1} x\,dx = \left[\frac{x^2}{2}\right]_{0}^{1} = \frac{1}{2}.$$

Now, $\mu_1 = 0$;

$$\mu_2 = \int_{-\infty}^{\infty} (x - \mu)^2 f(x)\,dx$$

$$= \int_{0}^{1} \left(x - \frac{1}{2}\right)^2 dx = \int_{0}^{1} \left(x^2 - x + \frac{1}{4}\right) dx$$

$$= \left[\frac{x^3}{3} - \frac{x^2}{2} + \frac{x}{4}\right]_{0}^{1} = \frac{1}{3} - \frac{1}{2} + \frac{1}{4} = \frac{1}{12};$$

$$\mu_3 = \int_{-\infty}^{\infty} (x-\mu)^3 f(x)\, dx$$

$$= \int_0^1 \left(x - \frac{1}{2}\right)^3 dx = \int_0^1 \left(x^3 - \frac{3x^2}{2} + \frac{3x}{4} - \frac{1}{8}\right) dx$$

$$= \left[\frac{x^4}{4} - \frac{x^3}{2} + \frac{3x^2}{8} - \frac{x}{8}\right]_0^1 = \frac{1}{4} - \frac{1}{2} + \frac{3}{8} - \frac{1}{8} = \frac{0}{8} = 0;$$

$$\mu_4 = \int_{-\infty}^{\infty} (x-\mu)^4 f(x)\, dx = \int_0^1 \left(x - \frac{1}{2}\right)^4 dx$$

$$= \int_0^1 \left(x^4 - 2x^3 + \frac{3x^2}{2} - \frac{x}{2} + \frac{1}{16}\right) dx$$

$$= \left[\frac{x^5}{5} - \frac{x^4}{2} + \frac{x^3}{2} - \frac{x^2}{4} + \frac{x}{16}\right]_0^1$$

$$= \frac{1}{5} - \frac{1}{2} + \frac{1}{2} - \frac{1}{4} + \frac{1}{16} = \frac{1}{80}.$$

The Mean Deviation about the mean is

$$M.D. = \int_{-\infty}^{\infty} |x - \mu|\, f(x)\, dx = \int_0^1 |x - 1/2|\, dx$$

$$= \int_0^{1/2} \left(\frac{1}{2} - x\right) dx + \int_{1/2}^1 \left(x - \frac{1}{2}\right) dx$$

$$= \left[\frac{x}{2} - \frac{x^2}{2}\right]_0^{1/2} + \left[\frac{x^2}{2} - \frac{x}{2}\right]_{1/2}^1$$

$$= \left(\frac{1}{4} - \frac{1}{8}\right) + \left\{\left(\frac{1}{2} - \frac{1}{2}\right) - \left\{\left(\frac{1}{8} - \frac{1}{4}\right)\right\}\right.$$

$$= \frac{1}{8} + \frac{1}{8} = \frac{1}{4}.$$

7.35. The total probability must be unity.

$$\int_{-\infty}^{\infty} f(x)\,dx = \frac{1}{2}\int_0^1 x\,dx + \frac{1}{2}\int_1^2 dx + \frac{1}{2}\int_2^3 (3-x)\,dx$$

$$= \frac{1}{2}\left[\frac{x^2}{2}\right]_0^1 + \frac{1}{2}\left[x\right]_1^2 + \frac{1}{2}\left[\frac{-(3-x)^2}{2}\right]_2^3$$

$$= \frac{1}{2}\left(\frac{1}{2}\right) + \frac{1}{2}(1) + \frac{1}{2}\left(\frac{1}{2}\right) = \frac{1}{4} + \frac{1}{2} + \frac{1}{4} = 1.$$

Now $\mu = \int_{-\infty}^{\infty} f(x)\,dx$

$$= \frac{1}{2}\int_0^1 x^2\,dx + \frac{1}{2}\int_1^2 x^2\,dx + \frac{1}{2}\int_2^3 x(3-x)\,dx$$

$$= \frac{1}{2}\left[\frac{x^3}{3}\right]_0^1 + \frac{1}{2}\left[\frac{x^2}{2}\right]_1^2 + \frac{1}{2}\left[\frac{3x^2}{2} - \frac{x^3}{3}\right]_2^3$$

$$= \frac{1}{2}\left(\frac{1}{3}\right) + \frac{1}{2}\left[2 - \frac{1}{2}\right] + \frac{1}{2}\left[\left(\frac{27}{2} - \frac{27}{3}\right) - \left(6 - \frac{8}{3}\right)\right]$$

$$= \frac{1}{6} + \frac{3}{4} + \frac{7}{12} = \frac{18}{12} = \frac{3}{2}.$$

$$\sigma^2 = \mu_2' - (\mu)^2 = \int_{-\infty}^{\infty} x^2 f(x)\,dx - (\mu)^2$$

$$= \left\{\frac{1}{2}\int_0^1 x^3 dx + \frac{1}{2}\int_1^2 x^2 dx + \frac{1}{2}\int_2^3 x^2(3-x)\,dx\right\} - \left(\frac{3}{2}\right)^2$$

$$= \left\{\frac{1}{2}\left[\frac{x^4}{2}\right]_0^1 + \frac{1}{2}\left[\frac{x^3}{3}\right]_1^2 + \frac{1}{2}\left[x^3 - \frac{x^4}{4}\right]_2^3\right\} - \frac{9}{4}$$

$$= \left\{\frac{1}{2}\left(\frac{1}{4}\right) + \frac{1}{2}\left(\frac{8}{3} - \frac{1}{3}\right) + \frac{1}{2}\left(\frac{27}{4} - \frac{4}{1}\right)\right\} - \frac{9}{4}$$

$$= \left\{\frac{1}{8} + \frac{7}{6} + \frac{11}{8}\right\} - \frac{9}{4} = \frac{8}{3} - \frac{9}{4} = \frac{5}{12}.$$

In order to find β_2, the moment measure of Kurtosis, we first find the third and fourth moments. Thus

$$\mu_3' = \int_{-\infty}^{\infty} x^3 \cdot f(x)\, dx$$

$$= \frac{1}{2}\int_0^1 x^4 dx + \frac{1}{2}\int_1^2 x^3 dx + \frac{1}{2}\int_2^3 x^3(3-x)\, dx$$

$$= \frac{1}{2}\left[\frac{x^5}{5}\right]_0^1 + \frac{1}{2}\left[\frac{x^4}{4}\right]_1^2 + \frac{1}{2}\left[\frac{3x^4}{4} - \frac{x^5}{5}\right]_2^3$$

$$= \frac{1}{10} + \frac{15}{8} + \frac{1}{2}\left(\frac{243}{20} - \frac{28}{5}\right) = \frac{1}{10} + \frac{15}{8} + \frac{131}{40} = \frac{21}{4}$$

$$\mu_4' = \int_{-\infty}^{\infty} x^4 \cdot f(x)\, dx$$

$$= \frac{1}{2}\int_0^1 x^5 dx + \frac{1}{2}\int_1^2 x^4 dx + \frac{1}{2}\int_2^3 x^4(3-x)\, dx$$

$$= \frac{1}{2}\left[\frac{x^6}{6}\right]_0^1 + \frac{1}{2}\left[\frac{x^5}{5}\right]_1^2 + \frac{1}{2}\left[\frac{3x^5}{5} - \frac{x^6}{6}\right]_2^3$$

$$= \frac{1}{2}\left(\frac{1}{6}\right) + \frac{1}{2}\left(\frac{31}{5}\right) + \frac{1}{2}\left(\frac{729}{30} - \frac{128}{15}\right)$$

$$= \frac{1}{12} + \frac{31}{10} + \frac{473}{60} = \frac{166}{15}.$$

Now $\mu_3 = \mu_3' - 3\mu_1'\mu_2' + 2(\mu_1')^3$

$$= \frac{21}{4} - 3\left(\frac{3}{2}\right)\left(\frac{8}{3}\right) + 2\left(\frac{3}{2}\right)^3 = \frac{21}{4} - \frac{12}{1} + \frac{27}{4} = 0, \text{ and}$$

$$\mu_4 = \mu_4' - 4\mu_1'\mu_3' + 6\mu_1'^2\mu_2' - 3\mu_1'^4$$

$$= \frac{166}{15} - 4\left(\frac{3}{2}\right)\left(\frac{21}{4}\right) + 6\left(\frac{3}{2}\right)^2\left(\frac{8}{3}\right) - 3\left(\frac{3}{2}\right)^4$$

$$= \frac{166}{15} - \frac{63}{2} + \frac{36}{1} - \frac{243}{16} = \frac{91}{240}$$

Hence $\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{91}{240} \times \frac{144}{25} = \frac{273}{125} = 2.184.$

7.36. Here the distribution is $f(x) = x^2(6-x)^2$ between $x=0$ and $x=6$. Therefore we have

$$\int_0^6 x^2(6-x)^2\,dx = \int_0^6 (x^4 - 12x^3 + 36x^2)\,dx$$

$$= \left[\frac{x^5}{5} - \frac{12x^4}{4} + \frac{36x^3}{3}\right]_0^6$$

$$= \frac{(6)^5}{5} - 3(6)^4 + 12(6)^3$$

$$= 216\left[\frac{36}{5} - 18 + 12\right] = 216 \times \frac{6}{5} = \frac{1296}{5}$$

To make the total frequency unity, we must multiply $f(x)$ by $\frac{5}{1296}$.

Now we calculate the moments about origin as:

$$\mu_1' = \int_{-\infty}^{\infty} xf(x)\,dx = \frac{5}{1296}\int_0^6 x^3(x^2 - 12x + 36)\,dx$$

$$= \frac{5}{1296}\left[\frac{x^6}{6} - \frac{12x^5}{5} + \frac{36x^4}{4}\right]_0^6$$

$$= \frac{5}{1296}\left[(6)^4\left\{\frac{36}{6} - \frac{72}{5} + 9\right\}\right] = \frac{5 \times 1296}{1296}\left\{\frac{3}{5}\right\} = 3;$$

$$\mu_2' = \int_{-\infty}^{\infty} x^2 f(x)\,dx = \frac{5}{1296}\int_0^6 x^4(x^2 - 12x + 36)\,dx$$

$$= \frac{5}{1296}\left[\frac{x^7}{7} - 2x^6 + \frac{36x^5}{5}\right]_0^6$$

$$= \frac{5}{1296} \times (6)^5 \left[\frac{36}{7} - 12 + \frac{36}{5}\right] = 30\left[\frac{12}{35}\right] = \frac{72}{7};$$

$$\mu_3' = \int_{-\infty}^{\infty} x^3 f(x)\, dx = \frac{5}{1296}\int_0^6 x^5(x^2 - 12x + 36)\, dx$$

$$= \frac{5}{1296}\left[\frac{x^8}{8} - \frac{12x^7}{7} + 6x^6\right]_0^6$$

$$= \frac{5}{1296}\left[x^6\left(\frac{x^2}{8} - \frac{12x}{7} + 6\right)\right]_0^6$$

$$= \frac{5}{1296}\left[1296 \times 36\left(\frac{9}{2} - \frac{72}{7} + 6\right)\right] = 180 \times \frac{3}{14} = \frac{270}{7}$$

$$\mu_4' = \int_{-\infty}^{\infty} x^4 f(x)\, dx = \frac{5}{1296}\int_0^6 x^6(x^2 - 12x + 36)\, dx$$

$$= \frac{5}{1296}\left[\frac{x^9}{9} - \frac{12x^8}{8} + \frac{36x^7}{7}\right]_0^6$$

$$= \frac{5}{1296}\left[x^7\left(\frac{x^2}{9} - \frac{3x}{2} + \frac{36}{7}\right)\right]_0^6$$

$$= \frac{5}{1296}\left[(1296 \times 216)\left(4 - 9 + \frac{36}{7}\right)\right]$$

$$= 1080 \times \frac{1}{7} = \frac{1080}{7}.$$

Hence the moments about the mean are:

$\mu_1 = 0;$

$$\mu_2 = \mu'_2 - (\mu'_1)^2 = \frac{72}{7} - (3)^2 = \frac{9}{7} = 1.29;$$

$$\mu_3 = \mu'_3 - 3\mu'_1\mu'_2 + 2(\mu'_1)^3$$

$$= \frac{270}{7} - 3(3)\left(\frac{72}{7}\right) + 2(3)^3 = \frac{270}{7} - \frac{648}{7} + 54 = 0;$$

$$\mu_4 = \mu'_4 - 4\mu'_1\mu'_3 + 6(\mu'_1)^2\mu'_2 - 3(\mu'_1)^4$$

$$= \frac{1080}{7} - \frac{3240}{7} + \frac{3888}{7} - 243 = \frac{27}{7} = 3.86;$$

and the kurtosis of the distribution is

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{27}{7} \times \left(\frac{7}{9}\right)^2 = \frac{7}{3} = 2.33.$$

7.37. To compute the marginal p.d.'s and the correlation co-efficient, the values are arranged in the tabular form as below:

X \ Y	1	2	3	$g(x)$
1	$\frac{2}{15}$	$\frac{4}{15}$	$\frac{3}{15}$	$\frac{9}{15}$
2	$\frac{1}{15}$	$\frac{1}{15}$	$\frac{4}{15}$	$\frac{6}{15}$
$h(y)$	$\frac{3}{15}$	$\frac{5}{15}$	$\frac{7}{15}$	1

The correlation co-efficient, ρ, is computed as

$$\rho = \frac{E(XY) - E(X)\,E(Y)}{\sqrt{[E(X^2) - \{(EX)\}^2]\,[E(Y^2) - \{E(Y)\}^2]}}, \text{ where}$$

$$E(X) = \sum_{i=1}^{2} x_i g(x) = 1 \times \frac{9}{15} + 2 \times \frac{6}{15} = \frac{9}{15} + \frac{12}{15} = \frac{21}{15};$$

$$E(Y) = \sum_{j=1}^{3} y_j h(y_j) = 1 \times \frac{3}{15} + 2 \times \frac{5}{15} + 3 \times \frac{7}{15}$$

$$= \frac{3}{15} + \frac{10}{15} + \frac{21}{15} = \frac{34}{15};$$

$$E(X^2) = \sum_{i=1}^{2} x_i^2 g(x_i) = 1 \times \frac{9}{15} + 4 \times \frac{6}{15} = \frac{9}{15} + \frac{24}{15} = \frac{33}{15};$$

$$E(Y^2) = \sum_{j=1}^{3} y_j^2 h(y_j) = 1 \times \frac{3}{15} + 4 \times \frac{5}{15} + 9 \times \frac{7}{15}$$

$$= \frac{3}{15} + \frac{20}{15} + \frac{63}{15} = \frac{86}{15}; \text{ and}$$

$$E(XY) = \sum_i \sum_j x_i y_j f(x_i, y_j) = 1 \times \frac{2}{15} + 2 \times \frac{4}{15} + 3 \times \frac{3}{15}$$

$$+ 2 \times \frac{1}{15} + 4 \times \frac{1}{15} + 6 \times \frac{1}{15}$$

$$= \frac{2}{15} + \frac{8}{15} + \frac{9}{15} + \frac{2}{15} + \frac{4}{15} + \frac{24}{15} = \frac{49}{15}$$

$$\therefore \rho = \frac{\frac{49}{15} - \left(\frac{21}{15}\right)\left(\frac{34}{15}\right)}{\sqrt{\left\{\frac{33}{15} - \left(\frac{21}{15}\right)^2\right\}\left\{\frac{86}{15} - \left(\frac{34}{15}\right)^2\right\}}} = \frac{\frac{49}{15} - \left(\frac{714}{225}\right)}{\sqrt{\left\{\frac{33}{15} - \frac{441}{225}\right\}\left\{\frac{86}{15} - \frac{1156}{225}\right\}}}$$

$$= \frac{\frac{735 - 714}{225}}{\sqrt{\left\{\frac{495 - 441}{225}\right\}\left\{\frac{1290 - 1156}{225}\right\}}} = \frac{\frac{21}{225}}{\sqrt{\left(\frac{54}{225}\right)\left(\frac{134}{225}\right)}}$$

$$= \frac{21}{85.06} = 0.25$$

7.38. (b) Given $f(x, y) = 2 - x - y, \; 0 \le x \le 1, \; 0 \le y \le 1$
$= 0, \text{ elsewhere.}$

Now $g(x) = \int_0^1 (2-x-y)\, dy = \left[2y - xy - \frac{y^2}{2}\right]_0^1 = \frac{3}{2} - x$; and

$$h(x) = \int_0^1 (2-x-y)\,dx = \left[2x - \frac{x^2}{2} - xy\right]_0^1 = \frac{3}{2} - y;$$

$$E(X) = \int_0^1 x\left(\frac{3}{2} - x\right) dx = \left[\frac{3x^2}{4} - \frac{x^3}{3}\right]_0^1 = \frac{5}{12};$$

$$E(Y) = \int_0^1 y\left(\frac{3}{2} - y\right) dy = \left[\frac{3y^2}{4} - \frac{y^3}{3}\right]_0^1 = \frac{5}{12};$$

$$E(X^2) = \int_0^1 x^2\left(\frac{3}{2} - x\right) dx = \left[\frac{3x^3}{6} - \frac{x^4}{4}\right]_0^1 = \frac{1}{4};$$

$$E(Y^2) = \int_0^1 y^2\left(\frac{3}{2} - y\right) dy = \left[\frac{3y^3}{6} - \frac{y^4}{4}\right]_0^1 = \frac{1}{4};$$

$$E(XY) = \int_0^1\int_0^1 xy(2-x-y)\,dx\,dy = \int_0^1 \left[\frac{2xy^2}{2} - \frac{x^2y^2}{2} - \frac{xy^3}{3}\right]_0^1 dx$$

$$= \int_0^1 \left(x - \frac{x^2}{2} - \frac{x}{3}\right) dx = \left[\frac{x^2}{2} - \frac{x^3}{6} - \frac{x^2}{6}\right]_0^1 = \frac{1}{6}.$$

$$\therefore \quad \rho = \frac{E(XY) - E(X)\,E(Y)}{\sqrt{[E(X^2) - \{E(X)\}^2]\,[E(Y^2) - \{E(Y)\}^2]}}$$

$$= \frac{\frac{1}{6} - \left(\frac{5}{12}\right)\left(\frac{5}{12}\right)}{\sqrt{\left[\frac{1}{4} - \left(\frac{5}{12}\right)^2\right]\left[\frac{1}{4} - \left(\frac{5}{12}\right)^2\right]}}$$

$$= \frac{\frac{1}{6} - \frac{25}{144}}{\frac{1}{4} - \frac{25}{144}} = \frac{-\frac{1}{144}}{\frac{11}{144}} = -\frac{1}{11} = -0.091.$$

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