Probability Distribution - I

Set Theory

Set :-

A well-define collection of distinct objects.

e.g.

$U = \{1,2,3,4,5,6,7,8,9,10\}$ → Set of first 10 natural no.

$A = \{2,4,6,8,10\}$ → Set of first 5 even numbers.

$B = \{1,3,5,7,9\}$ → Set of first 5 odd no.

$C = \{2,3,5,7\}$ → Set of first 4 Prime no.

$D = \{\ \} = \phi$ → Empty Set.

Subset :-

If all elements of set A are also the element of Set B then "A" is subset of "B".

e.g.

$U = \{1,2,3,4,5,6,7,8,9,10\}$

$A = \{2,4,6,8,10\}$

So,

$A \subset U$

or $U \supset A$

∴ Empty Set is a subset of every Set.

Empty Set (Null Set) :-

A Set having no element is called empty Set $\{\phi\}$.

Universal Set:

If all under study sets belong to a particular set "U", then "U" is a universal set.

e.g.

$U = \{1,2,3,4,5,6,7,8,9,10\}$

and $A = \{1,2,3\}$; $B = \{4,5,6\}$; $C = \{\phi\}$

So, "U" is a universal set.

Disjoint Sets:

Two sets "A" and "B" are said to be disjoint, if they have no common element.

$A \cap B = \phi$

e.g.

$A = \{2,4,6,8,10\}$; $B = \{1,3,5,7,9\}$

$A \cap B = \{\ \} = \phi$

Overlaping Sets:

Two Sets "A" and "B" are said to be overlaping set, if they have atleast one common element. $A \cap B \neq \phi$

e.g.

$A = \{2,3,5,7\}$; $B = \{2,4,6,8,10\}$

$A \cap B = \{2\} \neq \phi$

Operations on Sets:

i) Union of Sets

ii) Intersection of Sets

iii) Difference of Sets

iv) Complemtry Set

Union of Sets:

"A" and "B" contain all elements those are belongs to set "A" or Set "B" or both sets "A" and "B".

e.g.

$A = \{1, 2, 3, 5\}$; $B = \{2, 4, 5, 6, 7\}$

$A \cup B = \{1, 2, 3, 4, 5, 6, 7\}$

Intersection of Sets:

"A" and "B" contain all elements those are belongs to both "A" and "B".

$A = \{2, 4, 3, 1\}$; $B = \{2, 3, 9, 7\}$

$A \cap B = \{2, 3\}$

Difference of Sets:

Set of all elements of "A" which do not belong to "B", denoted by $A - B$.

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Complementory Set:

It is denoted by A', $\bar{A}$ or A^c.

It contain all elements those are belongs to the Universal Set but not in "A".

e.g.

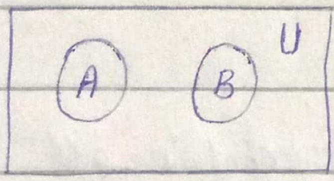
$U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

$A = \{2, 4, 6, 8, 10\}$

$\bar{A} = U - A = \{1, 3, 5, 7, 9\}$

Venn Diagram:

Presentation of Sets or operations of Sets, Known as Venn diagram.



Venn Diagram

Experiment:

Any Process for obtaining the observations or outcome is called experiment.

Random Experiment:

An experiment whose result cannot be predicted with certainty before the Performance of experiment.

A random experiment has three properties:

i) Repeated any number of time.
ii) Two or mor outcomes.
iii) Outcome is unpredictable.

Outcome:

A single possible result of random experiment is an outcome.

e.g.

A fair coin is toss, Possible outcomes are Head or Tail.

Sample Space:

A set of all possible outcomes of a random experiment.

e.g. A Coin is tossed, then Sample Space is $S = \{H, T\}$.

Event:

Any Subset of given Sample Space.

e.g.

$S = \{1, 2, 3, 4, 5, 6\}$

$A = \{2, 4, 6\}$ and $B = \{1, 3, 5\}$

Both "A" and "B" are events of "S".

Types of Events:

i) **Simple Event :** Contain only one sample point.

ii) **Compound Event:** Contain more than one outcomes.

e.g.

$\underbrace{A = \{2, 4, 6\} ; B = \{1, 3, 5\}}_{\text{Compound}}$

$\underbrace{A = \{1\} ; B = \{5\}}_{\text{Simple}}$

Mutually Exclusive Events: (disjoint)

Two or more events are mutually exclusive if they have no common point in the given sample space. OR If they cannot occur together.

e.g. $S = \{1, 2, 3, 4, 5, 6\}$

$\left.\begin{array}{l} A = \{2, 4, 6\} \\ B = \{1, 3, 5\} \end{array}\right]$ — Mutually Exclusive

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Equally Likely Events:

Two events "A" and "B" are said to be equally likely if they have equal number of Sample points or outcomes OR Same chances of Occurrance.

$S = \{1, 2, 3, 4, 5, 6\}$

$$\left.\begin{array}{l} A = \{2, 4, 6\} \\ B = \{1, 3, 5\} \end{array}\right\} \longrightarrow \text{Equally Likely}$$

Exhaustive Events:

Two events "A" and "B" are said to be exhaustive events if their union is equal to universal set.

$A = \{2, 4, 6\}$; $B = \{1, 3, 5\}$

$A \cup B = \{1, 2, 3, 4, 5, 6\}$ = Sample Space.

A and B are exhastive events.

Independent Events:

If occurence of one event does not effect on occurrence of another event from a given ~~spa~~ Sample Space.

e.g. With replacement selection.

Dependent Events:

If occurence of one effects on occurence of another event. **e.g.** Without replacement

Definition of Probability

i) Classical / Priori Defination

If an experiment consist of "n" mutually exclusive and equally likely and exhaustive outcomes out of which "m" are favourable of event "A" then:

$$P(A) = \frac{m}{n}$$

ii) Relative Frequency / Postriori Defination

If an experiment repeated a large number of times say "n" under uniform conditions and an event "A" occur "m" times then:

$$P(A) = \lim_{n \to \infty} \frac{m}{n}$$

iii) Mathematical / Axiomatic Approach

If an experiment consist of "n(S)" mutually exclusive, equally likely and exhoustive outcomes out of which "n(A)" are favourable of event "A" then

$$P(A) = \frac{n(A)}{n(S)}$$

If experiment follows the following conditions:
For any event "A".

i) $P(A) \geq 0$

ii) $P(A)$ lies b/w 0 to 1.

iii) $P(S) = 1$

iv) $P(A \cup B) = P(A) + P(B)$ ("A" and "B" are mutually exclusive)

Types of Probability

Objective

i) Classical

ii) Empirical

iii) Mathematical

Subjective

★ The Probability of an event based on personal experience is subjective Prob.

★ The Probability of an event based on mathematical reasoning is called objective Probability.

Difference b/w Outcome & Event

Outcome:

A single possible result of a random experiment.

e.g. H, T, (H,H), (H,T), 3, 6, (6,6)

Event:

Any subset of sample space.

e.g. $S = \{1, 2, 3, 4, 5, 6\}$

$A = \{\ \}$, $B = \{1, 2, 3, 4, 5, 6\}$, $C = \{6\}$

Here; A, B and C are events and sample points contains within these events are outcomes.

Power Set:

The power set of a set "A" is the set of all possible subsets of "A". It is denoted by 2^A or P.

Let $A = \{2, 4\}$ then

$P = 2^A = 2^2 = 4 = \{\phi, \{2\}, \{4\}, \{2, 4\}\}$

The $\{2, 4\}$ is sample space itself and is also an event. It always occurs and known as sure event.

The empty set is also an event, sometime known as impossible event.

Field (Algebra):

A field (Algebra) "C" is a non-empty set of sets that satisfies:

i) $\Omega \in C$

ii) if $A \in C$ then $\bar{A} \in C$

iii) if $A, B \in C$ then $A \cup B \in C$.

iv) if $A, B \in C$ then $A \cap B \in C$.

So, Algebra is a set containing Ω that is closed under finite unions, finite intersections, and complements. It contain both Ω and ϕ.

e.g.

A sample space for tossing two fair coins is:

$$\Omega = \{HH, HT, TH, TT\} \quad n(\Omega) = 4$$

Then field C over Ω is:

$$C = \{\phi, \{HH\}, \{HT\}, \{TH\}, \{TT\}, \ldots \{HH, HT, TH, TT\}\}$$

Let:

$$A = \{HH, HT\}, \quad B = \{HH, TT\}$$

i) $\Omega \in C$, Satisfies

ii) if $A \in C$ then $\bar{A} \in C$

$$A = \{HH, HT\} \in C$$

$$\bar{A} = \Omega - A = \{TH, TT\} \in C$$

iii) if $A, B \in C$ then $A \cup B \in C$.

$A; B \in C$

$A \cup B = \{HH, HT, TT\} \in C$

iv) if $A, B \in C$ then $A \cap B \in C$.

$A, B \in C$

$A \cap B = \{HH\} \in C$

σ-Field (σ-Algebra):

The only difference b/w algebra and σ-Algebra is that is third property sequence of elements of C is infinite in σ-algebra unlike algebra.

i) $\phi, \Omega \in F$

ii) if $A \in F$, then $\bar{A} \in F$

iii) if $A_1, A_2, \ldots$ is a sequence of element of $F \in C$, then $A_1 \cup A_2 \cup A_3, \ldots \cup A_k \in F$. $\bigcup_{n=1}^{\infty} A_i \in F$

e.g.

A fair die is thrown:

$\Omega = \{1, 2, 3, 4, 5, 6\}$

$F = \{\phi, \Omega, \{1\}, \{2\}, \{1, 2\}, \{2, 3, 4, 5, 6\}, \{1, 3, 4, 5, 6\}, \{3, 4, 5, 6\}\}$

i) Justify it as σ-Field?

ii) Generate more Field as you can?

Principles of Counting

The counting principle (also called the counting rule) is a way to figure out the number of outcomes in a probability problem.

Three basic counting rules are:

i) Multiplication

ii) Permutation

iii) Combination

Rule of Multiplication

If a compound experiment consists of two experiments such that the first experiment has exactly m distinct outcomes and, if corresponding to each outcome of the first experiment there can be n distinct outcomes of the second experiment, then the compound experiment has exactly mn outcomes.

e.g. Compound experiment of tossing a coin and rolling a die together.

$A = \{H, T\}$; $B = \{1, 2, 3, 4, 5, 6\}$

$n(A) = 2$ $\qquad$ $n(B) = 6$

All possible outcomes are $2 \times 6 = 12$.

Rule of Permutation:

A Permutation is any ordered arrangement of all whole part of a set or object.
The number of Permutation of "n" distinct objects taken "r" at a time is denoted by $P(n,r) = {}^{n}P_{r} = \frac{n!}{(n-r)!}$

Where:

n = Total no. of objects

r = Selective objects

Cases of Permutation

i) When all "n" distinct objects are taken at a time, then Permutation is:

$$P = n!$$

e.g.

How many ways six books are arranged on a shelf?

Sol:

$n = 6$ books

$n! = 6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1$

$= 720$ ways.

ii) The number of Permutation of "n" distinct objects arranged in a circle or ring.

a) When clock-wise and anti clock-wise arrangment is different, then $P = (n-1)!$

e.g.

How many ways the committe of six person be sitting around the table?

Sol:

$$P = (n-1)!$$
$$= (6-1)! = 5!$$
$$= 5 \times 4 \times 3 \times 2 \times 1 = 120$$

b) When clockwise and anti-clockwise arrangement are same, then $P = \frac{(n-1)!}{2}$

e.g.

One white, One red, One blue, One pink and One yellow beats are threaded in a ring. The Permutation is same, then the arrangments are:

Sol:

$$P = \frac{(n-1)!}{2} = \frac{(5-1)!}{2} = \frac{4!}{2}$$

$$= \frac{4 \times 3 \times 2 \times 1}{2} = 12$$

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iii) When some of the objects are same kind and remaining other are different, then number of Permutations are :

$$P = \frac{n!}{k!}$$

Where :

k = Number of same kind of objects

n = Total number of objects

e.g.

One blue, One white, One green and Four yellow beats are threaded. Find the number of arrangments ?

Sol:

$n = 7$ $\qquad$ $P = \frac{7!}{4!}$

$k = 4$

iv) The number of ways arrange "n" distinct objects from which "n_1" all of first type, "n_2" all of second type and "n_3" all of 3rd type ..... "n_k" all of kth type.

Then

$$P = \frac{n!}{n_1! n_2! \cdots n_k!} \qquad \because \sum_{i=1}^{k} n = n$$

e.g.

How many ways the word "STATISTICS" can be arranged?

Sol:

$n_1 = 3$

$n_2 = 3$ $\qquad P = \dfrac{n!}{n_1!\,n_2!\,n_3!\,n_4!\,n_5!}$

$n_3 = 1$

$n_4 = 2$ $\qquad = \dfrac{10!}{3!\,3!\,2!} = 50400$

$n_5 = 1$

Factorial:

The Product of first "n" natural numbers, denoted by $n! = n(n-1)(n-2) \cdots 3.2.1$

It Should be noted that:

$$0! = 1$$

Rule of Combination

The number of combination "n" distinct objects without any ordered arrangment taken "r" at a time is denoted by:

$$C(n, r) = {}^{n}C_{r} = \frac{n!}{r!\,(n-r)!} \quad : r \leq n$$

e.g.

A three person committee is to be formed from a list of four person. How many sample points are associated with the experiment?

Sol:

$n = 4$

$r = 3$

$C = {}^{n}C_{r} = {}^{4}C_{3} = \frac{4!}{3!(4-3)!} = 4$

Conditional Probability

Let we have any two events "A" and "B", then the P(B) occure when we know that the probability of other event has already occured, is called conditional probability, denoted by P(B/A).

For Dependent Events

i) $P(B/A) = \frac{P(A \cap B)}{P(A)}$ $P(A) \neq 0$ and $P(A) > 0$

ii) $P(A/B) = \frac{P(A \cap B)}{P(B)}$ $P(B) \neq 0$ and $P(B) > 0$

For Independent Events

i) $P(A/B) = P(A)$

ii) $P(B/A) = P(B)$

Explanation:-

Basically, Sample space reduce due to some conditional information and probabilities associated with such a reduce sample space is conditional Probability.

Let:

A die thrown experiment.

$S = \{1, 2, 3, 4, 5, 6\}$

we wish to know the probability of the outcome that the die shows 6.

$A = \{6\}$

Before seeing the outcome, we told that the die shows an even number of dots.

$B = \{2, 4, 6\}$

Now,

our sample space reduce to $\{2, 4, 6\}$ from $\{1, 2, 3, 4, 5, 6\}$.

And:

our desired probability in the reduce sample space is $\frac{1}{3}$. We say that $\frac{1}{3}$ is C.P of the event A.

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Example :- Random experiment : Tossing 2 Coins.

$S = \{HH, HT, TH, TT\}$

A : getting 2 Heads $\quad P(A) = 1/4$

B : at least 1 Head $\quad P(B) = 3/4$

$$P(A/B) = \frac{P(A \cap B)}{P(B)}$$

$$P(\text{getting 2 heads} \mid \text{at least one head}) = \frac{P(\text{getting 2 heds} \cap \text{at least 1})}{P(\text{at least one head})}$$

$A = \{HH\}$

$B = \{HT, TH, HH\} \quad P(B) = 3/4$

$A \cap B = \{HH\} \quad P(A \cap B) = 1/4$

$$P(A/B) = \frac{1/4}{3/4} = \frac{1}{3}$$

$\because$ Sample Space reduce from $\{HH, HT, TH, TT\}$ to $\{HT, TH, HH\}$, and directly we find probability of event A (getting two heads) from reduce Sample space, that is $\frac{1}{3}$.

Example 2:-

An instructor has a questions bank, 300 easy T/F, 200 difficult T/F, 500 easy MCQ and 400 difficult MCQ.

If a question is selected at random from the question bank, what is the Prob. it will be easy given that it is a MCQ.

Sol:-

	T/F	MCQ
Easy	300	500
Difficult	200	400

$$P(\text{easy} \mid MCQ) = \frac{P(\text{easy} \cap MCQ)}{P(MCQ)}$$

$$= \frac{500/1400}{900/1400} = \frac{5}{9}$$

Example 3:-

A Committee of 4 students selected at random from a group of 8 boys and 4 girls. Given that there is at least one girl in a Committee, calculate the Prob. that there are exactly 2 girls in the Committee.

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Solution:-

12 Students

8B 4G

4 students

G_1 G_2 B_1 B_2

$$P(2G, 2B \mid \text{at least } 1G) = \frac{P(2G, 2B \cap \text{at least } 1G)}{P(\text{at least one } G)}$$

at least one girl

1G, 3B

2G, 2B $P(B) = P(\text{at least one } G) = 1 - P(\text{all 4 Boys})$

3G, 1B $= 1 - \frac{\binom{8}{4}}{\binom{12}{4}} = \frac{85}{99}$

4G, 0B

$$P(A \cap B) = \frac{({}^4C_2)({}^8C_2)}{({}^{12}C_4)} = \frac{56}{165}$$

$$P(A \mid B) = \frac{56/165}{85/99} =$$

Properties of Conditional Probability

i) $0 \le P(A/B) \le 1$

ii) $P(S/B) = 1$ $\quad \because (S \cap B) = B, \because P(S/B) = \frac{P(S \cap B)}{P(B)} = \frac{P(B)}{P(B)}$

iii) $P(A_1 \cup A_2 /B) = P(A_1/B) + P(A_2/B)$

Bayes Theorem