Probability

Theory (2019)

Short Question

(i)

let A and B be the event with
$p(A \cup B) = 4/5$, $p(\bar{A}) = 2/3$, $p(A \cap B) = 1/3$
find $p(A \cap \bar{B})$.

$$P(A) = 1 - p(\bar{A}) = 1 - \frac{2}{3} = 1/3$$

$$P(A \cap B) = p(A) + p(B) - p(A \cup B)$$

$$\frac{1}{3} = \frac{1}{3} + p(B) - \frac{4}{5}$$

$$p(B) = 4/5$$

$$p(\bar{B}) = 1 - \frac{4}{5} = 1/5$$

1st method:-

$$P(A \cap \bar{B}) = p(A) \cdot p(\bar{B})$$

$$= \frac{1}{3} \cdot \frac{1}{5} = 1/15$$

2nd

$$p(A \cap \bar{B}) = p(A) - p(A \cap B)$$

$$= \frac{1}{3} - \frac{1}{3}$$

$$= 0$$

(ii)

A club consist of 10 members. In how many ways can:

(a) the three officer be chosen.

(b) A committe of 3 member be selected.

(a) $^{10}P_3 = \dfrac{10!}{(10-3)!}$

$= 720$

(b) $^{10}C_3 = \dfrac{10!}{3!\,(10-3)!}$

$= 120.$

(iii)

Define Subjective approach of probability.

Subjective probability:-

As its name suggests, the subjective or personalistic probability is a measure of strength of a person's belief regarding the occurance of an event A. It has disadvantage that two or more persons faced with the same evidence may arrive at different probability.

(iv)

prove that $E(cX) = cE(X)$ where c is a constant.

let the p.d. of the r.v. X be $[x_i, f(x_i)]$, $i = 1, 2, \dots, n$

$$E(cX) = \sum_{i=1}^{n} (cx_i) f(x_i)$$

$$= (cx_1) f(x_1) + (cx_2) f(x_2) + \dots + (cx_n) f x_n$$

$$= c[x_1 f x_1 + x_2 f x_2 + \dots + x_n f(x_n)]$$

$$= c \sum_{i=1}^{n} x_i f(x_i)$$

$$= c E(X).$$

(v)

let X have a binomial distribution with $n = 3$, $p = 0.6$. Find $P(X \geq 2)$.

$q = 1 - p = 1 - 0.6 = 0.4$

$$p(X \geq 2) = 1 - (P(X < 2))$$

$$= 1 - \left[\binom{3}{x} (0.6)^x (0.4)^{3-x}\right]$$

$$= 1 - \left[\binom{3}{0} (0.6)^0 (0.4)^3 + \binom{3}{1} (0.6)(0.4)^2\right]$$

$$= 1 - (0.064 + 0.288)$$

$$= 0.648.$$

(vi)

If $E(x) = 10$, $\sigma = 3$. Can X have a negative binomial distribution.

Since mean is greater than variance, so X cannot have a negative binomial distribution.

Negative binomial distribution is:-

$$p(X=x) = \binom{x-1}{k-1} p^k q^{x-k}$$

(vii)

If x is a poisson random variable with $\mu = 1.6$, find $p(x>2)$.

$$p(X=x) = \frac{e^{-\mu} \mu^x}{x!}$$

$$p(x>2) = 1 - (P(x \le 2))$$

$$p(X=0) = \frac{e^{-1.6} (1.6)^0}{0!}$$

$$= 0.2018$$

$$p(X=1) = \frac{e^{-1.6} (1.6)}{1!}$$

$$= 0.32288$$

$$p(X=2) = \frac{e^{-1.6} (1.6)^2}{2!}$$

$= 0.2584$

$P(X>2) = 1-[0.2018+0.32288+0.2584]$
$= 0.2168.$

(viii)

write down two properties of hypergeometric distribution.

properties:-

i) The outcome of each trial may be classified into one of two categories, Success and failure.

ii) The probability of Success changes on each trial.

iii) The Successive trials are dependent.

(ix)

Define distribution function.

Distribution function:-

The distribution function of a random variable x denoted by $F(x)$ is defined by:

$$F(x) = P(X \leq x)$$

The $F(x)$ gives the probability of the event that X take a value less than or equal to a specified value of x is called distribution function.

(x)

Discrete probability distribution?

A probability distribution in which the random variable x can take only specific values, is called discrete probability distribution.

Example:-

Binomial, Hypergeometric, poisson distribution.

(xi)

Define conditional probability?

let X and Y be two discrete r.v's with joint probability function $f(x,y)$. then the conditional prob. function for X given Y=y denoted as $f(x/y)$ is defined by:

$$f(x/y) = f(x,y) / h(y)$$

$$f(y/x) = f(x,y) / g(x)$$

(xii)

Define central Limit theorem.

Central Limit theorem states:

" the distribution of sample mean approximates a normal distribution as a sample size get larger regardless of their population distribution".

(xiii)

Define Bernouli trials?

If the probability of each outcome remains the same throughout the trials then such trials are called the Bernouli trials.

(xiv)

write down 3 properties of normal distribution.

↳ Area under the curve is always equal to 1.

↳ Mean, mode and median are equal in normal distribution.

↳ Mean deviation is approximately $\frac{4}{5}$ of its standard deviation.

$$M.D = \frac{4}{5}\sigma$$

(XV)

In a binomial distribution with $n = 5$, $p(X=0) = p(X=1)$. Find the variance.

$$p(X = x) = \binom{n}{x} p^x q^{n-x}$$

$$p(X=0) = q^n$$

$$p(X=1) = \binom{n}{1} p q^{n-1}$$

$$= npq^{n-1}$$

$$p(X=0) = p(X=1)$$

$$q^n = npq^{n-1}$$

$$1 = \frac{npq^{n-1}}{q^n}$$

$$1 = npq^{n-1-n}$$

$$npq^{-1} = 1$$

$$np = 1/q^{-1} = q$$

$$var(X) = npq$$

$$= (q)(q)$$

$$= q^2$$

(xvii)

State the formula of hypergeometric and poisson distribution.

→ Formula for hypergeometric distribution:-

$$P(X=x) = \frac{\binom{k}{x}\binom{N-k}{n-x}}{\binom{N}{n}} \qquad x = 0,1,\dots n \quad x = 0,1,\dots k.$$

→ Formula for poisson distribution:-

$$P(X=x) = \frac{e^{-\mu}\,\mu^{x}}{x!} \qquad x = 0,1,\dots \infty.$$

Long Question

Question #2

The probability that A will be alive after 10 years to come is 5/7 and for B it is 7/9. Find out the following probabilities:-

i) Both will die (ii) A will be alive and B dead (iii) B will be alive and A dead (iv) Both will alive

$P(A) = 5/7$, $P(B) = 7/9$

$P(\bar{A}) = 1 - 5/7$ $\quad$ $P(\bar{B}) = 1 - 7/9$

$= 2/7$ $\quad$ $= 2/9$

i) Both will die:-

$P(\bar{A} \cap \bar{B}) = P(\bar{A}) \cdot P(\bar{B})$

$= \frac{2}{7} \cdot \frac{2}{9}$

$= 4/63$

ii) B alive and A dead:-

$P(B \cap \bar{A}) = P(B) \cdot P(\bar{A})$

$= \frac{7}{9} \cdot \frac{2}{7} = 2/9$

iii) A alive and B dead:-

$P(A \cap \bar{B}) = P(A) \cdot P(\bar{B})$

$= \frac{5}{7} \cdot \frac{2}{9}$

$= 10/63$

iv) Both will alive:-

$P(A \cap B) = P(A) \cdot P(B)$

$= 5/7 \cdot 7/9$

$= 5/9$

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Question # 3

prove that $var(X \pm Y) = var(X) + var(Y)$.

let X and Y be two discrete r.v's. then then the variance of the r.v. X + Y is defined as:-

$\rightarrow$ $var(X+Y) = E[(X+Y) - E(X+Y)]^2$

$= E[\{X - E(X)\} + \{Y - E(Y)\}]^2$

$= E[X - E(X)]^2 + E(Y - E(Y))^2 +$

$2E[X - E(X)][Y - E(Y)]$

$= var(X) + var(Y) + 2cov(X,Y)$

If X and Y are independent, then

$cov(X,Y) = 0$, so

$var(X+Y) = var(X) + var(Y)$.

$\rightarrow$ $var(X-Y) = E[(X-Y) - E(X-Y)]^2$

$= E[\{X - E(X)\} - \{Y - E(Y)\}]^2$

$= E[X - E(X)]^2 + E(Y - E(Y))^2 - 2cov(X,Y)$

X & Y independent, then $cov(X,Y) = 0$

so,

$var(X-Y) = var(X) + var(Y)$

so,

$var(X \pm Y) = var(X) + var(Y)$

Hence proved.

Question # 4

Find the value of k so, that the function $f(x)$ defined as follows may be a density function.

$$f(x) = kx^3(1-x) \qquad 0<x<1$$

determine mean and variance.

For probability density function:

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_0^1 k(x^3 - x^4)\,dx$$

$$1 = k\int_0^1 x^3 - x^4\,dx$$

$$= k\left[\frac{1}{4}x^4 - \frac{1}{5}x^5\right]_0^1$$

$$= k\left[\frac{1}{4} - \frac{1}{5}\right]$$

$$= k\left[\frac{5-4}{20}\right]$$

$$= k\left[1/20\right]$$

$$k = 20$$

i) Mean:-

$$E(x) = \int_0^1 x\,f(x)\,dx$$

$$= 20\int_0^1 x \cdot x^3(1-x)\,dx$$

$$= 20\int_0^1 x^4 - x^5\,dx$$

$$= 20\left[\frac{1}{5}x^5 - \frac{1}{6}x^6\right]_0^1$$

$= 20 \left[\frac{1}{5} - \frac{1}{6} \right]$

$= 20 \left[\frac{6-5}{30} \right] = 20 \left(\frac{1}{30} \right)$

$E(x) = 2/3$

→ variance:-

$var(x) = E(x^2) - [E(x)]^2$

$E(x^2) = \int_0^1 x^2 f(x)\, dx$

$= 20 \int_0^1 x^5 (1-x)\, dx$

$= 20 \int_0^1 x^5 - x^6\, dx$

$= 20 \left(\frac{1}{6} x^6 - \frac{1}{7} x^7 \right]_0^1$

$= 20 \left[\frac{1}{6} - \frac{1}{7} \right]$

$= 20 \left[\frac{7-6}{42} \right]$

$= \frac{20}{42} = 10/21$

$var(x) = \frac{10}{21} - \frac{4}{9}$

$= \frac{30-28}{63} = 2/63$

Question #5

The experience of a house-agent indicates that he can provide suitable accommodation for 75% of clients. If on a particular occasion, 6 clients approach him independently. Calculate prob. (i) less than 4 clients (ii) exactly 4 clients (iii) atleast 5 will get satisfactory accommodation.

(i) $P(X<4) = \sum_{x=0}^{3} \binom{6}{x} \left(\frac{3}{4}\right)^x \left(\frac{1}{4}\right)^{6-x}$

$75\% = \frac{75}{100} = \frac{3}{4} = p$

$$P(x<4) = \binom{6}{0}\left(\frac{3}{4}\right)^0\left(\frac{1}{4}\right)^6 + \binom{6}{1}\left(\frac{3}{4}\right)\left(\frac{1}{4}\right)^5$$
$$+ \binom{6}{2}\left(\frac{3}{4}\right)^2\left(\frac{1}{4}\right)^4 + \binom{6}{3}\left(\frac{3}{4}\right)^3\left(\frac{1}{4}\right)^3$$

$$= \frac{1}{4096} + 6\left(\frac{3}{4}\right)\cdot\frac{1}{1024} + 15\cdot\frac{9}{16}\cdot\frac{1}{256}$$
$$+ 20\cdot\frac{27}{64}\cdot\frac{1}{64}$$

$$= \frac{1}{4096} + \frac{18}{4096} + \frac{135}{4096} + \frac{540}{4096}$$

$$= \frac{694}{4096} = 0.169$$

(ii) $p(X=4) = \binom{6}{4}\left(\frac{3}{4}\right)^4\left(\frac{1}{4}\right)^2$

$= \frac{15 \times 81}{4096} = \frac{1215}{4096}$

$= 0.297$

(iii) $p(X \geq 5) = \sum_{x=5}^{6} \binom{6}{x}\left(\frac{3}{4}\right)^x\left(\frac{1}{4}\right)^{6-x}$

$= \binom{6}{5}\left(\frac{3}{4}\right)^5\left(\frac{1}{4}\right) + \binom{6}{6}\left(\frac{3}{4}\right)^6\left(\frac{1}{4}\right)^0$

$= 6 \cdot \frac{243}{1024} \cdot \frac{1}{4} + (1) \cdot \frac{729}{4096}(1)$

$= \frac{1458}{4096} + \frac{729}{4096}$

$= \frac{2187}{4096}$

$= 0.5339.$

Question # 6

Suppose that weights of 2000 male students are normally distributed with mean 155 pounds and standard deviation 20 pounds. Find the no. of students with weights (i) less than or equal to 100 pounds (ii) b/w 120 and 130 pounds. (iii) b/w 150 and 175 pounds

(iv) greater than or equal to 200 pounds.

(i) $P(x \le 100)$

$$Z = \frac{x - \mu}{\sigma} = \frac{x - 155}{20}$$

$$Z = \frac{100 - 155}{20} = -2.75.$$

$$P(X \le 100) = P(Z \le -2.75)$$
$$= 0.5 - P(-2.75 \le Z \le 0)$$
$$= 0.5 - 0.4970$$
$$= 0.0030$$

Therefore no. of students with weights less than or equal to hundred pounds is :- $= 2000 \times 0.0030$

$= 6$

(ii) $P(120 \le X \le 130)$

$$Z = \frac{x - \mu}{\sigma} = \frac{x - 155}{20}$$

At $X = 120$

$$Z = \frac{120 - 155}{20} = -1.75$$

At $X = 130$

$$Z = \frac{130 - 155}{20} = -1.25$$

$$P(120 \le X \le 130) = P(-1.75 \le Z \le -1.25)$$
$$= P(-1.75 \le Z \le 0) - P(-1.25 \le Z \le 0)$$

$= 0.4599 - 0.3944$

$= 0.0655$

No. of Students with weights b/w 120 and 130 pounds is:-

$= 2000 \times 0.0655$

$= 131.$

(iii) weights b/w 150 and 175 pounds:-

$\cdot P(150 \le X \le 175)$

$Z = \frac{X - 155}{20}$ at 150.

$Z = \frac{150 - 155}{20} = -0.25.$

At $X = 175.$

$Z = \frac{175 - 155}{20} = 1.00$

$P(150 \le X \le 175) = P(-0.25 \le Z \le 1.00)$

$= P(-0.25 \le Z \le 0) + P(0 \le Z \le 1.00)$

$= 0.0987 + 0.3413$

$= 0.4400.$

No. of Students with weights b/w 150 and 175 pounds is:-

$= 2000 \times 0.4400$

$= 880$

(iv) weights greater then or equal to 200 pounds:

$P(X \geq 200)$. $\quad z = \frac{x-\mu}{\sigma}$

At $x = 200$ $\quad z = \frac{200-155}{20} = 2.25$

$P(X \geq 200) = P(z \geq 2.25)$

$= 0.5 - P(0 \leq z \leq 2.25)$

$= 0.5 - 0.4878$

$= 0.0122$

No. of students with weights greater then or equal to 200 pounds is:-

$= 2000 \times 0.0122$

$= 24$

Question # 7

State and prove chebyshev's inequality. Explain the significance of this inequality.

Chebyshev's Inequality:-

Statement:-

If X is a r.v. having mean μ and variance $\sigma^2 > 0$ and k

is any positive constant, then the probability that a value of X falls within k standard deviation of the mean is at least $(1-\frac{1}{k^2})$. That is:-

$$P(\mu - k\sigma < X < \mu + k\sigma) \geq 1 - \frac{1}{k^2}$$

or equivalently:-

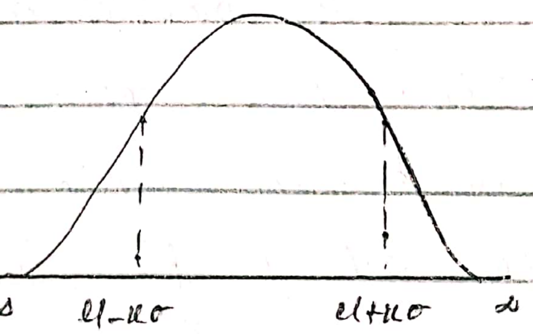
$$P(|x-\mu| \geq k\sigma) \leq \frac{1}{k^2}$$

Proof:-

By definition, we have:-

$$\sigma^2 = E(X-\mu)^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx$$

Dividing the range into 3 disjoint parts $(-\infty, \mu-k\sigma), (\mu-k\sigma, \mu+k\sigma); (\mu+k\sigma, \infty)$;



$$\sigma^2 = \int_{-\infty}^{\mu-k\sigma} (x-\mu)^2 f(x)\,dx + \int_{\mu-k\sigma}^{\mu+k\sigma} (x-\mu)^2 f(x)\,dx + \int_{\mu+k\sigma}^{\infty} (x-\mu)^2 f(x)\,d$$

Dropping the middle term.

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$$\sigma^2 \geq \int_{-\infty}^{\mu-k\sigma} (x-\mu)^2 f(x)\,dx + \int_{\mu+k\sigma}^{\infty} (x-\mu)^2 f(x)\,dx$$

Since $|x-\mu| \geq k\sigma \Rightarrow (x-\mu)^2 \geq k^2\sigma^2$

$$\sigma^2 \geq \int_{-\infty}^{\mu-k\sigma} k^2\sigma^2 f(x)\,dx + \int_{\mu+k\sigma}^{\infty} k^2\sigma^2 f(x)\,dx$$

or $$\int_{-\infty}^{\mu-k\sigma} f(x)\,dx + \int_{\mu+k\sigma}^{\infty} f(x)\,dx \leq \frac{1}{k^2}$$

Hence $$P(\mu-k\sigma < x < \mu+k\sigma) = \int_{\mu-k\sigma}^{\mu+k\sigma} f(x)(dx) \geq 1-\frac{1}{k^2}$$

Hence proved that:

$$P(\mu-k\sigma < X < \mu+k\sigma) \geq 1-\frac{1}{k^2}$$

Significance:-

It provides a mean of understanding how the variance measure variability about the mean of a r.v. It holds for all p.d.f's having finite mean and variance. In case of a discrete r.v., the proof is same with integrals being replaced by summation.