Chapter 12

CURVE FITTING BY LEAST SQUARES

12.1. (c) Let the equation of the straight line to be fitted to the data, be

$Y = a + bX$, where a and b are to be determined.

The two normal equations for determining a and b are

$$\sum Y = na + b\sum X, \quad \sum XY = a\sum X + b\sum X^2 .$$

The necessary calculations are shown in the following table:

	X	Y	XY	X^2	$Y' = a + bX$	$D = Y - Y'$
	1	2	2	1	3.18	−1.18
	2	6	12	4	4.84	1.16
	3	7	21	9	6.50	0.50
	4	8	32	16	8.16	−0.16
	5	10	50	25	9.82	0.18
	6	11	66	36	11.48	−0.48
$\sum$	21	44	183	91	--	0

Substituting these totals in the normal equations, we get

$6a + 21b = 44$ and $21a + 91b = 183$

Solving, we obtain $a = 1.52$ and $b = 1.66$

Hence the equation of the desired straight line is

$$Y = 1.52 + 1.66X.$$

The values of deviations D_i are given in the last column of the above table.

12.2. (c) Let the equation of the straight line to be fitted to the given data, be

$Y = a + bX$, where a and b are to be evaluated.

The two normal equations are

$$\sum Y = na + b\sum X, \quad \sum XY = a\sum X + b\sum X^2.$$

The computations involved are shown in the table below:

	X	Y	XY	X^2	Y'
	0	5	0	0	7.20
	1	11	11	1	8.48
	2	8	16	4	9.76
	3	14	42	9	11.04
	4	10	40	16	12.32
	5	16	80	25	13.60
	6	12	72	36	14.88
	7	20	140	49	16.16
	8	15	120	64	17.44
$\sum$	36	111	521	204	--

Putting these values in the normal equtions, we get

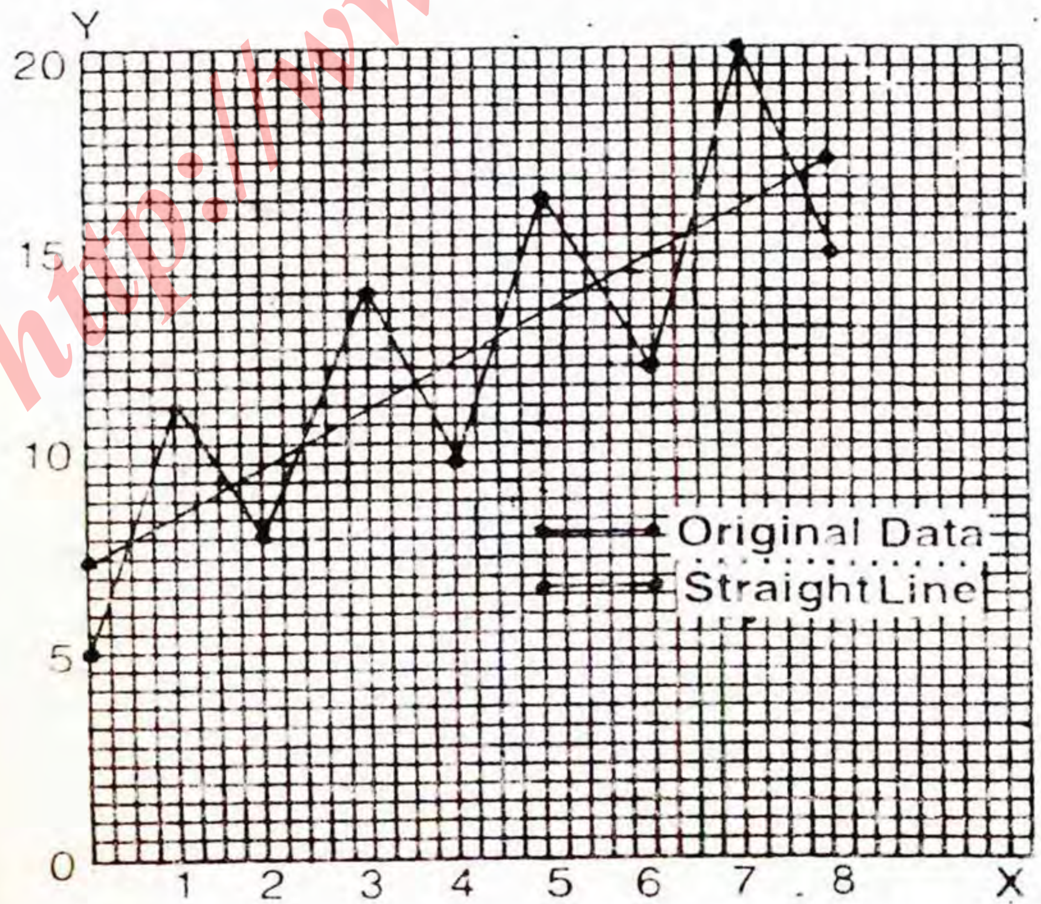
$$9a + 36b = 111, \; 36a + 204b = 521.$$

Solving them simultaneously, we obtain $a = 7.2$ and $b = 1.28$.

Hence the equation of the required straight line is $Y = 7.2 + 1.28X$.

The calculated values of $Y(=Y')$ from this equation are shown in the last column of the table.

Graph



12.3. (b) Let the equation to the least squares line be

$Y = a + bX$, where a and b are to be determined.

The two normal equations are

$$\sum Y = na + b\sum X,$$

$$\sum XY = a\sum X + b\sum X^2.$$

The computations involved are shown below:

	Year (X)	Output (Y)	XY	X^2	Y'	$Y-Y'$	$(Y-Y')^2$
	1	1	1	1	1.67	−0.67	0.4489
	2	3	6	4	2.17	0.83	0.6889
	3	2	6	9	2.67	−0.67	0.4489
	4	4	16	16	3.17	0.83	0.6889
	5	3	15	25	3.67	−0.67	0.4489
	6	5	30	36	4.17	0.83	0.6889
	7	4	28	49	4.67	−0.67	0.4489
	8	6	48	64	5.17	0.83	0.6889
	9	5	45	81	5.67	−0.67	0.4489
$\sum$	45	33	195	285	33.03	--	5.0001

Putting these values in the normal equations, we get

$$9a + 45b = 33$$

$$45a + 285b = 195$$

Solving them simultaneously, we get $a = 1.17$ and $b = 0.50$

Hence the required least squares line is $Y = 1.17 + 0.50X$

Now, we find the values of $Y(Y')$ from this line by substituting the values of X. These values appear in the above table in column 5 and other calculations are also shown there.

The sum of the squared deviations $= \sum(Y-Y')^2 = 5.0001$

12.4. (b) Let the equation of the straight line be

$Y = a + bX$, where a and b are to be determined.

The normal equations are $\sum Y = na + b\sum X$ and $\sum XY = a\sum X + b\sum X^2$.

The necessary computations are shown in the following table:

	X	Y	XY	X^2
	0	12	0	0
	5	15	75	25
	10	17	170	100
	15	22	330	225
	20	24	480	400
	25	30	750	625
$\sum$	75	120	1805	1375

The normal equations then become

$$6a + 75b = 120 \text{ and } 75a + 1375b = 1805$$

Solving them simultaneously, we get $a = 11.25$ and $b = 0.70$.

Hence the required equation of the straight line is $Y = 11.25 + 0.70X$.

12.5. (a) Let the least squares line be $Y = a + bX$, where

$$b = \frac{n\sum XY - \sum X\sum Y}{n\sum X^2 - (\sum X)^2} = \frac{(\sum XY)/n - \bar{X}\,\bar{Y}}{(\sum X^2)/n - \bar{X}^2},$$

$$= \frac{(404)/20 - (2)(8)}{(180)/20 - (2)^2} = \frac{4.2}{5} = 0.84, \text{ and}$$

$$a = \bar{Y} - b\bar{X} = 8 - (0.84)(2) = 6.32.$$

Hence the desired least squares line is

$$Y = 6.32 + 0.84X.$$

(b) The normal equations in respect of the straight line $Y = a + bX$ are

$$\sum Y = na + b\sum X,$$

$$\sum XY = a\sum X + b\sum X^2.$$

And the normal equations when the straight line is $X=c+dY$, are

$$\sum X = nc + d\sum Y,$$

$$\sum XY = c\sum Y + d\sum Y^2.$$

The computations involved are shown in the following table:

	X	Y	XY	X^2	Y^2
	1	8	8	1	64
	2	9	18	4	81
	3	13	39	9	169
	4	18	72	16	324
	5	27	135	25	729
$\sum$	15	75	272	55	1367

Straight line $Y = a + bX$
Substituting these values, we get

$$5a + 15b = 75$$
$$15a + 55b = 272$$

Solving them simultaneously, we obtain

$$a = 0.9 \text{ and } b = 4.7$$

Hence the required equation is

$$Y = 0.9 + 4.7X$$

When $X = 6$, then

$$Y = 0.9 + 4.7(6) = 29.1$$

Straight line $X = c + dY$
Substituting these values, we get

$$5c + 75d = 15$$
$$75c + 1367d = 272$$

Solving them simultaneously, we obtain

$$c = 0.09 \text{ and } d = 0.19$$

Hence the required equation is

$$X = 0.09 + 0.19Y$$

When $X = 6$, then

$$6 = 0.09 + 0.19Y$$

or $Y = 31.1$

12.6(c) As $\sum X = 0 = \sum X^3$, the normal equations reduce to

$$\sum Y = na + c\sum X^2,\ \sum XY = b\sum X^2,\ \sum X^2Y = a\sum X^2 + c\sum X^4.$$

The arithmetic involved in computing the necessary summations is shown in the following table:

X	Y	X^2	X^4	XY	X^2Y
−2	−5	4	16	10	−20
−1	−2	1	1	2	−2
0	1	0	0	0	0
1	2	1	1	2	2
2	1	4	16	2	4
0	−3	10	34	16	−16

Substituting these values, we get

$$5a + 10c = -3$$

$$10b = 16$$

$$10a + 34c = -16$$

Solving them, we get

$$a = 0.83, \quad b = 1.60 \text{ and } c = -0.71$$

Hence the equation of the fitted parabola is

$$Y = 0.83 + 1.60X - 0.71X^2$$

12.7. (b) Let the equation of the second degree parabola be

$$Y = a + bX + cX^2.$$

Then the normal equations are

$$\sum Y = na + b\sum X + c\sum X^2$$

$$\sum XY = a\sum X + b\sum X^2 + c\sum X^3$$

$$\sum X^2Y = a\sum X^2 + b\sum X^3 + c\sum X^4$$

Now $\sum X = n(\bar{X}) = (5)\,(2) = 10$, and

$$\sum Y = n(\bar{Y}) = (5)\,(15) = 75.$$

Substituting the values in the normal equations, we get

$$5a + 10b + 30c = 75 \qquad \dots A$$

$$10a + 30b + 100c = 242 \qquad \dots B$$

$$30a + 100b + 354c = 850 \quad \ldots C$$

Now 2A: $10a + 20b + 60c = 150 \quad \ldots D$

D−B: $10b + 40c = 92 \quad \ldots E$

Again 6A: $30a + 60b + 180c = 450 \quad \ldots F$

C−F: $40b + 174c = 400 \quad \ldots G$

Again 4E: $40b + 160c = 368 \quad \ldots H$

H−G: $14c = 32$ or $c = 2.286$

Putting c = 2.286 in E, we get

$10b + 40(2.286) = 92$, which gives

$$b = \frac{92 - 91.44}{10} = 0.056.$$

Now, putting $b = 0.056$ and $c = 2.286$ in A, we obtain

$$5a + 10(0.056) + 30(2.286) = 75$$

$$\therefore \quad 5a = 75 - 0.56 - 68.58$$

$$= 5.86, \text{ giving } a = 1.172$$

Hence the fitted second degree parabola is

$$Y = 1.172 + 0.056X + 2.286X^2.$$

(c) Let the equation of the parabola of the second degree fitting the data be $Y = a + bX + cX^2$, where a, b and c are to be determined. The three normal equations are

$$\sum Y = na + b\sum X + c\sum X^2$$

$$\sum XY = a\sum X + b\sum X^2 + c\sum X^3$$

$$\sum X^2Y = a\sum X^2 + b\sum X^3 + c\sum X^4$$

The necessary computations are given in the table below:

X	Y	X^2	X^3	X^4	XY	X^2Y
0	1	0	0	0	0	0
1	5	1	1	1	5	5
2	10	4	8	16	20	40
3	22	9	27	81	66	198
4	38	16	64	256	152	608
10	76	30	100	354	243	851

Substituting these values in the three normal equations, we get

$$5a + 10b + 30c = 76$$

$$10a + 30b + 100c = 243$$

$$30a + 100b + 354c = 851$$

Solving them simultaneously, we obtain

$$a = 1.428,\ b = 0.244 \text{ and } c = 2.214$$

Hence the equation of the required second degree parabola is

$$Y = 1.428 + 0.244X + 2.214X^2$$

12.8. The arithmetic may be simplified by taking $u = \dfrac{X-2.5}{0.5}$.

The equation of the second degree parabola then becomes

$$Y = a + bu + cu^2.$$

Since $\sum u = 0 = \sum u^3$, the normal equations are therefore reduced to

$$\sum Y = na + c\sum u^2,$$

$$\sum uY = b\sum u^2,$$

$$\sum u^2 Y = a\sum u^2 + c\sum u^4.$$

The necessary calculations are shown in the following table:

X	Y	$u = \dfrac{X-2.5}{0.5}$	u^2	u^4	uY	u^2Y
1.0	1.1	−3	9	81	−3.3	9.9
1.5	1.3	−2	4	16	−2.6	5.2
2.0	1.6	−1	1	1	−1.6	1.6
2.5	2.0	0	0	0	0	0
3.0	2.7	1	1	1	2.7	2.7
3.5	3.4	2	4	16	6.8	13.6
4.0	4.1	3	9	81	12.3	36.9
$\sum$	16.2	0	28	196	14.3	69.9

Substituting these values in the normal equations, we get

$$7a + 28c = 16.2$$
$$28b = 14.3$$
$$28a + 196c = 69.9$$

Solving, we get

$a = 2.071,\ b = 0.511$ and $c = 0.061$, giving

$$Y = 2.071 + 0.511u + 0.061u^2$$

Whence, by writing $\frac{X-2.5}{0.5}$ for u, we get

$$Y = 2.071 + 0.511\left(\frac{X-2.5}{0.5}\right) + 0.061\left(\frac{X-2.5}{0.5}\right)^2$$
$$= 2.071 + 0.511\,(2X-5) + 0.061\,(2X-5)^2$$
$$= 1.04 - 0.20X + 0.24X^2.$$

which is the required equation of the second degree parabola.

12.9. Taking $u = \frac{X-24}{4}$, **we find that** $\sum u = 0 = \sum u^3$.

Then the equation of the second degree parabola of Y on u is $Y = a + bu + cu^2$, and the normal equations become

$$\sum Y = na + c\sum u^2,\quad \sum uY = b\sum u^2$$
$$\sum u^2 Y = a\sum u^2 + c\sum u^4$$

The calculations involved are shown in the following table:

X	$u = \frac{X-24}{4}$	Y	u^2	u^4	uY	u^2Y
8	−4	2.4	16	256	−9.6	38.4
12	−3	4.8	9	81	−14.4	43.2
16	−2	8.3	4	16	−16.6	33.2
20	−1	9.5	1	1	−9.5	9.5
24	0	11.2	0	0	−50.1	0
28	1	24.3	1	1	24.3	24.3
32	2	22.2	4	16	44.4	88.8
36	3	21.2	9	81	63.6	190.8
40	4	25.4	16	256	101.6	406.4
$\sum$	0	129.3	60	708	233.9 183.8	834.6

Substituting these values, we get

$$9a + 60c = 129.3, \quad 60b = 183.8,$$

$$60a + 708c = 834.6$$

Solving these equations, we obtain

$a = 14.96$, $b = 3.063$ and $c = -0.089$, giving

$$Y = 14.96 + 3.063\,u - 0.089\,u^2.$$

Whence by writing $\frac{X-24}{4}$ for u, we get

$$Y = 14.96 + 3.063\left(\frac{X-24}{4}\right) - 0.089\left(\frac{X-24}{4}\right)^2$$

or $$Y = 14.96 + 3.063(0.25X - 6) - 0.089\,(0.25X-6)^2$$

$$= 14.96+0.766X-18.378-0.0056X^2+0.267X - 3.204$$

$$= -6.622 + 1.033X - 0.0056X^2,$$

which is the required equation of the second degree parabola.

12.10. Since $u = X - 3$ **(given), so we find that** $\sum u = 0 = \sum u^3$.

The normal equations are reduced to

$$\sum v = na + c\sum u^2, \text{ where } v = (Y-1650)/50,$$

$$\sum uv = b\sum u^2, \quad \sum u^2 v = a\sum u^2 + c\sum u^4$$

The necessary calculations are arranged in the following table:

X	$u = X-3$	Y	$v = \frac{Y-1650}{50}$	u^2	u^4	uv	u^2v
1	−2	1250	−8	4	16	16	−32
2	−1	1400	−5	1	1	5	−5
3	0	1650	0	0	0	0	0
4	1	1950	6	1	1	6	6
5	2	2300	13	4	16	26	52
$\sum$	---	---	6	10	34	53	21

Substituting these values in the normal equations, we get

$$5a + 10c = 6,$$

$$10b = 53,$$

$$10a + 34c = 21$$

Solving them, we find $a = -0.086$, $b = 5.30$ and $c = 0.643$.

Hence the equation of the required parabolic curve of v on u is

$$v = -0.086 + 5.30u + 0.643\, u^2.$$

Whence, by writing $X-3$ for u and $\frac{Y-1650}{50}$ for v, we get

$$\frac{Y-1650}{50} = -0.086 + 5.30(X-3) + 0.643\,(X-3)^2$$

Simplifying, we have

$$Y = 1140 + 72.14X + 32.14X^2$$

as the required parabolic curve of Y on X.

12.11. The arithmetic is simplified by taking $u = 2\left(\frac{X-37.5}{5}\right)$ as the number of pairs of values is even. As $\sum u = 0 = \sum u^3$, the normal equations become regarding the

(i) **Straight line:**

$$\sum Y = na \text{ and } \sum uY = b\sum u^2;$$

(ii) **Second degree parabola:**

$$\sum Y = na + c\sum u^2$$
$$\sum uY = b\sum u^2$$
$$\sum u^2Y = a\sum u^2 + c\sum u^4$$

The necessary computations are give in the following table:

X	Y	u	u^2	u^4	uY	u^2Y
20	240	−7	49	2401	−1680	11760
25	315	−5	25	625	−1575	7875
30	403	−3	9	81	−1209	3627
35	450	−1	1	1	−450	450
40	488	1	1	1	+488	488
45	520	3	9	81	1560	4680
50	525	5	25	625	2625	13125
55	532	7	49	2401	3724	26068
$\sum$	3473	0	168	6216	3483	68073

Substituting these values in the normal equations, we get

(i) **Straight line:**

$$8a = 3473 \text{ and } 168b = 3483 \text{ giving}$$

$$a = 434.125 \text{ and } b = 20.73$$

Thus $Y = 434.125 + 20.73\,u$

The straight line of Y on X is obtained by writing $\frac{2(X-37.5)}{5}$ for u. Then

$$Y = 434.125 + 20.73\left[\frac{2(X-37.5)}{5}\right]$$

$$= 434.125 + 20.73\,(0.4X - 15)$$

$$= 123.175 + 8.292X$$

which is the equation of the required straight line.

(ii) **Second degree parabola:**

$$8a + 168c = 3473$$

$$168b = 3483$$

$$168a + 6216c = 68073.$$

Solving them simultaneously, we find that

$a = 472.093$, $b = 20.73$ and $c = -1.808$, giving

$$Y = 472.093 + 20.73\,u - 1.808\,u^2.$$

Whence, by writing $2\,\frac{(X-37.5)}{5}$, *i.e.* $(0.4X-15)$ for u, we get

$$Y = 472.093 + 20.73\,(0.4X-15) - 1.808\,(0.4X-15)^2$$

$$= -245.657 + 29.988X - 0.289\,X^2,$$

which is the equation of the required second degree parabola of Y on X.

The computations needed to find the sums of squares of residuals are given below:

X	Y	X^2	XY	Y^2	X^2Y
20	240	400	4800	57600	96000
25	315	625	7875	99225	196875
30	403	900	12090	162409	362700
35	450	1225	15750	202500	551250
40	488	1600	19520	238144	780800
45	520	2025	23400	270400	1053000
50	525	2500	26250	275625	1312500
55	532	3025	29260	283024	1609300
$\sum$	3473	12300	138945	1588927	5962425

The sum of squres of residuals in case of straight line is

$$S = \sum(Y-a-bX)^2 = \sum Y^2 - a\sum Y - b\sum XY$$

$$= 1588927 - (123.175)(3473) - (8.292)(138945)$$

$$= 1588927 - 427786.77 - 1152131.9 = 9008.4;$$

and in case of 2nd degree parabola, is

$$S = \sum(Y-a-bX-cX^2)^2 = \sum Y^2 - a\sum Y - b\sum XY - c\sum X^2Y$$

$$= 1588927-(-245.657)(3473) - (29.988)(138945) - (-0.289)(5962425)$$

$$= 1588927 + 853166.76 - 4166682.06 + 172929.5$$

$$= 340.6.$$

12.12. Taking $u = X-2$, we find that $\sum u = 0 = \sum u^3 = \sum u^5$. Then the normal equations in case of

(i) *Straight line*, reduce to $\sum Y = na$ and $\sum uY = b\sum u^2$;

(ii) *Quadratic parabola*, reduce to

$$\sum Y = na + c\sum u^2, \quad \sum uY = b\sum u^2, \quad \sum u^2Y = a\sum u^2 + c\sum u^4;$$

(iii) *Cubic parabola,* reduce to

$$\sum Y = na + c\sum u^2, \quad \sum uY = b\sum u^2 + d\sum u^4,$$

$$\sum u^2 Y = a\sum u^2 + c\sum u^4, \; \sum u^3 Y = b\sum u^4 + d\sum u^6.$$

The necessary computations are given in the following table:

u	Y	u^2	u^4	u^6	uY	u^2Y	u^3Y
−2	1.0	4	16	64	−2.0	4.0	−8.0
−1	1.8	1	1	1	−1.8	1.8	−1.8
0	1.3	0	0	0	0	0	0
1	2.5	1	1	1	2.5	2.5	2.5
2	6.3	4	16	64	12.6	25.2	50.4
$\sum$	12.9	10	34	130	11.3	33.5	43.1

Substituting these values in the normal equations and solving them, we get

(i) **Straight line:**

$5a = 12.9$, and $10b = 11.3$

or $a = 2.58$ and $b = 1.13$.

Thus $Y = 2.58 + 1.13u$.

But the straight line of Y on X is obtained by writing $X-2$ for u. Therefore

$$\begin{aligned} Y &= 2.58 + 1.13(X-2) \\ &= 0.32 + 1.13X, \end{aligned}$$

which is the equation of the desired straight line.

(ii) **Parabola of second degree:**

$$5a + 10c = 12.9$$

$$10b = 11.3$$

$$10a + 34c = 33.5$$

Solving them simultaneously, we find that

$a = 1.48$, $b = 1.13$ and $c = 0.55$

Then $\quad Y = 1.48 + 1.13u + 0.55u^2.$

Writing $X-2$ for u, we get the quadratic parabola as

$$\begin{aligned} Y &= 1.48 + 1.13\,(X-2) + 0.55\,(X-2)^2 \\ &= 1.42 - 1.07X + 0.55X^2 \end{aligned}$$

(iii) **Cubic parabola**:

$$5a + 10c = 12.9, \qquad 10b + 34d = 11.3,$$

$$10a + 34c = 33.5, \quad 34b + 130d = 43.1.$$

Solving them simultaneously, we find that

$$a = 1.48,\ b = 0.025,\ c = 0.55 \text{ and } d = 0.325$$

$$\therefore \quad Y = 1.48 + 0.025u + 0.55u^2 + 0.325u^3$$

Replacing u by $X-2$, we get the desired parabola of third degree

$$\begin{aligned} Y &= 1.48 + 0.025\,(X-2) + 0.55\,(X-2)^2 + 0.325(X-2)^3 \\ &= 1.03 + 1.725X - 1.40X^2 + 0.325X^3. \end{aligned}$$

Next, we calculate the expected values $(\hat{Y})$ and the sums of squares of residuals as below:

X	Y	Straight line		Quadratic parabola		Cubic parabola	
		$\hat{Y} = 0.32 + 1.13X$	$(Y-\hat{Y})^2$	$\hat{Y} = 1.42 - 1.07X + 0.55X^2$	$(Y-\hat{Y})^2$	$\hat{Y} = 1.03 + 1.725X - 1.40X^2 + 0.325X^3$	$(Y-\hat{Y})^2$
0	1.0	0.32	0.4624	1.42	0.1764	1.03	0.0009
1	1.8	1.45	0.1225	0.90	0.8100	1.68	0.0144
2	1.3	2.58	1.6384	1.48	0.0324	1.48	0.0324
3	2.5	3.71	1.4641	3.16	0.4356	2.38	0.0144
4	6.3	4.84	2.1316	5.94	0.1296	6.33	0.0009
$\sum$	12.9	12.9	5.8190	12.9	1.5840	12.9	0.0630

Alternatively. The computations needed to find the sums of squares of residuals are given as follows:

X	Y	X^2	XY	Y^2	X^2Y	X^3Y
0	1.0	0	0	1.00	0	0
1	1.8	1	1.8	3.24	1.8	1.8
2	1.3	4	2.6	1.69	5.2	10.4
3	2.5	9	7.5	6.25	22.5	67.5
4	6.3	16	25.2	39.69	100.8	403.2
$\sum$	12.9	30	37.1	51.87	130.3	482.9

(i) *Straight line:*

$$S = \sum(Y-a-bX)^2 = \sum Y^2 - a\sum Y - b\sum XY$$
$$= 51.87 - (0.32)(12.9) - (1.13)(37.1)$$
$$= 51.87 - 4.128 - 41.932 = 5.82$$

(ii) *Quadratic parabola:*

$$S = \sum(Y-a-bX-cX^2)^2 = \sum Y^2 - a\sum Y - b\sum XY - c\sum X^2Y$$
$$= 51.87 - (1.42)(12.9) - (-1.07)(37.1) - (0.55)(130.3)$$
$$= 51.87 - 18.318 + 39.697 - 71.665 = 1.584.$$

(iii) *Cubic parabola:*

$$S = \sum(Y-a-bX-cX^2-dX^3)^2$$
$$= \sum Y^2 - a\sum Y - b\sum XY - c\sum X^2Y - d\sum X^3Y$$
$$= 51.87 - (1.03)(12.9) - (1.725)(37.1) - (-1.4)\times(130.3) - (0.325)(482.9)$$
$$= 51.87 - 13.287 - 63.9975 + 182.42 - 156.9425$$
$$= 234.29 - 234.227 = 0.063,$$

12.13. (b) To find a suitable curve, we construct a difference table as below:

X	Y	ΔY	$\Delta^2 Y$
0	10		
		7	
1	17		4
		11	
2	28		4
		15	
3	43		4
		19	
4	62		

As the second order differences are constant, therefore the suitable curve is the second degree parabola, *i.e.* $Y = a + bX + cX^2$. Then the three normal equations are

$$\sum Y = na + b\sum X + c\sum X^2$$

$$\sum XY = a\sum X + b\sum X^2 + c\sum X^3$$

$$\sum X^2Y = a\sum X^2 + b\sum X^3 + c\sum X^4.$$

The necessary computations are given below:

X	Y	X^2	X^3	X^4	XY	X^2Y
0	10	0	0	0	0	0
1	17	1	1	1	17	17
2	28	4	8	16	56	112
3	43	9	27	81	129	387
4	62	16	64	256	248	992
10	160	30	100	354	450	1508

Substituting these values in the normal equations, we get

$$5a + 10b + 30c = 160$$

$$10a + 30b + 100c = 450$$

$$30a + 100b + 354c = 1508$$

Solving them simultaneously, we find that

$$a = 10, \quad b = 5 \text{ and } c = 2$$

Hence the desired second degree parabolic curve is

$$Y = 10 + 5X + 2X^2.$$

12.14. (b) The given curve Y = ab^X **may be written as**

$$\log Y = \log a + X \log b$$

or $\quad Y' = A + BX'$ (Form of *a* st. line)

where $Y' = \log Y$, $A = \log a$ and $B = \log b$.

The normal equations are:

$$\sum Y' = nA + B\sum X$$

$$\sum XY' = A\sum X + B\sum X^2.$$

The calculations involved are shown in the following table.

	X	Y	X^2	$Y' = \log Y$	XY
	0	73	0	1.8633	0
	1	91	1	1.9590	1.9590
	2	112	4	2.0492	4.0984
	3	131	9	2.1173	6.3519
	4	162	16	2.2095	8.8380
$\sum$	10	–	30	10.1983	21.2473

Substituting these values in the normal equations, we get

$$5A + 10B = 10.1983$$

$$10A + 30B = 21.2473$$

Solving them, we find that $A = 1.8695$ and $B = 0.08507$

Thus $\quad a = \text{antilog of } A$

$= \text{antilog } (1.8695) = 74.04$, and

$b = \text{antilog of } B$

$= \text{antilog } (0.08507) = 1.216$

Hence the eqution of the required curve is

$$Y = 74.04\,(1.22)^X$$

Next, we estimate the values of Y when $X = 5$ and 6. The equation is $Y = 74.04(1.22)^X$. Putting $X = 5$ and 6, we get

$$Y_5 = 74.04\,(1.22)^5 = 200.11; \text{ and}$$

$$Y_6 = 74.04\,(1.22)^6 = 244.13.$$

12.15. The equaton of the curve $Y = ab^X$ may be written as

$$\log Y = \log a + X \log b$$

or $\quad Y' = A + BX$, (form of a st. line)

where $\quad Y' = \log Y,\; A = \log a$ and $B = \log b$

The two normal equations are

$$\sum Y' = nA + B\sum X$$

$$\sum XY' = A\sum X + B\sum X^2.$$

The necessary calculations are given in the table below:

	X	Y	X^2	$Y' = \log Y$	XY'
	0	32	0	1.5051	0
	1	47	1	1.6721	1.6721
	2	65	4	1.8129	3.6258
	3	92	9	1.9638	5.8914
	4	132	16	2.1206	8.4824
	5	190	25	2.2788	11.3940
	6	275	36	2.4393	14.6358
$\sum$	21	--	91	13.7926	45.7015

Substituting these values in the normal equations, we get

$$7A + 21B = 13.7926, \quad 21A + 91B = 45.7015$$

Solving them, we find that $A = 1.5072$ and $B = 0.1544$

Thus $\quad a = \text{Antilog of } A$

$$= \text{Antilog } (1.5072) = 32.15, \text{ and}$$

$$b = \text{Antilog of } B = \text{Antilog } (0.1544) = 1.427$$

Hence the least-squares curve is of the form

$$Y = 32.15\,(1.427)^X$$

Estimated value of Y when $X = 7$, is

$$Y = 32.15\,(1.427)^7 = 387.40$$

12.16. We may write the exponential equation in the logarithmic form as

$$\log Y = \log a + X \log b$$

i.e. $\quad Y' = A + BX$, (Form of a straight line)

where $Y' = \log Y$, $A = \log a$ and $B = \log b$.

The two normal equations are

$$\sum Y' = nA + B\sum X, \quad \sum XY' = A\sum X + B\sum X^2$$

The necessary calculatons are arranged in the following table:

	Day (X)	Height (Y)	X^2	$Y' = \log Y$	XY'
	0	0.75	0	−0.1249	0
	1	1.20	1	0.0792	0.0792
	2	1.75	4	0.2430	0.4860
	3	2.50	9	0.3979	1.1937
	4	3.45	16	0.5378	2.1512
	5	4.70	25	0.6721	3.3605
	6	6.20	36	0.7924	4.7544
	7	8.25	49	0.9165	6.4155
	8	11.50	64	1.0607	8.4856
$\sum$	36	--	204	4.5747	26.9261

Substituting these values, we get

$$9A + 36B = 4.5747$$

$$36A + 204B = 26.9261$$

Solving them simultaneously, we get

$$A = -0.0669 \text{ and } B = 0.1438$$

Whence, we find that

$$a = \text{Antilog of } A = \text{Antilog } (-0.0669)$$

$$= \text{Antilog } (\bar{1}.9331) = 0.8572, \text{ and}$$

$$b = \text{Antilog of } B$$

$$= \text{Antilog } (0.1438) = 1.393$$

Hence the equation of the required exponential curve is

$$Y = 0.86\,(1.39)^X.$$

12.17. The equation of the curve $Y = ab^X$ may be written as

$$\log Y = \log a + X \log b$$

or $$Y' = A + BX,$$

where $Y' = \log Y$, $A = \log a$ and $B = \log b$.

The two normal equations are

$$\sum Y' = nA + B\sum X$$

$$\sum XY' = A\sum X + B\sum X^2$$

The necessary calculatons are given in the table below:

	X	Y	X^2	$Y' = \log Y$	XY'
	1	10.0	1	1.0000	1.0000
	2	12.2	4	1.0864	2.1728
	3	14.5	9	1.1614	3.4842
	4	17.3	16	1.2380	4.9520
	5	21.0	25	1.3222	6.6110
	6	25.0	36	1.3979	8.3874
	7	29.0	49	1.4624	10.2368
$\sum$	28	---	140	8.6683	36.8442

Substituting these values in the normal equations, we get

$$7A + 28B = 8.6683$$

$$28A + 140B = 36.8442$$

Solving them, we find that $A = 0.9283$ and $B = 0.0775$

Thus a = Antilog of A

= Antilog (0.9283) = 8.478, and

b = Antilog of B = Antilog (0.0775) = 1.195.

Hence the equation of the desired curve is

$$Y = 8.478\,(1.195)^X$$

12.18. The equation of the curve $Y = ab^X$ may be written as

$$\log Y = \log a + X \log b$$

or $$Y' = A + BX$$

where $Y' = \log Y$, $A = \log a$ and $B = \log b$.

The two normal equations are

$$\sum Y' = nA + B\sum X$$

$$\sum XY' = A\sum X + B\sum X^2$$

The following table contains the necessary computations:

X (years)	Y	X^2	$Y' = \log Y$	XY'
1	304	1	2.4829	2.4829
2	341	4	2.5328	5.0656
3	393	9	2.5944	7.7832
4	457	16	2.6599	10.6396
5	548	25	2.7388	13.6940
6	670	36	2.8261	16.9566
7	882	49	2.9455	20.6185
28	--	140	18.7804	77.2404

Substituting these values in the normal equations, we get

$$7A + 28B = 18.7804$$

$$28A + 140B = 77.2404$$

Solving them, we find that $A = 2.380$ and $B = 0.076$

Thus $\quad a = \text{Antilog of } A = \text{Antilog } (2.380) = 239$

$\quad\quad b = \text{Antilog of } B = \text{Antilog of } (0.076) = 1.19$

Hence the least squares curve is of the form

$$Y = 239\,(1.19)^X$$

The expected enrolment 5 years from now $(X = 12)$ is

$$Y_{12} = 239\,(1.19)^{12} = 1954$$

12.19. (a) Considering $\sum X^3 = 582$ redundant, we may fit to the given values the curve $Y = ab^X$, which in terms of logs becomes

$$\log Y = \log a + X \log b$$

or $\quad \log Y = A + BX \quad (A = \log a,\ B = \log b)$

The two normal equations then are

$$\sum \log Y = nA + B\sum X,$$

$$\sum X \log Y = A\sum X + B\sum X^2$$

Substituting the values, we get

$$8\,A + 16\,B = 23$$

$$16\,A + 204\,B = 104$$

Solving, we obtain $A = 2.20$ and $B = 0.3372$

Whence $a = 158.5$ and $b = 2.17$

Hence the curve is $Y = 158.5\,(2.17)^X$

(b) The equation of the curve $Y = a + b\sqrt{X}$ may be written as

$$Y = a + bX', \text{ where } X' = \sqrt{X}$$

The two normal equations are

$$\sum Y = na + b\sum X'$$

$$\sum X'Y = a\sum X' + b\sum X'^2$$

The necessary calculations are given in the table below:

X	Y	$X' = \sqrt{X}$	X'^2	$X'Y$
1.20	6.33	1.0954	1.20	6.9339
2.50	8.03	1.5811	2.50	12.6962
3.40	8.95	1.8439	3.40	16.5029
4.70	10.09	2.1679	4.70	21.8741
5.30	10.56	2.3022	5.30	24.3112
$\sum$	43.96	8.9905	17.10	82.3183

Substituting these values in the normal equations, we get

$$5a + 8.9905\,b = 43.96,$$

$$8.9905\,a + 17.10\,b = 82.3183.$$

Solving them, we find that $a = 2.50$ and $b = 3.50$

Hence the least-squares curve is of the form

$$Y = 2.50 + 3.50\sqrt{X}$$

12.20. The equatoin of the cure $Y = aX^b$ may be written as

$$\log Y = \log a + b \log X$$

or $$Y' = A + bX',$$

where $Y' = \log Y$, $A = \log a$ and $X' = \log X$.

The two normal equation are

$$\sum Y' = nA + b\sum X'$$

$$\sum X'Y' = A\sum X' + b\sum X'^2$$

The necessary calculations are shown in the following table:

X	Y	$X' = \log X$	X'^2	$X' = \log X$	$X'Y'$
1	1200	0	0	3.0792	0
2	900	0.3010	0.090601	2.9542	0.889214
3	600	0.4771	0.227624	2.7782	1.325479
4	200	0.6021	0.362524	2.3010	1.385432
5	110	0.6990	0.488601	2.0414	1.426939
6	50	0.7782	0.605595	1.6990	1.322162
$\sum$	--	2.8574	1.774945	14.8530	6.349226

Substituting these values in the normal equations, we get

$$6A + 2.8574\,b = 14.8530$$

$$2.8574\,A + 1.774945\,b = 6.349226$$

Solving them, we find that $A = 3.3083$ and $b = -1.7488$

Thus $\quad a = \text{Antilog of } A$

$$= \text{Antilog of } (3.3083) = 2033.0$$

Hence the equation of the desired curve is

$$Y = 2033\,(X)^{-1.7488}$$

12.21. The equation of the curve $Y = aX^b$ may be written as

$$\log Y = \log a + b \log X,$$

i.e. $\quad Y' = A + bX', \quad$ (a straight line) where $Y' = \log Y$, $A = \log a$ and $X' = \log X$

Then the two normal equations are

$$\sum Y' = nA + b\sum X'$$

$$\sum X'Y' = A\sum X' + b\sum X'^2$$

The necessary calculations are given in the table below:

X	Y	$X' = \log X$	X'^2	$Y' = \log Y$	$X'Y'$
50	108	1.6990	2.886601	2.0334	3.454747
100	53	2.0000	4.000000	1.7243	3.448600
250	24	2.3979	5.749924	1.3802	3.309582
500	9	2.6990	7.284601	0.9542	2.575386
1000	5	3.0000	9.000000	0.6990	2.097000
$\sum$	--	11.7959	28.921126	6.7911	14.885315

Substituting these values in the normal equations, we get

$$5A + 11.7959\,b = 6.7911,$$

$$11.7959\,A + 28.921126\,b = 14.885315$$

Solving them, we find that $A = 3.81177$ and $b = -1.04$

Thus a = Antilog of A

= Antilog (3.81177) = 6483.1

Hence the equation of the required curve is

$$Y = 6483.1\,(X)^{-1.04}$$

To estimate the value of Y when $X = 400$, we use the equation $Y' = A + bX'$

Therefore $Y' = 3.81177 + (-1.04)(2.6021)$

$= 1.1056$

Hence taking Antilog, we get $Y = 12.76$

12.22. We may write the given relation in logs as

$$\log Y = \log a + n \log X$$

or $\quad Y' = A + nX'$; (a straight line) where $Y' = \log Y$, $A = \log a$ and $X' = \log X$

The two normal equations are

$$\sum Y' = kA + n\sum X' \quad (k = \text{no. of values})$$

$$\sum X'Y' = A\sum X' + n\sum X'^2$$

The computations involved are shown in the following table:

X	Y	$X' = \log X$	X'^2	$Y' = \log Y$	$X'Y'$
0.5	3.4	−0.3010	0.0906	0.5315	−0.15998
1.5	7.0	0.1761	0.0310	0.8451	+0.14882
2.5	12.8	0.3979	0.1582	1.1072	0.44055
5.0	29.8	0.6990	0.4886	1.4742	1.03047
10.0	68.2	1.0000	1.0000	1.8338	1.83380
Σ	--	1.9720	1.7684	5.7918	3.29366

Substituting these values in the normal equations, we get

$$5A + 1.9720\,n = 5.7918,$$

$$1.9720\,A + 1.7684\,n = 3.29366.$$

Solving them simultaneously, we find that

$$A = 0.7561 \text{ and } n = 1.02$$

Now $\quad a = \text{Antilog of } A$

$\quad = \text{Antilog}(0.7561) = 5.703$

Hence $a = 5.703$ and $n = 1.02$ are the required values.

12.23. The equation of the curve $v = ae^{bt}$ may be written as

$$\log v = \log a + (b \log e)\,t$$

or $\quad y = A + Bt,$

where $y = \log v$, $A = \log a$ and $B = b \log e$

Then the two normal equations are

$$\sum y = nA + B\sum t$$

$$\sum yt = A\sum t + B\sum t^2$$

The necessary calculations are given in the table below:

t	v	$y = \log v$	yt	t^2
0.5	9.1	0.9590	0.47950	0.25
0.8	8.5	0.9294	0.74352	0.64
1.4	7.5	0.8751	1.22514	1.96
2.0	6.7	0.8261	1.65220	4.00
2.5	6.1	0.7853	1.96325	6.25
7.2	--	4.3749	6.06361	13.10

Substituting these values in the normal equations, we get

$$5A + 7.2B = 4.3749$$

$$7.2A + 13.10B = 6.06361.$$

Solving them, we find that $A = 0.9995$ and $B = -0.0865$

Thus $\quad a = \text{Antilog of } A$

$$= \text{Antilog } (0.9995) = 9.998$$

$$b = B/\log e \text{ where } \log e = 0.4343$$

$$= -0.0865/0.4343 = -0.1992 = -0.20$$

Hence the equation of the curve to be fitted to the data is

$$v = 9.998 \; e^{-0.2t}$$

$$= 10 \, e^{-0.2t}$$

12.24.(a) The equation of the curve $Y = ae^{bX}$ may be written as

$$\log Y = \log a + (b \log e) X$$

or $\quad Y' = A + BX,$

where $\; Y' = \log Y, \; A = \log a$ and $B = b \log e$

Then the two normal equations are

$$\sum Y' = nA + B\sum X$$

$$\sum XY' = A\sum X + B\sum X^2$$

The computations needed to fit the curve are given in the table below:

X	Y	$Y' = \log Y$	XY'	X^2
1	27	1.4314	1.4314	1
2	73	1.8633	3.7266	4
3	200	2.3010	6.9030	9
4	545	2.7364	10.9456	16
5	1484	3.1715	15.8575	25
15	--	11.5036	38.8641	55

Substituting these values in the normal equations, we get

$$5A + 15B = 11.5036$$

$$15A + 55B = 38.8641$$

Solving them, we find that $A = 0.99473$ and $B = 0.43533$

Therefore $Y = 0.99473 + 0.43533X$

Now $\quad a = \text{Antilog of } A$

$$= \text{Antilog } (0.99473) = 9.88, \text{ and}$$

$$0.4343b = 0.43533 \quad (\because \log_{10}e = 0.4343)$$

or $\quad b = \dfrac{0.43533}{0.4343} = 1.002$

Hence the equation of the curve fitted to the data is

$$Y = 9.88\ e^{1.002X}$$

(b) The values of Y from the approximating line for various values of X are obtained below:

X	$Y' = 0.99473 + 0.43533X$	$\hat{Y} = \text{Antilog } Y'$	$Y - \hat{Y}$
1	1.43006	26.92	0.08
2	1.86539	73.34	−0.34
3	2.30072	199.81	0.19
4	2.73605	544.56	0.44
5	3.17138	1484.30	−0.30
Σ	--	--	0.07

The deviations of the estimated values of Y from the corresponding observed values add to 0.07. This slight difference is due to rounding off.

12.25. The Pareto curve $n = AX^{-\alpha}$ may be written as

$$\log n = \log A - \alpha \log X$$

or $\quad Y = a + bX'$, (a straight line)

where $Y = \log n$, $a = \log A$, $b = -\alpha$ and $X' = \log X$

The two normal equations are

$$\sum Y = Na + b\sum X'$$

$$\sum X'Y = a\sum X' + b\sum X'^2$$

The computations involved are shown in the following table:

Income (X)	$X' = \log X$	X'^2	No. (n)	$Y = \log n$	$X'Y$
150	2.1761	4.7354	14,000,000	7.1461	15.55063
500	2.6990	7.2846	825,000	5.9165	15.96763
1,000	3.0000	9.0000	173,000	5.2380	15.71400
2,000	3.3010	10.8966	35,500	4.5502	15.02021
$\sum$	11.1761	31.9166	---	22.8508	62.25247

Substituting these values, we get

$$4a + 11.1761\,b = 22.8508$$

$$11.1761\,a + 31.9166\,b = 62.25247$$

Multiplying the first equation by 11.1761 and the second by 4, we get

$$44.7044\,a + 124.9052\,b = 255.38283$$

$$44.7044\,a + 127.6664\,b = 249.00988$$

Subtraction gives

$$-2.7612\,b = 6.37295$$

$$\therefore \quad b = -2.31$$

Whence, we find that $\alpha = -b = 2.31$.

Putting $b = -2.31$ in the first normal equation and simplifying, we get $a = 12.1669$.

Thus A = Antilog of a

= Antilog (12.1669) = 1,469,000,000,000.

Hence the result.

12.26. Taking logs of both sides of the equation $pv^{\gamma}=c$, we obtain

$$\log p + \gamma \log v = \log c$$

or $\log v = \frac{1}{\gamma} \log c - \frac{1}{\gamma} \log p$, as p is to be taken as independent variable.

Calling $\log v = y$, and $\log p = x$, we may write it as

$$y = a + bx, \text{ where } a = \frac{1}{\gamma}\log c \text{ and } b = -\frac{1}{\gamma}$$

Then the two normal equations are

$$\sum y = na + b\sum x$$

$$\sum xy = a\sum x + b\sum x^2, \text{ giving}$$

$$b = \frac{n\sum xy - (\sum x)(\sum y)}{n\sum x^2 - (\sum x)^2}; \text{ and}$$

$$a = \frac{(\sum x^2)(\sum y) - (\sum x)(\sum xy)}{n\sum x^2 - (\sum x)^2}.$$

The computations involved are shown in the following table:

p	v	$x(=\log p)$	$y(=\log v)$	xy	x^2
0.5	1.62	−0.3010	0.2095	−0.06306	0.09060
1.0	1.00	0.0000	0.0000	0	0
1.5	0.75	0.1761	−0.1249	−0.02079	0.03101
2.0	0.62	0.3010	−0.2076	−0.06249	0.09060
2.5	0.52	0.3979	−0.2840	−0.11400	0.15832
3.0	0.46	0.4771	−0.3372	−0.16088	0.22762
$\sum$	--	1.0511	−0.7442	−0.42122	0.59815

Substitution gives

$$b = \frac{(6)(-0.42122) - (-0.7442)(1.0511)}{(6)(0.59815) - (1.0511)^2}$$

$$= \frac{-2.52732 + 0.78223}{3.58890 - 1.1048112} = \frac{-1.74509}{2.4840888} = -0.702509$$

Thus $\qquad -\frac{1}{\gamma} = -0.702509$

or $\qquad \gamma = \frac{1}{0.702509} = 1.42.$

12.27. (b) The equation of the curve $\frac{1}{Y} = a + bX$ may be written as

$$Y' = a + bX, \text{ where } Y' = \frac{1}{Y}.$$

The two normal equations are

$$\sum Y' = na + b\sum X$$

$$\sum XY' = a\sum X + b\sum X^2$$

The necessary calculations are given in the table below:

	X	Y	X^2	$Y' = 1/Y$	XY'
	0	10	0	0.100	0
	1	8	1	0.125	0.125
	4	5	16	0.200	0.800
	6	4	36	0.250	1.500
	12	2.5	144	0.400	4.800
	16	2	256	0.500	8.000
Σ	39	---	453	1.575	15.225

Substituting these values in the normal equations, we get

$$6a + 39b = 1.575$$

$$39a + 453b = 15.225$$

Solving them, we find that $a = 0.10$ and $b = 0.025$

Hence the required equation of the reciprocal curve is

$$\frac{1}{Y} = 0.10 + 0.025\,X$$

12.28. (a) The given linear equation is

$$Y = a + bX_1 + cX_2,$$

where a, b and c are to be determined.

The method of least-squres calls for the selection of those values a, b and c which make the sum of squares of deviations, S, a minimum. In other words, we have to minimize $S = \sum(Y-a-bX_1-cX_2)^2$.

For a minimum value, $\frac{\partial S}{\partial a}$, $\frac{\partial S}{\partial b}$ and $\frac{\partial S}{\partial c}$ must be zero.

Thus $$\frac{\partial S}{\partial a} = 2\sum (Y-a-bX_1-cX_2)(-1) = 0,$$

$$\frac{\partial S}{\partial b} = 2\sum (Y-a-bX_1-cX_2)(-X_1) = 0, \text{ and}$$

$$\frac{\partial S}{\partial c} = 2\sum (Y-a-bX_1-cX_2)(-X_2) = 0,$$

which on simplification become

$$\sum Y = na + b\sum X_1 + c\sum X_2$$

$$\sum X_1 Y = a\sum X_1 + b\sum X_1^2 + c\sum X_1 X_2$$

$$\sum X_2 Y = a\sum X_2 + b\sum X_1 X_2 + c\sum X_2^2$$

These are the required normal equations.

(b) The normal equations of $Y = a + bX_1 + cX_2$ are

$$\sum Y = na + b\sum X_1 + c\sum X_2$$

$$\sum X_1 Y = a\sum X_1 + b\sum X_1^2 + c\sum X_1 X_2$$

$$\sum X_2 Y = a\sum X_2 + b\sum X_1 X_2 + c\sum X_2^2$$

The necessary calculations are given in the table below:

	Y	X_1	X_2	X_1Y	X_2Y	X_1^2	X_2^2	X_1X_2
	2	8	0	16	0	64	0	0
	5	8	1	40	5	64	1	8
	7	6	1	42	7	36	1	6
	8	5	3	40	24	25	9	15
	5	3	4	15	20	9	16	12
Σ	27	30	9	153	56	198	27	41

Substituting these values in the normal equations, we get

$$5a + 30b + 9c = 27$$

$$30a + 198b + 41c = 153$$

$$9a + 41b + 27c = 56$$

Solving them, we find that $a = 4.49$, $b = -0.04$ and $c = 0.64$.

Hence the required equation is $Y = 4.49 - 0.04X_1 + 0.64X_2$.

12.32. (a) According to the principle of least squares, the sum to be minimized is given by

$$S = (X+7Y-17)^2 + (2X-Y-0)^2 + (3X-2Y+1)^2$$

The two normal equations are obtained as

$$\frac{\partial S}{\partial X} = 2(X+7Y-17)(1)+2(2X-Y-0)(2)+2(3X-2Y+1)(3)=0,$$

$$\frac{\partial S}{\partial Y} = 2(X+7Y-17)(7)+2(2X-Y-0)(-1)+2(3X-2Y+1)(-2)=0.$$

Simplifying, we get

$$14X - Y = 14, \text{ and } -X + 54Y = 121$$

Multiplying the first equation by 1 and second by 14, we have

$$14X - Y = 14, \text{ and } -14X + 756Y = 1694$$

Adding, we get

$$755Y = 1708 \text{ or } Y = 2.262$$

Putting $Y = 2.262$ in the first normal equation, we have

$$14X - 2.262 = 14 \quad \text{or} \quad 14X = 14 + 2.262$$

$$\therefore \quad X = \frac{16.262}{14} = 1.162$$

Hence the required solution is $X = 1.162$ and $Y = 2.262$

(b) The normal equation for X is obtianed by multiplying each equatioin by the co-efficient of X in it and adding them together as below:

$$4X + 2Y = 9.6$$

$$9X - 6Y = -6.3$$

$$+X - 3Y = -6.3$$

$$\underline{9X + 6Y = 24.0}$$

$$23X - Y = 21.0 \qquad \dots (A)$$

Similarly, the normal equation for Y is obtained as below:

$$2X + Y = 4.8$$

$$-6X + 4Y = 4.2$$

$$-3X + 9Y = 18.9$$

$$\underline{6X + 4Y = 16.0}$$

$$-X + 18Y = 43.9 \qquad \dots (B)$$

Now $18(A)$: $\quad 414X - 18Y = 378 \qquad \dots (C)$

$\therefore (B) + (C)$: $\quad 413X = 421.9$ or $X = 1.02$

Putting $X = 1.02$ in (B), we get

$$-1.02 + 18Y = 43.9 \quad \text{or} \quad Y = \frac{43.9 + 1.02}{18} = 2.50$$

Hence the required solution is $X = 1.02$ and $Y = 2.50$.

Substituting $X = 1.02$ and $Y = 2.50$ in the given set of equations, we get the residuals as

$$e_1 = -0.26, \; e_2 = 0.16, \; e_3 = 0.18, \; e_4 = 0.06$$

$\therefore$ Sum of squres of residuals is

$$\Sigma e_i^2 = (-0.26)^2 + (0.16)^2 + (0.18)^2 + (0.06)^2 = 0.1292$$

12.33. (a) Normal equation for X is

$$X + Y = 3.01$$
$$4X - 2Y = 0.06$$
$$X + 3Y = 7.02$$
$$\underline{9X + 3Y = 14.91}$$
$$15X + 5Y = 25.00 \qquad \ldots (A)$$

Normal equation for Y is

$$X + Y = 3.01$$
$$-2X + Y = -0.03$$
$$3X + 9Y = 21.06$$
$$\underline{3X + Y = 4.97}$$
$$5X + 12Y = 29.01 \qquad \ldots (B)$$

Now 3(B): $\quad 15X + 36Y = 87.03 \qquad \ldots (C)$

(C)–(A): $\quad 31Y = 62.03$

$\therefore \quad Y = 2.001$

Putting this value of Y in (A) and simplifying, we get

$$X = 0.9997$$

(b) Normal equation for X is

$$4X + 2Y = 8$$
$$X + 2Y = 5.02$$
$$9X - 3Y = 30.06$$
$$\underline{9X + 6Y = 2.91}$$
$$23X + 7Y = 45.99 \qquad \ldots (A)$$

Normal equation for Y is

$$2X + Y = 4$$
$$2X + 4Y = 10.04$$
$$-3X + Y = -10.02$$
$$\underline{6X + 4Y = 1.94}$$
$$7X + 10Y = 5.96 \qquad \ldots (B)$$

Now 10(A): $\quad 230X + 70Y = 459.9 \qquad \ldots (C)$

7(B): $\quad \underline{49X + 70Y = 41.72} \qquad \ldots (D)$

(C)–(D): $\quad 181X = 418.18$

$$\therefore \qquad X = \frac{418.18}{181} = 2.31$$

Substituting $X = 2.31$ in (B), we get

$$7(2.31) + 10Y = 5.96$$

or $$10Y = 5.96 - 16.17$$

or $$Y = \frac{-10.21}{10} = -1.02$$

Hence the best possible values are

$$X = 2.31 \text{ and } Y = -1.02$$

12.34. Normal equation for X is obtained below:

$$\begin{aligned} X + 2Y + Z &= 1 \\ X - Y - 2Z &= -3 \\ 4X + 2Y + 2Z &= 8 \\ \underline{16X + 8Y - 20Z} &\underline{= -28} \\ 22X + 11Y - 19Z &= -22 \qquad \dots (A) \end{aligned}$$

Normal equation for Y is obtained below:

$$\begin{aligned} 2X + 4Y + 2Z &= 2 \\ -X + Y + 2Z &= 3 \\ 2X + Y + Z &= 4 \\ \underline{8X + 4Y - 10Z} &\underline{= -14} \\ 11X + 10Y - 5Z &= -5 \qquad \dots (B) \end{aligned}$$

Normal equation for Z is obtained as below:

$$\begin{aligned} X + 2Y + Z &= 1 \\ -2X + 2Y + 4Z &= 6 \\ 2X + Y + Z &= 4 \\ \underline{-20X - 10Y + 25Z} &\underline{= 35} \\ -19X - 5Y + 31Z &= 46 \qquad \dots (C) \end{aligned}$$

Now $2(B)$: $$22X + 20Y - 10Z = -10 \qquad \dots (D)$$

$$(D)-(A): \quad 9Y + 9Z = 12$$

$$\text{or} \quad 3Y + 3Z = 4 \quad \ldots (E)$$

$$\text{Again } 19(B): \quad 209X + 190Y - 95Z = -95 \quad \ldots (F)$$

$$11(C): \quad \underline{-209X - 55Y + 341Z = 506} \quad \ldots (G)$$

$$(F)+(G): \quad 135Y + 246Z = 411 \quad \ldots (H)$$

Solving equations (E) and (H) simultaneously, we get

$$Y = -0.75 \text{ and } Z = 2.08$$

Putting these two values in (A), we get

$$22X + 11(-0.75) - 19(2.08) = -22$$

$$\text{or} \quad 22X = -22 + 8.25 + 39.52$$

$$\text{or} \quad X = \frac{25.77}{22} = 1.17$$

Hence the required solution is

$$X = 1.17, \; Y = -0.75, \text{ and } Z = 2.08$$

12.35. Normal equation for X is:

$$\begin{aligned} X - Y + 2Z &= 3 \\ 9X + 6Y - 15Z &= 15 \\ 16X + 4Y + 16Z &= 84 \\ \underline{X - 3Y - 3Z} &\underline{= -14} \\ 27X + 6Y &= 88 \quad \ldots (A) \end{aligned}$$

Normal equation for Y is:

$$\begin{aligned} -X + Y - 2Z &= -3 \\ 6X + 4Y - 10Z &= 10 \\ 4X + Y + 4Z &= 21 \\ \underline{-3X + 9Y + 9Z} &\underline{= 22} \\ 6X + 15Y + Z &= 70 \quad \ldots (B) \end{aligned}$$

Normal equation for Z is:

$$
\begin{aligned}
2X - 2Y + 4Z &= 6 \\
-15X - 10Y + 25Z &= -25 \\
16X + 4Y + 16Z &= 84 \\
\underline{-3X + 9Y + 9Z} &\underline{= 42} \\
Y + 54Z &= 107 \qquad \ldots (C)
\end{aligned}
$$

Solving these equations by determinants, we get

$$
\frac{X}{\begin{vmatrix} 88 & 6 & 0 \\ 70 & 15 & 1 \\ 107 & 1 & 54 \end{vmatrix}} = \frac{Y}{\begin{vmatrix} 27 & 88 & 0 \\ 6 & 70 & 1 \\ 0 & 107 & 54 \end{vmatrix}} = \frac{Z}{\begin{vmatrix} 27 & 6 & 88 \\ 6 & 15 & 70 \\ 0 & 1 & 107 \end{vmatrix}} = \frac{1}{\begin{vmatrix} 27 & 6 & 0 \\ 6 & 15 & 1 \\ 0 & 1 & 54 \end{vmatrix}}
$$

or $$\frac{X}{49154} = \frac{Y}{70659} = \frac{Z}{38121} = \frac{1}{19899}$$

$$
\therefore \quad X = \frac{49154}{19899} = 2.47;
$$

$$
Y = \frac{70659}{19899} = 3.55; \text{ and}
$$

$$
Z = \frac{38121}{19899} = 1.92.
$$

Hence the most plausible values are

$$X = 2.47, \; Y = 3.55 \quad \text{and} \quad Z = 1.92.$$

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