Discrete Mathematics

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Discrete Mathematics

COUNTING

One of the first things you learn in Mathematics is how to count. Now we want to count large collections of things quickly and precisely. For example:

- In a group of 10 people, if everyone shakes hands with everyone else exactly once, how many handshakes took place?
- How many ways can you distribute 10 cookies to 7 bays?
- Suppose that a password on a Computer consists of six characters. Each of these characters must be a digit or an alphabet. Each password must contain at least one digit. How many such your passwords are there?

Before tackling questions like this, let's look at the basics of Counting.

SUM RULE

"The sum rule states that if an event A can occur in m ways and event B can occur in n disjoint ways, then the event "A or B" can occur in m+ n ways."

Note: it is important that event be disjoint i.e. there is no way for A and B to happen both at the same time.

Example: How many two letter "words" start with either A or B?

Solution: Firstly, how many two letter word start with A $A\ _\ (26\ character\ are\ available\ )$

How many two letter word start with B $B\ _\ (26\ letters\ are\ available\ )$

There are 26 + 26 = 52 two letter words which start either with A or B.

Example: Suppose that either a member of mathematics faculty or a student from mathematics department is choosing as a representative to a university committee. How many different choices are there for this community if there are 37 faculty members and 83 students and no one is both a faculty member and a student.

Solution: There are 37 ways to choose a member of mathematics and 83 ways to choose as student. Since both events are disjoint, therefore there are 37 + 83 = 120 ways to the representative

Note: The sum rule also works with more than two events.

Example: How many two letter words start with one of the vowels?

Solution: There are 26 two letters world start with A, another 26 start with E, and so on so.

There are 26 + 26 + 26 + 26 + 26 = 26 x 5=130 two letter words that start with one of the vowels.

PRODUCT RULE

"Suppose that a procedure can be broken down into a sequence of two tasks. If there are m ways to do the first task and for each of these ways of doing The first task, there are n ways to do the Second task , then there are $m \times n$ ways to do the procedure."

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Example: A new company with just two employees, Sanchez and Patel, rents a floor of a building with 12 offices. How many ways are there to assign different offices to these two employees?

Solution: There are 12 choices to assign office to Sanchez. To office to Patel different from Sanchez, the assigning options are 11. So by the product rule, there are 12 · 11 = 132 ways to assign offices to these two employees.

Example: How many to license plated can you make if first 3 places are letters followed by 3 numeral digits.

Solution: There are six events: the first letter, the 2nd letter, the 3rd letter, the first digit, the 2nd digit, the 3rd digit. The first 3 event can each happen in 26 ways and the last 3 events can each happen in 10 ways. So the total number of license plates will be 26 x 26 x 26 x 10 x 10x 10 = 17576000 (by product rule).

Example: The chairs of an auditorium are to be labeled with an uppercase English letter followed by a positive integer not exceeding 100. What is the largest number of chairs that can be labeled differently?

Solutions: By product rule, there are 26 x 100 = 2600 different ways in which a chair can be labelled.

Example: There are 32 computers in a data center in the cloud. Each of these computers has 24 ports. How many different computer ports are there in this data center?

Solutions: The procedure of choosing a port consists of two tasks, first picking a computer and then picking a port on this computer. Because there are 32 ways to choose the computer and 24 ways to choose the port no matter which computer has been selected, the product rule shows that there are 32 · 24 = 768 ports. (by product rule)

Note: The multiplicative principle (product Rule) can be generalized to more than two events as discussed is license plates example)

Example: How many different Bit strings of length seven are there?

Solutions: Since each bit is either 0 or 1, therefore each of the Seven bits can be chosen in 2 ways. So there are $2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 2^7 = 128$ different bit strings of length seven.

We will look at the sum and product rules in a slightly different way. For this purpose, instead of thinking about event A and event B, we want to think of a set A and a set B. The sets will contain all the different ways the event can happen. We can rephrase the sum rule in terms of sets as follows:

SUM RULE (With Sets)

Given two sets A and B, if $A \cap B = \emptyset$ (i.e. if there is no element in common to both A and B), then $|A \cup B| = |A| + |B|$

Definition: *Given sets A and B, we can form the set* $A \times B = \{(x, y) | x \in A \wedge y \in B\}$ *to be the set of all ordered pairs (x, y), where* $x \in A,\ y \in B$ *. We call* $A \times B$ *the* <u>*Cartesian product*</u> *of A and B.*

Question: What is $|A \times B|$?

Solutions: $Set\ A = \{a_1, a_2, a_3 \ldots, a_m\}\ and\ B == \{b_1, b_2, b_3 \ldots, b_n\}. Then\ \ |A| = m\ and\ |B| = n. Now$

$$A \times B = \{(a_1, b_1), (a_1, b_2), (a_1, b_3), \ldots \ldots \ldots \ldots\ (a_1, b_n),$$

$$(a_2, b_1), (a_2, b_2), (a_2, b_3), \ldots \ldots \ldots \ldots\ (a_2, b_n),$$

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$$(a_3, b_1), (a_3, b_2), (a_3, b_3), \dots \dots \dots \dots \ (a_3, b_n),$$

$$\dots\dots\dots\dots\dots \qquad \dots\dots\dots$$

$$\dots\dots\dots\dots\dots \qquad \dots\dots\dots$$

$$(a_m, b_1), (a_m, b_2), (a_m, b_3), \dots . \dots \ (a_m, b_n)\}$$

Note that we have made m rows of n pairs. Each row above is $\{a_i\} \times B for\ some\ a_i \in A$. Then

$$A \times B = (\{a_1\} \times B) \cup (\{a_2\} \times B) \cup (\{a_3\} \times B) \cup \dots \dots \dots \cup (\{a_m\} \times B)$$

$$|A \times B| = |\{a_1\} \times B| + |\{a_2\} \times B| + |\{a_3\} \times B| + \cdots \dots \dots + |\{a_m\} \times B|$$

$$|A \times B| = n + n + n + n + \cdots + n\ (\ m\ times\ ) = m \times n \qquad \text{So we have}$$

PRODUCT RULE (with Sets)

Girven two set A & B, we have $|A \times B| = |A|.|B|$

Note: The above principle can be generalized to any number of sets.

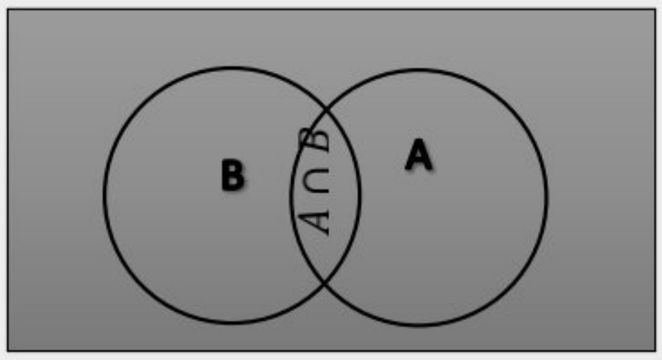
Question: What happens to the sum rule (with sets) if the sets are not disjoint?

<u>Solution</u>: Suppose that two sets A and B are not disjoint i.e. $A \cap B \neq \emptyset$. The sum rule says that

$$|A \cup B| = |A| + |B| \qquad \dots \dots \dots \dots \dots \dots . (1)$$

But when we take union, we do not repeat demands that are in both. In equation (1), we have counted every element in $A \cap B$ twice ; once in IAI and once in $|B|$. We correct the formula by subtracting $|A \cap B|$. In other words, we have:

Inclusion - Exclusion Principle



For any finite sets A and B, we have

$$|A \cup B| = |A| + |B| - \ |A \cap B|$$

Note: We can do something similar with three sets. (What?)

Example: In a group of 120 students, 84 can program in C and 66 can program in Java. If 45 can program in both Java & C, how many of the students can program in either of these languages?

<u>Solution</u>:$Let\ T = \{total\ students\}, C = \{C\ programing\ students\}, J = \{Java\ programing\ students\}$

$$we\ have\ to\ find,\ \ |C \cup J| = |C| + |J| - |C \cap J| = \ 84 + \ 66 - \ 45 = \ 105$$

Example: A computer company receives 350 applications from college graduates for a job planning a line of new web servers. Suppose that 220 of these applicants majored in computer science, 147 majored in business, and 51 majored both in computer science and in business. How many of these applicants majored neither in computer science nor in business?

<u>Solution:</u> $Let\ T = \ \{Total\ applicants\}, C = \{Computer\ Science\ majors], B = \ \{\ Business\ majors\}.$

$we\ have\ to\ find\ |C \cup B|$

$$|C \cup B| = |C| + |B| - |C \cap B| = 220 + 147 - 51 = 316$$

$$\Longrightarrow\ |\overline{C \cup B}| = |T| - |C \cup B| = 350 - 316 = 34$$

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Cardinality of a Union of 3 Sets

For any finite sets A, B and C, we have

$$|A \cup BUC| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$$

<u>Solution:</u> Let three finite sets A, B and C and by using inclusion-exclusion principle as

$$|(A \cup B)UC| = |A \cup B| + |C| - |(A \cup B) \cap C|$$

$$|(A \cup B)UC| = |A \cup B| + |C| - |(A \cap C) \cup (B \cap C)|$$

$$|(A \cup B)UC| = |A| + |B| - |A \cap B| + |C| - |(A \cap C) \cup (B \cap C)|$$

$$|(A \cup B)UC| = |A| + |B| - |A \cap B| + |C| - (|B \cap C| + |A \cap C| - |(A \cap C) \cap (B \cap C)|)$$

$$|(A \cup B)UC| = |A| + |B| - |A \cap B| + |C| - (|B \cap C| + |A \cap C| - |A \cap B \cap C|)$$

$$|(A \cup B)UC| = |A| + |B| - |A \cap B| + |C| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$$

$$|A \cup BUC| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$$

The Division Rule

"There are $\frac{n}{d}$ ways to do a task if it can be done using a procedure that can be carried out in n ways, and for every way w, exactly d of the n ways correspond to way w."

Explanation:

Suppose that we want to arrange $\{1, 1, 2, 3, 4\}$. We first assume that the two $1's$ are distinct, so we have 5! ways to arrange $\{1_a, 1_b, 2, 3, 4\}$. But for each of the 5! ways , we have over- counted some cases

$$(1, 1, 2, 3, 4)\left\{ \begin{array}{l} \{1_a, 1_b, 2, 3, 4\} \\ \{1_b, 1_a, 2, 3, 4\} \end{array} \right\}$$

$$(2, 1, 1, 3, 4)\left\{ \begin{array}{l} \{2, 1_a, 1_b, 3, 4\} \\ \{2, 1_b, 1_a, 3, 4\} \end{array} \right\}$$

....

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So to arrange $\{1, 1, 2, 3, 4\}$., we have $\frac{5!}{2!} = 60$

Image that you have a class of 24 students, which you want to arrange in a group of 4 for a project. The number of ways, you can arrange 4 students in $24.23.22.21$. This is n. The number of ways you can arrange those 4 students with in the group in 4! - This is d. So $\frac{n}{d} = \frac{24.23.22.21}{4!} = 10626$ gives the number of distinct.

In this case, each "way" is a different combination of students, and exactly d (4!) of the ways to project student corresponds to each unique combination of students.

Definition: $If\ f: A \longrightarrow B\ is\ a\ function, than\ f^{-1}(b) = \{a \in A\ |f(a) = b\}$

Definition: $A\ function\ f: A \longrightarrow B\ is\ d - to - 1,\ if\ |f^{-1}(b)| = d$

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Division Rule (Restatement)

Suppose that A and B are finite sets and $f: A \longrightarrow B \; is \; d - to - 1$, function. Then $|B| = \frac{|A|}{d} \; or \; |A| = d|B|$

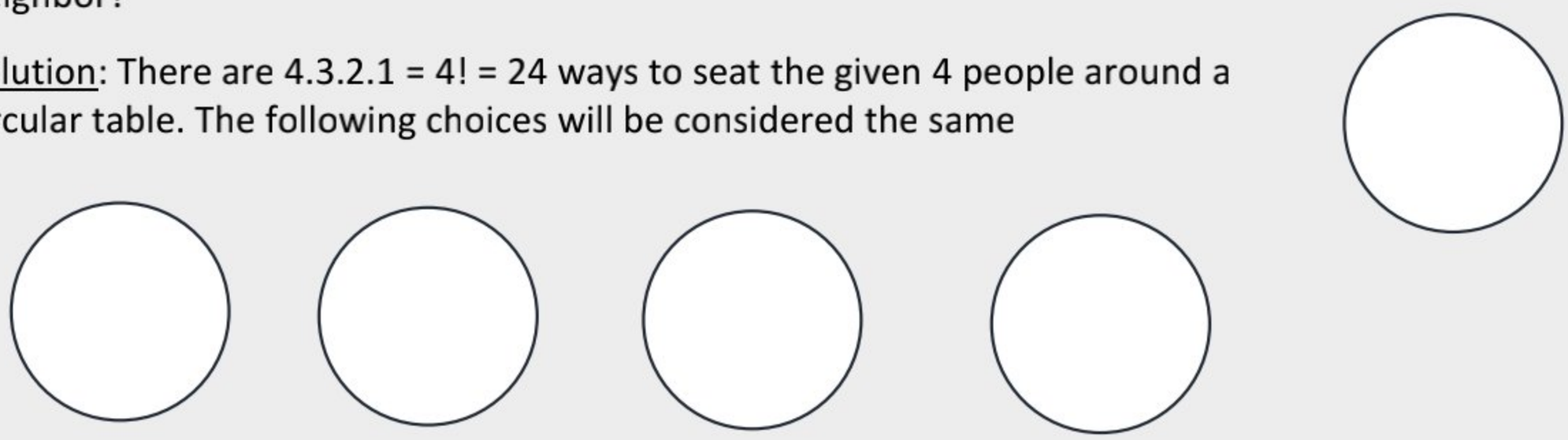
Example: Suppose that an automated system has been developed that counts the legs of cows in a pasture. Suppose that this system has determined that in a farmer's pasture there are exactly 572 legs. How many cows are there is this pasture, assuming that each cow has four legs and that there are no other animals present?

Solution: $n = 572 \; (No. of \; Legs), \; d = 4 \; (each \; cow \; has \; four \; legs)$

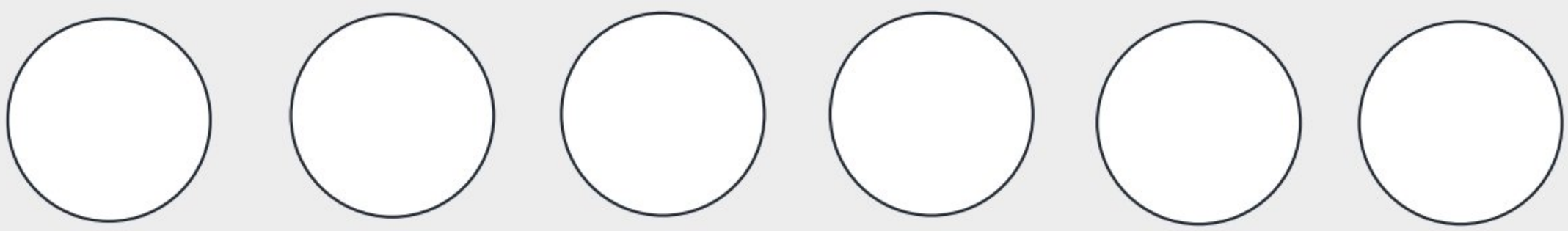
$There \; are \; n/d = 572/4 = 143 \; cows \; in \; the \; pasture \; (by \; division \; rule)$

Example: How many different ways are there to seat four people around a circular table, where two seatings are considered the same when each person has the same left neighbor and the same right neighbor?

Solution: There are 4.3.2.1 = 4! = 24 ways to seat the given 4 people around a circular table. The following choices will be considered the same



So by division rule, there are 24/4 = 6 ways to seat the 4 people round a circular table



Permutations

Suppose that we have n objects. In how many different orders can they be presented? The technical language for this question is " How many different permutations are there of n objects? " So a permutation is simply a reordering of objects. Permutations are used to find the number of ways to arrange a specified number of distinct elements of a set of particular size, where the order of these elements matters.

Example: In how many ways can we select three students from a group of five students to stand in line for a picture? In how many ways can we arrange all five of these students in a line for a picture?

Solution: First, note that the order in which we select the student's matters. There are five ways to select the first student to stand at the start of the line. Once this student has been selected, there are four ways to select the second student in the line. After the first and second students have been selected, there are three ways to select the third student in the line. By the product rule, there are $5 \cdot 4 \cdot 3 = 60$ ways to select three students from a group of five students to stand in line for a picture.

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To arrange all five students in a line for a picture, we select the first student in five ways, the second in four ways, the third in three ways, the fourth in two ways, and the fifth in one way. Consequently, there are $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$ ways to arrange all five students in a line for a picture.

Definition: *A permutation of a Set of district objects is an ordered arrangement of these objects. An ordered arrangement of r elements of a set is called an* <u>*r-permutation*</u>.

Example: Let $S = \{1,2,3\}$ the ordered arrangement 3, 2, 1 is a permutation and the order arrangement 3, 2 is a 2 – permutation.

The number of r-permutations of a set with n elements is denoted by $\boldsymbol{P(n,r)}$. We can find $P(n,r)$ using product rule as:

THEOREM If n is a positive integer and r is an integer with $1 \leq r \leq n$, then there are

$$P(n,r) = n(n-1)(n-2)\cdots(n-r+1)$$

r – permutations of a set with n distinct elements.

Proof: The 1st element of the permutation can be chosen in n ways because there are n elements in the set. There are n-I ways to pick the 2nd element of the permutation decrease there are n-1 elements left in the set. Similarly, there are n-2 ways to choose the 3rd element and so on. There are $n-(r-1) = n-r+1$ ways to choose the rth element . Consequently, by the product rule are

$$n(n-1)(n-2)\cdots(n-r+1)$$

r-permutations of the set.

Corollary: If n and r are integers with $0 \leq r \leq n$, then $P(n,r) = \frac{n!}{(n-r)!}$

Proof: For positive integers n and r with $1 \leq r \leq n$, we have

$$P(n,r) = n(n-1)(n-2)\cdots(n-r+1)$$

$$P(n,r) = \frac{n(n-1)(n-2)\cdots(n-r+1)(n-r)(n-r-1)\ldots\ldots.3.2.1}{(n-r)(n-r-1)\ldots\ldots.3.2.1} = \frac{n!}{(n-r)!}$$

Because $\frac{n!}{(n-0)!} = \frac{n!}{n!} = 1$, whenever n is a non- negative integer, we see that the formula $P(n,r) = \frac{n!}{(n-r)!}$ also holds for $r = 0$ This completes the proof.

Note: $P(n,r) = \frac{n!}{(n-0)!} = \frac{n!}{n!} = 1$ (So there is only one way to order zero i.e. there is exactly one list with no elements, namely the empty list.)

Example: How many ways are there to select a first-prize winner, a second-prize winner, and a third-prize winner from 100 different people who have entered a contest?

<u>Solution:</u> $Here\ n = 100\ and\ r = 3. We\ have\ to\ find\ P(100,3)$(Because matters which person wins which prize)

$$P(100,3) = \frac{100!}{(100-3)!} = \frac{100!}{(97)!} = \frac{100.99.98.97!}{(97)!} = 100.99.98 = 970200$$

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Example: Suppose that there are eight runners in a race. The winner receives a gold medal, the second place finisher receives a silver medal, and the third-place finisher receives a bronze medal. How many different ways are there to award these medals?

Solution: $Here\ n = 8\ and\ r = 3. We\ have\ to\ find\ P(8,3)$

$$P(8,3) = \frac{8!}{(8-3)!} = \frac{8!}{(5)!} = \frac{8.7.6.5!}{(5)!} = 8.7.6 = 336$$

Example: Suppose that a saleswoman has to visit eight different cities. She must begin her trip in a specified city, but she can visit the other seven cities in any order she wishes. How many possible orders can the saleswoman use when visiting these cities?

Solution: Because the 1st city has been determined, so there are $7! = 7.6.5.4.3.2.1 = 5040$ ways for the saleswoman to choose for her tour.

Example: How many permutations of the letters **ABCDEFGH** contain the string ABC?

Solution: Because the letters ABC must occur as a block, there are six objects namely ABC of the remaining letters D, E, F, G & H. So there are 6! = 720 permutations of the letters ABCDEFGH in which ABC occurs as a block.

Example: How many anagrams (permutations of the letter) are there of the following words?

(a) ANSWER **(b)** PERMUTE **(c)** LITTLE

Solution:

(a) Since ANSWER has 6 different letters, the number of permutations are $6! = 720$

(b) Note that interchanging two E's in a permutation of PERMUTE does not produce a new permutation. Since PERMUTE contains 7 letters, therefore there are $\frac{7!}{2} = 2520$ disjoint permutation of PERMUTE.

(c) There are $6!$ permutation of six letters, but again we need to take care of double counting. In each of these $6!$ arrangements, we can swap the T's or L's or both without obtaining new anagrams. So there are $\frac{6!}{4} = 180$ diferent anagrams of LITTLE .

Example: As part of a market Survey, you are Shown a list of 20 types of chocolate bar, and asked to list your fire favorite bars in order of preference. How many different responses to this question are there?

Solution: $Here\ n = 20\ and\ r = 5. We\ have\ to\ find\ P\ (20,5).\ So$

$$P(20,5) = \frac{20!}{(20-5)!} = \frac{20!}{(15)!} = \frac{20.19.18.17.16.15!}{(15)!} = 20.19.18.17.16 = 1860480$$

Example: How many functions $f: \{1,2,3,\ldots 8\} \longrightarrow \{1,2,3,\ldots 8\}$ are bijective.

Solution: Recall that a function is bijective if each element in the codomain must de the image of exactly one element of the domain. Using two-line notation, we can write one of these bijections as

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 1 & 5 & 8 & 7 & 6 & 2 & 4 \end{pmatrix}$$

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What are have to do is just rearrange the elements of Codomain. So we are creating a permutation of eight elements. So there are $8! = 40320$ bijective functions from {1, 2, 3, …8 } to itself.

Remark. " Permutation " is another term used to describe bijective functions from a set to itself.

Example: How many functions $f: \{1, 2,3\}\ \{1,2, 3, 6,5,6,7,8\}$ are injective?

Solution: Note that for an injective function, we cannot use an element of codomain more than once. We need to pick an element from the Codomain to be the image of 1. There are 8 choices. Then we need to pick one of the remaining 7 elements to be the image of 2. Finally, one of the remaining 6 elements must be the image of 3. So the total number of injective function is $8.7.6\ = P\ (8, 3)$

Note: $If\ |A| = k\ and\ |B| = n, then\ the\ number\ of\ injectives\ f: A \to B\ is\ P(n, k)$

Combinations

We investigated the problem of selecting a certain number of items from a set, where the order in which the items are selected matters (Permutations). A more common problem in practice is to find the numbers of ways in which a selection can be made without distinguishing between different orders. Problems of this kind are called COMBINATIONS PROBLEMS.

Example: Suppose that the rules of a lottery specify that entrants must select 8 different numbers from the natural numbers $1, 2, 3, \ldots, 45$. In how many different ways this selections can be made?

Solution: If order is taken into account, then the number of choices is $P(45, 8)$; but this is not the correct answer because we have counted the same selection many times ($8!\ times$). $P(45, 8)$ is larger than the correct answer by a factor of $8!$. So the correct answer to the problem $\frac{P(45,8)}{8!}\ =\ 215553195$.

In general, suppose we want to select r items from a set of n items without taking into account the order in which the selection is made. By previous example this can be done in $\frac{P(n,r)}{r!}$ ways i.e. in $\frac{n!}{(n-r)!r!}$ ways.

$$\therefore P(n, r) = \frac{n!}{(n-r)!}$$

The expression $\frac{n!}{(n-r)!r!}$ is denoted by $\binom{n}{r}\ or\ {}^{n}C_{r}$. It is the number of ways that r items can be chosen from a set a set of n items. Equivalently, it is the number of r-element subset of an n-element set.

Example: A test is to de generated by selecting six questions from a test bank consisting of 50 questions. How many different tests are possible, assuming that the order in which the questions appear on the test is not taken into account?

Solution: The number of different tests is

$$\binom{50}{6} = \frac{50!}{(50\ -\ 6)!\,6!} = \frac{50.49.48.47.46.45.44!}{44!, 6.5.4.3.2.1} = 15890700$$

Example: You decide to have a dinner party. You have 14 different friends but only have enough chairs to invite 6 of them.

a) How many choices do you have for which to friends to invite?
b) What if you need to decide not only which friends to invite but also where to beat them along your long take? How many choices do you have them?

Solution:

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a) You must simply choose 6 friends out of 14. This can be done in $\binom{14}{6} = \frac{14!}{6!.8!} = 3003$ ways.

b) Have you must count all the ways you can your mate 6 friends choose from a group of 14. So the answer is $P(14,6) = \frac{14!}{8!} = 2162160$.

Theorem: *Prove that* $\binom{n}{r} = \dfrac{n!}{(n-r)!\,r!}$

Proof: Let X be the set of n elements. Let A be the set of r-permutation of X and let B be the set of r-combinations of X. We have to find |B|. Let $f: A \longrightarrow B$ be the function-mapping r-permutations to r-combinations defined by

$$\begin{array}{ccc} f(x_1, x_2, x_3, \dots, x_r) & = & \{x_1, x_2, x_3, \dots, x_r\} \\ order\ matter & & order\ does\ not\ matter \\ (Permutation) & & (Combination) \end{array}$$

There $f\ is\ r! - to - 1$ function. In particular

$$f^{-1}(x_1, x_2, x_3, \dots, x_r) = \{(x_1, x_2, x_3, \dots, x_r), (x_2, x_1, x_3, \dots, x_r), \dots\dots\ all\ r!\ permutations\}$$

By the division rule

$$|A| = r!\,|B| \Rightarrow |B| = \frac{|A|}{r!} = \frac{P(n,r)}{r!} = \frac{n!}{(n-r)!\,r!}$$

Some Interesting Special Cases

$$\binom{n}{0} = \frac{n!}{(n-0)!.0!} = \frac{n!}{n!} = 1 \begin{pmatrix} i.e.\ there\ is\ a\ single\ size \\ 0\ subset\ of\ an\ n-set \end{pmatrix}$$

$$\binom{n}{n} = \frac{n!}{(n-n)!.n!} = \frac{n!}{n!} = 1 \begin{pmatrix} i.e.\ there\ is\ a\ single\ size \\ n\ subset\ of\ an\ n-set \end{pmatrix}$$

$$\binom{n}{1} = \frac{n!}{(n-1)!.1!} = \frac{n!}{(n-1)!} = n \begin{pmatrix} i.e.\ there\ is\ a\ single\ size \\ 1\ subset\ of\ an\ n-set \end{pmatrix}$$

$$\binom{n}{n-1} = \frac{n!}{(n-n+1)!.(n-1)!} = \frac{n!}{(n-1)!} = n \begin{pmatrix} i.e.\ there\ is\ a\ single\ size \\ n-1\ subset\ of\ an\ n-set \end{pmatrix}$$

Theorem: *For all* $n \geq 1$ *and* $0 \leq r \leq n$, *we have* $\binom{n}{r} = \binom{n}{n-r}$

Proof: $$\binom{n}{r} = \frac{n!}{(n-r)!.r!} = \frac{n!}{(n-r)!.(n-(n-r))!} = \binom{n}{n-r}$$

Theorem: Suppose *that* $1 \leq r \leq n$, *then* $\dfrac{\binom{n}{r}}{\binom{n}{r-1}} = \dfrac{n-r+1}{r}$

Proof: $$\frac{\binom{n}{r}}{\binom{n}{r-1}} = \frac{n!}{(n-r)!.r!} \cdot \frac{1}{\frac{n!}{(n-r+1)!.(r-1)!}} = \frac{n!}{(n-r)!.r!} \cdot \frac{(n-r+1)!.(r-1)!}{n!}$$

$$= \frac{n!}{(n-r)!.r(r-1)!} \cdot \frac{(n-r+1)(n-r)!.(r-1)!}{n!} = \frac{n-r+1}{r}$$

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Theorem: Suppose *that* $1 \le r \le n$, *then* $\binom{n}{r} = \frac{n}{r}\binom{n-1}{r-1}$

Proof: $\binom{n}{r} = \frac{n!}{(n-r)!.r!} = \frac{n(n-1)!}{(n-r)!.r(r-1)!} = \frac{n}{r}\frac{(n-1)!}{(n-r)!.(r-1)!} = \frac{n}{r}\frac{(n-1)!}{((n-1)-(r-1))!.(r-1)!} = \frac{n}{r}\binom{n-1}{r-1}$

Theorem: Suppose *that* $1 \le r \le n$, *then* $\binom{n-1}{r} + \binom{n-1}{r-1} = \binom{n}{r}$

Proof: $\binom{n-1}{r} + \binom{n-1}{r-1} = \frac{(n-1)!}{((n-1)-(r-1))!.(r-1)!} + \frac{(n-1)!}{((n-1)-r)!.r!} = \frac{(n-1)!}{(n-r)(n-r-1)!.(r-1)!} + \frac{(n-1)!}{(n-r-1)!.r(r-1)!}$

$$= \frac{(n-1)!}{(n-r-1)!.(r-1)!}\left(\frac{1}{(n-r)} + \frac{1}{r}\right) = \frac{(n-1)!}{(n-r-1)!.(r-1)!}\left(\frac{r+n-r}{r(n-r)}\right)$$

$$= \frac{(n-1)!}{(n-r-1)!.(r-1)!}\left(\frac{n}{r(n-r)}\right) = \frac{n(n-1)!}{(n-r)(n-r-1)!.r(r-1)!} = \frac{n!}{(n-r)!.r!} = \binom{n}{r}$$

Exercise (Page No. 416 of Discrete Mathematics and Its Applications, Eighth Edition)

1. There are 18 mathematics majors and 325 computer science majors at a college.

(**a**) In how many ways can two representatives be picked so that one is a mathematics major and the other is a computer science major?

(**b**) In how many ways can one representative be picked who is either a mathematics major or a computer science major?

Solution:

a) Two representatives can be picked by Using product rule. There are is ways to pick a math major and 325 for computer science major. So there are 18.225 = 5850 ways to choose thro representatives.

b) Sum rule is applicable here because we have two mutually exclusive ways to perform two tasks. There are 18+325 = 343 ways to pick either a math major or a Computer science major.

2. An office building contains 27 floors and has 37 offices on each floor. How many offices are in the building?

Solution:

To find total no. of flies in the building, we first select floor and then count offices. So product rule Says that there are 27 x 37 = 999 offices in this building.

3. A multiple-choice test contains 10 questions. There are four possible answers for each question.

(**a**) In how many ways can a student answer the questions on the test if the student answers every question?

(**b**) In how many ways can a student answer the questions on the test if the student can leave answers blank?

Solution:

a) The product rule applies, since the student will perform each of 10 tasks, one after the other. There are 4 ways to do each task. Therefore, there are $4 \cdot 4 \cdots 4 = 4^{10} = 1{,}048{,}576$ ways to answer the questions on the test.

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b) This is identical to part (a), except that now there are 5 ways to answer each question-give any of the 4 answers or give no answer at all. Therefore, there are $5^{10} = 9,765,625$ ways to answer the questions on the test.

4. A particular brand of shirt comes in 12 colors, has a male version and a female version, and comes in three sizes for each sex. How many different types of this shirt are made?

Solution:

Finding total no. of shirts can be found in 3 steps; 1st picking a color, 2nd picking gender and 3rd picking the side. So by product rule, there are 12 .2. 3= 72 different shirts.

5. Six different airlines fly from New York to Denver and seven fly from Denver to San Francisco. How many different pairs of airlines can you choose on which to book a trip from New York to San Francisco via Denver, when you pick an airline for the flight to Denver and an airline for the continuation flight to San Francisco?

Solution:

The product rule applies here, since a flight is determined by choosing an airline for the flight from New York to Denver (which can be done in 6 ways) and then choosing an airline for the flight from Denver to San Francisco (which can be done in 7 ways). Therefore, there are $6 \cdot 7 = 42$ different possibilities for the entire flight.

6. There are four major auto routes from Boston to Detroit and six from Detroit to Los Angeles. How many major auto routes are there from Boston to Los Angeles via Detroit?

Solution:

By the product rule there are $4 \cdot 6 = 24$ routes

7. How many different three-letter initials can people have?

Solution:

There are $26 \cdot 26 \cdot 26 = 17,576$ different three-letter initials people can have.

8. How many different three-letter initials with none of the letters repeated can people have?

Solution:

There are $26 \cdot 25 \cdot 24 = 15,600$ different three-letter initials with none of the letters repeated are possible.

9. How many different three-letter initials are there that begin with an A?

Solution:

There is only one way to specify the first initial, but as in Exercise 7, there are 26 ways to specify each of the other initials. Therefore, there are, by the product rule, $1 \cdot 26 \cdot 26 = 26^2 = 676$ possible three-letter initials beginning with A

10. How many bit strings are there of length eight?

Solution: There are 2 bits, either 0 of 1. There are 8 events and each event consists of picking a bit. So by product rule there are $2.2.2.2.2.2.2.2 = 2^8 = 256$ bit Strings of length 8 .

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11. How many bit strings of length ten both begin and end with a 1?

Solution:

A bit string is determined by choosing the bits in the string, one after another, so the product rule applies. We want to count the number of bit strings of length 10, except that we are not free to choose either the first bit or the last bit (they are mandated to be 1 's). Therefore, there are 8 choices to make, and each choice can be made in 2 ways (the bit can be either a 1 or a 0). Thus the product rule tells us that there are $2^8 = 256$ such strings.

12. How many bit strings are there of length six or less, not counting the empty string?

Solution:

We have to count bit strings of length 6, 5, 4, 3, 2, 1. Since there are two bits, 0 and 1, therefore

Bit strings of length 6 = $2.2.2.2.2.2 = 2^6 = 64$

Bit strings of length 5 = $2.2.2.2.2 = 2^5 = 32$

Bit strings of length 4 = $2.2.2.2 = 2^4 = 16$

Bit strings of length 3 = $2.2.2 = 2^3 = 08$

Bit strings of length 2 = $2.2 = 2^2 = 04$

Bit strings of length 1 = $2 = 2^1 = 02$

So by sum rule bit strings of length 6 or less are $= 64 + 32 + 16 + 8 + 4 + 2 = 126$

Permutations with Repetition

We now turn to permutations where elements are allowed to repeat. An "r-permutation with repetition" of a set S is the number of ways to choose r-elements from S with Repetition allowed S whose order matters. Fortunately, r-permutations with repetitions are easy to count using the product rule as is shown in the following example.

Example: How many strings of length I can be formed from the uppercase files of the English alphabet?

Solution: Because there are 26 uppercase alphabets and because each letter can be used repeatedly, therefore by product rule, there are

$$\underbrace{26.26....26}_{r-times} = 26^r$$

Strings of uppercase English letters of length r.

Theorem: The number of $r-$ permutations of a Set of n objects with repetition allowed is n^r

Proof: There are n ways to select an element of the set for each of the r positions in the r-permutation when repetition is allowed, because for each choice all n objects are available.

Hence, by the product rule there are n^r r-permutations when repetition is allowed.

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Permutations with limited Repetitions

Example: In how many ways can we arrange the letters in the word PEPPER?

Solution: we have to find 6 - permutations of the set {P, E, R] where P is repeated 3 times, E is repeated 2 times and R is repeated 1 time. First, we make all the letters distinct by adding subscripts $P_1E_1P_2P_3E_2R$ In this the case there are $6! = 720$ arrangements. Next, we erase the subscripts on E'S. This maps each arrangement of the letters $P_1E_1P_2P_3E_2R$ could be ordered in 2! ways before we erase, the mapping is 2! to 1. So by division rule there are $\frac{6!}{2!} = 360$ arrangement of the letters in $P_1E_1P_2P_3E_2R$. Finally, we erase subscripts on P's. This maps each arrangement of $P_1E_1P_2P_3E_2R$ to as arrangement to the letters PEPPER. So $P_1,\ P_2,\ P_3$ can be ordered in 3! ways before the erase, this mapping is 3! To 1. Therefore by division rule there are $\frac{6!}{2!.3!} = 60$ arrangement of the letter PEPPER.

We can prove a general theorem using the same arrangement as in the PEPPER problems.

Theorem: Set A be the set $\{a_1, a_2, a_3, \dots, a_n\}$and let $r_1, r_2, r_3, \dots, r_n$ be non $-$ negative integers.

Then the no. of permutations of tho set A where euch a_i is repeated exactly r_i times is

$$\frac{(r_1 + r_2 + r_3 + \cdots + r_n)!}{r_1!.r_2!.r_3!.\dots.r_n!}$$

Note: The theorem says that the no. of permutations of {P, B, R}, where P is repeated 3 times, B is repeated 2 times and R is repeated 1 time is $\frac{(3+2+1)!}{3!.2!.1!} = 60$. Which is same answer as we found before?

Proof of theorem: We initially make r_i distinct copies of each element a_i by adding subscripts. That the no of permutations where each element is repeated exactly once is $(r_1 + r_2 + r_3 + \cdots + r_n)!$. We then erase the subscripts on the distract copies of a_1. This defines an $r_1! - to - 1$ from old permutations to permutations where a_1 is repeated r_1 times and all other elements are repeated once. Therefore, the no. of new permutations is

$$\frac{(r_1 + r_2 + r_3 + \cdots + r_n)!}{r_1!}$$

Containing in this way with $a_2, a_3, \dots \dots$ we find the number of permutations, we find that the number of permutations where each element a_i is repeated exactly r_i times is

$$\frac{(r_1 + r_2 + r_3 + \cdots + r_n)!}{r_1!.r_2!.r_3!.\dots.r_n!}$$

The above theorem can also be rephrased as

Theorem: *The number of different permutations of n objects, where there are n_1 indistinguishable objects of type1 , n_2 indistinguishable objects of type 2 , …. and n_k indistinguishable objects of type k, is*

$$\frac{n!}{n_1!.n_2!.n_3!.\dots.n_k!}$$

Example: How many different strings can be made by reordering the letters of the word SUCCESS.

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Solution: There are $\frac{7!}{3!.2!.1!.1!} = 420$ strings made by ordering the letters of the word SUCCESS.

Definition: *The quantity* $\frac{(r_1+r_2+r_3+\cdots+r_n)!}{r_1!.r_2!.r_3!.\ldots.r_n!}$ *is called a multinomial coefficient and is denoted by*

$$\binom{r_1 + r_2 + r_3 + \cdots + r_n}{r_1, r_2, r_3, \ldots, r_n}$$

given a set with $r_1 + r_2 + r_3 + \cdots + r_n$ *elements, the multinomial coefficient* $\binom{r_1 + r_2 + r_3 + \cdots + r_n}{r_1, r_2, r_3, \ldots, r_n}$ *represents the no. of ways to choose* r_1 *elements, than* r_2 *of the remaining elements, and so forth.*

Questions:

1. In how many ways can five elements be selected in order from a set with 3 elements when repetition is allowed.

   Solution: We have to choose 5 elements in order from a set of 3 elements when repetition is allowed. We have 3 choices for each element. So there are $3.3.3.3.3 = 3^5 = 243$ ways in which 5 elements can be selected.

2. In how many different ways can five elements be selected in order from a set with 5 elements when repetition is allowed?

   Solution: We have 5 choices for each element to be selected. So there are $5.5.5.5.5 = 5^5 = 3125$ ways to choose 5 elements.

3. How many strings of six letters are there?

   Solution: Since we are considering strings, therefore order matters. There are 26 choices for each position in the string. So there are $26.26.26.26.26.26 = 26^6 = 308{,}915{,}776$ strings of six letters.

Combinations with Repetition

Consider the following counting problem

Example: You have 7 cookies to give to 4 kids. How many ways can you do this?

Solution: To tackle this problem, we need to decide after how many cookies do we stop giving cookies to the 1st kid and start giving cookies to the 2nd kid. And after how many do we switch to the 3rd kid? And after how many do we switch to the cookies to 4 kids as

$$\times\times\times|\times|\times|\times\times \quad \left(\begin{matrix}3\ cookies\ to\ the\ 1st\ kid, 1\ cookie\ to\ the\ 2nd\ kid,\\ one\ to\ the\ 3rd\ kid\ and\ 2\ to\ the\ 4th\ kid\end{matrix}\right)$$

We have used one star for each cookie, and one bar for each switch between the kids /so one fewer bars than there are kids)

So to count the number of ways to distribute of cookies to 4 kids, all we need to do is to count how many "stars and bar charts" are there. In each star and bar chart, there are to symbols (7 stars and 3 bars) and we must choose 3 of them to be bars.

So there are $\binom{10}{3}$ ways to distribute 7 cookies to 4 kids. From above example, we deduce the following theorem.

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Theorem: There are $\binom{n+r-1}{r}$ r − combinations from a set with n elements where repetition of elements is allowed.

Proof: Each r-combination of a set with n-elements when repetition is allowed can be represented by r starts and n-1 bars. To find r-combination of an n-set, we need to choose r symbols (from bar chart) from n-1+r symbols. This can be done in $\binom{n+r-1}{r}$ ways. This complete the proof.

Example: How many ways are there to Select 5 bills from a cash box containing $1 bills, $2 bills, $5 bills, $20 bills and $100 bills, assuming that the order in which bills the bills are chosen does not matter, and the bills are indistinguishable?

Solution: We have n = no. of bills = 7 and r = 5 So the no. of ways is which is 5 bills can be selected is

$$\binom{n+r-1}{r} = \binom{7+5-1}{5} = \binom{11}{5} = \frac{11!}{5!.\,6!} = 462$$

Example: Assume that a cookie shop has 4 different kinds of cookies. How many different ways can six cookies be chosen? Assume that only the type of cookie, and not the individual cookies or the order in which the cookies are chosen, makes.

Solutions: We have to find 6- combinations of a set with 4 elements with repetition allowed. Here n=4, r=6. So there are (($\binom{n+r-1}{r} = \binom{4+6-1}{6} = \binom{9}{6} = \frac{9!}{3!.6!} = 84$; ways to choose 6 cookies .

Example: How many solutions does the equation $x_1 + x_2 + x_3 = 11$ have, wherex_1, x_2 and x_3 are non-negative integer.

Solution: We need to find that in how many of the 11 units go to each of 3 Variables. A solution corresponds to a way of selecting 11 items from a set with 3 elements. So that $x_1$items of type one, x_2 items of type 2 and $x_3$items of type 3 are chosen. Hence the no. of solutions is equal to 11 - combinations with repetition allowed from a set with 3 elements. There are

$$\binom{n+r-1}{r} = \binom{3+11-1}{11} = \binom{13}{11} = \frac{13!}{11!.\,2!} = 78 \quad \text{Solution}$$

formulae for combinations and permutations with and without repetitions.

Type	Repetition Allowed?	Formula
r-permutations	No	$\frac{n!}{(n-r)!}$
r-combinations	No	$\frac{n!}{r!\ (n-r)!}$
r-permutations	Yes	n^r
r-combinations	Yes	$\frac{(n+r-1)!}{r!\ (n-1)!}$

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Exercise (Page No. 454 of Discrete Mathematics and Its Applications, Eighth Edition)

1. In how many different ways can five elements be selected in order from a set with three elements when repetition is allowed?

Solution:

Here r = 5, n = 3. we have to find 5- permutation of 3 elements when repetition is allowed. These are $n^r = 12 = 3^5 = 243$.

2. Every day a student randomly chooses a sandwich for lunch from a pile of wrapped sandwiches. If there are six kinds of sandwiches, how many different ways are there for the student to choose sandwiches for the seven days of a week if the order in which the sandwiches are chosen matters?

Solution:

There are 6 choices for the first day, 6 choices for the 2nd day and so on. By the product rule there are $6.6.6.6.6.6.6 = 6^7 = 279936$ ways to choose Sandwiches for 7 days.

3. How many ways are there to select five unordered elements from a set with three elements when repetition is allowed?

Solution:

We have to find 5- combinations of 3- element set when repetition is allowed. Here r = 5, n = 3. So. there are

$$\binom{n+r-1}{r} = \binom{3+5-1}{5} = \binom{7}{5} = \frac{7!}{5!\,2!} = 21$$

ways to select 5 unordered elements form a set with 3-elements when repetition is allowed.

4. How many different ways are there to choose a dozen donuts from the 21 varieties at a donut shop?

Solution:

Here r = 12 and n = 21. We have to find 12-combination of 21 elements when repetition is allowed. So there are

$$\binom{n+r-1}{r} = \binom{21+12-1}{12} = \binom{32}{12} = \frac{32!}{20!\,12!} = 225792840$$

ways to choose a dozen donuts from the 21 verities at a don't Shop

5. A croissant shop has plain croissants, cherry croissants, chocolate croissants, almond croissants, apple croissants, and broccoli croissants. How many ways are there to choose?
   a) a dozen croissants? b) three dozen croissants?
   c) two dozen croissants with at least two of each kind?
   d) two dozen croissants with no more than two broccoli croissants?
   e) two dozen croissants with at least five chocolate croissants and at least three almond croissants?
   f) two dozen croissants with at least one plain croissant, at least two cherry croissants, at least three chocolate croissants, at least one almond croissant, at least two apple croissants, and no more than three broccoli croissants?

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Solution:

a) We have r = 12, n = 6 (because there are 6 type of croissant namely, plain, cherry, chocolate, almond, apple, and broccoli). So repetition is allowed, required no of ways are

$$\binom{n+r-1}{r} = \binom{6+12-1}{12} = \binom{17}{12} = \frac{17!}{5!\,12!} = 6188$$

b) We have r = 36, n = 6, required no of ways are $\binom{36+6-1}{36} = \binom{41}{36} = \frac{41!}{5!36!} = 749396$

c) First we select 2 of each kind. Since there are 6 types of croissants, therefore there are 12 ways to select 2 croissants of each Kind. So 12 croissants have already been selected. We need to select the remaining 12 croissants (dozen total croissants to select are 24). So n = 6 r = 12 so the required no of ways is $\binom{6+12-1}{12} = \binom{17}{12} = \frac{17!}{5!12!} = 6188$

d) If no broccoli croissant is selected, then n = 5, r = 24. So in this case, the required no of ways is

$$\binom{5+24-1}{24} = \binom{28}{24} = \frac{28!}{4!24!} = 20435$$

If one broccoli croissant is Selected, then n = 6, r = 23. So the required no of ways is

$$\binom{6+23-1}{23} = \binom{28}{23} = \frac{28!}{5!\,24!} = 17550$$

If two broccoli croissant have been selected, then n = 6, r = 22. So the required no of ways is

$$\binom{6+22-1}{22} = \binom{27}{22} = \frac{27!}{5!\,22!} = 14950$$

The total no of ways (using sum rule) is 20475 + 17550 + 14950 = 52975

e) We first select 5 chocolate and 3 almond croissants. Since we have to select 2 dozen (24) croissants, we still need to select 24 - 5 - 3 = 16 croissants. So we have n = 5, r = 16. so required no. of ways is

$$\binom{16+6-1}{6} = \binom{21}{6} = \frac{21!}{15!\,6!} = 20349$$

f) We first select 1 plain, 2 cherries, 3 chocolates, 1 almond and 2 apple croissants. we then need to select 24-1-2-3- 1- 2 =15 croissants (Here are have not selected broccoli croissants). So n = 5, r = 15 So in this case the required no. of ways is

$$\binom{15+5-1}{15} = \binom{19}{15} = \frac{19!}{4!\,15!} = 3876$$

Now we select 1 plain, 2 cherries, 3 chocolates, 1 almond 2 apple and 1 broccoli croissants. we then need to select 24-1-2-3- 1- 2-1 =14 croissants, so n = 5, r = 14. So in this case the required no. of ways is

$$\binom{14+5-1}{14} = \binom{18}{14} = \frac{18!}{14!\,4!} = 3060$$

Now we select 1 plain, 2 cherries, 3 chocolates, 1 almond 2 apple and 2 broccoli croissants. we then need to select 24-1-2-3- 1- 2-2 =13 croissants, so n = 5, r = 13. So in this case the required no. of ways is

$$\binom{13+5-1}{13} = \binom{17}{13} = \frac{17!}{13!\,4!} = 2380$$

Now we select 1 plain, 2 cherries, 3 chocolates, 1 almond 2 apple and 3 broccoli croissants. we then need to select 24-1-2-3- 1- 2-3 =12 croissants from 5 remaining types. So n = 5, r = 12, so in this case the required no. of ways is

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$$\binom{12+5-1}{12} = \binom{16}{12} = \frac{16!}{12!\,4!} = 1820$$

So by sum rule total no. of ways are $3876 + 3060 + 2380 + 1820 = 11136$

6. How many different combinations of pennies, nickels, dimes, quarters, and half dollars can a piggy bank contain if it has 20 coins in it?

Solution:

We denote no. of pennies, nickels, dimes, quarters, and half dollars by a, b, c, d and e respectively. Then a + b + c + d + e = 20. Obviously $a \geq 0, b \geq 0, c \geq 0, d \geq 0, e \geq 0$. We need to select r = 20 coins from a set with n = 5 elements (pennies, nickels, dimes, quarters, and half dollars). So the required no of combinations is

$$\binom{n+r-1}{r} = \binom{5+20-1}{20} = \binom{24}{20} = \frac{24!}{4!\,20!} = 10626$$

7. A book publisher has 3000 copies of a discrete mathematics book. How many ways are there to store these books in their three warehouses if the copies of the book are indistinguishable?

Solution:

Have r = 300, n = 3, the no. of ways in which books can be stored in 3 warehouses is

$$\binom{n+r-1}{r} = \binom{3+300-1}{300} = \binom{302}{300} = \frac{302!}{2!\,300!} = 4504501$$

8. How many solutions are there to the equation $x_1 + x_2 + x_3 + x_4 + x_5 = 21$, where$x_I$ $i = 1, 2, 3, 4, 5$, is a nonnegative integer such that

a) $x_1 \geq 1$? b) $x_i \geq 2\ for\ i = 1, 2, 3, 4, 5$? c) $0 \leq x_1 \leq 10$?
d) $0 \leq x_1 \leq 3, 1 \leq x_2 < 4, and\ x_3 \geq 15$?

Solution:

a) Since $x_1 \geq 1$, therefore we can assume that $x_1 = x_1' + 1$, where $x_1' \geq 0$. Then the problem reduces to finding no. of nonnegative solutions to $(x'_1 + 1) + x_2 + x_3 + x_4 + x_5 = 21$ or $x'_1 + x_2 + x_3 + x_4 + x_5 = 20$ here we have to find 20 combination of a set with 5 elements when repetition is allowed which is given by $\binom{5+20-1}{20} = \binom{24}{20} = \frac{24!}{4!20!} = 10626$

b) We have $x_i \geq 2\ for\ i = 1, 2, 3, 4, 5$. So let's suppose that $x_i = x_i' + 1$, where $x_i' \geq 0\ \ for\ i = 1, 2, 3, 4, 5$. We need to fill no. of solutions of
$x_1' + 2 + x_2' + 2 + x_3' + 2 + x_4' + 2 + x_5' + 2 = 21$ or $x_1' + x_2' + x_3' + x_4' + x_5' = 11$
Takes r = 11, n = 5, the no. of solutions to this equation is

$$\binom{n+r-1}{r} = \binom{5+11-1}{11} = \binom{15}{11} = \frac{15!}{4!\,11!} = 1365$$

c) The no. of solutions to $x_1 + x_2 + x_3 + x_4 + x_5 = 21$, when there is no restriction on x_1, is $\binom{n+r-1}{r} = \binom{5+21-1}{21} = \binom{25}{21} = \frac{25!}{4!21!} = 12650$. The restriction on $x_1\ is\ 0 \leq x_1 \leq 10$. This conditions is violated when $x_1 \geq 11$. Supposing $x_1 = x'_1 + 11, x_1' \geq 0$, we find the no. of solution to $x'_1 + 11 + x_2 + x_3 + x_4 + x_5 = 21$ or $x'_1 + x_2 + x_3 + x_4 + x_5 = 10$. Under the violated condition, to be

$$\binom{n+r-1}{r} = \binom{5+10-1}{10} = \binom{14}{10} = \frac{14!}{4!\,10!} = 1001$$

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The no. of solution to the given equation under the condition $0 \leq x_1 \leq 10$ is 12650 - 1001 = 11649

d) First, we impose the restrictions that $x_3 \geq 15$ *and* $x_2 \geq 1$. So ,let $x_2 = x_2' + 1, x_3 = x_3' + 15$, where $x_2', x_3' \geq 0$. Then the problem is equivalent to counting the no. of solutions to $x_1 + x'_2 + x'_3 + x_4 + x_5 = 5$, subject to the constraints that $x_1 \leq 3$ *and* $x'_2 \leq 2$ (the latter coming from the original restriction that $x_2 < 4$). Note that these two restrictions cannot be violated simultaneously. Thus if we count the number of solutions to $x_1 + x'_2 + x'_3 + x_4 + x_5 = 5$, subtract the number of its solutions in which $x_1 \geq 4$ and subtract the numbers of its solutions in which $x'_2 \geq 3$, then we will have the answer. By Theorem 2 there are C(5 + 5 - 1, 5) = C(9, 5) = 126 solutions of the unrestricted equation. Applying the first restriction reduces the equation to $x'_1 + x'_2 + x'_3 + x_4 + x_5 = 1$, which has C(5 + 1 - 1, 1) = C(5, 1) = 5 solutions. Applying the second restriction reduces the equation to $x_1 + x''_2 + x'_3 + x_4 + x_5 = 2$, which has C(5 + 2 - l, 2) = C(6, 2) = 15 solutions. Therefore, the answer is 126 - 5 - 15 = 106.

The Pigeonhole Principle

"If more than n pigeons fly into n pigeonhole, than at least one pigeonhole will contain at least two pigeons."

This principle applies to other objects besides pigeons and pigeonholes.

Theorem: (Pigeonhole Principle) If k is a positive integer and k + 1 or more objects are *placed* into k boxes, then there is at least one box containing two or more of the objects

Proof: Suppose, on the contrary that each of the K boxes contains at most one object. Then, at most there will be K objects. But we assured that there are more than k objects. So this is not possible. This there must be at least one box containing two or more objects.

Example: Among any group of 367 people, there must be at least two with the same birthday, because there are only 366 possible birthdays.

Example: In any group of 27 English words, there must be at least two that begin with the same letter, because there are 26 letters in the English alphabet

Example: How many students must be in a class to guarantee that at least two students receive the same score on the final exam, if the exam is graded on a scale from 0 to 100 points?

Solution: There are 101 possible scores on the final. The pigeonhole principle shows that among any 102 students there must be at least 2 students with the same score.

Theorem: (The generalized Pigeonhole principle)If N objects are placed into k boxes, then there is at least one box containing at least $\lceil N/k \rceil$objects.

Proof: we prove it by contradiction. Suppose that none of the boxed contains more than [N / K] - 1 objects. Then total no. of objects is at most

$$K([N/K - 1) < K([N/K + 1] - 1) = N \qquad (\therefore [N/K] < N/K + 1).$$

which is a contradiction, because there are a total of N objects.

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Remarks:

1 - A common type of problem asks for the minimum number of objects must be in one of k boxes when these objects are distributed among the boxes.

2 - When we have N objects, the generalized pigeonhole principle tells us that there must be at least r objects in one of the boxes as long as$[\,N/\,K\,]\ \geq\ r$.

3 - The Smallest Integer N with $[N/K\,] >\ r-1$, namely N = K (r – 1) + 1is the smallest integer satisfying $[\,N/\,K\,]\ \geq\ r$,

Example: Among 100 people there are at least ⌈100⁄12⌉ = [8.333] = 9 who were born in the same month.

Example: What is the minimum number of students required in a discrete mathematics class to be sure that at least six will receive the same grade, if there are five possible grades, A, B, C, D, and F?

<u>Solution</u> The minimum number of students needed to ensure that at least six students receive the same grade is [N / K] = r i.e. [N / 5] = 6. The smallest such integer is N = K (r – 1) +1 = 5 · 5 + 1 = 26

Note: If you have only 25 students, it is possible for there to be five who have received each grade so that no six students have received the same grade.

Example: How many cards must be selected from a standard deck of 52 cards to guarantee that at least three cards of the same suit are selected? How many must be selected to guarantee that at least three hearts are selected?

<u>Solutions:</u> Suppose there are four boxes, one for each suit, and as cards are selected they are placed in the box reserved for cards of that suit. Using the generalized pigeonhole principle, we see that if N cards are selected, there is at least one box containing at least ⌈N⁄4⌉ cards. Consequently, we know that at least three cards of one suit are selected if ⌈N⁄4⌉ ≥ 3. The smallest integer N such that ⌈N⁄4⌉ ≥ 3 is N = 2 · 4 + 1 = 9, so nine cards suffice.

We do not use the generalized pigeonhole principle to answer this question, because we want to make sure that there are three hearts, not just three cards of one suit. Note that in the worst case, we can select all the clubs, diamonds, and spades, 39 cards in all, before we select a single heart. The next three cards will be all hearts, so we may need to select 42 cards to get three hearts

Example: What is the least number of area codes needed to guarantee that the 25 million phones in a state can be assigned distinct 10-digit telephone numbers? (Assume that telephone numbers are of the form NXX-NXX-XXXX, where the first three digits form the area code, N represents a digit from 2 to 9 inclusive, and X represents any digit.)

<u>Solutions:</u> There are eight million (8.10.10.10.10.10.10 = 8000000) different phone numbers of the form NXX-XXXX (by product rule). Hence, by the generalized pigeonhole principle, there are at least ⌈25,000,000⁄8,000,000⌉ = 4 of 25 million telephone which have identical phone numbers. Hence, at least four area codes are required to ensure that all 10-digit numbers are different.

Exercise (Page No. 454 of Discrete Mathematics and Its Applications, Eighth Edition)

1. Show that in any set of six classes, each meeting regularly once a week on a particular day of the week, there must be two that meet on the same day, assuming that no classes are held on weekends

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Solutions:

No. of classes = 6, No. of days= 5 (Since no class on weekends) by pigeonhole principle, there must be two classes which meet on the same day because no. of classes is more than no. of days.

2. Show that if there are 30 students in a class, then at least two have last names that begin with the same letter.

Solutions:

Since no. of students (20) is more than no. of alphabets (26), therefore by pigeonhole principle, there are at least two students who have last name that begin with same letter.

3. A drawer contains a dozen brown socks and a dozen black socks, all unmatched. A man takes socks out at random in the dark. How many socks must he take out to be sure that he has at least two socks of the same color? How many socks must he take out to be sure that he has at least two black socks?

Solutions:

There are 2 colors (pigeonholes) and are want to find least no. of pigeons needed to ensure that at least one of the pigeonholes contains two pigeons. By the pigeonholes principle the answer is 3.
It may happen that in 1st 12 Socks all are brown. So he needs to Select 14 balls to be sure that at least two socks are black.

4. A bowl contains 10 red balls and 10 blue balls. A woman selects balls at random without looking at them. How many balls must she select to be sure of having at least three balls of the same color? How many balls must she select to be sure of having at least three blue balls?

Solutions:

Red balls = 10, Blue balls = 10. There are two colors (of these as pigeonholes) So k = 2 . We have to find N such that [N / 2] = 3 → N = K (r – 1) + 1 = 2 . 2 + 1 = 5

Since total no of red balls is 10, so he must select at Least 13 balls to be sure that at least 3 blue balls are selected

5. Show that among any group of five (not necessarily consecutive) integers, there are two with the same remainder when divided by 4.

Solutions:

When an integer is divided by 4, there are four possible remainders (0, 1, 2, 3). These are the pigeonholes. So by pigeonholes principle among 5 given integers, there are at least 2 with same remainder.

6. What is the minimum number of students, each of whom comes from one of the 50 states, who must be enrolled in a university to guarantee that there are at least 100 who come from the same state?

Solutions:

Here the pigeons are the students and the pigeonholes are the states, which are 50 in number. We have to for a least N such that $\left[\frac{N}{50}\right] \geq 100 \Rightarrow N \geq 50(100-1)+1 = 4951$. Therefore we need at least 4951 students to guarantee that at least 100 come from the same State.

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7. Let d be a positive integer. Show that among any group of d + 1 (not necessarily consecutive) integers there are two with exactly the same remainder when they are divided by d.

Solutions:

There are only d possible remainders when an integer is divided by d, namely 0, 1, . . . , d − 1. By the pigeonhole principle, if we have d + 1 remainders, then at least two must be the same.

Binary Relations

A binary relation is used to descried the relationships between two objects. The word binary has means Simply that there are two objects involved.

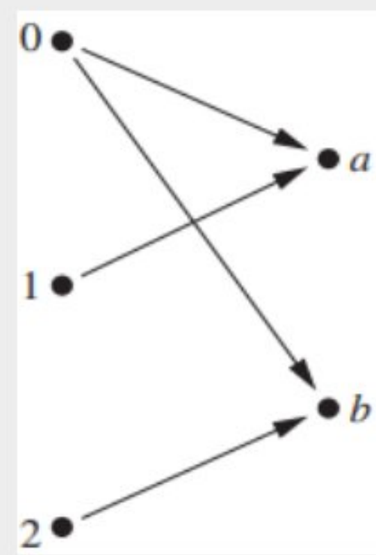
Definitions: "*Let A and B be sets. A binary relation from A to B is a subset of* $A \times B$."

In other words, a binary relation from A to B is a set R of ordered pairs, where the first element of each ordered pair comes from A and the second element comes from B.

Notation: aRb to denote that $(a, b) \in R$ and $a \not R b$ to denote that $(a, b) \notin R$. Moreover, when (a, b) belongs to R, a is said to be **related** to b by R

Example: Let A = {0, 1, 2} and B = {a, b}. Then {(0, a), (0, b), (1, a), (2, b)} is a relation from A to B.

Note that $(0, a) \in R$. Therefore, relations graphically using arrows and using tables. The above relation can be represented graphically as in tabular form as follows



R	a	b
0	×	×
1	×	
2		×

Functions as Relations

"Recall that a function f from a set A to a set B assigns exactly one element of B to each element of A."

This definition is problematic. The main difficulty is this use of the words "rule" and "assign". For example, consider

$$f(x) = \begin{Bmatrix} x^2 & \text{if there is life as we know if on Mars} \\ 3x - 5 & \text{if there is not life as we know on Mars} \end{Bmatrix}$$

Is this a function? Can we consider this a rule? Do we have to wait until we have found life on Mars (or not!) before are can consider this a function?

It is best to think about functions that avoids subjective words like " rule" and " assign "This is the motivation for our next definition.

Definition: "*A relation f from A to B in a relation R on a set A and B. such that*

i. *Every* $a \in A$ *is in the domain of* f
ii. *if* $(a, b) \in R$ *and* $(a, c) \in R$, *then* $b = c$ *of course we refer to A and B as the domain and the range of* f *respectively.*

Note: From above definition, we conclude that a function represents a relation where exactly one element of B is related to each element of A."

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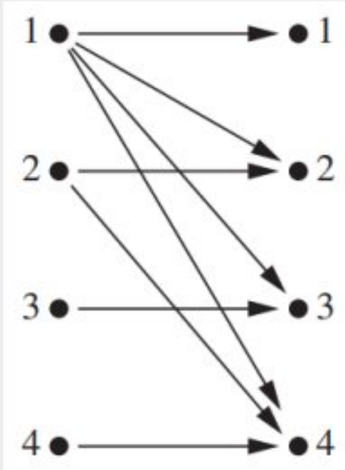
Definition: *"A relation on a set A is a relation from A to A."*

In other words, a relation on a set A is a subset of $A \times A$.

Example: Let A be the set $[1, 2, 3, 4\}$. Which ordered pairs are in the relation $R = \{(a, b) | a \text{ divides } b\}$.

R	1	2	3	4
1	×	×	×	×
2		×		×
3			×	
4				×



Solution: Note that $(a, b) \in R \Leftrightarrow a|b$. Then

$$R = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)\}$$

Example: How many relations are there on a set with n elements?

Solution: If A has n elements, then $A \times A$ as n^2 elements. Since a set with m elements has 2^m subsets therefore there are 2^{n^2} subset of $A \times A$. Hence there are 2^{n^2} relations on a set with n elements.

Example: Let $S = \{1, 2, 3\}$ and $T = [a, b, c, d, e\}$, then $f = \{1, b), (2, a), (3, c), (4, a), (5, b)\}$. This is a function because each element of the domain S is used once and only once. However, not all elements of the range are used. According to the definition f function, this is allowed.

Definition: *"A relation R on a set A is called reflexive if* $(a, a) \in R, \ \forall a \in A$"

This means that a relation on a set A is reflexive if every element of A is related to itself.

Example: Consider the following relations on [1,2, 3, 4].

$R_1 = \{(1,1), (1,2), (2,1), (2,2), (3,4), (4,1), (4,4)\}, R_2 = \{(1,1), (1,2), (2,1)\},$

$R_3 = \{(1,1), (1,2), (1,4), (2,1), (2,2), (3,3), (4,1), (4,4)\}, R_4 = \{(2,1), (3,1), (3,2), (4,1), (4,2), (4,3)\},$

$R_5 = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4), (4,4)\}, R_6 = \{(3,4)\}.$

Which of these are reflexive?

Solution: The relations $R_3$and R_5 are reflexive because they both contain all pairs of the form (a, a), namely, $(1,1), (2,2), (3,3), and\ (4,4)$.

Example: Consider the following relations on the set of integers Z:

$R_1 = \{(a,b) \mid a \leq b\}, R_2 = \{(a,b) \mid a > b\}, R_3 = \{(a,b) \mid a = b \text{ or } a = -b\},$

$R_4 = \{(a,b) \mid a = b\}, R_5 = \{(a,b) \mid a = b + 1\}, R_6 = \{(a,b) \mid a + b \leq 3\}.$

Which of these relations contain each of the pairs $(1,1), (1,2), (2,1), (1,-1), and\ (2,2)$? Which of the above relations are reflexive?

Solution: $(1,1) \in R_1, R_3, R_4, R_6$ $\quad (1,2) \in R_1\ R_6;$ $\quad (2,1) \in R_2, R_5, R_6;$

$(1,-1) \in R_2, R_3, R_6,$ $\quad (2,2) \in R_1, R_3, R_4$ $\quad R_1, R_3, R_4\ are\ reflexive$

Example: Is the relation "$aRb \Leftrightarrow a|b$" as reflexive on the set of positive integers?

Solution: Because $a|a$ $\forall$ positive integer, therefore $aRa\ \forall\ a \in Z^+$. Hence this relation is reflexive, if we replace set of positive integer with $\mathbb{Z}$ then the relation is not reflexive $(\therefore 0 \nmid 0)$

Definition*: "A relation R on a set A is called symmetric if* $(a, b) \in R \Rightarrow (b, a) \in R, \ \forall\ a, b \in A.$

A relation R on a set A is called antisymmetric if $(a, b) \in R$ *and* $(b, a) \in R \Rightarrow a = b, \ \forall\ a, b \in A."$

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Remark: Using quantifiers, we see that the relation R on the set A is symmetric if

$$\forall a, \forall b\ ((a,b) \in R \rightarrow (b,a) \in R)$$

Similarly, the relation R on the set A is antisymmetric if

$$\forall a, \forall b\ (((a,b) \in R) \wedge (b,a) \in R) \rightarrow (a = b)).$$

Example: Consider the following relations on the set of integers Z:

$$R_1 = \{(a,b) \mid a \leq b\}, R_2 = \{(a,b) \mid a > b\}, R_3 = \{(a,b) \mid a = b \text{ or } a = -b\},$$

$$R_4 = \{(a,b) \mid a = b\}, R_5 = \{(a,b) \mid a = b + 1\}, R_6 = \{(a,b) \mid a + b \leq 3\}.$$

Which of the relations from are symmetric and which are antisymmetric?

<u>Solution</u>

<u>Symmetric:</u>

a. if $(a,b) \in R_1$, then $a \leq b$. but $a \leq b \not\Rightarrow b \leq a$ so that $(b,a) \notin R_1$ is not symmetric.
b. If $(a,b) \in R_3$, then $a = b\ or\ a = -b$. Then $b = a\ \ or\ b = -a$ so that $(b,a) \in R_3. So\ R_3$ is symmetric
c. If $(a,b) \in R_2$, then $a > b$. but $a > b \not\Rightarrow b > a$ so that $(b,a) \notin R_2$. So R_2 is not symmetric.
d. If $(a,b) \in R_4$, then $a = b \Rightarrow b = a$. So that $(b,a) \in R_4. So\ R_4$ is symmetric
e. If $(a,b) \in R_5$, then $a = b + 1 \not\Rightarrow b = a + 1$. So that $(b,a)) \notin R_5$. So R_5 is not symmetric
f. If $(a,b) \in R_6$, then $a + b \leq 3 \Rightarrow b + a \leq 3$. So that $(b,a)) \in R_6$. So R_6 is symmetric

<u>Antisymmetric:</u>

a. if $(a,b) \in R_1$, then $(a,b) \in R_1$ if $a = b$ so that R_1 is antisymmetric.($if\ a \geq b$ and $b \geq a$, then $a = b$)
b. if $(a,b) \in R_2$, then $a < b$. So$(a,b) \notin R_2$. Hence R_2 is antisymmetric.
c. R_3 is not antisymmetric because $(1,-1)\ and\ (-1,1) \in R_3$, but $(1,-1) \neq (-1,1)$
d. If $(a,b) \in R_4$, then $a = b$ then $(b,a) \in R_4$. So $(a,b) \in R_4$ and $(b,a) \in R_4 \Rightarrow a = b$.
e. If $(a,b) \in R_5$, then $(b,a) \notin R_5. Since\ a = b + 1 \not\Rightarrow b = a + 1$. So R_5 is antisymmetric
f. Since $(1,2)$ and $(2,1) \in R_6$ and $(1,2) \neq (2,1)$, therefore. R_6 is not antisymmetric

Example: Is the "divides" relation on the set of positive integers symmetric? Is it antisymmetric?

<u>Solution:</u> This relation is not symmetric because $1|2$, but $2 \nmid 1$. However, it is antisymmetric. (i.e. $(1,2) \in R$ but $(2,1) \notin R$). This relation is antisymmetric because if $a|b$ and $b|a$, then $a = b$ (i.e. if $(a,b) \in R$ and $(b,a) \in R \Rightarrow a = b$).

To prove that $a|b$ and $b|a$, then $a = b$

If $a|b$, then a positive integer k_1, such that $b = k_1 a$. if $b|a$, then a positive integer k_2, such that $a = k_2 b$. so $b = k_1\ (k_2 b) = k_2 k_2 b$. so that $k_1 k_2 = 1$ which is possible only if $k_1 = k_2$ (k_1, k_2 are positive integer).

Definition*: "A relation R on a set A is called transitive if* $(a,b) \in R$ *and* $(b,c) \in R \Rightarrow (a,c) \in R, \forall\ a,b,c \in A$*."*

Remark*:* Using quantifiers we see that the relation R on a set A is transitive if we have

$$\forall a, \forall b, \forall c\ (((a,b) \in R \wedge (b,c) \in R) \rightarrow (a,c) \in R).$$

Example: Consider the following relations on [1,2, 3, 4].

$$R_1 = \{(1,1),(1,2),(2,1),(2,2),(3,4),(4,1),(4,4)\}, R_2 = \{(1,1),(1,2),(2,1)\},$$

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$R_3 = \{(1,1),(1,2),(1,4),(2,1),(2,2),(3,3),(4,1),(4,4)\}, R_4 = \{(2,1),(3,1),(3,2),(4,1),(4,2),(4,3)\},$

$R_5 = \{(1,1),(1,2),(1,3),(1,4),(2,2),(2,3),(2,4),(3,3),(3,4),(4,4)\}, R_6 = \{(3,4)\}.$

Which of these are transitive?

Solution: $(3,4) \in R_1$ and $(4,1) \in R_1 \nRightarrow (3,1) \in R_1$ so R_1 is not transitive.
$(1,2),(2,1) \in R_2$ but $(2,2) \notin R_2$ so R_2 is not transitive.
$(4,1),(1,2) \in R_3$ *but* $(4,2) \notin R_3$ so R_3 is not transitive.
$(3,2),(2,1) \in R_4 \Rightarrow (3,1) \in R_4, (4,2),(2,1) \in R_4 \Rightarrow (4,1) \in R_4, (4,3),(3,1) \in R_4 \Rightarrow (4,1) \in R_4.$
Hence R_4 is transitive.
In case of R_5 *and* R_6 , $(a,b) \in R_5, R_6$ *but* $(b,c) \notin R_5, R_6$ thus both are transitive.

Example: Which of the following relations on the set of integers Z are transitive?

$R_1 = \{(a,b) \mid a \le b\}, R_2 = \{(a,b) \mid a > b\}, R_3 = \{(a,b) \mid a = b \text{ or } a = -b\},$

$R_4 = \{(a,b) \mid a = b\}, R_5 = \{(a,b) \mid a = b + 1\}, R_6 = \{(a,b) \mid a + b \le 3\}.$

Solution:

R_1:if $a \le b$ and $b \le c \Rightarrow a \le c$ so if $(a,b),(b,c) \in R_1$ then $(a,c) \in R_1$. So that R_1 transitive.

R_2: $a > b$ and $b > c \Rightarrow a > c$ i.e. $(a,b),(b,c) \in R_2$ then $(a,c) \in R_2$. So that R_2 transitive.

R_3:if $a = \pm b$ and $b = \pm c \Rightarrow a = \pm c$ so that $(a,b),(b,c) \in R_3$ then $(a,c) \in R_3$. So R_3 transitive.

R_4:if $a = b$ and $b = c \Rightarrow a = c$ i.e. $(a,b),(b,c) \in R_4$ then $(a,c) \in R_4$. So R_4transitive.

R_5: $(2,1),(1,0) \in R_5$ but $(2,0) \notin R_5$. So R_5 is not transitive.

R_6: $(2,1),(1,2) \in R_6$ but $(2,2) \notin R_6$. So R_6 is not transitive.

Example: How many reflexive relations are there on a set with n elements?

Solution: A relation R on a set A is a subset of $A \times A$. If R is reflexive, each of the n ordered pairs (a,a) for $a \in A$ must be in R. Each of the other $n(n-1)$ ordered pairs of the form (a,b), where $a \neq b$, may or may not be in R. Hence, by the product rule for counting, there are $2^{n(n-1)}$ reflexive relations.

Combining Relations

"Because relations from A to B are subsets of A × B, two relations from A to B can be combined in any way two sets can be combined."

Example: Let $A = \{1,2,3\}$ and $B = \{1,2,3,4\}$. The relations $R_1 = \{(1,1),(2,2),(3,3)\}$ and $R_2 = \{(1,1),(1,2),(1,3),(1,4)\}$ can be combined to obtain

$R_1 \cup R_2 = \{(1,1),(1,2),(1,3),(1,4),(2,2),(3,3)\},$

$R_1 \cap R_2 = \{(1,1)\},$

$R_1 - R_2 = \{(2,2),(3,3)\},$

$R_2 - R_1 = \{(1,2),(1,3),(1,4)\}.$

Example: Let A and B be the set of all students and the set of all courses at a school, respectively. Suppose that R_1 consists of all ordered pairs (a,b), where a is a student who has taken course b, and R_2 consists of all ordered pairs (a,b), where a is a student who requires course b to graduate. What are the relations $R_1 \cup R_2, R_1 \cap R_2, R_1 \oplus R_2, R_1 - R_2$, and $R_2 - R_1$?

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Solution: The relation $R_1 \cup R_2$ consists of all ordered pairs (a, b), where a is a student who either has taken course b or needs course b to graduate, and $R_1 \cap R_2$ is the set of all ordered pairs (a, b), where a is a student who has taken course b and needs this course to graduate. Also, $R_1 \oplus R_2$ consists of all ordered pairs (a, b), where student a has taken course b but does not need it to graduate or needs course b to graduate but has not taken it. $R_1 - R_2$ is the set of ordered pairs (a, b), where a has taken course b but does not need it to graduate; that is, b is an elective course that a has taken. $R_2 - R_1$ is the set of all ordered pairs (a, b), where b is a course that a needs to graduate but has not taken.

Example: Let R_1 be the less than relation on the set of real numbers and let R_2 be the greater than relation on the set of real numbers, that is, $R_1 = \{(x, y) \mid x < y\}$ *and* $R_2 = \{(x, y) \mid x > y\}$. What are $R_1 \cup R_2, R_1 \cap R_2, R_1 \oplus R_2, R_1 - R_2$, and $R_2 - R_1$?

Solution: We note that $(x, y) \in R_1 \cup R_2$ if and only if $(x, y) \in R_1$ or $(x, y) \in R_2$. Hence, $(x, y) \in R_1 \cup R_2$ if and only if $x < y$ *or* $x > y$. Because the condition $x < y$ *or* $x > y$ is the same as the condition $x \neq y$, it follows that $R_1 \cup R_2 = \{(x, y) \mid x \neq y\}$. In other words, the union of the less than relation and the greater than relation is the not equals relation. Next, note that it is impossible for a pair (x, y) to belong to both R_1 and R_2 because it is impossible that $x < y$ and $x > y$. It follows that $R_1 \cap R_2 = \emptyset$. We also see that $R_1 - R_2 = R_1$, $R_2 - R_1 = R_2$, and $R_1 \oplus R_2 = R_1 - R_2 - R_1 - R_2 = \{(x, y) \mid x \neq y\}$.

Definition: *"Let R be a relation from a set A to a set B and S a relation from B to a set C. The composite of R and S is the relation consisting of ordered pairs (a, c), where* $a \in A, c \in C$, *and for which there exists an element* $b \in B$ *such that* $(a, b) \in R$ *and* $(b, c) \in S$. *We denote the composite of R and S by* $S \circ R$."

Remarks: To compute the composite of two relations requires that we find elements that are the second element of ordered pairs in the first relation and the first element of ordered pairs in the second relation

Example: What is the composite of the relations R and S, where R is the relation from $\{1, 2, 3\}$ *to* $\{1, 2, 3, 4\}$ with $R = \{(1, 1), (1, 4), (2, 3), (3, 1), (3, 4)\}$ and S is the relation from $\{1, 2, 3, 4\}$ *to* $\{0, 1, 2\}$ with $S = \{(1, 0), (2, 0), (3, 1), (3, 2), (4, 1)\}$?

Solution: $S \circ R$ is constructed using all ordered pairs in R and ordered pairs in S, where the second element of the ordered pair in R agrees with the first element of the ordered pair in S. For example, the ordered pairs (2, 3) in R and (3, 1) in S produce the ordered pair (2, 1) in $S \circ R$. Computing all the ordered pairs in the composite, we find $S \circ R = \{(1, 0), (1, 1), (2, 1), (2, 2), (3, 0), (3, 1)\}$

Example: Composing the parent relation with Itself Let R be the relation on the set of all people such that $(a, b) \in R$ if person a is a parent of person b. Then $(a, c) \in R \circ R$ if and only if there is a person b such that $(a, b) \in R$ and $(b, c) \in R$, that is, if and only if there is a person b such that a is a parent of b and b is a parent of c. In other words, $(a, c) \in R \circ R$ if and only if a is a grandparent of c.

Definition: *"Let R be a relation on the set A. The powers* $R^n, n = 1, 2, 3, \ldots$, *are defined recursively by*

$$R^1 = R \text{ and } R^{n+1} = R^n \circ R."$$

Remarks: The definition shows that $R^2 = R \circ R, R^3 = R^2 \circ R = (R \circ R) \circ R$, and so on.

Theorem: *A relations R on a set A is transitive if and only if* $R^n \subseteq R$ *for* $n = 1,2,3 \ldots$.

Proof: Suppose that $R^n \subseteq R$ for $n = 1,2,3 \ldots$. In particular $R^2 \subseteq R$. We have to prove that R is transitive. Let $(a, b), (b, c) \in R$, *then* $(a, c) \in R^2 = R \circ R$. But $R^2 \subseteq R$. So that $(a, c) \in R$. This prove that R is transitive.

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Conversely, suppose that R is transitive. We have to prove that $R^{n} \subseteq R \; for \; n = 1,2,3$ …. Will use mathematical induction to prove it. Note that the result is true for $n = 1$

Assume that $R^{n} \subseteq R$ for some integer n. We have to prove that $R^{n+1} \subseteq R$. Let $(a, b) \in R^{n+1} = R \circ R$. Then $\exists x \in A$ such that $(n, x) \in R$ and $(x, b) \in R^{n} \subseteq R$(by hypothesis) since R is transitive , therefore, $(a, x), (x, b) \in R \; implies \; (a, b) \in R$. Hence $R^{n+1} \subseteq R$.This complete the proof.

Example: Determine whether the relation R on the set of all people is reflexive, symmetry, antisymmetric, and/or transitive, where $(a, b) \in R$ if and only if

a) a is taller than b
b) a and b were born on the same day
c) a has the same first name as b
d) a and b have a common grandparent.

<u>Solution</u>**:**

a) R is not reflexive because $(a, a) \notin R$ since a person a can not be taller then a

   If $(a, b) \in R$, then $(b, a) \notin R$(if a is taller then b then b cannot be taller then a). So R is not symmetry.

   R is antisymmetric because if $(a, b) \in R$, then $(b, a) \notin R$.

   If a is taller than b and b is taller than c, then a is taller than c. That is if $(a, b), (c, d) \in R$, then $(a, c) \in R$.

b) R is obviously reflexive because $(a, a) \in R$
   If a and b were born on the same day, then $(a, b) \in R$. But this means that b and a were born on the same day, so that $(b, a) \in R$. Hence R is symmetric.
   R is not antisymmetric because $(a, b), (b, a) \in R$, then $a \neq b$.
   If a and b were born on the same day and b and c were born on the same day, then it means that a and c were born on the same day. Hence $(a, b), (b, c) \in R \Rightarrow (a, c) \in R$. So R is transitive.

c) R is obviously reflexive.
   R is symmetric because if a and b have same first name, then b and a have same first name.
   R is not antisymmetric because $(a, b), (b, a) \in R$ does not imply $a = b$.
   If a and b have same first name and b and c have same first name, then a and c will have same first name i.e. $(a, b), (b, c) \in R \Rightarrow (a, c) \in R$. So R is transitive.

d) R is obviously reflexive
   If a and b has same grandparents, then b and a have same grandparents i.e. $(a, b) \in R, (b, a) \in R$.
   R is not antisymmetric because $(a, b) \in R \; and \; (b, a) \in R \nRightarrow a = b$.
   If a and b have common grandparents and b and c have common grandparents, then a and c may or may not have common grandparent. So R is not transitive.

Example: Determine whether the relation R on the set of all real number is reflexive, symmetric, antisymmetric, and/or transitive, whether $(x, y) \in R$, iff

a) x+y=0
b) x=±y
c) x-y is a rational number.
d) x=2y
e) xy≥0
f) xy=0
g) x=1
h) x=1 or y=1

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Solution:

a) R is not reflexive because $x + x \neq 0$. So that $(x, x) \notin R$. if $x + y = 0$ (i.e. if$(x, y) \in R$), then $y + x = 0$, so that $(y, x) \in R$. This means that R is symmetric. R is not antisymmetric because if $x + y = 0$ and $y + x = 0$, then it is not necessary that $x = y$. if $x + y = 0$ and $y + z = 0$, then $x = -y$ and $y = -z$. so $x = -y = -(-z) = z$ and $x + z \neq 0$. So R is not transitive.
b) R is reflexive because $x = x$ and $(x, x) \in R$. If $(x, y) \in R$, then $x = \pm y$ so that $y = \pm x$ and we have $(y, x) \in R$. So R is symmetric. If $x = \pm y$ and $y = \pm x$ then $x = y$ does not hold always. So R is not antisymmetric. R is transitive because if $x = \pm y$ & $y = \pm z$, then $x = \pm z$. so $(x, y), (y, z) \in R$ $\Rightarrow n(x, z) \in R$.
c) Since $x - x = 0$, a rational number, therefore $(x, x) \in K$ and hence R is reflexive. If $x - y = 0$, then $y - x = 0$. This means that $(x, y) \in R \Rightarrow (y, x) \in R$. If $x - y = 0$ and $y - x = 0$, then $x \neq y$. so that R is not antisymmetric. If $x - y = 0$ and $y - x = 0$, then by addition, $x - z = 0$. So R is transitive.
d) $(x, x) \notin R$ because $x \neq 2x$, so that R is not reflexive. $x = 2y \nRightarrow y = 2x$. so that R is not symmetric. If $x = 2y$ and $y = 2x$, then $x = 2(2x)$ =4x which is not possible. So if $(x, y) \in R$, hence $(y, x) \notin R$. hence R is antisymmetric. Now if $x = 2y$ and $y = 2z$, then $x = 4z$. this means that $(x, y), (y, z) \in R \nRightarrow (x, z) \in R$. So that R is not transitive.
e) Since $xx \geq 0 \ \forall x \in R$, therefore $(x, x) \in R \ \forall x \in R$. So R is reflexive. If $xy \geq 0$, then $yx \geq 0$. Hence R is symmetric. If $xy \geq 0$ and $yx \geq 0$, then this does not imply $x = y$. so R is not antisymmetric. If $x = -2, y = 0, z = 5$, then $xy \geq 0$ and $yz \geq 0$ but $xz \leq 0$ meaning that $(x, y), (y, z) \in R \nRightarrow (x, z) \in R$. So R is not transitive.
f) $xx \neq 0 \ \forall x \in R|\{0\}$, so that $(x, x) \notin R \ \forall \ x \in R$. Hence R is not reflexive. If $xy = 0$, then $yx = 0$ meaning that R is not symmetric. If $xy = 0$ and $yx = 0$ then this does not mean tat $x = y$. so R is not antisymmetric. If $x = +2, y = 0, z = 3$, then $xy = 0$ and $yz = 0$ but $xz \neq 0$ meaning that $(x, y), (y, z) \in R \nRightarrow (x, z) \in R$. So R is not transitive.
g) $(x, x) \notin R \ \forall x \in R$ because $(0,0) \notin R$, so that R is not reflexive. $(1,2) \in R$ but $(2,1) \notin R$, so that R is not symmetric. If $(x, y) \in R$ & $(y, x) \in R$, then by definition of R, $x = 1 = y$ i.e. $x = y$. so R is antisymmetic. Let $(x, y), (y, z) \in R, then\ x = 1, y = 1. So\ (x, z) \in R$ and hence R is transitive.
h) R is not reflexive because $(0,0) \notin R$, then $x = 1\ or\ y = 1$ and hence $(y, x) \in R$. So R is symmetric. If $(x, y) \in R$ & $(y, x) \in R$, then this does not mean that $x = y$, so R is not antisymmetric. $(x, y), (y, z) \in R \nRightarrow (x, z) \in R$ because $(2,1), (1,3) \in R$ but $(2,3) \notin R$. hence R is not transitive.

Example: Given an example of a relation on a set that is

(a) both symmetric & antisymmetric (b) neither symmetric nor antisymmetric.

Solution:

a) Let $A = \{1,2,3,4\}$, then define a relation R on A by $R = \{(1,1), (2,2), (3,3)\}$ Then R is symmetric as well as antisymmetric
b) Define R on A by $R = \{(1,3), (2,4), (4,2)\}$ Then R is not symmetric because $(1,3) \in R$ but $(3,1) \notin R$. also R is not antisymmetric because $(2,4), (4,2) \in R$ but $4 \neq 2$.

Example: Let R be the set of real numbers. Define the following relations on R by

$$R_1 = \{(a, b) \in R^2 | a > b\}, \quad R_2 = \{(a, b) \in R^2 | a \geq b\}, \quad R_3 = \{(a, b) \in R^2 | a < b\},$$
$$R_4 = \{(a, b) \in R^2 | a \leq b\}, \quad R_5 = \{(a, b) \in R^2 | a = b\}, \quad R_6 = \{(a, b) \in R^2 | a \neq b\}$$

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Find, $(a) R^1 U R^3 \quad (b) R^1 U R^5 \quad (c) R^2 \cap R^4 \quad (d) R^3 \cap R^5$

$(e)\ R_1 - R_2 \quad (f)\ R_2 - R_1 \quad (g)\ R_1 \oplus R_3 \quad (h) R_2 \oplus R_4$

<u>Solution</u>:

a) $R^1 U R^3 = \{(a,b) \in R^2 | a>b\} U \{(a,b) \in R^2 | a<b\}$
$= \{(a,b) \in R^2 \ | a>b \text{ or } a<b\} = \{(a,b) \ \in R^2 \ | a \neq b\} = R_6$

b) $R_1 U R_5 = \ \{(a,b) \ \in R^2 | a>b\} \ U \ \{(a,b) \ \in R^2 | a = b\} = \{(a,b) \ \in R^2 \ | a \geq b\} = \ R_2$

c) $R^2 \cap R^4 \ = \{(a,b) \in R^2 | a \geq b\} \cap \ \{(a,b) \in R^2 | a \leq b\}$
$= \{(a,b) \ \in R^2 \ | a \geq b \ \cap \ a \leq b\} = \{(a,b) \ \in R^2 \ | \ a = b\} = R_5$

d) $R^3 \cap R^5 = \{(a,b) \ \in R^2 | a<b\} \ \cap \ \{(a,b) \ \in R^2 | a = b\} = \{(a,b) \ \in R^2 \ | a<b \ \cap \ a = b\} = \Phi$

e) $R_1 - R_2 \ = \{(a,b) \ \in R^2 | a>b\} \ - \ \{(a,b) \ \in R^2 | a \geq b\}$
$R_1 \cap R^c{}_2 = \{(a,b) \ \in R^2 | a>b\} \ \cap \ \{(a,b) \ \in R^2 | a \ngeq b\} = \{(a,b) \ \in R^2 | a>b\} \ \cap \ \{(a,b) \ \in R^2 | a<b\}$
$= \{(a,b) \ \in R^2 | a>b \ \cap \ a<b\} = \Phi$

f) $R_2 - R_1 = \ R_2 \cap R^c{}_1 = \{(a,b) \ \in R^2 | a \geq b\} \ \cap \{(a,b) \ \in R^2 | a \leq b\} = \{(a,b) \in R^2 | a>b \ \cap \ a \leq b\}$
$= \ \{(a,b) \ \in R^2 | a = b\} \ = \ R_5$

g) $R_1 \oplus R_3 = R_1 U R_3 \ - \ (R_1 \cap R_3) \ = \ \{(a,b) \ \in R^2 | a>b \text{ or } a<b\} \ - \ \{(a,b) \ \in R^2 | a>b \ \cap \ a<b\}$
$= \{(a,b) \ \in R^2 | a \neq b\}$ - $\Phi \ = \ \{(a,b) \ \in R^2 | a \neq b\} \ = R_6$

h) $R_2 \oplus R_4 = R_2 U R_4 \ - \ R_2 \cap R_4 = \{(a,b) \ \in R^2 | a \geq b \text{ or } a \leq b\} \ - \ \{(a,b) \ \in R^2 | a \geq b \ \cap \ a \leq b\}$
$= \{(a,b) \ \in R^2 | a \geq b \text{ or } a \leq b\} - \{(a,b) \in R^2 | a = b\} \ = \ \{(a,b) \in R^2 | a \geq b \ \cap \ a \leq b\}$
$= \{(a,b) \ \in R^2 | a \neq b\} \ = R_6$

Example: In previous example find

$(a) R_2 \circ R_1 \quad (b) R_2 \circ R_2 \quad (c) R_3 \circ R_5 \quad (d) R_4 \circ R_1$

$(e) R_5 \circ R_3 \quad (f) R_3 \circ R_6 \quad (g) R_4 \circ R_6 \quad (h) R_6 \circ R_6$

<u>Solution</u>:

a) $R_2 \circ R_1 = \{(a,b) \in R^2 | a \geq b\} \circ \{(a,b) \in R^2 | a>b\} = \{(a,c) \in R^2 | \exists\, b \in R^1 : (a,b) \in R^1 \ \& \ (b,c) \in R_2\}$
$= \{(a,c) \ \in R^2 | \exists\, b \in R_1 : \ a>b \text{ and } b \geq c\} = \{(a,c) \ \in R^2 | a>c\} \ = R_1$

b) $R_2 \circ R_2 = \{(a,c) \ \in R^2 | \exists\, b \in R_2 : \ (a,b) \ \in R_2{}^2 \text{ and } (b,c) \ \in R_2{}^2\}$
$= \{(a,c) \ \in R^2 | \exists\, b \in R_2 : \ a \geq b \text{ and } (b \geq c\} = \{(a,c) \ \in R^2 | a \geq c\} = R_2$

c) $R_3 \circ R_5 = \ \{(a,c) \ \in R^2 | \exists\, b \in R_5 : \ (a,b) \ \in R_5 \text{ and } (b,c) \ \in R_3\}$
$= \{(a,c) \ \in R^2 | \exists\, b \in R_5 : \ (a = b) \ \text{ and } (b \geq c)\} = \{(a,c) \ \in R^2 | a<c\} = R_3$

d) $R_4 \circ R_1 = \{(a,c) \ \in R^2 | \exists\, b \in R_1 : \ (a,b) \ \in R_1 \text{ and } (b,c) \ \in R_4\}$
$= \{(a,c) \ \in R^2 | \exists\, b \in R_1 : \ (a>b) \text{ and } (b \geq c)\} \ = \mathbb{R}^2$

e) $R_5 \circ R_3 = \{(a,c) \ \in R^2 | \exists\, b \in R_3 : \ (a,b) \ \in R_3 \text{ and } (b,c) \ \in R_5\}$
$= \{(a,c) \ \in R^2 | \exists\, b \in R_1 : \ (a<b) \text{ and } (b = c)\} \ = \{(a,c) \ \in R^2 | a<c\} = R_3$

f) $R_3 \circ R_6 = \ \{(a,c) \ \in R^2 | \exists\, b \in R_6 : \ (a,b) \ \in R_6 \text{ and } (b,c) \ \in R_3\}$
$= \{(a,c) \ \in R^2 | \exists\, b \in R_5 : \ (a \neq b) \text{and } (b<c)\} \ = \ \mathbb{R}^2$

g) $R_4 \circ R_6 = \{(a,c) \ \in R^2 | \exists\, b \in R_6 : \ (a,b) \ \in R_6 \text{ and } (b,c) \ \in R_4\}$
$= \{(a,c) \ \in R^2 | \exists\, b \in R_1 : \ (a \neq b) \text{and } (b \leq c)\} \ = \mathbb{R}^2$

h) $R_6 \circ R_6 = \{(a,c) \ \in R^2 | \exists\, b \in R_6 : \ (a,b) \ \in R_6 \text{ and } (b,c) \ \in R_6\}$
$= \ \{(a,c) \ \in R^2 | \exists\, b \in R_6 : \ (a \neq b) \text{and } (b \neq c)\} \ = \mathbb{R}^2$

(e.g.(2,2)$\in \mathbb{R}^2$, because we can chose b=2.1)

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Example: Let R be a relation $\{(1,2),(1,3),(2,3),(2,4),(3,1)\}$ and let S be a relation $\{(2,1),(3,1),(3,2),(4,2)\}$ find $S \circ R$

Solution: $S \circ R = \{(1,1),(1,2),(2,1),(2,2)\}$

We can use properties of relation to classify them into different types. We will be considering two important types of relations, equivalence relations and partial orders.

Definition: "*An equivalence relation is a relation that is reflexive, symmetric and transitive.*"

Definition: "*Two elements a and b that were relative by an equivalence relation are called equivalent. The notation* $a \sim b$ *is often used to denote that a and b are equivalent elements with respect to a particular equivalence relation.*"

Example: Let R be a relation on the set of integers such that $aRb \Leftrightarrow a = b \text{ or } a = (-b)$ is an equivalence relation.(check it)

Example: Let R be a relation on the set of real numbers such that $aRb \Leftrightarrow a - b$ is an integer. Is R an equivalence relation.

Solution: aRa $\forall$ real numbers because $a - a = 0$ is an integer, $\forall a. if\ a - b$ is an integer, then so is $b - a$. hence R is symmetric. If $a - b$ and $b - c$ are integers, then $(a - b) + (b - c) = a - c$ is also an integer. i.e if aRb & bRc, then aRc. So R is transitive. Being reflexive, symmetric and transitive. R ia an equivalence relation.

Definition: "*If a and b are integers, then we say that a=b (mod m) if m|a-b. a=b (mod m) is proved as "a is equivalent to b modulo m "or "a is congruent to b modulo m". An equivalent formulation says that a=b(mod m) if a=b+Km for some integer K.*"

Theorem: *Congruence modulo m is an equivalence relation.*

Proof: Let $R = \{(a,b) | a \equiv b(mod\ m)$ Clearly $m|a - a = 0$ So, $a \equiv a(mod\ m)\ i.e\ aRa$. If aRb, then $a = b + Km$ So that $b = a + (-K)m$, which means that $m|b - a$. So, bRa. Hence R is symmetric. Now suppose that $a \equiv b(mod\ m)$ & $b \equiv c(mod\ m)$. Then $m|a - b$ & $m|b - c$. So, $a - b = K_1 m$ and $b - c = K_2 m$. So, $(a - b) + (b - c) = (K_1 + K_2)m$. i.e. $a - c = Km$, where $K = K_1 + K_2$. So, $m|a - c$ i.e. aRa. This process that R is transitive. Hence R is an equivalence relation.

Lemma: If $a \equiv x(mod\ m)$ and $b \equiv y(mod\ m)$, then $(a + b) \equiv (x + y)(mod\ m)$.

Proof: Suppose that $a \equiv x(mod\ m)$ and $b \equiv y(mod\ m)$. Then $m|a - x$ & $m|b - y$. So $m|(a - x) + (b - y)$ i.e. $m|(a + b) - (x + y)$, i.e. $a + b \equiv x + y(mod\ m)$

Example: Suppose that R is a relation on the set of strings of English letters such that $aRb \Leftrightarrow l(a) = l(b)$, where $l(x)$ is the length of stving x. Is R an equivalence relation?

Solution: Since $l(a) = l(b)$, therefore aRa so that R is reflexive. Now suppose that aRb, then $l(a) = l(b)$. Then $l(b) = l(a)$ and we have bRa. So R is symmetric. Suppose that aRb & bRa i.e. $l(a) = l(b)$ and $l(b) = l(c)$). Then $l(a) = l(c)$ meaning that aRc. So R is transitive and hence R is an equivalence relation.

Do example 5 on page 609 of the text book.

Example: Show that that the divides relation on the set of positive integers is not an equivalence relation.

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Solution: We have aRb iff $a|b$. since $a|a\ \forall a$, therefore R is reflexive. Since $2|4$ but $4 \nmid 2$ is not, therefore $aRb \not\Rightarrow bRa$. So R is not symmetric and hence not an equivalence relation.

Example: Let R be the relation on the set of real numbers such that xRy iff x and y differ by less than 1, i.e. $|x-y|<1$. Show that R is not an equivalence relation.

Solution: Since |x-x|=0<1 ∀x, therefore xRx so that R is reflexive. If $|x-y|<1$,then $|y-x|=|x-y|<1$. i.e. if xRy, then yRx. However R is not transitive because if we take $x=2.8, y=1.9$ and $z=1.1$, how $|x-y|=0.9<1$ & $|y-z|=0.8<1$ but $|x-z|=1.7>1$. So xRy & yRz but xRz. So R is not an equivalence relation.

Definition: *"A partition of a set A is a collection of subsets {A_1,A_2,.....A_K} such that any two of them are disjoint (i.e. for any i≠j, $A_i \cap A_j = \Phi$) and such that their union is A."*

Definition: *"Let R be an equivalence relation on a set A. The set of all elements that are selected to an element a of A is called the equivalence class of a. The equivalence class of a with respect to R is denoted by [a]When only one relation is under consideration, we can delete the subscript Rand white [a] for this equivalence class."*

In other words, if R is an equivalence relation on a set A, the equivalence class of the element a is

$$[a]_R = \{s|(a,s) \in R\}$$

If $b \in [a]_R$, then b is called a representative of this equivalence class, any element of a class can be used as a representative of this class. That there is nothing special about this particular element chosen as the representative of the class.

Lemma: Let R be an equivalence relation a set A, then for every $a, b \in A$, either $[a]=[b] or [a] \cap [b] = \Phi$

Proof: Consider some arbitrary $a, b \in A$. if either $[a]=[b]\ or\ [a] \cap [b] = \Phi$, then we are done. So suppose not. Let c be the element that is in one of the sets $[a]$ and $[b]$ but not the other. Without loss of generality, we can assume that $c \in [b]-[a]$. Let $d \in [a] \cap [b]$. then aRd ($\because d \in [a]$). Note that $d=a$ is possible but this is because R is reflexive. Also dRb and bRc($\because c, d \in [b]$ and R is symmetric). This implies by transitivity that dRc. Since we have aRd and dRc , therefore by transitivity, aRc (i.e $c \in [a]$, a contradiction. Hence $d \in [a] \cap [b]$ is wrong. So $[a] \cap [b] = \Phi$. This completes the proof.

Lemma: Any partition { A_1,A_2,.....A_K} of A defines an equivalence relation by letting aRb iff a and b are same A_i

Proof: Reflecting for all a, we know that $a \in A_i$ for some i, by definition of partition , clearly a and b are in the same A_i (i.e aRa).

Symmetry: *Let aRb and bRc, then a and b are in the same A_i. Also b and a are in same A_i and hence bRa.*

Transitivity: *Let aRb and bRc, then a and b are in same A_i foe some i and b and c arevin same A_j for some j, but by definition of partition b can not be in two different A_i's. so it must beA_i=A_j and a and c are in the same A_i proving that aRc.*

Example: Let R be the relation on the set of integers such that aRb iff $a=b\ or\ a=(-b)$. What is the equivalence class of an integer for the equivalence relation R.

Solution: Let $a \in Z$, then$[a]=\{s|(a,s) \in R\}=\{a,-a\}$. This set contain two distinct integers unless $a=0$. For instance $[7]=\{7,-7\}, [-5]=\{-5,5\}$ and $[0]=\{0\}$

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Example: What are the equivalence class of 0 and 1 for the congruence modulo 4?

Solution: $[0] = \{s|\ s = 0(mod\ 4)\} = \{\ all\ integers\ having\ divisor\ 4\} = \{\dots -8, -4,0,4,8, \dots\}$

$[1] = \{s|\ s = 1(mod\ 4)\}\ = \{\ all\ integers\ having\ reminder\ 1, when\ divided\ by\ 4\}$

$= \{\dots -7, -3,1,5,9, \dots\}$

Note: The above example can be generalized by replacing 4 with any positive integer m. The equivalence classes of relation congruence modulo m are called the congruence closed modulo m. the congruence class of an integer m is denoted by $[a]_m$. So

$[a]m = \{\dots a - 2m, a - m, a, a + m, a + 2m \dots\}$

$[0]_4\ =\ \{\dots -8, -4,0,4,8, \dots\}$ $\quad$ $[1]_4\ = \{\dots -7, -3,1,5,9 \dots\}$

Theorem: *Let R be an equivalence relation on a set A. Then statements are equivalent*:

$$(i)\ aRb \qquad (ii)\ [a] = [b] \qquad (iii)\ [a] \cap [b] = \Phi$$

Proof: (i)⇒(ii): Assume that aRb. We have to prove $[a] = [b]$, by showing $[a] \subseteq [b]$ and $[b] \subseteq [a]$. Suppose $c \in [a]$. Then cRa. Since R is symmetric, therefore bRa and aRc. By transitivity of R, bRc. So $c \in\ [b]$, and we have $[a] \subseteq [b]$. By similar argument, we can how that $[b] \subseteq [a]$. Hence $[a] = [b]$.
(ii) ⇒ (iii): Assume that $[a] = [b]$. It follows that $[a] \cap [b] = \Phi$, because $[a]$ is nonempty(because $a \in\ [a]$ because R is reflexive).
(iii)⇒(i): Suppose that $[a] \cap [b] \neq \Phi$, then ∃ an element c such that $c \in\ [a]$ & $c \in\ [b]$. Then aRc and bRc. By symmetric property cRb. Then by transitive, because aRc & cRb, we have aRb. Because (i)⇒(ii), (ii)⇒(iii), (iii)⇒(i), the 3 elements (i), (ii), (iii) are equivalent.

Example: Suppose that $S = \{1,2,3,4,5,6\}$. The collection of sets $A_1 = \{1,2,3\}, A_2 = \{4,5\}, A_3 = \{6\}$, from a partition of S, because these we disjoint and their union is S.

To Show that an equivalence relation partitions a set

Let R be an equivalence relation on a set A. The union of equivalence closed of R is all of A because an element of A is in its own equivalence class, Namely $[a]_R$. In other words

$$\cup_{a \in A}[a]_R = A$$

In addition, we have proved in a previous lemma that equivalence closed are either equal or disjoint. So,

$$[a]_R \cap [b]_R = \Phi \text{ When } [a]_R \neq [b]_R$$

These two observations show that the equivalence classes from a partition of A because they split A into disjoint subset. Conversely, every partition of a set can be used to from an equivalence relation. Two elements are equivalence with respect to this relation if they are in the same subset of the partition.

To see this ,assume that $\{A_i|i \in I\}$ is a partition of S. Let R be the relation on S consisting of the pairs (x, y) such that x & y belong to the same subsets A_i of the partition. To see That R is an equivalence relation, we must show that R is reflexive, symmetric and transitive. Note that $(a, a)\ \in R$ for every $a \in S$, because a is in the same subset as itself. Hence R is reflexive. If $(a, b)\ \in\ R$, then b & a are in same subset of the partition so that $(b, a)\ \in R$. Hence R is symmetric. If $(a, b)\ \in R$ & $(b, c)\ \in R$, then a & b are in the same subset, say X in the partition & b and c are in the same subset, say Y of the partition , because the subsets of partition are disjoint & b belongs to X & Y, its follow that $X = Y$. Conversely a & c belong to the same subset of the partition so that $(a, c)\ \in R$. Thus R is transitive. It follows that R is an equivalence.

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The following theorem summarize the connections we have established b\w equivalence relations and partitions.

Theorem: *Let R be an equivalence relation on a set S. This the equivalence classes of R from a partition of S. Conversely, given a partition* $\{A_i, i \in I\}$ *of the sets S there is an equivalence relation R that has the sets* $A_i, i \in I$ *as the equivalence classes.*

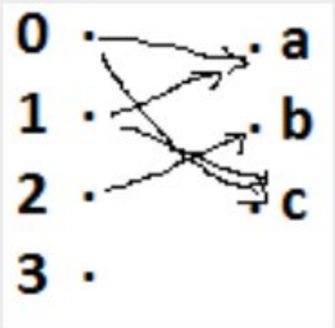
The following example shows how to constant an equivalence relation R from a partition.

Example: List the ordered pair in the equivalence relation R produced by the partition $A_1 = \{1,2,3\},\ A_2 = \{4,5\},\ A_3 = \{6\}\ of\ S = \{1,2,3,4,5,6\}$.

Solution: the subset in the partition are the equivalence classes of R. the pair $(a, b) \in R$ iff a & b are in the same subset of the partition. The pairs (1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3) belong to R because $A_1 = \{1,2,3\}$ is an equivalence. The pairs (4,4), (4,5), (5,4), (5,5) belongs to R because $A_2 = \{4,5\}$ is an equivalence class. Finally, the pair (6,6) belong to R because $A_3 = \{6\}$ is an equivalence class. No pairs other than those listed belongs to R.

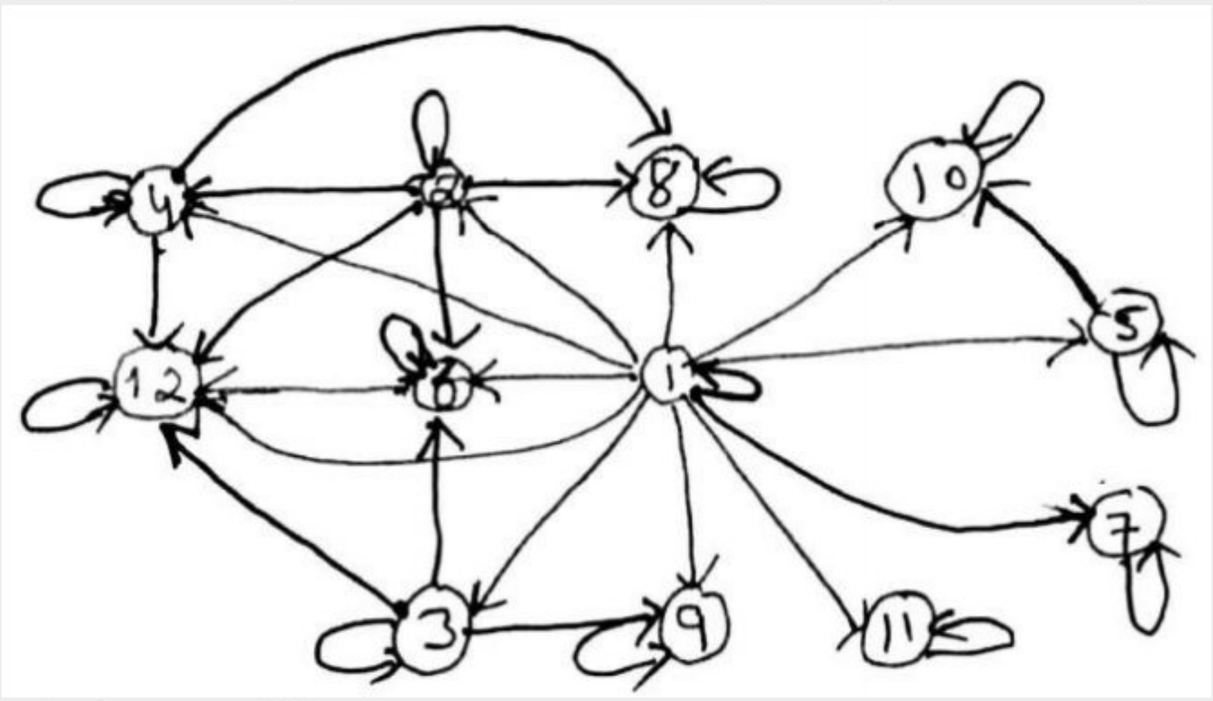
Definition: *"We can draw a picture of relation* $R \subseteq A \times B$ *by drawing a dot for every element of B and an arrow for* $a \in A$ *to* $b \in B$ *iff* aRb*. Such a picture is called a* <u>*directed graph or graph for short*</u>*."*

Example: The relation $\{(0, a), (0, c), (1, c), (2, b), (1, a)\}\ from\ \{0,1,2,3\}\ to\ \{a, b, c\}$ can be represented by digraph as



Diagraph are mainly used for relation where A=B. The diagraph may contain self-loops i.e. an arrow from a dot to itself, associated to the elements a such that (a,a) is in the relation. To represent relations on a set A as a diagraph, we draw a dot (vertex) for each element of A, and draw an arrow from 1st element to 2nd element of each pair in the relation.

Example: the divisibility relation over {1,2…12} is represented by the diagraph as



Reflexivity: All nodes have self-loops.

Symmetry: All edges are bidirectional.

Transitivity: Short-circuits for any sequence of consecutive, there is a single arrow from the 1st to the last node.

Definition: *"If R is a relation as A×B, then* R^{-1} *is a relation on B×A given by* $R^{-1} = \{(b, a) \mid (a, b) \in R\}$

It's just the relation "turned backward".

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Note: Given a graph for R, we get the graph for R^{-1} by reversing every edge in the original graph.

Definition: *"A path in a relation R is a sequence* $a_0, a_1, \dots \dots \dots . a_k$ *, with* $K \geq 0$ *such that* $(a_i + a_{i+1}) \in R$ *for every i<k. we call k the length of the path."*

Remarks: In a diagraph model, a path is something you can trace out by following arrows from vertex to vertex, without lifting your pen. Note that a single vertex is a length 0 path (this is just for convenience). A simple path is a path with no repeated vertices.

Lemma: $R^n = \{(a,b) \mid there\ is\ a\ length\ n\ path\ from\ a\ to\ b\ in\ R\}$

Proof: The proof is by induction. Let P(n): $R^n = \{(a,b) \mid there\ is\ a\ length\ n\ path\ from\ a\ to\ b\ in\ R\}$

<u>To prove P(1) is true:</u> There is exactly one edge from a to b for every $(a,b) \in R$ and there is exactly one pair for every edge from a to b. So $P(1)$ is true.

<u>To prove P(n)→P(n+1):</u> Suppose that P(n) is true. First consider a path that $a_0, \dots\dots a_{n+1}$ in R. This is a path $a_0, \dots . a_n$ in R followed by a pair $(a_n, a_{n+1}) \in R$. Since P(n) is true, we can assume that $(a_0, a_n) \in R^n$ and we have already mentioned that $(a_n, a_{n+1}) \in R$. Therefore $(a_0, a_{n+1}) \in R^{n+1}$ by definition of composition. Then every path of length n+1 corresponds to a relation in R^{n+1}. Now consider a pair (a, b) in R^{n+1}. By definition of composition ,∃ such that $(a, c) \in R^n$ & $(c, b) \in R$. Since P(n) is true(inductive hypothesis), we can assume that (a, c) correspondents to a length in path from a to c and since $(c, b) \in R$, here is an edge from c to b. thus there is a length n+1 path from a to b. Hence P(n)→P(n+1).

Closure

A closure of a relation R with respect to a property P is the relation S that

1. contains R
2. has the property P, and
3. is contained in any relation satisfying 1 & 2

That is, S is the smallest relation satisfying 1 & 2

Remarks: There are two ways to construct a closure of R with respect to property P:

1. We can start with R and add a few pairs as possible until the new relation has property P.
2. We can start with the largest possible relation (which is A×A for a relation on A) and remove as many not-m-R pairs as possible while preserving the property P.

The reflexive closure:

The relation $R = \{(1,1), (1,2), (2,1), (3,2)\}$ on the set $A = \{1,2,3\}$ is not reflexive. How can we produce a reflexive relation containing R that ia as small as possible? This can be done by adding (2,2) and (3,3) to R because those are the only pairs of the form (a, a) that are not in R. Clearly this is new relation (1) contains R, (2) is reflexive and (3) is contained in any relation satisfying (1) & (2) . Hence by definition this new relation $S = \{(1,1), (1,2), (2,1), (2,3)\}$.

Lemma: Let R be a relation on the set A. the reflexive closure of R is $S = R \cup \{(a,a) \mid \forall\, a \in A\}$.

Proof: S (1) contains R and (2) is reflexive by design. Furthermore, any relation satisfying both (1) & (2) must contain S. Hence S is reflexive closure of R by definition.

Example: Let $R = \{(a,b), (a,c), (b,d), (d,e)\}$ on $\{a,b,c,d,e\}$. Then the reflexive closure of R is

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$$= \{(a,b),(a,c),(b,d),(d,e),(a,a),(b,b),(c,c),(d,d),(e,e)\}$$

Example: What is the reflexive closure of the relation $R = \{(a,b) | a < b\}$ on the set of integers?

Solution: The reflexive closure of R is

$$S = R \cup \{(a,a) | a \in Z\} = \{(a,b) | a < b\} \cup \{(a,a) | a \in Z\} = \{(a,b) | a \leq b\}$$

Symmetric closure:

Lemma: Let R be a relation on set A, the symmetric closure of R is $S = R \cup R^{-1}$
Proof: The relation $S = R \cup R^{-1}$a)contains R (b)symmetric. It is also smallest such relation. To see thus suppose we have some symmetric relation T with R ⊆T. Consider (a, b) ∈R, then (a, b) ∈T so that (b, a) ∈T (by symmetry). It follows that $R^{-1} \subseteq T. So, S = R \cup R^{-1} \subseteq T$. Hence, S is smallest symmetric relation containing R. Hence S is symmetric closure of R.

Example: Let $R = \{(a,b),(a,c),(b,d),(d,e)\}$ be a relation. Then symmetric closure of R is

$$S = R \cup R^{-1} = \{(a,b),(a,c),(b,d),(d,e)\}\, U\, \{(b,a)(c,a),(d,b),(e,d)\}$$

$$= \{(a,b),(a,c),(b,d),(d,e),(b,a)(c,a),(d,b),(e,d)\}$$

Example: What is symmetric closure of relation $R = \{(a,b) | a > b\}$ on set of positive integers ?

Solution: The symmetric closure of R is relation $S = RUR^{-1} = \{(a,b) | a > b\} \cup \{(b,a) | a < b\} = \{(a,b) | a \neq b\}$

Transitive Closure:

Let R be a relation on set A the connectivity relation R* consists of pairs (a, b) such that there is path of length at least one from a to b.

Note: Because R^n consists of pairs (a, b) such that there is path of length n from a to b. It follows that R* is union of all sets R^n i.e. $R^* = \bigcup_{n=1}^{\infty} R^n$.

Example: Let R be a relation on set of all people in world that contains (a, b) if a has met b. What is R^n where n is a positive integer greater than one? What is R*?

Solution: The relation R^2 contains (a, b) if there is a person c such that(a, c) ∈ R and (c, b) ∈ R, that is, if there is a person c such that a has met c and c has met b. Similarly, Rn consists of those pairs (a, b) such that there are people $x_1, x_2, \ldots, x_{n-1}$ such that a has met x_1, x_1 has met $x_2, \ldots, and\ x_{n-1}$ has met b. The relation R∗ contains (a, b) if there is a sequence of people, starting with a and ending with b, such that each person in the sequence has met the next person in the sequence.

Example: Let R be the relation on the set of all subway stops in New York City that contains (a, b) if it is possible to travel from stop a to stop b without changing trains. What is R^n when n is a positive integer? What is R^*?

Solution: The relation R^n contains (a, b) if it is possible to travel from stop a to stop b by making at most n−1 changes of trains. The relation R^* consists of the ordered pairs (a, b) where it is possible to travel from stop a to stop b making as many changes of trains as necessary.

Example: Let R be the relation on the set of all states in the United States that contains (a, b) if state a and state b have a common border. What is R^n, where n is a positive integer? What is R^*?

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Solution: The relation R^n consists of the pairs (a, b), where it is possible to go from state a to state b by crossing exactly n state borders. R^* consists of the ordered pairs (a, b), where it is possible to go from state a to state b crossing as many borders as necessary.

Following theorem shows that transitive closure of relation and associated connectivity relation are same.

Theorem: *The transitive closure of a relation R equals the connectivity relation R^*.*

Proof: First we need to show that R^* is smallest transitive relation that contains R. i.e. we want to show the following

$R \subseteq R^*$ $\quad$ R^* is transitive $\quad$ If S is any other transitive relation containing R, then $R^* \subseteq S$

$R \subseteq R^*$: By definition of R^*

If $(a, b) \in R^*$ and $(b, c) \in R^*$ then there are path from a to b and from b to c in R. We obtain a path from a to c by starting with path from a to b and following it with path from b to c. Hence $(a, c) \in R^*$. It follows that R^* is transitive.

Consider transitive relation S containing R, then $R^* \subseteq S$. Assume for contradiction that $R^* \not\subset S$. This means that some pairs $(a, b) \in R^*$ but $(a, b) \notin S$. In other words a path $a_0, \ldots\ldots, a_k = S$ in R where $k \geq 1$ such that $(a_0, a_k) \notin S$. Call this a missing path. Set M be the set of missing paths. Then $M \neq \varphi$. Let $s_0, \ldots\ldots, s_m$ be the shortest missing path.

Case 1: If m=1 then s_0, s_1 is path in R so that $(s_0, s_1) \in R \subseteq S$. Thus $(s_0, s_1) \in S$, a contradiction

Case 2: If m >1 then $s_0 \ldots\ldots, s_{m-1}$ is path in R(m-1>0) But it is shorter than our original shortest missing path. So cannot be missing. Thus $(s_0, s_{m-1}) \in S$. Also we have $(s_{m-1}, s_m) \in S$. Since $R \subseteq S$. Thus by transitivity of S, $(s_0, s_m) \in S$ a contradiction. We get a contradiction either way. So, our a and s assumptions $R^* \not\subset S$ is false. This completes the proof.

Zero-one matrices:

All elements of zero-one matrix are either 0 or 1 representing false and true respectively.

Definition: "*The join of A, B (both m×n zero-one matrices) is*

$$A \vee B = [a_{ij} \vee b_{ij}]$$

And meet of A,B is $A \wedge B = [a_{ij} \wedge b_{ij}] = [a_{ij}\ b_{ij}]$"

Example: Let $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$

Then $A \vee B = \begin{bmatrix} 1 \vee 0 & 0 \vee 1 & 1 \vee 0 \\ 0 \vee 1 & 1 \vee 1 & 0 \vee 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ $\quad$ $A \wedge B = \begin{bmatrix} 1 \wedge 0 & 0 \wedge 1 & 1 \wedge 0 \\ 0 \wedge 1 & 1 \wedge 1 & 0 \wedge 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

Boolean Product of zero-one matrices:

The Boolean product of matrices A and B is like normal matrix multiplication but using $\vee$ instead of + and using $\wedge$ instead of ×.

Example: Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ Then

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$$A \circledcirc B = \begin{bmatrix} (1 \wedge 1) \vee (0 \wedge 0) & (1 \wedge 1) \vee (0 \wedge 1) & (1 \wedge 0) \vee (0 \wedge 1) \\ (0 \wedge 1) \vee (1 \wedge 0) & (0 \wedge 1) \vee (1 \wedge 1) & (0 \wedge 0) \vee (1 \wedge 1) \\ (1 \wedge 1) \vee (0 \wedge 0) & (1 \wedge 1) \vee (0 \wedge 1) & (1 \wedge 0) \vee (0 \wedge 1) \end{bmatrix}$$

$$= \begin{bmatrix} 1 \vee 0 & 1 \vee 0 & 0 \vee 0 \\ 0 \vee 0 & 0 \vee 1 & 0 \vee 1 \\ 1 \vee 0 & 1 \vee 0 & 0 \vee 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

We know that transitive closure of relation R is equal to its connectivity relation $R^* = \bigcup_{n=1}^{\infty} R^n$. But there is a problem. There is infinity many terms. The following lemma shows that it is sufficient to consider paths of length of most n, namely $R \cup R^2 \cup \ldots\ldots \cup R^n$, where n is number of elements in set.

Lemma: Suppose that A has n elements and that R is relation on A. Let a and b be elements of A. Suppose that there is path from a to b in R. Then there is path of length at most n from a to b.

Proof: Consider the shortest path $a = a_0, \ldots\ldots, a_k$ from a to b. Suppose k>n. Then some elements appears twice in this list ($\therefore$ With more than n list entries, one must be repeat). This means that path is at some point circling back to where it was before. We can cut out this cycle from path and get a shorter path. So we cannot have k >n. So there is path of length at most n from a to b.

Remark: From above lemma it follows that we do not need infinitely many terms. It is enough to take paths of length at most n namely $R \cup R^2 \cup \ldots\ldots \cup R^n$, so transitive closure of R is $R^* = R \cup R^2 \cup \ldots\ldots \cup R^n$, because there is path in R* between two vertices iff there is path between these vertices in R^i, for some positive integer i with i ≤n.

Theorem: Let M_R be the zero–one matrix of the relation R on a set with n elements. Then the zero–one matrix of the transitive closure R^* is

$$M_{R*} = M_R \vee M_R^{[2]} \vee M_R^{[3]} \vee \cdots \vee M_R^{[n]}$$

Example: Find the zero–one matrix of the transitive closure of the relation R where $M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$

<u>Solution</u>: $M_{R*} = M_R \vee M^{[2]}{}_R \vee M^{[3]}{}_R$, where $M_R^{[2]} = M_R \circledcirc M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \circledcirc \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$

$$= \begin{bmatrix} (1 \wedge 1) \vee (0 \wedge 0) \vee (1 \wedge 1) & (1 \wedge 0) \vee (0 \wedge 1) \vee (1 \wedge 1) & (1 \wedge 1) \vee (0 \wedge 1) \vee (1 \wedge 0) \\ (0 \wedge 1) \vee (1 \wedge 0) \vee (0 \wedge 1) & (0 \wedge 0) \vee (1 \wedge 1) \vee (0 \wedge 1) & (0 \wedge 1) \vee (1 \wedge 0) \vee (0 \wedge 0) \\ (1 \wedge 1) \vee (1 \wedge 0) \vee (0 \wedge 1) & (1 \wedge 0) \vee (1 \wedge 1) \vee (0 \wedge 1) & (1 \wedge 1) \vee (1 \wedge 0) \vee (0 \wedge 0) \end{bmatrix}$$

$$= \begin{bmatrix} 1 \vee 0 \vee 1 & 0 \vee 0 \vee 1 & 1 \vee 0 \vee 0 \\ 0 \vee 0 \vee 0 & 0 \vee 1 \vee 0 & 0 \vee 0 \vee 0 \\ 1 \vee 0 \vee 1 & 0 \vee 1 \vee 0 & 1 \vee 0 \vee 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$M^{[3]}{}_R = M_R^{[2]} \circledcirc M_R = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \circledcirc \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 \vee 0 \vee 1 & 0 \vee 1 \vee 1 & 1 \vee 0 \vee 0 \\ 0 \vee 0 \vee 0 & 0 \vee 1 \vee 0 & 0 \vee 0 \vee 0 \\ 1 \vee 0 \vee 1 & 0 \vee 1 \vee 1 & 1 \vee 0 \vee 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$M_R{}^* = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \vee \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \vee \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

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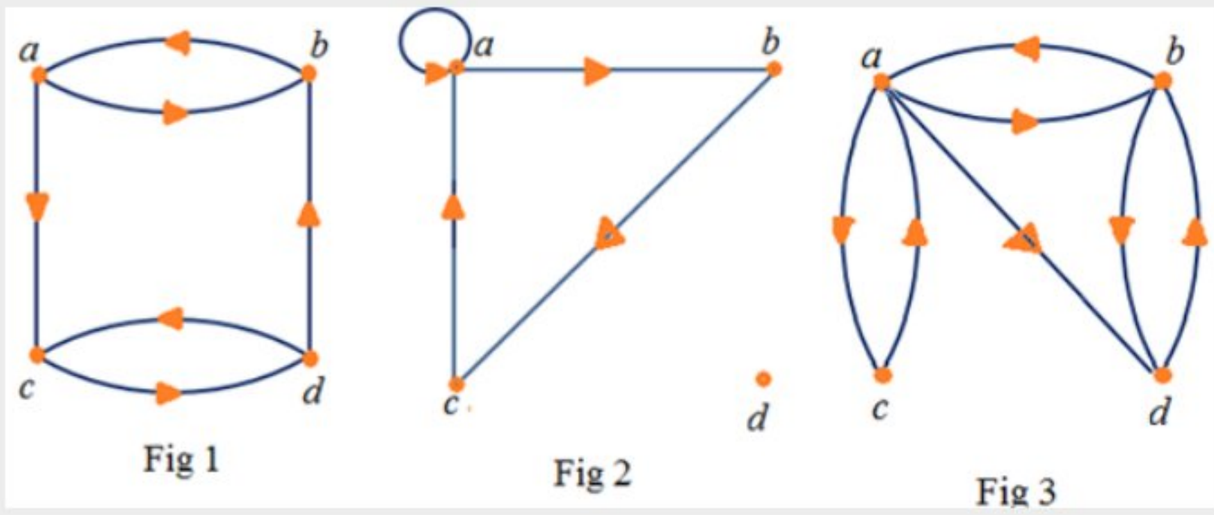
Procedure for computing transitive closure:

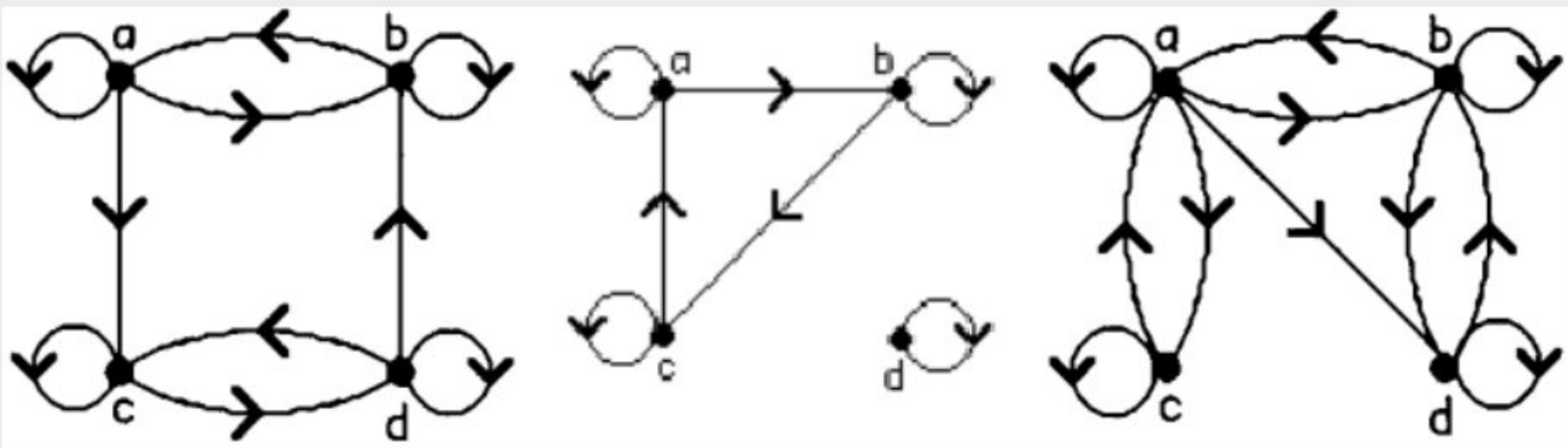
Find $M_{R^{*}*}$, the successive Boolean powers of M_R up to nth power are computed. As each power is calculated, its join with the join of all smaller powers is formed. When this is done with nth power, the matrix for R^* has been computed. This procedure is displayed as follows:

Algorithm 1: Procedure transitive closure (M_R: Zero-one n × n matrix)
A:= M_R , B:=A , (for i := 2 to n)
A := A ⓪ M_R, B := B ∨ A
return B{B is the zero–one matrix for R*}

Question: Draw derived graph of reflexive closure of relation with following directed graphs.



Solution:



Question: Use algorithm 1 to find transitive closure of relation $\{(2,1), (2,3), (3,1), (3,4), (4,1), (4,3)\}$ on set $\{1,2,3,4\}$

Solution: Algorithm 1 find transitive closure by computing successive powers of M_R and taking there join. We have

$$M_R = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \qquad M_R^{[2]} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} ⓪ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

$$M^{[3]}_R = M_R^{[2]} ⓪ M_R = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} ⓪ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

$$M^{[4]}_R = M_R^{[3]} ⓪ M_R = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} ⓪ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

$$M_{R^*} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \vee \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \vee \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \vee \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

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Warshall's Algorithm:

Warshall's algorithm named after Stephen Warshall is an efficient method for computing the transitive closure of relation. Suppose that R is relation on set with n elements. Let $v_1, v_2, \ldots, v_n$ be an arbitrary listing of these n elements. The concept of **interior vertices** of path is used in Warshall's algorithm. If a, $x_1, x_2, \ldots, x_{m-1}$, b is a path, its interior vertices are $x_1, x_2, \ldots, x_{m-1}$, that is, all the vertices of the path that occur somewhere other than as the first and last vertices in the path. For instance, the interior vertices of a path a, c, d, f, g, h, b, j in a directed graph are c, d, f, g, h, and b. The interior vertices of a, c, d, a, f, b are c, d, a, and f. (Note that the first vertex in the path is not an interior vertex unless it is visited again by the path, except as the last vertex. Similarly, the last vertex in the path is not an interior vertex unless it was visited previously by the path, except as the first vertex.) Warshall's algorithm is based on the construction of a sequence of zero–one matrix. These matrices are $W_0, W_1, \ldots, W_n$, where $W_0 = M_R$ is the zero–one matrix of this relation, and $W_k = [w_{ij}^{(k)}]$, where $w_{ij}^{(k)} = 1$ if there is a path from v_i to v_j such that all the interior vertices of this path are in the set $\{v_1, v_2, \ldots, v_n\}$ (the first k vertices in the list) and is 0 otherwise. Note that $W_n = M_{R^*}$ because the (i, j)th entry of M_{R^*} is 1 if and only if there is a path from v_i to v_j, with all interior vertices in the set $\{v_1, v_2, \ldots, v_n\}$ (but these are the only vertices in the directed graph). In short on kth interaction, the algorithm determines if a path exists between two vertices v_i and v_j. Using the j not vertices among v_1, to v_k, allowed as intermediate i.e.

$$w^{[k]}_{ij,} = \begin{cases} w_{ij}^{[k-1]} & \text{path just using } v_1, \ldots\ldots, v_{k-1} \\ w_{ik}^{[k-1]} \wedge w_{kj}^{k-1} & \text{path from } v_i \text{ to } v_k \textit{ and from } v_k \textit{ to } v_j \textit{ using just } v_1, \ldots\ldots, v_{k-1} \end{cases}$$

Warshall's algorithm (matrix generation):

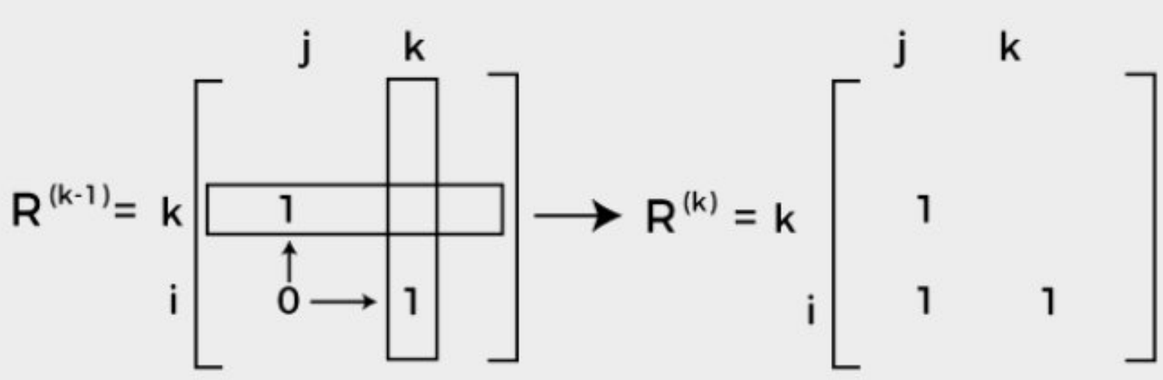
The recurrence relation relating elements $w^{[k]}_{ij,}$ to $w^{[k-1]}_{ij,}$ is

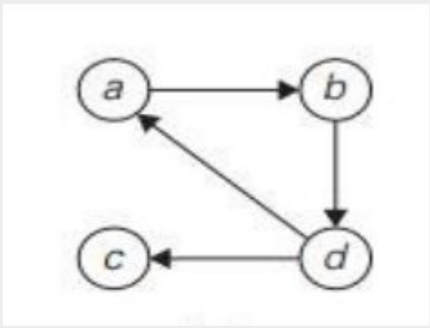
$$w^{[k]}_{ij,} = w^{[k-1]}_{ij,} \text{ or } w_{ik}^{[k-1]} \wedge w_{kj}^{[k-1]},$$

it implies the following rules for generating w_k from w_{k-1}

Rule 1. if an element in row I and column j is zero in w_{k-1}, It remains 1 in w_k.

Rule 2; if an element in row I and column j is zero in w_{k-1}, it has to be changed to 1 in w_k, if and only if the elements in row I and column j are both 1's in w_{k-1}





Example: Use Warshall's algorithm to find transitive closure of relation R with following directed graph.

<u>Solution:</u> $w_0 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}, w_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}, w_2 = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix},$

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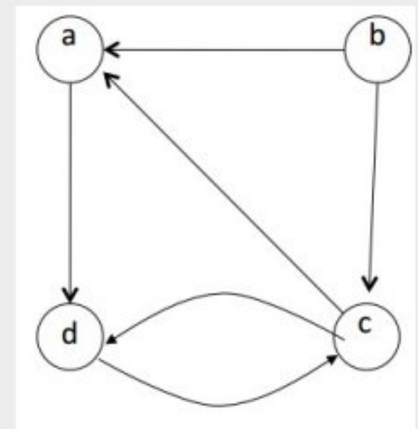
$$w_3 = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}, w_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} = M_{R^*} \text{ (transitive closure of R)}$$

Example: Use Warshall's algorithm to find transitive closure of relation R with following directed graph.

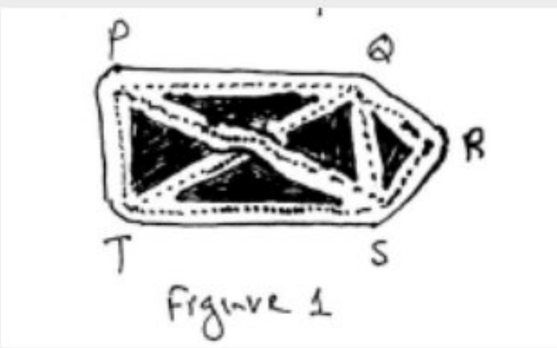


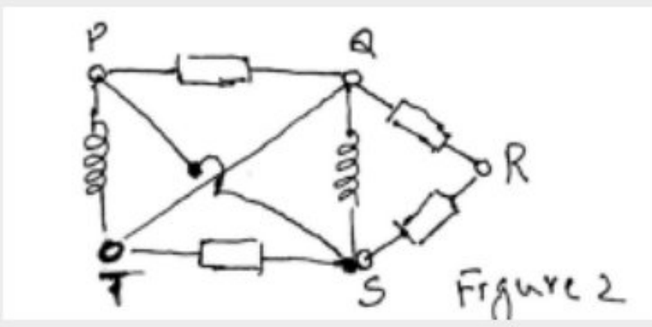
<u>Solution:</u> $w_0 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, w_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

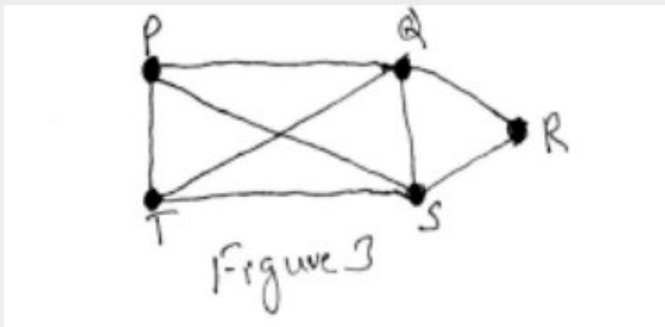
$$w_2 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, w_3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}, w_4 = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix} = M_{R^*} \text{ (transitive closure of R)}$$

what is graph?

We begin by considering fig 1 which depicts part of a road map. And figure 2 which depicts part of an electric network Either of above situations can be represented diagrammatically by means of points and lines as in fig 3 .

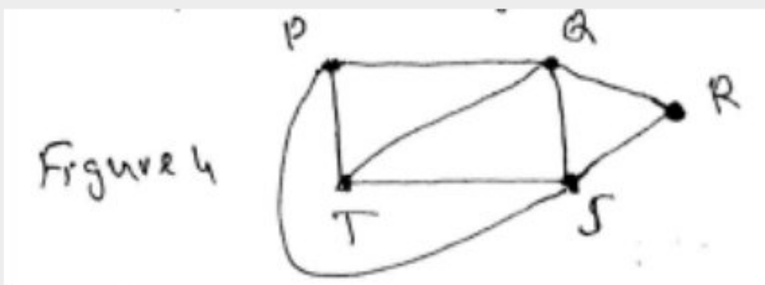




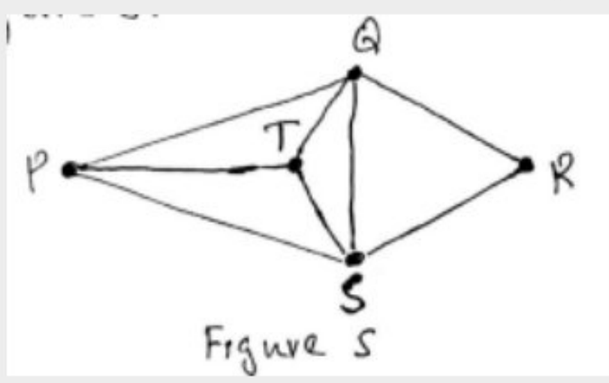


The points P, Q, R, S, T are called <u>vertices</u> and lines are called <u>edges</u> and whole diagram is called <u>graph</u>. Note that intersection of lines PS and PQ is not a vertex since it does not correspond to cross-roads or to meeting of two wires. The degree of vertex is number of edges with that vertex as an end point it corresponds to number of roads at an intersection in fig 1. For example, the degree of vertex Q is 4.

The graph in fig3 can also represent other situations. For example, if P, Q, R, S, T represents football teams, then existence of an edge might correspond to playing of game between the teams at its end points. Thus, in fig 3, team P has played against teams, team Q, S, T but not against team R. In this representation, the degree of a vertex is number of games played by corresponding team. Another way of depicting these situations is by graph in fig 4.



Here we have removed the crossing of lines PS and QT by drawing line PS outside the rectangle PQST. The resulting graph still tells us whether there is a direct road from one intersection to another, how the electric network is wired up and which football teams have played which. The only information we have lost concerns, material, properties such as length of a road and straightness of wire. Thus, graph is representation of set of points and of how they are joined up and any metrical properties are irrelevant. From this point of view, any graph that represent the same situation such as those of fig 3 and fig 4 are regarded as same graph.



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More generally, two graphs are same if two vertices are joined by an edge in one graph iff the corresponding vertices are joined by an edge in other. Awe need another graph that is same as graphs in figure 3 and figure 4 is shown in fig 5.

Here all the ideas of space and distance have gone although we can still tell a glance which points are joined by a road or a wire.

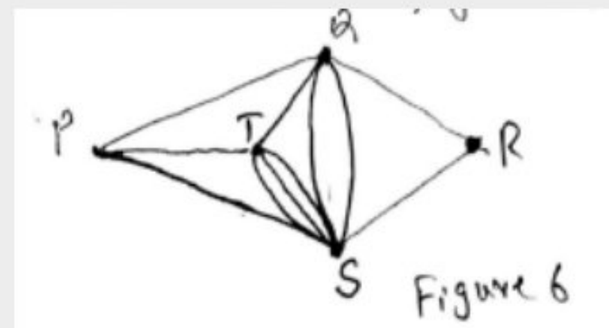

Figure 6

In the above graph there is almost one edge joining each pair of vertices. Suppose now that the road joining Q and S and S and T have too much traffic to carry. Then the situation is eased by building extra roads and the resulting diagram looks like fig 6.

The edges joining Q and S or S and T are called **MULTIPLE EDGES.** If in addition, we need a car park at P, then we indicate this by drawing an edge from P to itself, called a loop (see fig 7).

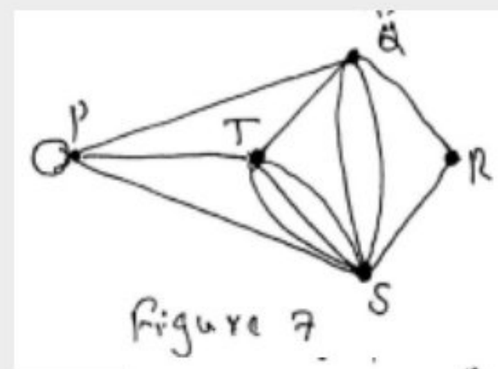

Figure 7

A graph may contain loops and multiple edges. Graphs with no loops or multiple edges such as graph in fig 5, are called **SIMPLE GRAPHS**.

The study of **DIRECTED GRAPH** (or digraphs as we abbreviate them) arises from making the roads into one-way streets. An example of digraph is shown in fig 8

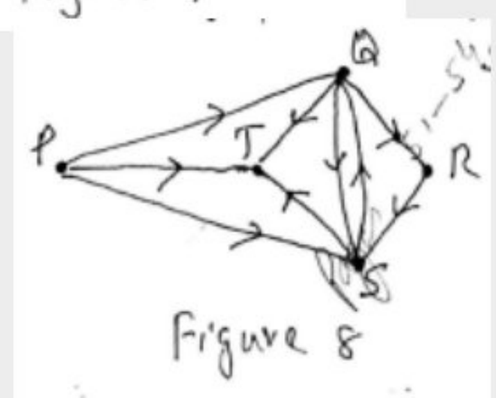

Figure 8

Definition; A **WALK** is a way of getting from one vertex to another and consists of a sequence of edges one following after another. For example, in the following figure P →Q→ R is a walk of length 2, P→ S→Q→T→S→R is walk of length 5

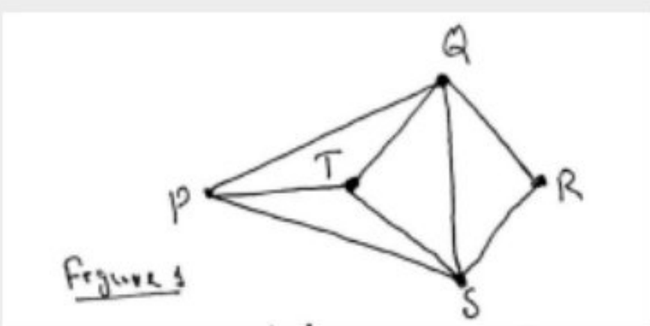

Figure 1

A walk in which no vertex appears more than once is called a **PATH.** For example, P→T→S→R is a path. A walk of form Q→S→T→Q is called a **CYCLE.** Some groups are in two or more paths. For example, consider graph where vertices are stations of London underground and New York Sub Wang and whose edges are lines joining them.

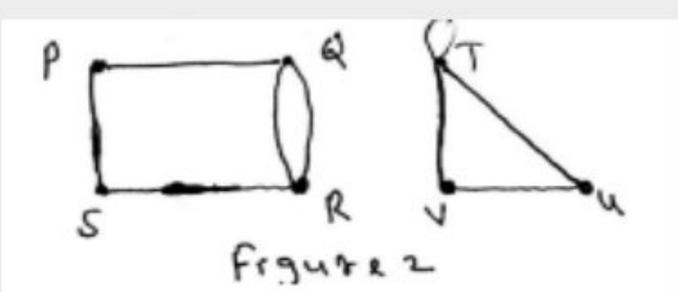

Figure 2

A graph that is in one piece, so that any two vertices are connected by path, is a **CONNECTED GRAPH.**A graph in more than one piece is a **DISCONNECTED GRAPH**. (see fig2). Now we lay foundations of a proper study of graph theory. We first formalize some of basic definitions of previous lecture.

Definition; A **SIMPLE GRAPH G** consists of non-empty finite set V(G) of elements called vertices (or nodes) and a finite set E(G) of distinct unordered pairs of distinct elements of V(G) called **EDGES.** We call V(G) the VERTEX SET and E(G) the EDGE SET of G. An edge $\{v, w\}$is said to join the vertices v and w and is usually abbreviated to vw. For example, fig 3 represents the simple graph G whose vertex set V(G) is {u, v, w, z} and whose edge set E(G) consists of edges uv, uw, vw, wz.

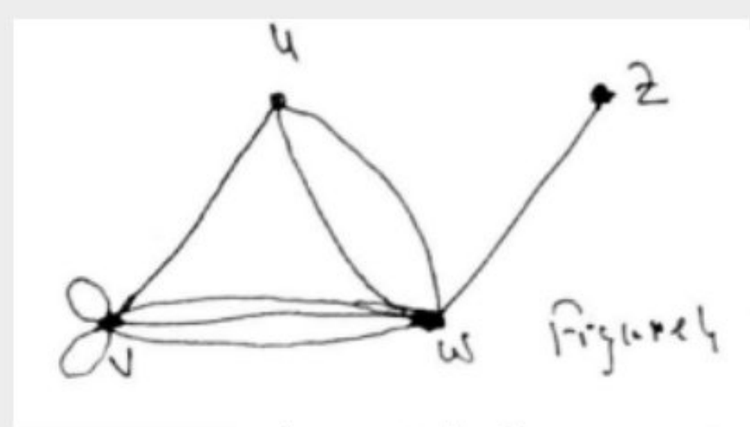

Figure 3

Remark; In any simple graph there is almost one edge joining a given pair of vertices. We may remove the restriction that an edge joins two distinct vertices, and allow loops-edges joining a vertex itself. The resulting object in which loops and multiple edges are

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allowed is called a **GENERAL GRAPH** or simply a graph. Thus, every simple graph is a graph but not every graph is a simple graph.

Definition; Two graphs G_1 and G_2 are **isomorphic** if there is one-to-one correspondence between vertices of G_1 and those of G_2 s.t the number of edges joining any two vertices of G_1 is equal to number of edges joining corresponding vertices of G_2. Thus, the two graphs in fig 5 are isomorphic under the correspondence u ↔l, v ↔m, w ↔n, x ↔p, y ↔q, z ↔r

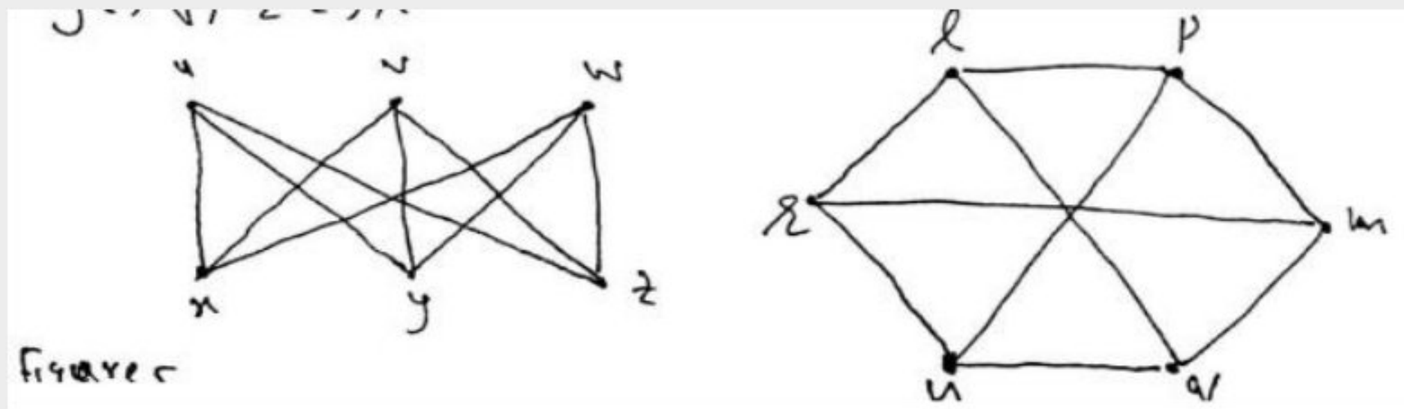

Figure 5

For many problems, the labels on the vertices are unnecessary and we drop them. We then say that two unlabeled graphs are isomorphic if we can assign labels so that the resulting labelled graphs are isomorphic.

For example, the unlabeled graphs in fig 6 are isomorphic since we can label the vertices as in fig 5

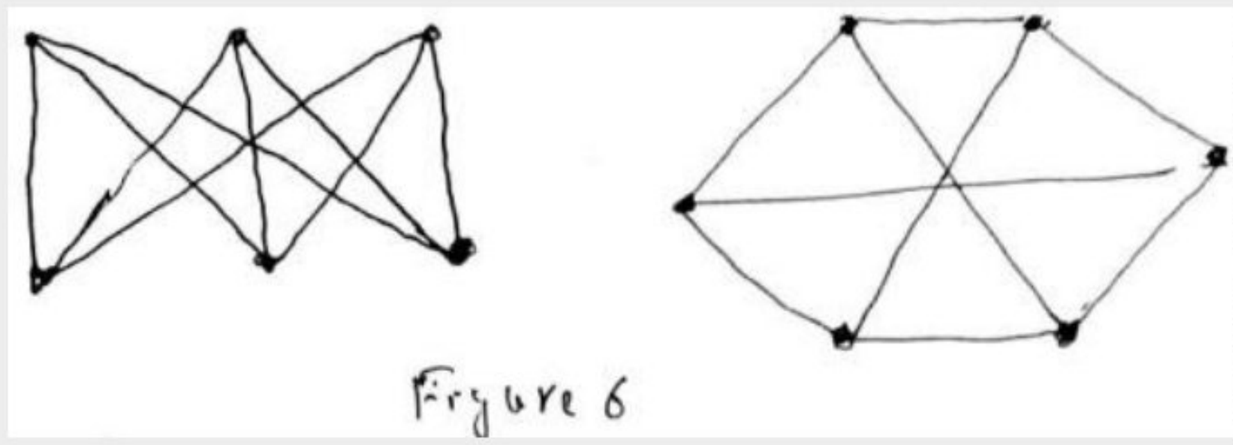
Figure 6

The difference between labeled and unlabeled graphs becomes more apparent when we try to count them. For example, if we restrict ourselves to graphs with 3 vertices, then there are up to isomorphism 8 different labeled graphs

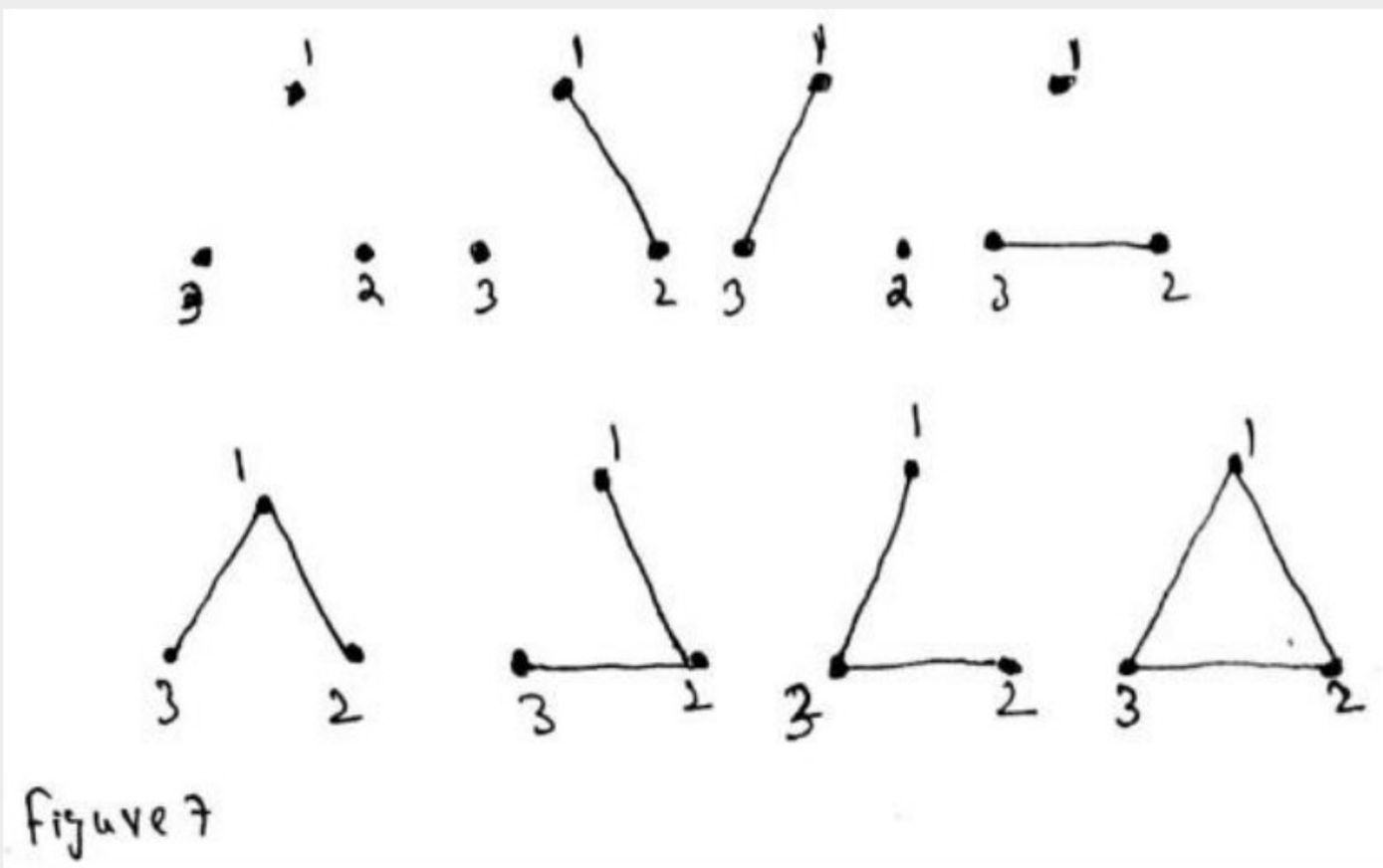

Figure 7

But only for unlabeled

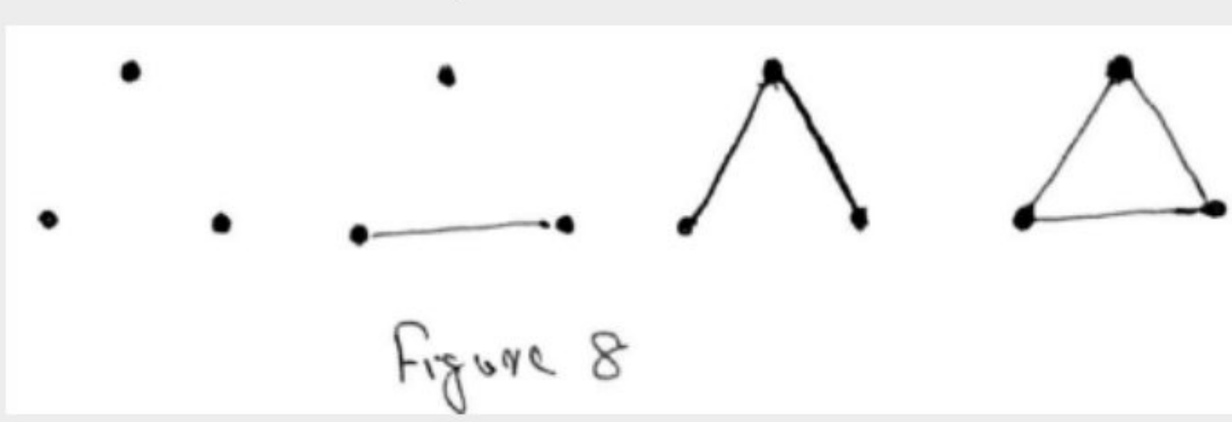
Figure 8

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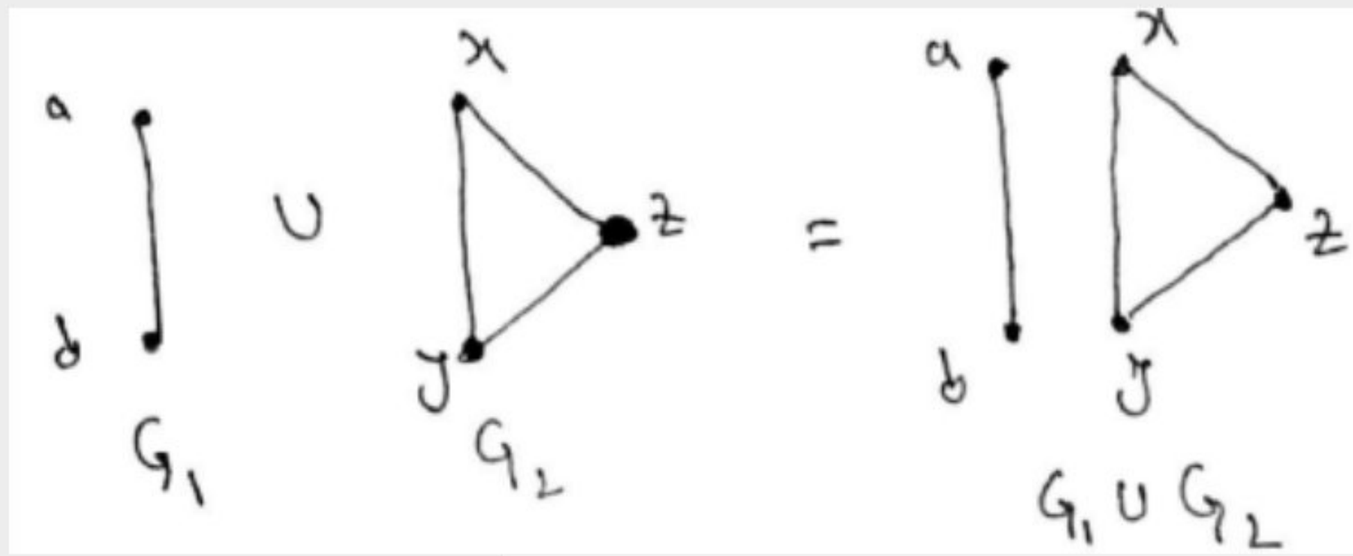
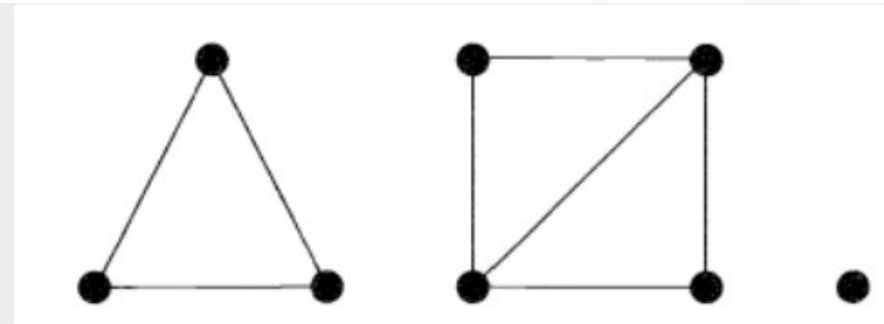
Connectedness:

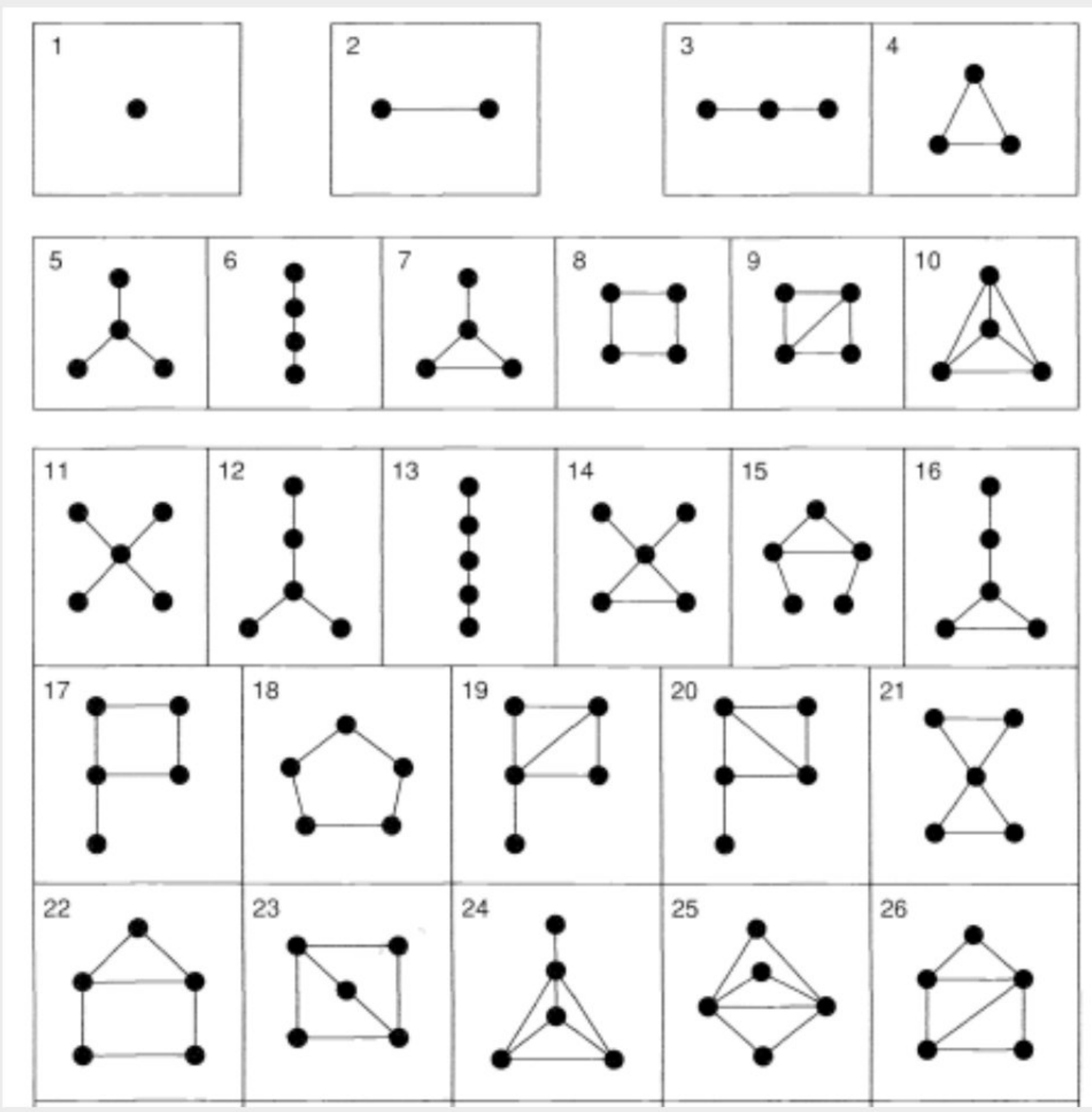
We can combine two graphs to make a larger graph. If two graphs are $G_1=(V(G_1), E(G_1))$ and $G_2=(V(G_2), E(G_2))$, where $V(G_1)$ and $V(G_2)$ are disjoint, then their union $G_1 \cup G_2$ is the graph with vertex set $V(G_1) \cup V(G_2)$ and edge family $E(G_1) \cup E(G_2)$.

A group is connected if it cannot be expressed as union of two groups, and disconnected otherwise. Clearly any disconnected graph G can be expressed as union of connected graphs each of which is a component of G. For example, a graph with three components is



When proving results about graphs in general, we can often obtain the corresponding results for connected graphs and then apply them to each component separately. A table of all the connected unlabeled graphs with up to five vertices is given in Fig.





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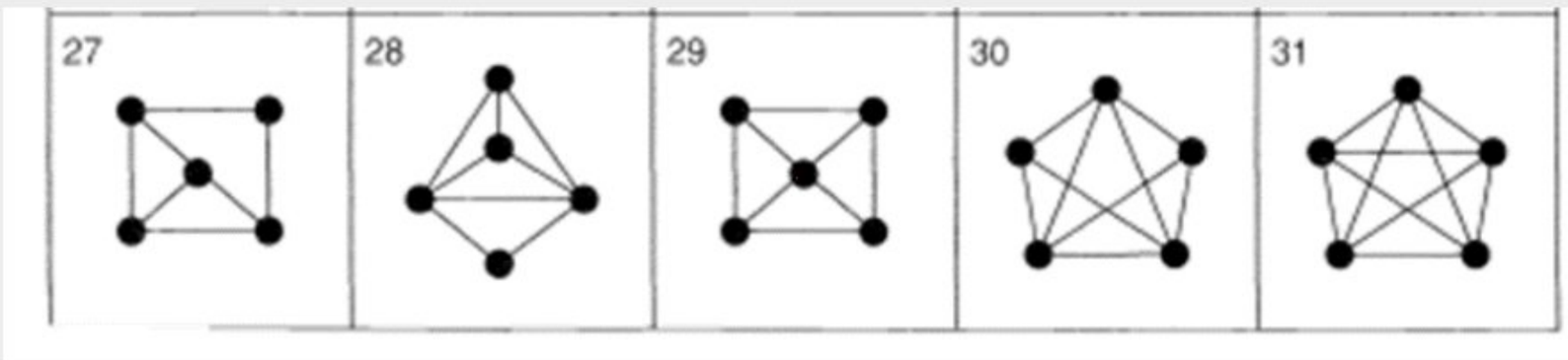
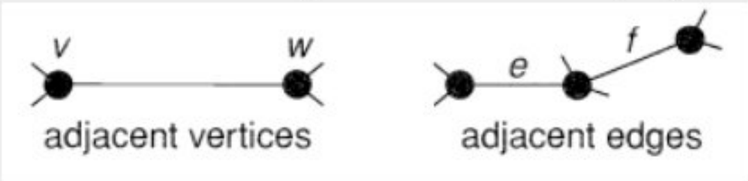
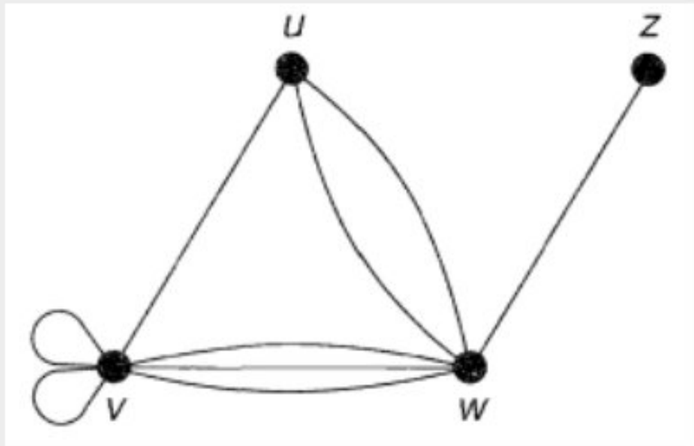


Adjacency

We say that two vertices v and w of a graph G are adjacent if there is an edge vw joining them, and the vertices v and w are then incident with such an edge. Similarly, two distinct edges e and f are adjacent if they have a vertex in common (see Fig.) The degree of a vertex v of G is the number of edges incident with v, and is written deg(v); in calculating the degree of v, we usually make the convention that a loop at v contributes 2 (rather than 1) to the degree of v. A vertex of degree 0 is an isolated vertex and a vertex of degree 1 is an end-vertex. Thus each of the two graphs in Fig .11 has two end-vertices and three vertices of degree 2, while the graph in Fig. 12 has one end-vertex, one vertex of degree 3, one of degree 6 and one of degree 8. The degree sequence of a graph consists of the degrees written in increasing order, with repeats where necessary. For example, the degree sequences of the graphs in Figs. 11 and 12 are (1, 1, 2, 2, 2) and (1, 3, 6, 8).

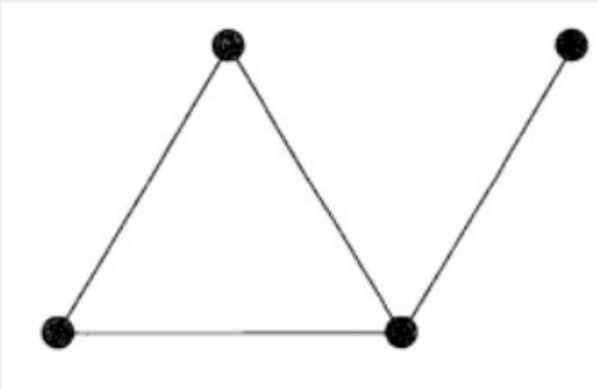


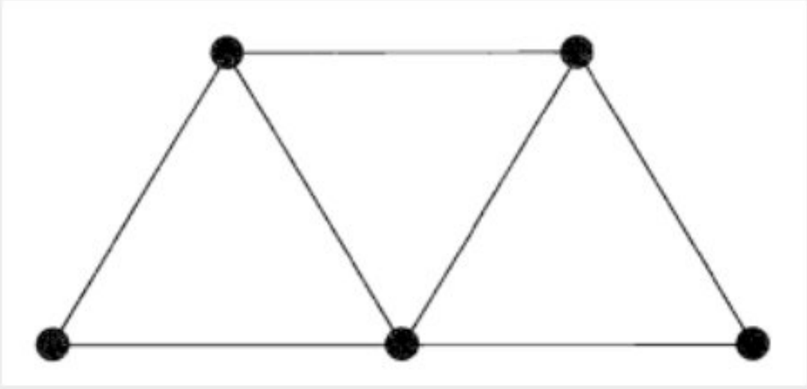


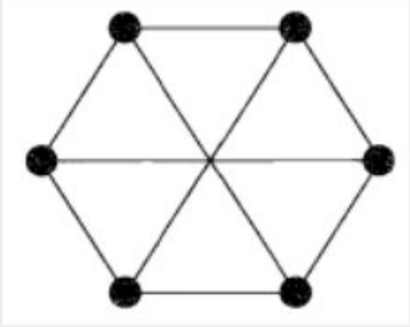
Note that in any graph the sum of all the vertex-degrees is an even number - in fact, twice the number of edges, since each edge contributes exactly 2 to the sum. This result, due essentially to Leonhard Euler in 1736, is called the handshaking lemma. It implies that if several people shake hands, then the total number of hands shaken must be even - precisely because just two hands are involved in each handshake. An immediate corollary of the handshaking lemma is that in any graph the number of vertices of odd degree is even.

Subgraphs

A subgraph of a graph G is a graph, each of whose vertices belongs to V(G) and each of whose edges belongs to E(G). Thus the graph in Fig. 13 is a subgraph of the graph in Fig. 14, but is not a subgraph of the graph in Fig. 15, since the latter graph contains no 'triangle'







We can obtain subgraphs of a graph by deleting edges and vertices. If e is an edge of a graph G, we denote by G - e the graph obtained from G by deleting the edge e. More generally, if F is any set of edges in G, we denote by G - F the graph obtained by deleting the edges in F. Similarly, if v is a vertex of G, we denote by G - v the graph obtained from G by deleting the vertex v together with the edges incident with

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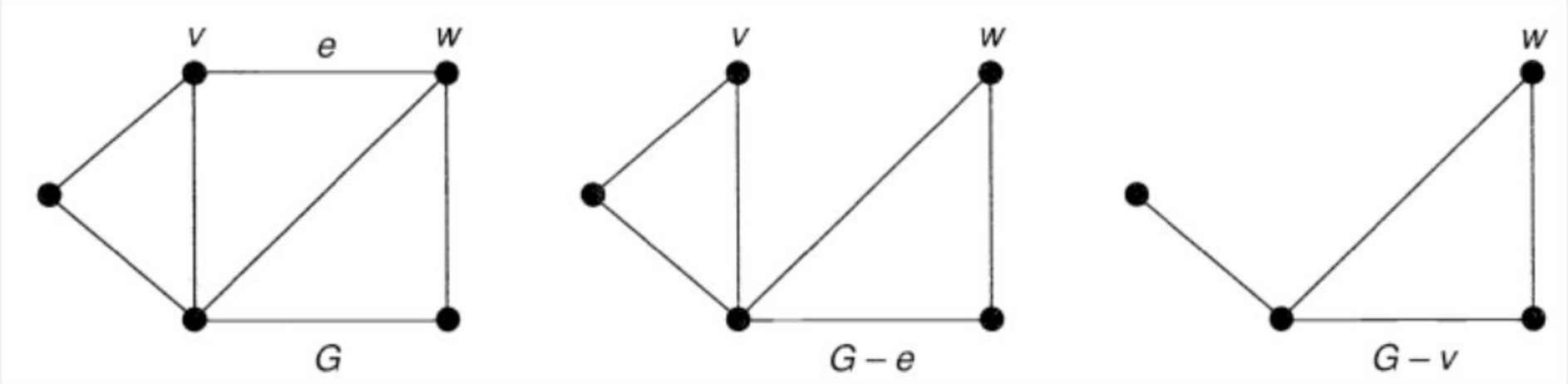
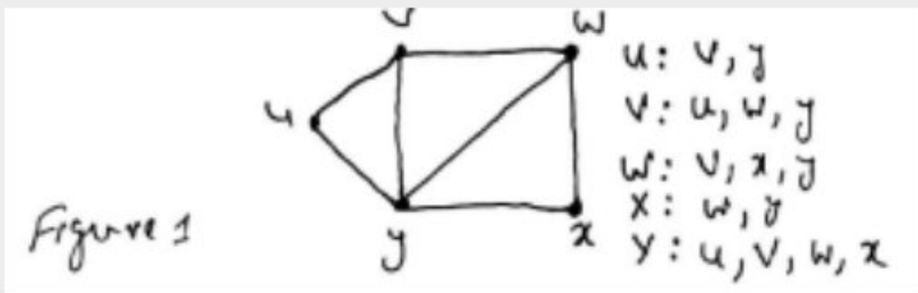
v. More generally, if S is any set of vertices in G, we denote by G - S the graph obtained by deleting the vertices in S and all edges incident with any of them. Some examples are shown in Fig. 16.

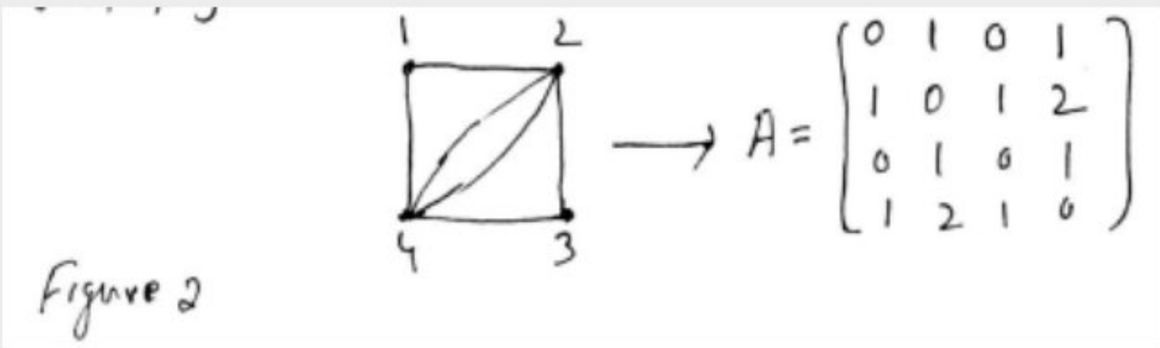


Matrix Representation;

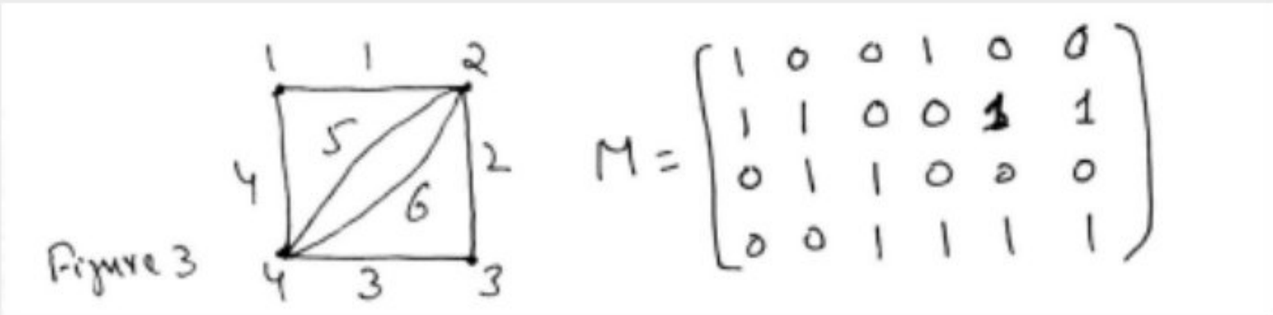
Although it is convenient to represent a graph by diagram of points joined by lines, such a representation may be unsuitable if we wish to store a large graph in computer. One way of storing a simple graph is by listing the vertices adjacent to each vertex of graph. An example of this is given in fig 1



Other useful representations involve matrices. If G is a graph with vertices labeled $\{1,2, \dots n\}$, its adjacency matrix A is n **x** n matrix whose ij-th entry is the number of edges joining vertex I and vertex j.



If, in addition, the edges are labeled $\{1,2, \dots ., n\}$ its incidence matrix M is the n **x** m matrix whose ij-th entry is 1 if vertex I is incident to edge j, and 0 otherwise.



Definition; A graph whose edge set is empty is a **NULL GRAPH**. We denote a null graph with n vertices by N_n : N_4, is shown in fig 4

Note that each vertex of null graph is isolated. Null graphs are not very interesting.

Definition; A graph in which each pair of distinct vertices are adjacent is a **COMPLETE GRAPH**. We denote the complete graph as n vertices by K_n, ; K_4 and K_5, are shown in fig 5

Note: K_4, has $\frac{n(n-1)}{2}$ edges.

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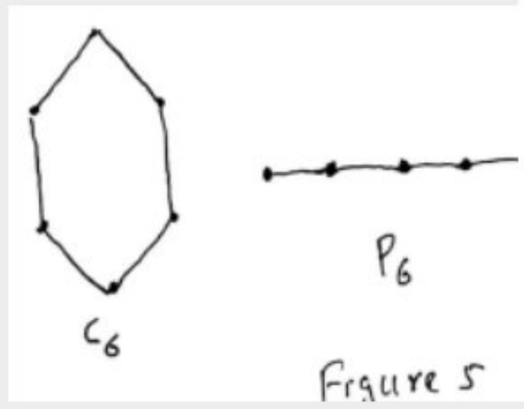
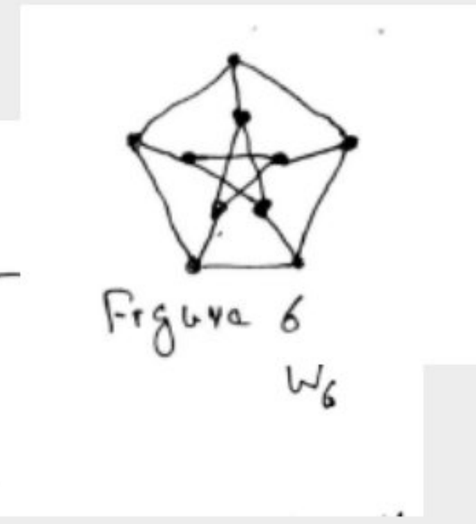
Definition; A connected graph that is regular (a graph in which each vertex has the same degree) of degree 2 is a **CYCLE GRAPH.** We denote the cycle graph on n vertices as C_n A graph obtained from C_n, by removing an edge is the **PATH GRAPH**, on n vertices and is denoted by P_n. The graph obtained from C_{n-1}, by joining each vertex to a new vertex v is the **WHEEL** on n vertices, denoted by W_n, The graphs C_6, P_6 and W_6, are shown in fig 5

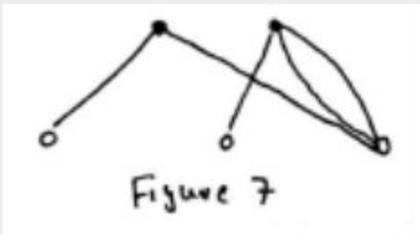


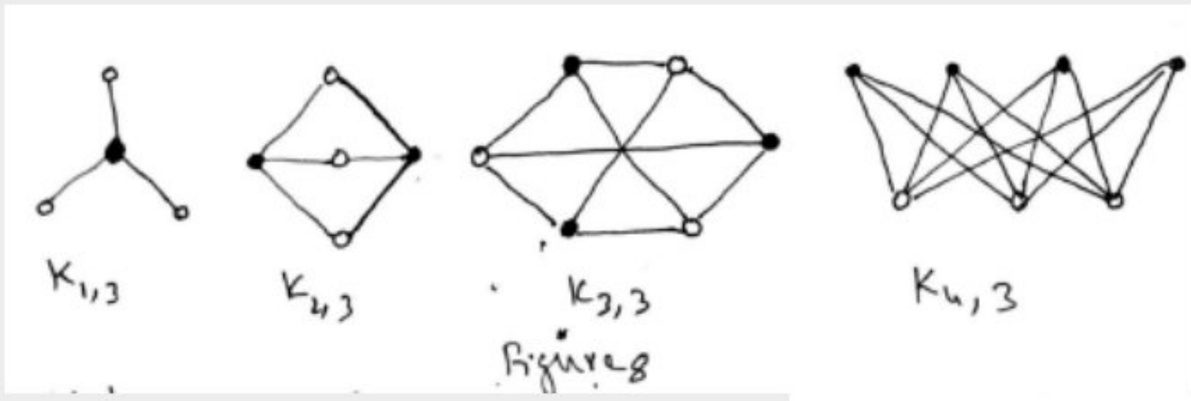


Definition; A regular graph of degree 3 is called **CUBIC GRAPH**. An example of cubic graph is the Petersen Graph, shown in fig 6

NOTE: The null graph N_n, is regular of degree 0, the cycle graph C_n is regular of degree 2 and the complete graph K_n, is regular of degree n-1

Definition; If vertex set of graph G can be split into two disjoint sets A and B so that each edge of G joins a vertex of A and a vertex of B, then G is called **BIPARTITE GRAPH**. Alternatively, bipartite graph is one whose vertices can be colored black and white in such a way that each edge joins a black vertex (in A) and a white vertex (in B). We denote the bipartite graph with black vertices and white vertices by $K_{r,s.}$ A compete bipartite graph is a bipartite graph in which each vertex in A is joined to each vertex in B.

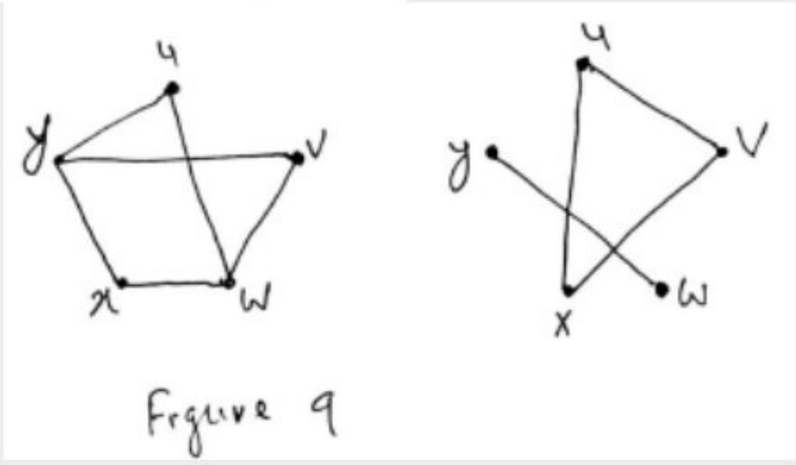




NOTE: $K_{r,s}$, has r+s vertices and rs edges.

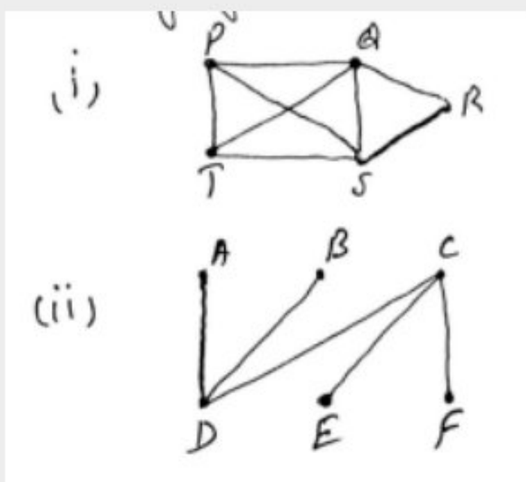
Definition; If G is a simple graph with vertex set V(G), its complete $\overline{G}$ is the simple graph with vertex set V(G) in which two vertices are adjacent iff they are not adjacent in G.

Note that complement of a complete graph is a null graph.



Question

1. Write down the number of vertices, the number of edges and degree of each vertex in following graphs



Solution:

A. There are 5 vertices and 8 edges.

Degree of vertex P =3 Degree of vertex Q = 4

Degree of vertex R = 2 Degree of vertex S = 4

Degree of vertex T = 3

B. There are 6 vertices and 5 edges.

The vertices A, B, E, F have degree 1 The vertices C and D have degree 3

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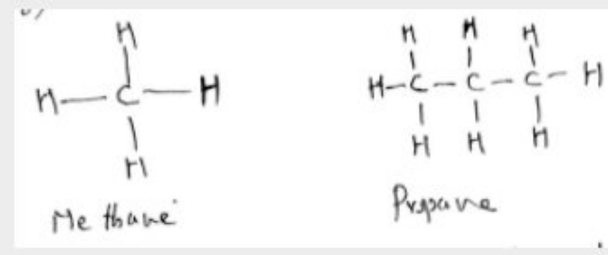
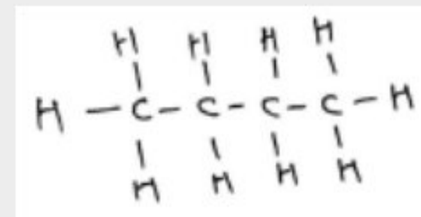
2. The following figure represents the chemical molecules of methane (CH_4) and propane (C_3H_8). Regarding these diagrams as graphs, what can you say about vertices representing carbon atoms (C) and hydrogen atoms (H). Draw graph corresponding to molecules ($C_4H_{10,}$)

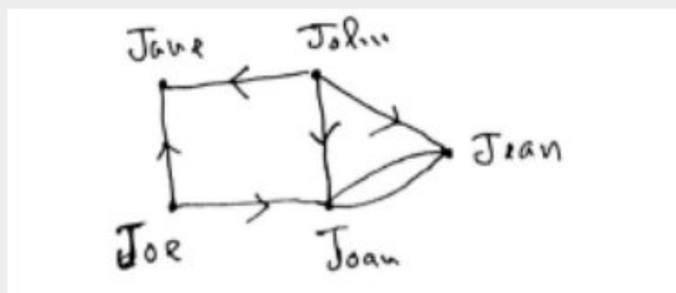


Solution: Each carbon atom has degree 4 and each hydrogen vertex has degree 1.

3. John likes joan, Jean, Jane. Joe likes Jane and Joan. Jean and Joan like each other. Draw a graph illustrating these relationships between John, Joan, Jane and Joe.



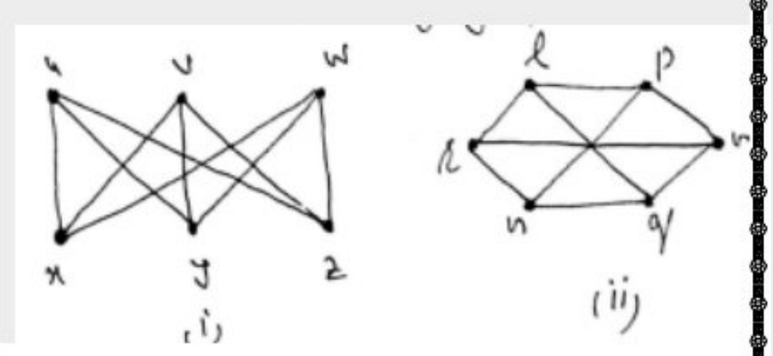
Solution:



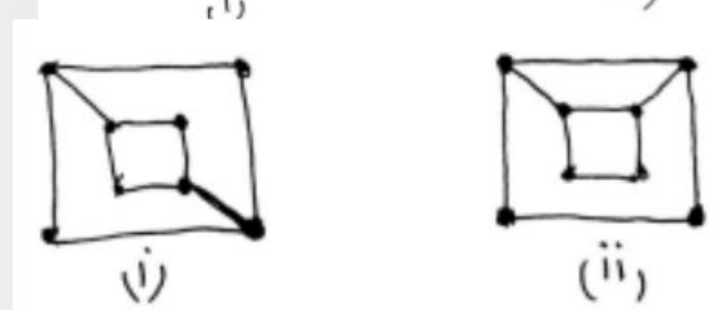
4. Write down the vertex set and edge set of each of following graphs.

Solution:

1. V(G)= $\{u, v, w, x, y, z\}$ E(G)= $\{ux, uy, uz, vx, vy, vz, wx, wy, wz\}$
2. V(G)= $\{l, m, n, p, q, r\}$ E(G)= $\{lr, lv, mp, mv, mr, np, nv, nr\}$

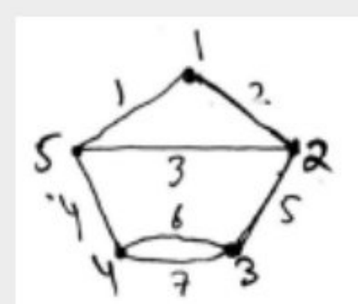


5. Explain why following graphs are not isomorphic.



Solution: In graph 1. No vertices of degree 2 are adjacent. In graph 2, they are adjacent. Since isomorphism preserves adjacency of vertices, the graphs are not isomorphic.

6. Write down the adjacency and incidence matrices of following graph.



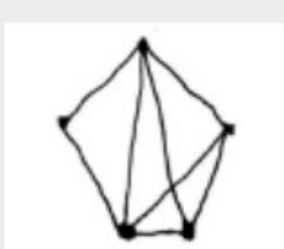
Solution:

$$A=\begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 2 & 0 & 1 \end{bmatrix} \qquad M=\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

7. Draw;

1. A simple graph
2. A non-simple graph with no loops
3. a non-simple graph with no multiple edges each with 5 vertices and 8 edges.

Solution:

1.  at most one edge for 0 vertices.

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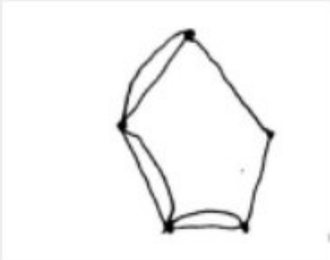
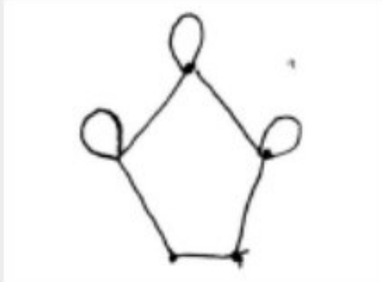
2.  non-simple because of multiple edges.

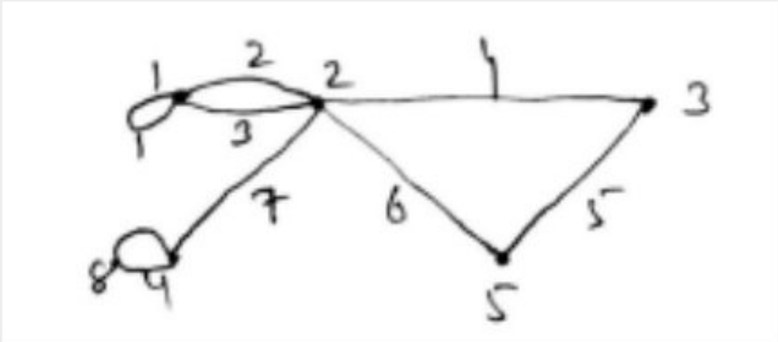
3. 

8. Classify the following statements as true or false.

1. Any two isomorphic graphs have same degree sequence.
2. Any two graphs with the same degree sequence are isomorphic.

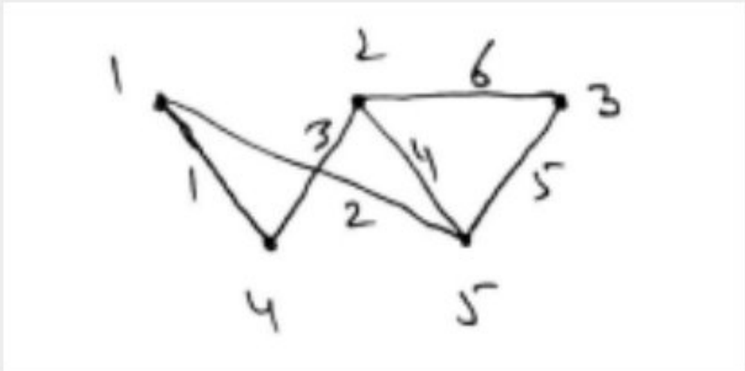
Solution: 1. True 2. False

9. Write down the incidence matrix of following graph.



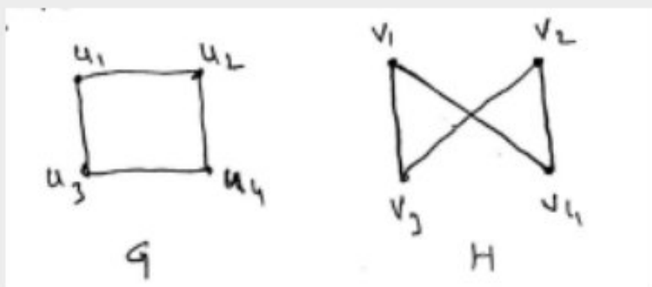
Solution: $M=\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}$

10. Write down the incidence matrix of following graph.



Solution: $M=\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}$

Definition; The simple graphs $G_1=(V(G_1), E(G_1))$ and $G_2 = (V(G_2), E(G_2))$ are isomorphic if there exists a one-to-one and onto function $f: V(G_1) \rightarrow V(G_2)$ with the property that a and b are adjacent in G_1 iff f(a) and f(b) are adjacent in G_2 for all a and b in $V(G_1)$. Such a function is called isomorphism. Two simple graphs that are not isomorphic are called non-isomorphic. In other words, when two simple graphs are isomorphic there is one-to-one correspondence between vertices of two graphs that preserves the adjacency relationship.



Example: Show that the following graphs are isomorphic

Solution: The adjacent vertices in G are (u_1, u_2), (u_1, u_3), (u_2, u_4), (u_3, u_4). Define a function $f: V(G) \rightarrow V(H)$ by $f(u_1) = v_1$, $f(u_3) = v_3$, $f(u_2) = v_4$ and $f(u_4) = v_2$. Thus, f is a one-to-one correspondence. This correspondence preserves adjacency because corresponding to each pair of adjacent vertices in G there exists a pair of adjacent vertices in H. Hence graph G and H are isomorphic.

Remark; We can show that two graphs are not isomorphic if we can find a property only one of two graphs has, but that is preserves by isomorphism.

Definition; A property preserved by isomorphism's of graphs is called a **GRAPH INVARIANT.** For example, isomorphic simple graphs must have the same number of vertices, because there is one-to-one correspondence between set of vertices of graphs.

Isomorphic graphs also must have the same number of edges, because one-to-one correspondence between vertices establishes a one-to-one correspondence between edges. In addition, the degrees of vertices in isomorphic simple graphs must be same. That is, a vertex V of degree d in G

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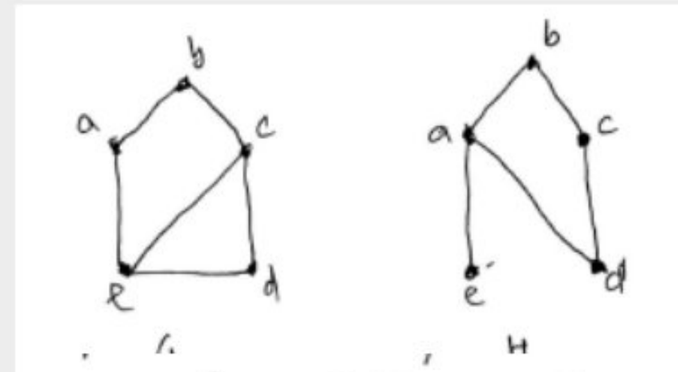
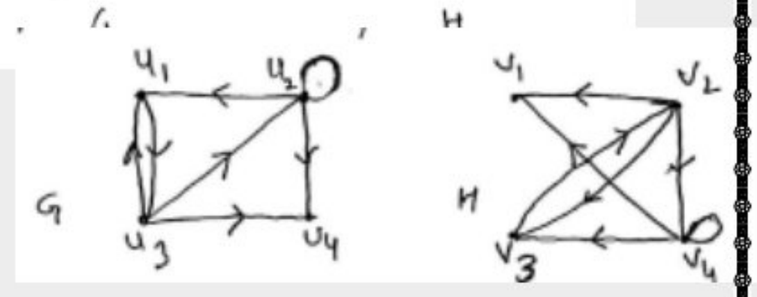
must correspond to a vertex f(V) of degree d in H because a vertex W in G is adjacent to V iff f(V) and f(W) are adjacent in H.

Example: Show that the graphs displayed below are not isomorphic



Solution: Graph H has a vertex of degree 1 (namely vertex e) but graph G has no vertex of degree 1. Thus, graph G and H are not isomorphic.

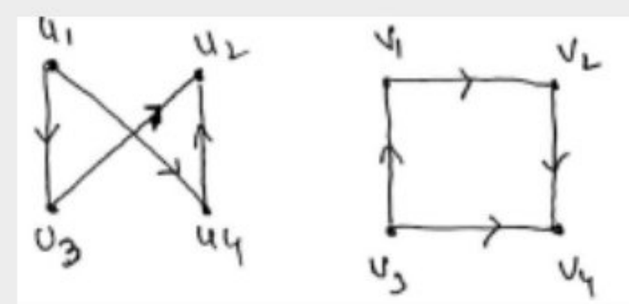
Example: Determine whether the following pairs of directed graphs are isomorphic.



Solution: Define f: V(G) →V(H) by $f(u_1) = v_3$, $f(u_2) = v_4$, $f(u_3) = v_2$, $f(u_4) = v_1$.

The edge (u_1, u_3) corresponds to $(f(u_1), f(u_3)) = (v_3, v_2)$
The edge (u_3, u_1) corresponds to $(f(u_3), f(u_1)) = (v_2, v_3)$
The edge (u_2, u_1) corresponds to $(f(u_2), f(u_1)) = (v_4, v_3)$
The edge (u_2, u_4) corresponds to $(f(u_2), f(u_4)) = (v_4, v_1)$
The edge (u_2, u_2) corresponds to $(f(u_2), f(u_2)) = (v_4, v_4)$
The edge (u_3, u_4) corresponds to $(f(u_3), f(u_4)) = (v_2, v_1)$
The edge (u_3, u_2) corresponds to $(f(u_3), f(u_2)) = (v_2, v_4)$

Example: Determine whether the following graphs are isomorphic.



Solution: To solve this, we need the concept of in-degree and out-degree of a vertex in a directed graph. The IN-DEGREE OF A VERTEX (deg^-) is the number of edges that have the vertex as their terminal vertex. The OUT-DEGREE OF A VERTEX (deg^+) is the number of edges that have the vertex as their initial vertex. Two directed graphs are isomorphic if the adjacent vertices have the same in and out degree in two graphs.In and out degrees of each vertex in left graph:

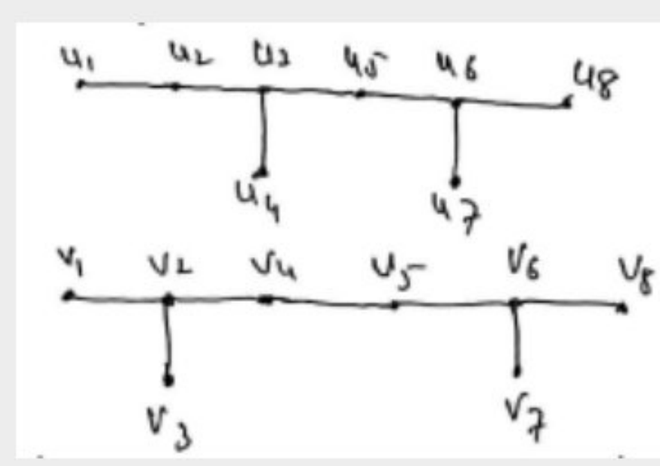
$deg^-(u_1) = 0$	$deg^-(u_2) = 2$	$deg^-(u_3) = 1$	$deg^-(u_4) = 1$
$deg^+(u_1) = 2$	$deg^+(u_2) = 0$	$deg^+(u_3) = 1$	$deg^+(u_4) = 1$

In and out degrees of each vertex in right graph

$deg^-(v_1) = 1$	$deg^-(v_2) = 1$	$deg^-(v_3) = 0$	$deg^-(v_4) = 2$
$deg^+(v_1) = 1$	$deg^+(v_2) = 1$	$deg^+(v_3) = 2$	$deg^+(v_4) = 0$

The vertex u_1 ($deg^- = 0$, $deg^+ = 2$) is connected with vertices with in and out degree 1, whereas the corresponding vertex v_3 ($deg^- = 0$, $deg^+ = 2$) is connected with a vertex of out degree 0. The graphs are therefore not isomorphic because the adjacent vertices must have the same in and out degree in two graphs.

Example: Determine whether the following graphs are isomorphic.



Solution: These graphs are not isomorphic because in first graph the vertices of degree 3 are adjacent to a common vertex but this is not the case for second graph.

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