Exercise 4.4

①

Q#1, 2 — Solution same as Example 3

Q #3 let $f = x^2+1$

$g = x^2-1$

$h = 2x-1$

(i) we need to show $\{f, g, h\}$ Span P_2

let $p \in P_2$

$\Rightarrow p = a_0 + a_1 x + a_2 x^2$

Now $\{f, g, h\}$ Spans P_2 if

$$p = c_1 f + c_2 g + c_3 h$$

has a solution for every p.

now putting values we have

$$a_0 + a_1 x + a_2 x^2 = c_1(x^2+1) + c_2(x^2-1) + c_3(2x-1)$$

$$a_0 + a_1 x + a_2 x^2 = (c_1 - c_2 - c_3) + 2c_3 x + (c_1 + c_2) x^2$$

Comparing Coefficients

$$c_1 - c_2 - c_3 = a_0$$

$$2c_3 = a_1$$

$$c_1 + c_2 = a_2$$

OR

$$\begin{bmatrix} 1 & -1 & -1 \\ 0 & 0 & 2 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}$$

$AX = b$

This System has a solution iff $\det A \neq 0$

Now $\det A = \begin{vmatrix} 1 & -1 & -1 \\ 0 & 0 & 2 \\ 1 & 1 & 0 \end{vmatrix} = (-1)^{2+3}(2)\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix}$ expanding w.r.t R_2

$= -2(1-(-1))$

$= -2(2)$

$= -4 \neq 0$

(ii) To show $\{f, g, h\}$ is linearly independent. (2)

Let $c_1 f + c_2 g + c_3 h = 0$

or $c_1(x^2+1) + c_2(x^2-1) + c_3(2x-1) = 0$

or $c_1 - c_2 - c_3 = 0$

$2c_3 = 0$

$c_1 + c_2 = 0$

OR

$$\begin{bmatrix} 1 & -1 & -1 \\ 0 & 0 & 2 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

It has only trivial solution i.e $(c_1, c_2, c_3) = (0, 0, 0)$

iff $\det A \neq 0$ where $A = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 0 & 2 \\ 1 & 1 & 0 \end{bmatrix}$

As $\det A = -1$

$\Rightarrow$ Vectors are linearly independent.

Now from (i) & (ii)

$1+x^2, 1-x^2, 2x-1$ form a basis for P_2.

Q#4: Same as 3

Q#5: $A = \begin{bmatrix} 3 & 6 \\ 3 & -6 \end{bmatrix}$; $B = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$; $C = \begin{bmatrix} 0 & -8 \\ -12 & -4 \end{bmatrix}$; $D = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$

Now let $M \in M_{22} \Rightarrow M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

To show that $\{A, B, C, D\}$ is a basis; We must prove

(i) $M = c_1 A + c_2 B + c_3 C + c_4 D$ has a solution for any M. (So that $\{A, B, C, D\}$ is a spanning set)

(ii) $c_1 A + c_2 B + c_3 C + c_4 D = 0 \Rightarrow c_1 = c_2 = c_3 = c_4 = 0$

i.e $\{A, B, C, D\}$ is linearly independent set.

OR

(i) $\Rightarrow$ we must have solution for this system

i.e
$$3c_1+0c_2+0c_3+1c_4=a$$
$$6c_1-c_2-8c_3+0c_4=b$$
$$3c_1-c_2-12c_3-c_4=c$$
$$-6c_1+0c_2-4c_3+2c_4=d$$

has a solution iff

$\det A \neq 0$, where

$$A=\begin{bmatrix}3 & 0 & 0 & 1\\ 6 & -1 & -8 & 0\\ 3 & -1 & -12 & -1\\ -6 & 0 & -4 & 2\end{bmatrix}$$

check $\det A$ yourself

(ii) $\Rightarrow$ we must have only trivial solution for homogenous system

$$3c_1+0c_2+0c_3+1c_4=0$$
$$6c_1-c_2-8c_3+0c_4=0$$
$$3c_1-c_2-12c_3-c_4=0$$
$$-6c_1+0c_2-4c_3+2c_4=0$$

which is possible iff $\det A \neq 0$

where
$$A=\begin{bmatrix}3 & 0 & 0 & 1\\ 6 & -1 & -8 & 0\\ 3 & -1 & -12 & -1\\ -6 & 0 & -4 & 2\end{bmatrix}$$

Q#6: Same as 5

Q#7 (Hint) Just check whether given vectors are linearly independent or not?

(i) If they are not linearly dependent.
$\Rightarrow$ they can't form a basis

(ii) If they are linearly independent then prove that they do not span $\mathbb{R}^3$.

Q#8, 9 (same as 7)

Q#10 Let $V=\mathrm{Span}\{v_1, v_2, v_3\}$

(a) since $\cos 2x = \cos^2 x - \sin^2 x$

$\Rightarrow$ $v_3 = v_1 - v_2$

or $v_3 - v_1 + v_2 = 0$

i.e v_1, v_2, v_3 are linearly dependent.

Hence they do not form a basis for V.

④

Q#10(b)

$V = \text{Span}\{v_1, v_2, v_3\}$

$= \text{Span}\{v_1, v_2\}$ $(\because v_3 = v_1 - v_2)$

$\boxed{\text{[using theorem 4.2-6].}}$

Now v_1, v_2 span V.

Also v_1, v_2 are linearly independent.

(verify through Wronskian).

$\Rightarrow \{v_1, v_2\}$ is a basis for V.

Q#11

Let $S = \{u_1, u_2\}$ and

let $w = c_1u_1 + c_2u_2$ —— (i)

i.e $(1, 1) = c_1(2, -4) + c_2(3, 8)$

$$\Rightarrow \quad 2c_1 + 3c_2 = 1$$

$$-4c_1 + 8c_2 = 1$$

$$\begin{bmatrix} 2 & 3 \\ -4 & 8 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$A \quad x = B \quad \text{—— (1)}$$

$\det A = 16 - (-12) = 16 + 12 = 28$

$$\text{Adj } A = \begin{bmatrix} 8 & -3 \\ 4 & 2 \end{bmatrix}$$

Now $A^{-1} = \frac{1}{|A|}\text{Adj } A = \frac{1}{28}\begin{bmatrix} 8 & -3 \\ 4 & 2 \end{bmatrix}$

using in (1)

$$X = A^{-1}B$$

$$= \frac{1}{28}\begin{bmatrix} 8 & -3 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{28}\begin{bmatrix} 8-3 \\ 4+2 \end{bmatrix} = \frac{1}{28}\begin{bmatrix} 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 5/28 \\ 6/28 \end{bmatrix} = \begin{bmatrix} 5/28 \\ 3/14 \end{bmatrix}$$

Scanned with CamScanner

using in 1

$$w = \frac{5}{28}u_1 + \frac{3}{14}u_2$$

Thus $(w)_S = \left(\frac{5}{28}, \frac{3}{14}\right)$

Q#11(b) Do yourself

Q#12, 13, 14, 15 (Same Concept as 11(a))

Q#17, 18 (method of Q#8 + Q#14)

Q#19 (Do yourself by using previous techniques)

Q#21

$e_1 = (1,0,0)$
$e_2 = (0,1,0)$
$e_3 = (0,0,1)$

(a) $T_A(e_1) = Ae_1 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -3 \\ -1 & 2 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

$T_A(e_2) = Ae_2 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$

$T_A(e_3) = Ae_3 = \begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix}$

Now check by usual technique for if $T_A(e_1), T_A(e_2), T_A(e_3)$ are linearly independent or not?

(b) Same as a

Q#22 (Same as 21)