space
a set
is called
following

endent.

a basis
v_1 then
can be
$v =$
exactly

is basis
ns V

can be
mbination

ny that
tion way
hen v_1
c_n

eq ① − eq (ii).

$0 = (c_1 - k_1)v_1 + (c_2 - k_2)v_2 + \dots + (c_n - k_n)v_n$

$\because c_1 - k_1 = 0$ because it is linearly independent.

~~so~~ $c_1 = k_1$, $c_2 - k_2 = 0$

$c_2 = k_2$

$\therefore c_n - k_n = 0$

$c_n = k_n$.

* Theorem:

If A & B are invertible matrix of same size then AB is invertible and

$(AB)^{-1} = B^{-1}A^{-1}$

* Proof:

$\because (AB)(AB)^{-1} = I$ (I stand for identity matrix)

pre multiplicating A^{-1} both side

$A^{-1}(AB)(AB)^{-1} = A^{-1}I$

$(A^{-1}A)B(AB)^{-1} = A^{-1}$

$(I)B(AB)^{-1} = A^{-1}$

$B(AB)^{-1} = A^{-1}$

pre multiplicating B^{-1} on b.s

$B \cdot B^{-1}(AB)^{-1} = A^{-1}B^{-1}$ or $B^{-1}A^{-1}$

$I(AB)^{-1} = A^{-1}B^{-1}$ or $B^{-1}A^{-1}$ $\because I \cdot (AB)^{-1} = (AB)^{-1}$

$(AB)^{-1} = A^{-1}B^{-1}$ or $B^{-1}A^{-1}$

A set S with two are more vector is

(a) linearly ~~dependent~~ depent if and only if at least one of the vector in S expressible as a linear combination of the other vector S.

Proof:

let $S = \{v_1, v_2, \ldots v_r\}$

Assume that S is linearly dependent

$k_1 v_1 + k_2 v_2 + \cdots k_r v_r = 0$

(and not all scalar are zero)

: let $k_1 \neq 0$

Divide by each term by k_1

$v_1 + \frac{k_2}{k_1} v_2 + \cdots + \frac{k_r}{k_1} v_r = 0$

$v_1 = \left(-\frac{k_2}{k_1}\right) v_2 + \cdots + \left(-\frac{k_r}{k_1}\right) v_r$

In similar manner if $k_i \neq 0$

then $v_i = \left(\frac{-k_2}{k_i}\right) v_2$

Conversely ... is at le... S that ... in the ... combinat...

v_1 = ...

$v_1 - c_1 v$...

$k_1 = 1$ ...

S is ...

Theorem:

A finite ... contain ... dependent.

Proof:

let the ... $S = \{v_1$ ... linearly ...

$0v_1 + 0v$ ...

because ... linear ... all ...

b) hold ve
is subspace

ice the vector
v is a vector

o (theorem).

i.e.

$v + (-v) = -v + v = 0$

Linear combination.

W is vector space

$k_1 v_1 + k_2 v_2 + \dots + k_n v_n$ is a linear combination of W

System of homogeneous linear eq:-

$Ax = 0$ if (Right side is zero or if left side is zero)

Theorem:-

if $Ax = 0$ is a homogenious linear system of m equation and n-variables then the set of solution vector is a subspace of R^n.

Proof:-

let W is the solution set of system of linear equation of m-equation and n-variable

let x and x' be the sol of $Ax = 0$

i.e $x, x' \in W$

$Ax = 0$ — (i) $Ax' = 0$ — (ii)

By add (i) and (ii).

$Ax + Ax' = 0 + 0$

$x = A^{-1} b$

To show there is one solution we have

let x_0 is solution of given system.

$Ax_0 = b.$

$x_0 = A^{-1} b.$

$\frac{1}{|A|} \text{adj}A$

$Ax = B$

$x = A^{-1}B$

$x = A^{-1}B$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \Big[\quad \Big]$$

$\cancel{x =}$

$x = \frac{|A_x|}{|A|}$

...s

matrix

...ix b,

has

...lat

...cide

2-) 2nd

Theorem 5.1:-

let V be a vector space U a vector in V and k is ~~the~~ a ~~vector~~ Scalar then.

(a) $0U = 0$

(b) $k0 = 0$

(c) $(-1)U = -U$

(d) If $kU = 0$ then $k = 0$ or $U = 0$.

Proof : (a).

By axiom (8)

$0U + 0U = (0+0)U$

$= 0U$ by property of number

By axiom (5)

$0U + 0U + (-0U) = 0U + (-0U)$

$0U = 0$

3rd axiom

$0U + (0U + (-0U)) = 0U + (-0U)$

$0U = 0$ (By additive identity)

(c) Proof

$(-1)U = -U$

We have to prove that $(-1)U = -U$

$(-1)U + U = 0$

By axiom (10) we have.

$\therefore 1 \cdot U = U$

Taking $(-1)U + U$

consider the vector.

$\hat{i} = (1, 0, 0)$

$\hat{j} = (0, 1, 0)$

$\hat{k} = (0, 0, 1)$

$S = \{\hat{i}, \hat{j}, \hat{k}\}$.

$k_1\hat{i} + k_2\hat{j} + k_3\hat{k}$

$(k_1, k_2, k_3) = (0, 0, 0)$

$k_1\hat{i} + k_2\hat{j} + k_3\hat{k} = 0$

it is linearly independent

$v_1 = (1, -2, 3)$, $v_2 = (5, 6, -1)$,

$v_3 = (3, 2, 1)$

$k_1 v_1 + k_2 v_2 + k_3 v_3 = 0$

$k_1(1, -2, 3) + k_2(5, 6, -1) + k_3(3, 2, 1) = 0$

$(k_1, -2k_1, 3k_1) + (5k_2, 6k_2, -k_2) + (3k_3, 2k_3, k_3)$

$= (0, 0, 0)$

$k_1 + 5k_2 + 3k_3 = 0$

$-2k_1 + 6k_2 + 2k_3 = 0$

$3k_1 - k_2 + k_3 = 0$

* linear combination:-

If $v_1, v_2, v_3 \cdots v_r$ are vector in vector space V

(a) The set W of all linear combination of $v_1, v_2 \cdots v_r$ is a subspace of V.

* Proof-

we have to show that W is subspace of V.

The linear combination of $v_1, v_2, \cdots v_r$.

$$k_1 v_1 + k_2 v_2 + \cdots + k_r v_r$$

$$0v_1 + 0v_2 + \cdots + 0v_r$$

$$0 \in W$$

~~$u+v = ($~~ $\quad) + (\quad)$

~~$u+v = ($~~

let $u, v \in W$

$$u = k_1 v_1 + k_2 v_2 + \cdots + k_r v_r$$

$$v = c_1 v_1 + c_2 v_2 + \cdots + c_r v_r$$

$$u+v = (k_1 v_1 + k_2 v_2 + \cdots + k_r v_r) + (c_1 v_1 + c_2 v_2 + \cdots + c_r v_r)$$

$$= (k_1 + c_1) v_1 + (k_2 + c_2) v_2 + \cdots + (k_r + c_r) v_r$$

$$(k_1 + c_1) = k_1'$$

$$u+v = k_1' v_1 + k_2' v_2 + \cdots + k_r' v_r$$

$$u+v \in W.$$

* Real vector space:-

let V be a arbitrary non empty set with two operation addition (+) and multiplication then following assumption should be.

(i) if $u, v \in V \Rightarrow u+v \in V$.

(ii) $u+v = v+u \quad \forall\, u, v \in V$

(iii) $u+(v+w) = (u+v)+w$

(iv) $\exists\, 0 \in V \quad u+0 = 0+u = u$.

~~if to~~

(v) $u \in V \quad \exists\, -u$

$u-u = 0$ or $u+u = 0$.

(vi) k be any scalar and $u \in V$.

(vii) $k(u+v) = ku + kv$.

(viii) $(k+m)u = ku + mu$

(ix) $k(mu) = (km)u$.

(x) $1 \cdot u = u$.

Check weather the vector $V = \{x, y, z \in R^3 : 2x + 3y^3 - 4z^2 = 0\}$ is a vector space or not.

$(2, 0, 1), (8, 0, 2) \in V$

* Basis:-

If V is any vector space and $S = \{v_1, v_2, \cdots v_n\}$ is a set of vector in V then S is called a basis for V if following full filled

(a) S is linearly independent.

(b) S span V.

if $S = \{v_1, v_2, \cdots v_n\}$ is a basic for a vector space V, then every vector v in V can be expressed in the form $v = c_1v_1 + c_2v_2 + \cdots + c_nv_n$ in exactly one way.

* Proof:-

Since given that S is basis of V. It mean S spans V by defination.

So every vector in v can be expressed as linear combination of vectors of S.

On contrary we say that we can write in two way.

$v = c_1v_1 + c_2v_2 + \cdots + c_nv_n$

$v = k_1v_1 + k_2v_2 + \cdots + k_nv_n$

---

eq ①

$0 = (c_1 - k_1)$

$= c_1 - k_1$

is linear

$\Rightarrow c_1 = k$

$\because c_n - k_n$

$c_n = k$

* Theorem:

If A &

of same

and

$(AB)^{-1}$

* Proof

$\because (AB)($

pre mul

$A^{-1}(AB)$

$(A^{-1}A)B$

$(I)B$

$B\ (A$

pre m

$B \cdot B^{-1}($

$I(AB)$

$(AB)^{-1}$

AP - 1

* Show that

$(A^{-1})^{-1} = A$

Sol:

$\because (A A^{-1}) = I$

Post multiplicating $(A^{-1})^{-1}$ on b.s.

$(A A^{-1})(A^{-1})^{-1} = I (A^{-1})^{-1}$

$A(A^{-1}(A^{-1})^{-1}) = (A^{-1})^{-1}$

$A(I) = (A^{-1})^{-1}$

$A = (A^{-1})^{-1}$

v.v. Imp Theorem

If A is an invertible $n \times n$ matrix then for each $n \times 1$ matrix b, the system of eq $Ax = b$ has exactly one solution namely

$x = A^{-1} b$.

* Proof:

$A(A^{-1} b) = b$

$\Rightarrow x = A^{-1} b$ is a solution of $Ax = b$.

$Ax = b$

Since A is non singular pre multiplying A^{-1} both side

$A^{-1}(Ax) = A^{-1} b$.

$(A^{-1} A) x = A^{-1} b$.

$\because A^{-1} A = I$ $\quad Ix = A^{-1} b$

$x = A^{-1} b$.

$$w(x) = f_1(x) \; f_2(x) \; \cdots\cdots \; f_n(x)$$
$$= f_1'(x) \; f_2'(x) \; \cdots\cdots \; f_n'(x)$$
$$= f_1''(x) \; f_2''(x) \; \cdots\cdots \; f_n''(x)$$
$$\vdots$$
$$= f_1^{n-1}(x) \quad f_2^{n-1}(x) \quad f_n^{n-1}(x)$$

$f_1(x) = x, \quad f_2(x) = e^x \quad f_3(x) = e^{2x}$

$$w(x) = \begin{vmatrix} x & e^x & e^{2x} \\ 1 & e^x & 2e^{2x} \\ 0 & e^x & 4e^{2x} \end{vmatrix}$$

$$= x \begin{vmatrix} e^x & 2e^{2x} \\ e^x & 4e^{2x} \end{vmatrix} - 1 \begin{vmatrix} e^x & e^{2x} \\ e^x & 4e^{2x} \end{vmatrix} + 0$$

$$= x(4e^{3x} - 2e^{3x}) - 1(4e^{3x} - e^{3x})$$
$$= x(2e^{3x}) - 1$$

$f_1(x) = 1 \quad f_2(x)$

$w(x) = \Big| \; 1 \; \ldots$
0
0

$= 1 \,(4e^{3x} -$
$4e^{3x} -$
$=$
$= 2e^{3x}$

if ans is n...
linearly indep...

$f_1(x) = x$ ...

$w(x) = \Big| \; x$
1

$w(x) = x$ ...
if $x = 0$ it ...
else it ...

→ Proof :-

if each vector in S is a linear combination of S'

span $S \subseteq$ span S'

span $(v_1, v_2, \ldots v_k) \subseteq$ span $(w_1, \ldots w_k)$

if each vector in S' is linear combination of S.

span $S' \subseteq$ span S.

span $(w_1, w_2 \ldots w_k) \subseteq$ span $(v_1 \ldots v_k)$

span S' = span S.

conversely assume that

span S = span S'

if $v_i \neq a_1 w_1 + a_2 w_2 + \cdots a_k w_k$.

For all scalars $a_1, a_2 \ldots a_k$.

$v_i \in$ span(S) but $v_i \notin$ span (S').

$k_1 v_1 + k_2 v_2 - - - - - - k_n v_n = 0$.

if $(k_1 = k_2 = \ldots k_n = 0)$ or

it is linearly independent if

and if $(k_1 = k_2 \ldots = k_n = 0)$ results zero

it is linearly dependent

consider th

$\hat{i} = (1, 0, 0$

$\hat{j} = (0, 1,$

$\hat{k} = (0, 0,$

$S = \{\hat{i}, \hat{j},$

$k_1 \hat{i} + k_2 \hat{j} + k$

$(k_1, k_2, k_3$

$k_1 \hat{i} + k_2 \hat{j}$

It is

$v_1 = (1, -2,$

v_3

$k_1 v_1 + k_2$

$k_1 (1, -2, 3)$

$(k_1, -2k_1, 3k_1)$

$k_1 + 5k_2 + 3k$

$-2k_1 + 6k_2$

$3k_1 - k_2 +$

$(-1)u + (1 \cdot u)$

$(-1+1)u$

$0u$

0

Subspace:-

If W is a set of one or more vectors from a vector space V, then W is a subspace of V if and only if the following conditions hold.

(a) If u and v are vectors in W, then $u+v$ is in W

(b) if k is any scalar and u is any vector in W, then ku is in W

Proof:-

Suppose that W is subspace of V let $u, v \in W$

Since W is subspace then

$\Rightarrow u+v \in W$ (a fulfill)

if k is any scalar and

$u \in W$

$\because W$ is subspace

then $ku \in W$ (b fulfill)

Conversely,

Assume that (a) & (b) hold we have to prove that W is subspace of V.

let $u, v \in W$

$\Rightarrow u+v \in W$ (by using a).

(ii) let $u, v \in W$

$u+v = v+u$ (bcz u, v are the vector from V and V is a vector space).

(iii) $u, v, w \in W$

$u+(v+w) = (u+v)+w$.

(iv) by part (b)

let $u \in W$

Taking $k=0$

$0u \in W$ $\because 0u = 0$ (Theorem).

$0 \in W$.

So we can write

$u+0 = 0+u = u$.

let $u \in W$

by part b.

$ku \in W$

$k=-1$

$-1 \cdot u \in W$

By theorem.

$-1 \cdot u = -u$

$-u \in W$

Let

$u \in W$

$u = k_1 v_1 + k_2 v_2 + \cdots + k_r v_r$

$ku = k(k_1 v_1 + k_2 v_2 + \cdots + k_r v_r)$

$= (kk_1)v_1 + (kk_2)v_2 + \cdots + (kk_r)v_r$

By associative property.

$\because kk_1 = k_1'$

$kv = k_1' v_1 + k_2' v_2 + \cdots + k_r' v_r$

$ku \in W$.

Hence W is a subspace of V.

$S = \{v_1, v_2, \cdots v_r\}$

$W = \text{span}(S)$ or $W = \text{span}\{v_1, v_2, \cdots v_r\}$

Imp Theorem 5.2.4:-

if $S = (v_1, v_2, \cdots v_k)$ and

$S' = (w_1, w_2, \cdots w_k)$ are two sets of vector in vector space V. then $\text{span}(v_1, v_2, \cdots v_r) = \text{span}(w_1, w_2, \cdots w_r)$ if and only if each vector in S is linear combination S' and each vector in S' is linear in S.

Conversely assume that there is at least one vector in S that can be written in the form of linear combination of other vectors.

$v_1 = c_2 v_2 + c_3 v_3 + \cdots + c_r v_r$

$v_1 - c_2 v_2 - c_3 v_3 - \cdots - c_r v_r = 0$

$k_1 = 1$, $k_2 = -c_2$, $k_3 = -c_3$

$\cdots\cdots k_r = -c_r$

S is linearly dependent

Theorem:

A finite set of vectors that contain the zero vector is linearly dependent.

Proof:

For the vectors $v_1, v_2 \cdots v_n$

$S = \{v_1, v_2 \cdots v_n, 0\}$ is linearly dependent.

$0v_1 + 0v_2 + 0v_3 + \cdots + 1 \cdot 0 = 0$

because 0 is written in linear combination in which all scalar are not zero.

$f_1(x) = 1 \quad f_2(x) = e^x \quad f_3(x) = e^{2x}$

$$w(x) = \begin{vmatrix} 1 & e^x & e^{2x} \\ 0 & e^x & 2e^{2x} \\ 0 & e^x & 4e^{2x} \end{vmatrix}$$

$$= 1(4e^{3x} - 2e^{3x})$$
$$= 4e^{3x} - 2e^{3x}$$
$$= 2e^{3x}$$

if ans is not zero it is linearly independent.

$f_1(x) = x, \quad f_2(x) = \sin x$

$$w(x) = \begin{vmatrix} x & \sin x \\ 1 & \cos x \end{vmatrix}$$

$w(x) = x\cos x - \sin x$

if $x = 0$ it is linearly dependent

else it is linearly independent.

$k_1 + 6k_2 = 9$

$2 = 2k_1 + 4k_2$

$18 = 2k_1 + 12k_2$
$2 = 2k_1 + 4k_2$

$16 = 8k_2$
$k_2 = 16/8 = 2$
$9 = k_1 + 6k_2$
$9 = k_1 + 12$
$9 - 12 = k_1$
$k_1 = -3$

$w = -3u + 2v$

4

$A(x+x') = 0$

$x+x' \in W$

let $x \in W$

$Ax = 0$

x by k on b.s

$kAx = 0$

$A(kx) = 0$

$kx \in W$

W is subspace of R^n.

* linear combination –

$u(1, 2, -1)$

$v(6, 4, 2) \in R^3$

show that $w = (9, 2, 7)$ is linear combination of u & v

$w(4, -1, 8)$ is not linear combination of u & v.

Sol:

$w = k_1 u + k_2 v$

$(9, 2, 7) = k_1(1, 2, -1) + k_2(6, 4, 2)$

$= (k_1 + 2k_1 - k_1) + (6k_2 + 4k_2 + 2$

$(9, 2, 7) = (k_1 + 6k_2) + (2k_1 + 4k_2) + (-k_1 + 2k_2)$

$=$

$k_1 + 6k_2 = 9$

$2 = 2k_1 +$

$18 = 2k_1 +$

$2 = 2k_1$

$16 = 8k$

$k_2 =$

$9 = k_1 + 6$

$9 = k_1 +$

$9 - 12 = k_1$

$k_1 = -$

$w = -3u +$

9

R^n is a vector space.

A vector space is 2×2 matrices.

$$u = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}, \quad v = \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix}$$

$$u + v = \begin{bmatrix} u_{11}+v_{11} & u_{12}+v_{12} \\ u_{21}+v_{21} & u_{22}+v_{22} \end{bmatrix}$$

A plane through origin is a vector space.

$$ax + by + cz = 0.$$

check weather the vector.

$V = \{ x, y, z \in R^3 : 2x + 3y^3 - 4z^2 = 0 \}$.

let $u = (2, 0, 1)$, $v = (8, 0, 2)$.

$u + v = (10, 0, 3)$.

$= 2(10) + 3(0) - 4(3)^2$

$= 20 + 0 - 36$

$= -16 \neq 0$

$V = \{ (a, b, c) \in R^3 : (a-b)c = 0 \}$ is

$u = (3, 0, 2)$, $v = (4, 0, 3)$

$u + v = (12, 0, 5)$

$(12 - 0)5 = 0$

$(12)5 = 0$

$60 \neq 0.$