Ahmad Hojatimalekshah HW6

3. Lets take $f(x) = \log x \quad \forall x > 0$.

The second derivative of $f(x)$ is a concave function $\left(f''(x) = -\frac{1}{x^2} < 0 \right)$.

Then according to the Jensen's inequality we have:

$$f\left(\frac{\sum_{i=1}^{n} a_i}{n} \right) \geq \frac{\sum_{i=1}^{n} f(a_i)}{n}$$

$$\Rightarrow \log\left(\frac{\sum_{i=1}^{n} a_i}{n} \right) \geq \frac{\sum_{i=1}^{n} \log(a_i)}{n} = \log\left[\left(\prod_{i=1}^{n} a_i \right)^{1/n} \right]$$

$$\Rightarrow \boxed{\frac{\sum_{i=1}^{n} a_i}{n} \geq \left(\prod_{i=1}^{n} a_i \right)^{1/n}} \quad ① \quad \Rightarrow AM \geq GM$$

from ①: $$\frac{\sum_{i=1}^{n} \frac{1}{a_i}}{n} \geq \left(\prod_{i=1}^{n} \frac{1}{a_i} \right)^{1/n} = \frac{1}{\left(\prod_{i=1}^{n} a_i \right)^{1/n}}$$

$$\Rightarrow \boxed{\frac{n}{\sum_{i=1}^{n} \frac{1}{a_i}} \leq \left(\prod_{i=1}^{n} a_i \right)^{1/n}} \quad \Rightarrow GM \geq HM \quad ②$$

①, ② $\Rightarrow$ $\boxed{AM \geq GM \geq HM}$