CHE260: Heat Transfer

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Contents

1 Introduction

- How is heat transfer different from thermodynamics? In thermodynamics, we assume quasi-equilibrium processes i.e. the time was not an important parameter. In heat transfer, time is an important parameter and we are interested in the rate of heat transfer.
- What is the relationship between $\dot{Q}$ and ΔT? What are the mechanisms of heat transfer?

- Conduction: Transfer of heat through a medium that is stationary.
- Convection: Transfer of heat from a solid surface and an adjacent fluid that is moving. Example: a fan blowing air over a hot plate. There is heat transfer from the hot plate into the fluid.
- Radiation: Energy emitted by matter in the form of electromagnetic waves.
- Radiation does not need a medium. In a vacuum, we can have radiation but not convection or condcution.
- Different mechanisms of heat transfer can take place simultaneously.
- Applications
  - Power Generation
    * Power plant: steam generation, condenser
    * Automobiles: engine cooling, space heating/cooling
  - Buildings
    * Heating / Cooling
    * Hot water
  - Refrigeration
  - Manufacturing
    * Casting / Heat treatment
    * Injection Moulding

2 Electronic Cooling

- $>$ 99 % of the electrical energy supplied to a circuit is dissipated as heat
- Heat has to be dissipated to the environment while keeping the temperature of the chip in a certain range
- Heat is lost from the surface of the chip
- Important parameter is heat flux $= \frac{\text{Heat Transfer Rate}}{\text{Unit Area}}$ (in $\frac{W}{\text{cm}^2}$)
- To reduce heat flux, we can reduce heat generation and increase the surface area
- As size increases, it becomes more difficult to lose heat
- Water cooling is more efficient for large systems compared to air cooling

3 Radiation

- Radiation is energy emitted by all matter in the form of e.m. radiation
- Thermal radiation is emitted by all bodies at a finite temperature
- Opaque objects emit only from the surface
- Amount of radiation depends on the surface temperature. Summarized by the Stefan Boltzmann Law: $\dot{Q}_{emit} = \sigma A T_s^4$ where σ is the Boltzmann constant ($5.67 \times 10^{-8} \ \frac{W}{m^2 k^4}$), T_s is the surface temperature in Kelvin and A is the surface area.
- A surface that emits as much radiation as this is called a "Blackbody". A real surface emits less than this.: $\dot{Q}_{emit} = \epsilon \sigma A T_s^4$ where ϵ is the emissivity and $0 \leq \epsilon \leq 1$
- Black paint has $\epsilon = 0.99$ which is very close to 1. Aluminum foil has a low emissivity of around 0.07.
- If radiation is incident on a surface some will be absorbed. The fraction absorbed is a surface property known as the absorptivity α such that $\dot{Q}_{absorbed} = \alpha \cdot \dot{Q}_{incident}$ and $\dot{Q}_{reflected} = (1 - \alpha) \cdot \dot{Q}_{incident}$
- Kirchoff's law says that $\alpha = \epsilon$
- Note: α and ϵ vary over different wavelengths
- Consider a special case of radiation
  - Small surface which is completely surrounded by a much larger surface
  - T_s, A_s are temperature and area of the small surface (which is also the boundary), T_{surr} is the temperature of the surrounding surface. Both surfaces are emitting and we are interested in the net emission
  - $\dot{Q}_{rad} = \epsilon \sigma A_s (T_s^4 - T_{surr}^4)$
- Example
  - Chip with an area of $15 \times 15mm$, $\epsilon = 0.6$, $T_{surr} = 25$.
  - Two methods of heat transfer
    - Natural convection
      - $h = c(T_s - T_\infty)^{\frac{1}{4}}$
      - $c = 4.2 \frac{W}{m^2 K^{\frac{5}{4}}}$
      - $q_{conv} = hA(T_S - T_\infty)$
      - $q_{rad} = \epsilon A(T_s^4 - T_{surr}^4)$
    - Forced convetion: h is constant at $250 \ \frac{W}{m^2 K}$
      - $q_{conv} = hA(T_S - T_\infty)$

4 Heat Conduction

- Heat Conduction Equation
  - x, y, z components of $\dot{Q}$
  - T is a function of (x, y, z, t)
  - $\vec{\dot{Q}} = \dot{Q}_x\hat{i} + \dot{Q}_y\hat{j} + \dot{Q}_z\hat{k}$
  - $\dot{Q}_x = -kA_x\frac{dT}{dx}$ (similar expressions for $\dot{Q}_y$ and $\dot{Q}_z$
- One dimensional heat conductivity can model more complicated situations. For example if $\Delta x << \Delta y, z$, $\frac{dT}{dx} >> \frac{dT}{dy}, \frac{dT}{dz}$ so that $\dot{Q}_y$ and $\dot{Q}_z$ can be neglected
- One dimensional heat conduction
  - Cross sectional area is $A(x)$ where x is the coordinate along which heat transfer occurs
  - $\dot{Q}_x$ at the entry and $\dot{Q}_{x+\Delta x}$ at the exit
  - Want to find $T(x)$ inside the object
  - Rate of increase of enthalpy $= mc_p\frac{\partial T}{\partial t} = \rho V c_p\frac{\partial T}{\partial t} = \rho c_p A\Delta x\frac{\partial T}{\partial t}$
  - Energy balance:
    * $\rho c_p A\Delta x\frac{\partial T}{\partial t} = \dot{Q}_x - \dot{Q}_{x+\Delta x}$
    * After simplifying and taking the limit as Δx approaches 0, we get $\rho c_p\frac{\partial T}{\partial t} = \frac{-1}{A}\frac{\partial(\dot{q}A)}{\partial x}$
    * A depends on the coordinate system and we use Fourier's law for $\dot{q}$: $\dot{q} = -k\frac{dT}{dx}$
- Cartesian Coordinates
  - A is a constant
  - $pc_p\frac{\partial T}{\partial t} = \frac{\partial}{\partial x}[k\frac{\partial T}{\partial x}]$
  - Assume k is constant. Then $\frac{\partial T}{\partial t} = \alpha\frac{\partial^2 T}{\partial x^2}$ where $\alpha = \frac{k}{pc_p}$.
  - If steady state i.e. $\frac{\partial T}{\partial t} = 0$ then $\frac{d^2T}{dx^2} = 0$.
  - Units of $\alpha = \frac{k}{pc_p}$, the thermal diffusivity is $\frac{m^2}{s}$.
  - High k means the material conducts well. High pc_p means that the material stores energy
- Cylindrical Coordinates
  - Heat being conducted radially so $\dot{q} = -k\frac{\partial T}{\partial r}$ and $A = 2\pi rL$
  - $pc_p\frac{\partial T}{\partial t} = \frac{-1}{2\pi rL}[\frac{\partial}{\partial r}\cdot\frac{\partial T}{\partial r}]$
  - $\frac{1}{\alpha}\frac{\partial T}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}(r\cdot\frac{\partial T}{\partial r})$
  - At steady state, $\frac{d}{dr}(r\frac{dT}{dr}) = 0$.

- Spherical Coordinates
  - $A = 4\pi r^2$ and $\dot{q} = -k\frac{\partial T}{\partial r}$ where r is the radial spherical coordinate
  - $\frac{1}{\alpha}\frac{\partial T}{\partial t} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial T}{\partial r})$
- In general $\frac{1}{r^n}\frac{\partial}{\partial r}(r^n\frac{\partial T}{\partial r}) = \frac{1}{\alpha}\frac{\partial T}{\partial t}$ where cartesian has $n = 0$, cylindrical has $n = 1$ and spherical has $n = 2$.

5 Thermal Resistance

- At steady state, $\frac{d^2T}{dx^2} = 0$
- Heat flux: $\dot{q} = -k\frac{dT}{dx}$.
- Heat flux is a constant
- Heat transer rate: $\dot{Q} = \dot{q}A = \frac{-kA(T_2-T_1)}{L}$
- $\dot{Q} = \frac{T_1-T_2}{R_{wall}}$ where T_1 and T_2 are the temperatures of the walls
- $R_{cond} = R_{wall} = \frac{L}{kA}$
- Similar to current with voltage and Resistance
- $\dot{Q}_{conv} = hA(T_s - T_\infty)$
- $R_{conv} = \frac{T_s-T_\infty}{Q_{conv}} = \frac{1}{hA}$
- Radiation is more complicated. $\dot{Q}_{rad} = \epsilon\sigma A(T_s^4 - T_{sur}^4)$
- We need to define a heat transfer coefficient for radiation. $h_{rad} = \frac{\epsilon\sigma A(T_s^4-T_{sur}^4)}{A(T_s-T_{sur})}$
- $h_{rad} = \epsilon\sigma(T_s^2 + T_{sur}^2)(T_s + T_{sur})$
- Can treat it as a resistance. $R_{rad} = \frac{T_s-T_{sur}}{\dot{Q}_{rad}} = \frac{1}{h_{rad}A}$
- Multilayer Plane Wall
  - Each layer has the same surface Area
  - Layers have thicknesses L_i
  - Temperature varies as T_1 on the outside, T_2, ..., T_{n+1} where n is the number of surfaces
  - Treat each layer seperately as resistances in series
  - For first wall $\dot{Q} = \frac{T_1-T_2}{R_1}$ so $T_1 - T_2 = \dot{Q}R_1$. In general, $T_i - T_{i+1} = \dot{Q}R_i$
  - Summing all of them gives $T_1 - T_{n+1} = \dot{Q}(R_1 + ... + R_n)$ so that $\dot{Q} = \frac{T_1-T_4}{R_{total}}$
  - Therefore you can sum up resistances similar to electric circuits

- We can find each R_i as $\frac{L_i}{k_i A}$

- $\dot{Q} = UA(T_{\infty,1} - T_{\infty,4})$ where U is the overall heat transfer

- Example: Refrigerator Wall

  - 1 mm thick insulation on the outside and width of refrigerator is L.
  - $T_{room} = 25C$ and $T_{refrig} = 3C$
  - $h_0 = 9\frac{W}{m^2C}$ and $h_i = 4\frac{W}{m^2C}$
  - $k_{steel} = 15.1\frac{W}{m^2C}$ and $k = 0.035\frac{W}{m^2C}$ inside the refrigerator
  - Constraint is that $T_{s,out} > 20$. We assume heat transfer from the outside room to the inside of the refrigerator
  - What is L to ensure $T_{s,out} > 20$ to prevent condensation on the outside of the refrigerator
  - Thermal circuit consists of a convective resistance outside the refrigerator followed by 3 conductive resistance on the surfaces and then one convective resistance at the end
  - INSERT PICTURE FROM LEC NOTES
  - $\dot{Q} = \frac{T_{room} - T_{s,out}}{R_{conv,0}} = \frac{T_{room} - T_{s,out}}{\frac{1}{h_0 A}}$
  - Consider unit area. Then $\dot{Q} = h_0(T_{room} - T_{s,out}) = 9(25 - 20) = 45W$
  - $R_{total} = \frac{1}{h} + (\frac{L}{k})_{metal} + (\frac{L}{k})_{insulation} + (\frac{L}{k})_{metal} + \frac{1}{h_i} = \frac{1}{9} + \frac{10^{-3}}{15.1} + \frac{L_2}{0.035} + \frac{10^{-3}}{15.1} + \frac{1}{4} = 0.361 + \frac{L_2}{0.035}$
  - $\dot{Q} = \frac{T_{room} - T_{refrig}}{R_{total}} \rightarrow 45(0.361 + \frac{L_2}{0.035}) = 25 - 3$ so that $L = 45mm$

6 Thermal Resistance Networks

- Multiple layers e.g. in an electric chip each with different thermal properties

  - We are interested in $\dot{Q} = \dot{Q}_1 + \dot{Q}_2 + \dot{Q}_3 = \frac{T_1 - T_2}{R_1} + \frac{T_1 - T_2}{R_2} + \frac{T_1 - T_2}{R_3}$
  - $= (T_1 - T_2)(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}) = \frac{T_1 - T_2}{R_{total}}$
  - This is the electrical analog to parallel resistances

- Thermal contact resistance

  - So far we have been assuming perfect contact between different boundaries
  - In reality, there is a rough surface at the boundary
  - We can always define a thermal contact resistance $R_c = \frac{T_2 - T_1}{\dot{q}}$ (units are $\frac{m^2C}{W}$ (Note: Resistanc per unit area)
  - The reciprocal of R_C is known as the thermal contact conductance h_c.

- $h = \frac{1}{R_c} = \frac{\dot{q}}{\Delta T}$ so that $\dot{q} = h_c \Delta T$
- h_c is thus similar to the heat transfer coefficient

- Heat conduction in cylinders & spheres
  - INSERT DRAWING FROM THE SLIDEs
  - For a long pipe, the main temperature gradient is in the radial direction i.e. $\frac{dT}{dx} << \frac{dT}{dR}$
  - Therefore we can assume 1-D radial conduction
  - INSERT r_1, r_2 diagram
  - Solve heat conduction equation in cylindrical coordinates to get $T(r)$
  - Steady state: $\frac{d}{dr}(r\frac{dT}{dr}) = 0$, at $r = r_1, T = T_1$ and at $r = r_2, T = T_2$
  - Integrate this to get $T(r) = c_1 \ln r + c_2$
  - Using the boundary conditions gives $c_1 = \frac{T_1 - T_2}{\ln(\frac{r_1}{r_2})}$ and $c_2 = T_2 - \frac{T_1 - T_2}{\ln(\frac{r_1}{r_2})} \ln r_2$

- Define thermal resistance of a cylinder
  - $R_{cyl} = \frac{T_1 - T_2}{\dot{Q}_{cond}} = \frac{\ln(\frac{r_2}{r_1})}{2\pi LK}$

7 Conduction in cylinders and spheres, Insulation

- Inner temperature and then two surface layers
- $R_C = \frac{1}{hA} = \frac{1}{2\pi rLh}$
- $R_{total} = R_{c,1} + R_{cond} + R_{c,2} = \frac{1}{2\pi r_1 L h_1} + \frac{\ln(\frac{r_2}{r_!})}{2\pi LR} + \frac{1}{2\pi r_2 L h_2}$
- Insulation
  - $R_{total} = R_{c,1} + R_{cyl,1} + R_{cyl,2} + R_{c,2}$ where $R_{cyl,2}$ is for the insulation
  - For a sphere
    * Insulation around a spherical metal tank
    * $R_{total} = R_{c,1} + R_{sph,1} + R_{sph,2} + R_{c,2}$
    * $= \frac{1}{4\pi r_1^2 h_1} + \frac{r_2 - r_1}{4\pi r_1 r_2 k_1} + \frac{r_3 - r_2}{4\pi r_3 r_2 k} + \frac{1}{4\pi r_3^2 h_2}$
- R-value
  - Thermal resistance. Thickness L, Surface area A and thermal conductivity k
  - $R = \frac{L}{k}$ is the R-value
  - $\dot{Q} = \frac{\Delta T}{R} \times A$
  - Units here are in imperial units i.e. L is in feet, k is in $\frac{Btu}{hftF}$

- Critical Radius of insulation
  - Consider the area for heat loss
  - Insulation: increase thickness, increasing conduction resistance and decreasing convective resistance
  - Can we increase heat transfer?
    * Plot $\dot{Q}$ against r_2 to find the critical value of Resistance
    * Equivalently set $\frac{d\dot{Q}}{dr_2} = 0$ to find the critical radius

8 Heat Transfer from Finned Surfaces

- Read Chapter 17.6
- How to find $R_{heatsink}$ for finned surfaces e.g. heat sinks in computers
- Add diagram from notes: We have a cylinder with cross sectional area $A_c(x)$ and heat transfer coeff h.
  - Consider a thin slice of this with thickness Δx some distance x away from the end
  - Energy balance $\dot{Q}_{cond,x}$ in and $\dot{Q}_{cond,x+\Delta x}$ out
  - Energy in = Energy out: $\dot{Q}_{cond,x} = \dot{Q}_{cond,x+\Delta x} + \dot{Q}_{conv}$
  - Let the perimeter of fin be P. Then surface area of element is $P\Delta x$ so that $\dot{Q}_{conv} = hP\Delta x(T - T_\infty)$
  - Simplifying the energy balance by taking the limit as $\Delta x \to 0$: $\frac{d\dot{Q}_{cond}}{dx} + hP(T - T_\infty) = 0$
  - Using Fourier's law $\dot{Q}_{cond} = -kA_C\frac{dT}{dx}$: $\frac{d}{dx}(kA_c\frac{dT}{dx}) - hP(T - T_\infty) = 0$
  - Assuming A_C, k, P constant: $\frac{d^2T}{dx^2} - \frac{hP}{kA_C}(T - T_\infty) = 0$
  - Define $\Theta = T - T_\infty$ and $a^2 = \frac{hP}{kA_c}$ (constant) so that $\frac{d^2\Theta}{dx^2} - a^2\Theta = 0$ where the solution is $\Theta(x) = c_1e^{ax} + c_2e^{-ax}$
  - Boundary conditions: $T = T_b$ at the left end while on the right end as $L \to \infty$, $T = T_\infty$
  - Therefore we simplify by having $T = T_\infty$ at $x = L$
  - In terms of Θ: At $x = 0$, $\Theta = T_b - T_\infty = \Theta_b$ and $\Theta(\infty) = 0$
  - This gives $c_1 = 0$ and $c_2 = \Theta_b$ so that the solution is $\Theta(x) = \Theta_b e^{-ax}$
- What is the heat loss from the fin? $\dot{Q}_b = -kA_c\frac{dT}{dx}|_{x=0}\dot{Q}_{fin}$
- $\dot{Q}_{fin,long} = \sqrt{hPkA_C}(T_b - T_\infty)$
- Finite fin length: What is the boundary condition at the open end

- We can heat transfer is negligible so that adiabatic and $\frac{dT}{dx} = 0$ at the boundary
- At $x = L$, $\frac{dT}{dx} = 0$ so that $\frac{d\Theta}{dx} = 0$: $c_1 e^{aL} - c_2 e^{-aL} = 0$
- At $x = 0$, $\Theta = \Theta_b$
- Solve for c_1 & c_2 and the following solution will be obtained: $\frac{T(x) - T_\infty}{T_b - T_\infty} = \frac{\cosh a(L-x)}{\cosh aL}$

- Can do the same for $\dot{Q}_{fin,insulated} = -kA_C \frac{dT}{dx}|_{x=0}$ which will give $\dot{Q}_{fin,insulated} = \sqrt{hPkA_c}(T_b - T_\infty)\tanh(aL)$ where $a = \sqrt{\frac{hP}{kA_c}}$
- To account for heat transfer from the tip, we can add a length ΔL at the end and the area there will be $A_c = \Delta LP$ (P is the perimeter of the fin) so that the corrected length is $L_c = L + \frac{A_c}{P}$

9 Heat transfer from finned surfaces (contd)

- Fin efficiency
  - ΔT given by the difference between the fin temperature and the surrounding
  - Most efficient fin would have a uniform temperature T_b everywhere
  - This would imply an infinite thermal conductivity
  - In this case, $\dot{Q} = hA_{fin}(T_b - T_\infty) = hPL(T_b - T_\infty)$
  - We define $\eta_{fin} = \frac{dotQ_{fin}}{\dot{Q}_{fin,max}}$
  - For an infinitely long fin, $\eta_{fin,long} = \frac{\sqrt{hPkA_c}(T_b - T_\infty)}{hPL(T_b - T_\infty)} = \frac{1}{L}\sqrt{\frac{kA_c}{hP}} = \frac{1}{aL}$ which is in terms of physical properties
  - $\dot{Q}_{fin} = \eta_{fin}\dot{Q}_{fin,max} = \eta_{fin} hA_{fin}(T_b - T_\infty) = h\eta_{fin}A_{fin}(T_b - T_\infty)$
  - $\eta_{fin}A_{fin}$ can be treated as the corrected area
  - For an insulated tip, perform the same steps with the original definition to get $\eta_{insulatedtip} = \frac{\tanh aL}{aL}$
- Fin effectiveness
  - How much has the fin increased heat transfer by?
  - $\epsilon_{fin} = \frac{\dot{Q}_{fin}}{\dot{Q}_{nofin}}$
  - $\dot{Q}_{nofin} = hA_c(T_b - T_\infty)$
  - $\dot{Q}_{longfin} = \sqrt{hPkA_c}(T_b - T_\infty)$ so that $\epsilon_{longfin} = \sqrt{\frac{kP}{hA_c}}$
  - To increase the effectiveness, make k as large as possible and maximize $\frac{P}{A_c}$
  - Fins are most effective with low h so they are used for gases, hot liquids
  - Generally we use fins if $\epsilon \geq 2$

- When can we assume fins are infinitely long?
  - $\frac{\dot{Q}_{fin,insulated}}{\dot{Q}_{fin,long}} = \tanh(aL)$. tanh asymptotically approaches 1 as aL approaches ∞
  - In practice, if $aL \geq 5$ we can assume an infinitely long fin. But even $aL = 1$ has tanh $= 0.76$ so it gives 76 % of heat transfer of an infinitely long fin. Therefore $L = \frac{1}{a}$ is a reasonable length for a fin

- Designing a heat sink
  - $\dot{Q}_{total} = \dot{Q}_{unfinned} + \dot{Q}_{fin}$
  - From the definition of efficience $\dot{Q}_{fin} = \eta_{fin} \cdot hA_{fin}(T_b - T_\infty)$
  - $\Rightarrow \dot{Q}_{total} = hA_{unfinned}(T_b - T_\infty) + h\eta_{fin}A_{fin}(T_b - T_\infty) = h\left[A_{unfinned} + \eta_{fin}A_{fin}\right](T_b - T_\infty)$
  - We can define a thermal resistance $R_{fin} = \frac{T_b - T_\infty}{\dot{Q}_{total}} = \frac{1}{h\left[A_{unfinned} + \eta_{fin}A_{fin}\right]}$

10 Transient Heat Conduction

- Consider a solid at temperture T_i and a liquid at $T_\infty < T_i$. The solid is dropped into the liquid. How does T vary over time?
  - We would expect the temperature T to asymptotically approach T_∞
  - Heat Conduction equation: $\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$
  - We set the first three terms equal to 0 by using the lumped capacitance approximation i.e. no temperature gradient in the body
  - We would expect this to be valid when the object is small and has a high thermal conductivity
  - Using an energy balance: $\dot{E}_{store} = -\dot{Q}_{conv}$
  - $\dot{E}_{store} = mc_p\frac{dT}{dt} = \rho V c_p \frac{dT}{d}$ and $\dot{Q}_{conv} = hA(T - T_\infty)$
  - Equating the two, we get $\frac{d(T-T_\infty)}{T-T_\infty} = -\frac{hA}{\rho V c_p}dt$ so that $\ln(T - T_\infty) = \frac{-hA}{\rho V c_p}t + C_1$
  - Using $T = T_i$ at $t = 0$, we get $\ln\left[\frac{T-T_\infty}{T_i-T_\infty}\right] = \frac{-hA}{\rho V c_p}t$
  - We define a "time constant" $\tau = \frac{\rho V c_p}{hA}$ so that $\frac{T-T_\infty}{T_i-T_\infty} = \exp\left[\frac{-t}{\tau}\right]$
  - The LHS starts at 1 and decays to 0 as $t \to \infty$. Moreoever at $t = \tau$, the value is $\frac{1}{e} \approx 0.368$

- The response time of a thermometer is usually 3τ. However it is important to note that τ is a function of h so it varies in different environments

- Moreover τ depends on $\frac{V}{A} = \frac{r}{3}$ for a sphere so to get a fast response time you would make it very thin

- When is a lumped capacitance valid?

- At steady state, the conduction in a solid must be equal to the convection in a fluid i.e. $kA\frac{(T_1-T_2)}{L} = hA(T_2 - T_\infty)$ i.e. $\frac{T_1-T_2}{T_2-T_\infty} = \frac{hL}{k}$
- $\frac{hL}{k}$ is a dimensionless number and is known as a Biot number.
- Note: k is the thermal conductivity of the solid, L is the length scale in the direction of conduction
- Suppose the Biot number is large i.e. $>> 1$ so that $T_1 - T_2 >> T_2 - T_\infty$
- If Biot number is very small, then $T_1 - T_2 << T_2 - T_\infty$ so we can neglect T change inside the body (we assume uniform temp in the body) and use the lumped capacitance model
- By $<<$ we typically mean a Biot value < 0.1
- In an irregular body, the length scale used is $L = \frac{V}{A}$

11 Transient Heat Conduction in 2 and 3 Dimensions

- A ball of volume V,mass m and SA A and heat transfer coefficient h is droppped into a fluid
- Assume that the temperature T is uniform in the body
- We had previously assumed that if $Bi < 0.1$ we have a lumped capacitance
- Example: Steel shaft, $k = 51.2$, $\rho = 7832$, $c = 541$ and $T_i = 300$ is placed into a furnace with $T_\infty = 1200$.
  - How long before the shaft temperature reaches 800?
  - We first calculate the Biot number as $Bi = \frac{hL}{k}$ where $L = \frac{V}{A} = \frac{\pi r^2 L}{2\pi r L} = \frac{r}{2}$
  - Then $Bi = \frac{h\frac{r}{2}}{k} = \frac{100\times\frac{0.05}{2}}{51.2} = 0.05$ so we can apply the lumped value
  - $\frac{T-T_\infty}{T_i-T_\infty} = \exp\left[-\frac{hA}{\rho V c}t\right]$ where $\frac{A}{V} = \frac{2}{r}$
  - This gives $\ln\left[\frac{800-1200}{300-1200}\right] = \ldots$ and solving gives $t = 859s$
- Transient heat conduction in 3 dimensions e.g. plane walls, cylinders, spheres
  - What happens if $Bi > 0.1$?
  - In this case we cannot neglect the temperature gradients inside the body
  - Have to solve the complete heat conduction equation
  - Consider a solid wall with temperature T_i on one side which then instantly becomes lowered to a temperature T_∞ as it is placed into a fluid.
  - As time increases, the temperature inside the wall e.g. at the center decreases
  - So in this case, T is a function of x and t
  - For a plane wall $\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$ where $\alpha = \frac{k}{\rho c_p}$ is the thermal diffusivity

- Second order wrt x so two boundary conditions are needed there. First order wrt t so one initial condition is neeeded there
- This can be solved analytically but we will not do that
- We instead consider the lumped capacitance solution $\ln\left[\frac{T-T_\infty}{T_i-T_\infty}\right] = \frac{-hA}{\rho V c_p}t$
- We take the characteristic length $L = \frac{V}{A}$.
- $\frac{hA}{\rho V c_p}t = \frac{h}{\rho L c_p}t = (\frac{h}{\rho L c_p}t)(\frac{L}{L}\cdot\frac{k}{k}) = (\frac{hL}{k})(\frac{k}{\rho c_p})(\frac{t}{L^2}) = (\frac{hL}{k})(\frac{\alpha t}{L^2})$ where $Bi = \frac{hL}{k}$ which is unitless
- We define the Fourier number $Fo = \frac{\alpha t}{L^2}$ which is also dimensionless
- The dimensionless temperature $\Theta = \frac{T-T_\infty}{T_i-T_\infty}$ so that the lumped capacitance solution can be written as $\Theta = \exp(-Bi \cdot Fo)$
- A physical interpreation of the Fourier number can be found by considering a cube with side length L. Then $\dot{Q}_{cond} = kA\frac{\partial T}{\partial x} = kL^2\frac{\Delta T}{L} = kL\Delta T$
- $\dot{Q}_{store} = mc_p\frac{\partial T}{\partial t} = \rho L^3 c_p\frac{\Delta T}{t}$ so that $\frac{\dot{Q}_{cond}}{\dot{Q}_{store}} = \frac{k}{\rho c_p}\cdot\frac{t}{L^2} = \frac{\alpha t}{L^2} = Fo$
- Even when we cannot assume lumped capacitance and get an exact solution of the heat conduction equation, the solution is of the form $\Theta = \Theta(Bi, Fo)$
- We can define $\Theta_0 = \frac{T_0-T_\infty}{T_i-T_\infty}$ so that the solution is of the form $\Theta_0 = A_1 e^{-\lambda_1^2 Fo}$ where A_1, λ_1 are function of Bi

• Example: Carbon steel plate with $T_i = 440$ is placed in a furnace at $T_\infty = 600$. We need to heat to a minimum temperature of 520. What is the time t required.

- $h = 200$, $k = 40$ and $\alpha = 8 \times 10^{-6}$
- $Bi = \frac{hL}{k} = \frac{200\times 0.04}{40} = 0.2$
- Since $Bi > 0.1$, we cannot use the lumped capacitance and instead use the analytical solution which we obtain from tables
- From Table 18.2, $Bi = 0.2 \implies \lambda_1 = 0.4328, A_1 = 1.0311$
- $\Theta_0 = \frac{T_0-T_\infty}{T_i-T_\infty} = A_1 e^{-\lambda_1^2 Fo}$
- $\Theta_0 = \frac{520-600}{440-600} = 0.5$ so that $0.5 = 1.0311\exp(-(0.4328)^2 Fo)$ and so $Fo = 3.864$
- $Fo = \frac{\alpha t}{L^2} \implies t = \frac{FoL^2}{\alpha} = \frac{3.864\times(0.04)^2}{8\times 10^{-6}} = 773s$

12 Transient Heat Conduction in Semi-Infinite Solids

• The general problem consists of a body at temperature T_i with the surface temperature suddenly being changed to T_s.

- Temperature variation would go from T_s down to T_i. δ corresponds to a skin depth and after that we are in the core
- How does the skin depth δ vary with time?

- Heat conduction equation with one dimension: $\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$
- An estimate of $\frac{\partial^2 T}{\partial x^2}$ can be found as $\frac{(\frac{\partial T}{\partial x})_{x\sim\delta} - (\frac{\partial T}{\partial x})_{x\sim 0}}{\delta}$
- $(\frac{\partial T}{\partial x})_{x=0} \sim \frac{T_i - T_s}{\delta}$ so that $\frac{\partial^2 T}{\partial x^2} \sim 0 - \frac{T_i - T_s}{\delta^2}$, $\frac{\partial T}{\partial t} \sim \frac{T_s - T_i}{t}$
- From the heat conduction equation $-\frac{T_i - T_s}{\delta^2} \sim \frac{1}{\alpha}\frac{T_s - T_i}{t}$
- Time for effect to be felt throughout the body is $t_c \sim \frac{r_0^2}{\alpha}$
- $\delta \sim \sqrt{\alpha t}$
- For a short time $t << t_c$, we can treat the body as being semi infinite
- The exact solution is given by $\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$
- δ grows as a function of time
- Boundary conditions: At $x = 0$, $T(0,t) = T_s$ and as $x \to \ \infty T(\infty, t) = T_i$
- Initial conditions: $T(x,0) = T_i$
- Define a similarity variable $\eta = \frac{x}{\delta}$ such that $0 < \eta < 1$
- $\eta = \frac{x}{2\sqrt{\alpha t}}$ so that $\frac{\partial T}{\partial t} = \frac{dT}{d\eta} \cdot \frac{\partial \eta}{\partial t} = \frac{d}{dt}\left[\frac{-x}{4t\sqrt{\alpha t}}\right]$
- $\frac{\partial T}{\partial x} = \frac{dT}{d\eta} \cdot \frac{\partial \eta}{\partial x} = \frac{dT}{d\eta}\left[\frac{1}{2\sqrt{\alpha t}}\right]$
- $\frac{\partial^2 T}{\partial x^2} = \frac{d}{d\eta}(\frac{\partial T}{\partial \eta}) \cdot \frac{\partial \eta}{\partial x} = \frac{d^2 T}{d\eta} = \frac{1}{4\alpha t}$
- Transforming the heat conduction equation, we get $\frac{d^2 T}{d\eta^2} = \frac{1}{\alpha}\frac{dT}{d\eta}(\frac{-x}{4t\sqrt{\alpha t}}) = -2\eta\frac{dT}{d\eta}$
- The PDE is now an ODE
- At $x = 0$, $\eta = 0$, $T(0) = T_s$ and as $x \to \infty$, $\eta \to \infty$, $T(\infty) = T_i$
- Let $w = \frac{dT}{d\eta}$ so that $\frac{dw}{d\eta} = -2\eta w$ which gives $\ln w = -\eta^2 + c_0$
- $\frac{dT}{d\eta} = w = c_0 e^{-\eta^2}$ and integrating, $T = c_0 \int_0^\eta e^{-u^2}\, du + c_1$
- Boundary conditions give $T_i = c_0 \int_0^\infty e^{-u^2}\, du + T_s$ which gives the solution $\frac{T - T_s}{T_i - T_s} = \frac{2}{\sqrt{\pi}} \int_0^\eta e^{-u^2}\, du = erf(\eta)$ where the RHS is the error function
- Can also be written as $1 - \frac{T - T_s}{T_i - T_s} = 1 - erf(\eta)$ or $\frac{T - T_i}{T_s - T_i} = erfc(\eta)$

- Heat Flux at Surface
  - $\dot{q}_s = -k\frac{\partial T}{\partial x}|_{x=0} = 0k\frac{\partial T}{\partial \eta}\frac{\partial \eta}{\partial x}|_{\eta=0}$
  - $\frac{d\eta}{dx} = \frac{1}{2\sqrt{\alpha t}}$
  - $\dot{q} = \frac{k(T_s - T_i)}{\sqrt{\pi \alpha t}}$
- Contact of two semi-infinite bodies
  - One body with temperature $T_{A,i}$ and conductivity k_A and the other with $T_{B,i}$ and k_B

- We require that $T_{S,A} = T_{S,B}$
- The heat flux must be the same i.e. $\dot{q}_{S,A} = \dot{q}_{S,B}$
- $-\frac{k_A(T_s - T_{A,i})}{\sqrt{\pi(\alpha_a t)}} = \frac{k_B(T_s - T_{B,i})}{\sqrt{\pi(\alpha_B t)}}$
- $\implies \frac{T_{A,i} - T_s}{T_s - T_{B,i}} = \frac{\sqrt{(k\rho c)_B}}{\sqrt{(k\rho c)_A}}$
- Define effusivity as $\gamma = \sqrt{k\rho c}$