

# CHE260: Heat Transfer

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## 1 Introduction

- How is heat transfer different from thermodynamics? In thermodynamics, we assume quasi-equilibrium processes i.e. the time was not an important parameter. In heat

transfer, time is an important parameter and we are interested in the rate of heat transfer.

- What is the relationship between  $\dot{Q}$  and  $\Delta T$ ? What are the mechanisms of heat transfer?
- Conduction: Transfer of heat through a medium that is stationary.
- Convection: Transfer of heat from a solid surface and an adjacent fluid that is moving. Example: a fan blowing air over a hot plate. There is heat transfer from the hot plate into the fluid.
- Radiation: Energy emitted by matter in the form of electromagnetic waves.
- Radiation does not need a medium. In a vacuum, we can have radiation but not convection or conduction.
- Different mechanisms of heat transfer can take place simultaneously.
- Applications
  - Power Generation
    - \* Power plant: steam generation, condenser
    - \* Automobiles: engine cooling, space heating/cooling
  - Buildings
    - \* Heating / Cooling
    - \* Hot water
  - Refrigeration
  - Manufacturing
    - \* Casting / Heat treatment
    - \* Injection Moulding

## 2 Electronic Cooling

- > 99 % of the electrical energy supplied to a circuit is dissipated as heat
- Heat has to be dissipated to the environment while keeping the temperature of the chip in a certain range
- Heat is lost from the surface of the chip
- Important parameter is heat flux =  $\frac{\text{Heat Transfer Rate}}{\text{Unit Area}}$  (in  $\frac{W}{\text{cm}^2}$ )
- To reduce heat flux, we can reduce heat generation and increase the surface area
- As size increases, it becomes more difficult to lose heat
- Water cooling is more efficient for large systems compared to air cooling

### 3 Radiation

- Radiation is energy emitted by all matter in the form of e.m. radiation
- Thermal radiation is emitted by all bodies at a finite temperature
- Opaque objects emit only from the surface
- Amount of radiation depends on the surface temperature. Summarized by the Stefan Boltzmann Law:  $\dot{Q}_{emit} = \sigma AT_s^4$  where  $\sigma$  is the Boltzmann constant ( $5.67 \times 10^{-8} \frac{W}{m^2 K^4}$ ),  $T_s$  is the surface temperature in Kelvin and  $A$  is the surface area.
- A surface that emits as much radiation as this is called a "Blackbody". A real surface emits less than this.:  $\dot{Q}_{emit} = \epsilon \sigma AT_s^4$  where  $\epsilon$  is the emissivity and  $0 \leq \epsilon \leq 1$
- Black paint has  $\epsilon = 0.99$  which is very close to 1. Aluminum foil has a low emissivity of around 0.07.
- If radiation is incident on a surface some will be absorbed. The fraction absorbed is a surface property known as the absorptivity  $\alpha$  such that  $\dot{Q}_{absorbed} = \alpha \cdot \dot{Q}_{incident}$  and  $\dot{Q}_{reflected} = (1 - \alpha) \cdot \dot{Q}_{incident}$
- Kirchoff's law says that  $\alpha = \epsilon$
- Note:  $\alpha$  and  $\epsilon$  vary over different wavelengths
- Consider a special case of radiation
  - Small surface which is completely surrounded by a much larger surface
  - $T_s, A_s$  are temperature and area of the small surface (which is also the boundary),  $T_{surr}$  is the temperature of the surrounding surface. Both surfaces are emitting and we are interested in the net emission
  - $\dot{Q}_{rad} = \epsilon \sigma A_s (T_s^4 - T_{surr}^4)$
- Example
  - Chip with an area of  $15 \times 15 mm$ ,  $\epsilon = 0.6$ ,  $T_{surr} = 25$ .
  - Two methods of heat transfer
    - \* Natural convection
      - $h = c(T_s - T_\infty)^{\frac{1}{4}}$
      - $c = 4.2 \frac{W}{m^2 K^{\frac{5}{4}}}$
      - $q_{conv} = hA(T_s - T_\infty)$
      - $q_{rad} = \epsilon A(T_s^4 - T_{surr}^4)$
    - \* Forced convection:  $h$  is constant at  $250 \frac{W}{m^2 K}$ 
      - $q_{conv} = hA(T_s - T_\infty)$

## 4 Heat Conduction

- Heat Conduction Equation

- $x, y, z$  components of  $\dot{Q}$
- $T$  is a function of  $(x, y, z, t)$
- $\vec{\dot{Q}} = \dot{Q}_x \hat{i} + \dot{Q}_y \hat{j} + \dot{Q}_z \hat{k}$
- $\dot{Q}_x = -kA_x \frac{dT}{dx}$  (similar expressions for  $\dot{Q}_y$  and  $\dot{Q}_z$ )

- One dimensional heat conductivity can model more complicated situations. For example if  $\Delta x \ll \Delta y, z, \frac{dT}{dx} \gg \frac{dT}{dy}, \frac{dT}{dz}$  so that  $\dot{Q}_y$  and  $\dot{Q}_z$  can be neglected

- One dimensional heat conduction

- Cross sectional area is  $A(x)$  where  $x$  is the coordinate along which heat transfer occurs
- $\dot{Q}_x$  at the entry and  $\dot{Q}_{x+\Delta x}$  at the exit
- Want to find  $T(x)$  inside the object
- Rate of increase of enthalpy  $= mc_p \frac{\partial T}{\partial t} = \rho V c_p \frac{\partial T}{\partial t} = \rho c_p A \Delta x \frac{\partial T}{\partial t}$
- Energy balance:
  - \*  $\rho c_p A \Delta x \frac{\partial T}{\partial t} = \dot{Q}_x - \dot{Q}_{x+\Delta x}$
  - \* After simplifying and taking the limit as  $\Delta x$  approaches 0, we get  $\rho c_p \frac{\partial T}{\partial t} = \frac{-1}{A} \frac{\partial(\dot{q}A)}{\partial x}$
  - \*  $A$  depends on the coordinate system and we use Fourier's law for  $\dot{q}$ :  $\dot{q} = -k \frac{dT}{dx}$

- Cartesian Coordinates

- $A$  is a constant
- $\rho c_p \frac{\partial T}{\partial t} = \frac{\partial}{\partial x} [k \frac{\partial T}{\partial x}]$
- Assume  $k$  is constant. Then  $\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$  where  $\alpha = \frac{k}{\rho c_p}$ .
- If steady state i.e.  $\frac{\partial T}{\partial t} = 0$  then  $\frac{d^2 T}{dx^2} = 0$ .
- Units of  $\alpha = \frac{k}{\rho c_p}$ , the thermal diffusivity is  $\frac{m^2}{s}$ .
- High  $k$  means the material conducts well. High  $\rho c_p$  means that the material stores energy

- Cylindrical Coordinates

- Heat being conducted radially so  $\dot{q} = -k \frac{\partial T}{\partial r}$  and  $A = 2\pi r L$
- $\rho c_p \frac{\partial T}{\partial t} = \frac{-1}{2\pi r L} [\frac{\partial}{\partial r} \cdot \frac{\partial T}{\partial r}]$
- $\frac{1}{\alpha} \frac{\partial T}{\partial t} = \frac{1}{r} \frac{\partial}{\partial r} (r \cdot \frac{\partial T}{\partial r})$
- At steady state,  $\frac{d}{dr} (r \frac{dT}{dr}) = 0$ .

- Spherical Coordinates

–  $A = 4\pi r^2$  and  $\dot{q} = -k \frac{\partial T}{\partial r}$  where  $r$  is the radial spherical coordinate

–  $\frac{1}{\alpha} \frac{\partial T}{\partial t} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \frac{\partial T}{\partial r})$

- In general  $\frac{1}{r^n} \frac{\partial}{\partial r} (r^n \frac{\partial T}{\partial r}) = \frac{1}{\alpha} \frac{\partial T}{\partial t}$  where cartesian has  $n = 0$ , cylindrical has  $n = 1$  and spherical has  $n = 2$ .

## 5 Thermal Resistance

- At steady state,  $\frac{d^2 T}{dx^2} = 0$

- Heat flux:  $\dot{q} = -k \frac{dT}{dx}$ .

- Heat flux is a constant

- Heat transfer rate:  $\dot{Q} = \dot{q}A = \frac{-kA(T_2 - T_1)}{L}$

- $\dot{Q} = \frac{T_1 - T_2}{R_{wall}}$  where  $T_1$  and  $T_2$  are the temperatures of the walls

- $R_{cond} = R_{wall} = \frac{L}{kA}$

- Similar to current with voltage and Resistance

- $\dot{Q}_{conv} = hA(T_s - T_\infty)$

- $R_{conv} = \frac{T_s - T_\infty}{\dot{Q}_{conv}} = \frac{1}{hA}$

- Radiation is more complicated.  $\dot{Q}_{rad} = \epsilon \sigma A(T_s^4 - T_{sur}^4)$

- We need to define a heat transfer coefficient for radiation.  $h_{rad} = \frac{\epsilon \sigma A(T_s^4 - T_{sur}^4)}{A(T_s - T_{sur})}$

- $h_{rad} = \epsilon \sigma (T_s^2 + T_{sur}^2)(T_s + T_{sur})$

- Can treat it as a resistance.  $R_{rad} = \frac{T_s - T_{sur}}{\dot{Q}_{rad}} = \frac{1}{h_{rad}A}$

- Multilayer Plane Wall

– Each layer has the same surface Area

– Layers have thicknesses  $L_i$

– Temperature varies as  $T_1$  on the outside,  $T_2, \dots, T_{n+1}$  where  $n$  is the number of surfaces

– Treat each layer separately as resistances in series

– For first wall  $\dot{Q} = \frac{T_1 - T_2}{R_1}$  so  $T_1 - T_2 = \dot{Q}R_1$ . In general,  $T_i - T_{i+1} = \dot{Q}R_i$

– Summing all of them gives  $T_1 - T_{n+1} = \dot{Q}(R_1 + \dots + R_n)$  so that  $\dot{Q} = \frac{T_1 - T_{n+1}}{R_{total}}$

– Therefore you can sum up resistances similar to electric circuits

- We can find each  $R_i$  as  $\frac{L_i}{k_i A}$
- $\dot{Q} = UA(T_{\infty,1} - T_{\infty,4})$  where  $U$  is the overall heat transfer
- Example: Refrigerator Wall
  - 1 mm thick insulation on the outside and width of refrigerator is  $L$ .
  - $T_{room} = 25C$  and  $T_{refrig} = 3C$
  - $h_0 = 9 \frac{W}{m^2C}$  and  $h_i = 4 \frac{W}{m^2C}$
  - $k_{steel} = 15.1 \frac{W}{m^2C}$  and  $k = 0.035 \frac{W}{m^2C}$  inside the refrigerator
  - Constraint is that  $T_{s,out} > 20$ . We assume heat transfer from the outside room to the inside of the refrigerator
  - What is  $L$  to ensure  $T_{s,out} > 20$  to prevent condensation on the outside of the refrigerator
  - Thermal circuit consists of a convective resistance outside the refrigerator followed by 3 conductive resistance on the surfaces and then one convective resistance at the end
  - INSERT PICTURE FROM LEC NOTES
  - $\dot{Q} = \frac{T_{room} - T_{s,out}}{R_{conv,0}} = \frac{T_{room} - T_{s,out}}{\frac{1}{h_0 A}}$
  - Consider unit area. Then  $\dot{Q} = h_0(T_{room} - T_{s,out}) = 9(25 - 20) = 45W$
  - $R_{total} = \frac{1}{h} + (\frac{L}{k})_{metal} + (\frac{L}{k})_{insulation} + (\frac{L}{k})_{metal} + \frac{1}{h_i} = \frac{1}{9} + \frac{10^{-3}}{15.1} + \frac{L_2}{0.035} + \frac{10^{-3}}{15.1} + \frac{1}{4} = 0.361 + \frac{L_2}{0.035}$
  - $\dot{Q} = \frac{T_{room} - T_{refrig}}{R_{total}} \rightarrow 45(0.361 + \frac{L_2}{0.035}) = 25 - 3$  so that  $L = 45mm$

## 6 Thermal Resistance Networks

- Multiple layers e.g. in an electric chip each with different thermal properties
  - We are interested in  $\dot{Q} = \dot{Q}_1 + \dot{Q}_2 + \dot{Q}_3 = \frac{T_1 - T_2}{R_1} + \frac{T_1 - T_2}{R_2} + \frac{T_1 - T_2}{R_3}$
  - $= (T_1 - T_2)(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}) = \frac{T_1 - T_2}{R_{total}}$
  - This is the electrical analog to parallel resistances
- Thermal contact resistance
  - So far we have been assuming perfect contact between different boundaries
  - In reality, there is a rough surface at the boundary
  - We can always define a thermal contact resistance  $R_c = \frac{T_2 - T_1}{\dot{q}}$  (units are  $\frac{m^2C}{W}$ )  
(Note: Resistanc per unit area)
  - The reciprocal of  $R_C$  is known as the thermal contact conductance  $h_c$ .

- $h = \frac{1}{R_c} = \frac{\dot{q}}{\Delta T}$  so that  $\dot{q} = h_c \Delta T$
- $h_c$  is thus similar to the heat transfer coefficient
- Heat conduction in cylinders & spheres
  - INSERT DRAWING FROM THE SLIDEs
  - For a long pipe, the main temperature gradient is in the radial direction i.e.  $\frac{dT}{dx} \ll \frac{dT}{dr}$
  - Therefore we can assume 1-D radial conduction
  - INSERT  $r_1, r_2$  diagram
  - Solve heat conduction equation in cylindrical coordinates to get  $T(r)$
  - Steady state:  $\frac{d}{dr}(r \frac{dT}{dr}) = 0$ , at  $r = r_1, T = T_1$  and at  $r = r_2, T = T_2$
  - Integrate this to get  $T(r) = c_1 \ln r + c_2$
  - Using the boundary conditions gives  $c_1 = \frac{T_1 - T_2}{\ln(\frac{r_1}{r_2})}$  and  $c_2 = T_2 - \frac{T_1 - T_2}{\ln(\frac{r_1}{r_2})} \ln r_2$
- Define thermal resistance of a cylinder
  - $R_{cyl} = \frac{T_1 - T_2}{\dot{Q}_{cond}} = \frac{\ln(\frac{r_2}{r_1})}{2\pi L K}$

## 7 Conduction in cylinders and spheres, Insulation

- Inner temperature and then two surface layers
- $R_C = \frac{1}{hA} = \frac{1}{2\pi r L h}$
- $R_{total} = R_{c,1} + R_{cond} + R_{c,2} = \frac{1}{2\pi r_1 L h_1} + \frac{\ln(\frac{r_2}{r_1})}{2\pi L K} + \frac{1}{2\pi r_2 L h_2}$
- Insulation
  - $R_{total} = R_{c,1} + R_{cyl,1} + R_{cyl,2} + R_{c,2}$  where  $R_{cyl,2}$  is for the insulation
  - For a sphere
    - \* Insulation around a spherical metal tank
    - \*  $R_{total} = R_{c,1} + R_{sph,1} + R_{sph,2} + R_{c,2}$
    - \*  $= \frac{1}{4\pi r_1^2 h_1} + \frac{r_2 - r_1}{4\pi r_1 r_2 k_1} + \frac{r_3 - r_2}{4\pi r_3 r_2 k} + \frac{1}{4\pi r_3^2 h_2}$
- R-value
  - Thermal resistance. Thickness  $L$ , Surface area  $A$  and thermal conductivity  $k$
  - $R = \frac{L}{k}$  is the R-value
  - $\dot{Q} = \frac{\Delta T}{R} \times A$
  - Units here are in imperial units i.e.  $L$  is in feet,  $k$  is in  $\frac{Btu}{h ft F}$

- Critical Radius of insulation
  - Consider the area for heat loss
  - Insulation: increase thickness, increasing conduction resistance and decreasing convective resistance
  - Can we increase heat transfer?
    - \* Plot  $\dot{Q}$  against  $r_2$  to find the critical value of Resistance
    - \* Equivalently set  $\frac{d\dot{Q}}{dr_2} = 0$  to find the critical radius

## 8 Heat Transfer from Finned Surfaces

- Read Chapter 17.6
- How to find  $R_{heatsink}$  for finned surfaces e.g. heat sinks in computers
- Add diagram from notes: We have a cylinder with cross sectional area  $A_c(x)$  and heat transfer coeff  $h$ .
  - Consider a thin slice of this with thickness  $\Delta x$  some distance  $x$  away from the end
  - Energy balance  $\dot{Q}_{cond,x}$  in and  $\dot{Q}_{cond,x+\Delta x}$  out
  - Energy in = Energy out:  $\dot{Q}_{cond,x} = \dot{Q}_{cond,x+\Delta x} + \dot{Q}_{conv}$
  - Let the perimeter of fin be  $P$ . Then surface area of element is  $P\Delta x$  so that  $\dot{Q}_{conv} = hP\Delta x(T - T_\infty)$
  - Simplifying the energy balance by taking the limit as  $\Delta x \rightarrow 0$ :  $\frac{d\dot{Q}_{cond}}{dx} + hP(T - T_\infty) = 0$
  - Using Fourier's law  $\dot{Q}_{cond} = -kA_C \frac{dT}{dx}$ :  $\frac{d}{dx}(kA_c \frac{dT}{dx}) - hP(T - T_\infty) = 0$
  - Assuming  $A_C, k, P$  constant:  $\frac{d^2T}{dx^2} - \frac{hP}{kA_C}(T - T_\infty) = 0$
  - Define  $\Theta = T - T_\infty$  and  $a^2 = \frac{hP}{kA_C}$  (constant) so that  $\frac{d^2\Theta}{dx^2} - a^2\Theta = 0$  where the solution is  $\Theta(x) = c_1e^{ax} + c_2e^{-ax}$
  - Boundary conditions:  $T = T_b$  at the left end while on the right end as  $L \rightarrow \infty$ ,  $T = T_\infty$
  - Therefore we simplify by having  $T = T_\infty$  at  $x = L$
  - In terms of  $\Theta$ : At  $x = 0$ ,  $\Theta = T_b - T_\infty = \Theta_b$  and  $\Theta(\infty) = 0$
  - This gives  $c_1 = 0$  and  $c_2 = \Theta_b$  so that the solution is  $\Theta(x) = \Theta_b e^{-ax}$
- What is the heat loss from the fin?  $\dot{Q}_b = -kA_c \frac{dT}{dx}|_{x=0} \dot{Q}_{fin}$
- $\dot{Q}_{fin,long} = \sqrt{hPkA_C}(T_b - T_\infty)$
- Finite fin length: What is the boundary condition at the open end



- We can heat transfer is negligible so that adiabatic and  $\frac{dT}{dx} = 0$  at the boundary
- At  $x = L$ ,  $\frac{dT}{dx} = 0$  so that  $\frac{d\Theta}{dx} = 0$ :  $c_1 e^{aL} - c_2 e^{-aL} = 0$
- At  $x = 0$ ,  $\Theta = \Theta_b$
- Solve for  $c_1$  &  $c_2$  and the following solution will be obtained:  $\frac{T(x)-T_\infty}{T_b-T_\infty} = \frac{\cosh a(L-x)}{\cosh aL}$
- Can do the same for  $\dot{Q}_{fin,insulated} = -kA_C \frac{dT}{dx}|_{x=0}$  which will give  $\dot{Q}_{fin,insulated} = \sqrt{hPkA_c}(T_b - T_\infty) \tanh(aL)$  where  $a = \sqrt{\frac{hP}{kA_c}}$
- To account for heat transfer from the tip, we can add a length  $\Delta L$  at the end and the area there will be  $A_c = \Delta LP$  ( $P$  is the perimeter of the fin) so that the corrected length is  $L_c = L + \frac{A_c}{P}$

## 9 Heat transfer from finned surfaces (contd)

- Fin efficiency
  - $\Delta T$  given by the difference between the fin temperature and the surrounding
  - Most efficient fin would have a uniform temperature  $T_b$  everywhere
  - This would imply an infinite thermal conductivity
  - In this case,  $\dot{Q} = hA_{fin}(T_b - T_\infty) = hPL(T_b - T_\infty)$
  - We define  $\eta_{fin} = \frac{\dot{Q}_{fin}}{\dot{Q}_{fin,max}}$
  - For an infinitely long fin,  $\eta_{fin,long} = \frac{\sqrt{hPkA_c}(T_b-T_\infty)}{hPL(T_b-T_\infty)} = \frac{1}{L} \sqrt{\frac{kA_c}{hP}} = \frac{1}{aL}$  which is in terms of physical properties
  - $\dot{Q}_{fin} = \eta_{fin} \dot{Q}_{fin,max} = \eta_{fin} hA_{fin}(T_b - T_\infty) = h\eta_{fin} A_{fin}(T_b - T_\infty)$
  - $\eta_{fin} A_{fin}$  can be treated as the corrected area
  - For an insulated tip, perform the same steps with the original definition to get  $\eta_{insulatedtip} = \frac{\tanh aL}{aL}$
- Fin effectiveness
  - How much has the fin increased heat transfer by?
  - $\epsilon_{fin} = \frac{\dot{Q}_{fin}}{\dot{Q}_{nofin}}$
  - $\dot{Q}_{nofin} = hA_c(T_b - T_\infty)$
  - $\dot{Q}_{longfin} = \sqrt{hPkA_c}(T_b - T_\infty)$  so that  $\epsilon_{longfin} = \sqrt{\frac{kP}{hA_c}}$
  - To increase the effectiveness, make  $k$  as large as possible and maximize  $\frac{P}{A_c}$
  - Fins are most effective with low  $h$  so they are used for gases, hot liquids
  - Generally we use fins if  $\epsilon \geq 2$

- When can we assume fins are infinitely long?

$$- \frac{\dot{Q}_{fin,insulated}}{\dot{Q}_{fin,long}} = \tanh(aL). \text{ tanh asymptotically approaches 1 as } aL \text{ approaches } \infty$$

- In practice, if  $aL \geq 5$  we can assume an infinitely long fin. But even  $aL = 1$  has  $\tanh = 0.76$  so it gives 76 % of heat transfer of an infinitely long fin. Therefore  $L = \frac{1}{a}$  is a reasonable length for a fin

- Designing a heat sink

$$- \dot{Q}_{total} = \dot{Q}_{unfinned} + \dot{Q}_{fin}$$

$$- \text{From the definition of efficiency } \dot{Q}_{fin} = \eta_{fin} \cdot hA_{fin}(T_b - T_{\infty})$$

$$- \Rightarrow \dot{Q}_{total} = hA_{unfinned}(T_b - T_{\infty}) + h\eta_{fin}A_{fin}(T_b - T_{\infty}) = h[A_{unfinned} + \eta_{fin}A_{fin}](T_b - T_{\infty})$$

$$- \text{We can define a thermal resistance } R_{fin} = \frac{T_b - T_{\infty}}{\dot{Q}_{total}} = \frac{1}{h[A_{unfinned} + \eta_{fin}A_{fin}]}$$

## 10 Transient Heat Conduction

- Consider a solid at temperature  $T_i$  and a liquid at  $T_{\infty} < T_i$ . The solid is dropped into the liquid. How does  $T$  vary over time?

- We would expect the temperature  $T$  to asymptotically approach  $T_{\infty}$

$$- \text{Heat Conduction equation: } \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = \frac{1}{\alpha} \frac{\partial T}{\partial t}$$

- We set the first three terms equal to 0 by using the lumped capacitance approximation i.e. no temperature gradient in the body

- We would expect this to be valid when the object is small and has a high thermal conductivity

$$- \text{Using an energy balance: } \dot{E}_{store} = -\dot{Q}_{conv}$$

$$- \dot{E}_{store} = mc_p \frac{dT}{dt} = \rho V c_p \frac{dT}{dt} \text{ and } \dot{Q}_{conv} = hA(T - T_{\infty})$$

$$- \text{Equating the two, we get } \frac{d(T - T_{\infty})}{T - T_{\infty}} = -\frac{hA}{\rho V c_p} dt \text{ so that } \ln(T - T_{\infty}) = \frac{-hA}{\rho V c_p} t + C_1$$

$$- \text{Using } T = T_i \text{ at } t = 0, \text{ we get } \ln \left[ \frac{T - T_{\infty}}{T_i - T_{\infty}} \right] = \frac{-hA}{\rho V c_p} t$$

$$- \text{We define a "time constant" } \tau = \frac{\rho V c_p}{hA} \text{ so that } \frac{T - T_{\infty}}{T_i - T_{\infty}} = \exp \left[ \frac{-t}{\tau} \right]$$

- The LHS starts at 1 and decays to 0 as  $t \rightarrow \infty$ . Moreover at  $t = \tau$ , the value is  $\frac{1}{e} \approx 0.368$

- The response time of a thermometer is usually  $3\tau$ . However it is important to note that  $\tau$  is a function of  $h$  so it varies in different environments

- Moreover  $\tau$  depends on  $\frac{V}{A} = \frac{\tau}{3}$  for a sphere so to get a fast response time you would make it very thin

- When is a lumped capacitance valid?

- At steady state, the conduction in a solid must be equal to the convection in a fluid i.e.  $kA\frac{(T_1-T_2)}{L} = hA(T_2 - T_\infty)$  i.e.  $\frac{T_1-T_2}{T_2-T_\infty} = \frac{hL}{k}$
- $\frac{hL}{k}$  is a dimensionless number and is known as a Biot number.
- Note:  $k$  is the thermal conductivity of the solid,  $L$  is the length scale in the direction of conduction
- Suppose the Biot number is large i.e.  $\gg 1$  so that  $T_1 - T_2 \gg T_2 - T_\infty$
- If Biot number is very small, then  $T_1 - T_2 \ll T_2 - T_\infty$  so we can neglect  $T$  change inside the body (we assume uniform temp in the body) and use the lumped capacitance model
- By  $\ll$  we typically mean a Biot value  $< 0.1$
- In an irregular body, the length scale used is  $L = \frac{V}{A}$

## 11 Transient Heat Conduction in 2 and 3 Dimensions

- A ball of volume  $V$ , mass  $m$  and SA  $A$  and heat transfer coefficient  $h$  is dropped into a fluid
- Assume that the temperature  $T$  is uniform in the body
- We had previously assumed that if  $Bi < 0.1$  we have a lumped capacitance
- Example: Steel shaft,  $k = 51.2$ ,  $\rho = 7832$ ,  $c = 541$  and  $T_i = 300$  is placed into a furnace with  $T_\infty = 1200$ .
  - How long before the shaft temperature reaches 800?
  - We first calculate the Biot number as  $Bi = \frac{hL}{k}$  where  $L = \frac{V}{A} = \frac{\pi r^2 L}{2\pi r L} = \frac{r}{2}$
  - Then  $Bi = \frac{h\frac{r}{2}}{k} = \frac{100 \times \frac{0.05}{2}}{51.2} = 0.05$  so we can apply the lumped value
  - $\frac{T-T_\infty}{T_i-T_\infty} = \exp\left[-\frac{hA}{\rho V c} t\right]$  where  $\frac{A}{V} = \frac{2}{r}$
  - This gives  $\ln\left[\frac{800-1200}{300-1200}\right] = \dots$  and solving gives  $t = 859s$
- Transient heat conduction in 3 dimensions e.g. plane walls, cylinders, spheres
  - What happens if  $Bi > 0.1$ ?
  - In this case we cannot neglect the temperature gradients inside the body
  - Have to solve the complete heat conduction equation
  - Consider a solid wall with temperature  $T_i$  on one side which then instantly becomes lowered to a temperature  $T_\infty$  as it is placed into a fluid.
  - As time increases, the temperature inside the wall e.g. at the center decreases
  - So in this case,  $T$  is a function of  $x$  and  $t$
  - For a plane wall  $\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha} \frac{\partial T}{\partial t}$  where  $\alpha = \frac{k}{\rho c_p}$  is the thermal diffusivity

- Second order wrt  $x$  so two boundary conditions are needed there. First order wrt  $t$  so one initial condition is needed there
- This can be solved analytically but we will not do that
- We instead consider the lumped capacitance solution  $\ln \left[ \frac{T - T_\infty}{T_i - T_\infty} \right] = \frac{-hA}{\rho V c_p} t$
- We take the characteristic length  $L = \frac{V}{A}$ .
- $\frac{hA}{\rho V c_p} t = \frac{h}{\rho L c_p} t = \left( \frac{h}{\rho L c_p} t \right) \left( \frac{L}{L} \cdot \frac{k}{k} \right) = \left( \frac{hL}{k} \right) \left( \frac{k}{\rho c_p} \right) \left( \frac{t}{L^2} \right) = \left( \frac{hL}{k} \right) \left( \frac{\alpha t}{L^2} \right)$  where  $Bi = \frac{hL}{k}$  which is unitless
- We define the Fourier number  $Fo = \frac{\alpha t}{L^2}$  which is also dimensionless
- The dimensionless temperature  $\Theta = \frac{T - T_\infty}{T_i - T_\infty}$  so that the lumped capacitance solution can be written as  $\Theta = \exp(-Bi \cdot Fo)$
- A physical interpretation of the Fourier number can be found by considering a cube with side length  $L$ . Then  $\dot{Q}_{cond} = kA \frac{\partial T}{\partial x} = kL^2 \frac{\Delta T}{L} = kL \Delta T$
- $\dot{Q}_{store} = mc_p \frac{\partial T}{\partial t} = \rho L^3 c_p \frac{\Delta T}{t}$  so that  $\frac{\dot{Q}_{cond}}{\dot{Q}_{store}} = \frac{k}{\rho c_p} \cdot \frac{t}{L^2} = \frac{\alpha t}{L^2} = Fo$
- Even when we cannot assume lumped capacitance and get an exact solution of the heat conduction equation, the solution is of the form  $\Theta = \Theta(Bi, Fo)$
- We can define  $\Theta_0 = \frac{T_0 - T_\infty}{T_i - T_\infty}$  so that the solution is of the form  $\Theta_0 = A_1 e^{-\lambda_1^2 Fo}$  where  $A_1, \lambda_1$  are function of  $Bi$
- Example: Carbon steel plate with  $T_i = 440$  is placed in a furnace at  $T_\infty = 600$ . We need to heat to a minimum temperature of 520. What is the time  $t$  required.
  - $h = 200$ ,  $k = 40$  and  $\alpha = 8 \times 10^{-6}$
  - $Bi = \frac{hL}{k} = \frac{200 \times 0.04}{40} = 0.2$
  - Since  $Bi > 0.1$ , we cannot use the lumped capacitance and instead use the analytical solution which we obtain from tables
  - From Table 18.2,  $Bi = 0.2 \implies \lambda_1 = 0.4328, A_1 = 1.0311$
  - $\Theta_0 = \frac{T_0 - T_\infty}{T_i - T_\infty} = A_1 e^{-\lambda_1^2 Fo}$
  - $\Theta_0 = \frac{520 - 600}{440 - 600} = 0.5$  so that  $0.5 = 1.0311 \exp(-(0.4328)^2 Fo)$  and so  $Fo = 3.864$
  - $Fo = \frac{\alpha t}{L^2} \implies t = \frac{Fo L^2}{\alpha} = \frac{3.864 \times (0.04)^2}{8 \times 10^{-6}} = 773s$

## 12 Transient Heat Conduction in Semi-Infinite Solids

- The general problem consists of a body at temperature  $T_i$  with the surface temperature suddenly being changed to  $T_s$ .
  - Temperature variation would go from  $T_s$  down to  $T_i$ .  $\delta$  corresponds to a skin depth and after that we are in the core
  - How does the skin depth  $\delta$  vary with time?

- Heat conduction equation with one dimension:  $\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha} \frac{\partial T}{\partial t}$
- An estimate of  $\frac{\partial^2 T}{\partial x^2}$  can be found as  $\frac{(\frac{\partial T}{\partial x})_{x \sim \delta} - (\frac{\partial T}{\partial x})_{x \sim 0}}{\delta}$
- $(\frac{\partial T}{\partial x})_{x=0} \sim \frac{T_i - T_s}{\delta}$  so that  $\frac{\partial^2 T}{\partial x^2} \sim 0 - \frac{T_i - T_s}{\delta^2}$ ,  $\frac{\partial T}{\partial t} \sim \frac{T_s - T_i}{t}$
- From the heat conduction equation  $-\frac{T_i - T_s}{\delta^2} \sim \frac{1}{\alpha} \frac{T_s - T_i}{t}$
- Time for effect to be felt throughout the body is  $t_c \sim \frac{r_0^2}{\alpha}$
- $\delta \sim \sqrt{\alpha t}$
- For a short time  $t \ll t_c$ , we can treat the body as being semi infinite
- The exact solution is given by  $\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha} \frac{\partial T}{\partial t}$
- $\delta$  grows as a function of time
- Boundary conditions: At  $x = 0$ ,  $T(0, t) = T_s$  and as  $x \rightarrow \infty$ ,  $T(\infty, t) = T_i$
- Initial conditions:  $T(x, 0) = T_i$
- Define a similarity variable  $\eta = \frac{x}{\delta}$  such that  $0 < \eta < 1$
- $\eta = \frac{x}{2\sqrt{\alpha t}}$  so that  $\frac{\partial T}{\partial t} = \frac{dT}{d\eta} \cdot \frac{\partial \eta}{\partial t} = \frac{d}{dt} \left[ \frac{-x}{4t\sqrt{\alpha t}} \right]$
- $\frac{\partial T}{\partial x} = \frac{dT}{d\eta} \cdot \frac{\partial \eta}{\partial x} = \frac{dT}{d\eta} \left[ \frac{1}{2\sqrt{\alpha t}} \right]$
- $\frac{\partial^2 T}{\partial x^2} = \frac{d}{d\eta} \left( \frac{\partial T}{\partial \eta} \right) \cdot \frac{\partial \eta}{\partial x} = \frac{d^2 T}{d\eta^2} = \frac{1}{4\alpha t}$
- Transforming the heat conduction equation, we get  $\frac{d^2 T}{d\eta^2} = \frac{1}{\alpha} \frac{dT}{d\eta} \left( \frac{-x}{4t\sqrt{\alpha t}} \right) = -2\eta \frac{dT}{d\eta}$
- The PDE is now an ODE
- At  $x = 0$ ,  $\eta = 0$ ,  $T(0) = T_s$  and as  $x \rightarrow \infty$ ,  $\eta \rightarrow \infty$ ,  $T(\infty) = T_i$
- Let  $w = \frac{dT}{d\eta}$  so that  $\frac{dw}{d\eta} = -2\eta w$  which gives  $\ln w = -\eta^2 + c_0$
- $\frac{dT}{d\eta} = w = c_0 e^{-\eta^2}$  and integrating,  $T = c_0 \int_0^\eta e^{-u^2} du + c_1$
- Boundary conditions give  $T_i = c_0 \int_0^\infty e^{-u^2} du + T_s$  which gives the solution  $\frac{T - T_s}{T_i - T_s} = \frac{2}{\sqrt{\pi}} \int_0^\eta e^{-u^2} du = \text{erf}(\eta)$  where the RHS is the error function
- Can also be written as  $1 - \frac{T - T_s}{T_i - T_s} = 1 - \text{erf}(\eta)$  or  $\frac{T - T_i}{T_s - T_i} = \text{erfc}(\eta)$

• Heat Flux at Surface

- $\dot{q}_s = -k \frac{\partial T}{\partial x} |_{x=0} = 0k \frac{\partial T}{\partial \eta} \frac{\partial \eta}{\partial x} |_{\eta=0}$
- $\frac{d\eta}{dx} = \frac{1}{2\sqrt{\alpha t}}$
- $\dot{q} = \frac{k(T_s - T_i)}{\sqrt{\pi \alpha t}}$

• Contact of two semi-infinite bodies

- One body with temperature  $T_{A,i}$  and conductivity  $k_A$  and the other with  $T_{B,i}$  and  $k_B$

- We require that  $T_{S,A} = T_{S,B}$
- The heat flux must be the same i.e.  $\dot{q}_{S,A} = \dot{q}_{S,B}$
- $$-\frac{k_A(T_s - T_{A,i})}{\sqrt{\pi(\alpha_A t)}} = \frac{k_B(T_s - T_{B,i})}{\sqrt{\pi(\alpha_B t)}}$$
- $$\implies \frac{T_{A,i} - T_s}{T_s - T_{B,i}} = \frac{\sqrt{(k\rho c)_B}}{\sqrt{(k\rho c)_A}}$$
- Define effusivity as  $\gamma = \sqrt{k\rho c}$

## 13 Forced convection, Velocity and Thermal boundary layers, Reynolds, Prandtl and Nusselt numbers

- Forced Convection

- Convection is the heat transfer from a surface to a moving fluid
- Forced part means that motion is imposed by external means
- A surface at a temperature  $T_s$  and a fluid with velocity  $V_\infty$  and temperature  $T_\infty$
- $\dot{Q}_{conv} = hA(T_s - T_\infty)$
- How do we determine  $h$ ?
  - \* Quite complicated
  - \* Depends on physical properties of fluid: viscosity  $\mu$ , density  $\rho$ , thermal conductivity  $k$  and specific heat  $c_p$
  - \* Depends on the fluid velocity  $V_\infty$
  - \* Depends on the shape and size of body: Characteristic length i.e. for a plate it is the length  $L$  whereas for a cylinder or sphere it is the diameter  $D$ . Differences for objects of irregular shapes
  - \* Type of flow - laminar, turbulent

- Velocity Boundary layer

- Incoming flow at velocity  $V_\infty$  and temperature  $T_\infty$  and a solid plate
- There is a no slip condition at the interface between the fluid and the solid. i.e. the velocity at the interface must be 0 but if you move away, it will go back to  $V_\infty$
- We can define a boundary layer thickness by choosing where the velocity is  $0.99V_\infty$
- At  $y = 0$  (plate surface),  $V = 0$ . The heat transfer there is thus by conduction only so that Fourier's law applies
- Here  $\dot{q}_{cond} = -k_{fluid} \frac{\partial T}{\partial y}|_{y=0}$ . We have also defined  $\dot{q}_{conv} = h(T_s - T_\infty)$
- At  $y = 0$ ,  $\dot{q}_{conv} = \dot{q}_{cond}$  so that  $h = \frac{-k_{fluid} \frac{\partial T}{\partial y}|_{y=0}}{T_s - T_\infty}$

- Therefore  $h$  depends on  $k_{fluid}$  and the temperature gradient  $\frac{\partial T}{\partial y}|_{y=0}$ . This leads to thermal boundary layers
- Thermal Boundary layer
  - Temperature of the surface is  $T_s > T_\infty$
  - Fluid coming at temperature  $T_\infty$
  - When the fluid comes in contact with the plate, there cannot be a discontinuity so it must be equal to  $T_s$  at the contact point
  - INSERT PIC FROM NOTES. We define the thermal boundary layer in a similar way to be the point where  $T - T_s = 0.99(T_\infty - T_s)$
  - $\frac{\partial T}{\partial y}|_{y=0}$  is the temperature gradient.
  - $h$  changes with position as the local  $\frac{\partial T}{\partial y}|_{y=0}$  changes
  - We define a local heat transfer coefficient  $h(x)$
  - We can average:  $\bar{h} = \frac{1}{L} \int_0^L h(x) dx$
  - Instead of worrying about local variations, we use the average value of  $h$
- Velocity Boundary layer flow
  - The difference between the boundary and the surfaces increases linearly as a straight path in laminar flow
  - There is a transition region then where the path will start becoming unstable which eventually becomes a turbulent region with turbulent flow
  - Fluid exerts a drag on the plate. Measured in terms of the shear stress (force per unit area)
  - $\tau = \mu \frac{\partial V}{\partial y}|_{y=0}$  where  $\mu$  is the fluid viscosity ( $\frac{kg}{ms}$ )
  - Velocity gradient can be solved but it is very complicated and practically we define a friction coefficient
  - We imagine a fluid coming to rest at a stagnation point. Using Bernoulli's equation,  $\frac{P}{\rho} = \frac{V_\infty^2}{2} + \frac{P_\infty}{\rho} \implies P - P_\infty = \frac{\rho V_\infty^2}{2}$ . This is the pressure rise and the force felt by plate
  - We define the friction coefficient  $c_f$  so that  $\tau = c_f \frac{\rho v_\infty^2}{2}$
  - Generally we expect  $c_f \sim 1$  but this depends on the shape
- Laminar & Turbulent Flow
  - Heat transfer is greater in turbulent flow i.e. with an increased velocity
  - However when the heat transfer goes up, shear goes up and so bigger fans (more energy) are needed
  - The transition to turbulence depends on the ratio of fluid inertia to viscosity

- A high inertia drives random motion and therefore turbulence
- A high viscosity damps turbulence
- Consider a mass with diameter  $D$  and a fluid coming in around it at velocity  $V_\infty$ 
  - \* There is an inertial force  $F_i$  and a viscous force  $F_v$
  - \* Take the characteristic distance to be  $D$
  - \* The inertial force  $F_i = ma$  where the mass  $m = \rho D^3$  and the acceleration  $a = \frac{V_\infty^2}{D}$
  - \* The viscous force  $F_v = \tau A = \mu \frac{\partial V}{\partial y} \cdot A$
  - \*  $F_v \sim \mu \frac{V_\infty}{D} \cdot D^2 \implies \frac{F_i}{F_v} = \frac{\rho V_\infty D}{\mu}$
  - \* This number is known as the Reynolds number:  $Re = \frac{\rho V_\infty D}{\mu}$

## 14 Forced Convection Currents

- Fluid with velocity  $V_\infty$  and density  $\rho_{mu}$ . What forces are exerted onto the body (with characteristic length being the diameter  $D$ )
  - Inertial force  $F_i = ma$ 
    - \*  $m \sim \rho D^3$
    - \* The fluid starts with velocity  $V_\infty$  and is brought to rest over a distance  $D$
    - \*  $t = \frac{D}{V_\infty}$
    - \*  $\Delta V = V_\infty$  so  $a \sim \frac{\Delta v}{t} = \frac{V_\infty}{\frac{D}{V_\infty}} = \frac{V_\infty^2}{D}$
    - \* Thus  $F_i \sim \rho D^3 \left(\frac{V_\infty^2}{D}\right) = \rho D^2 V_\infty^2$
  - Viscous Force
    - \*  $F_v = \tau A = \mu \frac{dV}{dy} \cdot A$
    - \*  $F_v \sim \mu \frac{V_\infty}{D} \cdot D^2$
  - We are interested in the ratio
    - \*  $\frac{F_i}{F_v} \sim \frac{\rho V_\infty D}{\mu}$
    - \* This is a very important number called the Reynolds Number  $Re = \frac{\rho V_\infty D}{\mu}$
    - \* The kinematic viscosity is defined as  $\nu = \frac{\mu}{\rho}$  so that the Reynolds number can also be written as  $Re = \frac{V_\infty D}{\nu}$
    - \* For small  $Re$ , viscous forces are dominant. Fluctuations in the flow are damped. This leads to laminar flow.
    - \* For large  $Re$ , inertial forces are dominant. Fluctuations in the flow become amplified and this leads to turbulent flow
    - \* For every geometry, there is a critical value of  $Re$  at which a transition to turbulence occurs e.g.  $Re_{critical, flatplate} = 5 \times 10^5$
- Two boundary layers are developing - velocity, thermal



- The velocity goes from  $V_\infty$  down to 0 and the temperature goes down from  $T_\infty$  to  $T_s$
- Let  $\delta_v$  be the velocity boundary layer and  $\delta_t$  be the thermal boundary layer
- $\delta_t$  may be smaller or larger than  $\delta_v$ . How do we tell?
  - \* This depends on the physical properties of the fluid
- Fluids with high viscosity  $\nu$  (oils) have thick velocity BL i.e.  $\delta_v$  is a large fraction of  $V_\infty$
- Fluids with high thermal diffusivity ( $\alpha = \frac{k}{\rho c_p}$ ) have thick thermal BL
- The ratio  $\frac{\delta_v}{\delta_t}$  is given by the ratio  $\frac{\nu}{\alpha}$
- The Prandtl Number is defined to be this ratio:  $Pr = \frac{\nu}{\alpha} = \frac{\mu c_p}{k}$
- $Pr$  is a fluid property
- For  $Pr \ll 1$  (e.g. a liquid metal)  $\delta_v \ll \delta_t$
- For  $Pr \gg 1$  (e.g. oils)  $\delta_v \gg \delta_t$
- For  $Pr \sim 1$  (e.g. gases)  $\delta_t \sim \delta_v$
- We have two dimensionless parameters  $Re = \frac{\rho V_\infty D}{\mu}$  and  $Pr = \frac{\nu}{\alpha}$
- Need to non dimensionalize  $h$ 
  - $\dot{Q}_{conv} = hA(T_s - T_\infty) \sim hD^2(T_s - T_\infty)$
  - Suppose the fluid was not moving
  - Then heat transfer is by conduction only
  - $\dot{Q}_{cond} = k_{fluid}A \frac{dT}{dr} \sim k_{fluid}D^2 \frac{T_s - T_\infty}{D}$
  - How much is heat transfer enhanced due to convection
  - This is given by the ratio  $\frac{\dot{Q}_{conv}}{\dot{Q}_{cond}} = \frac{hD}{k_{fluid}}$  which is a dimensionless number
  - This ratio is known as the Nusselt number  $Nu = \frac{hD}{k_{fluid}}$
  - Do not confuse with  $Bi = \frac{hD}{k_{solid}}$
- We started with  $h = f(D, V_\infty, \rho, \mu, c_p, k)$  and dependent on the geometry
- This can now be written as  $\frac{hD}{k} = f\left(\frac{\rho D V_\infty}{\mu}, \frac{\mu c_p}{k}\right)$  and dependent on the geometry
- $\implies Nu = f(Re, Pr)$  and geometry
- Can do experiments
  - Consider a plate with a uniform heat flux due to an electrical current through it
  - Can put it into a wind tunnel with  $V_\infty$  and  $T_\infty$
  - We know that  $\dot{Q} = hA(T_s - T_\infty) = P = EI$  (electrical power)

- We know the other variables and can thus solve for  $h$  and plot it as a function of  $V_\infty$
- Repeat for different plate sizes, different  $T_s$  and different  $T_\infty$
- Can plot  $Nu$  vs  $Re$  and all the data should fall on the same curve
- Repeat for different fluids to get different curves for different fluids
- Can then plot  $\ln(\frac{Nu}{Pr^n})$  against  $\ln Re$  and the data for different fluids will all lie on the same line
- $\ln(\frac{Nu}{Pr^n}) = m \ln Re + C$  or equivalently  $Nu = cRe^m Pr^n$  where  $n, m, C$  are determined experimentally for each geometry