

CHE260: Heat Transfer

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Contents

1	Introduction	1
2	Electronic Cooling	2
3	Radiation	3
4	Heat Conduction	4
5	Thermal Resistance	5
6	Thermal Resistance Networks	6
7	Conduction in cylinders and spheres, Insulation	7
8	Heat Transfer from Finned Surfaces	8
9	Heat transfer from finned surfaces (contd)	9
10	Transient Heat Conduction	10
11	Transient Heat Conduction in 2 and 3 Dimensions	11

1 Introduction

- How is heat transfer different from thermodynamics? In thermodynamics, we assume quasi-equilibrium processes i.e. the time was not an important parameter. In heat transfer, time is an important parameter and we are interested in the rate of heat transfer.
- What is the relationship between \dot{Q} and ΔT ? What are the mechanisms of heat transfer?
- Conduction: Transfer of heat through a medium that is stationary.

- Convection: Transfer of heat from a solid surface and an adjacent fluid that is moving.
Example: a fan blowing air over a hot plate. There is heat transfer from the hot plate into the fluid.
- Radiation: Energy emitted by matter in the form of electromagnetic waves.
- Radiation does not need a medium. In a vacuum, we can have radiation but not convection or conduction.
- Different mechanisms of heat transfer can take place simultaneously.
- Applications
 - Power Generation
 - * Power plant: steam generation, condenser
 - * Automobiles: engine cooling, space heating/cooling
 - Buildings
 - * Heating / Cooling
 - * Hot water
 - Refrigeration
 - Manufacturing
 - * Casting / Heat treatment
 - * Injection Moulding

2 Electronic Cooling

- > 99 % of the electrical energy supplied to a circuit is dissipated as heat
- Heat has to be dissipated to the environment while keeping the temperature of the chip in a certain range
- Heat is lost from the surface of the chip
- Important parameter is heat flux = $\frac{\text{Heat Transfer Rate}}{\text{Unit Area}}$ (in $\frac{W}{\text{cm}^2}$)
- To reduce heat flux, we can reduce heat generation and increase the surface area
- As size increases, it becomes more difficult to lose heat
- Water cooling is more efficient for large systems compared to air cooling

3 Radiation

- Radiation is energy emitted by all matter in the form of e.m. radiation
- Thermal radiation is emitted by all bodies at a finite temperature
- Opaque objects emit only from the surface
- Amount of radiation depends on the surface temperature. Summarized by the Stefan Boltzmann Law: $\dot{Q}_{emit} = \sigma AT_s^4$ where σ is the Boltzmann constant ($5.67 \times 10^{-8} \frac{W}{m^2 K^4}$), T_s is the surface temperature in Kelvin and A is the surface area.
- A surface that emits as much radiation as this is called a "Blackbody". A real surface emits less than this.: $\dot{Q}_{emit} = \epsilon \sigma AT_s^4$ where ϵ is the emissivity and $0 \leq \epsilon \leq 1$
- Black paint has $\epsilon = 0.99$ which is very close to 1. Aluminum foil has a low emissivity of around 0.07.
- If radiation is incident on a surface some will be absorbed. The fraction absorbed is a surface property known as the absorptivity α such that $\dot{Q}_{absorbed} = \alpha \cdot \dot{Q}_{incident}$ and $\dot{Q}_{reflected} = (1 - \alpha) \cdot \dot{Q}_{incident}$
- Kirchoff's law says that $\alpha = \epsilon$
- Note: α and ϵ vary over different wavelengths
- Consider a special case of radiation
 - Small surface which is completely surrounded by a much larger surface
 - T_s, A_s are temperature and area of the small surface (which is also the boundary), T_{surr} is the temperature of the surrounding surface. Both surfaces are emitting and we are interested in the net emission
 - $\dot{Q}_{rad} = \epsilon \sigma A_s (T_s^4 - T_{surr}^4)$
- Example
 - Chip with an area of $15 \times 15 mm$, $\epsilon = 0.6$, $T_{surr} = 25$.
 - Two methods of heat transfer
 - * Natural convection
 - $h = c(T_s - T_\infty)^{\frac{1}{4}}$
 - $c = 4.2 \frac{W}{m^2 K^{\frac{5}{4}}}$
 - $q_{conv} = hA(T_s - T_\infty)$
 - $q_{rad} = \epsilon A(T_s^4 - T_{surr}^4)$
 - * Forced convection: h is constant at $250 \frac{W}{m^2 K}$
 - $q_{conv} = hA(T_s - T_\infty)$

4 Heat Conduction

- Heat Conduction Equation

- x, y, z components of \dot{Q}
- T is a function of (x, y, z, t)
- $\vec{\dot{Q}} = \dot{Q}_x \hat{i} + \dot{Q}_y \hat{j} + \dot{Q}_z \hat{k}$
- $\dot{Q}_x = -kA_x \frac{dT}{dx}$ (similar expressions for \dot{Q}_y and \dot{Q}_z)

- One dimensional heat conductivity can model more complicated situations. For example if $\Delta x \ll \Delta y, z, \frac{dT}{dx} \gg \frac{dT}{dy}, \frac{dT}{dz}$ so that \dot{Q}_y and \dot{Q}_z can be neglected

- One dimensional heat conduction

- Cross sectional area is $A(x)$ where x is the coordinate along which heat transfer occurs
- \dot{Q}_x at the entry and $\dot{Q}_{x+\Delta x}$ at the exit
- Want to find $T(x)$ inside the object
- Rate of increase of enthalpy $= mc_p \frac{\partial T}{\partial t} = \rho V c_p \frac{\partial T}{\partial t} = \rho c_p A \Delta x \frac{\partial T}{\partial t}$
- Energy balance:
 - * $\rho c_p A \Delta x \frac{\partial T}{\partial t} = \dot{Q}_x - \dot{Q}_{x+\Delta x}$
 - * After simplifying and taking the limit as Δx approaches 0, we get $\rho c_p \frac{\partial T}{\partial t} = \frac{-1}{A} \frac{\partial(\dot{q}A)}{\partial x}$
 - * A depends on the coordinate system and we use Fourier's law for \dot{q} : $\dot{q} = -k \frac{dT}{dx}$

- Cartesian Coordinates

- A is a constant
- $\rho c_p \frac{\partial T}{\partial t} = \frac{\partial}{\partial x} [k \frac{\partial T}{\partial x}]$
- Assume k is constant. Then $\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$ where $\alpha = \frac{k}{\rho c_p}$.
- If steady state i.e. $\frac{\partial T}{\partial t} = 0$ then $\frac{d^2 T}{dx^2} = 0$.
- Units of $\alpha = \frac{k}{\rho c_p}$, the thermal diffusivity is $\frac{m^2}{s}$.
- High k means the material conducts well. High ρc_p means that the material stores energy

- Cylindrical Coordinates

- Heat being conducted radially so $\dot{q} = -k \frac{\partial T}{\partial r}$ and $A = 2\pi r L$
- $\rho c_p \frac{\partial T}{\partial t} = \frac{-1}{2\pi r L} [\frac{\partial}{\partial r} \cdot \frac{\partial T}{\partial r}]$
- $\frac{1}{\alpha} \frac{\partial T}{\partial t} = \frac{1}{r} \frac{\partial}{\partial r} (r \cdot \frac{\partial T}{\partial r})$
- At steady state, $\frac{d}{dr} (r \frac{dT}{dr}) = 0$.

- Spherical Coordinates

– $A = 4\pi r^2$ and $\dot{q} = -k \frac{\partial T}{\partial r}$ where r is the radial spherical coordinate

– $\frac{1}{\alpha} \frac{\partial T}{\partial t} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \frac{\partial T}{\partial r})$

- In general $\frac{1}{r^n} \frac{\partial}{\partial r} (r^n \frac{\partial T}{\partial r}) = \frac{1}{\alpha} \frac{\partial T}{\partial t}$ where cartesian has $n = 0$, cylindrical has $n = 1$ and spherical has $n = 2$.

5 Thermal Resistance

- At steady state, $\frac{d^2 T}{dx^2} = 0$

- Heat flux: $\dot{q} = -k \frac{dT}{dx}$.

- Heat flux is a constant

- Heat transfer rate: $\dot{Q} = \dot{q}A = \frac{-kA(T_2 - T_1)}{L}$

- $\dot{Q} = \frac{T_1 - T_2}{R_{wall}}$ where T_1 and T_2 are the temperatures of the walls

- $R_{cond} = R_{wall} = \frac{L}{kA}$

- Similar to current with voltage and Resistance

- $\dot{Q}_{conv} = hA(T_s - T_\infty)$

- $R_{conv} = \frac{T_s - T_\infty}{\dot{Q}_{conv}} = \frac{1}{hA}$

- Radiation is more complicated. $\dot{Q}_{rad} = \epsilon \sigma A(T_s^4 - T_{sur}^4)$

- We need to define a heat transfer coefficient for radiation. $h_{rad} = \frac{\epsilon \sigma A(T_s^4 - T_{sur}^4)}{A(T_s - T_{sur})}$

- $h_{rad} = \epsilon \sigma (T_s^2 + T_{sur}^2)(T_s + T_{sur})$

- Can treat it as a resistance. $R_{rad} = \frac{T_s - T_{sur}}{\dot{Q}_{rad}} = \frac{1}{h_{rad}A}$

- Multilayer Plane Wall

– Each layer has the same surface Area

– Layers have thicknesses L_i

– Temperature varies as T_1 on the outside, T_2, \dots, T_{n+1} where n is the number of surfaces

– Treat each layer separately as resistances in series

– For first wall $\dot{Q} = \frac{T_1 - T_2}{R_1}$ so $T_1 - T_2 = \dot{Q}R_1$. In general, $T_i - T_{i+1} = \dot{Q}R_i$

– Summing all of them gives $T_1 - T_{n+1} = \dot{Q}(R_1 + \dots + R_n)$ so that $\dot{Q} = \frac{T_1 - T_{n+1}}{R_{total}}$

– Therefore you can sum up resistances similar to electric circuits

- We can find each R_i as $\frac{L_i}{k_i A}$
- $\dot{Q} = UA(T_{\infty,1} - T_{\infty,4})$ where U is the overall heat transfer
- Example: Refrigerator Wall
 - 1 mm thick insulation on the outside and width of refrigerator is L .
 - $T_{room} = 25C$ and $T_{refrig} = 3C$
 - $h_0 = 9 \frac{W}{m^2C}$ and $h_i = 4 \frac{W}{m^2C}$
 - $k_{steel} = 15.1 \frac{W}{m^2C}$ and $k = 0.035 \frac{W}{m^2C}$ inside the refrigerator
 - Constraint is that $T_{s,out} > 20$. We assume heat transfer from the outside room to the inside of the refrigerator
 - What is L to ensure $T_{s,out} > 20$ to prevent condensation on the outside of the refrigerator
 - Thermal circuit consists of a convective resistance outside the refrigerator followed by 3 conductive resistance on the surfaces and then one convective resistance at the end
 - INSERT PICTURE FROM LEC NOTES
 - $\dot{Q} = \frac{T_{room} - T_{s,out}}{R_{conv,0}} = \frac{T_{room} - T_{s,out}}{\frac{1}{h_0 A}}$
 - Consider unit area. Then $\dot{Q} = h_0(T_{room} - T_{s,out}) = 9(25 - 20) = 45W$
 - $R_{total} = \frac{1}{h} + (\frac{L}{k})_{metal} + (\frac{L}{k})_{insulation} + (\frac{L}{k})_{metal} + \frac{1}{h_i} = \frac{1}{9} + \frac{10^{-3}}{15.1} + \frac{L_2}{0.035} + \frac{10^{-3}}{15.1} + \frac{1}{4} = 0.361 + \frac{L_2}{0.035}$
 - $\dot{Q} = \frac{T_{room} - T_{refrig}}{R_{total}} \rightarrow 45(0.361 + \frac{L_2}{0.035}) = 25 - 3$ so that $L = 45mm$

6 Thermal Resistance Networks

- Multiple layers e.g. in an electric chip each with different thermal properties
 - We are interested in $\dot{Q} = \dot{Q}_1 + \dot{Q}_2 + \dot{Q}_3 = \frac{T_1 - T_2}{R_1} + \frac{T_1 - T_2}{R_2} + \frac{T_1 - T_2}{R_3}$
 - $= (T_1 - T_2)(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}) = \frac{T_1 - T_2}{R_{total}}$
 - This is the electrical analog to parallel resistances
- Thermal contact resistance
 - So far we have been assuming perfect contact between different boundaries
 - In reality, there is a rough surface at the boundary
 - We can always define a thermal contact resistance $R_c = \frac{T_2 - T_1}{\dot{q}}$ (units are $\frac{m^2C}{W}$)
(Note: Resistanc per unit area)
 - The reciprocal of R_C is known as the thermal contact conductance h_c .

- $h = \frac{1}{R_c} = \frac{\dot{q}}{\Delta T}$ so that $\dot{q} = h_c \Delta T$
- h_c is thus similar to the heat transfer coefficient
- Heat conduction in cylinders & spheres
 - INSERT DRAWING FROM THE SLIDES
 - For a long pipe, the main temperature gradient is in the radial direction i.e. $\frac{dT}{dx} \ll \frac{dT}{dr}$
 - Therefore we can assume 1-D radial conduction
 - INSERT r_1, r_2 diagram
 - Solve heat conduction equation in cylindrical coordinates to get $T(r)$
 - Steady state: $\frac{d}{dr}(r \frac{dT}{dr}) = 0$, at $r = r_1, T = T_1$ and at $r = r_2, T = T_2$
 - Integrate this to get $T(r) = c_1 \ln r + c_2$
 - Using the boundary conditions gives $c_1 = \frac{T_1 - T_2}{\ln(\frac{r_1}{r_2})}$ and $c_2 = T_2 - \frac{T_1 - T_2}{\ln(\frac{r_1}{r_2})} \ln r_2$
- Define thermal resistance of a cylinder
 - $R_{cyl} = \frac{T_1 - T_2}{\dot{Q}_{cond}} = \frac{\ln(\frac{r_2}{r_1})}{2\pi L K}$

7 Conduction in cylinders and spheres, Insulation

- Inner temperature and then two surface layers
- $R_C = \frac{1}{hA} = \frac{1}{2\pi r L h}$
- $R_{total} = R_{c,1} + R_{cond} + R_{c,2} = \frac{1}{2\pi r_1 L h_1} + \frac{\ln(\frac{r_2}{r_1})}{2\pi L K} + \frac{1}{2\pi r_2 L h_2}$
- Insulation
 - $R_{total} = R_{c,1} + R_{cyl,1} + R_{cyl,2} + R_{c,2}$ where $R_{cyl,2}$ is for the insulation
 - For a sphere
 - * Insulation around a spherical metal tank
 - * $R_{total} = R_{c,1} + R_{sph,1} + R_{sph,2} + R_{c,2}$
 - * $= \frac{1}{4\pi r_1^2 h_1} + \frac{r_2 - r_1}{4\pi r_1 r_2 k_1} + \frac{r_3 - r_2}{4\pi r_3 r_2 k} + \frac{1}{4\pi r_3^2 h_2}$
- R-value
 - Thermal resistance. Thickness L , Surface area A and thermal conductivity k
 - $R = \frac{L}{k}$ is the R-value
 - $\dot{Q} = \frac{\Delta T}{R} \times A$
 - Units here are in imperial units i.e. L is in feet, k is in $\frac{Btu}{h ft F}$

- Critical Radius of insulation
 - Consider the area for heat loss
 - Insulation: increase thickness, increasing conduction resistance and decreasing convective resistance
 - Can we increase heat transfer?
 - * Plot \dot{Q} against r_2 to find the critical value of Resistance
 - * Equivalently set $\frac{d\dot{Q}}{dr_2} = 0$ to find the critical radius

8 Heat Transfer from Finned Surfaces

- Read Chapter 17.6
- How to find $R_{heatsink}$ for finned surfaces e.g. heat sinks in computers
- Add diagram from notes: We have a cylinder with cross sectional area $A_c(x)$ and heat transfer coeff h .
 - Consider a thin slice of this with thickness Δx some distance x away from the end
 - Energy balance $\dot{Q}_{cond,x}$ in and $\dot{Q}_{cond,x+\Delta x}$ out
 - Energy in = Energy out: $\dot{Q}_{cond,x} = \dot{Q}_{cond,x+\Delta x} + \dot{Q}_{conv}$
 - Let the perimeter of fin be P . Then surface area of element is $P\Delta x$ so that $\dot{Q}_{conv} = hP\Delta x(T - T_\infty)$
 - Simplifying the energy balance by taking the limit as $\Delta x \rightarrow 0$: $\frac{d\dot{Q}_{cond}}{dx} + hP(T - T_\infty) = 0$
 - Using Fourier's law $\dot{Q}_{cond} = -kA_C \frac{dT}{dx}$: $\frac{d}{dx}(kA_c \frac{dT}{dx}) - hP(T - T_\infty) = 0$
 - Assuming A_C, k, P constant: $\frac{d^2T}{dx^2} - \frac{hP}{kA_C}(T - T_\infty) = 0$
 - Define $\Theta = T - T_\infty$ and $a^2 = \frac{hP}{kA_C}$ (constant) so that $\frac{d^2\Theta}{dx^2} - a^2\Theta = 0$ where the solution is $\Theta(x) = c_1e^{ax} + c_2e^{-ax}$
 - Boundary conditions: $T = T_b$ at the left end while on the right end as $L \rightarrow \infty$, $T = T_\infty$
 - Therefore we simplify by having $T = T_\infty$ at $x = L$
 - In terms of Θ : At $x = 0$, $\Theta = T_b - T_\infty = \Theta_b$ and $\Theta(\infty) = 0$
 - This gives $c_1 = 0$ and $c_2 = \Theta_b$ so that the solution is $\Theta(x) = \Theta_b e^{-ax}$
- What is the heat loss from the fin? $\dot{Q}_b = -kA_c \frac{dT}{dx}|_{x=0} \dot{Q}_{fin}$
- $\dot{Q}_{fin,long} = \sqrt{hPkA_C}(T_b - T_\infty)$
- Finite fin length: What is the boundary condition at the open end

- We can heat transfer is negligible so that adiabatic and $\frac{dT}{dx} = 0$ at the boundary
- At $x = L$, $\frac{dT}{dx} = 0$ so that $\frac{d\Theta}{dx} = 0$: $c_1 e^{aL} - c_2 e^{-aL} = 0$
- At $x = 0$, $\Theta = \Theta_b$
- Solve for c_1 & c_2 and the following solution will be obtained: $\frac{T(x)-T_\infty}{T_b-T_\infty} = \frac{\cosh a(L-x)}{\cosh aL}$
- Can do the same for $\dot{Q}_{fin,insulated} = -kA_C \frac{dT}{dx}|_{x=0}$ which will give $\dot{Q}_{fin,insulated} = \sqrt{hPkA_c}(T_b - T_\infty) \tanh(aL)$ where $a = \sqrt{\frac{hP}{kA_c}}$
- To account for heat transfer from the tip, we can add a length ΔL at the end and the area there will be $A_c = \Delta LP$ (P is the perimeter of the fin) so that the corrected length is $L_c = L + \frac{A_c}{P}$

9 Heat transfer from finned surfaces (contd)

- Fin efficiency
 - ΔT given by the difference between the fin temperature and the surrounding
 - Most efficient fin would have a uniform temperature T_b everywhere
 - This would imply an infinite thermal conductivity
 - In this case, $\dot{Q} = hA_{fin}(T_b - T_\infty) = hPL(T_b - T_\infty)$
 - We define $\eta_{fin} = \frac{\dot{Q}_{fin}}{\dot{Q}_{fin,max}}$
 - For an infinitely long fin, $\eta_{fin,long} = \frac{\sqrt{hPkA_c}(T_b-T_\infty)}{hPL(T_b-T_\infty)} = \frac{1}{L} \sqrt{\frac{kA_c}{hP}} = \frac{1}{aL}$ which is in terms of physical properties
 - $\dot{Q}_{fin} = \eta_{fin} \dot{Q}_{fin,max} = \eta_{fin} hA_{fin}(T_b - T_\infty) = h\eta_{fin} A_{fin}(T_b - T_\infty)$
 - $\eta_{fin} A_{fin}$ can be treated as the corrected area
 - For an insulated tip, perform the same steps with the original definition to get $\eta_{insulatedtip} = \frac{\tanh aL}{aL}$
- Fin effectiveness
 - How much has the fin increased heat transfer by?
 - $\epsilon_{fin} = \frac{\dot{Q}_{fin}}{\dot{Q}_{nofin}}$
 - $\dot{Q}_{nofin} = hA_c(T_b - T_\infty)$
 - $\dot{Q}_{longfin} = \sqrt{hPkA_c}(T_b - T_\infty)$ so that $\epsilon_{longfin} = \sqrt{\frac{kP}{hA_c}}$
 - To increase the effectiveness, make k as large as possible and maximize $\frac{P}{A_c}$
 - Fins are most effective with low h so they are used for gases, hot liquids
 - Generally we use fins if $\epsilon \geq 2$

- When can we assume fins are infinitely long?

$$- \frac{\dot{Q}_{fin,insulated}}{\dot{Q}_{fin,long}} = \tanh(aL). \text{ tanh asymptotically approaches 1 as } aL \text{ approaches } \infty$$

- In practice, if $aL \geq 5$ we can assume an infinitely long fin. But even $aL = 1$ has $\tanh = 0.76$ so it gives 76 % of heat transfer of an infinitely long fin. Therefore $L = \frac{1}{a}$ is a reasonable length for a fin

- Designing a heat sink

$$- \dot{Q}_{total} = \dot{Q}_{unfinned} + \dot{Q}_{fin}$$

$$- \text{From the definition of efficiency } \dot{Q}_{fin} = \eta_{fin} \cdot hA_{fin}(T_b - T_{\infty})$$

$$- \Rightarrow \dot{Q}_{total} = hA_{unfinned}(T_b - T_{\infty}) + h\eta_{fin}A_{fin}(T_b - T_{\infty}) = h[A_{unfinned} + \eta_{fin}A_{fin}](T_b - T_{\infty})$$

$$- \text{We can define a thermal resistance } R_{fin} = \frac{T_b - T_{\infty}}{\dot{Q}_{total}} = \frac{1}{h[A_{unfinned} + \eta_{fin}A_{fin}]}$$

10 Transient Heat Conduction

- Consider a solid at temperature T_i and a liquid at $T_{\infty} < T_i$. The solid is dropped into the liquid. How does T vary over time?

- We would expect the temperature T to asymptotically approach T_{∞}

$$- \text{Heat Conduction equation: } \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = \frac{1}{\alpha} \frac{\partial T}{\partial t}$$

- We set the first three terms equal to 0 by using the lumped capacitance approximation i.e. no temperature gradient in the body

- We would expect this to be valid when the object is small and has a high thermal conductivity

$$- \text{Using an energy balance: } \dot{E}_{store} = -\dot{Q}_{conv}$$

$$- \dot{E}_{store} = mc_p \frac{dT}{dt} = \rho V c_p \frac{dT}{dt} \text{ and } \dot{Q}_{conv} = hA(T - T_{\infty})$$

$$- \text{Equating the two, we get } \frac{d(T - T_{\infty})}{T - T_{\infty}} = -\frac{hA}{\rho V c_p} dt \text{ so that } \ln(T - T_{\infty}) = \frac{-hA}{\rho V c_p} t + C_1$$

$$- \text{Using } T = T_i \text{ at } t = 0, \text{ we get } \ln \left[\frac{T - T_{\infty}}{T_i - T_{\infty}} \right] = \frac{-hA}{\rho V c_p} t$$

$$- \text{We define a "time constant" } \tau = \frac{\rho V c_p}{hA} \text{ so that } \frac{T - T_{\infty}}{T_i - T_{\infty}} = \exp \left[\frac{-t}{\tau} \right]$$

- The LHS starts at 1 and decays to 0 as $t \rightarrow \infty$. Moreover at $t = \tau$, the value is $\frac{1}{e} \approx 0.368$

- The response time of a thermometer is usually 3τ . However it is important to note that τ is a function of h so it varies in different environments

- Moreover τ depends on $\frac{V}{A} = \frac{\tau}{3}$ for a sphere so to get a fast response time you would make it very thin

- When is a lumped capacitance valid?

- At steady state, the conduction in a solid must be equal to the convection in a fluid i.e. $kA\frac{(T_1-T_2)}{L} = hA(T_2 - T_\infty)$ i.e. $\frac{T_1-T_2}{T_2-T_\infty} = \frac{hL}{k}$
- $\frac{hL}{k}$ is a dimensionless number and is known as a Biot number.
- Note: k is the thermal conductivity of the solid, L is the length scale in the direction of conduction
- Suppose the Biot number is large i.e. $\gg 1$ so that $T_1 - T_2 \gg T_2 - T_\infty$
- If Biot number is very small, then $T_1 - T_2 \ll T_2 - T_\infty$ so we can neglect T change inside the body (we assume uniform temp in the body) and use the lumped capacitance model
- By \ll we typically mean a Biot value < 0.1
- In an irregular body, the length scale used is $L = \frac{V}{A}$

11 Transient Heat Conduction in 2 and 3 Dimensions

- A ball of volume V , mass m and SA A and heat transfer coefficient h is dropped into a fluid
- Assume that the temperature T is uniform in the body
- We had previously assumed that if $Bi < 0.1$ we have a lumped capacitance
- Example: Steel shaft, $k = 51.2$, $\rho = 7832$, $c = 541$ and $T_i = 300$ is placed into a furnace with $T_\infty = 1200$.
 - How long before the shaft temperature reaches 800?
 - We first calculate the Biot number as $Bi = \frac{hL}{k}$ where $L = \frac{V}{A} = \frac{\pi r^2 L}{2\pi r L} = \frac{r}{2}$
 - Then $Bi = \frac{h\frac{r}{2}}{k} = \frac{100 \times \frac{0.05}{2}}{51.2} = 0.05$ so we can apply the lumped value
 - $\frac{T-T_\infty}{T_i-T_\infty} = \exp\left[-\frac{hA}{\rho V c} t\right]$ where $\frac{A}{V} = \frac{2}{r}$
 - This gives $\ln\left[\frac{800-1200}{300-1200}\right] = \dots$ and solving gives $t = 859s$
- Transient heat conduction in 3 dimensions e.g. plane walls, cylinders, spheres
 - What happens if $Bi > 0.1$?
 - In this case we cannot neglect the temperature gradients inside the body
 - Have to solve the complete heat conduction equation
 - Consider a solid wall with temperature T_i on one side which then instantly becomes lowered to a temperature T_∞ as it is placed into a fluid.
 - As time increases, the temperature inside the wall e.g. at the center decreases
 - So in this case, T is a function of x and t
 - For a plane wall $\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha} \frac{\partial T}{\partial t}$ where $\alpha = \frac{k}{\rho c_p}$ is the thermal diffusivity

- Second order wrt x so two boundary conditions are needed there. First order wrt t so one initial condition is needed there
- This can be solved analytically but we will not do that
- We instead consider the lumped capacitance solution $\ln \left[\frac{T-T_\infty}{T_i-T_\infty} \right] = \frac{-hA}{\rho V c_p} t$
- We take the characteristic length $L = \frac{V}{A}$.
- $\frac{hA}{\rho V c_p} t = \frac{h}{\rho L c_p} t = \left(\frac{h}{\rho L c_p} t \right) \left(\frac{L}{L} \cdot \frac{k}{k} \right) = \left(\frac{hL}{k} \right) \left(\frac{k}{\rho c_p} \right) \left(\frac{t}{L^2} \right) = \left(\frac{hL}{k} \right) \left(\frac{\alpha t}{L^2} \right)$ where $Bi = \frac{hL}{k}$ which is unitless
- We define the Fourier number $Fo = \frac{\alpha t}{L^2}$ which is also dimensionless
- The dimensionless temperature $\Theta = \frac{T-T_\infty}{T_i-T_\infty}$ so that the lumped capacitance solution can be written as $\Theta = \exp(-Bi \cdot Fo)$
- A physical interpretation of the Fourier number can be found by considering a cube with side length L . Then $\dot{Q}_{cond} = kA \frac{\partial T}{\partial x} = kL^2 \frac{\Delta T}{L} = kL \Delta T$
- $\dot{Q}_{store} = mc_p \frac{\partial T}{\partial t} = \rho L^3 c_p \frac{\Delta T}{t}$ so that $\frac{\dot{Q}_{cond}}{\dot{Q}_{store}} = \frac{k}{\rho c_p} \cdot \frac{t}{L^2} = \frac{\alpha t}{L^2} = Fo$
- Even when we cannot assume lumped capacitance and get an exact solution of the heat conduction equation, the solution is of the form $\Theta = \Theta(Bi, Fo)$
- We can define $\Theta_0 = \frac{T_0-T_\infty}{T_i-T_\infty}$ so that the solution is of the form $\Theta_0 = A_1 e^{-\lambda_1^2 Fo}$ where A_1, λ_1 are function of Bi
- Example: Carbon steel plate with $T_i = 440$ is placed in a furnace at $T_\infty = 600$. We need to heat to a minimum temperature of 520. What is the time t required.
 - $h = 200$, $k = 40$ and $\alpha = 8 \times 10^{-6}$
 - $Bi = \frac{hL}{k} = \frac{200 \times 0.04}{40} = 0.2$
 - Since $Bi > 0.1$, we cannot use the lumped capacitance and instead use the analytical solution which we obtain from tables
 - From Table 18.2, $Bi = 0.2 \implies \lambda_1 = 0.4328, A_1 = 1.0311$
 - $\Theta_0 = \frac{T_0-T_\infty}{T_i-T_\infty} = A_1 e^{-\lambda_1^2 Fo}$
 - $\Theta_0 = \frac{520-600}{440-600} = 0.5$ so that $0.5 = 1.0311 \exp(-(0.4328)^2 Fo)$ and so $Fo = 3.864$
 - $Fo = \frac{\alpha t}{L^2} \implies t = \frac{Fo L^2}{\alpha} = \frac{3.864 \times (0.04)^2}{8 \times 10^{-6}} = 773s$