MAT292 Notes

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Contents

1 Existence and Uniqueness Theorem

1. We need $f(t, y)$ continuous in the rectangle to get existence
2. We need $f_y(t, y)$ continuous in the rectangle to get uniqueness
3. E & U Theorem is sufficient but not necessary. i.e. these conditions imply solution but not having these conditions doesnt mean there is no solution

2 Autonomous Equations and Population Dynamics

2.1 Logistic Growth

If uninhibited, we assume exp. growth however in the long run,population is limited to K

Model: $y' = rh(y)y$

We want $h(y) \approx 1$ if y is small, $h(y) < 1$ if $y < k$, $h(y) = 0$ if $y = k$ and $h(y) < 0$ if $y > K$

This can thus be modelled as $y' = r(1 - \frac{y}{k})y$. This has two equilibria namely at $y = 0$ and yk. The inflection points can be found by setting the derivative y'' to 0.

3 Direction Fields and Orbits

3.1 Reducing non homogeneous systems to homogeneous systems

Lets take a solution x and write it as $x = \phi + v$ where v is a constant. Then $x' = Ax + b \rightarrow \phi = A(\phi + v) + b$. Since $x_{eq} = A^{-1}b$, $Av + b = 0$ by the equilibrium condition $(\phi' - A\phi)$ we have that $\phi' = A\phi$. So that $x = \phi + x_{eq}$ where ϕ is a solution of the homogeneous system.

Every solution of the non homogeneous problem can be written as a solution of the homogeneous problem plus the equilibrium.

4 Laplace Transform

- Remark: The laplace transform will allow us to reduce solving an ODE to solving an algebraic equation
- Solve algebraic equation and use the inverse laplace transform to get the solution to the ODE
- Definition: If f is defined on $[0, \infty]$, the Laplace Transform is defined as $F(s) = \int_0^\infty e^{-st} f(t)\, dt$
- We write $F = \mathcal{L}\{f\}$
- We use uppercase letters for Laplace transform e.g. $G(s)$ is the LT of $g(t)$
- Example: For $f(t) = e^{at}$, we get $F(s) = \mathcal{L}\{f\}(s) = \int_0^\infty e^{-st}e^{at}\, dt = \lim_{b\to\infty} \int_0^b e^{(a-s)t}\, dt = \lim_{b\to\infty} \frac{1}{a-s}\left(e^{(a-s)b} - 1\right) = \frac{1}{s-a}$ if $s > a$
- $\mathcal{L}\{1\} = \frac{1}{s}$
- Theorem: $\mathcal{L}\{c_1 f_1 + c_2 f_2\} = c_1\mathcal{L}\{f_1\} + c_2\mathcal{L}\{f_2\}$
- To find $\mathcal{L}\{\sin(at)\}$, write $\sin(at) = \frac{1}{2i}(e^{iat} - e^{-iat})$ and use the theorem above
- This will give $\frac{1}{2i}\left(\frac{1}{s-ia}\right) - \frac{1}{2i}\left(\frac{1}{s+ia}\right) = \frac{a}{s^2+a^2}$ for $s > 0$

- Example: LT of $f(t) = e^{2t}$ for $0 \leq t < 1$ and $f(t) = 4$ for $1 \leq t$
- Divide the integral into two seperate parts and evaluate it
- Exponential order: A function $f(t)$ is of exponential order for $M > 0$, $K > 0$ and $a \in \mathbb{R}$ if $|f(t)| \leq Ke^{at}$ for $t \geq M$ i.e. f eventually becomes between two exponential functions
- Theorem: Every bounded function is of exponential order
- A function $f(t)$ is piecewise continuous on $[a, b]$ iff there are finitely many "jump points" between a and b $a \leq t_0 < t_1 < \cdots < t_{k-1} < t_k = b$ such that f is continuous on each of the intervals (t_i, t_{i+1}) and f has finite limits at the jump points.
- Theorem: If for a function $f(t)$, we have that f is piecewise continuous on $[0, A]$ $\forall A \geq 0$ and f is of exponential order for M, k and a. Then $\mathcal{L}\{f\}$ exists for all $s > a$.
- Theorem: If $f(t)$ is of exponential order then we have: $F(s) \to 0$ as $s \to \infty$ where $F(s)$ is the L.T. of f
- Theorem: If f is continuous and f' is piecewise continuous on any interval $[0, A]$ and f, f' are of exponential order for M, k, a then $\mathcal{L}\{f'\}(s) = s\mathcal{L}\{f\}(s) - f(0)$ for $s > a$. Under the same conditions for n derivatives, $\mathcal{L}\{f^{(n)}\}(s) = s^n\mathcal{L}\{f\}(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \ldots sf^{(n-2)}(0) - f^{(n-1)}(0)$
- Proof: $\mathcal{L}\{f'\}(s) = \int_0^\infty e^{-st} f'(t)\, dt = \lim_{b\to\infty} \left(\int_0^b e^{-st} f'(t)\, dt\right)$

  $= \lim_{b\to\infty} \left([e^{-st}f(t)]_0^b + \int_0^b f(t)se^{-st}\, dt\right) = \lim_{b\to\infty} \left(e^{-bs}f(b) - f(0) + s\int_0^b f(t)e^{-st}\, dt\right)$
  $= s\mathcal{L}\{f\}(s) - f(0)$ where $s > a$ (by definition of exponential order)

5 Inverse Laplace Transform

- Theorem: If $f(t), g(t)$ are piecewise continuous and of exponential order, then $\mathcal{L}\{f\} = \mathcal{L}\{g\} \implies f(t) = g(t)$
- Technicality: Take $f(t) = e^t$, $g(t) = \begin{cases} e^t & t \neq 5 \\ 0 & t = 5 \end{cases}$. Clearly $\mathcal{L}\{f\} = \mathcal{L}\{g\}$ but $f(t) \neq g(t)\, \forall t$
- Convention: We write $f(t) = g(t)$ as long as they are the same whenever they are continuous
- Definition: If f is piecewise continuous and of exponential order and $\mathcal{L}\{f\}(s) = F(s)$, then we call $f(t) = \mathcal{L}^{-1}\{F\}(t)$
- There is a complex analysis formula (Mellin Transform) to find $\mathcal{L}^{-1}\{F\}$. However this is rarely used in practice and we instead use tables

6 Solving ODEs with Laplace Transform

- Lets solve the IVP $y'' + 2y' + 5y = e^{-t}$, $y(0) = 1$, $y'(0) = -3$
  - Use the Laplace transform: $\mathcal{L}\{y'' + 2y' + 5y\} = \mathcal{L}\{e^{-t}\}$
  - $s^2Y(s) - sy(0) - y'(0) + 2(sY(s) - y(0)) + 5Y(s) = \frac{1}{s+1}$
  - $s^2Y(s) - s \cdot 1 - (-3) + 2(sY(s) - 1) + 5Y(s) = \frac{1}{s+1}$ using the initial conditions
  - Solving this gives $Y(s) = \frac{s^2}{(s+1)(s^2+2s+5)}$
  - Simplifying and using partial fractions, $Y(s) = \frac{1}{4}\frac{1}{s+1} + \frac{3}{4}\frac{s+1}{(s+1)^2+4} - \frac{2}{(s+1)^2+4}$
  - $y(t) = \mathcal{L}^{-1}\{Y(s)\} = \frac{1}{4}\mathcal{L}^{-1}\{\frac{1}{s+1}\} + \frac{3}{4}\mathcal{L}^{-1}\{\frac{s+1}{(s+1)^2+4}\} - 2\mathcal{L}^{-1}\{\frac{1}{(s+1)^2+4}\}$
  - $= \frac{1}{4}e^{-t} + \frac{3}{4}e^{-t}\cos(2t) - e^{-t}\sin(2t)$
- Lets solve the IVP $y'''' + 2y' + y = 0$, $y(0) = 1$, $y'(0) = -1$, $y''(0) = 0$, $y'''(0) = 2$
  - Applying the Laplace transform: $s^4Y(s) - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0) + 2(s^2Y(s) - sy(0) - y'(0)) + Y(s) = 0$
  - Using the initial conditions, $s^4Y(s) - s^3 + s^2 - 2 + 2(s^2Y(s) - s + 1) + Y(s) = 0$
  - Solving gives $Y(s) = \frac{s^3-s^2+2s}{(s^2+1)^2}$
  - We use a repeated partial fraction decomposition to write $Y(s)$ as $\frac{As+B}{s^2+1} + \frac{s+1}{(s^2+1)^2}$
  - We know $y(t) = \mathcal{L}^{-1}\{\frac{s}{s^2+1} - \frac{1}{s^2+1} + \frac{s}{(s^2+1)^2} + \frac{1}{(s^2+1)^2}\}$
  - For the third term, we use $\mathcal{L}\{t \cdot f(t)\} = \frac{-d}{ds}F(s)$ and then get $y(t) = \cos t - \sin t + \frac{1}{2}t\sin t + \frac{1}{2}\sin t - \frac{1}{2}t\cos t$

7 Discontinuous Forcing Functions

- Finding $\mathcal{L}\{t\} = \int_0^\infty e^{-st}t\,dt = \frac{1}{s^2}$ (IBP) or using $\mathcal{L}\{t\} = \mathcal{L}\{t \cdot 1\}$ and $\mathcal{L}\{t \cdot 1\} = -\frac{d}{ds}(\frac{1}{s}) = \frac{1}{s^2}$
- Consider $y'' + 4y = g(t)$ with $y(0) = 0$, $y'(0) = 0$ and $g(t) = \begin{cases} 0 & 0 < t < 5 \\ \frac{t-5}{5} & t = 5 \le t < 10 \\ t & 10 \le t \end{cases}$
  - Can split it up into 3 parts and find relevant boundary conditions in each case to be used in the next case
- Piecewise defined functions can simulate the activation of a signal
  - $u_c(t)$ is a step function that is 0 till $x = c$ and then 1 afterwards
  - $u_{cd}(t)$ is an indicator function which is 1 between c and d and 0 everywhere else
  - Use a combination of step and indicator functions $= u_{cd}(t)\frac{t-c}{d-c} + u_d(t)$ which is 0 less than c, increasing between c and d and 1 for values greater than d

- We can find the Laplace transforms of these functions
  - $\mathcal{L}\{u_c\} = \int_0^\infty e^{-st}u_c(t)\,dt = \int_c^\infty e^{-st}\cdot 1\,dt = \frac{e^{-cs}}{s}$
  - $\mathcal{L}\{u_{cd}\} = \mathcal{L}\{u_c - u_d\} = \frac{e^{-cs}}{s} - \frac{e^{-ds}}{s}$
- The function $g(t)$ from earlier can be written as $u_{5,10}(t)\cdot\frac{t-5}{5} + u_{10}(t)\cdot 1$
  - Rearrange this as $\frac{1}{5}\left[u_{5,10}(t)(t-5) + 5u_{10}\right] = \frac{1}{5}\left[(u_5(t) - u_{10}(t))(t-5) + 5u_{10}(t)\right] = \frac{1}{5}\left[u_5(t)(t-5) - u_{10}(t)(t-10)\right]$
  - $h(t) = u_5(t)(t-5)$ is the time shift of f by $t=5$
- Theorem (Laplace transform of time-shift): If $F = \mathcal{L}\{f\}$ exists for $s > a$ and $c \geq 0$, then $\mathcal{L}\{u_c(t)f(t-c)\} = \int_0^\infty e^{-st}u_c(t)f(t-c)\,dt = \int_c^\infty e^{-st}f(t-c)\,dt = \int_0^\infty e^{-s(u+c)}f(u)\,du = e^{-sc}\int_0^\infty e^{-su}f(u)\,du = e^{-sc}\mathcal{L}\{f\}$
- Consider the IVP $y(0) = 0$, $y'(0) = 0$ and $y'' + 4y = u_1(t)$
  - $\mathcal{L}\{y'' + 4y\} = \mathcal{L}\{u_1(t)\} \to s^2Y(s) - sy(0) + 4Y(s) = \frac{e^{-s\cdot 1}}{s}$
  - $(s^2+4)Y(s) = \frac{e^{-s}}{s} \to Y(s) = \frac{e^{-s}}{s(s^2+4)}$
  - $Y(s) = e^{-s}\left(\frac{1}{4s} + \frac{-\frac{1}{4}s}{s^2+4}\right) = \frac{1}{4}e^{-s}\left(\frac{1}{s} - \frac{s}{s^2+4}\right)$
  - Let $H(s) = \frac{1}{s} - \frac{s}{s^2+4}$ so that $h(t) = 1 - \cos(2t)$. $y(t) = \frac{1}{4}u_1(t)h(t-1)$ by the previous theorem
- Consider $y'' + 4y = g(t)$ with $y(0) = 0$, $y'(0) = 0$ and $g(t) = \begin{cases} 0 & 0 < t < 5 \\ \frac{t-5}{5} & t = 5 \leq t < 10 \\ t & 10 \leq t \end{cases}$ as before
  - $g(t)$ is equivalent to $\frac{1}{5}(u_5(t)(t-5) - u_{10}(t)(t-10))$
  - $\mathcal{L}\{y'' + 4y\} = \mathcal{L}\{g(t)\} \to (s^2+4)Y(s) = \frac{1}{5}\left[e^{-5s}\cdot\frac{1}{s^2} - e^{-10s}\cdot\frac{1}{s^2}\right]$
  - $Y(s) = \frac{1}{5}\left[e^{-5s}\frac{1}{s^2(s^2+4)} - e^{-10s}\frac{1}{s^2(s^2+4)}\right]$ so that $y(t) = \frac{1}{5}\left[u_5(t)h(t-5) - u_{10}(t)h(t-10)\right]$ where $h(t) = \mathcal{L}^{-1}\{\frac{1}{s^2(s^2+4)}\} = \frac{1}{4}t - \frac{1}{8}\sin(2t)$

8 Impulse Functions

- $\delta_\epsilon(t) = \frac{1}{\epsilon}\cdot\left[u_0(t) - u_\epsilon(t)\right]$
- Note: $\int_0^\infty \delta_\epsilon(t)\,dt = 1$
- Equivalently $\delta_\epsilon(t) = \begin{cases} \frac{1}{\epsilon} & 0 \leq t < \epsilon \\ 0 & \text{else} \end{cases}$
- Let $\delta = \lim_{\epsilon\to 0}\delta$ such that $\delta(t) = 0$ when $t \neq 0$

- Then $\delta(t) = \begin{cases} \infty & t = 0 \\ 0 & t \neq 0 \end{cases}$

- $\int_0^\infty \delta(t)\,dt = \lim_{\epsilon\to 0} 1 = 1$

- For any function $f(t)$ continuous on $a \leq 0 < b$, we have $\int_a^b f(t)\delta(t)\,dt = \lim_{\epsilon\to 0}\int_a^b f(t)\delta_\epsilon(t)\,dt = \lim_{\epsilon\to 0}\int_0^\epsilon f(t)\cdot\frac{1}{\epsilon}\,dt = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_0^\epsilon f(t)\,dt = \lim_{\epsilon\to 0}\frac{1}{\epsilon}(\epsilon - 0)f(t*)$ for some point $t*$ s.t. $0 \leq t* \leq \epsilon$ (MVT for integrals)

- $= \lim_{\epsilon\to 0} f(t*) = f(0)$ since $t* \to 0$ as $\epsilon \to 0$

- δ is a generalized function (aka distribution). It is also called the Dirac delta function

- We can also shift the impulse function i.e. $\delta(t - t_0) = 0$ if $t \neq t_0$

- If $f(t)$ is continuous on $a \leq t_0 < b$, $\int_a^b f(t)\delta(t - t_0)\,dt = f(t_0)$

- Theorem $\mathcal{L}\{\delta(t - t_0)\} = \int_0^\infty e^{-st}\delta(t - t_0)\,dt = f(t_0) = e^{-st_0}$

- Consider an undamped oscillator $y'' + y = I_0\delta(t)$ $y(0) = 0$, $y'(0) = 0$
  - Laplace: $s^2Y(s) + Y(s) = I_0e^{-s\cdot 0} \implies Y(s) = I_0\frac{1}{s^2+1}$
  - Inverse LT: $y(t) = I_0\sin(t)$
  - However this gives $y'(0) = I_0$ but since we only consider $t \geq 0$, $y(t)$ is actually $u_0(t)I_0\sin(t)$
  - $\lim_{t\to 0^-} y'(t) = 0$

9 Convolution Integrals and Their Applications

- If f and g are piecewise continuous on $[0, \infty)$, we define their convolution as $(f * g)(t) = \int_0^t f(t - \tau)g(\tau)\,d\tau$

- Example: $f(t) = t$ and $g(t) = e^{-2t}$
  - $(f * g)(t) = \int_0^t (t - \tau)e^{-2\tau}\,d\tau = [(t - \tau)(\frac{-1}{2})e^{-2\tau}]_{\tau=0}^{\tau=t} + \int_0^t(-\frac{1}{2})e^{-2\tau}\,d\tau = \frac{t}{2} + \frac{1}{4}e^{-2t} - \frac{1}{4}$

- Rules for Convolutions
  - $f * g = g * f$
  - $f * (g + h) = f * g + f * h$
  - $f * 0 = 0$
  - $(cf) * g = c(f * g)$
  - Note: $f * 1 = f$

- Proof that the convolution is commutative:

- $f * g = \int_0^t f(t-\tau)g(\tau)\,d\tau = -\int_t^0 f(u)g(t-u)\,du = \int_0^t g(t-u)f(u)\,du = g * f$

- Convolution Theorem: If $F(s) = \mathcal{L}\{f(t)\}$ and $G(s) = \mathcal{L}\{g(t)\}$ both exist for $s > a \geq 0$ then $F(s)G(s) = \mathcal{L}\{(f * g)(t)\}$ and equivalently $\mathcal{L}^{-1}\{F(s)G(s)\} = (f * g)(t)$

10 Introduction to Partial Differential Equations

- The Heat Equation
  - Consider a metal rod of length L. Let $u(x,t)$ be the temperature in the cross section at location x and time t
  - $u_t(x,t)$ represents the change in temperature over time for a fixed slice
  - $u_x(x,t)$ represents the change over the rod for a fixed time
  - $u_{xx}(x,t)$ represents how $u_x(t)$ changes over the rod for a fixed time
  - The Heat Equation is given by $u_t = \alpha^2 u_{xx}$ where a^2 is the thermal diffusivity

- Example: Solving the Heat Equation
  - We can solve this by the seperation of variables
  - Consider a metal rod with length 50cm, insulated on the ends and with an initial temp of 20 C throughout with the ends maintained at 0C
  - The homogeneous problem is $u_t = \alpha^2 u_{xx}$ with $u(0,T) = 0$, $u(L,t) = 0$ for $t > 0$ and $u(x,0) = 20$
  - Assume that we can write $u(x,t) = X(x)T(t)$. This is a strong assumption since many functions cannot be written in this form e.g. $u(x,t) = \sin(xt)$
  - Using this assumption, we can rewrite the homogeneous problem as $X(x)T'(t) = \alpha^2 X''(x)T(t)$, $X(0)T(t) = 0$, $X(L)T(t) = 0$ and $X(x)T(0) = 20$
  - This leads to $X(0) = 0$ and $X(L) = 0$ since $T(t)$ cannot always be 0
  - We rearrange the equation above as $\frac{T'(t)}{\alpha^2 T(t)} = \frac{X''(x)}{X(x)} = -\lambda$
  - Since the two sides are equal, this expression cannot change over x or t so we let the expression be equal to $-\lambda$ where λ is a constant
  - We can find a solution for $X''(x)$ since $\frac{X''(x)}{X(x)} = -\lambda$ which has a solution $X(x) = c_1\cos(\sqrt{\lambda}x) + c_2\sin(\sqrt{\lambda}x)$ and using $X(0) = 0$ and $X(L) = 0$, we get $c_1 = 0$ and $0 = c_2\sin(\sqrt{\lambda}\cdot L)$ so that $\sqrt{\lambda}\cdot L = n\cdot\pi$. Equivalently $\lambda = \frac{n^2\cdot\pi^2}{L^2}$
  - We can similarly find a solution for $T(t)$ where the ODE is $\frac{T'(t)}{\alpha^2 T(t)} = -\lambda$
  - $T'(t) = -\frac{n^2\pi^2}{L^2}\alpha^2 T(t) \implies T(t) = e^{-\frac{n^2\pi^2}{L^2}\alpha^2 t}$
  - Combining into solutions of the heat equation: $u(x,t) = X(x)T(t) = \sin(\frac{n\pi}{L}x)e^{-\frac{n^2\pi^2}{L^2}\alpha^2 t}$ (there can be any constant factor at the front)
  - Fundamental solutions to the Heat Equation: $u_n(x,t) = \sin(\frac{n\pi x}{L})e^{-\frac{n^2\pi^2}{L^2}\alpha^2 t}$

- Theorem (Superposition Principle): If $a(x,t)$ and $b(x,t)$ solve the heat equation, $c_1 a(x,t) + c_2 b(x,t)$ solves the heat equation $\forall\, c_1, c_2 \in \mathbb{R}$
- General solution to the heat equation is $u(x,t) = \sum_{n=1}^{\infty} c_n \sin(\frac{n\pi x}{L}) e^{-\frac{n^2\pi^2\alpha^2}{L^2}t}$
- Using the initial conditions, we have $20 = u(x,0) = \sum_{n=1}^{\infty} c_n \sin(\frac{n\pi x}{L})$
- This will work for $c_n = \frac{2}{L}\int_0^L 20 \cdot \sin(\frac{n\pi x}{L})$ (Fourier Series)

11 Fourier Series

- We define an inner product on $PC[a,b]$ (piecewise continuous from a to b) by $< f, g >= \int_a^b f(x)g(x)\,dx$
- We say f and g are orthogonal if $< f, g >= 0$
- Consider $f(x) = \cos(x)$ and $g(x) = \sin(x)$ in $PC[-\pi,\pi]$. $< \cos x, \sin x >= \int_{-\pi}^{\pi} \cos x \sin x\,dx = 0 \implies \cos x, \sin x$ are orthogonal on $PC[-\pi,\pi]$
- Consider $f(x) = \cos(x)$ and $g(x) = x^2$ in $PC[-\pi,\pi]$ $< \cos x, x^2 >= \int_{-\pi}^{\pi} \cos x x^2\,dx = -4\pi$
- Theorem: The following set of functions is an orthogonal family in $PC[-L,L]$: $\left\{\frac{1}{2}, \sin(\frac{m\pi x}{L}), \cos(\frac{m\pi x}{L}) \ : m = 1,2,3,\dots\right\}$
- Theorem (Fourier's Theorem): Suppose f is periodic with period $2L$ and both f, f' belonging to $PC[-L,L]$. Then f has a Fourier Series:

  $f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty}\left(a_m \cos\frac{m\pi x}{L} + b_m \sin\frac{m\pi x}{L}\right)$, where $a_0 = \frac{2}{L} < f(x), \frac{1}{2} >= \frac{1}{L}\int_{-L}^{L} f(x)\,dx$, $a_m = \frac{1}{L} < f(x), \cos\frac{m\pi x}{L} >= \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{m\pi x}{L}\,dx$ and $b_m = \frac{1}{L} < f(x), \sin\frac{m\pi x}{L} >= \int_{-L}^{L} f(x)\sin\frac{m\pi x}{L}\,dx$
- The Fourier series converges to $f(x)$ at all points where f is continuous and to the midway point between left and right limit at all points where f is discontinuous