MAT292 Notes

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Contents

1 Existence and Uniqueness Theorem

1. We need $f(t, y)$ continuous in the rectangle to get existence
2. We need $f_y(t, y)$ continuous in the rectangle to get uniqueness
3. E & U Theorem is sufficient but not necessary. i.e. these conditions imply solution but not having these conditions doesnt mean there is no solution

2 Autonomous Equations and Population Dynamics

2.1 Logistic Growth

If uninhibited, we assume exp. growth however in the long run,population is limited to K

Model: $y' = rh(y)y$

We want $h(y) \approx 1$ if y is small, $h(y) < 1$ if $y < k$, $h(y) = 0$ if $y = k$ and $h(y) < 0$ if $y > K$

This can thus be modelled as $y' = r(1 - \frac{y}{k})y$. This has two equilibria namely at $y = 0$ and yk. The inflection points can be found by setting the derivative y'' to 0.

3 Direction Fields and Orbits

3.1 Reducing non homogeneous systems to homogeneous systems

Lets take a solution x and write it as $x = \phi + v$ where v is a constant. Then $x' = Ax + b \rightarrow \phi = A(\phi + v) + b$. Since $x_{eq} = A^{-1}b$, $Av + b = 0$ by the equilibrium condition $(\phi' - A\phi)$ we have that $\phi' = A\phi$. So that $x = \phi + x_{eq}$ where ϕ is a solution of the homogeneous system.

Every solution of the non homogeneous problem can be written as a solution of the homogeneous problem plus the equilibrium.

4 Laplace Transform

- Remark: The laplace transform will allow us to reduce solving an ODE to solving an algebraic equation
- Solve algebraic equation and use the inverse laplace transform to get the solution to the ODE
- Definition: If f is defined on $[0, \infty]$, the Laplace Transform is defined as $F(s) = \int_0^\infty e^{-st} f(t)\, dt$
- We write $F = \mathcal{L}\{f\}$
- We use uppercase letters for Laplace transform e.g. $G(s)$ is the LT of $g(t)$
- Example: For $f(t) = e^{at}$, we get $F(s) = \mathcal{L}\{f\}(s) = \int_0^\infty e^{-st} e^{at}\, dt = \lim_{b\to\infty} \int_0^b e^{(a-s)t}\, dt = \lim_{b\to\infty} \frac{1}{a-s}\left(e^{(a-s)b} - 1\right) = \frac{1}{s-a}$ if $s > a$
- $\mathcal{L}\{1\} = \frac{1}{s}$
- Theorem: $\mathcal{L}\{c_1 f_1 + c_2 f_2\} = c_1\mathcal{L}\{f_1\} + c_2\mathcal{L}\{f_2\}$
- To find $\mathcal{L}\{\sin(at)\}$, write $\sin(at) = \frac{1}{2i}(e^{iat} - e^{-iat})$ and use the theorem above
- This will give $\frac{1}{2i}\left(\frac{1}{s-ia}\right) - \frac{1}{2i}\left(\frac{1}{s+ia}\right) = \frac{a}{s^2+a^2}$ for $s > 0$
- Example: LT of $f(t) = e^{2t}$ for $0 \le t < 1$ and $f(t) = 4$ for $1 \le t$
- Divide the integral into two seperate parts and evaluate it
- Exponential order: A function $f(t)$ is of exponential order for $M > 0$, $K > 0$ and $a \in \mathbb{R}$ if $|f(t)| \le Ke^{at}$ for $t \ge M$ i.e. f eventually becomes between two exponential functions
- Theorem: Every bounded function is of exponential order

- A function $f(t)$ is piecewise continuous on $[a, b]$ iff there are finitely many "jump points" between a and b $a \leq t_0 < t_1 < \cdots < t_{k-1} < t_k = b$ such that f is continuous on each of the intervals (t_i, t_{i+1}) and f has finite limits at the jump points.

- Theorem: If for a function $f(t)$, we have that f is piecewise continuous on $[0, A]$ $\forall A \geq 0$ and f is of exponential order for M, k and a. Then $\mathcal{L}\{f\}$ exists for all $s > a$.

- Theorem: If $f(t)$ is of exponential order then we have: $F(s) \to 0$ as $s \to \infty$ where $F(s)$ is the L.T. of f

- Theorem: If f is continuous and f' is piecewise continuous on any interval $[0, A]$ and f, f' are of exponential order for M, k, a then $\mathcal{L}\{f'\}(s) = s\mathcal{L}\{f\}(s) - f(0)$ for $s > a$. Under the same conditions for n derivatives, $\mathcal{L}\{f^{(n)}\}(s) = s^n\mathcal{L}\{f\}(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \ldots sf^{(n-2)}(0) - f^{(n-1)}(0)$

- Proof: $\mathcal{L}\{f'\}(s) = \int_0^\infty e^{-st}f'(t)\,dt = \lim_{b\to\infty}\left(\int_0^b e^{-st}f'(t)\,dt\right)$

  $= \lim_{b\to\infty}\left([e^{-st}f(t)]_0^b + \int_0^b f(t)se^{-st}\,dt\right) = \lim_{b\to\infty}\left(e^{-bs}f(b) - f(0) + s\int_0^b f(t)e^{-st}\,dt\right)$
  $= s\mathcal{L}\{f\}(s) - f(0)$ where $s > a$ (by definition of exponential order)

5 Inverse Laplace Transform

- Theorem: If $f(t), g(t)$ are piecewise continuous and of exponential order, then $\mathcal{L}\{f\} = \mathcal{L}\{g\} \implies f(t) = g(t)$

- Technicality: Take $f(t) = e^t$, $g(t) = \begin{cases} e^t & t \neq 5 \\ 0 & t = 5 \end{cases}$. Clearly $\mathcal{L}\{f\} = \mathcal{L}\{g\}$ but $f(t) \neq g(t)\ \forall t$

- Convention: We write $f(t) = g(t)$ as long as they are the same whenever they are continuous

- Definition: If f is piecewise continuous and of exponential order and $\mathcal{L}\{f\}(s) = F(s)$, then we call $f(t) = \mathcal{L}^{-1}\{F\}(t)$

- There is a complex analysis formula (Mellin Transform) to find $\mathcal{L}^{-1}\{F\}$. However this is rarely used in practice and we instead use tables

6 Solving ODEs with Laplace Transform

- Lets solve the IVP $y'' + 2y' + 5y = e^{-t}$, $y(0) = 1$, $y'(0) = -3$
  - Use the Laplace transform: $\mathcal{L}\{y'' + 2y' + 5y\} = \mathcal{L}\{e^{-t}\}$
  - $s^2Y(s) - sy(0) - y'(0) + 2(sY(s) - y(0)) + 5Y(s) = \frac{1}{s+1}$
  - $s^2Y(s) - s \cdot 1 - (-3) + 2(sY(s) - 1) + 5Y(s) = \frac{1}{s+1}$ using the initial conditions

- Solving this gives $Y(s) = \frac{s^2}{(s+1)(s^2+2s+5)}$
- Simplifying and using partial fractions, $Y(s) = \frac{1}{4}\frac{1}{s+1} + \frac{3}{4}\frac{s+1}{(s+1)^2+4} - \frac{2}{(s+1)^2+4}$
- $y(t) = \mathcal{L}^{-1}\{Y(s)\} = \frac{1}{4}\mathcal{L}^{-1}\{\frac{1}{s+1}\} + \frac{3}{4}\mathcal{L}^{-1}\{\frac{s+1}{(s+1)^2+4}\} - 2\mathcal{L}^{-1}\{\frac{1}{(s+1)^2+4}\}$
- $= \frac{1}{4}e^{-t} + \frac{3}{4}e^{-t}\cos(2t) - e^{-t}\sin(2t)$

- Lets solve the IVP $y'''' + 2y' + y = 0$, $y(0) = 1$, $y'(0) = -1$, $y''(0) = 0$, $y'''(0) = 2$
  - Applying the Laplace transform: $s^4Y(s) - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0) + 2(s^2Y(s) - sy(0) - y'(0)) + Y(s) = 0$
  - Using the initial conditions, $s^4Y(s) - s^3 + s^2 - 2 + 2(s^2Y(s) - s + 1) + Y(s) = 0$
  - Solving gives $Y(s) = \frac{s^3-s^2+2s}{(s^2+1)^2}$
  - We use a repeated partial fraction decomposition to write $Y(s)$ as $\frac{As+B}{s^2+1} + \frac{s+1}{(s^2+1)^2}$
  - We know $y(t) = \mathcal{L}^{-1}\{\frac{s}{s^2+1} - \frac{1}{s^2+1} + \frac{s}{(s^2+1)^2} + \frac{1}{(s^2+1)^2}\}$
  - For the third term, we use $\mathcal{L}\{t \cdot f(t)\} = \frac{-d}{ds}F(s)$ and then get $y(t) = \cos t - \sin t + \frac{1}{2}t\sin t + \frac{1}{2}\sin t - \frac{1}{2}t\cos t$

7 Discontinuous Forcing Functions

- Finding $\mathcal{L}\{t\} = \int_0^\infty e^{-st}t\,dt = \frac{1}{s^2}$ (IBP) or using $\mathcal{L}\{t\} = \mathcal{L}\{t \cdot 1\}$ and $\mathcal{L}\{t \cdot 1\} = -\frac{d}{ds}(\frac{1}{s}) = \frac{1}{s^2}$

- Consider $y'' + 4y = g(t)$ with $y(0) = 0$, $y'(0) = 0$ and $g(t) = \begin{cases} 0 & 0 < t < 5 \\ \frac{t-5}{5} & t = 5 \leq t < 10 \\ t & 10 \leq t \end{cases}$
  - Can split it up into 3 parts and find relevant boundary conditions in each case to be used in the next case

- Piecewise defined functions can simulate the activation of a signal
  - $u_c(t)$ is a step function that is 0 till $x = c$ and then 1 afterwards
  - $u_{cd}(t)$ is an indicator function which is 1 between c and d and 0 everywhere else
  - Use a combination of step and indicator functions $= u_{cd}(t)\frac{t-c}{d-c} + u_d(t)$ which is 0 less than c, increasing between c and d and 1 for values greater than d

- We can find the Laplace transforms of these functions
  - $\mathcal{L}\{u_c\} = \int_0^\infty e^{-st}u_c(t)\,dt = \int_c^\infty e^{-st} \cdot 1\,dt = \frac{e^{-cs}}{s}$
  - $\mathcal{L}\{u_{cd}\} = \mathcal{L}\{u_c - u_d\} = \frac{e^{-cs}}{s} - \frac{e^{-ds}}{s}$

- The function $g(t)$ from earlier can be written as $u_{5,10}(t) \cdot \frac{t-5}{5} + u_{10}(t) \cdot 1$

- Rearrange this as $\frac{1}{5}\left[u_{5,10}(t)(t-5)+5u_{10}\right]=\frac{1}{5}\left[(u_5(t)-u_{10}(t))(t-5)+5u_{10}(t)\right]=\frac{1}{5}\left[u_5(t)(t-5)-u_{10}(t)(t-10)\right]$
- $h(t)=u_5(t)(t-5)$ is the time shift of f by $t=5$

- Theorem (Laplace transform of time-shift): If $F=\mathcal{L}\{f\}$ exists for $s>a$ and $c\geq 0$, then $\mathcal{L}\{u_c(t)f(t-c)\}=\int_0^\infty e^{-st}u_c(t)f(t-c)\,dt=\int_c^\infty e^{-st}f(t-c)\,dt=\int_0^\infty e^{-s(u+c)}f(u)\,du=e^{-sc}\int_0^\infty e^{-su}f(u)\,du=e^{-sc}\mathcal{L}\{f\}$