UMARU MUSA YAR'ADUA UNIVERSITY KATSINA
DEPARTMENT OF MATHEMATICS AND STATISTICS

MTH 1311 (ALGEBRA AND TRIGONOMETRY) LECTURE NOTE

Prepared by
Lawal Aliyu Yantumaki
Nuraddeen Sayyadi
Typed by Dr. Ibrahim Mu'awiyya Idris

Course Contents:

- Elementary Set Theory
  - Subsets
  - Set Operations
  - Algebra of sets
  - Venn Diagram
- Number System
  - Natural numbers
  - Integers
  - Rational and Irrational numbers
  - Real numbers
  - Complex numbers
- Real Sequences and Series
- Theory of Quadratic Equations
- Binomial Theorem
- Trigonometry
  - Angular measurement
  - Trigonometric functions
  - Trigonometric functions of any magnitude
  - Trigonometric formulae

Recommended Reading Books:

MODULE 1

1 Sets and basic operations on sets

The concept of sets appears in all branches of mathematics, the concept formalizes the idea of grouping objects together and viewing them as a single entity.

Definition 1.1 *Set is a well-defined collection of distinct objects, the objects are called elements or members of the set.*

If a set has no element, it is called an **empty set**, or **null set** denoted by the symbol $\emptyset$

Examples:

1. The odd numbers between zero and ten {1, 3, 5, 7, 9}
2. The states in Nigeria {katsina, kano, kaduna, ...}
3. The english alphabets {a, b, c, d, e, f, ...}
4. The roots of the equation $x^2 - 3x + 2 = 0$
5. The list of students absent from school
6. The rivers in Nigeria

Observe that the set in examples $1-3$ above are each defined by listing it's elements, while examples $4-6$ are each defined by stating the properties of it's elements. Conventionally, a set is denoted by a capital letter or a capital Greek letter such as A, B, X, Y, Ω, and the members denoted by small letters or small Greeks such as a, b, x, y, ω. A set is completely specified by denoting the set with a capital letter or symbol and specifying it's elements by either listing distinctly or stating it's basic properties. For instance, the set of english vowels can be written in any of the two ways below

$$A = \{a,\, e,\, i,\, o,\, u\}$$

$$A = \{vowel \,|\, english\ vowels\}$$

NOTE:

1. The first set A above is a set of elements containing english vowels listed distinctly and separated by commas enclosed in braces {...}, this type of specifying a set is called **"roster method"**.
2. The second set A is the set of elements containing english vowels specified by stating the basic properties enclosed in braces {...}, this type of specifying a set is called **"set-builder method"**. The set-builder method is the more mathematical way of specifying sets no matter the size of it's elements, consider the following example

   $$B = \{x : x\, is\ an\ even\, integer,\ x > 0\}.$$

   The above expression denotes set B whose elements are the positive even integers, specifying such a set using the roster method would be a tedious exercise or rather one can say is almost impossible. Using set-builder method, usually small letter x is used to denote a typical member of the set followed by a "colon" which means "such that".

1.1 Equality of sets

Two sets A and B are said to be equal written as $A = B$ if both sets have the same elements, that is, if every element which belongs to set A also belongs to set B and vice versa. Two sets A and B are not equal if at least one element which is a member of one is not a member of the other one, written as $A \neq B$. The statement "p is an element of A" or equivalently, the statement "p belongs to A" is written as $p \in A$. One can also write $a,\ b \in A$ to state that both a and b belongs to set A, whereas the statement that p is not an element of set A is written as $p \notin A$. If every element of set A is also an element of set B, then we say that A is a **subset** of B written as $A \subseteq B$

Examples 1.0:

1. The set of english vowels can be written as

   $A = \{x : x \ is\ an\ english\ alphabets,\ x\ is\ a\ vowels\}.$

   Observed that $b \notin A$, but $e,\ i,\ a\ \in A$ and $p \notin A$.

2. Let $E = \{x : x^2-3x+2 = 0\}$, E consists of numbers that solve the equation $x^2-3x+2 = 0$, or in other word the solution set of the given equation. Since the solution set is given by the integers 1 and 2, then one can write $E = \{1, 2\}$.

3. Let $E = \{x : x^2 - 3x + 2 = 0\}$, $F = \{2, 1\}$ and $G = \{1, 2, 2, 1, \frac{6}{3}\}$. Then $E = F = G$ since each contains the same elements 1 and 2. Observe here that a set does not depend on the way in which the elements are arranged, and a set remains the same even if some elements are repeated.

NOTE:

The followings are some important sets of numbers that occurs often, and thus special symbols are attached to them. Unless otherwise specified, we let

- $\mathbb{N}$ = The set of natural numbers
- $\mathbb{Z}$ = The set of integers (positive and non-positive inclusive)
- $\mathbb{Z}^+$ = The set of positive integers
- $\mathbb{Z}^-$ = The set of positive integers
- $\mathbb{Q}$ = The set of rational numbers
- $\mathbb{R}$ = The set of real numbers
- $\mathbb{C}$ = The set of complex numbers

1.2 Universal set

All sets under consideration are assumed to be extracted from a larger fixed set known as the **universal set**. For example, in plane geometry, the universal set consists of all the points in the plane, and in the study of human population, the universal set consist of all people in the universe. Conventionally, the universal set is often denoted by the capital letter $\mathbb{U}$ unless otherwise specified.

Note: Null set $\emptyset$ is an element of universal set $\mathbb{U}$

1.3 Venn diagram

A venn diagram is a pictorial representation of sets as circles enclosed in a rectangle, for example $A \subset B$ can be represented pictorially as

1.4 Complement of a set

Given a set A which is a proper subset of a designed universal set $\mathbb{U}$ written as $A \subset \mathbb{U}$, it is therefore possible to find the set of elements in $\mathbb{U}$ that are not in A known as A complement written as

$$A^c = \{x : x \in \mathbb{U}, x \notin A\}$$

Note, some text denote complement with A' or $\overline{A}$.

Example 1.1: Let $\mathbb{U} = \{a,\ b,\ c,\ ...,y,\ z\}$ be the universal set of english alphabets, and let $A = \{a,\ b,\ c,\ d,\ e\}$, $B = \{e,\ f,\ g\}$, $V = \{a,\ e,\ o,\ u,\ i\}$. Then

1. $A^c = \{f,\ g,\ h,\ ...,y,\ z\}$
2. $B^c = \{a,\ b,\ c,\ d,\ h,\ i,...,y,\ z\}$
3. $V^c = \{all\ non - vowels,\ consonants\}$

1.5 Subsets

Suppose every element of set A is also an element of set B, then set A is called a **subset** of set B. In other word, we can also say that set A is contained in set B or that set B contains set A and mathematically written as $A \subset B$ or $B \supset A$. Set A is not a subset of set B if at least one element of set A is not an element of set B, and written as $A \not\subset B$ or $B \not\supset A$.

Example 1.2:

1. Consider the sets $A = \{1,\ 3,\ 5,\ 8,\ 9\}$, $B = \{1,\ 2,\ 3,\ 5,\ 7\}$, $C = \{1,\ 5\}$. Then
   - $C \subset A$
   - $C \subset B$
   - $B \not\subset A$

Furthermore, since the elements in sets A, B and C must also belong to the universal set $\mathbb{U}$, then it is clear that $\mathbb{U}$ must at least contain the set

$$\mathbb{U} = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

2. Set $E = \{2, 4, 6\}$ is a subset of a set $F = \{6, 2, 4\}$, since each element which belong to E also belong to F. In fact $E = F$, this shows that every set is a subset of itself.

Proper subset: set A is said to be a proper subset of set B denoted as $A \subset B$. Moreover, A is said to be a proper subset of B if and only if $A \neq B$, that is, B contained all elements of A and some other elements

For example, consider the three sets $A = \{1, 3\}$, $B = \{1, 2, 3\}$ and $C = \{1, 3, 2\}$. Then B is an improper subset of C, while A is a proper subset of C.

1.6 Operations on set

1.6.1 Intersection of Sets:

If A and B are sets, the intersection of A and B denoted as $A \cap B$ is the set of elements that belongs to both A and B. Mathematically written as

$$A \cap B = \{x : x \in A, x \in B\}$$

as depicted in the following venn diagram

1.6.2 Union of Sets:

If A and B are sets, the union of A and B denoted as $A \cup B$ is the set of elements that belongs to at least A or B or both.

$$A \cup B = \{x : x \in A \, or \, x \in B \, or \, x \in A \text{ } and \text{ } B\}$$

as depicted in the following venn diagram

Example 1.3:

1. Let $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 5, 6, 7\}$ and $C = \{2, 3, 8, 9\}$. Then
   - $A \cup B = \{1, 2, 3, 4, 5, 6, 7\}$
   - $A \cap B = \{3, 4\}$
   - $A \cup C = \{1, 2, 3, 4, 8, 9\}$
   - $A \cap C = \{2, 3\}$
   - $B \cup C = \{2, 3, 4, 5, 6, 7, 8, 9\}$
   - $B \cap C = \{3\}$
2. Let $\mathbb{U}$ denote the set of students at a university, and let M and F denote respectively the set of male and female students at the university, then

$$M \cup F = \mathbb{U},$$

since each student in $\mathbb{U}$ is either in M or in F, therefore

$$M \cap F = \emptyset,$$

since no element belongs to both M and F (no student is a male and female at the same time).

The following properties for the operations of union and intersection of sets should be noted:

- Every element x in $A \cap B$ belongs to both A and B; hence x belongs to A and x belongs to B. Thus $A \cap B$ is a subset of A and of B, i.e,

$$A \cap B \subseteq A$$

and

$$A \cap B \subseteq B$$

- An element x belongs to the union $A \cup B$ if x belongs to A or x belongs to B; hence every element in A belongs to $A \cup B$, and also every element in B belongs to $A \cup B$, i.e,

$$A \subseteq A \cup B$$

and

$$B \subseteq A \cup B$$

1.7 Disjoint:

Two sets A and B are said to be disjoint if they have no element in common

For example, suppose that $A = \{1, 2\}$, $B = \{2, 4, 6\}$, $C = \{4, 5, 6\}$. Then sets A and B are not disjoint since both sets intersect at element 2. Similarly, sets B and C are not disjoint having intersected at elements 4 and 6, but sets A and C are disjoint having no element in common. In addition, if A and B are two arbitrary sets, then it is possible that some elements are in A but not in B, some elements are in B but not in A, some are in both A and B, and some are neither in A nor B. In general, we represent sets A and B as

1.8 Difference and Symmetric Difference

Let A and B be sets. The relative complement of B with respect to A or simply, the difference of A and B denoted by $A \backslash B$ is the set of elements which belongs to A but do not belongs to B, i.e,

$$A \backslash B = \{x : x \in A,\, x \notin B\}$$

The symmetric difference of sets A and B denoted by $A \oplus B$ consists of those elements which belongs to A or B but not to both A and B, i.e,

$$A \oplus B = (A \cup B) \backslash (A \cap B)$$

or

$$A \oplus B = (A \backslash B) \cup (B \backslash A)$$

Example 1.4: Consider the sets $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 5, 6, 7\}$ and $C = \{6, 7, 8, 9\}$. Then

a) $A \backslash B = \{1, 2\}$

b) $B \backslash C = \{3, 4, 5\}$

c) $B \backslash A = \{5, 6, 7\}$

d) $C \backslash B = \{8, 9\}$

Also,

e) $A \oplus B = \{1, 2, 5, 6, 7\}$

f) $B \oplus C = \{3, 4, 5, 8, 9\}$

Note that A and C are disjoints, i.e, $A \backslash C = A$, $C \backslash A = C$ and $A \oplus C = A \cup C$

1.9 Algebra of Sets

Sets under the operations of union, intersection and complement satisfies some laws or identities. If A, B and C are sets, and $\mathbb{U}$ is the universal set, then

1. Idempotent laws
   a) $A \cup A = A$
   b) $A \cap A = A$
2. Associative laws
   a) $(A \cup B) \cup C = A \cup (B \cup C)$
   b) $(A \cap B) \cap B = A \cap (B \cap C)$
3. Commutative laws
   a) $A \cup B = B \cup A$
   b) $A \cap B = B \cap A$
4. Distributive Laws
   a) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
   b) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
5. Identity Laws
   a) $A \cup \emptyset = A$

b) $A \cap \mathbb{U} = A$

c) $A \cup \mathbb{U} = \mathbb{U}$

d) $A \cap \emptyset = \emptyset$

6. Involution Law

$(A^c)^c = A$

7. Complement Laws

a) $A \cup A^c = \mathbb{U}$

b) $A \cap A^c = \emptyset$

c) $\mathbb{U}^c = \emptyset$

d) $\emptyset^c = \mathbb{U}$

8. De-Morgans Laws

a) $(A \cup B)^c = A^c \cap B^c$

b) $(A \cap B)^c = A^c \cup B^c$

1.10 Finite Sets

A set is said to be finite if it contains exactly m distinct elements, where m denotes some non-negative integer. On the other hand, a set is set to be infinite if it contains an infinitely many elements. For example, the empty set $\{\emptyset\}$ and the set of letters of the english alphabets are all finite sets, whereas the set of even positive integers $\{2, 4, 6, ...\}$ is infinite. The notation $n(A)$ or $|A|$ will denote the number of elements in a finite set A. Suppose that A and B are finite disjoint sets, then $A \cup B$ is a finite set written as

$$n(A \cup B) = n(A) + n(B).$$

There also exist a formula for $n(A \cup B)$ even when A and B are not disjoint called the inclusion principle.

Theorem: Suppose A and B are two finite sets, then $A \cap B$ and $A \cup B$ are finite

$$n(A \cup B) = n(A) + n(B) - n(A \cap B),$$

also for three finite sets A, B and C we have

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C).$$

Example 1.5: Consider the following data for 152 mathematics students, of which 65 study French, 45 study German, 42 study Russian, 20 study at least French and German, 25 study at least French and Russian, 15 study at least German and Russian, while 8 students study all the three languages.

a) Find the number of students that study at least one of the three languages.

b) Fill in the correct number of students in each of the eight regions of the venn diagram.

c) Find the number k of students that study:

1. Exactly one language.
2. Exactly two languages.

Solution:

a) The number of students that study at least one of three languages is given by

$$n(F \cup G \cup R) = n(F) + n(G) + n(R) - n(F \cap G) - n(F \cap R) - n(G \cap R) + n(F \cap G \cap R)$$

$$n(F \cup G \cup R) = 65 + 45 + 42 - 20 - 15 - 25 + 8 = 100$$

b) Using the number of students that study all the three languages $(F \cap G \cap R) = 8$ and the number of students that study at least one of the three languages $(F \cup G \cup R) = 100$, the remaining seven regions of the required Venn diagram are obtained as:

1. $20 - 8 = 12$ students study French and German
2. $25 - 8 = 17$ students study French and Russian
3. $15 - 8 = 7$ students study German and Russian
4. $65 - 37 = 28$ students study only French
5. $45 - 18 = 18$ students study only German
6. $42 - 32 = 10$ students study only Russian
7. $152 - 100 = 52$ students do not study any of the three languages

Using Venn diagram the problem is represented as

c) Use Venn diagram to obtain:

1. $k = 28 + 18 + 10 = 56$
2. $k = 12 + 17 + 7 = 36$

1.11 Power Set

The set of all subsets of a set A is called the power set of A, denoted by $P(A)$. For instance, suppose that $A = \{1,\, 2,\, 3\}$, then

$$P(A) = \{\emptyset,\, \{1\}, \{2\}, \{3\}, \{1,\, 2\}, \{1,\, 3\}, \{2,\, 3\}, A\}$$

Note that the empty set $\emptyset$ and the set A itself belongs to $P(A)$. For a set A with finite number of elements, the number of elements in $P(A)$ is raised to the power of $n(A)$, i.e,

$$n(P(A)) = 2^{n(A)}.$$

GENERAL PROBLEMS

1. Which of the following sets are equal
   $\{r,\, s,\, t,\, s\}$, $\{s,\, t,\, r,\, s\}$, $\{t,\, s,\, t,\, r\}$ and $\{s,\, r,\, s,\, t\}$?

   The sets are all equal because order and repetition do not change a set

2. List the elements of the following sets where $P = \{1,\, 2,\, 3,\, ...\}$

   a) $A = \{x : x \in P,\, 3 < x < 12\}$
   b) $B = \{x : x \in P,\, x \; is \; even,\, x < 15\}$
   c) $C = \{x : x \in P,\, 4 + x = 3\}$
   d) $D = \{x : x \in P,\, x \; is \; a \; multiple \; of \; 5\}$

3. Consider the following sets: $\{\emptyset\}$, $A = \{1\}$, $B = \{1,\, 3\}$, $C = \{1,\, 5,\, 9\}$, $D = \{1,\, 2,\, 3,\, 4,\, 5\}$ and $E = \{1,\, 3,\, 5,\, 7,\, 9\}$, such that the universal set is given by $\mathbb{U} = \{1,\, 2,\, 3, ...,\, 8,\, 9\}$. Insert the correct symbol $\subset$ or $\not\subseteq$ between each pair of sets and state your reason:

   a) $\emptyset,\, A$
   b) $A,\, B$
   c) $B,\, C$
   d) $B,\, E$
   e) $C,\, D$
   f) $C,\, E$
   g) $D,\, E$
   h) $D,\, \mathbb{U}$

4. Consider the universal set $\mathbb{U} = \{1,\, 2,\, 3, ...,\, 9\}$ and the sets: $A = \{1,\, 2,\, 3,\, 4,\, 5\}$, $B = \{4,\, 5,\, 6,\, 7\}$, $C = \{5,\, 6,\, 7,\, 8,\, 9\}$, $D = \{1,\, 3,\, 5,\, 7,\, 9\}$, $E = \{2,\, 4,\, 6,\, 8\}$ and $F = \{1,\, 5,\, 9\}$. Find:

   a) $A \cup B$ and $A \cap B$
   b) $B \cup D$ and $B \cap D$
   c) $E \cup E$ and $E \cap E$
   d) $D \cup F$ and $D \cap F$
   e) A^c, B^c, D^c and E^c

f) $\mathbb{U}^c$, $\emptyset^c$

g) $A\backslash B$, $B\backslash A$, $D\backslash E$ and $F\backslash D$

h) $A\cap(B\cup E)$

i) $(A\backslash E)^c$

j) $(A\cap D)\backslash B$

5. Show that $(A\cup B)\backslash(A\cap B) = (A\backslash B)\cup(B\backslash A)$.

6. In a science class, 18 students read Physics, 25 read Mathematics, 23 read Chemistry, 9 read Physics and Mathematics, 10 read Mathematics and Chemistry, 6 read Physics and Chemistry. If there were 50 students all together and 5 students did not read any of the three subjects, then how many students read:

   (i) All the three subjects

   (ii) Only Mathematics

   (iii) Chemistry but not Mathematics

   (iv) Physics and Chemistry but not Mathematics

MODULE 2

2 Number System

The concept of numbers has evolved over a long period of time, different types of numbers were invented by mathematicians for different purposes. The following are the set of numbers under which all numbers are named and classified:

(i) $\mathbb{N}$atural numbers

(ii) $\mathbb{I}$ntegers

(iii) $\mathbb{R}$ational numbers

(iv) $\mathbb{I}$rrational numbers

(v) $\mathbb{R}$eal numbers

(vi) $\mathbb{C}$omplex numbers

1. $\mathbb{N}$**atural numbers:** This is the set of basic numbers which is also called the counting numbers. Natural numbers are numbers in the set $\{1, 2, 3, 4, ...\}$, the three dots which is called an ellipsis indicates that the pattern extends indefinitely. We denote the set of natural numbers by the symbol $\mathbb{N}$.

   Remarks:

   (i) The set of Natural numbers $\mathbb{N}$ is closed under the operations of addition $(+)$ and multiplication $(\times)$.

   (ii) The set of Natural numbers $\mathbb{N}$ is not closed under the operations of division $(\div)$ and subtraction $(-)$.

   (iii) The set of Natural numbers $\mathbb{N}$ is countable.

2. $\mathbb{I}$**ntegers:** Since the set of Natural numbers are not adequate for some applications, then more numbers are needed to be invented for other needs. For example, in trade if the set of Natural numbers could not be used for both credit and debt, because debt is running in negative. The integers are the numbers in the set $\{..., -3, -2, -1, 0, 1, 2, 3, ...\}$, and the set of integers is denoted by the symbol $\mathbb{Z}$ after a German mathematician Zahlen.

   Remarks:

   (i) The components of integer are easily expressible in a number line

   (ii) The set of integers $\mathbb{Z}$ is closed under the operations of addition $(+)$, multiplication $(\times)$ and subtraction $(-)$.

(iii) The set of integers $\mathbb{Z}$ is not closed under the operation of division $(\div)$

3. **$\mathbb{R}$ational numbers:** Over the time the set of integers $\mathbb{Z}$ was found to be inadequate for some purpose. Therefore, the set of integers $\mathbb{Z}$ had to be expanded by inventing new set of numbers called the rational numbers. The rational numbers which are denoted by the symbol $\mathbb{Q}$ are numbers in the set

$$\mathbb{Q} = \{x : x = \frac{a}{b},\ a, b \in \mathbb{Z},\ b \neq 0\}.$$

In other word, the set of rational numbers $\mathbb{Q}$ are the quotient numbers such as $\frac{1}{2}, \frac{3}{4}, -\frac{5}{2}, \frac{10}{1}$.

Remarks:

(i) The set of Integers $\mathbb{Z}$ is a proper subset of the set of Rational numbers $\mathbb{Q}$, thus $\mathbb{Z} \subset \mathbb{Q}$.

(ii) The set of Rational numbers $\mathbb{Q}$ is closed under the operations of addition $(+)$, multiplication $(\times)$, subtraction $(-)$ and division $(\div)$.

(iii) Every rational number may be represented as decimals with either terminating or repeating digits.

Note:

(a) Any decimal number is a rational number, e.g

$$\begin{aligned} 12.478 &= 12.478 \times \frac{10^3}{10^3} \\ &= \frac{12.478 \times 1000}{1000} \\ &= \frac{12478}{1000} \end{aligned}$$

(b) Any repeating (recurring) and terminating decimals are rational numbers, e.g

$$\frac{1}{3} = 0.333333333... \qquad \frac{7}{66} = 0.106060606... \qquad \frac{5}{2} = 2.5 \qquad \frac{3}{4} = 0.75$$

(c) Any decimal number that is not rational is irrational.

4. **Irrational numbers:** The set of Irrational numbers are those decimals that neither terminate nor repeats, and cannot be expressed as a quotient of two integers $(\frac{a}{b})$ where $b \neq 0$. Examples of irrational numbers are Surds, i.e, $\sqrt{2}$, $2\sqrt{3}$, π, and the set of Irrational numbers have no symbol representing it.

Remark: Irrational numbers are not countable.

Example: Prove that $\sqrt{3}$ is not a rational number

Solution: We prove by contradiction as follows:

Suppose that $\sqrt{3} = \frac{p}{q}$ where $(p,\ q)$ are co-prime with GCD equal 1, and $(p, q) \in \mathbb{Z}$

taking the square of both sides, we've $3 = \frac{p^2}{q^2}$ which implies that

$$3q^2 = p^2$$

then dividing both sides by 3 we have

$$\tfrac{p^2}{3} = q^2$$

by theorem, since 3 is a prime number, if 3 divides p^2 then 3 also divides p

therefore 3 is a factor of p which implies that

$$p = 3c, \qquad c \in \mathbb{Z}$$

taking $p = 3c$ implies that

$$\tfrac{(3c)^2}{3} = q^2$$

$$\tfrac{q^2}{3} = c^2$$

which similarly implies 3 is also a factor of q, but this contradict the earlier assumption that p, q are co-prime, therefore $\sqrt{3}$ is not a rational number

5. **Real Numbers:** Irrational numbers are not closed under the operations $(+,\ \times,\ -,\ \div)$, e.g. $\sqrt{2} \times \sqrt{8}$ is not irrational. Therefore, the system of rational numbers $\mathbb{Q}$ is enlarged by combining it with irrational numbers to form the real number system. In other word, the set of Real numbers is the union of the set of rational and irrational numbers. The symbolic representation for the set of Real numbers is the Greek letter $\mathbb{R}$ such that

$$\mathbb{R} = \{x | x^2 \geq 0, x \in \mathbb{Q} \text{ or } x \in \mathbb{Q}'\}, \text{ where } \mathbb{Q}' \text{ is an irrational number}$$

Remarks:

(i) In many real life applications, the set of real numbers $\mathbb{R}$ are found to be adequately sufficient for research. Mathematicians usually deals with $\mathbb{R}$.

(ii) $\mathbb{R}$ is everywhere continuous.

(iii) $\mathbb{R}$ is closed under rational operations $(+,\ \times,\ -,\ \div)$, except division by zero.

6. **Complex numbers:** Unfortunately, the real numbers $\mathbb{R}$ are found to be not sufficient in some situations. In solving quadratic equations using almighty formula, there seemed to be difficulty when the discriminant $[b^2 - 4ac]$ is negative. For instance, consider the solution of the following equation

$$x^2 + 2x + 2 = 0.$$

Thus, we have the following solution using almighty formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

$$x = \frac{-2 \pm \sqrt{-4}}{2}.$$

Therefore, $\sqrt{-4}$ is not a real number. Indeed, the square root of any negative number is not a real number.

Therefore, the development leads to the invention of new set of numbers called the complex numbers. A complex number is any number of the form $[a + bi]$ where $a,\ b \in \mathbb{R}$, and $i = \sqrt{-1}$ is called the imaginary part. Examples of complex numbers are $2 + 3i,\ 7 - 8i,\ -4 + \sqrt{3}i$.

Note:

(a) $\sqrt{-1} = i$, so $i^2 = -1$

(b) $\sqrt{-4} = \sqrt{4(-1)} = \sqrt{4} \times \sqrt{-1} = 2i$

(c) $\sqrt{-81} = \sqrt{81(-1)} = \sqrt{81} \times \sqrt{-1} = 9i$

(d) $\sqrt{-3} = \sqrt{3(-1)} = \sqrt{3} \times \sqrt{-1} = \sqrt{3}i$

Remark:

(i) The pictorial representation of complex numbers in the $xy-$plane is called an Argand diagram

where the complex number $3 + 2i$ is represented by the point $(3, 2)$ in the plane.

(ii) The x-axis represent the real part, while the y-axis represent the imaginary part.

(iii) For complex numbers, any number other than zero has:

  (a) 2 square root ($\sqrt[2]{..}$)
  (b) 3 cube root ($\sqrt[3]{..}$)
  (c) 4 forth root ($\sqrt[4]{..}$)
  (d) n number has n^{th} roots ($\sqrt[n]{..}$)

STRUCTURE OF COMPLEX NUMBER SYSTEM

Complex number system has the following structures;

(a) Addition of two complex numbers is component by component, i.e., $(a_1 + ib_1) + (a_2 + ib_2) = (a_1 + a_2) + (b_1 + b_2)i$.

  Example 2.0: $(2 + 3i) + (7 - 4i) = (2 + 7) + (3 - 4)i = 9 - i$

(b) Two complex numbers are said to be equal if and if they are equal component wise, i.e., $a_1 + ib_1 = a_2 + ib_2$ then $a_1 = a_2$ and $b_1 = b_2$.

(c) Subtraction of two complex numbers is also component wise, i.e., $(a_1 + ib_1) - (a_2 + ib_2) = (a_1 - a_2) + (b_1 - b_2)i$.

  Example 2.1: $(2 + 3i) - (7 - 4i) = (2 - 7) + (3 + 4)i = -5 + 7i$

(d) Multiplication of two complex numbers is carried out by the usual method of expansion in real numbers, i.e.,

$$(a_1 + ib_1)(a_2 + ib_2) = a_1 \times a_2 + a_1 \times b_2 i + b_1 i \times a_2 + ib_1 \times ib_2$$

  Example 2.2: $(5 + 7i)(3 - i) = 5 \times 3 + 5 \times (-i) + 7i \times 3 + 7i \times (-i) = 22 + 16i$

(e) Division is carried out by rationalization

$$\frac{a_1 + ib_1}{a_2 + ib_2} = \frac{a_1 + ib_1}{a_2 + ib_2} \times \frac{a_2 - ib_2}{a_2 - ib_2}$$

Example 2.3: Evaluate the following expression

$$\begin{aligned} \frac{4+2i}{3+2i} &= \frac{4+2i}{3+2i} \times \frac{3-2i}{3-2i} \\ &= \frac{16}{13} + \frac{2}{13}i \end{aligned}$$

Note:

(i) $i = \sqrt{-1}$

(ii) $i^2 = -1$

(iii) $i^3 = i^2 \times i = -i$

(iv) $i^4 = i^2 \times i^2 = -1 \times -1 = 1$

(v) $i^5 = i^2 \times i^2 \times i = -1 \times -1 \times i = i$

Example 2.4: Find x and y satisfying the equation $2 + 3i = (x + i) + (3 - yi)$

EXERCISES

1. State which of the following is define under the operations of Addition, subtraction, multiplication and division
   (i) Natural numbers
   (ii) Integers
   (iii) Rational numbers
   (iv) Real numbers

2. State the simplest set to which each of the following numbers belongs
   (i) -3
   (ii) $2\frac{1}{2}$
   (iii) $\sqrt{7}$
   (iv) 1.76
   (v) 20000
   (vi) $1+\sqrt{3}i$
   (vii) $0.3333...$
   (viii) -0.0037

3. Assuming $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Q}'$, $\mathbb{Z}$ and $\mathbb{Z}^+$ denote respectively the sets of Real numbers, Rational numbers, Irrational numbers, Integers and positive Integers. State whether each of the following is true of false and stating reasons for your answer.
   (i) $-7 \in \mathbb{Z}^+$
   (ii) $\sqrt{2} \in \mathbb{Q}'$
   (iii) $4 \in \mathbb{Z}$
   (iv) $3\pi \in \mathbb{Q}$
   (v) $\sqrt[3]{8} \in \mathbb{Z}^+$
   (vi) $-2 \in \mathbb{Z}$
   (vii) $\pi^2 \in \mathbb{R}$
   (viii) $\sqrt{\frac{9}{4}} \in \mathbb{Q}'$

4. Prove that $\sqrt{2}$ is not a rational number.

5. Given that $Z_1 = 3+5i$, $Z_2 = 4-i$ and $Z_3 = 2+i$, find
   (i) $Z-1+Z_2+Z_3$
   (ii) $Z_1 - Z_2$
   (iii) $Z_1 + Z_2 - Z_3$
   (iv) $Z_1 Z_2$
   (vi) $\frac{Z_1}{Z_2}$

3 Sequences and series

3.1 Sequence

A sequence is a set of quantities u_1, u_2, $u_3, \ldots$ stated in a definite order and each term is formed according to a fixed pattern, i.e., $u_n = f(n)$. We shall used a subscripted lower case letters such as u_1 to represent the first term, u_2 to represent the second term, u_3 to represent the third term, ..., u_n to represent the n^{th}-term, and u_{n+1} to represent the $(n+1)^{th}$ term of a sequence respectively. The n^{th}-term or general term of a sequence is a rule for determining the terms of the sequence.

Example 3.0:

(i) 1, 3, 5, 7 is a sequence, and the next term will be 9.

(ii) 2, 6, 18, $54, \ldots$ is a sequence, and the next term will be 162.

(iii) 1^2, -2^2, 3^2, $-4^2, \ldots$ is a sequence, and the next term will be 5^2.

(iv) 1, -5, 37, 6 is a sequence, but its pattern is more involved and the next term cannot be anticipated.

A sequence is said to be finite if it contains only a finite number of terms, and an infinite sequence contains an infinite number of terms. Identify the following sequences as either finite or infinite

(a) All the Natural numbers 1, 2, 3, $4, \ldots$

(b) The page numbers of a book

(c) The telephone numbers in a directory

Example 3.1: Write the first 4 terms for each of the following sequences

(i) $u_n = n^{-1}$
$u_1 = 1^{-1} = 1$
$u_2 = 2^{-1} = \frac{1}{2}$
$u_3 = 3^{-1} = \frac{1}{3}$
$u_4 = 4^{-1} = \frac{1}{4}$

(ii) $u_n = 3^{n-1}$
$u_1 = 3^{1-1} = 3^0 = 1$
$u_2 = 3^{2-1} = 3^1 = 3$
$u_3 = 3^{3-1} = 3^2 = 9$
$u_4 = 3^{4-1} = 3^3 = 27$

(iii) $u_n = (-1)^n \times n$
$u_1 = (-1)^1 \times 1 = -1 \times 1 = -1$
$u_2 = (-1)^2 \times 2 = 1 \times 2 = 2$
$u_3 = (-1)^3 \times 3 = -1 \times 3 = -3$
$u_4 = (-1)^4 \times 4 = 1 \times 4 = 4$

Example 3.2: Deduce the n^{th}-term of the following sequences

(i) 1, 8, 27, 64, ... the n^{th}-term is given by $u_n = n^3$

(ii) 1, 3, 5, 7, 9, ... the n^{th}-term is given by $u_n = 2n - 1$

(iii) 2, 4, 6, 8, 10, ... the n^{th}-term is given by $u_n = 2n$

(iv) 7, 11, 15, 19, 23, ... the n^{th}-term is given by $u_n = 4n + 3$

(v) 9, 7, 5, 3, 1, ...

(vi) 3, −6, 12, −24, ...

3.1.1 Arithmetic Progression

Arithmetic progression is a sequence in which the difference between any successive terms is a fixed number called the common difference. If a_1, a_2, ..., a_n, ... represent successive terms of a sequence, then

Definition 3.1 *A sequence is said to be an arithmetic sequence if its successive terms may be define recursively as* $a_1 = a,\ a_n - a_{n-1} = d$ *such that*

$$a_n = a_{n-1} + d$$

where $a_1 = a$ and d are both real numbers. The number a is called the **first term**, and the number d is called the **common difference**. It follows that the successive terms of an arithmetic progression with first term a_1 and common difference d will follow the pattern

$$a_1,\ a_1 + d,\ a_2 + d,\ a_3 + d, ...$$

Example 3.3: The sequence 1, 3, 5, 7, ... is arithmetic since the difference between any two successive terms is the common number 2, the first term $a_1 = 1$ and the common difference is $d = 2$.

Example 3.4: Find the first term and the common difference of the arithmetic progression $2n + 1$. Hence write out the first five terms of the progression.

3.1.2 n^{th}-term of Arithmetic Progression

The n^{th}-term or general term of an arithmetic progression is given by the formula

$$u_n = a_1 + (n - 1)d,$$

where $a_1 = a$ is the first term, d is the common difference, $n =$ input numbers, and $u_n =$ is the last term.

Example 3.5:

(a) Find the 15^{th}-term of the arithmetic sequence −3, 2, 7, ...

Solution: The n^{th}-term of an arithmetic progression is given by

$$u_n = a_1 + (n - 1)d$$

where $a_1 = -3$, $d = 2 - (-3) = 5$, $n = 15$, $u_n =?$. Therefore

$$u_{15} = -3 + (15 - 1) \times 5 = 67$$

(b) Find the 100^{th}-term for each of the following sequences

(i) $5,\ 3\frac{1}{2},\ 2,\ \frac{1}{2},...$

(ii) $-5,\ -1,\ 3,\ 7,...$

(iii) $5,\ 8,\ 11,\ 14,...$

Solution:

$$5,\ 3\tfrac{1}{2},\ 2,\ \tfrac{1}{2},...$$

$$u_n = a_1 + (n-1)d,$$

$$\text{where } a_1 = 5,\ (d = \tfrac{7}{2} - 5 = -\tfrac{3}{2}),\ n = 100,\ u_n = ?$$

$$u_{100} = 5 + (100-1) \times \left(-\frac{3}{2}\right) = -143\frac{1}{2}.$$

(c) Find the number of terms n if $n+3$, $2n+6$, and 8 are three consecutive terms of an arithmetic progression (A.P).

The common difference d is given as

$$d = (2n+6) - (n+3) = n+3,$$

or

$$d = 8 - (2n+6) = 2 - 2n,$$

Therefore, since the sequence is an A.P, then the following equality is true

$$\begin{aligned} n + 3 &= 2n - 2, \\ n &= -\frac{1}{3}, \end{aligned}$$

and the common difference $d = \frac{8}{3}$.

(d) The 4^{th}-term of an A.P is 18, and the common difference is -5. Find the 1^{st}-term and the 9^{th}-term of the sequence.

Solution: Using the formula for the n^{th}-term of an A.P

$$u_n = a_1 + (n-1) \times d,$$

where $n = 4$, $u_4 = 18$, $a_1 = ?$, $d = -5$, therefore

$$\begin{aligned} 18 &= a_1 + (4-1) \times (-5), \\ 18 &= a_1 - 15, \\ a_1 &= 33. \end{aligned}$$

Therefore, the 9^{th}-term is given by the data $n = 9$, $a_1 = 33$, $d = -5$, $u_9 = ?$

$$\begin{aligned} u_9 &= 33 + (9-1) \times (-5), \\ u_9 &= -7 \end{aligned}$$

(e) If the 4^{th}-term of an A.P is 13 and the 10^{th}-term is 31, find the sequence.

Solution: This type of problem leads to two equations to be solved simultaneously. Thus, using the formula for the n^{th}-term of an A.P

$$u_n = a + (n-1)d.$$

The first equation is given by the data $u_4 = 13$, $n = 4$, $a =?$, $d =?$, therefore

$$13 = a + (4-1) \times d,$$

$$13 = a + 3d. \tag{1}$$

The second equation is given by the data $u_{10} = 31$, $n = 10$, $a =?$, $d =?$, similarly we have

$$31 = a + (10-1) \times d,$$

$$31 = a + 9d. \tag{2}$$

Solving Eq's. (1) and (2) simultaneously, we have

$$\begin{aligned} -6d &= -18, \\ d &= 3. \end{aligned}$$

Substituting $d = 3$ in Eq. (1), we have

$$\begin{aligned} 13 &= a + 9, \\ a &= 4. \end{aligned}$$

The sequence is given by a, $a + d$, $a + 2d$, $a + 3d, ...,$ therefore

$$4,\ 7,\ 10,\ 13, ...$$

3.2 Series

A series is formed by summing the terms of a sequence, e.g, if $1,\ 3,\ 5,\ 7, ...$ is a sequence, then $1+3+5+7+...$ is a series. The sum of the first n^{th}-term of a sequence is denoted by S_n, then the sum of the first 5 terms of a sequence will be indicated by S_5.

3.2.1 Sum of an Arithmetic Progression (A.P)

The general formula for the sum of the n^{th}-term of an A.P is given by

$$S_n = \frac{n}{2}\{2a + (n-1)d\}$$

where

- a= first term
- d= common difference
- n= number of terms
- S_n= sum of the n^{th}-term

or equivalently

$$S_n = \frac{n}{2}(a + l)$$

where l is the last term of the sequence.

Example 3.6:

(a) Find the following sum of an A.P

(i) $1 + 3 + 5 + ... + 101$

(ii) $-11 - 7 - 3 - ... + 50$

(iii) $x + 3x + 5x + ... + 21x$

Solution: Considering the fact that the last term is given in each case, we use the second formula. Therefore, for the series $1 + 3 + 5 + ... + 101$

$$S_n = \frac{n}{2}(a + l),$$

where $n =?$, $S_n =?$, $a = 1$, $l = 101$. The number of terms n must be determined in order to proceed, using the n^{th}-term formula

$$u_n = a + (n - 1)d$$

where $u_n = l = 101$, $a = 1$, $n =?$ and $d = 2$. Thus, substituting for n we have

$$\begin{aligned} 101 &= 1 + (n - 1) \times 2, \\ n &= 51. \end{aligned}$$

Using $n = 51$, find the sum of the A.P

$$\begin{aligned} S_n &= \frac{51}{2} \times (1 + 101), \\ S_n &= 2601. \end{aligned}$$

Similarly for the series $x + 3x + 5x + ... + 21x$, we first find the number of terms n using the formula

$$u_n = a + (n - 1)d$$

where $u_n = l = 21x$, $a = x$, $d = 2x$, $n =?$, substituting and solving for n we have

$$\begin{aligned} 21x &= x + (n - 1) \times 2x, \\ n &= 11. \end{aligned}$$

The sum of the A.P is given by

$$S_n = \frac{n}{2}(a + l),$$

where $u_n = l = 21x$, $a = x$, $n = 11$ and $d = 2x$, thus

$$\begin{aligned} S_{11} &= \frac{11}{2} \times (x + 21x), \\ S_{11} &= 121x. \end{aligned}$$

(b) Find the sum of first twenty terms of the following arithmetic progressions A.P

(i) $-4,\ -1,\ 2, ...$
(ii) $2,\ 3\frac{1}{2},\ 5, ...$
(iii) $-2,\ -5,\ -8,\ -11, ...$

Solution: Since no last term is given, then the most straight forward formula to use is the first formula

$$S_n = \frac{n}{2}\{2a + (n-1)d\}.$$

For the first A.P $\{-4,\ -1,\ 2, ...\}$, where $n = 20$, $a = -4$, $d = 3$, $S_{20} =?$

$$S_{20} = \frac{20}{2}\{2 \times (-4) + (20 - 1) \times 3\},$$
$$S_{20} = 490$$

(c) The first term of an A.P is -12, and the last term is 40. If the sum of the progression is 196, find the number of terms and the common difference.

Solution: Since the last term is given with no common difference, then the straight forward formula to use is the second formula

$$S_n = \frac{n}{2}(a + l),$$

where $a = -12$, $l = 40$, $S_n = 196$, $n =?$

$$196 = \frac{n}{2} \times (-12 + 40),$$
$$n = 14.$$

The common difference d can be found in two different ways that is by either using the formula for the n^{th}-term or by using the first formula for the sum of an A.P

$$u_n = a_1 + (n-1)d$$

where $a_1 = -12$, $n = 14$, $u_{14} = 40$ and $d =?$, thus

$$40 = -12 + (14 - 1)d,$$
$$d = 4,$$

or equivalently

$$S_n = \frac{n}{2}\{2a_1 + (n-1)d\}.$$

where $a_1 = -12$, $n = 14$, $S_{14} = 196$, $d =?$

$$196 = \frac{14}{2}\{2 \times (-12) + (14 - 1)d\},$$
$$d = 4.$$

(d) The second term of an A.P is 15, and the fifth term is 21. Find the common difference, the first term and the sum of the first ten terms.

Solution: Using the formula for the n^{th}-term of an A.P

$$u_n = a + (n-1)d,$$

where $a = ?$, $n = 2$, $d = ?$, $u_2 = 15$

$$15 = a + (2-1) \times d,$$

$$15 = a + d, \tag{3}$$

similarly

$$21 = a + (5-1) \times d,$$

$$21 = a + 4d. \tag{4}$$

Solving Eq's (3) and (4) simultaneously, we have

$$d = 2.$$

Substitute $d = 2$ into Eq. (3)

$$\begin{aligned} 15 &= a + 2, \\ a &= 13. \end{aligned}$$

The sum of the A.P is determined from the formula

$$S_n = \frac{n}{2}\{2a + (n-1)d\}.$$

where $a = 13$, $d = 2$, $n = 10$, $S_{10} = ?$

$$\begin{aligned} S_{10} = \frac{10}{2}\{2 \times 13 + (10-1)2\}&, \\ S_{10} = 220&. \end{aligned}$$

3.2.2 Arithmetic mean (AM)

If a, b, c are three consecutive terms of an A.P, then b is called the arithmetic mean of a and c. The common difference of the progression is given by

$$b - a \text{ or } c - b,$$

$$\begin{aligned} b - a &= c - b, \\ 2b &= a + c, \\ b &= \frac{a+c}{2}. \end{aligned}$$

In other word, b is the average of a and c.

Example 3.7:

(a) Find the arithmetic mean of 4 and 64

$$a.m = \frac{4+64}{2} = 34$$

(b) Insert four arithmetic mean between -12 and 13.

Solution: Let the arithmetic mean be $x,\ y,\ z,\ w$. Moreover, one will have $-12,\ x,\ y,\ z,\ w,\ 13$ where $a = -12$, $d = x + 12$, $n = 6$, $u_6 = 13$. Therefore

$$\begin{aligned} 13 &= -12 + (6-1) \times (x + 12), \\ x &= -7, \end{aligned}$$

using the value $x = -7$, other values will be obtained

$$\begin{aligned} d &= -7 + 12 = 5 \\ y &= -7 + 5 = -2 \\ z &= -2 + 5 = 3 \\ w &= 3 + 5 = 8 \end{aligned}$$

The A.P is $-12,\ -7,\ -2,\ 3,\ 8,\ 13$.

(c) If $x + 1$, $2x - 1$ and $x + 5$ is an A.P, find the value of x.

Solution: Using the formula for the n^{th}-term of an A.P

$$u_n = a + (n-1)d,$$

where $a = x + 1$, $d = (2x - 1) - (x + 1) = x - 2$, $n = 3$, $u_3 = x + 5$, thus

$$\begin{aligned} x + 5 &= x + 1 + (3-1) \times (x - 2), \\ x &= 4. \end{aligned}$$

3.2.3 Geometric Progression (G.P)

Geometric progression is a sequence in which the difference between the successive terms is a common ratio, or is a sequence in which the terms increase or decrease by a common ratio.

Definition 3.2 *A sequence is said to be geometric if its successive terms may be define recursively as*

$$a_1 = a, \qquad \frac{a_n}{a_{n-1}} = r,$$

such that

$$a_n = a_{n-1}r$$

where $a_1 = a$ and $r \neq 0$ are real numbers. The number a_1 is called the first term and the non-zero number r is called the common ratio. Then, it follows that successive terms of a geometric progression with first term a_1 and common ratio r will follow the pattern

$$a_1,\ a_1r,\ a_1r^2,\ a_1r^3, \ldots$$

Example 3.8: The following sequences are geometric since the difference between any two successive terms is the common ratio 2 and 3 respectively

(i) $1,\ 2,\ 4,\ 8,\ 16, \ldots$

(ii) $3,\ 9,\ 27,\ 81, \ldots$

Example 3.9: Show that the sequence $\{a_n\} = \{3^{-n}\}$ is geometric, hence find the first term and the common ratio.

3.2.4 n^{th}-term of Geometric Progression

The n^{th}-term or general term of a geometric progression is given by the formula

$$a_n = a_1 r^{n-1}, \qquad r \neq 0$$

where

$a_1 =$ first term

$r =$ common ratio

$n =$ number of terms

$a_n = n^{th}$-term or last term

Example 3.10:

(a) Given the geometric sequence 128, 64, 32, ..., find

(i) the $12^{th}-$term of the sequence

(ii) the formula for the n^{th}-term

Solution: The $12^{th}-$term of the sequence is $a_{12} = \frac{1}{16}$, and the formula for the n^{th}-term is $\{a_n\} = \{\frac{128}{2^{n-1}}\}$. Students to check and verify the results.

(b) The $5^{th}-$term of a geometric progression is $20\frac{1}{4}$, find the common ration r given that the $1^{th}-$term $a_1 = 4$.

Solution: The common ration is $r = \frac{3}{2}$. Students to check and verify the result.

(c) The $2^{nd}-$term of a G.P is 9 and the $4^{th}-$term is 81, find the $9^{th}-$term of the sequence.

Solution:

(i) Case 1: The $9^{th}-$term of the sequence is $a_9 = 19683$ if the common ratio $r = 3$.

(ii) Case 2: The $9^{th}-$term of the sequence is $a_9 = -19683$ if the common ratio $r = -3$. Students to check and verify results.

3.2.5 Sum of a Geometric Progression (G.P)

The general formula for the sum of the $n^{th}-$term of a geometric progression denoted as S_n is given by

$$S_n = \frac{a_1(1 - r^n)}{1 - r} \qquad r < 1$$

and

$$S_n = \frac{a_1(r^n - 1)}{r - 1} \qquad r > 1$$

Proof: The successive terms of a geometric progression G.P with first term a_1 and non-zero common ratio $r \neq 0$ follows the pattern

$$a_1,\ a_1 r,\ a_1 r^2,\ a_1 r^3, ...,\ a_1 r^{n-1}$$

then it follows that the sum of the sequence is given as

$$S_n = a_1 + a_1 r + a_1 r^2 + a_1 r^3 + ... + a_1 r^{n-1} = \sum_{k=1}^{n} a_1 r^{k-1} \tag{5}$$

multiplying Eq. (5) by r, we've

$$rS_n = a_1 r + a_1 r^2 + a_1 r^3 + a_1 r^4 + ... + a_1 r^{n-1}. \tag{6}$$

Subtracting Eq. (6) from Eq. (5), we've the expression

$$S_n - rS_n = a_1 - a_1 r^n$$

$$S_n(1 - r) = a_1(1 - r^n).$$

Therefore, making S_n the subject, we've

$$S_n = \frac{a_1(1 - r^n)}{1 - r} \qquad r \neq 0,\ 1$$

Example 3.11:

(a) Find the sum of the first ten terms of the sequence $\frac{1}{8}, \frac{1}{4}, \frac{1}{2}, ...$

Solution: The sequence is geometric with $a_1 = \frac{1}{8}$, $r = 2$, $n = 10$ and $S_{10} = ?$, then the appropriate formula to use is the second formula since $r > 1$

$$S_n = a_1 \frac{(r^n - 1)}{r - 1} \qquad r > 1$$

therefore

$$S_{10} = \frac{1}{8}\left[\frac{(2^{10} - 1)}{2 - 1}\right],$$

$$S_{10} = \frac{1023}{8}.$$

(b) The $3^{rd}-$term and the $6^{th}-$term of a geometric progression are 108 and -32 respectively. Find the sum of the first 7 terms.

Solution: The formula to find the $n^{th}-$term of a geometric progression is given by

$$a_n = a_1 r^{n-1}.$$

The solution procedure leads to two equations which are to be solved simultaneously, based on the formula for the $n^{th}-$term of a G.P, we've:

Part 1: Plugging the values $a_1 = ?$, $r = ?$, $n = 3$ and $a_n = 108$ into the formula for the $n^{th}-$term, we've

$$108 = a_1 r^2, \tag{7}$$

Part 2: Plugging the values $a_1 = ?$, $r = ?$, $n = 6$ and $a_n = -32$ into the formula for the $n^{th}-$term, we've

$$-32 = a_1 r^5. \tag{8}$$

Thus, dividing Eq. (8) by Eq. (7), we've

$$\frac{-32 = a_1 r^5}{108 = a_1 r^2},$$

$$r = -\frac{2}{3},$$

substitute the value of r in Eq. (7) to find a_1, thus

$$108 = a_1 \times \left(-\frac{2}{3}\right)^2,$$

therefore

$$a_1 = 243.$$

To find the sum of the first 7 terms, the appropriate formula to use is the first formula since the common ratio $r < 1$

$$S_n = \frac{a_1(1 - r^n)}{1 - r} \qquad r < 1,$$

plugging the values $a_1 = 243,\ r = -\frac{2}{3},\ n = 7$, we've

$$S_7 = 243 \times \left[\frac{(1 - (-\frac{2}{3})^7)}{1 + \frac{2}{3}}\right] = \frac{463}{3}$$

3.2.6 Sum of an Infinite Geometric progression

A geometric progression G.P with first term a_1 and common ratio r is said to be infinite if the sequence extends indefinitely

$$a_1,\ a_1 r,\ a_1 r^2,\ a_1 r^3, ...,\ a_1 r^{n-1}, ...$$

The sum of an infinite geometric progression G.P is given as

$$a_1 + a_1 r + a_1 r^2 + a_1 r^3 + ... + a_1 r^{n-1} + ... = \sum_{n=k=1}^{\infty} a_1 r^{k-1}$$

from the sum of the first n terms of a geometric progression

$$S_n = \frac{a_1(1 - r^n)}{1 - r} \qquad r \neq 0, 1$$

then as n turns to infinity ($n \to \infty$), we have

$$S_\infty = \frac{a_1(1 - r^\infty)}{1 - r} = \frac{a_1}{1 - r} - \frac{a_1 r^\infty}{1 - r}$$

The sum of an infinite geometric progression S_∞ converges to a value if $r < 1$, and diverges if $r > 1$.

Therefore, the sum of a convergent infinite geometric progression G.P is given by

$$S_\infty = \frac{a_1}{1 - r}.$$

Example 3.12: Verify if the following geometric progressions converge, hence find the sum.

(i) $1,\ \frac{1}{3},\ \frac{1}{9},\ \frac{1}{27},\ \frac{1}{81}, ...$

(ii) $1,\ 2,\ 4,\ 8,\ 16, ...$

(iii) $\frac{2}{3},\ \frac{4}{9},\ \frac{16}{81}, ...$

4 Binomial Theorem

An expressions for the expansion of $(a+b)^n$ for $n = 2, 3, 4$ are straight forward

$$(a+b)^2 = a^2 + 2ab + b^2$$

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

Features of binomial expansion $(a+b)^n$**:**

(i) The expansion is homogeneous in a and b, i.e., each expansion begins with a^n and ends with b^n

(ii) As you go from left to right, the powers of a decreases by 1 while the powers of b increases by 1

(iii) For each expansion, the number of terms equals $n+1$

Thus, consider the binomial expansion of $(a+b)^n$ for $n = 0, 1, 2, 3, 4$

$$\begin{aligned}
(a+b)^0 &= 1 \\
(a+b)^1 &= a+b \\
(a+b)^2 &= a^2 + 2ab + b^2 \\
(a+b)^3 &= a^3 + 3a^2b + 3ab^2 + b^3 \\
(a+b)^4 &= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4 \\
(a+b)^5 &= a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5
\end{aligned}$$

Notice that the coefficients in each expansion can be arrange in a triangular pattern below

$$\begin{array}{lccccccccccc}
(a+b)^0 = & & & & & & 1 & & & & & \\
(a+b)^1 = & & & & & 1 & & 1 & & & & \\
(a+b)^2 = & & & & 1 & & 2 & & 1 & & & \\
(a+b)^3 = & & & 1 & & 3 & & 3 & & 1 & & \\
(a+b)^4 = & & 1 & & 4 & & 6 & & 4 & & 1 & \\
(a+b)^5 = & 1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}$$

The above display of numbers is called **"the pascal triangle"**, named after a French mathematician Blaise Pascal (1623-1662).

Features of Pascal Triangle:

(i) The triangle is symmetrical about the center

(ii) Each row contains $(n+1)$ terms

(iii) The pascal triangle has $1'$s down the sides, and all the intermediate values in each row can be obtained by adding the two nearest entries in the row above it

Binomial expression $(a+b)^n$ is said to be expanded n times when it is written as a series of $n+1$ terms, and the series is called it's expansion.

Example 4.0:

1. Expand $(x+1)^4$ in descending powers of x using coefficients from the pascal triangle.

   Solution:

   (i) There will be 5 terms in the expansion

   (ii) The homogeneity in x and 1 is given by

   $$(x+1)^4 = x^4 \times 1^0 + x^3 \times 1^1 + x^2 \times 1^2 + x \times 1^3 + x^0 \times 1^4$$

   $$(x+1)^4 = x^4 + x^3 + x^2 + x + 1$$

   (iii) The coefficients from the pascal triangle are $1,\ 4,\ 6,\ 4,\ 1$. Therefore the expansion of $(x+1)^4$ in descending powers of x is

   $$(x+1)^4 = x^4 + 4x^3 + 6x^2 + 4x + 1$$

2. Expand $(2x+3y)^3$ in descending powers of x using coefficient from the pascal triangle.

   Solution:

3. Use the pascal triangle to expand the following:

   (i) $(x+y)^3$

   (ii) $(x+2y)^4$

   (iii) $(x-\frac{1}{x})^5$

   (iv) $(2x+\frac{1}{3})^3$

   (v) $(2x-\frac{1}{2})^4$

   Solution: Re-write question (iii) as

   $$\left(x-\frac{1}{x}\right)^5 = \left(x+\left(-\frac{1}{x}\right)\right)^5$$

   where $a = x,\ b = -\frac{1}{x}$, and there will be a total of six terms in the expansion

   $$\left(x+\left(-\frac{1}{x}\right)\right)^5 = x^5 + x^4\left(-\frac{1}{x}\right) + x^3\left(-\frac{1}{x}\right)^2 + x^2\left(-\frac{1}{x}\right)^3 + x\left(-\frac{1}{x}\right)^4 + \left(-\frac{1}{x}\right)^5$$

   $$= x^5 - x^3 + x - \frac{1}{x} + \frac{1}{x^3} - \frac{1}{x^5}$$

   The coefficients from the pascal triangle are: $1,\ 5,\ 10,\ 10,\ 5,\ 1$, thus

   $$\left(x-\frac{1}{x}\right)^5 = x^5 - 5x^3 + 10x - \frac{10}{x} + \frac{5}{x^3} - \frac{1}{x^5}$$

4. Use pascal triangle to obtain the value of $(1.002)^5$ correct to six places of decimal.

   Solution:

   $1.002 = 1 + 0.002$
   $a = 1$ and $b = 0.002$
   There will be a total of six terms
   The homogeneity of 1 and 0.002 is given by

   $$(1 + 0.002)^5 = 1^5 + 1^4(0.002) + 1^3(0.002)^2 + 1^2(0.002)^3 + 1(0.002)^4 + (0.002)^5$$

   The coefficients from the pascal triangle are: 1, 5, 10, 10, 5, 1, thus

   $$(1.002)^5 = 1 + 5(0.002) + 10(0.002)^2 + 10(0.002)^3 + 5(0.002)^4 + (0.002)^5$$

   $$(1.002)^5 = 1.010040 \text{ correct to 6 p.d}$$

Definition 4.1 *Binomial theorem is a formula for the expansion of $(a + b)^n$ for any positive integers n*

$$\begin{aligned}(a+b)^n = \begin{pmatrix} n \\ 0 \end{pmatrix} a^n + \begin{pmatrix} n \\ 1 \end{pmatrix} a^{n-1}b + \begin{pmatrix} n \\ 2 \end{pmatrix} a^{n-2}b^2 + ... + \begin{pmatrix} n \\ j \end{pmatrix} a^{n-j}b^j \\ +... + \begin{pmatrix} n \\ n \end{pmatrix} b^n\end{aligned} \tag{9}$$

where the symbols $\begin{pmatrix} n \\ j \end{pmatrix}$ for $j = 0, 1, 2, ..., n$ are numbers to be determined, and are given by

$$\begin{pmatrix} n \\ j \end{pmatrix} = \frac{n!}{j!(n-j)!}$$

The following special cases should be noted

$$\begin{pmatrix} n \\ 0 \end{pmatrix} = 1, \quad \begin{pmatrix} n \\ 1 \end{pmatrix} = n, \quad \begin{pmatrix} n \\ n-1 \end{pmatrix} = n, \quad \begin{pmatrix} n \\ n \end{pmatrix} = 1$$

In other word, the symbol $\begin{pmatrix} n \\ j \end{pmatrix}$ or equivalently C_j^n is the combination of n objects taking j at a time.

Factorial:

The factorial expression of an arbitrary natural number n denoted as $n!$ is given by

$$n! = n \times (n-1) \times (n-2) \times (n-3) \times ... \times 2 \times 1$$

Example 4.1:

$5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$

$9! = 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 302880$

$3! = 3 \times 2 \times 1 = 6$

$2! = 2 \times 1 = 2$

$1! = 1$

Note that $0! = 1$

Example 4.2: Evaluate each of the followings

(i) $6!$

(ii) $(7-2)!$

(iii) $\frac{8!}{3!}$

(iv) $\frac{3!}{0!}$

(v) $\frac{5!}{(5-2)!2!}$

(vi) $\begin{pmatrix} 7 \\ 5 \end{pmatrix} \times \begin{pmatrix} 9 \\ 6 \end{pmatrix}$

It follows that the binomial formula given in Eq. (9) can also be stated for any positive integer n as

$$(a+b)^n = a^n + \frac{n}{1!}a^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}b^3 + ... + b^n.$$

Moreover, for the case where n is negative or fractional, the series of terms goes to infinity

$$(a+b)^n = a^n + \frac{n}{1!}a^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}b^3 + ...$$

Features of Binomial Expansion:

(i) Each expansion of $(a+b)^n$ begins with a^n and ends with b^n as you move from left to right

(ii) All the terms in the expansion are n homogeneous, i.e., the sum of powers of a and b in each term equals n

(iii) The powers of a decreases by 1 while the powers of b increases by 1 as you move from left to right

(iv) The coefficient for each term is given by $\begin{pmatrix} n \\ j \end{pmatrix}$

(v) There will be a total of $n+1$ number of terms

Note: To find a particular coefficient in the expansion of $(a+b)^n$ without writing the entire terms, use the following expression

$$\boxed{\begin{pmatrix} n \\ j \end{pmatrix} a^{n-j}b^j} \text{ or } \boxed{\begin{pmatrix} n \\ n-j \end{pmatrix} a^j b^{n-j}} \text{ which ever one is more appropriate.}$$

Example 4.3:

1. Use binomial theorem to expand the following binomial expressions

   (i) $(x+y)^5$

   (ii) $(x+2y)^4$

   (iii) $(2x+\frac{1}{3})^3$

   (iv) $(x-\frac{1}{x})^5$

   Solutions:

   For $(x+y)^5$, then from binomial theorem $a = x$ and $b = y$

   $$(x+y)^5 = \begin{pmatrix} 5 \\ 0 \end{pmatrix} x^5y^0 + \begin{pmatrix} 5 \\ 1 \end{pmatrix} x^4y^1 + \begin{pmatrix} 5 \\ 2 \end{pmatrix} x^3y^2 + \begin{pmatrix} 5 \\ 3 \end{pmatrix} x^2y^3 + \begin{pmatrix} 5 \\ 4 \end{pmatrix} x^1y^4 + \begin{pmatrix} 5 \\ 5 \end{pmatrix} x^0y^5$$

   therefore, evaluating the numbers $\begin{pmatrix} n \\ j \end{pmatrix}$ in each term yields

   $$(x+y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5.$$

   For $(2x+\frac{1}{3})^3$, then from binomial theorem $a = 2x$ and $b = \frac{1}{3}$

   $$\left(2x+\frac{1}{3}\right)^3 = (2x)^3 + \begin{pmatrix} 3 \\ 1 \end{pmatrix} (2x)^2\left(\frac{1}{3}\right) + \begin{pmatrix} 3 \\ 2 \end{pmatrix} 2x\left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^3$$

   thus, we've

   $$\left(2x+\frac{1}{3}\right)^3 = 8x^3 + 4x^2 + \frac{2x}{3} + \frac{1}{27}.$$

2. Find the coefficient of x^{10} in the expansion of $(2x-3)^{14}$

   Solution: Using the expression

   $$\begin{pmatrix} n \\ n-j \end{pmatrix} a^j b^{n-j},$$

   Then the coefficient of x^{10} in the expansion of $(2x-3)^{14}$ will be contained in the term

   $$\begin{pmatrix} 14 \\ 4 \end{pmatrix} (2x)^{10}(-3)^4$$

such that

$$\begin{pmatrix} 14 \\ \\ 4 \end{pmatrix} 2x^{10}(-3)^4 = \frac{14!}{10!4!} 82944x^{10}$$

$$\begin{pmatrix} 14 \\ \\ 4 \end{pmatrix} 2x^{10}(-3)^4 = \frac{14 \times 13 \times 12 \times 11 \times 10!}{10! \times 4 \times 3 \times 2} 82944x^{10}$$

$$\begin{pmatrix} 14 \\ \\ 4 \end{pmatrix} 2x^{10}(-3)^4 = 83026944x^{10}$$

Therefore, the coefficient of x^{10} in the expansion of $(2x - 3)^{14}$ is 83026944

3. Write down and simplify the 4^{th} term in the expansion of $(2 - \frac{x}{2})^{12}$ in ascending powers of x.

   Solution:

4. Obtain the first four terms of $(1 + \frac{x}{2})^{10}$. Hence evaluate the first four terms of $(1.005)^{10}$ correct to 4 places of decimal.

   Solution: The first four terms are:

$$\left(1 + \frac{x}{2}\right)^{10} = 1^{10} + \begin{pmatrix} 10 \\ \\ 1 \end{pmatrix} 1^9 \frac{x}{2} + \begin{pmatrix} 10 \\ \\ 2 \end{pmatrix} 1^8 \left(\frac{x}{2}\right)^2 + \begin{pmatrix} 10 \\ \\ 3 \end{pmatrix} 1^7 \left(\frac{x}{2}\right)^3$$

$$= 1 + \frac{10!}{9!1!}\frac{x}{2} + \frac{10!}{8!2!}\left(\frac{x}{2}\right)^2 + \frac{10!}{7!3!}\left(\frac{x}{2}\right)^3$$

$$= 1 + 5x + \frac{45}{4}x^2 + 15x^3$$

   To evaluate first four terms of $(1.005)^{10}$, we let

$$(1.005)^{10} = (1 + 0.005)^{10}$$

   such that $\frac{x}{2} = 0.005$ which further implies that $x = 0.01$, thus

$$(1 + 0.005)^{10} = 1 + 5(0.01) + \frac{45}{5}(0.01)^2 + 15(0.01)^3$$

$$(1 + 0.005)^{10} = 1.0511$$

5. Obtain the expansion of the following as far as the term in x^3

   (i) $(1 + x - 2x^2)^8$

   (ii) $(3 - 2x + x^2)^7$

Solution: The polynomial expressions may be considered as binomial as

$$(1+x-2x^2)^8 = [1+(x-2x^2)]^8$$

which can be expressed in terms of the binomial theorem as

$$[1+(x-2x^2)]^8 = 1 + \begin{pmatrix} 8 \\ \\ 1 \end{pmatrix}(x-2x^2) + \begin{pmatrix} 8 \\ \\ 2 \end{pmatrix}(x-2x^2)^2 + ... + (x-2x^2)^8$$

$$= 1 + 8(x-2x^2) + \frac{8!}{6!2!}(x-2x^2)^2 + \frac{8!}{5!3!}(x-2x^2)^3$$

$$= 1 + 8(x-2x^2) + 28(x^2-2x^3+4x^4) + 168(13x^3 + \ \textit{other terms})$$

$$= 1 + 8x - 16x^2 + 28x^2 - 56x^3$$

Student should try $(3-2x+x^2)^7$ as an exercise.

6. Find the first four terms in the expansion of $(1+x)^{\frac{1}{2}}$ in ascending order of x.

   Solution: Since $n = \frac{1}{2}$ is fractional, we use the formula

$$(a+b)^n = a^n + \frac{n}{1!}a^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}b^3 + ...$$

   therefore

$$(1+x)^{\frac{1}{2}} = 1^{\frac{1}{2}} + \frac{\frac{1}{2}}{1!}1^{\frac{1}{2}-1}x + \frac{\frac{1}{2}(\frac{1}{2}-1)}{2!}1^{\frac{1}{2}-2}x^2 + \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{3!}1^{\frac{1}{2}-3}x^3 + ...$$

$$(1+x)^{\frac{1}{2}} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{3x^3}{48} + ...$$

7. Obtain the first five terms in the expansion of $\sqrt[3]{(1+2x)}$ in ascending powers of x.

   Solution:

8. Use binomial theorem to expand the following in ascending powers of x as far as the term x^3:

   (i) $\frac{1}{1-x}$

   (ii) $(2+x)^{-2}$

   Solution:

9. Find the value of n if the coefficients of x^3 and x^4 in the expansion of $(2+x)^n$ are in the ratio 4 : 11

   Solution:

The coefficient of x^3 in the expansion of $(2+x)^n$ is given by

$$\begin{pmatrix} n \\ \\ 3 \end{pmatrix} 2^{n-3}x^3$$

similarly the coefficient of x^4 in the expansion of $(2+x)^n$ is given by

$$\begin{pmatrix} n \\ \\ 4 \end{pmatrix} 2^{n-4}x^4$$

Therefore, the ratio $4:11$ is thus given by

$$\begin{pmatrix} n \\ \\ 3 \end{pmatrix} 2^{n-3}x^3 : \begin{pmatrix} n \\ \\ 4 \end{pmatrix} 2^{n-4}x^4 = 4:11$$

$$\frac{\begin{pmatrix} n \\ \\ 4 \end{pmatrix} 2^{n-4}x^4}{\begin{pmatrix} n \\ \\ 3 \end{pmatrix} 2^{n-3}x^3} = \frac{11}{4}$$

$$\frac{\frac{n!}{(n-4)!4!}2^{n-3}}{\frac{n!}{(n-3)!3!}2^{n-4}} = \frac{11}{4}$$

$$\frac{n!}{(n-4)!4!} \times \frac{(n-3)!3!}{n!}2^{-1} = \frac{11}{4}$$

$$\frac{(n-3)(n-4)!3!}{(n-4)! \times 4 \times 3!}2^{-1} = \frac{11}{4}$$

$\therefore$ $(n-3) \times 2^{-1} = 11$, and hence, $n = 25$

EXERCISES

1. Expand the following in descending powers of x
   (i) $(2x+3y)^4$
   (ii) $(2x+\frac{1}{3})^3$
2. Use pascal triangle to expand the following
   (i) $(a-2b)^3$
   (ii) $(\frac{x}{2}+\frac{2}{x})^4$
   (iii) $(x-\frac{1}{x})^6$
   (iv) $(a^2-b^2)^5$

3. Write down the expansion of $(2+x)^3$ in ascending powers of x. Use the expanded result to find the value of $(2.001)^3$ correct to five places of decimal.

4. Use pascal triangle to expand $(1+\frac{1}{4}x)^4$. Taking the first term of the expansion, find the value of $(1.025)^4$ correct to 3 decimal places.

5. Simplify, leaving surds in your answers where appropriate

   (i) $(1+\sqrt{2})^3+(1-\sqrt{2})^3$

   (ii) $(\sqrt{2}+\sqrt{3})^4+(\sqrt{2}-\sqrt{3})^4$

   (iii) $(\sqrt{6}+\sqrt{2})^3-(\sqrt{6}-\sqrt{2})^3$

6. Obtain the coefficient of x^8 in the expansion of $(2x+4y)^{17}$

5 Theory of Quadratic Equation

Quadratic equation is a polynomial function of degree 2 written as

$$p(x) = a_2x^2 + a_1x + a_0,$$

where the coefficients $a_2 \neq 0$, a_1, a_0 are real numbers called the quadratic coefficient, linear coefficient and constant respectively. The standard form of a quadratic equation is given as

$$ax^2 + bx + c = 0.$$

It follows that the coefficients $a \neq 0$, b and c are real numbers that determines the shape, location and position of a quadratic equation along the $xy-$plane. Moreover, the quadratic function opens upward **"concave up"** if the quadratic coefficient is greater than zero $a > 0$, and the quadratic function opens downward **"concave down"** if the quadratic coefficient is less than zero $a < 0$ as illustrated in the following figures.

Moreover, a quadratic equation without a linear coefficient is called a pure quadratic equation

$$ax^2 + c = 0$$

5.1 Roots of Quadratic Equation

There are four methods for finding roots (solutions) of quadratic equations

1. **Factorization method:** The method requires expressing the left hand side of quadratic equation as product of two linear algebraic expressions

   Case 1: For quadratic equations with quadratic coefficient $a \neq 1$, factorization has the following form

   $$ax^2 + bx + c \equiv (px + q)(rx + s) = 0$$

   where p, q, r and s are values to be determined by inspection that will make the two forms equivalent to one another

Case 2: For quadratic equations with quadratic coefficient $a = 1$, factorization has the following form

$$ax^2 + bx + c \equiv (x + q)(x + s) = 0$$

where q and s are values to be determined such that the sum $q + s = b$ and the product $q \times s = c$

Finally, if a quadratic equation $ax^2 + bx + c = 0$ is successfully written as product of two linear algebraic expressions as in the two cases above, the zero-product property of real numbers states that each linear expression equals zero. Therefore, solving the two linear equations provides the roots (solution) to the equation

Example 5.0: Find the roots of the following quadratic equations by method of factorization

(a) $3x^2 + 5x - 2 = 0$

(b) $x^2 + 5x + 6 = 0$

2. **Completing the Square Method:** The method requires expressing quadratic equations as a perfect square making use of the following algebraic identity

$$(x + h)^2 = x^2 + 2hx + h^2$$

Completing the square represent a well-defined algorithm that can be used to solve any quadratic equation.

Completing the square algorithm: For any standard quadratic equation

$$ax^2 + bx + c = 0, \qquad a \neq 0.$$

Completing the square algorithm requires the following steps:

(i) Make the quadratic coefficient equals 1 (i.e., $a = 1$) by diving through by a

(ii) Subtract the constant term $\frac{c}{a}$ from both sides of the equation

(iii) Add the square of one half of the coefficient of x to both sides of the equation, this complete the square converting the left-side into a perfect square

(iv) Write the left-side as a perfect square and simplify the right-side if necessary

(v) Produce two linear equations by taking the square root of both sides of the equation

(vi) Solve the two linear equations for x, this yields the roots of the quadratic equation

Example 5.1: Find the roots of the following quadratic equations by method of completing the square

(a) $3x^2 + 5x - 2 = 0$

(b) $x^2 + 5x + 6 = 0$

3. **Quadratic Formula:** A general formula for solving quadratic equations is derived using completing the square method and is given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

where the two roots are respectively given as

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

The expression under the square root sign is called the discriminant, and is usually denoted by a capital letter D, thus

$$D = b^2 - 4ac$$

5.2 Types of Roots of Quadratic Equation

The discriminant $[D = b^2 - 4ac]$ is an important factor that determines the nature of solution (roots) of a quadratic equation. It is therefore important to note the following:

(i) If $D > 0$, the equation will have two distinct real roots.

(ii) If $D = 0$, the equation will have two equal real roots.

(iii) If $D < 0$, the equation will have two distinct complex roots that are conjugate of one another.

Example 5.2: If equation $x^2 - 3x + 1 = p(x - 3)$ has equal roots, find the possible values of p.

Solution:

Re-writing the equation in standard form

$x^2 - x(3 + p) + (1 + 3p) = 0,$

where by the general formula $a = 1$, $b = -(3 + p)$ and $c = (1 + 3p)$ respectively.

For equal roots, $D = 0$ that is $b^2 - 4ac = 0$.

Hence $(3 + p)^2 - 4 \times 1 \times (1 + 3p) = 9 + 6p + p^2 - 4 + 12p = p^2 - 6p + 5 = 0$

Therefore $(p - 5)(p - 1) = 0$, where $p = 5$ or $p = 1$.

Example 5.3: If equation $x^2 - 3x + 1 = p(x - 3)$ has unequal roots, find an expression for the possible solution of p.

5.3 Sum and Product of Roots:

Many problems concerning the roots of quadratic equations can be solved without actually finding them. For example, we can find the sum and the product of the roots directly from the coefficients in the equation. It is usual to call the roots α and β, and if a quadratic equation $ax^2 + bx + c = 0$ has roots α and β, then

$$\begin{aligned} \text{Either } x = \alpha \text{ or } x = \beta &\implies x - \alpha = 0 \text{ or } x - \beta = 0 \\ &\implies (x-\alpha)(x-\beta) = 0 \end{aligned}$$

Expanding $(x-\alpha)(x-\beta) = x^2 - (\alpha+\beta)x + \alpha\beta = 0$

Now compare the two equations

$$x^2 + \left(\frac{b}{a}\right)x + \frac{c}{a} = x^2 - (\alpha+\beta)x + (\alpha\beta),$$

then the followings are true

$$(\alpha+\beta) = -\frac{b}{a}, \qquad (\alpha\beta) = \frac{c}{a}.$$

Hence, for any quadratic equation $ax^2 + bx + c = 0$ with roots α and β

$$\boxed{(\alpha+\beta) = -\frac{b}{a}, \qquad (\alpha\beta) = \frac{c}{a}}$$

The sum and product of the roots can also be derived directly (but not so neatly) from the formula for the roots. Let

$$\alpha = \frac{-b+\sqrt{D}}{2a}, \qquad \beta = \frac{-b-\sqrt{D}}{2a},$$

then

$$\alpha+\beta = \frac{-b+\sqrt{D}-b-\sqrt{D}}{2a} = -\frac{b}{a}$$

$$\alpha\beta = \left[\frac{-b+\sqrt{D}}{2a}\right]\left[\frac{-b-\sqrt{D}}{2a}\right] = \frac{c}{a}$$

Example 5.4:

1. If the roots of $3x^2 - 4x - 1 = 0$ are α and β, find $\alpha+\beta$ and $\alpha\beta$.

   Solution: Comparing the given equation with the standard form $ax^2 + bx + c = 0$, we have $a = 3,\ b = -4$ and $c = -1$. Therefore

   $$\alpha+\beta = -\left(-\frac{4}{3}\right) = \frac{4}{3},$$

   $$\alpha\beta = -\frac{1}{3}.$$

   As the equation $x^2 + (\frac{b}{a})x + \frac{c}{a} = 0$ is equivalent to $x^2 - (\alpha+\beta)x + \alpha\beta = 0$ where α and β are the roots. Therefore any quadratic equation can be written in the form:

   $$\boxed{x^2 - (SUM\ of\ the\ roots\ \alpha+\beta)x + (PRODUCT\ of\ the\ roots\ \alpha\beta) = 0}$$

   and this form is very useful.

2. Construct an equation with roots $\sqrt{2}+1$, $\sqrt{2}-1$.

   Solution:

   The sum of the roots $\alpha+\beta = (\sqrt{2}+1)+(\sqrt{2}-1) = 2\sqrt{2}$
   The product of the roots $\alpha\beta = (\sqrt{2}+1)(\sqrt{2}-1) = 1$
   Hence the equation is $x^2-(2\sqrt{2})x+1=0$

5.4 Symmetric Functions of the Roots:

Knowing the values of $\alpha+\beta$ and $\alpha\beta$ for a given quadratic equation, values of other functions of α and β can be calculated provided that they are symmetric.

Definition 5.1 (Symmetric Function) *A symmetric function of α and β is one in which if α and β are interchanged, the function is the same or is multiplied by -1.*

For example, $\alpha^2+\beta^2$ is symmetric function as it become $\beta^2+\alpha^2$, also $\alpha^2-\beta^2$ is symmetric since interchanging means multiplying by -1 that is $-(\beta^2-\alpha^2)$, but $3\alpha^2+\beta^2$ is not symmetric function. The values of a symmetric functions of α and β can be found without knowing the values of α and β.

Example 5.5:

1. If α, β are the roots of $2x^2-x-2=0$, find the values of

   (a) $\alpha^2+\beta^2$
   (b) $\alpha-\beta$
   (c) $\alpha^2-\beta^2$
   (d) $\frac{1}{\alpha}+\frac{1}{\beta}$
   (e) $\alpha^3+\beta^3$
   (f) $\alpha^3-\beta^3$

   Solutions: From the given equation, it is known that $\alpha+\beta=\frac{1}{2}$ and $\alpha\beta=-1$, therefore, each of the functions (a) to (f) must be expressed in terms of $\alpha+\beta$ and $\alpha\beta$ only.

   (a) $\alpha^2+\beta^2 = (\alpha+\beta)^2-2\alpha\beta = \frac{1}{4}+2 = \frac{9}{4}$
   (b) $\alpha-\beta$ cannot be found directly, but using $(\alpha-\beta)^2 = (\alpha+\beta)^2-4\alpha\beta = \frac{1}{4}+4 = \frac{17}{4}$.
   Hence $\alpha-\beta = \pm\frac{\sqrt{17}}{2}$ depending on whether $\alpha>\beta$ or $\alpha<\beta$.
   (c) $\alpha^2-\beta^2 = (\alpha+\beta)(\alpha-\beta) = \frac{1}{2}\times\frac{\sqrt{17}}{2} = \frac{\sqrt{17}}{4}$ (taking $\alpha>\beta$).
   (d) $\frac{1}{\alpha}+\frac{1}{\beta} = \frac{(\alpha+\beta)}{\alpha\beta} = \frac{\frac{1}{2}}{-1} = -\frac{1}{2}$.
   (e) $\alpha^3\pm\beta^3$, these can be factorized. Note the results for future use
   $\alpha^3+\beta^3 = (\alpha+\beta)(\alpha^2-\alpha\beta+\beta^2)$
   $\alpha^3-\beta^3 = (\alpha-\beta)(\alpha^2+\alpha\beta+\beta^2)$
   Hence $\alpha^3+\beta^3 = (\alpha+\beta)(\alpha^2-\alpha\beta+\beta^2) = (\alpha+\beta)[(\alpha+\beta)^2-3\alpha\beta]$, therefore
   $\alpha^3+\beta^3 = (\frac{1}{2})[(\frac{1}{2})^2+3] = \frac{13}{8}$.
   (f) $\alpha^3-\beta^3 = (\alpha-\beta)(\alpha^2+\alpha\beta+\beta^2) = (\alpha-\beta)[(\alpha+\beta)^2-\alpha\beta]$, therefore

$\alpha^3 - \beta^3 = \frac{\sqrt{17}}{2}[(\frac{1}{2})^2 + 1] = \frac{5\sqrt{17}}{8}$.

2. If α, β are roots of $3x^2 + 5x - 1 = 0$, construct equations whose roots are:

   (a) 5α, 5β

   (b) α^2, β^2

   (c) $\frac{1}{\alpha}$, $\frac{1}{\beta}$

   (d) $\frac{\alpha+1}{\beta}$, $\frac{\beta+1}{\alpha}$

Solutions: $\boxed{x^2 - (SUM\ of\ the\ roots\ \alpha+\beta)x + (PRODUCT\ of\ the\ roots\ \alpha\beta) = 0}$ is used to derive the required equation. In each case the **SUM** and **PRODUCT** of the new roots are expressed in terms of $\alpha+\beta$ and $\alpha\beta$. Therefore from the given quadratic equation $\alpha + \beta = -\frac{5}{3}$, $\alpha\beta = -\frac{1}{3}$.

(a) Sum of roots $= 5\alpha + 5\beta = 5(\alpha+\beta) = -\frac{25}{3}$.

Product of roots $= (5\alpha)(5\beta) = 25\alpha\beta = -\frac{25}{3}$.

Hence the required equation is given as

$$x^2 + \frac{25}{3}x - \frac{25}{3} = 0,$$

$$3x^2 + 25x - 25 = 0.$$

(b) Sum of roots $= \alpha^2 + \beta^2 = (\alpha+\beta)^2 - 2\alpha\beta = \frac{25}{9} + \frac{2}{3} = \frac{31}{9}$.

Product of roots $= (\alpha^2)(\beta^2) = \frac{1}{9}$.

Therefore, the required equation is given as

$$x^2 - \frac{31}{9}x + \frac{1}{9} = 0,$$

$$9x^2 - 31x + 1 = 0.$$

(c) Sum of roots $= \frac{1}{\alpha} + \frac{1}{\beta} = \frac{(\alpha+\beta)}{\alpha\beta} = \frac{-\frac{5}{3}}{-\frac{1}{3}} = 5$.

Product of roots $= (\frac{1}{\alpha})(\frac{1}{\beta}) = \frac{1}{\alpha\beta} = -3$.

The new equation is given as

$$x^2 - 5x - 3 = 0.$$

(d) Sum of roots $= \frac{\alpha+1}{\beta} + \frac{\beta+1}{\alpha} = \frac{(\alpha+\beta)^2 - \alpha\beta}{\alpha\beta} = \frac{(-\frac{5}{3})+\frac{1}{3}}{-\frac{1}{3}} = -\frac{28}{3}$.

Product of roots $= \left(\frac{\alpha+1}{\beta}\right)\left(\frac{\beta+1}{\alpha}\right) = \frac{(\alpha+1)(\beta+1)}{\alpha\beta} = \frac{\alpha\beta+(\alpha+\beta)+1}{\alpha\beta} = \frac{-\frac{1}{3}-\frac{5}{3}+1}{(-\frac{1}{3})(-\frac{5}{3})} = -\frac{9}{5}$.

The required equation is given as

$$x^2 - \frac{28}{3}x - \frac{9}{5} = 0,$$

$$x^2 - 140x - 27 = 0.$$

In each of the above, the new roots were symmetric functions of α and β. In other cases, it would be necessary to know the actual values of α and β. Moreover, if there is a given relationship between the roots, this can be used to determine an unknown coefficient in the equation. Consider the next example:

3. One root of the equation $27x^2 + bx + 8 = 0$ is known to be the square of the other, find b.

   Solution:

EXERCISES

1. If α and β are the roots of the following equations, find the values of $\alpha+\beta$, $\alpha\beta$ and $\alpha^2+\beta^2$ in each case:
   (i) $x^2 - x + 1 = 0$
   (ii) $3x^2 = x + 1$
   (iii) $px^2 = q$
   (iv) $x^2 + \sqrt{3}x + 1 = 0$
   (v) $2y^2 - (a+3)y + a^2 = 0$
   (vi) $pt^2 - qt - r = 0$
2. Construct and simplify equations whose roots are:
   (i) -3, 1
   (ii) $\frac{1}{2}$, 2
   (iii) $\sqrt{3} - 2$, $\sqrt{3} + 2$
   (iv) $\frac{1}{(1+\sqrt{2})}$, $\frac{1}{(1-\sqrt{2})}$
   (v) $\sin^2\theta$, $\cos^2\theta$
3. A student is asked to solve the equation $3x^2 - x + 1 = 0$. He knows that if the roots equals α and β, then $\alpha + \beta = \frac{1}{3}$ and $\alpha\beta = \frac{1}{3}$. He then attempts to solve these two equations simultaneously, what does he find?
4. It is known that $p + q = \frac{1}{6}$ and $pq = -\frac{1}{3}$. By forming a quadratic equation whose roots are p, q and solving it, find the value of p and q.
5. If α, β are roots of $2x^2 - 2x - 1 = 0$, form the equation whose roots are $\frac{\alpha}{\beta}$ and $\frac{\beta}{\alpha}$.
6. One root of the equation $2x^2 - x + c = 0$ is double the other root, find c.
7. If one root of the equation $ax^2 + bx + c = 0$ is double the other root, show that:
   (a) The roots are $-\frac{b}{3a}$ and $-\frac{2b}{3a}$
   (b) $2b^2 = 9ac$
8. If the roots of $ax^2 + bx + c = 0$ differ by 1, show that they are $\frac{(a-b)}{2a}$ and $-\frac{(a+b)}{2a}$. Hence prove that $b^2 = a(a + 4c)$.
9. One root of the equation $2x^2 + bx + c = 0$ is three times the other root. Prove that $3b^2 = 32c$.
10. Find the value of p for which the equation $(x-2)(x-3) = p$ has roots which differ by 2.

6 Angles and Their Measurement

6.1 Angles

A half-line is that portion of a line that starts at a point P on the line and extends indefinitely in a given direction, and the starting point P is called the vertex of the half-line. If two half-lines are drawn with common vertex, they form an angle. We call one half-line of the angle the **initial side** and the other **terminal side**

The angle formed is denoted by indicating the direction and amount of rotation from the **initial side** to the **terminal side**. Moreover, if the rotation is in the anti-clockwise direction, the angle is positive, and if the rotation is clockwise the angle is negative as indicated in the figures above. Conventionally, lowercase Greek letters such as α (alpha), β (beta), γ (gamma), and θ (theta) are often used to denote angles.

6.2 Angular Measurement

Angles are measured by determining the amount of rotation needed for the initial side to become coincident with the terminal side. The two prominent methods for measuring angles are degrees and radians.

6.2.1 Degrees

An angle formed by rotating the initial side exactly once in the anti-clockwise direction until it coincides with itself is said to measure 360 degrees (1 revolution), abbreviated 360^0. Therefore, **one degree** 1^0 is $\frac{1}{360}$ revolution.

Special Angles:

(1) A right angle is an angle that measures 90^0, or $\frac{1}{4}$ revolution.

(2) Straight angle is an angle that measures 180^0, or $\frac{1}{2}$ revolution.

(3) Any angle less than 90^0 is called **acute angle**.

(4) Any angle greater than 90^0 but less 180^0 is called **obtuse angle**.

(5) Any angle greater than 180^0 is called **reflex angle**.

Example 6.0: Roughly sketch the following angles: 60^0, -45^0, 225^0, 405^0.

Conversion: Although Subdivisions of a degree may be obtained by using decimals, subdivisions of a degree may be obtained using the notion of minutes and seconds. Moreover, conversion between degrees to decimal, degrees to minutes and seconds is straight forward. Consider the following:

(i) 1 anti-clockwise revolution 360^0

(ii) $1^0 = 60'$ and $1' = 60''$

(iii) **One minute**, denoted by $1'$ is defined as $\frac{1}{60}$ degrees, written as

$$1' = \left(\frac{1}{60}\right)^0$$

(iv) **One second**, denoted by $1''$ is defined as $\frac{1}{60}$ minutes, or equivalently $\frac{1}{3600}$ degrees

$$1'' = \left(\frac{1}{60}\right)' = \left(\frac{1}{60 \times 60}\right)^0$$

Note: An angle of 53 degrees $29'$ and $7''$ is written as $53^0 29' 7''$.

Example 6.1:

(a) Convert $253^0 18'$ to decimal degree form.

Solution:

$$\begin{aligned} 253^0 18' &= 253^0 + 18 \times 1' \\ &= 253^0 + 18 \times \left(\tfrac{1}{60}\right)^0 \\ &= 253^0 + 0.3^0 \\ &= 253.3^0 \end{aligned}$$

(b) Convert $53^0 29' 7''$ to decimal degree form leaving your answer in 3 decimal places.

Solution:

$53^0 29' 7'' = 53^0 + 29 \times 1' + 7 \times 1''$

where $1' = \left(\frac{1}{60}\right)^0$ and $1'' = \left(\frac{1}{60 \times 60}\right)^0$, therefore we have

$$\begin{aligned} 53^0 29' 7'' &= 53^0 + 29 \times \left(\tfrac{1}{60}\right)^0 + 7 \times \left(\tfrac{1}{3600}\right)^0 \\ &= 53^0 + 0.483^0 + 0.00194^0 \\ &= 53.485^0 \text{ to 3 decimal places} \end{aligned}$$

(c) Convert 45.605^0 to degree, minute, second ($D^0 M' S''$) format.

Solution:

$$\begin{aligned} 45.605^0 &= 45^0 + 0.605^0 \\ &= 45^0 + 0.605 \times 1^0 \end{aligned}$$

where $1^0 = 60'$, therefore

$$45.605^0 = 45^0 + 0.605 \times 60'$$
$$= 45^0 + 36.3'$$
$$= 45^0 + 36' + 0.3 \times 1'$$

where $1' = 60''$, therefore we have

$$= 45^0 + 36' + 0.3 \times 60''$$
$$= 45^0 + 36' + 18''$$
$$45.605^0 = 45^0 36' 18''$$

(d) Convert 21.256^0 to degree, minute, second ($D^0 M' S''$) format.

Solution:

6.2.2 Radians

Radian is an alternative angular measurement used in advanced mathematics, which is based on measuring an angle with its radius. A central angle is a positive angle whose vertex is at the center of a circle and whose initial side coincides with the positive $x-$axis, such that the initial side and the terminal side subtends an arc on the circle. If the radius of the circle is r and the length of the arc subtended by the central angle say θ is also r, then angle θ is said to measure 1 **radian**.

Definition 6.1 *1 radian is the angle subtended at the center of a circle by an arc equal in length to the radius of the circle.*

Then it follows that for a circle of radius r, a central angle of θ radians subtends an arc whose length s is given by

$$s = r\theta. \tag{10}$$

Conversion: It is imperative to be able to convert between the two ways of measuring angles. It follows from Eq. (10) that the circumference of a circle of radius r is equal to $2\pi r$, therefore substituting this amount into Eq. (10), hence

$$2\pi r = r\theta,$$

solving for θ, we have

$$\theta = 2\pi \text{ radians}$$

It follows that 1 revolution $= 2\pi$ radians, and since 1 revolution $= 360^0$ then we have

$360^0 = 2\pi$ radians

dividing both sides of the above equation by 2 we have

$\boxed{180^0 = \pi \ radians}$

dividing both sides of the above equation firstly by 180^0, and secondly by π yields the two conversion formulas as follows

$$\boxed{1^0 = \frac{\pi}{180} \ radians, \qquad 1 \ radian = \frac{180^0}{\pi}, \qquad where \ \pi = 3.1416}$$

Examples:

1. Express the following angles in radians.
   (a) 30^0
   (b) $-143^0 10'$
   (c) 107^0
   Solutions:
   (a) $30^0 = 30 \times 1^0 = 30 \times \frac{\pi}{180} = \frac{\pi}{6}$ radians
   (b) $-143^0 10' = -142.8^0 = -142.8 \times \frac{\pi}{180}$
   (c) $107^0 = 107 \times \frac{\pi}{180}$

2. Express the following angles in degrees and minutes correct to the nearest minute.
   (a) 2.1 radians
   (b) $\frac{\pi}{12}$ radians
   Solutions:
   (a) 2.1 radians $= 2.1 \times \frac{180^0}{\pi} = 2.1 \times \frac{180^0}{3.1416}$
   (b) $\frac{\pi}{12}$ radians $= \frac{\pi}{12} \times \frac{180^0}{\pi}$

7 TRIGONOMETRIC FUNCTION OF ANGLES OF ANY MAGNITUDE

7.1 General Angle:

Consider a wheel which is force to rotate about a fixed axis, and suppose that one of the spoke is marked with a thin line of paint. If the wheel starts from rest and makes one revolution, then the marked spoke turns through 360^o and if the wheel makes another one revolution, the marked spoke turned through 360^o again. Therefore one can say that the wheel has turned through 720^o thereby using angles greater that 360^o, thus, the number of revolutions may be specified as well as the position of the marked spoke

Now consider the $x-$axis of the $xy-$plane, the positive direction is usually taken to the right and the negative direction is taken to the left. Similarly, if a wheel is rotating in an anti-clockwise direction, then the rotation is sensed to be positive and the clockwise rotation is thus considered negative

7.2 Angles:

As pointed earlier, angles are measured from the positive $x-$axis. Angles measured from the $x-$axis in an anti-clockwise direction are positive, while angles measured in a clockwise direction are negative

Trigonometric ratios of angles of any magnitude are required in connection with oscillatory bodies and rotation about an axis

7.3 Quadrants:

The axes in an $xy-$plane divides the plane into four quadrants, i.e., one complete rotation is divided into four quadrants. The quadrants are used for conveniently locating and measuring angles rotated in an anti-clockwise direction from the positive $x-$axis

Consider the following figures with a point (x, y) and its coordinates will be given a suffix corresponding to the quadrant it lies in

1. An acute angle in the first quadrant $(0 < \theta < 90^o)$ or $(0 < \theta < \frac{\pi}{2})$

2. An obtuse angle in the second quadrant $(90^o < \theta < 180^o)$ or $(\frac{\pi}{2} < \theta < \pi)$

3. A reflex angle in the third quadrant $(180^o < \theta < 270^o)$ or $(\pi < \theta < \frac{3\pi}{2})$

4. A reflex angle in the fourth quadrant $(270^o < \theta < 360^o)$ or $(\frac{3\pi}{2} < \theta < 2\pi)$

7.4 Trigonometric Ratios of Angles

7.4.1 Acute Angle in the First Quadrant

Consider an acute angle in the first quadrant

Given the right-angle triangle AOB with angle θ_1 at vertex zero where side AB is opposite θ_1, side OB is adjacent to θ_1 and side OA is called the hypotenuse. The main trigonometric ratios of acute angle are determined using the slogan "$SOHCAHTOA$" as follows

(i) Sine of angle $\theta_1 = \sin\theta_1 = \dfrac{\text{opposite}}{\text{hypotenuse}} = \frac{+y}{r}$

(ii) Cosine of angle $\theta_1 = \cos\theta_1 = \dfrac{\text{adjacent}}{\text{hypotenuse}} = \frac{+x}{r}$

(iii) Tangent of angle $\theta_1 = \tan\theta_1 = \dfrac{\text{opposite}}{\text{adjacent}} = \frac{+y}{+x}$

Note: The hypotenuse r is the length of the longest side OA, and is taken to be strictly positive $r > 0$

7.4.2 Acute Angle in the Second Quadrant

Consider an acute angle θ_2 in the second quadrant, the main trigonometric ratios are determined accordingly

7.4.3 Acute Angle in the Third Quadrant

Consider an acute angle θ_3 in the third quadrant, the main trigonometric ratios are determined accordingly

7.4.4 Acute Angle in the Fourth Quadrant

Consider an acute angle θ_4 in the fourth quadrant, the main trigonometric ratios are determined accordingly

The results can be summarized by writing which ratios are positive in each quadrant

or equivalently in a tabular format below

Ratios	$1^{st}Quad$	$2^{nd}Quad$	$3^{rd}Quad$	$4^{th}Quad$
$Sine$	$+ve$	$+ve$	$-ve$	$-ve$
$Cosine$	$+ve$	$-ve$	$-ve$	$+ve$
$Tangent$	$+ve$	$-ve$	$+ve$	$-ve$

7.4.5 Other Trigonometric Ratios (Reciprocal Ratios)

In addition to the three main trigonometric ratios, there three reciprocal ratios. The relationship between the three main trigonometric ratios (sin, cos, tan) and the reciprocal ratios is that the reciprocal ratios are reciprocals of the main trigonometric ratios respectively

$$\sec\theta = \frac{1}{\cos\theta}$$

$$cosec\theta = \frac{1}{\sin\theta}$$

$$\cot\theta = \frac{1}{\tan\theta}$$

Suppose $\sin y = x$, this means that the argument y is the number of radians 0r degrees in the angle whose sin is x. This can be written as $y = \sin^{-1} x$, and y is called the inverse of x which implies that the statements $\sin y = x$ and $y = \sin^{-1} x$ are equivalent, and by extension this applies to the cosine and tangent functions respectively. In advanced trigonometry, it is imperative to have a standard abbreviation for the phrase "the angle whose sin is x" etc, the arcsin, arccos, arctan or $\sin^{-1}$, $\cos^{-1}$, $\tan^{-1}$ are used

7.5 Reference Angle

The trigonometric ratios of any angle can be deduced from the ratios of an acute angle. The acute angle is the reference angle and it is the angle the given angle makes with the $x-$axis. Let ϕ denote a reference angle, then

(i) In the 1^{st} quadrant, the reference angle ϕ to angle θ are the same

$$\sin\theta = \sin\phi$$

$$\cos\theta = \cos\phi$$

$$\tan\theta = \tan\phi$$

(ii) In the 2^{nd} quadrant, the reference angle ϕ to angle θ is given

$$\sin\theta = \pm\sin(180^o - \theta) = \pm\sin\phi$$
$$\cos\theta = \pm\cos(180^o - \theta) = \pm\cos\phi$$
$$\tan\theta = \pm\tan(180^o - \theta) = \pm\tan\phi$$

(iii) In the 3^{nd} quadrant, the reference angle ϕ to angle θ is given

$$\sin\theta = \pm\sin(\theta - 180^o) = \pm\sin\phi$$
$$\cos\theta = \pm\cos(\theta - 180^o) = \pm\cos\phi$$
$$\tan\theta = \pm\tan(\theta - 180^o) = \pm\tan\phi$$

(iv) In the 4^{th} quadrant, the reference angle ϕ to angle θ is given

$$\sin\theta = \pm\sin(360^o - \theta) = \pm\sin\phi$$
$$\cos\theta = \pm\cos(360^o - \theta) = \pm\cos\phi$$
$$\tan\theta = \pm\tan(360^o - \theta) = \pm\tan\phi$$

Example: Express the following angles in terms of the trigonometric ratios of an acute angle

(i) $\sin 170^o = \sin(180^o - 170^o) = \sin 10^o$

(ii) $\tan 300^o = -\tan(360^o - 300^o) = -\tan 60^o$

(iii) $\cos 293^o = \cos(360^o - 293^o) = \cos 67^o$

(iv) $\sec 142^o = \frac{1}{\cos 142^o} = \frac{1}{-\cos(180^o - 142^o)} = \frac{1}{-\cos 38^o} = -\sec 38^o$

(v) $cosec\, 230^o = \frac{1}{\sin 230^o} = \frac{1}{-\sin(230^o - 180^o)} = \frac{1}{-\sin 50^o} = -cosec\ 50^o$

(vi) $\cot 156^o = \frac{1}{\tan 156^o} = \frac{1}{-\tan(180^o - 156^o)} = \frac{1}{-\tan 24^o} = -\cot 24^o$

7.6 Negative Angles

The rotation in the clockwise direction forms negative $-ve$ angles. In finding trigonometric ratios for negative angles $(-\theta)$ of any magnitude, 360^o or its multiple are added until the first positive angle $(0 < \theta < 360^o)$ is obtained. Hence, for angles $\theta > 90^o$, the concept of reference angle is then used to find the six trigonometric ratios in terms of acute angles

Example: Express the following angles in terms of trigonometric ratios of acute angle

(i) $\sin(-50^o) = \sin(-50^o + 360^o)$

$$= \sin 310^o$$

$$= -\sin(360^o - 310^o)$$

$$= -\sin 50^o$$

(ii) $\cos(-20^o)$

(iii) $\tan(-420^o)$

(iv) $\sec(-172^o) = \sec(-172^o + 360^o)$

$$= \sec 188^o$$

$$= \frac{1}{\cos 188^o}$$

$$= \frac{1}{-\cos(188^o - 180^o)}$$

$$= \frac{1}{-\cos 8^o}$$

$$= -\sec 8^o$$

(v) $cosec\ (-510^o)$

(vi) $\cot(-780^o)$

7.7 Trigonometric Ratios of Special Angles (0^o, 30^o, 45^o, 60^o, 90^o)

Consider the right-angle triangle below

the six trigonometric ratios of the acute angle θ are given as

$$\sin\theta = \frac{y}{r},\ \cos\theta = \frac{x}{r},\ \tan\theta = \frac{y}{x},\ \sec\theta = \frac{r}{x},\ cosec\ \theta = \frac{r}{y},\ \cot\theta = \frac{x}{y}$$

7.7.1 Trigonometric Ratios of Angles 0^o and 90^o

The trigonometric ratios of angles 0^o and 90^o can be obtained from the figure below

For angle $\theta = 0^o$, it implies that $y = 0$ and $x = r = 1$

$\sin 0^o = \frac{y}{r} = \frac{0}{1} = 0$

$\cos 0^o = \frac{x}{r} = \frac{1}{1} = 1$

$\tan 0^o = \frac{y}{x} = \frac{0}{1} = 0$

For angle $\theta = 90^o$, it implies that $x = 0$ and $y = r = 1$

$\sin 90^o = \frac{y}{r} = \frac{1}{1} = 1$

$\cos 90^o = \frac{x}{r} = \frac{0}{1} = 0$

$\tan 90^o = \frac{y}{x} = \frac{1}{0} = \infty$

7.7.2 Trigonometric Ratios of Angles 30^o and 60^o

Consider an equilateral triangle with all sides equal to 2 below

taking the right-angle triangle ACD

Using Pythagoras theorem, the side AC of the triangle is computed below

$$|AD|^2 = |AC|^2 + |CD|^2$$

$$2^2 = |AC|^2 + 1^2$$

$$\therefore |AC| = \sqrt{3}$$

Consider angle $\theta = 30^o$, then side CD is opposite while side AC is adjacent to angle θ respectively, thus the exact values of the trigonometric ratios are

$$\sin 30^o = \tfrac{1}{2}$$

$$\cos 30^o = \tfrac{\sqrt{3}}{2}$$

$$\tan 30^o = \tfrac{1}{\sqrt{3}}$$

Consider angle $\theta = 60^o$, then side AC is opposite while side CD is adjacent to angle θ respectively, thus the exact values of the trigonometric ratios are

$$\sin 60^o = \tfrac{\sqrt{3}}{2}$$

$$\cos 60^o = \tfrac{1}{2}$$

$$\tan 60^o = \sqrt{3}$$

7.7.3 Trigonometric Ratios of Angle 45^o

Consider the right-angle isosceles triangle with two equal sides, and let the longest side denote the hypotenuse as given below

Using Pythagoras theorem, the side AC of the triangle is computed below

$$|AB|^2 = |AC|^2 + |BC|^2$$

$$|AB|^2 = 1^2 + 1^2$$

$$\therefore |AB| = \sqrt{2}$$

therefore the exact values for the trigonometric ratios of angle $\theta = 45^o$ are

$$\sin 45^o = \tfrac{1}{\sqrt{2}}$$

$$\cos 45^o = \tfrac{1}{\sqrt{2}}$$

$$\tan 45^o = 1$$

EXERCISES

1. Express the following angles in radians

   (i) $\theta = 65^o$

   (ii) $\theta = 120^o$

   (iii) $\theta = 225^o$

   (iv) $\theta = -150^o$

2. Express the following angles in degrees

   (i) $\theta = 45\text{rad}$

   (ii) $\theta = \frac{2\pi}{3}$

   (iii) $\theta = \frac{3\pi}{5}$

3. Express the following angles in terms of the trigonometric ratios of acute angle

   (i) $\cos 293^o$

   (ii) $\sin 250^o$

   (iii) $\tan \frac{3\pi}{2}$

   (iv) $\sin(-230^o)$

   (v) $cosec\,(-53^o)$

   (vi) $\tan 143^o$

   (vii) $\sec(-172^o)$

4. Write down the exact values of the following angles leaving surds in your answers

   (i) $\sin 405^o$

   (ii) $\sin(-270^o)$

   (iii) $\tan 120^o$

   (iv) $\cos(-120^o)$

   (v) $\cos 225^o$

   (vi) $\tan 150^o$

5. (a) If $y = b\cos\theta$, simplify

   (i) $b^2 + y^2$

   (ii) $y\sqrt{b^2 + y^2}$

   (b) If $z = a\sec\theta$, simplify

   (i) $\frac{1}{\sqrt{z^2-a^2}}$

   (ii) $\frac{\sqrt{z^2-a^2}}{z}$

   (c) If $c = \cos\theta$, simplify

   (i) $\frac{\sqrt{1-c^2}}{c}$

   (ii) $\frac{c}{1-c^2}$

   (d) If $c = cosec\,\theta$, simplify

   (i) $\sqrt{c^2 - 1}$

(ii) $\frac{\sqrt{c^2-1}}{c}$

6. Eliminate θ from the following equations

   (i) $x = 1 - \sin\theta, \qquad y = 1 + \cos\theta$

   (ii) $x = 1 + \tan\theta, \qquad y = \cos\theta$

   (iii) $x = a\sec\theta, \qquad y = b + c\cos\theta$

   (iv) $x = a\sec\theta, \qquad y = b\cot\theta$

   (v) $x = \sin\theta + \cos\theta, \qquad y = \tan\theta$

7. Prove the following identities

   (i) $(\sec\theta + \tan\theta)(\sec\theta - \tan\theta) = 1$

   (ii) $\cos^4\theta - \sin^4\theta = \cos^2\theta - \sin^2\theta$

   (iii) $\sqrt{\sec^2\theta - \tan^2\theta} + \sqrt{cosec^2\theta - \cot^2\theta}$

   (iv) $\frac{1-\cos^2\theta}{\sec^2\theta} = 1 - \sin^2\theta$

   (v) $\frac{2\tan\theta}{1+\tan^2\theta} = 2\sin\theta\cos\theta$

7.8 Compound Angles

7.8.1 Addition Formulas

If A and B are any two angles, the expressions $\sin(A+B)$, $\cos(A+B)$ and $\tan(A+B)$ can be obtained where

$$\sin(A+B) \neq \sin A + \sin B$$

$$\cos(A+B) \neq \cos A + \cos B$$

$$\tan(A+B) \neq \tan A + \tan B$$

To obtain the correct addition formulas, consider the figure below

Now let $R\hat{P}Q = O\hat{P}Q - O\hat{P}M$, then in terms of complimentary angles we have

$$R\hat{P}Q = (90^o - B) - (90^o - (A+B)) = A$$

$\therefore\ R\hat{P}Q = A$

$$\sin(A+B) = \frac{PM}{OP} = \frac{MR+RP}{OP} = \frac{MR}{OP} + \frac{RP}{OP} = \frac{QN}{OP} + \frac{RP}{OP}$$

because $\frac{MR}{OP}$ is not a trigonometric ratio, then using a logical 1 technique we have

$$\sin(A+B) = \frac{QN}{OQ} \times \frac{OQ}{OP} + \frac{RP}{PQ} \times \frac{PQ}{OP}$$

Therefore by interpreting the above equation in terms of trigonometric ratios of acute angle, we have the following

1. $\sin(A+B) = \sin A \cos B + \cos A \sin B$
2. $\sin(A-B) = \sin A \cos B - \cos A \sin B$

   Note: $\cos(-\theta) = \cos\theta$ implying that cosine is an even function

   $\sin(-\theta) = -\sin\theta$ implying that sine is an odd function

   $\tan(-\theta) = -\tan\theta$ implying that tangent is an odd function

   Similarly, for the cosine function we have

   $$\cos(A+B) = \frac{OM}{OP} = \frac{ON-MN}{OP} = \frac{ON}{OP} - \frac{MN}{OP} = \frac{ON}{OP} - \frac{RQ}{OP}$$

   again using the logical 1 technique we have

   $$\cos(A+B) = \frac{ON}{OQ} \times \frac{OQ}{OP} - \frac{RQ}{PQ} \times \frac{PQ}{OP}$$

   interpreting the above equation in terms of trigonometric ratio of acute angle

3. $\cos(A+B) = \cos A \cos B - \sin A \sin B$
4. $\cos(A-B) = \cos A \cos B + \sin A \sin B$ for the tangent function of angle θ, the addition formula is derived from the following identity

   $$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

   therefore

   $$\tan(A+B) = \frac{\sin(A+B)}{\cos(A+B)} = \frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B}$$

   dividing both the numerator and denominator by the factor $(\cos A \cos B)$, we have

5. $\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$
6. $\tan(A-B) = \frac{\tan A + \tan(-B)}{1 - \tan \tan(-B)} = \frac{\tan A - \tan B}{1 + \tan A \tan B}$

Example: Evaluate the following angles without using table or calculator

(i) $\sin(30^o + 45^o)$

Using addition formula $\sin(A+B) = \sin A \cos B + \sin B \cos A$

$$\sin(30^o + 45^o) = \sin 30^o \cos 45^o + \sin 45^o \cos 30^o$$

$$= \frac{1}{2} \times \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2}$$

$$= \frac{\sqrt{2}+\sqrt{6}}{4}$$

(ii) $\sin(120^o + 45^o) = \sin 120^o \cos 45^o + \sin 45^o \cos 120^o$

$$= \sin 60^o \cos 45^o + \sin 45^o \cos(-60^o)$$

$$= \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \times -\frac{1}{2}$$

$$= \frac{\sqrt{3}-1}{2\sqrt{2}}$$

(iii) $cosec\, 75^o =$

(iv) $\cos(45^o - 30^o) = \cos 45^o \cos 30^o + \sin 45^o \sin 30^o$

$$= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \times \frac{1}{2}$$

$$= \frac{\sqrt{3}+1}{2\sqrt{2}}$$

(v) $\tan(120^o + 45^o) = \frac{\tan 120^o + \tan 45^o}{1 - \tan 120^o \tan 45^o}$

$$= \frac{-\tan 60^o + \tan 45^o}{1-(-\tan 60^o)\tan 45^o}$$

$$= \frac{-\sqrt{3}+1}{1-(-\sqrt{3})\times 1}$$

$$= \sqrt{3} - 2$$

(vi) $\sin 15^o = \sin(45^o - 30^o)$

$$= \sin 45^o \cos 30^o - \sin 30^o \cos 45^o$$

$$= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} - \frac{1}{2} \times \frac{1}{\sqrt{2}}$$

$$= \frac{\sqrt{6}-\sqrt{2}}{4}$$

Example: If $\sin A = \frac{3}{5}$ and $\sin B = \frac{5}{13}$ where A and B are acute angles, find without using table the value of

(i) $\sin(A+B)$

(ii) $\cot(A+B)$

Solution: Let the following figures represent the acute angles A and B respectively

from triangle $\triangle POQ$

$$|OP|^2 = |OQ|^2 + |PQ|^2$$

$$5^2 = |OQ|^2 + 3^2$$

$$|OQ| = 4$$

similarly from triangle $\triangle ROS$

$$|OR|^2 = |OS|^2 + |RS|^2$$

$$13^2 = |OS|^2 + 5^2$$

$$|OS| = 12$$

(i) $\sin(A+B) = \sin A \cos B + \sin B \cos A$

$$= \tfrac{3}{5} \times \tfrac{12}{13} + \tfrac{5}{13} \times \tfrac{4}{5}$$

$$= \tfrac{56}{65}$$

(ii) $\cot(A+B) = \frac{1}{\tan(A+B)}$

$$= \frac{1-\tan A \tan B}{\tan A + \tan B}$$

$$= \frac{1-\left(\frac{3}{4}\right)\left(\frac{5}{12}\right)}{\frac{3}{4}+\frac{5}{12}}$$

$$= \tfrac{33}{56}$$

Example: If $\tan\theta = \frac{7}{24}$ and θ is reflex, evaluate without using table the following

(i) $\sec\theta$

(ii) $cosec\,\theta$

(iii) $\cot\theta$

Example: If $\tan(x + 45^o) = 2$, find without using table the value of $\tan x$

Example: Express the following as single trigonometric ratios

(i) $\frac{1}{2}\cos x - \frac{\sqrt{3}}{2}\sin x$

(ii) $\frac{\sqrt{3}+\tan x}{1-\sqrt{3}\tan x}$

(iii) $\cos 16^o \sin 42^o - \sin 16^o \cos 42^o$

(iv) $\frac{1}{2}\cos 75^o + \frac{\sqrt{3}}{2}\sin 75^o$

Example: Find without using tables the value of

(i) $\cos 75^o \cos 15^o + \sin 75^o \sin 15^o$

(ii) $\frac{1}{\sqrt{2}}\cos 15^o - \frac{1}{\sqrt{2}}\sin 15^o$

(iii) $\frac{1-\tan 15^o}{1+\tan 15^o}$

7.8.2 Double Angle

The basic trigonometric identities and the compound angles formulas can be used ot derived the double angle formulas as follows

1. $\sin 2A = \sin(A + A)$

$$= \sin A \cos A + \sin A \cos A$$

$$= 2 \sin A \cos A$$

2. $\cos 2A = \cos(A + A)$

$$= \cos A \cos A + \sin A \sin A$$

$$= \cos^2 A - \sin^2 A$$

$$= \cos^2 A - (1 - \cos^2 A)$$

$$= 2\cos^2 A - 1$$

or equivalently

$\cos 2A = \cos^2 A - \sin^2 A$

$$= (1 - \sin^2 A) - \sin^2 A$$

$$= 1 - 2\sin^2 A$$

For further mathematical applications, it is worthy to note the following

$\cos^2 A = \frac{1}{2}(1 + \cos 2A)$

$\sin^2 A = \frac{1}{2}(1 - \cos 2A)$

Example: Without using table, find the value of

(i) $2 \sin 15^o \cos 15^o$

recall that $\sin 2A = 2 \sin A \cos A$, therefore

$2 \sin 15^o \cos 15^o = \sin 2(15^o)$

$$= \sin 30^o = \frac{1}{2}$$

(ii) $\cos^2 105^o - \sin^2 105^o$

recall that $\cos 2A = \cos^2 A - \sin^2 A$, therefore

$\cos^2 105^0 - \sin^2 105^o = \cos 2(105^o)$

$$= \cos 210^o$$

$$= -\cos 30^o$$

$$= -\frac{\sqrt{3}}{2}$$

(iii) $1 - 2\sin^2 15^o$

recall that $\cos 2A = 1 - \sin^2 A$, therefore

$$1 - 2\sin^2 15^o = \cos 2(15^o)$$
$$= \cos 30^o$$
$$= \tfrac{\sqrt{3}}{2}$$

(iv) $\sin 165^o \cos 165^o$

recall that $\sin 2A = 2\sin A \cos A$, therefore

$$\sin 165^o \cos 165^o = \tfrac{1}{2}(2\sin 165^o \cos 165^o)$$
$$= \tfrac{1}{2}(\sin 2(165^o))$$
$$= \tfrac{1}{2}\sin 330^o$$
$$= \tfrac{1}{2}(-\sin 30^o)$$
$$= \tfrac{1}{2}\left(-\tfrac{1}{2}\right) = -\tfrac{1}{4}$$

(v) $\frac{2\tan\frac{3\pi}{8}}{1-\tan^2\frac{3\pi}{8}}$

recall that $\tan 2A = \frac{2\tan A}{1-\tan^2 A}$, therefore

$$\frac{2\tan\frac{3\pi}{8}}{1-\tan^2\frac{3\pi}{8}} = \tan\tfrac{3\pi}{4}$$
$$= \tan 3(45^o)$$
$$= \tan 135^o$$
$$= -\tan 45^o$$
$$= -1$$

Example: Prove the following

(i) $\frac{1-\cos 2\theta}{\sin 2\theta} = \tan\theta$

$$\frac{1-\cos 2\theta}{\sin 2\theta} = \frac{1-(1-2\sin^2\theta)}{2\sin\theta\cos\theta}$$
$$= \frac{2\sin^2\theta}{2\sin\theta\cos\theta} = \frac{\sin\theta}{\cos\theta} = \tan\theta$$

(ii) $\cos^4\theta - \sin^4\theta = \cos 2\theta$

$$\cos^4\theta - \sin^4\theta = (\cos^2\theta)^2 - (\sin^2\theta)^2$$
$$= (\cos^2\theta + \sin^2\theta)(\cos^2\theta - \sin^2\theta)$$
$$= \cos 2\theta$$

(iii) $\frac{\sin 2A}{1+\cos 2A} = \tan A$

(iv) $\sin 3A = 3\sin A - 4\sin^3 A$

EXERCISES

1. Without using tables, find the value of the following leaving surds in your answer

   (i) $\tan(60^o + 30^o)$

   (ii) $\cos(60^o + 45^o)$

   (iii) $\tan 15^o$

   (iv) $\cos 105^o$

   (v) $\sin 165^o$

2. If $\sin A = \frac{4}{5}$ and $\cos B = \frac{12}{13}$ where A is obtuse and B is acute, find without using table the values of

   (i) $\sin(A + B)$

   (ii) $\tan(A - B)$

   (iii) $\cos(A - B)$

3. If $\cos A = \frac{3}{5}$ and $\tan B = \frac{12}{5}$ where A and B are both reflex angles, find without using table the value of

   (i) $\sec(A - B)$

   (ii) $\cot(A + B)$

   (iii) $cosec\,(A - B)$

4. If $\tan(A + B) = \frac{1}{7}$ and $\tan A = 3$, find without using table the value of $\tan B$

5. Express as single trigonometric ratios

   (i) $\frac{1}{\sqrt{2}} \sin x + \frac{1}{\sqrt{2}} \cos x$

   (ii) $\frac{1}{\cos 24^o \cos 15^o - \sin 24^o \sin 15^o}$

6. Find without using table the values of

   (i) $\sin 50^o \cos 20^o - \cos 50^o \sin 20^o$

   (ii) $\frac{\sqrt{3}}{2} \cos 15^o - \frac{1}{2} \sin 15^o$

   (iii) $\cos 15^o + \sin 15^o$

   (iv) $\frac{\tan 10^o + \tan 20^o}{1 - \tan 10^o \tan 20^o}$

7. Simplify the following

   (i) $2 \sin 17^o \cos 17^o$

   (ii) $2 \cos^2 42^0 - 1$

   (iii) $2 \sin \frac{1}{2}\theta \cos \frac{1}{2}\theta$

   (iv) $\sin x \cos x$

8. Evaluate without using table

   (i) $2 \cos^2 75^o - 1$

   (ii) $\frac{2 \tan 22\frac{1}{2}^o}{1 - \tan^2 22\frac{1}{2}^o}$

(iii) $1 - 2\sin^2 67\frac{1}{2}^o$

(iv) $\frac{1-2\cos^2 25^o}{1-2\sin^2 65^o}$

(v) $\frac{1-\tan^2 15^o}{\tan 15^o}$

9. Eliminate θ from the equations

   (i) $x = 2\sin\theta, \qquad y = 3\cos\theta$

   (ii) $x = 2\sec\theta, \qquad y = \cos 2\theta$

10. Prove the following identities

    (i) $\frac{\cos 2A}{\cos A+\sin A} = \cos A - \sin A$

    (ii) $\frac{\sin A}{\sin B} + \frac{\cos A}{\cos B} = \frac{2\sin(A+B)}{\sin 2B}$

    (iii) $\frac{1}{\cos A+\sin A} + \frac{1}{\cos A-\sin A} = \tan 2A\, cosec\, A$

11. Prove that $\cos 3A = 4\cos^2 A - 3\cos A$

12. prove the following identities

    (i) $\frac{1-\cos 2\theta}{1+\cos 2\theta} = \tan^2\theta$

    (ii) $(\sin\theta + \cos\theta)^2 = 1 + \sin 2\theta$

    (iii) $(2\cos\theta + 1)(2\cos\theta - 1) = 2\cos\theta - 1$

    (iv) $2\sin\left(\frac{\pi}{4} + \theta\right)\sin\left(\frac{\pi}{4} + \theta\right) = \cos 2\theta$

7.9 Trigonometric Equations

The trigonometric equations in which the trigonometric ratios of a known quantity occur can be solved using the idea of solving algebraic equations. Even though there are limited number of solutions to an algebraic equation, there are indeed unlimited number of solutions to trigonometric equations

Example: Solve the equation $\cos\theta = \frac{1}{2}$

Solution: The cosine function is positive in the 1^{st} and 4^{th} quadrants

$\theta = 60^o,\ 300^o,\ 420^o,\ 660^o,\ 780^o,\ \ldots$

$\theta = -60^o,\ -300^o,\ -420^o,\ -660^o,\ -780^o,\ \ldots$

Example: Find the solution to the equation $\sin x = 0.515$ which lies in the range $0^o \leq x \leq 360^o$

Solution:

$\sin x = 0.515$

$x = \sin^{-1}(0.515)$

$x = 31^o$

The sine function is positive in the 1^{st} and 2^{nd} quadrants

$\therefore\ x = 31^o,\ 149^o$

Example: Find the solution to the equation $2\cos x - 3\sin x = 0$ which lies in the range $0^o \leq x \leq 360^o$

Solution:

$2\cos x - 3\sin x = 0$

$\frac{2}{3} = \frac{\sin x}{\cos x} = \tan x$

$x = \tan^{-1}\left(\frac{2}{3}\right)$

The tangent function is positive in the 1^{st} and 3^{rd} quadrants

$\therefore\ x = 33^o41',\ 213^o41'$

Example: Solve $\tan\theta = 2\sin\theta$ for $-180^o \leq \theta \leq 180^o$

Solution:

$\frac{\sin\theta}{\cos\theta} = 2\sin\theta$

$\sin\theta = 2\sin\theta\cos\theta$

$\sin\theta - 2\sin\theta\cos\theta = 0$

$\sin\theta(1 - 2\cos\theta) = 0$

from the zero product property of real numbers either $\sin\theta = 0$ or $1 - 2\cos\theta = 0$

for $\sin\theta = 0$

$\theta = \sin^{-1}(0)$

$\therefore\ \theta = 0^o,\ 180^o,\ -180^o$

for $1 - 2\cos\theta = 0$

$\cos\theta = \frac{1}{2}$

$\theta = \cos^{-1}\left(\frac{1}{2}\right)$

then θ lies in the 1^{st} or 4^{th} quadrants

$\therefore\ \theta = 60^o,\ -60^o$

Overall $\theta = 0^o,\ 180^o,\ -180^o,\ 60^o,\ -60^o$

Example: Solve $2\sin^2\theta + \sin\theta = 0$ for $0^o \leq \theta \leq 180^o$

Solution:

$2\sin^2\theta + \sin\theta = \sin\theta(2\sin\theta + 1) = 0$

either $\sin\theta = 0$ or $2\sin\theta + 1 = 0$ or both are zero

for $\sin\theta = 0$

$\theta = \sin^{-1}(0)$

$\therefore\ \theta = 0^o,\ 180^o,\ 360^o$

for $\sin\theta = -\frac{1}{2}$

quadrant: 3^{rd} and 4^{th} reference angle 30^o

$\therefore\ \theta = 210^o,\ 330^o$

Overall $\theta = 0^o,\ 180^o,\ 210^o,\ 330^o,\ 3660^o$

Example: Find θ in the range $0^o \leq \theta \leq 480^o$ if $2\cos^2\theta = 2 - \sin\theta$

Solution:

Example: Solve the equation $3\cos 2\theta + \sin\theta = 1$ for $0^o \leq \theta \leq 360^o$

Solution:

MORE EXERCISES

1. Solve the following equations within the range $0^o \leq \theta \leq 360^o$
   - (i) $\cos 2\theta + 5\cos\theta = 2$
   - (ii) $\tan 2\theta + \tan\theta = 0$
   - (iii) $4\cos\theta - 3\sin\theta = 1$
2. Solve the equation $\sin 2\theta \cos\theta + 3\sin^2\theta = 3$ for values $0^o \leq \theta \leq 360^o$