2

Here is a list of basic derivatives and the basic integrals that go with them:

1	$\dfrac{d}{dx}(x^n) = nx^{n-1}$	$\therefore \displaystyle\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad \left\{\begin{matrix}\text{provided} \\ n \neq -1\end{matrix}\right\}$
2	$\dfrac{d}{dx}(\ln x) = \dfrac{1}{x}$	$\therefore \displaystyle\int \frac{1}{x}\,dx = \ln x + C$
3	$\dfrac{d}{dx}(e^x) = e^x$	$\therefore \displaystyle\int e^x\,dx = e^x + C$
4	$\dfrac{d}{dx}(e^{kx}) = ke^{kx}$	$\therefore \displaystyle\int e^{kx}\,dx = \frac{e^{kx}}{k} + C$
5	$\dfrac{d}{dx}(a^x) = a^x \ln a$	$\therefore \displaystyle\int a^x\,dx = \frac{a^x}{\ln a} + C$
6	$\dfrac{d}{dx}(\cos x) = -\sin x$	$\therefore \displaystyle\int \sin x\,dx = -\cos x + C$
7	$\dfrac{d}{dx}(\sin x) = \cos x$	$\therefore \displaystyle\int \cos x\,dx = \sin x + C$
8	$\dfrac{d}{dx}(\tan x) = \sec^2 x$	$\therefore \displaystyle\int \sec^2 x\,dx = \tan x + C$
9	$\dfrac{d}{dx}(\cosh x) = \sinh x$	$\therefore \displaystyle\int \sinh x\,dx = \cosh x + C$
10	$\dfrac{d}{dx}(\sinh x) = \cosh x$	$\therefore \displaystyle\int \cosh x\,dx = \sinh x + C$
11	$\dfrac{d}{dx}(\sin^{-1} x) = \dfrac{1}{\sqrt{1-x^2}}$	$\therefore \displaystyle\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + C$
12	$\dfrac{d}{dx}(\cos^{-1} x) = \dfrac{-1}{\sqrt{1-x^2}}$	$\therefore \displaystyle\int \frac{-1}{\sqrt{1-x^2}}\,dx = \cos^{-1} x + C$
13	$\dfrac{d}{dx}(\tan^{-1} x) = \dfrac{1}{1+x^2}$	$\therefore \displaystyle\int \frac{1}{1+x^2}\,dx = \tan^{-1} x + C$
14	$\dfrac{d}{dx}(\sinh^{-1} x) = \dfrac{1}{\sqrt{x^2+1}}$	$\therefore \displaystyle\int \frac{1}{\sqrt{x^2+1}}\,dx = \sinh^{-1} x + C$
15	$\dfrac{d}{dx}(\cosh^{-1}) = \dfrac{1}{\sqrt{x^2-1}}$	$\therefore \displaystyle\int \frac{1}{\sqrt{x^2-1}}\,dx = \cosh^{-1} x + C$
16	$\dfrac{d}{dx}(\tanh^{-1} x) = \dfrac{1}{1-x^2}$	$\therefore \displaystyle\int \frac{1}{1-x^2}\,dx = \tanh^{-1} x + C$

Spend a little time copying this list carefully into your record book as a reference list.

Differentiation

$(cu)' = cu' \qquad (c \text{ constant})$

$(u + v)' = u' + v'$

$(uv)' = u'v + uv'$

$$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$$

$$\frac{du}{dx} = \frac{du}{dy} \cdot \frac{dy}{dx} \qquad \text{(Chain rule)}$$

$(x^n)' = nx^{n-1}$

$(e^x)' = e^x$

$(e^{ax})' = ae^{ax}$

$(a^x)' = a^x \ln a$

$(\sin x)' = \cos x$

$(\cos x)' = -\sin x$

$(\tan x)' = \sec^2 x$

$(\cot x)' = -\csc^2 x$

$(\sinh x)' = \cosh x$

$(\cosh x)' = \sinh x$

$$(\ln x)' = \frac{1}{x}$$

$$(\log_a x)' = \frac{\log_a e}{x}$$

$$(\arcsin x)' = \frac{1}{\sqrt{1 - x^2}}$$

$$(\arccos x)' = -\frac{1}{\sqrt{1 - x^2}}$$

$$(\arctan x)' = \frac{1}{1 + x^2}$$

$$(\operatorname{arccot} x)' = -\frac{1}{1 + x^2}$$

Integration

$$\int uv'\,dx = uv - \int u'v\,dx \text{ (by parts)}$$

$$\int x^n\,dx = \frac{x^{n+1}}{n + 1} + c \qquad (n \neq -1)$$

$$\int \frac{1}{x}\,dx = \ln|x| + c$$

$$\int e^{ax}\,dx = \frac{1}{a} e^{ax} + c$$

$$\int \sin x\,dx = -\cos x + c$$

$$\int \cos x\,dx = \sin x + c$$

$$\int \tan x\,dx = -\ln|\cos x| + c$$

$$\int \cot x\,dx = \ln|\sin x| + c$$

$$\int \sec x\,dx = \ln|\sec x + \tan x| + c$$

$$\int \csc x\,dx = \ln|\csc x - \cot x| + c$$

$$\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \arctan \frac{x}{a} + c$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin \frac{x}{a} + c$$

$$\int \frac{dx}{\sqrt{x^2 + a^2}} = \operatorname{arcsinh} \frac{x}{a} + c$$

$$\int \frac{dx}{\sqrt{x^2 - a^2}} = \operatorname{arccosh} \frac{x}{a} + c$$

$$\int \sin^2 x\,dx = \tfrac{1}{2}x - \tfrac{1}{4}\sin 2x + c$$

$$\int \cos^2 x\,dx = \tfrac{1}{2}x + \tfrac{1}{4}\sin 2x + c$$

$$\int \tan^2 x\,dx = \tan x - x + c$$

$$\int \cot^2 x\,dx = -\cot x - x + c$$

$$\int \ln x\,dx = x \ln x - x + c$$

$$\int e^{ax} \sin bx\,dx = \frac{e^{ax}}{a^2 + b^2}(a \sin bx - b \cos bx) + c$$

$$\int e^{ax} \cos bx\,dx = \frac{e^{ax}}{a^2 + b^2}(a \cos bx + b \sin bx) + c$$

Basic Properties and Facts

Arithmetic Operations

$ab + ac = a(b+c)$ $\quad$ $a\left(\frac{b}{c}\right) = \frac{ab}{c}$

$\frac{\left(\frac{a}{b}\right)}{c} = \frac{a}{bc}$ $\quad$ $\frac{a}{\left(\frac{b}{c}\right)} = \frac{ac}{b}$

$\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$ $\quad$ $\frac{a}{b} - \frac{c}{d} = \frac{ad-bc}{bd}$

$\frac{a-b}{c-d} = \frac{b-a}{d-c}$ $\quad$ $\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}$

$\frac{ab+ac}{a} = b+c,\ a \neq 0$ $\quad$ $\frac{\left(\frac{a}{b}\right)}{\left(\frac{c}{d}\right)} = \frac{ad}{bc}$

Exponent Properties

$a^n a^m = a^{n+m}$ $\quad$ $(ab)^n = a^n b^n$

$(a^n)^m = a^{nm}$ $\quad$ $a^0 = 1,\ a \neq 0$

$\frac{a^n}{a^m} = a^{n-m} = \frac{1}{a^{m-n}}$ $\quad$ $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$

$a^{\frac{n}{m}} = \left(a^{\frac{1}{m}}\right)^n = (a^n)^{\frac{1}{m}}$ $\quad$ $\frac{1}{a^{-n}} = a^n$

$\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n = \frac{b^n}{a^n}$ $\quad$ $a^{-n} = \frac{1}{a^n}$

Properties of Radicals

$\sqrt[n]{a} = a^{\frac{1}{n}}$ $\quad$ $\sqrt[n]{ab} = \sqrt[n]{a}\,\sqrt[n]{b}$

$\sqrt[m]{\sqrt[n]{a}} = \sqrt[nm]{a}$ $\quad$ $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$

$\sqrt[n]{a^n} = a$ if n is odd

$\sqrt[n]{a^n} = |a|$ if n is even

Properties of Inequalities

If $a < b$ then $a + c < b + c$ and $a - c < b - c$

If $a < b$ and $c > 0$ then $ac < bc$ and $\frac{a}{c} < \frac{b}{c}$

If $a < b$ and $c < 0$ then $ac > bc$ and $\frac{a}{c} > \frac{b}{c}$

Properties of Absolute Value

$$|a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0 \end{cases}$$

$|a| \geq 0$ $\quad$ $|-a| = |a|$

$|ab| = |a|\,|b|$ $\quad$ $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$

$|a+b| \leqslant |a| + |b|$ Triangle Inequality

Distance Formula

If $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ are two points the distance between them is

$$d(P_1, P_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Complex Numbers

$i = \sqrt{-1}$ $\quad$ $i^2 = -1$ $\quad$ $\sqrt{-a} = i\sqrt{a},\ a \geq 0$

$(a+bi) + (c+di) = a + c + (b+d)i$

$(a+bi) - (c+di) = a - c + (b-d)i$

$(a+bi)(c+di) = ac - bd + (ad+bc)i$

$(a+bi)(a-bi) = a^2 + b^2$

$|a+bi| = \sqrt{a^2+b^2}$ Complex Modulus

$\overline{(a+bi)} = a - bi$ Complex Conjugate

$\overline{(a+bi)}(a+bi) = |a+bi|^2$

Logarithms and Log Properties

Definition
$y = \log_b(x)$ is equivalent to $x = b^y$

Example
$\log_5(125) = 3$ because $5^3 = 125$

Special Logarithms
$\ln(x) = \log_e(x)$ natural log

$\log(x) = \log_{10}(x)$ common log

where $e = 2.718281828\ldots$

Logarithm Properties
$\log_b(b) = 1$ $\quad\log_b(1) = 0$

$\log_b(b^x) = x$ $\quad b^{\log_b(x)} = x$

$\log_b(x^r) = r\log_b(x)$

$\log_b(xy) = \log_b(x) + \log_b(y)$

$\log_b\left(\frac{x}{y}\right) = \log_b(x) - \log_b(y)$

The domain of $\log_b(x)$ is $x > 0$

Factoring and Solving

Factoring Formulas

$x^2 - a^2 = (x+a)(x-a)$

$x^2 + 2ax + a^2 = (x+a)^2$

$x^2 - 2ax + a^2 = (x-a)^2$

$x^2 + (a+b)x + ab = (x+a)(x+b)$

$x^3 + 3ax^2 + 3a^2x + a^3 = (x+a)^3$

$x^3 - 3ax^2 + 3a^2x - a^3 = (x-a)^3$

$x^3 + a^3 = (x+a)\left(x^2 - ax + a^2\right)$

$x^3 - a^3 = (x-a)\left(x^2 + ax + a^2\right)$

$x^{2n} - a^{2n} = (x^n - a^n)(x^n + a^n)$

If n is odd then,

$x^n - a^n = (x-a)\left(x^{n-1} + ax^{n-2} + \cdots + a^{n-1}\right)$

$x^n + a^n =$
$(x+a)\left(x^{n-1} - ax^{n-2} + a^2x^{n-3} - \cdots + a^{n-1}\right)$

Quadratic Formula

Solve $ax^2 + bx + c = 0$, $a \neq 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

If $b^2 - 4ac > 0$ – Two real unequal solns.

If $b^2 - 4ac = 0$ – Repeated real solution.

If $b^2 - 4ac < 0$ – Two complex solutions.

Square Root Property

If $x^2 = p$ then $x = \pm\sqrt{p}$

Absolute Value Equations/Inequalities

If b is a positive number

$|p| = b \quad\Rightarrow\quad p = -b$ or $p = b$

$|p| < b \quad\Rightarrow\quad -b < p < b$

$|p| > b \quad\Rightarrow\quad p < -b$ or $p > b$

Completing the Square

Solve $2x^2 - 6x - 10 = 0$

(1) Divide by the coefficient of the x^2

$$x^2 - 3x - 5 = 0$$

(2) Move the constant to the other side.

$$x^2 - 3x = 5$$

(3) Take half the coefficient of x, square it and add it to both sides

$$x^2 - 3x + \left(-\frac{3}{2}\right)^2 = 5 + \left(-\frac{3}{2}\right)^2 = 5 + \frac{9}{4} = \frac{29}{4}$$

(4) Factor the left side

$$\left(x - \frac{3}{2}\right)^2 = \frac{29}{4}$$

(5) Use Square Root Property

$$x - \frac{3}{2} = \pm\sqrt{\frac{29}{4}} = \pm\frac{\sqrt{29}}{2}$$

(6) Solve for x

$$x = \frac{3}{2} \pm \frac{\sqrt{29}}{2}$$

Functions and Graphs

Constant Function

$y = a$ or $f(x) = a$

Graph is a horizontal line passing through the point $(0, a)$.

Line/Linear Function

$y = mx + b$ or $f(x) = mx + b$

Graph is a line with point $(0, b)$ and slope m.

Slope

Slope of the line containing the two points (x_1, y_1) and (x_2, y_2) is

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\text{rise}}{\text{run}}$$

Slope – intercept form

The equation of the line with slope m and y-intercept $(0, b)$ is

$$y = mx + b$$

Point – Slope form

The equation of the line with slope m and passing through the point (x_1, y_1) is

$$y = y_1 + m(x - x_1)$$

Parabola/Quadratic Function

$y = a(x-h)^2 + k$ $f(x) = a(x-h)^2 + k$

The graph is a parabola that opens up if $a > 0$ or down if $a < 0$ and has a vertex at (h, k).

Parabola/Quadratic Function

$y = ax^2 + bx + c$ $f(x) = ax^2 + bx + c$

The graph is a parabola that opens up if $a > 0$ or down if $a < 0$ and has a vertex at $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$.

Parabola/Quadratic Function

$x = ay^2 + by + c$ $g(y) = ay^2 + by + c$

The graph is a parabola that opens right if $a > 0$ or left if $a < 0$ and has a vertex at $\left(g\left(-\frac{b}{2a}\right), -\frac{b}{2a}\right)$.

Circle

$(x-h)^2 + (y-k)^2 = r^2$

Graph is a circle with radius r and center (h, k).

Ellipse

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

Graph is an ellipse with center (h, k) with vertices a units right/left from the center and vertices b units up/down from the center.

Hyperbola

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

Graph is a hyperbola that opens left and right, has a center at (h, k), vertices a units left/right of center and asymptotes that pass through center with slope $\pm\frac{b}{a}$.

Hyperbola

$$\frac{(y-k)^2}{b^2} - \frac{(x-h)^2}{a^2} = 1$$

Graph is a hyperbola that opens up and down, has a center at (h, k), vertices b units up/down from the center and asymptotes that pass through center with slope $\pm\frac{b}{a}$.

Common Algebraic Errors

Error	Reason/Correct/Justification/Example
$\frac{2}{0} \neq 0$ and $\frac{2}{0} \neq 2$	Division by zero is undefined!
$-3^2 \neq 9$	$-3^2 = -9$, $(-3)^2 = 9$ Watch parenthesis!
$\left(x^2\right)^3 \neq x^5$	$\left(x^2\right)^3 = x^2 x^2 x^2 = x^6$
$\frac{a}{b+c} \neq \frac{a}{b} + \frac{a}{c}$	$\frac{1}{2} = \frac{1}{1+1} \neq \frac{1}{1} + \frac{1}{1} = 2$
$\frac{1}{x^2 + x^3} \neq x^{-2} + x^{-3}$	A more complex version of the previous error.
$\frac{\cancel{a} + bx}{\cancel{a}} \neq 1 + bx$	$\frac{a+bx}{a} = \frac{a}{a} + \frac{bx}{a} = 1 + \frac{bx}{a}$ Beware of incorrect canceling!
$-a(x-1) \neq -ax - a$	$-a(x-1) = -ax + a$ Make sure you distribute the "-"!
$(x+a)^2 \neq x^2 + a^2$	$(x+a)^2 = (x+a)(x+a) = x^2 + 2ax + a^2$
$\sqrt{x^2 + a^2} \neq x + a$	$5 = \sqrt{25} = \sqrt{3^2 + 4^2} \neq \sqrt{3^2} + \sqrt{4^2} = 3 + 4 = 7$
$\sqrt{x+a} \neq \sqrt{x} + \sqrt{a}$	See previous error.
$(x+a)^n \neq x^n + a^n$ and $\sqrt[n]{x+a} \neq \sqrt[n]{x} + \sqrt[n]{a}$	More general versions of previous three errors.
$2(x+1)^2 \neq (2x+2)^2$	$2(x+1)^2 = 2\left(x^2 + 2x + 1\right) = 2x^2 + 4x + 2$ $(2x+2)^2 = 4x^2 + 8x + 4$ Square first then distribute!
$(2x+2)^2 \neq 2(x+1)^2$	See the previous example. You can not factor out a constant if there is a power on the parenthesis!
$\sqrt{-x^2 + a^2} \neq -\sqrt{x^2 + a^2}$	$\sqrt{-x^2 + a^2} = \left(-x^2 + a^2\right)^{\frac{1}{2}}$ Now see the previous error.
$\frac{a}{\left(\frac{b}{c}\right)} \neq \frac{ab}{c}$	$\frac{a}{\left(\frac{b}{c}\right)} = \frac{\left(\frac{a}{1}\right)}{\left(\frac{b}{c}\right)} = \left(\frac{a}{1}\right)\left(\frac{c}{b}\right) = \frac{ac}{b}$
$\frac{\left(\frac{a}{b}\right)}{c} \neq \frac{ac}{b}$	$\frac{\left(\frac{a}{b}\right)}{c} = \frac{\left(\frac{a}{b}\right)}{\left(\frac{c}{1}\right)} = \left(\frac{a}{b}\right)\left(\frac{1}{c}\right) = \frac{a}{bc}$

USEFUL TRIGONOMETRIC IDENTITIES

Definitions

$$\tan x = \frac{\sin x}{\cos x}$$

$$\sec x = \frac{1}{\cos x} \qquad \operatorname{cosec} x = \frac{1}{\sin x} \qquad \cot x = \frac{1}{\tan x}$$

Fundamental trig identity

$$(\cos x)^2 + (\sin x)^2 = 1$$

$$1 + (\tan x)^2 = (\sec x)^2$$

$$(\cot x)^2 + 1 = (\operatorname{cosec} x)^2$$

You can get this one from the top one if you divide by $(\cos x)^2$

You can get this one from the top one if you divide by $(\sin x)^2$

Odd and even properties

$$\cos(-x) = \cos(x) \qquad \sin(-x) = -\sin(x) \qquad \tan(-x) = -\tan(x)$$

cos is an even function

sin and tan are odd functions

Double angle formulas

$$\sin(2x) = 2\sin x \cos x \qquad \cos(2x) = (\cos x)^2 - (\sin x)^2$$

$$\cos(2x) = 2(\cos x)^2 - 1$$

$$\cos(2x) = 1 - 2(\sin x)^2$$

You get this from the top one if you sub in $(\sin x)^2 = 1-(\cos x)^2$

You get this from the top one if you sub in $(\cos x)^2 = 1-(\sin x)^2$

Half angle formulas

$$\left[\sin(\tfrac{1}{2}x)\right]^2 = \tfrac{1}{2}(1-\cos x)$$

$$\left[\cos(\tfrac{1}{2}x)\right]^2 = \tfrac{1}{2}(1+\cos x)$$

These come from rearranging the cos double angle formulas and then replacing x with $\frac{1}{2}x$

Sums and differences of angles

If you sub $A=x$, $B=x$ into these you get the double angle formulas

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$

$$\cos(A-B) = \cos A \cos B + \sin A \sin B$$

$$\sin(A+B) = \sin A \cos B + \cos A \sin B$$

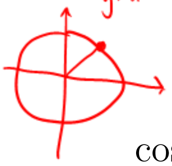
$$\sin(A-B) = \sin A \cos B - \cos A \sin B$$

You can get these by replacing B with $-B$ in the other two and using the odd and even properties

** See other side for more identities **

USEFUL TRIGONOMETRIC IDENTITIES

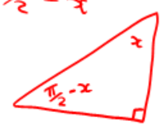
You can find these by drawing a diagram of the unit circle



Unit circle properties

$$
\begin{aligned}
\cos(\pi - x) &= -\cos(x) & \sin(\pi - x) &= \sin(x) & \tan(\pi - x) &= -\tan(x)\\
\cos(\pi + x) &= -\cos(x) & \sin(\pi + x) &= -\sin(x) & \tan(\pi + x) &= \tan(x)\\
\cos(2\pi - x) &= \cos(x) & \sin(2\pi - x) &= -\sin(x) & \tan(2\pi - x) &= -\tan(x)\\
\cos(2\pi + x) &= \cos(x) & \sin(2\pi + x) &= \sin(x) & \tan(2\pi + x) &= \tan(x)
\end{aligned}
$$

You can find these by drawing a right-angled triangle with small angles x and $\pi/2 - x$



Right-angled triangle properties

$$
\cos\left(\tfrac{\pi}{2} - x\right) = \sin(x) \qquad \sin\left(\tfrac{\pi}{2} - x\right) = \cos(x)
$$

These are a combination of the above two sets of formulas and the odd/even properties

Shifting by $\frac{\pi}{2}$

$$
\begin{aligned}
\cos(x) &= \cos(x) & \cos(x) &= \cos(x) & \cos(-x) &= \cos(x)\\
\cos(x + \tfrac{\pi}{2}) &= -\sin(x) & \cos(x - \tfrac{\pi}{2}) &= \sin(x) & \cos(\tfrac{\pi}{2} - x) &= \sin(x)\\
\cos(x + \pi) &= -\cos(x) & \cos(x - \pi) &= -\cos(x) & \cos(\pi - x) &= -\cos(x)\\
\cos(x + \tfrac{3\pi}{2}) &= \sin(x) & \cos(x - \tfrac{3\pi}{2}) &= -\sin(x) & \cos(\tfrac{3\pi}{2} - x) &= -\sin(x)\\
\cos(x + 2\pi) &= \cos(x) & \cos(x - 2\pi) &= \cos(x) & \cos(2\pi - x) &= \cos(x)
\end{aligned}
$$

$$
\begin{aligned}
\sin(x) &= \sin(x) & \sin(x) &= \sin(x) & \sin(-x) &= -\sin(x)\\
\sin(x + \tfrac{\pi}{2}) &= \cos(x) & \sin(x - \tfrac{\pi}{2}) &= -\cos(x) & \sin(\tfrac{\pi}{2} - x) &= \cos(x)\\
\sin(x + \pi) &= -\sin(x) & \sin(x - \pi) &= -\sin(x) & \sin(\pi - x) &= \sin(x)\\
\sin(x + \tfrac{3\pi}{2}) &= -\cos(x) & \sin(x - \tfrac{3\pi}{2}) &= \cos(x) & \sin(\tfrac{3\pi}{2} - x) &= -\cos(x)\\
\sin(x + 2\pi) &= \sin(x) & \sin(x - 2\pi) &= \sin(x) & \sin(2\pi - x) &= -\sin(x)
\end{aligned}
$$

$$
\begin{aligned}
\tan(x) &= \tan(x) & \tan(x) &= \tan(x) & \tan(-x) &= -\tan(x)\\
\tan(x + \tfrac{\pi}{2}) &= -\cot(x) & \tan(x - \tfrac{\pi}{2}) &= -\cot(x) & \tan(\tfrac{\pi}{2} - x) &= \cot(x)\\
\tan(x + \pi) &= \tan(x) & \tan(x - \pi) &= \tan(x) & \tan(\pi - x) &= -\tan(x)\\
\tan(x + \tfrac{3\pi}{2}) &= -\cot(x) & \tan(x - \tfrac{3\pi}{2}) &= -\cot(x) & \tan(\tfrac{3\pi}{2} - x) &= \cot(x)\\
\tan(x + 2\pi) &= \tan(x) & \tan(x - 2\pi) &= \tan(x) & \tan(2\pi - x) &= -\tan(x)
\end{aligned}
$$

** See other side for more identities **

First Order Ordinary Differential Equation

mccp-richard-1

Introduction

Prerequisites: You will need to know about trigonometry, differentiation, integration, complex numbers in order to make the most of this teach-yourself resource.

We are looking at equations involving a function $y(x)$ and its first derivative:

$$\frac{dy}{dx} + P(x)y = Q(x) \tag{1}$$

We want to find $y(x)$, either explicitly if possible, or otherwise implicitly.

Direct Integration

This method is used to solve ODEs in the form:

$$\frac{dy}{dx} = f(x)$$

These ODEs can be solved as follows:

$$\begin{aligned} dy &= f(x)dx \\ \text{so: } y &= \int f(x)dx \end{aligned}$$

Example:

$$\begin{aligned} \frac{dy}{dx} &= 3x^2 - 6x + 5 \\ y &= x^3 - 3x^2 + 5x + C \text{ with } C \text{ constant of integration} \end{aligned}$$

The constant of integration can be anything, unless you have boundary conditions. In the previous example, if you have y(0)=0 then:

$$y(0) = C = 0$$

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Separation of Variables

This method is used to solve ODEs in the form:

$$\frac{dy}{dx} = f(x)g(y)$$

These ODEs can be solved as follows:

$$\begin{aligned}\frac{dy}{g(y)} &= f(x)dx \\ \int \frac{dy}{g(y)} &= \int f(x)dx\end{aligned}$$

Example:

$$\begin{aligned}\frac{dy}{dx} &= \frac{2x}{y+1} \\ (y+1)dy &= 2xdx \\ \int (y+1)dy &= \int 2xdx \\ \frac{1}{2}y^2 + y &= x^2 + C \text{ with C constant of integration}\end{aligned}$$

Homogeneous Equations

A first order homogeneous differential equation is a differential equation in the form:

$$\frac{dy}{dx} = f(\frac{y}{x})$$

These ODEs can be solved by making the substitution $y(x) = v(x) \cdot x$ where v is a function of x. Then we have:

$$\begin{aligned}\text{Using the product rule:} \quad & \frac{dy}{dx} = \frac{dv}{dx} \cdot x + v \\ \text{Inserting in the ODE:} \quad & \frac{dv}{dx} \cdot x + v = f(v) \\ \text{Re-arranging:} \quad & \frac{dv}{f(v) - v} = \frac{dx}{x} \\ & \int \frac{dv}{f(v) - v} = \int \frac{dx}{x}\end{aligned}$$

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Example

$$\frac{dy}{dx} = \frac{x+3y}{2x}$$

Rearranging, we get: $\frac{dy}{dx} = \frac{1}{2} + \frac{3}{2}\frac{y}{x}$

$$x\frac{dv}{dx} + v = \frac{1}{2} + \frac{3}{2}v$$

$$x\frac{dv}{dx} = \frac{1}{2} + \frac{1}{2}v$$

$$\frac{dv}{1+v} = \frac{dx}{2x}$$

$$\ln(1+v) = \frac{1}{2}\ln(x) + C = \ln(x^{1/2}) + C \text{ with} C \text{ constant of integration}$$

Taking the log on both sides: $v = e^C x^{1/2} - 1$

with $y = v \cdot x$, $y = kx^{3/2} - x$, with $k = e^C$

Integrating Factor

This method is used to solve ODEs in the form:

$$\frac{dy}{dx} + P(x)y = Q(x)$$

These ODEs can be solved as follows: multiply both sides of the equation by **the integrating factor:** $e^{\int P(x)dx}$. Then:

$$e^{\int P(x)dx}\frac{dy}{dx} + P(x)e^{\int P(x)dx}y = Q(x)e^{\int P(x)dx}$$

Now we see that, using the product rule and the chain rule:

$$e^{\int P(x)dx}\frac{dy}{dx} + P(x)e^{\int P(x)dx}y = \frac{d}{dx}\left(e^{\int P(x)dx}y\right)$$

Therefore:

$$\begin{aligned} \frac{d}{dx}\left(e^{\int P(x)dx}y\right) &= Q(x)e^{\int P(x)dx} \\ y &= e^{-\int P(x)dx}\int Q(x)e^{\int P(x)dx}dx \end{aligned}$$

Example

$$\frac{dy}{dx} - y = x \text{ with } P(x) = -1 \text{ and } Q(x) = x$$

The integrating factor is $e^{\int -dx} = e^{-x}$ and:

$$e^{-x}\frac{dy}{dx} - e^{-x}y = xe^{-x}$$

$$\frac{d}{dx}\left(ye^{-x}\right) = xe^{-x}$$

Using integration by part: $ye^{-x} = -e^{-x}(1+x) + C$ with C constant of integration

$$y = -(1+x) + Ce^x$$

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Bernouilli Equations

Bernouilli equations are of the form:

$$\frac{dy}{dx} + P(x)y = Q(x)y^n$$

These equations are solved with the following method:

- Multiply both sides of the equation by y^{-n}, the equation becomes:

$$y^{-n}\frac{dy}{dx} + P(x)y^{1-n} = Q(x)$$

- Make the change of variable $v = y^{1-n}$, then:

$$\frac{dv}{dx} = (1-n)y^{-n}\frac{dy}{dx}$$

$$\text{so} \quad \frac{1}{1-n}\frac{dv}{dx} + P(x)v = Q(x)$$

The latest form of the equation can be solved for v using the method of the integrating factor. Finally, y can be found from the relationship $y = v^{n-1}$.

Example:

$$\begin{aligned}
\frac{dy}{dx} + \frac{1}{x}y &= xy^2 \\
y^{-2}\frac{dy}{dx} + \frac{1}{x}y^{-1} &= x \\
v = y^{-1} \quad &, \quad \frac{dv}{dx} = -\frac{1}{y^2}\frac{dy}{dx} \\
\frac{dv}{dx} - \frac{v}{x} &= -x \\
\text{The integrating factor is IF} &= e^{\int -\frac{1}{x}dx} = e^{-\ln x} = \frac{1}{x} \\
\frac{1}{x}\frac{dv}{dx} - \frac{1}{x^2}v &= -1 \\
\frac{1}{x}v &= \int -dx \\
v &= -x^2 + Cx \\
y &= \frac{1}{-x^2 + Cx}
\end{aligned}$$

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Exercises

(a) $x\dfrac{dy}{dx} = 5x^3 + 4$

(b) $x(y-3)\dfrac{dy}{dx} = 4y$

(c) $(2y-x)\dfrac{dy}{dx} = 2x + y$ with $y(2) = 3$

(d) $x\dfrac{dy}{dx} - y = x^3 + 3x^2 - 2x$

(e) $(1+x^2)\dfrac{dy}{dx} + 3xy = 5x$ with $y(1) = 2$

(f) $2\dfrac{dy}{dx} + y = y^3(x-1)$

Answers

(a) $y = \dfrac{5}{3}x^3 + 4\ln x + C$

(b) $y - 3\ln y = 4\ln x + C$

(c) $(\dfrac{y}{x})^2 - \dfrac{y}{x} - 1 + x^{-2} = 0$

(d) $y = \dfrac{1}{2}x^3 + 3x^2 - 2x\ln x + Cx$

(e) $y = \dfrac{5}{3} + \dfrac{\sqrt{8}}{3}(1+x^2)^{-3/2}$

(f) $y = (x + Ce^{-x})^{-1/2}$

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Second Order Ordinary Differential Equations

Introduction

Prerequisites: In order to make the most of this resource, you need to know about trigonometry, differentiation, integration and complex numbers.

We are looking at equations involving a function $y(x)$, its first derivative and second derivative:

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x) \tag{1}$$

We will only look at equations where the coefficients a, b and c are constant; we will not treat in this handout the case of coefficients which are functions of x.

Homogeneous Equations

If $f(x) = 0$ then the ODE is called an homogeneous equation. To solve a second order homogeneous ODE, we look at the **characteristic equation**, obtained by replacing $\frac{d^2y}{dx^2}$, $\frac{dy}{dx}$ and y by r^2, r and 1 in the ODE:

$$ar^2 + br + c = 0$$

We distinguish between 3 cases: the case when the roots of the characteristic equation are distinct and real, complex or equal.

Case 1: real and distinct roots r_1 and r_2

Then the solutions of the homogeneous equation are of the form:

$$y(x) = Ae^{r_1x} + Be^{r_2x}$$

The constants A and B can be anything you like if there are no boundary conditions. If you have boundary conditions, e.g. you know that $y(x_0) = \alpha$ and $y'(x_0) = \beta$, then A and B will be uniquely defined by:

$$\begin{aligned} Ae^{r_1x_0} + Be^{r_2x_0} &= \alpha \\ Ar_1e^{r_1x_0} + Br_2e^{r_2x_0} &= \beta \end{aligned}$$

Example

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 0$$

The characteristic equation is: $r^2 + 5r + 6 = 0$ and the roots are $\dfrac{-5 \pm \sqrt{25 - 4 \times 6}}{2} = -3$ or -2. Therefore the solutions of the ODE are:

$$y(x) = Ae^{-3x} + Be^{-2x}$$

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Case 2: complex roots

If the roots are complex then they can be written as $r + js$ and $r - js$ (with j the imaginary number, $j^2 = -1$) and the solutions of the homogeneous equations are of the form:

$$\begin{aligned} y(x) &= e^{rx}\left(Ae^{jsx} + Be^{-jsx}\right) \\ \text{which can also be written as} &= e^{rx}\left(C\cos(sx) + D\sin(sx)\right) \end{aligned}$$

As before, the constants A and B (or C and D) will be defined by the boundary conditions.

Example

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 9y = 0$$

The characteristic equation is: $r^2 + 4r + 9 = 0$ and the roots are $\dfrac{-4 \pm \sqrt{16 - 4 \times 9}}{2} = -2 + \sqrt{5}j$ or $-2 - \sqrt{5}j$. Therefore the solutions of the ODE are:

$$\begin{aligned} y(x) &= e^{-2x}\left(Ae^{\sqrt{5}xj} + Be^{-\sqrt{5}xj}\right) \\ \text{or} &= e^{-2x}\left(C\cos(\sqrt{5}x) + D\sin(\sqrt{5}x)\right) \end{aligned}$$

Case 3: equal roots $r_1 = r_2 = r$

If the characteristic equation has one root only then the solutions of the homogeneous equation are of the form:

$$y(x) = Ae^{rx} + Bxe^{rx}$$

Example

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = 0$$

The characteristic equation is: $r^2 + 4r + 4 = 0$ e.g. $(r + 2)^2 = 0$ and its root is -2. Therefore the solutions of the ODE are:

$$y(x) = Ae^{-2x} + Bxe^{-2x}$$

Second Order ODEs with Right-Hand Side

If the right-hand side in Equation (1) is not 0, then the solutions can be found as follows:

- First, find the form of the solution of the corresponding homogeneous equation **keeping the constants** A **and** B **as such**: this is called the complementary solution $y_c(x)$;
- Second, find a particular integral of the ODE $y_p(x)$.

Then the solutions of the ODE are of the form: $y(x) = y_c(x) + y_p(x)$. **At this point only**, you may determine the constants A and B from the boundary conditions.

There are two methods to find a particular integral of the ODE: the method of undetermined coefficients and the method of variation of parameters.

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Undetermined coefficients

This method consists in making an educated guess as to what the particular integral should look like. The following table can be used:

f(x)	**particular integral**
k	C
kx	$Cx + D$
kx^2	$Cx^2 + Dx + E$
$k\sin x$ or $k\cos x$	$C\cos x + D\sin x$
$k\sinh x$ or $k\cosh x$	$C\cosh x + D\sinh x$
e^{kx}	Ce^{kx}
e^{rx}, where r is a root of the characteristic equation	Cxe^{rx} or Cx^2e^{rx}

The constants C and D are found by 'plugging' the particular integral in the ODE, which will lead to conditions that define C and D.

Example

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 2\sin 4x$$

We first find the complementary solution of the ODE. The characteristic equation is $r^2 - 5r + 6 = 0$ and the roots are $\dfrac{5 \pm \sqrt{25 - 4 \times 6}}{2} = 3$ or 2. Therefore the complementary solution is:

$$y_c(x) = Ae^{3x} + Be^{2x}$$

Then, we find a particular integral of the ODE. Since the right-hand side contains a $\sin 4x$, we look for a particular integral in the form $y_p(x) = C\cos 4x + D\sin 4x$. We want y_p to be solution of the ODE so we must have:

$$\frac{d^2y_p}{dx^2} - 5\frac{dy_p}{dx} + 6y_p = 2\sin 4x$$

We have:

$\dfrac{dy_p}{dx} = -4C\sin 4x + 4D\cos 4x$

$\dfrac{d^2y_p}{dx^2} = -16C\cos 4x - 16D\sin 4x$

Putting back in the ODE:

$(-16C\cos 4x - 16D\sin 4x) - 5\,(-4C\sin 4x + 4D\cos 4x) + 6\,(C\cos 4x + D\sin 4x) = 2\sin 4x$

Re-arranging cos and sin:

$(-16C - 20D + 6C)\cos 4x + (-16D + 20C + 6D)\sin 4x = 2\sin 4x$

$(-10C - 20D)\cos 4x + (-10D + 20C)\sin 4x = 2\sin 4x$

The last equation must be true for any value of x, so we must have:

$$\begin{cases} -10C - 20D = 0 \\ 20C - 10D = 2 \end{cases}$$

So:

$$\begin{cases} C = \frac{2}{25} \\ D = -\frac{1}{25} \end{cases}$$

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So a particular integral of the ODE is $y_p(x) = \frac{2}{25}\cos 4x - \frac{1}{25}\sin 4x$ and the general solutions of the ODE are of the form:

$$y(x) = \frac{2}{25}\cos 4x - \frac{1}{25}\sin 4x + Ae^{3x} + Be^{2x}$$

Variation of parameters

This method is more general and will work for any function $f(x)$ in the right-hand side of Equation (1), although it may look intimidating at first sight! First let's rewrite the complementary solution of the ODE in the form:

$$y_c(x) = Ay_1(x) + By_2(x)$$

with $y_1(x) = e^{r_1 x}$, $y_2(x) = e^{r_2 x}$ or $xe^{r_1 x}$ if $r_1 = r_2$ with r_1, r_2 roots of the characteristic equation

Then a particular integral of Equation (1) is:

$$\begin{aligned} y_p(x) &= -y_1(x)\int \frac{y_2(x)f(x)}{W(y_1,y_2)}dx + y_2(x)\int \frac{y_1(x)f(x)}{W(y_1,y_2)}dx \\ \text{with } W \text{ the Wronskian: } W(y_1,y_2) &= y_1(x)y_2'(x) - y_1'(x)y_2(x) \end{aligned}$$

Example

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = \frac{e^x}{x^2+1}$$

First, let's find the complementary solution of the ODE. The characteristic equation is $r^2-2r+1 = 0$, e.g. $(r-1)^2 = 0$, so there is one root which is 1. The complementary solution is of the form:

$$y_c(x) = Ae^x + Bxe^x$$

To find a particular integral of the ODE, we calculate the Wronskian:

$$\begin{aligned} \text{with: } y_1(x) = e^x \quad \text{and:} &\quad y_2(x) = xe^x \\ W(y_1,y_2) &= y_1(x)y_2'(x) - y_1'(x)y_2(x) \\ &= e^x(1+x)e^x - e^x xe^x = e^{2x} \end{aligned}$$

Then a particular integral of the ODE is:

$$\begin{aligned} y_p(x) &= -e^x\int \frac{xe^x}{e^{2x}}\frac{e^x}{x^2+1}dx + xe^x\int \frac{e^x}{e^{2x}}\frac{e^x}{x^2+1}dx \\ &= -e^x\int \frac{x}{x^2+1}dx + xe^x\int \frac{1}{1+x^2}dx \\ \int \frac{x}{1+x^2}dx = \frac{1}{2}\ln(1+x^2) \quad \text{and} &\quad \int \frac{1}{1+x^2}dx = \arctan x \\ &= -e^x\cdot\frac{1}{2}\ln(1+x^2) + xe^x\cdot\arctan x \end{aligned}$$

The general solution of the ODE is:

$$y(x) = Ae^x + Bxe^x - \frac{1}{2}e^x\ln(1+x^2) + xe^x\arctan x$$

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Exercises

(a) $\dfrac{d^2y}{dx^2} + 7y = 0$

(b) $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + y = e^{-2x}$

(c) $3\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} - y = 2x - 3$

(d) $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + 5y = 2e^{-2x}$ with $y(0) = 1,\ \dfrac{dy}{dx}(0) = -2$

(e) $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + 4y = 2\cos^2 x$

(f) $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + y = 4\sinh x$

Answers

(a) $y = Ae^{i\sqrt{7}x} + Be^{-i\sqrt{7}x}$
or $y = C\cos(\sqrt{7}x) + D\sin(\sqrt{7}x)$

(b) $y = Ae^{-x} + Bxe^{-x} + e^{-2x}$

(c) $y = Ae^{x} + Be^{-1/3x} - 2x + 7$

(d) $y = e^{-2x}(2 - \cos x)$

(e) $y = (A + Bx)e^{-2x} + \dfrac{1}{4} + \dfrac{1}{8}\sin(2x)$

(f) $y = (A + Bx - x^2)e^{-x} + \dfrac{1}{2}e^{x}$

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First-order differential equations

Learning outcomes

When you have completed this Programme you will be able to:

- ☐ Recognize the order of a differential equation
- ☐ Appreciate that a differential equation of order *n* can be derived from a function containing *n* arbitrary constants
- ☐ Solve certain first-order differential equations by direct integration
- ☐ Solve certain first-order differential equations by separating the variables
- ☐ Solve certain first-order homogeneous differential equations by an appropriate substitution
- ☐ Solve certain first-order differential equations by using an integrating factor
- ☐ Solve Bernoulli's equation

Introduction

1

A *differential equation* is a relationship between an independent variable, x, a dependent variable y, and one or more derivatives of y with respect to x.

e.g. $x^2 \dfrac{dy}{dx} = y \sin x = 0$

$$xy\frac{d^2y}{dx^2} + y\frac{dy}{dx} + e^{3x} = 0$$

Differential equations represent dynamic relationships, i.e. quantities that change, and are thus frequently occurring in scientific and engineering problems.

The *order* of a differential equation is given by the highest derivative involved in the equation.

$x\dfrac{dy}{dx} - y^2 = 0$ is an equation of the 1st order

$xy\dfrac{d^2y}{dx^2} - y^2 \sin x = 0$ is an equation of the 2nd order

$\dfrac{d^3y}{dx^3} - y\dfrac{dy}{dx} + e^{4x} = 0$ is an equation of the 3rd order

So that $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + 10y = \sin 2x$ is an equation of the .... order.

2

$\boxed{\text{second}}$

Because

In the equation $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + 10y = \sin 2x$, the highest derivative involved is $\dfrac{d^2y}{dx^2}$.

Similarly:

(a) $x\dfrac{dy}{dx} = y^2 + 1$ is a ....order equation

(b) $\cos^2 x\dfrac{dy}{dx} + y = 1$ is a ....order equation

(c) $\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} + 2y = x^2$ is a ....order equation

(d) $(y^3 + 1)\dfrac{dy}{dx} - xy^2 = x$ is a ....order equation

On to Frame 3

3

(a) first	(b) first	(c) second	(d) first

Next frame

Formation of differential equations

4

Differential equations may be formed in practice from a consideration of the physical problems to which they refer. Mathematically, they can occur when arbitrary constants are eliminated from a given function. Here are a few examples.

Example 1

Consider $y = A\sin x + B\cos x$, where A and B are two arbitrary constants.

If we differentiate, we get:

$$\frac{dy}{dx} = A\cos x - B\sin x$$

and $$\frac{d^2y}{dx^2} = -A\sin x - B\cos x$$

which is identical to the original equation, but with the sign changed.

i.e. $\dfrac{d^2y}{dx^2} = -y \qquad \therefore \dfrac{d^2y}{dx^2} + y = 0$

This is a differential equation of the .... order.

5

second

Example 2

Form a differential equation from the function $y = x + \dfrac{A}{x}$.

We have $y = x + \dfrac{A}{x} = x + Ax^{-1}$

$$\therefore \frac{dy}{dx} = 1 - Ax^{-2} = 1 - \frac{A}{x^2}$$

From the given equation, $\dfrac{A}{x} = y - x \quad \therefore A = x(y - x)$

$$\therefore \frac{dy}{dx} = 1 - \frac{x(y-x)}{x^2}$$

$$= 1 - \frac{y-x}{x} = \frac{x - y + x}{x} = \frac{2x - y}{x}$$

$$\therefore x\frac{dy}{dx} = 2x - y$$

This is an equation of the .... order.

6

first

Now one more.

Example 3

Form the differential equation for $y = Ax^2 + Bx$.

We have $$y = Ax^2 + Bx \qquad (1)$$

$$\therefore \frac{dy}{dx} = 2Ax + B \qquad (2)$$

$$\therefore \frac{d^2y}{dx^2} = 2A \qquad (3) \quad A = \frac{1}{2}\frac{d^2y}{dx^2}$$

Substitute for $2A$ in (2): $\dfrac{dy}{dx} = x\dfrac{d^2y}{dx^2} + B$

$$\therefore B = \frac{dy}{dx} - x\frac{d^2y}{dx^2}$$

Substituting for A and B in (1), we have:

$$y = x^2 . \frac{1}{2}\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx} - x\frac{d^2y}{dx^2}\right)$$

$$= \frac{x^2}{2} . \frac{d^2y}{dx^2} + x . \frac{dy}{dx} - x^2 . \frac{d^2y}{dx^2}$$

$$\therefore y = x\frac{dy}{dx} - \frac{x^2}{2} . \frac{d^2y}{dx^2}$$

and this is an equation of the .... order.

7

second

If we collect our last few results together, we have:

$y = A\sin x + B\cos x$ gives the equation $\dfrac{d^2y}{dx^2} + y = 0$ (2nd order)

$y = Ax^2 + Bx$ gives the equation $y = x\dfrac{dy}{dx} - \dfrac{x^2}{2} . \dfrac{d^2y}{dx^2}$ (2nd order)

$y = x + \dfrac{A}{x}$ gives the equation $x\dfrac{dy}{dx} = 2x - y$ (1st order)

If we were to investigate the following, we should also find that:

$y = Axe^x$ gives the differential equation $x\dfrac{dy}{dx} - y(1 + x) = 0$ (1st order)

$y = Ae^{-4x} + Be^{-6x}$ gives the differential equation $\dfrac{d^2y}{dx^2} + 10\dfrac{dy}{dx} + 24y = 0$ (2nd order)

Some of the functions give 1st-order equations: some give 2nd-order equations. Now look at the five results above and see if you can find any distinguishing features in the functions which decide whether we obtain a 1st-order equation or a 2nd-order equation in any particular case.

When you have come to a conclusion, move on to Frame 8

8

A function with 1 arbitrary constant gives a 1st-order equation.
A function with 2 arbitrary constants gives a 2nd-order equation.

Correct, and in the same way:

A function with 3 arbitrary constants would give a 3rd order equation.

So, without working each out in detail, we can say that:

(a) $y = e^{-2x}(A + Bx)$ would give a differential equation of .... order.

(b) $y = A\dfrac{x - 1}{x + 1}$ would give a differential equation of .... order.

(c) $y = e^{3x}(A\cos 3x + B\sin 3x)$ would give a differential equation of .... order.

9

(a) 2nd (b) 1st (c) 2nd

Because

(a) and (c) each have 2 arbitrary constants,

while (b) has only 1 arbitrary constant.

Similarly:

(a) $x^2 \dfrac{dy}{dx} + y = 1$ is derived from a function having .... arbitrary constants.

(b) $\cos^2 x \dfrac{dy}{dx} = 1 - y$ is derived from a function having .... arbitrary constants.

(c) $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + y = e^{2x}$ is derived from a function having .... arbitrary constants.

10

(a) 1 (b) 1 (c) 2

So, from all this, the following rule emerges:

A 1st-order differential equation is derived from a function having 1 arbitrary constant.

A 2nd-order differential equation is derived from a function having 2 arbitrary constants.

An nth-order differential equation is derived from a function having n arbitrary constants.

Copy this last statement into your record book. It is important to remember this rule and we shall make use of it at various times in the future.

Then on to Frame 11

Solution of differential equations

11

To solve a differential equation, we have to find the function for which the equation is true. This means that we have to manipulate the equation so as to eliminate all the derivatives and leave a relationship between y and x. The rest of this particular Programme is devoted to the various methods of solving *first-order differential equations*. Second-order equations will be dealt with in the next Programme.

So, for the first method, move on to Frame 12

12

Method 1: *By direct integration*

If the equation can be arranged in the form $\frac{dy}{dx} = f(x)$, then the equation can be solved by simple integration.

Example 1

$$\frac{dy}{dx} = 3x^2 - 6x + 5$$

$$\text{Then} \quad y = \int (3x^2 - 6x + 5)dx = x^3 - 3x^2 + 5x + C$$

$$\text{i.e.} \quad y = x^3 - 3x^2 + 5x + C$$

As always, of course, the constant of integration must be included. Here it provides the one arbitrary constant which we always get when solving a first-order differential equation.

Example 2

$$\text{Solve} \quad x\frac{dy}{dx} = 5x^3 + 4$$

In this case, $\frac{dy}{dx} = 5x^2 + \frac{4}{x}$ So, $y = \dots\dots\dots\dots$

13

$$\boxed{y = \frac{5x^3}{3} + 4\ln x + C}$$

As you already know from your work on integration, the value of C cannot be determined unless further information about the function is given. In this present form, the function is called the *general solution* (or *primitive*) of the given equation.

If we are told the value of y for a given value of x, C can be evaluated and the result is then a *particular solution* of the equation.

Example 3

Find the particular solution of the equation $e^x \frac{dy}{dx} = 4$, given that $y = 3$ when $x = 0$.

First rewrite the equation in the form $\frac{dy}{dx} = \frac{4}{e^x} = 4e^{-x}$.

$$\text{Then} \quad y = \int 4e^{-x}\, dx = -4e^{-x} + C$$

Knowing that when $x = 0$, $y = 3$, we can evaluate C in this case, so that the required particular solution is $y = \dots\dots\dots\dots$

14

$$\boxed{y = -4e^{-x} + 7}$$

Method 2: *By separating the variables*

If the given equation is of the form $\frac{dy}{dx} = f(x, y)$, the variable y on the right-hand side prevents solving by direct integration. We therefore have to devise some other method of solution.

Let us consider equations of the form $\frac{dy}{dx} = f(x).F(y)$ and of the form $\frac{dy}{dx} = \frac{f(x)}{F(y)}$, i.e. equations in which the right-hand side can be expressed as products or quotients of functions of x or of y.

A few examples will show how we proceed.

Example 1

Solve $\frac{dy}{dx} = \frac{2x}{y+1}$

We can rewrite this as $(y+1)\frac{dy}{dx} = 2x$

Now integrate both sides with respect to x:

$$\int (y+1)\frac{dy}{dx}\,dx = \int 2x\,dx \quad \text{i.e.}$$

$$\int (y+1)\,dy = \int 2x\,dx$$

and this gives $\frac{y^2}{2} + y = x^2 + C$

15

Example 2

Solve $\frac{dy}{dx} = (1+x)(1+y)$

$$\frac{1}{1+y}\frac{dy}{dx} = 1 + x$$

Integrate both sides with respect to x:

$$\int \frac{1}{1+y}\frac{dy}{dx}\,dx = \int (1+x)\,dx \qquad \therefore \int \frac{1}{1+y}\,dy = \int (1+x)\,dx$$

$$\ln(1+y) = x + \frac{x^2}{2} + C$$

The method depends on our being able to express the given equation in the form $F(y).\frac{dy}{dx} = f(x)$. If this can be done, the rest is then easy, for we have

$$\int F(y).\frac{dy}{dx}\,dx = \int f(x)dx \qquad \therefore \int F(y)\,dy = \int f(x)\,dx$$

and we then continue as in the examples.

Let us see another example, so move on to Frame 16

16

Example 3

Solve $\frac{dy}{dx} = \frac{1+y}{2+x}$ (1)

This can be written as $\frac{1}{1+y}\frac{dy}{dx} = \frac{1}{2+x}$

Integrate both sides with respect to x:

$$\int \frac{1}{1+y}\frac{dy}{dx}\,dx = \int \frac{1}{2+x}\,dx$$

$$\therefore \int \frac{1}{1+y}\,dy = \int \frac{1}{2+x}\,dx \qquad (2)$$

$$\therefore \ln(1+y) = \ln(2+x) + C$$

It is convenient to write the constant C as the logarithm of some other constant A:

$$\ln(1+y) = \ln(2+x) + \ln A = \ln A(2+x)$$
$$\therefore\ 1+y = A(2+x)$$

Note: We can, in practice, get from the given equation (1) to the form of the equation in (2) by a simple routine, thus:

$$\frac{dy}{dx} = \frac{1+y}{2+x}$$

First multiply across by dx:

$$dy = \frac{1+y}{2+x}dx$$

Now collect the 'y-factor' with the dy on the left, i.e. divide by $(1+y)$:

$$\frac{1}{1+y}\,dy = \frac{1}{2+x}\,dx$$

Finally, add the integral signs:

$$\int \frac{1}{1+y}\,dy = \int \frac{1}{2+x}\,dx$$

and then continue as before.

This is purely a routine which enables us to sort out the equation algebraically, the whole of the work being done in one line. Notice, however, that the RHS of the given equation must be expressed as 'x-factors' and 'y-factors'.

Now for another example, using this routine.

Example 4

Solve $\quad \dfrac{dy}{dx} = \dfrac{y^2 + xy^2}{x^2y - x^2}$

First express the RHS in 'x-factors' and 'y-factors':

$$\frac{dy}{dx} = \frac{y^2(1+x)}{x^2(y-1)}$$

Now rearrange the equation so that we have the 'y-factors' and dy on the LHS and the 'x-factors' and dx on the RHS.

So we get . . . .

17

$$\boxed{\frac{y-1}{y^2}\,dy = \frac{1+x}{x^2}\,dx}$$

We now add the integral signs:

$$\int \frac{y-1}{y^2}\,dy = \int \frac{1+x}{x^2}\,dx$$

and complete the solution:

$$\int\left\{\frac{1}{y} - y^{-2}\right\}dy = \int\left\{x^{-2} + \frac{1}{x}\right\}dx$$
$$\therefore\ \ln y + y^{-1} = \ln x - x^{-1} + C$$
$$\therefore\ \ln y + \frac{1}{y} = \ln x - \frac{1}{x} + C$$

Here is another.

Example 5

Solve

$$\frac{dy}{dx} = \frac{y^2 - 1}{x}$$

$$dy = \frac{y^2 - 1}{x} dx$$

Rearranging, we have

$$\frac{1}{y^2 - 1} dy = \frac{1}{x} dx$$

$$\int \frac{1}{y^2 - 1} dy = \int \frac{1}{x} dx$$

which gives ....

18

$$\boxed{\frac{1}{2} \ln \frac{y-1}{y+1} = \ln x + C}$$

$$\therefore \ \ln \frac{y-1}{y+1} = 2 \ln x + \ln A$$

$$\therefore \ \frac{y-1}{y+1} = Ax^2$$

$$y - 1 = Ax^2(y+1)$$

You see they are all done in the same way. Now here is one for you to do:

Example 6

Solve $\quad xy \dfrac{dy}{dx} = \dfrac{x^2 + 1}{y + 1}$

First of all, rearrange the equation into the form:

$$F(y) dy = f(x) dx$$

i.e. arrange the 'y-factors' and dy on the LHS and the 'x-factors' and dx on the RHS.

What do you get?

19

$$\boxed{y(y+1) dy = \frac{x^2 + 1}{x} dx}$$

Because

$$xy \frac{dy}{dx} = \frac{x^2 + 1}{y + 1} \qquad \therefore \ xy\, dy = \frac{x^2 + 1}{y + 1} dx \qquad \therefore \ y(y+1)\, dy = \frac{x^2 + 1}{x} dx$$

So we now have

$$\int (y^2 + y)\, dy = \int \left(x + \frac{1}{x}\right) dx$$

Now finish it off, then move on to the next frame.

20

$$\boxed{\frac{y^3}{3}+\frac{y^2}{2}=\frac{x^2}{2}+\ln x+C}$$

Provided that the RHS of the equation $\frac{dy}{dx}=f(x,y)$ can be separated into 'x-factors' and 'y-factors', the equation can be solved by the method of *separating the variables*. Now do this one entirely on your own.

Example 7

Solve $\quad x\frac{dy}{dx}=y+xy$

When you have finished it completely, move on to Frame 21 and check your solution

21

Here is the result. Follow it through carefully, even if your own answer is correct.

$$x\frac{dy}{dx}=y+xy \qquad \therefore\ x\frac{dy}{dx}=y(1+x)$$

$$x\,dy=y(1+x)\,dx \qquad \therefore\ \frac{dy}{y}=\frac{1+x}{x}\,dx$$

$$\therefore\ \int\frac{1}{y}\,dy=\int\left(\frac{1}{x}+1\right)dx$$

$$\therefore\ \ln y=\ln x+x+C$$

At this stage we have eliminated the derivatives and so we have solved the equation. However, we can express the result in a neater form, thus:

$$\ln y-\ln x=x+C$$

$$\therefore\ \ln\left\{\frac{y}{x}\right\}=x+C$$

$$\therefore\ \frac{y}{x}=e^{x+c}=e^x.e^c \quad \text{Now } e^c \text{ is a constant; call it } A.$$

$$\therefore\ \frac{y}{x}=Ae^x \qquad \therefore\ y=Ax\,e^x$$

Next frame

22

This final example looks more complicated, but it is solved in just the same way. We go through the same steps as before. Here it is.

Example 8

Solve $\quad y\tan x\frac{dy}{dx}=(4+y^2)\sec^2 x$

First separate the variables, i.e. arrange the 'y-factors' and dy on one side and the 'x-factors' and dx on the other.

So we get . . . .

23

$$\boxed{\frac{y}{4+y^2}\,dy=\frac{\sec^2 x}{\tan x}\,dx}$$

Adding the integral signs, we get:

$$\int\frac{y}{4+y^2}\,dy=\int\frac{\sec^2 x}{\tan x}\,dx$$

Now determine the integrals, so that we have . . . .

24

$$\boxed{\frac{1}{2}\ln(4+y^2) = \ln\tan x + C}$$

This result can now be simplified into:

$\ln(4+y^2) = 2\ln\tan x + \ln A$ (expressing the constant $2C$ as $\ln A$)

$\therefore\ 4+y^2 = A\tan^2 x$

$\therefore\ y^2 = A\tan^2 x - 4$

So there we are. Provided we can factorize the equation in the way we have indicated, solution by separating the variables is not at all difficult. So now for a short review exercise to wind up this part of the Programme.

Move on now to Frame 25

Review exercise

25 Work all the exercise before checking your results.

Find the general solutions of the following equations:

1 $\dfrac{dy}{dx} = \dfrac{y}{x}$

2 $\dfrac{dy}{dx} = (y+2)(x+1)$

3 $\cos^2 x\dfrac{dy}{dx} = y+3$

4 $\dfrac{dy}{dx} = xy - y$

5 $\dfrac{\sin x}{1+y}\cdot\dfrac{dy}{dx} = \cos x$

When you have finished them all, move to Frame 26 and check your answers with the solutions given there

26

1 $\dfrac{dy}{dx} = \dfrac{y}{x}\quad\therefore\ \displaystyle\int\frac{1}{y}\,dy = \int\frac{1}{x}\,dx$

$\therefore\ \ln y = \ln x + C$

$= \ln x + \ln A$

$\therefore\ y = Ax$

2 $\dfrac{dy}{dx} = (y+2)(x+1)$

$\therefore\ \displaystyle\int\frac{1}{y+2}\,dy = \int(x+1)\,dx$

$\therefore\ \ln(y+2) = \dfrac{x^2}{2} + x + C$

3 $\cos^2 x\dfrac{dy}{dx} = y+3$

$\therefore\ \displaystyle\int\frac{1}{y+3}\,dy = \int\frac{1}{\cos^2 x}\,dx$

$= \displaystyle\int\sec^2 x\,dx$

$\ln(y+3) = \tan x + C$

4 $\dfrac{dy}{dx} = xy - y \quad \therefore \; \dfrac{dy}{dx} = y(x-1)$

$$\therefore \; \int \frac{1}{y}\,dy = \int (x-1)\,dx$$

$$\therefore \; \ln y = \frac{x^2}{2} - x + C$$

5 $\dfrac{\sin x}{1+y} \cdot \dfrac{dy}{dx} = \cos x$

$$\int \frac{1}{1+y}\,dy = \int \frac{\cos x}{\sin x}\,dx$$

$$\begin{aligned} \therefore \; \ln(1+y) &= \ln\sin x + C \\ &= \ln\sin x + \ln A \\ 1 + y &= A\sin x \\ \therefore \; y &= A\sin x - 1 \end{aligned}$$

If you are quite happy about those, we can start the next part of the Programme, so move on now to Frame 27

27

Method 3: *Homogeneous equations – by substituting y = vx*

Here is an equation:

$$\frac{dy}{dx} = \frac{x+3y}{2x}$$

This looks simple enough, but we find that we cannot express the RHS in the form of 'x-factors' and 'y-factors', so we cannot solve by the method of separating the variables.

In this case we make the substitution $y = vx$, where v is a function of x. So $y = vx$. Differentiate with respect to x (using the product rule):

$$\therefore \; \frac{dy}{dx} = v.1 + x\frac{dv}{dx} = v + x\frac{dv}{dx}$$

Also $\quad \dfrac{x+3y}{2x} = \dfrac{x+3vx}{2x} = \dfrac{1+3v}{2}$

The equation now becomes $v + x\dfrac{dv}{dx} = \dfrac{1+3v}{2}$

$$\begin{aligned} \therefore \; x\frac{dv}{dx} &= \frac{1+3v}{2} - v \\ &= \frac{1+3v-2v}{2} = \frac{1+v}{2} \\ \therefore \; x\frac{dv}{dx} &= \frac{1+v}{2} \end{aligned}$$

The given equation is now expressed in terms of v and x, and in this form we find that we can solve by separating the variables. Here goes:

$$\int \frac{2}{1+v}\,dv = \int \frac{1}{x}\,dx$$

$$\therefore \; 2\ln(1+v) = \ln x + C = \ln x + \ln A$$

$$(1+v)^2 = Ax$$

But $y = vx \quad \therefore \; v = \left\{\dfrac{y}{x}\right\} \quad \therefore \; \left(1 + \dfrac{y}{x}\right)^2 = Ax$

which gives $(x+y)^2 = Ax^3$

Note: $\dfrac{dy}{dx} = \dfrac{x + 3y}{2x}$ is an example of a *homogeneous differential equation*.

This is determined by the fact that the total degree in x and y for each of the terms involved is the same (in this case, of degree 1). The key to solving every homogeneous equation is to substitute $y = vx$ where v is a function of x. This converts the equation into a form which we can solve by separating the variables.

Let us work some examples, so move on to Frame 28

28

Example 1

Solve $\dfrac{dy}{dx} = \dfrac{x^2 + y^2}{xy}$

Here, all terms of the RHS are of degree 2, i.e. the equation is homogeneous. $\therefore$ We substitute $y = vx$ (where v is a function of x)

$$\therefore \frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$\text{and} \quad \frac{x^2 + y^2}{xy} = \frac{x^2 + v^2x^2}{vx^2} = \frac{1 + v^2}{v}$$

The equation now becomes:

$$v + x\frac{dv}{dx} = \frac{1 + v^2}{v}$$

$$\therefore x\frac{dv}{dx} = \frac{1 + v^2}{v} - v$$

$$= \frac{1 + v^2 - v^2}{v} = \frac{1}{v}$$

$$\therefore x\frac{dv}{dx} = \frac{1}{v}$$

Now you can separate the variables and get the result in terms of v and x.

Off you go: when you have finished, move on to Frame 29

29

$$\boxed{\frac{v^2}{2} = \ln x + C}$$

Because

$$\int v\,dv = \int \frac{1}{x}\,dx$$

$$\therefore \frac{v^2}{2} = \ln x + C$$

All that remains now is to express v back in terms of x and y. The substitution we used was $y = vx \quad \therefore v = \dfrac{y}{x}$

$$\therefore \frac{1}{2}\left(\frac{y}{x}\right)^2 = \ln x + C$$

$$y^2 = 2x^2(\ln x + C)$$

Now, what about this one?

Example 2

Solve $\dfrac{dy}{dx} = \dfrac{2xy + 3y^2}{x^2 + 2xy}$

Is this a homogeneous equation? If you think so, what are your reasons?

When you have decided, move on to Frame 30

30

$$\boxed{\text{Yes, because the degree of each term is the same}}$$

Correct. They are all, of course, of degree 2.

So we now make the substitution, $y = \dots\dots\dots\dots$

31

$$\boxed{y = vx, \text{ where } v \text{ is a function of } x}$$

Right. That is the key to the whole process.

$$\frac{dy}{dx} = \frac{2xy + 3y^2}{x^2 + 2xy}$$

So express each side of the equation in terms of v and x:

$$\frac{dy}{dx} = \dots\dots\dots\dots$$

and $$\frac{2xy + 3y^2}{x^2 + 2xy} = \dots\dots\dots\dots$$

When you have finished, move on to the next frame

32

$$\boxed{\begin{aligned} \frac{dy}{dx} &= v + x\frac{dv}{dx} \\ \frac{2xy + 3y^2}{x^2 + 2xy} &= \frac{2vx^2 + 3v^2x^2}{x^2 + 2vx^2} = \frac{2v + 3v^2}{1 + 2v} \end{aligned}}$$

So that $v + x\dfrac{dv}{dx} = \dfrac{2v + 3v^2}{1 + 2v}$

Now take the single v over to the RHS and simplify, giving:

$$x\frac{dv}{dx} = \dots\dots\dots\dots$$

33

$$\boxed{\begin{aligned} x\frac{dv}{dx} &= \frac{2v + 3v^2}{1 + 2v} - v \\ &= \frac{2v + 3v^2 - v - 2v^2}{1 + 2v} \\ x\frac{dv}{dx} &= \frac{v + v^2}{1 + 2v} \end{aligned}}$$

Now you can separate the variables, giving $\dots\dots\dots\dots$

34

$$\boxed{\int \frac{1+2v}{v+v^2}\,\mathrm{d}v = \int \frac{1}{x}\,\mathrm{d}x}$$

Integrating both sides, we can now obtain the solution in terms of v and x. What do you get?

35

$$\boxed{\begin{aligned} &\ln(v+v^2) = \ln x + C = \ln x + \ln A \\ &\therefore\ v+v^2 = Ax \end{aligned}}$$

We have almost finished the solution. All that remains is to express v back in terms of x and y.

Remember the substitution was $y = vx$, so that $v = \dfrac{y}{x}$

So finish it off.

Then move on

36

$$\boxed{xy + y^2 = Ax^3}$$

Because

$$v + v^2 = Ax \text{ and } v = \frac{y}{x} \qquad \therefore\ \frac{y}{x} + \frac{y^2}{x^2} = Ax$$

$$xy + y^2 = Ax^3$$

And that is all there is to it.

Move to Frame 37

37 Here is the solution of the previous equation, all in one piece. Follow it through again.

To solve $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{2xy+3y^2}{x^2+2xy}$

This is homogeneous, all terms of degree 2. Put $y = vx$

$$\therefore\ \frac{\mathrm{d}y}{\mathrm{d}x} = v + x\frac{\mathrm{d}v}{\mathrm{d}x}$$

$$\frac{2xy+3y^2}{x^2+2xy} = \frac{2vx^2+3v^2x^2}{x^2+2vx^2} = \frac{2v+3v^2}{1+2v} \qquad \therefore\ v + x\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{2v+3v^2}{1+2v}$$

$$x\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{2v+3v^2}{1+2v} - v = \frac{2v+3v^2-v-2v^2}{1+2v}$$

$$\therefore\ x\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{v+v^2}{1+2v} \qquad \therefore\ \int \frac{1+2v}{v+v^2}\,\mathrm{d}v = \int \frac{1}{x}\,\mathrm{d}x$$

$$\therefore\ \ln(v+v^2) = \ln x + C = \ln x + \ln A$$

$$v + v^2 = Ax$$

$$\text{But } y = vx \qquad \therefore\ v = \frac{y}{x} \qquad \therefore\ \frac{y}{x} + \frac{y^2}{x^2} = Ax \qquad \therefore\ xy + y^2 = Ax^3$$

Now, in the same way, you do this one. Take your time and be sure that you understand each step.

Example 3

Solve $\quad (x^2+y^2)\dfrac{dy}{dx} = xy$

When you have completely finished it, move to Frame 38 and check your solution

---

Here is the solution in full. **38**

$$(x^2+y^2)\frac{dy}{dx} = xy \qquad \therefore \frac{dy}{dx} = \frac{xy}{x^2+y^2}$$

Put $\quad y = vx \quad \therefore \dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

and $\quad \dfrac{xy}{x^2+y^2} = \dfrac{vx^2}{x^2+v^2x^2} = \dfrac{v}{1+v^2}$

$$\therefore v + x\frac{dv}{dx} = \frac{v}{1+v^2}$$

$$x\frac{dv}{dx} = \frac{v}{1+v^2} - v$$

$$x\frac{dv}{dx} = \frac{v-v-v^3}{1+v^2} = \frac{-v^3}{1+v^2}$$

$$\therefore \int \frac{1+v^2}{v^3}\,dv = -\int \frac{1}{x}\,dx$$

$$\therefore \int \left(v^{-3} + \frac{1}{v}\right)dv = -\ln x + C$$

$$\therefore \frac{-v^{-2}}{2} + \ln v = -\ln x + \ln A$$

$$\ln v + \ln x + \ln K = \frac{1}{2v^2} \qquad (\ln K = -\ln A)$$

$$\ln Kvx = \frac{1}{2v^2}$$

But $\quad v = \dfrac{y}{x} \qquad \therefore \ln Ky = \dfrac{x^2}{2y^2}$

$$2y^2 \ln Ky = x^2$$

This is one form of the solution: there are of course other ways of expressing it.

Now for a short review exercise on this part of the work, move on to Frame 39

Review exercise

Solve the following: **39**

1 $\quad (x-y)\dfrac{dy}{dx} = x + y$

2 $\quad 2x^2\dfrac{dy}{dx} = x^2 + y^2$

3 $\quad (x^2+xy)\dfrac{dy}{dx} = xy - y^2$

When you have finished all three, move on and check your results

40 The solution of equation **1** can be written as:

$$\boxed{\tan^{-1}\left\{\frac{y}{x}\right\} = \ln A + \ln x + \frac{1}{2}\ln\left\{1 + \frac{y^2}{x^2}\right\}}$$

Did you get that? If so, move straight on to Frame 41. If not, check your working with the following.

1 $(x-y)\dfrac{dy}{dx} = x + y \quad \therefore \dfrac{dy}{dx} = \dfrac{x+y}{x-y}$

Put $y = vx \quad \therefore \dfrac{dy}{dx} = v + x\dfrac{dv}{dx} \qquad \dfrac{x+y}{x-y} = \dfrac{1+v}{1-v}$

$$\therefore v + x\frac{dv}{dx} = \frac{1+v}{1-v} \quad \therefore x\frac{dv}{dx} = \frac{1+v}{1-v} - v = \frac{1+v-v+v^2}{1-v} = \frac{1+v^2}{1-v}$$

$$\therefore \int \frac{1-v}{1+v^2}dv = \int \frac{1}{x}dx \quad \therefore \int\left\{\frac{1}{1+v^2} - \frac{v}{1+v^2}\right\}dv = \ln x + C$$

$$\therefore \tan^{-1} v - \frac{1}{2}\ln(1+v^2) = \ln x + \ln A$$

But $v = \dfrac{y}{x} \quad \therefore \tan^{-1}\left\{\dfrac{y}{x}\right\} = \ln A + \ln x + \dfrac{1}{2}\ln\left(1 + \dfrac{y^2}{x^2}\right)$

This result can, in fact, be simplified further.

Now on to Frame 41

41 Equation **2** gives the solution: $\boxed{\dfrac{2x}{x-y} = \ln x + C}$

If you agree, move straight on to Frame 42. Otherwise, follow through the working. Here it is.

2 $2x^2\dfrac{dy}{dx} = x^2 + y^2 \quad \therefore \dfrac{dy}{dx} = \dfrac{x^2+y^2}{2x^2}$

Put $y = vx \quad \therefore \dfrac{dy}{dx} = v + x\dfrac{dv}{dx}; \quad \dfrac{x^2+y^2}{2x^2} = \dfrac{x^2+v^2x^2}{2x^2} = \dfrac{1+v^2}{2}$

$$\therefore v + x\frac{dv}{dx} = \frac{1+v^2}{2} \quad \therefore x\frac{dv}{dx} = \frac{1+v^2}{2} - v = \frac{1-2v+v^2}{2} = \frac{(v-1)^2}{2}$$

$$\therefore \int \frac{2}{(v-1)^2}dv = \int \frac{1}{x}dx \quad \therefore -2\frac{1}{v-1} = \ln x + C$$

But $v = \dfrac{y}{x}$ and $\dfrac{2}{1-v} = \ln x + C \quad \therefore \dfrac{2x}{x-y} = \ln x + C$

On to Frame 42

42

One form of the result for equation **3** is: $\boxed{xy = Ae^{x/y}}$. Follow through the working and check yours.

3 $(x^2 + xy)\dfrac{dy}{dx} = xy - y^2 \quad \therefore \dfrac{dy}{dx} = \dfrac{xy - y^2}{x^2 + xy}$

Put $y = vx \quad \therefore \dfrac{dy}{dx} = v + x\dfrac{dv}{dx}; \quad \dfrac{xy - y^2}{x^2 + xy} = \dfrac{vx^2 - v^2x^2}{x^2 + vx^2} = \dfrac{v - v^2}{1 + v}$

$$\therefore v + x\frac{dv}{dx} = \frac{v - v^2}{1 + v}$$

$$x\frac{dv}{dx} = \frac{v - v^2}{1 + v} - v = \frac{v - v^2 - v - v^2}{1 + v} = \frac{-2v^2}{1 + v}$$

$$\therefore \int \frac{1 + v}{v^2}dv = \int \frac{-2}{x}dx$$

$$\int \left(v^{-2} + \frac{1}{v}\right)dv = -\int \frac{2}{x}dx$$

$$\therefore \ln v - \frac{1}{v} = -2\ln x + C \qquad \text{Let } C = \ln A$$

$$\ln v + 2\ln x = \ln A + \frac{1}{v}$$

$$\ln\left\{\frac{y}{x}.x^2\right\} = \ln A + \frac{x}{y} \quad \therefore xy = Ae^{x/y}$$

Now move to the next frame

Method 4: *Linear equations – use of integrating factor*

43

Consider the equation $\dfrac{dy}{dx} + 5y = e^{2x}$

This is clearly an equation of the first order, but different from those we have dealt with so far. In fact, none of our previous methods could be used to solve this one, so we have to find a further method of attack.

In this case, we begin by multiplying both sides by e^{5x}. This gives

$$e^{5x}\frac{dy}{dx} + y5e^{5x} = e^{2x}.e^{5x} = e^{7x}$$

We now find that the LHS is, in fact, the derivative of $y.e^{5x}$.

$$\therefore \frac{d}{dx}\left\{y.e^{5x}\right\} = e^{7x}$$

Now, of course, the rest is easy. Integrate both sides with respect to x:

$$\therefore y.e^{5x} = \int e^{7x}\,dx = \frac{e^{7x}}{7} + C \quad \therefore y = \dots\dots\dots\dots$$

44

$$\boxed{y = \frac{e^{2x}}{7} + Ce^{-5x}}$$

Did you forget to divide the C by the e^{5x}? It is a common error so watch out for it.

The equation we have just solved is an example of a set of equations of the form $\dfrac{dy}{dx} + Py = Q$, where P and Q are functions of x (or constants). This equation is called a *linear equation of the first order* and to solve any such equation, we multiply both sides by an *integrating factor* which is always $e^{\int P dx}$. This converts the LHS into the derivative of a product.

In our previous example, $\dfrac{dy}{dx} + 5y = e^{2x}$, $P = 5$

$\therefore \int P dx = 5x$ and the integrating factor was therefore e^{5x}.

Note: In determining $\int P\,dx$, we do not include a constant of integration. This omission is purely for convenience, for a constant of integration here would in practice give a constant factor on both sides of the equation, which would subsequently cancel. This is one of the rare occasions when we do not write down the constant of integration.

So: *To solve a differential equation of the form*

$$\frac{dy}{dx} + Py = Q$$

where P and Q are constants or functions of x, multiply both sides by the integrating factor $e^{\int P dx}$

This is important, so copy this rule down into your record book.

Then move on to Frame 45

45

Example 1

To solve $\dfrac{dy}{dx} - y = x$

If we compare this with $\dfrac{dy}{dx} + Py = Q$, we see that in this case

$$P = -1 \text{ and } Q = x.$$

The integrating factor is always $e^{\int P dx}$ and here $P = -1$.

$\therefore \int P\,dx = -x$ and the integrating factor is therefore …………

46

$$\boxed{e^{-x}}$$

We therefore multiply both sides by e^{-x}.

$$\therefore e^{-x}\frac{dy}{dx} - ye^{-x} = xe^{-x}$$

$$\frac{d}{dx}\left\{e^{-x}y\right\} = xe^{-x} \quad \therefore ye^{-x} = \int xe^{-x}\,dx$$

The RHS integral can now be determined by integrating by parts:

$$ye^{-x} = x(-e^{-x}) + \int e^{-x}\,dx = -xe^{-x} - e^{-x} + C$$

$$\therefore y = -x - 1 + Ce^{x}$$

$$\therefore y = Ce^{x} - x - 1$$

The whole method really depends on:

(a) being able to find the integrating factor

(b) being able to deal with the integral that emerges on the RHS.

Let us consider the general case.

47

Consider $\frac{dy}{dx} + Py = Q$ where P and Q are functions of x. Integrating factor, IF $= e^{\int P dx}$

$$\therefore \frac{dy}{dx}.e^{\int P dx} + Pye^{\int P dx} = Qe^{\int P dx}$$

You will now see that the LHS is the derivative of $ye^{\int P dx}$

$$\therefore \frac{d}{dx}\left\{ye^{\int P dx}\right\} = Qe^{\int P dx}$$

Integrate both sides with respect to x:

$$ye^{\int P dx} = \int Qe^{\int P dx}.dx$$

This result looks far more complicated than it really is. If we indicate the integrating factor by IF, this result becomes:

$$y.\text{IF} = \int Q.\text{IF}\, dx$$

and, in fact, we remember it in that way.

So, the solution of an equation of the form

$$\frac{dy}{dx} + Py = Q \text{ (where } P \text{ and } Q \text{ are functions of } x\text{)}$$

is given by $\quad y.\text{IF} = \int Q.\text{IF}\, dx \qquad$ where IF $= e^{\int P dx}$

Copy this into your record book.

Then move to Frame 48

48

So if we have the equation:

$$\frac{dy}{dx} + 3y = \sin x \qquad \left[\frac{dy}{dx} + Py = Q\right]$$

then in this case:

(a) $P = \ldots\ldots\ldots\ldots$ (b) $\int P\, dx = \ldots\ldots\ldots\ldots$ (c) IF $= \ldots\ldots\ldots\ldots$

49

(a) $P = 3$ (b) $\int P\, dx = 3x$ (c) IF $= e^{3x}$

Before we work through any further examples, let us establish a very useful piece of simplification, which we can make good use of when we are finding integrating factors. We want to simplify $e^{\ln F}$, where F is a function of x.

Let $y = e^{\ln F}$

Then, by the very definition of a logarithm, $\ln y = \ln F$

$\therefore y = F \quad \therefore F = e^{\ln F} \quad$ i.e. $e^{\ln F} = F$

This means that $e^{\ln(\text{function})} = $ function. Always!

$$e^{\ln x} = x$$

$$e^{\ln \sin x} = \sin x$$

$$e^{\ln \tanh x} = \tanh x$$

$$e^{\ln(x^2)} = \ldots\ldots\ldots\ldots$$

50

$$\boxed{x^2}$$

Similarly, what about $e^{k \ln F}$? If the log in the index is multiplied by any external coefficient, this coefficient must be taken inside the log as a power.

e.g.
$$e^{2\ln x} = e^{\ln(x^2)} = x^2$$
$$e^{3\ln\sin x} = e^{\ln(\sin^3 x)} = \sin^3 x$$
$$e^{-\ln x} = e^{\ln(x^{-1})} = x^{-1} = \frac{1}{x}$$
and $e^{-2\ln x} = \dots\dots\dots\dots$

51

$$\boxed{\frac{1}{x^2}} \quad \text{because } e^{-2\ln x} = e^{\ln(x^{-2})} = x^{-2} = \frac{1}{x^2}$$

So here is the rule once again: $e^{\ln F} = F$

Make a note of this rule in your record book.

Then on to Frame 52

52 Now let us see how we can apply this result to our working.

Example 2

Solve $x\dfrac{dy}{dx} + y = x^3$

First we divide through by x to reduce the first term to a single $\dfrac{dy}{dx}$

i.e. $\dfrac{dy}{dx} + \dfrac{1}{x}.y = x^2$

Compare with $\left[\dfrac{dy}{dx} + Py = Q\right]$

$\therefore P = \dfrac{1}{x}$ and $Q = x^2$

$\text{IF} = e^{\int P dx} \qquad \displaystyle\int P\,dx = \int \frac{1}{x}\,dx = \ln x$

$\therefore \text{IF} = e^{\ln x} = x$

$\therefore \text{IF} = x$

The solution is $y.\text{IF} = \displaystyle\int Q.\text{IF}\,dx$ so

$$yx = \int x^2.x\,dx = \int x^3\,dx = \frac{x^4}{4} + C$$

$$\therefore xy = \frac{x^4}{4} + C$$

Move to Frame 53

53

Example 3

Solve $\dfrac{dy}{dx} + y\cot x = \cos x$

Compare with $\left[\dfrac{dy}{dx} + Py = Q\right] \quad \therefore \begin{cases} P = \cot x \\ Q = \cos x \end{cases}$

$\text{IF} = e^{\int P dx} \qquad \int P\,dx = \int \cot x\,dx = \int \dfrac{\cos x}{\sin x}\,dx = \ln \sin x$

$\therefore \text{IF} = e^{\ln \sin x} = \sin x$

$y.\text{IF} = \int Q.\text{IF}\,dx \quad \therefore y\sin x = \int \sin x \cos x\,dx = \dfrac{\sin^2 x}{2} + C$

$\therefore y = \dfrac{\sin x}{2} + C\ x$

Now here is another.

Example 4

Solve $(x+1)\dfrac{dy}{dx} + y = (x+1)^2$

The first thing is to ....

54

$\boxed{\text{Divide through by } (x+1)}$

Correct, since we must reduce the coefficient of $\dfrac{dy}{dx}$ to 1.

$\therefore \dfrac{dy}{dx} + \dfrac{1}{x+1}.y = x + 1$

Compare with $\dfrac{dy}{dx} + Py = Q$

In this case $P = \dfrac{1}{x+1}$ and $Q = x + 1$

Now determine the integrating factor, which simplifies to

IF = ...

55

$\boxed{\text{IF} = x + 1}$

Because

$\int P\,dx = \int \dfrac{1}{x+1}\,dx = \ln(x+1)$

$\therefore \text{IF} = e^{\ln(x+1)} = (x+1)$

The solution is always $y.\text{IF} = \int Q.\text{IF}\,dx$

and we know that, in this case, IF $= x + 1$ and $Q = x + 1$.

So finish off the solution and then move on to Frame 56

56

$$\boxed{y = \frac{(x+1)^2}{3} + \frac{C}{x+1}}$$

Here is the solution in detail:

$$y.(x+1) = \int (x+1)(x+1)\,\mathrm{d}x$$
$$= \int (x+1)^2\,\mathrm{d}x$$
$$= \frac{(x+1)^3}{3} + C$$
$$\therefore y = \frac{(x+1)^2}{3} + \frac{C}{x+1}$$

Now let us do another one.

Example 5

Solve $x\frac{\mathrm{d}y}{\mathrm{d}x} - 5y = x^7$

In this case, $P = \ldots\ldots\ldots\ldots$ $Q = \ldots\ldots\ldots\ldots$

57

$$\boxed{P = -\frac{5}{x} \quad Q = x^6}$$

Because if

$$x\frac{\mathrm{d}y}{\mathrm{d}x} - 5y = x^7$$
$$\therefore \frac{\mathrm{d}y}{\mathrm{d}x} - \frac{5}{x}.y = x^6$$

Compare with $\left[\frac{\mathrm{d}y}{\mathrm{d}x} + Py = Q\right]$ $\therefore P = -\frac{5}{x};\quad Q = x^6$

So integrating factor, IF $= \ldots\ldots\ldots\ldots$

58

$$\boxed{\mathrm{IF} = x^{-5} = \frac{1}{x^5}}$$

Because

$$\mathrm{IF} = e^{\int P\mathrm{d}x} \qquad \int P\,\mathrm{d}x = -\int \frac{5}{x}\,\mathrm{d}x = -5\ln x$$
$$\therefore \mathrm{IF} = e^{-5\ln x} = e^{\ln(x^{-5})} = x^{-5} = \frac{1}{x^5}$$

So the solution is:

$$y.\frac{1}{x^5} = \int x^6.\frac{1}{x^5}\,\mathrm{d}x$$
$$\frac{y}{x^5} = \int x\,\mathrm{d}x = \frac{x^2}{2} + C \qquad \therefore y = \ldots\ldots\ldots\ldots$$

59

$$\boxed{y = \frac{x^7}{2} + Cx^5}$$

Did you remember to multiply the C by x^5?

Fine. Now you do this one entirely on your own.

Example 6

Solve $\quad (1 - x^2)\dfrac{dy}{dx} - xy = 1.$

When you have finished it, move to Frame 60

60

$$\boxed{y\sqrt{1 - x^2} = \sin^{-1} x + C}$$

Here is the working in detail. Follow it through.

$$(1 - x^2)\frac{dy}{dx} - xy = 1$$

$$\therefore \frac{dy}{dx} - \frac{x}{1 - x^2}.y = \frac{1}{1 - x^2}$$

$$\text{IF} = e^{\int P dx} \qquad \int P\,dx = \int \frac{-x}{1 - x^2}\,dx = \frac{1}{2}\ln(1 - x^2)$$

$$\therefore \text{IF} = e^{\frac{1}{2}\ln(1 - x^2)} = e^{\ln\{(1 - x^2)^{\frac{1}{2}}\}} = (1 - x^2)^{\frac{1}{2}}$$

$$\text{Now } y.\text{IF} = \int \text{Q.IF}\,dx$$

$$\therefore y\sqrt{1 - x^2} = \int \frac{1}{1 - x^2}\sqrt{1 - x^2}.dx$$

$$= \int \frac{1}{\sqrt{1 - x^2}}\,dx = \sin^{-1} x + C$$

$$y\sqrt{1 - x^2} = \sin^{-1} x + C$$

Now on to Frame 61

61

In practically all the examples so far, we have been concerned with finding the general solutions. If further information is available, of course, particular solutions can be obtained. Here is one final example for you to do.

Example 7

Solve the equation

$$(x - 2)\frac{dy}{dx} - y = (x - 2)^3$$

given that $y = 10$ when $x = 4$.

Off you go then. It is quite straightforward.

When you have finished, move on to Frame 62 and check your solution

62

$$\boxed{2y = (x-2)^3 + 6(x-2)}$$

Here it is:

$$(x-2)\frac{dy}{dx} - y = (x-2)^3$$

$$\frac{dy}{dx} - \frac{1}{x-2}.y = (x-2)^2$$

$$\int P\,dx = \int \frac{-1}{x-2}\,dx = -\ln(x-2)$$

$$\therefore \text{ IF} = e^{-\ln(x-2)} = e^{\ln\left\{(x-2)^{-1}\right\}} = (x-2)^{-1}$$

$$= \frac{1}{x-2}$$

$$\therefore y.\frac{1}{x-2} = \int (x-2)^2.\frac{1}{(x-2)}\,dx$$

$$= \int (x-2)\,dx$$

$$= \frac{(x-2)^2}{2} + C$$

$$\therefore y = \frac{(x-2)^3}{2} + C(x-2) \qquad \text{General solution}$$

When $x = 4$, $y = 10$:

$$10 = \frac{8}{2} + C.2 \quad \therefore 2C = 6 \quad \therefore C = 3$$

$$\therefore 2y = (x-2)^3 + 6(x-2)$$

Review exercise

63 Finally, for this part of the Programme, here is a short revision exercise.

Solve the following:

1 $\dfrac{dy}{dx} + 3y = e^{4x}$

2 $x\dfrac{dy}{dx} + y = x\sin x$

3 $\tan x\dfrac{dy}{dx} + y = \sec x$

Work through them all: then check your results with those given in Frame 64

64

1 $y = \dfrac{e^{4x}}{7} + Ce^{-3x}$ $\quad$ $(\text{IF} = e^{3x})$

2 $xy = \sin x - x\cos x + C$ $\quad$ $(\text{IF} = x)$

3 $y\sin x = x + C$ $\quad$ $(\text{IF} = \sin x)$

There is just one other type of equation that we must consider. Here is an example: let us see how it differs from those we have already dealt with.

To solve $\dfrac{dy}{dx}+\dfrac{1}{x}.y=xy^2$

Note that if it were not for the factor y^2 on the right-hand side, this equation would be of the form $\dfrac{dy}{dx}+Py=Q$ that we know of old.

To see how we deal with this new kind of equation, we will consider the general form, so move on to Frame 65.

Bernoulli's equation

65

The differential equation:

$$\frac{dy}{dx}+P(x)y=Q(x)y^n$$

is known as Bernoulli's equation and it is solved as follows:

(a) Divide both sides by y^n. This gives:

$$y^{-n}\frac{dy}{dx}+Py^{1-n}=Q$$

(b) Now put $z=y^{1-n}$

so that, differentiating, $\dfrac{dz}{dx}=$ ....

66

$$\boxed{\frac{dz}{dx}=(1-n)y^{-n}\frac{dy}{dx}}$$

So we have:

$$\frac{dy}{dx}+Py=Qy^n \qquad (1)$$

$$\therefore\ y^{-n}\frac{dy}{dx}+Py^{1-n}=Q \qquad (2)$$

Put $z=y^{1-n}$ so that $\dfrac{dz}{dx}=(1-n)y^{-n}\dfrac{dy}{dx}$

If we now multiply (2) by $(1-n)$ we shall convert the first term into $\dfrac{dz}{dx}$.

$$(1-n)y^{-n}\frac{dy}{dx}+(1-n)Py^{1-n}=(1-n)Q$$

Remembering that $z=y^{1-n}$ and that $\dfrac{dz}{dx}=(1-n)y^{-n}\dfrac{dy}{dx}$, this last line can now be written $\dfrac{dz}{dx}+P_1z=Q_1$ with P_1 and Q_1 functions of x.

This we can solve by use of an integrating factor in the normal way.

Finally, having found z, we convert back to y using $z=y^{1-n}$.

Let us see this routine in operation – so on to Frame 67

67

Example 1

Solve $\frac{dy}{dx} + \frac{1}{x}y = xy^2$

(a) Divide through by y^2, giving ....

68

$$\boxed{y^{-2}\frac{dy}{dx} + \frac{1}{x}.y^{-1} = x}$$

(b) Now put $z = y^{1-n}$, i.e. in this case $z = y^{1-2} = y^{-1}$

$$z = y^{-1} \quad \therefore \frac{dz}{dx} = -y^{-2}\frac{dy}{dx}$$

(c) Multiply through the equation by (-1), to make the first term $\frac{dz}{dx}$.

$$-y^{-2}\frac{dy}{dx} - \frac{1}{x}y^{-1} = -x$$

so that $\frac{dz}{dx} - \frac{1}{x}z = -x$ which is of the form $\frac{dz}{dx} + Pz = Q$ so that you can now solve the equation by the normal integrating factor method. What do you get?

When you have done it, move on to the next frame

69

$$\boxed{y = (Cx - x^2)^{-1}}$$

Check the working:

$$\frac{dz}{dx} - \frac{1}{x}z = -x$$

$$\text{IF} = e^{\int P\,dx} \qquad \int P\,dx = \int -\frac{1}{x}\,dx = -\ln x$$

$$\therefore \text{IF} = e^{-\ln x} = e^{\ln(x^{-1})} = x^{-1} = \frac{1}{x}$$

$$z.\text{IF} = \int Q.\text{IF}\,dx \quad \therefore z\frac{1}{x} = \int -x.\frac{1}{x}\,dx$$

$$\therefore \frac{z}{x} = \int -1\,dx = -x + C$$

$$\therefore z = Cx - x^2$$

But $z = y^{-1} \quad \therefore \frac{1}{y} = Cx - x^2 \quad \therefore y = (Cx - x^2)^{-1}$

Right! Here is another.

Example 2

Solve $x^2y - x^3\frac{dy}{dx} = y^4\cos x$

First of all, we must rewrite this in the form $\frac{dy}{dx} + Py = Qy^n$

So, what do we do?

70

$$\boxed{\text{Divide both sides by } (-x^3)}$$

giving $\dfrac{dy}{dx} - \dfrac{1}{x}.y = -\dfrac{y^4 \cos x}{x^3}$

Now divide by the power of y on the RHS giving . . . .

71

$$\boxed{y^{-4}\frac{dy}{dx} - \frac{1}{x}y^{-3} = -\frac{\cos x}{x^3}}$$

Next we make the substitution $z = y^{1-n}$ which, in this example, is $z = y^{1-4} = y^{-3}$

$\therefore\ z = y^{-3}$ and $\therefore\ \dfrac{dz}{dx} =$ . . . .

72

$$\boxed{\frac{dz}{dx} = -3y^{-4}\frac{dy}{dx}}$$

If we now multiply the equation by (-3) to make the first term into $\dfrac{dz}{dx}$, we have:

$$-3y^{-4}\frac{dy}{dx} + 3\frac{1}{x}.y^{-3} = \frac{3\cos x}{x^3}$$

$$\text{i.e.}\quad \frac{dz}{dx} + \frac{3}{x}z = \frac{3\cos x}{x^3}$$

This you can now solve to find z and so back to y.

Finish it off and then check with the next frame

73

$$\boxed{y^3 = \frac{x^3}{3\sin x + C}}$$

Because

$$\frac{dz}{dx} + \frac{3}{x}.z = \frac{3\cos x}{x^3}$$

$$\text{IF} = e^{\int P\,dx} \qquad \int P\,dx = \int \frac{3}{x}\,dx = 3\ln x$$

$$\therefore\ \text{IF} = e^{3\ln x} = e^{\ln(x^3)} = x^3$$

$$z.\text{IF} = \int Q.\text{IF}\,dx$$

$$\therefore\ zx^3 = \int \frac{3\cos x}{x^3}x^3\,dx = \int 3\cos x\,dx$$

$$\therefore\ zx^3 = 3\sin x + C$$

But, in this example, $z = y^{-3}$

$$\therefore\ \frac{x^3}{y^3} = 3\sin x + C$$

$$\therefore\ y^3 = \frac{x^3}{3\sin x + C}$$

Let us look at the complete solution as a whole, so on to Frame 74

74 Here it is:

To solve $x^2y - x^3\dfrac{dy}{dx} = y^4\cos x$

$$\frac{dy}{dx} - \frac{1}{x}y = -\frac{y^4\cos x}{x^3}$$

$$\therefore\ y^{-4}\frac{dy}{dx} - \frac{1}{x}y^{-3} = -\frac{\cos x}{x^3}$$

Put $z = y^{1-n} = y^{1-4} = y^{-3}$

$$\therefore\ \frac{dz}{dx} = -3y^{-4}\frac{dy}{dx}$$

Equation becomes

$$-3y^{-4}\frac{dy}{dx} + \frac{3}{x}.y^{-3} = \frac{3\cos x}{x^3}$$

i.e. $\dfrac{dz}{dx} + \dfrac{3}{x}.z = \dfrac{3\cos x}{x^3}$

$\text{IF} = e^{\int P\,dx}$ $\qquad \displaystyle\int P\,dx = \int \frac{3}{x}\,dx = 3\ln x$

$$\therefore\ \text{IF} = e^{3\ln x} = e^{\ln(x^3)} = x^3$$

$$\therefore\ zx^3 = \int \frac{3\cos x}{x^3}x^3\,dx$$

$$= \int 3\cos x\,dx$$

$$\therefore\ zx^3 = 3\sin x + C$$

But $\quad z = y^{-3}$

$$\therefore\ \frac{x^3}{y^3} = 3\sin x + C$$

$$\therefore\ y^3 = \frac{x^3}{3\sin x + C}$$

They are all done in the same way. Once you know the trick, the rest is very straightforward.

On to the next frame

75 Here is one for you to do entirely on your own.

Example 3

Solve $\quad 2y - 3\dfrac{dy}{dx} = y^4e^{3x}$

Work through the same steps as before. When you have finished, check your working with the solution in Frame 76.

76

$$\boxed{y^3 = \frac{5e^{2x}}{e^{5x} + A}}$$

Solution in detail:

$$2y - 3\frac{dy}{dx} = y^4 e^{3x}$$

$$\therefore\ \frac{dy}{dx} - \frac{2}{3}y = -\frac{y^4 e^{3x}}{3}$$

$$\therefore\ y^{-4}\frac{dy}{dx} - \frac{2}{3}y^{-3} = -\frac{e^{3x}}{3}$$

Put $z = y^{1-4} = y^{-3}$ $\quad\therefore\ \dfrac{dz}{dx} = -3y^{-4}\dfrac{dy}{dx}$

Multiplying through by (-3), the equation becomes:

$$-3y^{-4}\frac{dy}{dx} + 2y^{-3} = e^{3x}$$

$$\text{i.e.}\quad \frac{dz}{dx} + 2z = e^{3x}$$

$$\text{IF} = e^{\int P\,dx} \qquad \int P\,dx = \int 2\,dx = 2x \quad \therefore\ \text{IF} = e^{2x}$$

$$\therefore\ ze^{2x} = \int e^{3x}e^{2x}dx = \int e^{5x}dx$$

$$= \frac{e^{5x}}{5} + C$$

But $z = y^{-3}$ $\quad\therefore\ \dfrac{e^{2x}}{y^3} = \dfrac{e^{5x} + A}{5}$

$$\therefore\ y^3 = \frac{5e^{2x}}{e^{5x} + A}$$

On to Frame 77

77

Finally, one example for you, just to be sure.

Example 4

Solve $\quad y - 2x\dfrac{dy}{dx} = x(x+1)y^3$

First rewrite the equation in standard form $\dfrac{dy}{dx} + Py = Qy^n$

This gives . . . .

78

$$\boxed{\frac{dy}{dx} - \frac{1}{2x}.y = -\frac{(x+1)y^3}{2}}$$

Now off you go and complete the solution. When you have finished, check with the working in Frame 79.

79

$$\boxed{y^2 = \frac{6x}{2x^3 + 3x^2 + A}}$$

Working:

$$\frac{dy}{dx} - \frac{1}{2x}.y = -\frac{(x+1)y^3}{2}$$

$$\therefore\ y^{-3}\frac{dy}{dx} - \frac{1}{2x}.y^{-2} = -\frac{(x+1)}{2}$$

Put $z = y^{1-3} = y^{-2}$ $\quad\therefore\ \dfrac{dz}{dx} = -2y^{-3}\dfrac{dy}{dx}$

Equation becomes:

$$-2y^{-3}\frac{dy}{dx} + \frac{1}{x}.y^{-2} = x + 1$$

$$\text{i.e.}\quad \frac{dz}{dx} + \frac{1}{x}.z = x + 1$$

$$\text{IF} = e^{\int P dx} \qquad \int P\,dx = \int \frac{1}{x}\,dx = \ln x$$

$$\therefore\ \text{IF} = e^{\ln x} = x$$

$$z.\text{IF} = \int Q.\text{IF}\,dx \qquad \therefore\ zx = \int (x+1)x\,dx$$

$$= \int (x^2 + x)\,dx$$

$$\therefore\ zx = \frac{x^3}{3} + \frac{x^2}{2} + C$$

$$\text{But } z = y^{-2} \qquad \therefore\ \frac{x}{y^2} = \frac{2x^3 + 3x^2 + A}{6} \qquad (A = 6C)$$

$$\therefore\ y^2 = \frac{6x}{2x^3 + 3x^2 + A}$$

There we are. You have now reached the end of this Programme, except for the **Can You?** checklist and the **Test exercise** which follow. Before you tackle them, however, read down the **Review summary** presented in the next frame. It will remind you of the main points that we have covered in this Programme on first-order differential equations.

Move on then to Frame 80

Review summary

80

1 The *order* of a differential equation is given by the highest derivative present. An equation of *order n* is derived from a function containing *n arbitrary constants*.

2 *Solution of first-order differential equations*

(a) By direct integration: $\dfrac{dy}{dx} = f(x)$

gives $y = \int f(x)\,dx$

(b) By separating the variables: $F(y).\dfrac{dy}{dx} = f(x)$

gives $\int F(y)\,dy = \int f(x)\,dx$

(c) Homogeneous equations: Substituting $y = vx$

gives $v + x\dfrac{dv}{dx} = F(v)$

(d) Linear equations: $\dfrac{dy}{dx} + Py = Q$

Integrating factor, $\text{IF} = e^{\int P dx}$

and remember that $e^{\ln F} = F$

gives $y\text{IF} = \int Q.\text{IF}\,dx$

(e) Bernoulli's equation: $\dfrac{dy}{dx} + Py = Qy^n$

Divide by y^n: then put $z = y^{1-n}$

Reduces to type (d) above.

Can You?

Checklist 25 **81**

Check this list before and after you try the end of Programme test.

On a scale of 1 to 5 how confident are you that you can: Frames

- Recognize the order of a differential equation? 1 to 3

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Appreciate that a differential equation of order n can be derived from a function containing n arbitrary constants? 4 to 10

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Solve certain first-order differential equations by direct integration? 11 to 13

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Solve certain first-order differential equations by separating the variables? 14 to 26

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Solve certain first-order homogeneous differential equations by an appropriate substitution? 27 to 42

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Solve certain first-order differential equations by using an integrating factor? 43 to 64

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Solve Bernoulli's equation 65 to 79

  Yes ☐ ☐ ☐ ☐ ☐ *No*

Test exercise 25

82

The questions are similar to the equations you have been solving in the Programme. They cover all the methods, but are quite straightforward. Do not hurry: take your time and work carefully and you will find no difficulty with them.

Solve the following differential equations:

1 $x\frac{dy}{dx} = x^2 + 2x - 3$

2 $(1+x)^2\frac{dy}{dx} = 1 + y^2$

3 $\frac{dy}{dx} + 2y = e^{3x}$

4 $x\frac{dy}{dx} - y = x^2$

5 $x^2\frac{dy}{dx} = x^3 \sin 3x + 4$

6 $x\cos y\frac{dy}{dx} - \sin y = 0$

7 $(x^3 + xy^2)\frac{dy}{dx} = 2y^3$

8 $(x^2 - 1)\frac{dy}{dx} + 2xy = x$

9 $\frac{dy}{dx} + y\tanh x = 2\sinh x$

10 $x\frac{dy}{dx} - 2y = x^3\cos x$

11 $\frac{dy}{dx} + \frac{y}{x} = y^3$

12 $x\frac{dy}{dx} + 3y = x^2y^2$

Further problems 25

83

Solve the following equations.

I. *Separating the variables*

1 $x(y-3)\frac{dy}{dx} = 4y$

2 $(1+x^3)\frac{dy}{dx} = x^2y$, given that $x = 1$ when $y = 2$.

3 $x^3 + (y+1)^2\frac{dy}{dx} = 0$

4 $\cos y + (1 + e^{-x})\sin y\frac{dy}{dx} = 0$, given that $y = \pi/4$ when $x = 0$.

5 $x^2(y+1) + y^2(x-1)\frac{dy}{dx} = 0$

II. *Homogeneous equations*

6 $(2y - x)\frac{dy}{dx} = 2x + y$, given that $y = 3$ when $x = 2$.

7 $(xy + y^2) + (x^2 - xy)\frac{dy}{dx} = 0$

8 $(x^3 + y^3) = 3xy^2\frac{dy}{dx}$

9 $y - 3x + (4y + 3x)\frac{dy}{dx} = 0$

10 $(x^3 + 3xy^2)\frac{dy}{dx} = y^3 + 3x^2y$

III. *Integrating factor*

11 $x\dfrac{dy}{dx} - y = x^3 + 3x^2 - 2x$

12 $\dfrac{dy}{dx} + y\tan x = \sin x$

13 $x\dfrac{dy}{dx} - y = x^3\cos x$, given that $y = 0$ when $x = \pi$.

14 $(1 + x^2)\dfrac{dy}{dx} + 3xy = 5x$, given that $y = 2$ when $x = 1$.

15 $\dfrac{dy}{dx} + y\cot x = 5e^{\cos x}$, given that $y = -4$ when $x = \pi/2$.

IV. *Transformations*. Make the given substitutions and work in much the same way as for first-order homogeneous equations.

16 $(3x + 3y - 4)\dfrac{dy}{dx} = -(x + y)$ Put $x + y = v$

17 $(y - xy^2) = (x + x^2y)\dfrac{dy}{dx}$ Put $y = \dfrac{v}{x}$

18 $(x - y - 1) + (4y + x - 1)\dfrac{dy}{dx} = 0$ Put $v = x - 1$

19 $(3y - 7x + 7) + (7y - 3x + 3)\dfrac{dy}{dx} = 0$ Put $v = x - 1$

20 $y(xy + 1) + x(1 + xy + x^2y^2)\dfrac{dy}{dx} = 0$ Put $y = \dfrac{v}{x}$

V. *Bernoulli's equation*

21 $\dfrac{dy}{dx} + y = xy^3$

22 $\dfrac{dy}{dx} + y = y^4 e^x$

23 $2\dfrac{dy}{dx} + y = y^3(x - 1)$

24 $\dfrac{dy}{dx} - 2y\tan x = y^2\tan^2 x$

25 $\dfrac{dy}{dx} + y\tan x = y^3\sec^4 x$

VI. *Miscellaneous.* Choose the appropriate method in each case.

26 $(1 - x^2)\dfrac{dy}{dx} = 1 + xy$

27 $xy\dfrac{dy}{dx} - (1 + x)\sqrt{y^2 - 1} = 0$

28 $(x^2 - 2xy + 5y^2) = (x^2 + 2xy + y^2)\dfrac{dy}{dx}$

29 $\dfrac{dy}{dx} - y\cot x = y^2\sec^2 x$, given $y = -1$ when $x = \pi/4$.

30 $y + (x^2 - 4x)\dfrac{dy}{dx} = 0$

VII. *Further examples*

31 Solve the equation $\frac{dy}{dx} - y\tan x = \cos x - 2x\sin x$, given that $y = 0$ when $x = \pi/6$.

32 Find the general solution of the equation $\frac{dy}{dx} = \frac{2xy + y^2}{x^2 + 2xy}$.

33 Find the general solution of $(1 + x^2)\frac{dy}{dx} = x(1 + y^2)$.

34 Solve the equation $x\frac{dy}{dx} + 2y = 3x - 1$, given that $y = 1$ when $x = 2$.

35 Solve $x^2\frac{dy}{dx} = y^2 - xy\frac{dy}{dx}$, given that $y = 1$ when $x = 1$.

36 Solve $\frac{dy}{dx} = e^{3x-2y}$, given that $y = 0$ when $x = 0$.

37 Find the particular solution of $\frac{dy}{dx} + \frac{1}{x}.y = \sin 2x$, such that $y = 2$ when $x = \pi/4$.

38 Find the general solution of $y^2 + x^2\frac{dy}{dx} = xy\frac{dy}{dx}$.

39 Obtain the general solution of the equation $2xy\frac{dy}{dx} = x^2 - y^2$.

40 By substituting $z = x - 2y$, solve the equation $\frac{dy}{dx} = \frac{x - 2y + 1}{2x - 4y}$, given that $y = 1$ when $x = 1$.

41 Find the general solution of $(1 - x^3)\frac{dy}{dx} + x^2y = x^2(1 - x^3)$.

42 Solve $\frac{dy}{dx} + \frac{y}{x} = \sin x$, given $y = 0$ when $x = \pi/2$.

43 Solve $\frac{dy}{dx} + x + xy^2 = 0$, given $y = 0$ when $x = 1$.

44 Determine the general solution of the equation $\frac{dy}{dx} + \left\{\frac{1}{x} - \frac{2x}{1 - x^2}\right\}y = \frac{1}{1 - x^2}$

45 Solve $(1 + x^2)\frac{dy}{dx} + xy = (1 + x^2)^{3/2}$.

46 Solve $x(1 + y^2) - y(1 + x^2)\frac{dy}{dx} = 0$, given $y = 2$ when $x = 0$.

47 Solve $\frac{r\tan\theta}{a^2 - r^2}\cdot\frac{dr}{d\theta} = 1$, given $r = 0$ when $\theta = \pi/4$.

48 Solve $\frac{dy}{dx} + y\cot x = \cos x$, given $y = 0$ when $x = 0$.

49 Use the substitution $y = \frac{v}{x}$, where v is a function of x only, to transform the equation $\frac{dy}{dx} + \frac{y}{x} = xy^2$ into a differential equation in v and x. Hence find y in terms of x.

50 The rate of decay of a radioactive substance is proportional to the amount A remaining at any instant. If $A = A_0$ at $t = 0$, prove that, if the time taken for the amount of the substance to become $\frac{1}{2}A_0$ is T, then $A = A_0 e^{-(t \ln 2)/T}$. Prove also that the time taken for the amount remaining to be reduced to $\frac{1}{20}A_0$ is $4{\cdot}32T$.

Now visit the companion website for this book at www.palgrave.com/stroud for more questions applying this mathematics to science and engineering. **84**

Second-order differential equations

Learning outcomes

When you have completed this Programme you will be able to:

- ☐ Use the auxiliary equation to solve certain second-order homogeneous equations
- ☐ Use the complementary function and the particular integral to solve certain second-order inhomogeneous equations

Homogeneous equations

Many practical problems in engineering give rise to second-order differential equations of the form 1

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=f(x)$$

where a, b and c are constant coefficients and $f(x)$ is a given function of x. By the end of this Programme you will have no difficulty with equations of this type.

Let us first take the case where $f(x)=0$, so that the equation becomes

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0$$

This is called a linear, constant coefficient, second-order **homogeneous** differential equation. We shall now look at the solutions to this equation. Let $y=u$ and $y=v$ (where u and v are functions of x) be two solutions of the equation:

$$\therefore\ a\frac{d^2u}{dx^2}+b\frac{du}{dx}+cu=0 \quad\text{and}\quad a\frac{d^2v}{dx^2}+b\frac{dv}{dx}+cv=0$$

Adding these two lines together, we get:

$$a\left(\frac{d^2u}{dx^2}+\frac{d^2v}{dx^2}\right)+b\left(\frac{du}{dx}+\frac{dv}{dx}\right)+c(u+v)=0$$

Now $\dfrac{d}{dx}(u+v)=\dfrac{du}{dx}+\dfrac{dv}{dx}$ and $\dfrac{d^2}{dx^2}(u+v)=\dfrac{d^2u}{dx^2}+\dfrac{d^2v}{dx^2}$, therefore the equation can be written

$$a\frac{d^2}{dx^2}(u+v)+b\frac{d}{dx}(u+v)+c(u+v)=0$$

which is our original equation with y replaced by $(u+v)$.

i.e. If $y=u$ and $y=v$ are solutions of the equation

$a\dfrac{d^2y}{dx^2}+b\dfrac{dy}{dx}+cy=0$, so also is $y=u+v$.

This is an important result and we shall be referring to it later, so make a note of it in your record book.

Move on to Frame 2

Our equation was $a\dfrac{d^2y}{dx^2}+b\dfrac{dy}{dx}+cy=0$. If $a=0$, we get the first-order equation 2
of the same family:

$$b\frac{dy}{dx}+cy=0 \quad\text{i.e.}\quad \frac{dy}{dx}+ky=0 \quad\text{where } k=\frac{c}{b}$$

Solving this by the method of separating variables, we have

$$\frac{dy}{dx}=-ky \quad\therefore\quad \int\frac{dy}{y}=-\int k\,dx$$

which gives . . . .

3

$$\boxed{\ln y = -kx + c}$$

$\therefore\ y = e^{-kx+c} = e^{-kx}.e^{c} = Ae^{-kx}$ (since e^c is a constant)

i.e. $y = Ae^{-kx}$

If we write the symbol m for $-k$, the solution is $y = Ae^{mx}$

In the same way, $y = Ae^{mx}$ will be a solution of the second-order equation $a\dfrac{d^2y}{dx^2} + b\dfrac{dy}{dx} + cy = 0$, if it satisfies this equation.

Now, if $\quad y = Ae^{mx}$

$$\frac{dy}{dx} = Ame^{mx}$$

$$\frac{d^2y}{dx^2} = Am^2e^{mx}$$

and substituting these expressions for the differential coefficients in the left-hand side of the equation, we get ....

On to Frame 4

4

$$\boxed{aAm^2e^{mx} + bAme^{mx} + cAe^{mx} = 0}$$

Right. So dividing both sides by Ae^{mx} we obtain

$am^2 + bm + c = 0$

which is a quadratic equation giving two values for m. Let us call these

$m = m_1$ and $m = m_2$

i.e. $y = Ae^{m_1x}$ and $y = Be^{m_2x}$ are two solutions of the given equation.

Now we have already seen that if $y = u$ and $y = v$ are two solutions so also is $y = u + v$.

$\therefore$ If $y = Ae^{m_1x}$ and $y = Be^{m_2x}$ are solutions so also is

$y = Ae^{m_1x} + Be^{m_2x}$

Note: This contains the necessary two arbitrary constants for a second-order differential equation, so there can be no further solution.

Move to Frame 5

5

The solution, then, of $a\dfrac{d^2y}{dx^2} + b\dfrac{dy}{dx} + cy = 0$ is seen to be

$y = Ae^{m_1x} + Be^{m_2x}$

where A and B are two arbitrary constants and m_1 and m_2 are the roots of the quadratic equation $am^2 + bm + c = 0$.

This quadratic equation is called the *auxiliary equation* and is obtained directly from the equation $a\dfrac{d^2y}{dx^2} + b\dfrac{dy}{dx} + cy = 0$, by writing m^2 for $\dfrac{d^2y}{dx^2}$, m for $\dfrac{dy}{dx}$, 1 for y.

Example

For the equation $2\dfrac{d^2y}{dx^2} + 5\dfrac{dy}{dx} + 6y = 0$, the auxiliary equation is $2m^2 + 5m + 6 = 0$.

In the same way, for the equation $\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} + 2y = 0$, the auxiliary equation is

. . . .

Then on to Frame 6

6

$$\boxed{m^2 + 3m + 2 = 0}$$

Since the auxiliary equation is always a quadratic equation, the values of m can be determined in the usual way.

i.e. if $m^2 + 3m + 2 = 0$

$(m+1)(m+2) = 0 \qquad \therefore\ m = -1$ and $m = -2$

$\therefore$ the solution of $\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} + 2y = 0$ is $y = Ae^{-x} + Be^{-2x}$

In the same way, if the auxiliary equation were $m^2 + 4m - 5 = 0$, this factorizes into $(m+5)(m-1) = 0$ giving $m = 1$ or -5, and in this case the solution would be

. . . .

7

$$\boxed{y = Ae^{x} + Be^{-5x}}$$

The type of solution we get depends on the roots of the auxiliary equation.

1 *Real and different roots* to the auxiliary equation

Example 1

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 0$$

Auxiliary equation: $m^2 + 5m + 6 = 0$

$\therefore\ (m+2)(m+3) = 0 \quad \therefore\ m = -2$ or $m = -3$

$\therefore$ Solution is $y = Ae^{-2x} + Be^{-3x}$

Example 2

$$\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 12y = 0$$

Auxiliary equation: $m^2 - 7m + 12 = 0$

$(m-3)(m-4) = 0 \quad \therefore\ m = 3$ or $m = 4$

So the solution is . . . .

Move to Frame 8

8

$$\boxed{y = Ae^{3x} + Be^{4x}}$$

Here you are. Do this one.

Solve the equation $\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} - 10y = 0$

When you have finished, move on to Frame 9

9

$$\boxed{y = Ae^{2x} + Be^{-5x}}$$

Now consider the next case.

2 *Real and equal roots* to the auxiliary equation

Let us take $\dfrac{d^2y}{dx^2} + 6\dfrac{dy}{dx} + 9y = 0.$

The auxiliary equation is: $m^2 + 6m + 9 = 0$

$\therefore\ (m+3)(m+3) = 0 \quad \therefore\ m = -3$ (twice)

If $m_1 = -3$ and $m_2 = -3$ then these would give the solution $y = Ae^{-3x} + Be^{-3x}$ and their two terms would combine to give $y = Ce^{-3x}$. But every second-order differential equation has two arbitrary constants, so there must be another term containing a second constant. In fact, it can be shown that $y = Kxe^{-3x}$ also satisfies the equation, so that the complete general solution is of the form $y = Ae^{-3x} + Bxe^{-3x}$

i.e. $y = e^{-3x}(A + Bx)$

In general, if the auxiliary equation has real and equal roots, giving $m = m_1$ twice, the solution of the differential equation is

$$y = e^{m_1 x}(A + Bx)$$

Make a note of this general statement and then move on to Frame 10

10 Here is an example:

Example 1

Solve $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + 4y = 0$

Auxiliary equation: $m^2 + 4m + 4 = 0$

$(m+2)(m+2) = 0 \quad \therefore\ m = -2$ (twice)

The solution is: $y = e^{-2x}(A + Bx)$

Here is another:

Example 2

Solve $\dfrac{d^2y}{dx^2} + 10\dfrac{dy}{dx} + 25y = 0$

Auxiliary equation: $m^2 + 10m + 25 = 0$

$(m+5)^2 = 0 \quad \therefore\ m = -5$ (twice)

$y = e^{-5x}(A + Bx)$

Example 3

Now here is one for you to do:

Solve $\dfrac{d^2y}{dx^2} + 8\dfrac{dy}{dx} + 16y = 0$

When you have done it, move on to Frame 11

11

$$\boxed{y = e^{-4x}(A + Bx)}$$

Because if $\frac{d^2y}{dx^2} + 8\frac{dy}{dx} + 16y = 0$

the auxiliary equation is

$m^2 + 8m + 16 = 0$

$\therefore\ (m+4)^2 = 0 \quad \therefore\ m = -4$ (twice)

$\therefore\ y = e^{-4x}(A + Bx)$

So, for *real and different roots* $m = m_1$ and $m = m_2$ the solution is

$y = Ae^{m_1x} + Be^{m_2x}$

and for *real and equal roots* $m = m_1$ (twice) the solution is

$y = e^{m_1x}(A + Bx)$

Just find the values of m from the auxiliary equation and then substitute these values in the appropriate form of the result.

Move to Frame 12

12

3 *Complex roots* to the auxiliary equation

Now let us see what we get when the roots of the auxiliary equation are complex.

Suppose $m = \alpha \pm j\beta$, i.e. $m_1 = \alpha + j\beta$ and $m_2 = \alpha - j\beta$. Then the solution would be of the form:

$$\begin{aligned} y &= Ce^{(\alpha+j\beta)x} + De^{(\alpha-j\beta)x} = Ce^{\alpha x}.e^{j\beta x} + De^{\alpha x}.e^{-j\beta x} \\ &= e^{\alpha x}\left\{Ce^{j\beta x} + De^{-j\beta x}\right\} \end{aligned}$$

Now from our previous work on complex numbers, we know that:

$$e^{jx} = \cos x + j\sin x$$

$$e^{-jx} = \cos x - j\sin x$$

and that $\begin{cases} e^{j\beta x} = \cos\beta x + j\sin\beta x \\ e^{-j\beta x} = \cos\beta x - j\sin\beta x \end{cases}$

Our solution above can therefore be written:

$$\begin{aligned} y &= e^{\alpha x}\{C(\cos\beta x + j\sin\beta x) + D(\cos\beta x - j\sin\beta x)\} \\ &= e^{\alpha x}\{(C+D)\cos\beta x + j(C-D)\sin\beta x\} \\ y &= e^{\alpha x}\{A\cos\beta x + B\sin\beta x\} \end{aligned}$$

where $A = C + D$ and $B = j(C - D)$

$\therefore$ If $m = \alpha \pm j\beta$, the solution can be written in the form:

$$y = e^{\alpha x}\left\{A\cos\beta x + B\sin\beta x\right\}$$

Here is an example: If $m = -2 \pm j3$

then $y = e^{-2x}\left\{A\cos 3x + B\sin 3x\right\}$

Similarly, if $m = 5 \pm j2$ then $y = \dots\dots\dots\dots$

13

$$\boxed{y = e^{5x}[A\cos 2x + B\sin 2x]}$$

Here is one of the same kind:

Solve $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + 9y = 0$

Auxiliary equation: $m^2 + 4m + 9 = 0$

$$\therefore m = \frac{-4 \pm \sqrt{16-36}}{2} = \frac{-4 \pm \sqrt{-20}}{2} = \frac{-4 \pm 2j\sqrt{5}}{2} = -2 \pm j\sqrt{5}$$

In this case $\alpha = -2$ and $\beta = \sqrt{5}$

Solution is: $y = e^{-2x}(A\cos\sqrt{5}x + B\sin\sqrt{5}x)$

Now you can solve this one: $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + 10y = 0$

When you have finished it, move on to Frame 14

14

$$\boxed{y = e^{x}(A\cos 3x + B\sin 3x)}$$

Just check your working:

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 10y = 0$$

Auxiliary equation: $m^2 - 2m + 10 = 0$

$$m = \frac{2 \pm \sqrt{4-40}}{2} = \frac{2 \pm \sqrt{-36}}{2} = 1 \pm j3$$

$$y = e^{x}(A\cos 3x + B\sin 3x)$$

Move to Frame 15

15 Here is a *summary* of the work so far.

Equations of the form $a\dfrac{d^2y}{dx^2} + b\dfrac{dy}{dx} + cy = 0$

Auxiliary equation: $am^2 + bm + c = 0$

1 *Roots real and different* $m = m_1$ and $m = m_2$

Solution is $y = Ae^{m_1x} + Be^{m_2x}$

2 *Real and equal roots* $m = m_1$ (twice)

Solution is $y = e^{m_1x}(A + Bx)$

3 *Complex roots* $m = \alpha \pm j\beta$

Solution is $y = e^{\alpha x}(A\cos\beta x + B\sin\beta x)$

In each case, we simply solve the auxiliary equation to establish the values of m and substitute in the appropriate form of the result.

On to Frame 16

16

We shall now consider equations of the form $\frac{d^2y}{dx^2} \pm n^2y = 0$

This is a special case of the equation

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0 \text{ when } b = 0$$

$$\text{i.e.} \quad a\frac{d^2y}{dx^2} + cy = 0 \quad \text{i.e.} \quad \frac{d^2y}{dx^2} + \frac{c}{a}y = 0$$

which can be written as $\frac{d^2y}{dx^2} \pm n^2y = 0$ to cover the two cases when the coefficient of y is positive or negative.

(a) If $\frac{d^2y}{dx^2} + n^2y = 0$, $\quad m^2 + n^2 = 0 \quad \therefore m^2 = -n^2 \quad \therefore m = \pm jn$

(This is like $m = \alpha \pm j\beta$, when $\alpha = 0$ and $\beta = n$)

$\therefore y = A\cos nx + B\sin nx$

(b) If $\frac{d^2y}{dx^2} - n^2y = 0$, $\quad m^2 - n^2 = 0 \quad \therefore m^2 = n^2 \quad \therefore m = \pm n$

$\therefore y = Ce^{nx} + De^{-nx}$

This last result can be written in another form which is sometimes more convenient, so move on to the next frame and we will see what it is.

17

You will remember from your work on hyperbolic functions that

$$\cosh nx = \frac{e^{nx} + e^{-nx}}{2} \qquad \therefore e^{nx} + e^{-nx} = 2\cosh nx$$

$$\sinh nx = \frac{e^{nx} - e^{-nx}}{2} \qquad \therefore e^{nx} - e^{-nx} = 2\sinh nx$$

Adding these two results: $\quad 2e^{nx} = 2\cosh nx + 2\sinh nx$

$\therefore e^{nx} = \cosh nx + \sinh nx$

Similarly, by subtracting: $\quad e^{-nx} = \cosh nx - \sinh nx$

Therefore, the solution of our equation $y = Ce^{nx}$ can be written:

$$\begin{aligned} y &= C(\cosh nx + \sinh nx) + D(\cosh nx - \sinh nx) \\ &= (C + D)\cosh nx + (C - D)\sinh nx \end{aligned}$$

i.e. $\quad y = A\cosh nx + B\sinh nx$

Note: In this form the two results are very much alike:

(a) $\frac{d^2y}{dx^2} + n^2y = 0 \qquad y = A\cos nx + B\sin nx$

(b) $\frac{d^2y}{dx^2} - n^2y = 0 \qquad y = A\cosh nx + B\sinh nx$

Make a note of these results in your record book.

Then – next frame

18

Here are some examples:

Example 1

$\frac{d^2y}{dx^2} + 16y = 0 \quad \therefore m^2 = -16 \quad \therefore m = \pm j4$

$\therefore y = A\cos 4x + B\sin 4x$

Example 2

$$\frac{d^2y}{dx^2} - 3y = 0 \quad \therefore\ m^2 = 3 \quad \therefore\ m = \pm\sqrt{3}$$

$$y = A\cosh\sqrt{3}x + B\sinh\sqrt{3}x$$

Similarly

Example 3

$$\frac{d^2y}{dx^2} + 5y = 0$$

$$y = \ldots\ldots\ldots\ldots$$

Then move on to Frame 19

19

$$\boxed{y = A\cos\sqrt{5}x + B\sin\sqrt{5}x}$$

And now this one:

Example 4

$$\frac{d^2y}{dx^2} - 4y = 0 \quad \therefore\ m^2 = 4 \quad \therefore\ m = \pm 2$$

$$y = \ldots\ldots\ldots\ldots$$

20

$$\boxed{y = A\cosh 2x + B\sinh 2x}$$

Now before we go on to the next section of the Programme, here is a Review exercise on what we have covered so far. The questions are set out in the next frame. Work them all before checking your results.

So on you go to Frame 21

Review exercise

21 Solve the following:

1 $\frac{d^2y}{dx^2} - 12\frac{dy}{dx} + 36y = 0$

2 $\frac{d^2y}{dx^2} + 7y = 0$

3 $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 3y = 0$

4 $2\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 3y = 0$

5 $\frac{d^2y}{dx^2} - 9y = 0$

For the answers, move to Frame 22

22

Here are the answers:

1 $y = e^{6x}(A + Bx)$

2 $y = A\cos\sqrt{7}x + B\sin\sqrt{7}x$

3 $y = Ae^{x} + Be^{-3x}$

4 $y = e^{-x}\left(A\cos\dfrac{x}{\sqrt{2}} + B\sin\dfrac{x}{\sqrt{2}}\right)$

5 $y = A\cosh 3x + B\sinh 3x$

By now we are ready for the next section of the Programme, so move on to Frame 23

Inhomogeneous equations

23

So far we have considered equations of the form:

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x) \text{ for the case where } f(x) = 0$$

If $f(x) = 0$, then $am^2 + bm + c = 0$ giving $m = m_1$ and $m = m_2$ and the solution is in general $y = Ae^{m_1x} + Be^{m_2x}$.

The equation $a\dfrac{d^2y}{dx^2} + b\dfrac{dy}{dx} + cy = f(x)$ where $f(x) \neq 0$ is called an **inhomogeneous** equation and the substitution $y = Ae^{m_1x} + Be^{m_2x}$ would make the left-hand side zero. Therefore, there must be a further term in the solution which will make the LHS equal to $f(x)$ and not zero. The complete solution will therefore be of the form

$y = Ae^{m_1x} + Be^{m_2x} + X$, where X is the extra function yet to be found.

$y = Ae^{m_1x} + Be^{m_2x}$ is called the *complementary function* (CF)

$y = X$ (a function of x) is called the *particular integral* (PI)

Note: The complete general solution to the inhomogeneous equation is given by:

general solution = complementary function + particular integral

Our main problem at this stage is how are we to find the particular integral for any given equation? This is what we are now going to deal with.

So on then to Frame 24

24

To solve an equation $a\dfrac{d^2y}{dx^2} + b\dfrac{dy}{dx} + cy = f(x)$

(I) The *complementary function* is obtained by solving the equation with $f(x) = 0$, as in the previous part of this Programme. This will give one of the following types of solution:

(a) $y = Ae^{m_1x} + Be^{m_2x}$

(b) $y = e^{m_1x}(A + Bx)$

(c) $y = e^{\alpha x}(A\cos\beta x + B\sin\beta x)$

(d) $y = A\cos nx + B\sin nx$

(e) $y = A\cosh nx + B\sinh nx$

(II) The *particular integral* is found by assuming the general form of the function on the right-hand side of the given equation, substituting this in the equation, and equating coefficients. An example will make this clear:

Solve $\dfrac{d^2y}{dx^2} - 5\dfrac{dy}{dx} + 6y = x^2$

(1) *To find the CF* solve LHS = 0, i.e. $m^2 - 5m + 6 = 0$

$\therefore (m-2)(m-3) = 0 \quad \therefore m = 2$ or $m = 3$

$\therefore$ Complementary function is $y = Ae^{2x} + Be^{3x}$ (1)

(2) *To find the PI* we assume the general form of the RHS which is a second-degree function. Let $y = Cx^2 + Dx + E$.

Then $\dfrac{dy}{dx} = 2Cx + D$ and $\dfrac{d^2y}{dx^2} = 2C$

Substituting these in the given equation, we get:

$$2C - 5(2Cx + D) + 6(Cx^2 + Dx + E) = x^2$$
$$2C - 10Cx - 5D + 6Cx^2 + 6Dx + 6E = x^2$$
$$6Cx^2 + (6D - 10C)x + (2C - 5D + 6E) = x^2$$

Equating coefficients of powers of x, we have:

$[x^2]$ $6C = 1$ $\qquad\therefore C = \dfrac{1}{6}$

$[x]$ $6D - 10C = 0$ $\qquad\therefore 6D = \dfrac{10}{6} = \dfrac{5}{3}$ $\qquad\therefore D = \dfrac{5}{18}$

[CT] $2C - 5D + 6E = 0$ $\qquad\therefore 6E = \dfrac{25}{18} - \dfrac{2}{6} = \dfrac{19}{18}$ $\qquad\therefore E = \dfrac{19}{108}$

$\therefore$ Particular integral is $y = \dfrac{x^2}{6} + \dfrac{5x}{18} + \dfrac{19}{108}$ (2)

Complete general solution = CF + PI

General solution is $y = Ae^{2x} + Be^{3x} + \dfrac{x^2}{6} + \dfrac{5x}{18} + \dfrac{19}{108}$

This frame is quite important, since all equations of this type are solved in this way.

On to Frame 25

25

We have seen that to find the particular integral, we assume the general form of the function on the RHS of the equation and determine the values of the constants by substitution in the whole equation and equating coefficients. These will be useful:

If $f(x) = k \ldots \quad \ldots$	Assume $y = C$
$f(x) = kx \ldots \quad \ldots$	$y = Cx + D$
$f(x) = kx^2 \ldots \quad \ldots$	$y = Cx^2 + Dx + E$
$f(x) = k\sin x$ or $k\cos x$	$y = C\cos x + D\sin x$
$f(x) = k\sinh x$ or $k\cosh x$	$y = C\cosh x + D\sinh x$
$f(x) = e^{kx} \ldots \quad \ldots$	$y = Ce^{kx}$

This list covers all the cases you are likely to meet at this stage.

So if the function on the RHS of the equation is $f(x) = 2x^2 + 5$, you would take as the assumed PI:

$y = \ldots\ldots\ldots\ldots$

26

$$\boxed{y = Cx^2 + Dx + E}$$

Correct, since the assumed PI will be the general form of the second-degree function.

What would you take as the assumed PI in each of the following cases:

1 $f(x) = 2x - 3$

2 $f(x) = e^{5x}$

3 $f(x) = \sin 4x$

4 $f(x) = 3 - 5x^2$

5 $f(x) = 27$

6 $f(x) = 5\cosh 4x$

When you have decided all six, check you answers with those in Frame 27

27

Here are the answers:

1	$f(x) = 2x - 3$	PI is of the form	$y = Cx + D$
2	$f(x) = e^{5x}$		$y = Ce^{5x}$
3	$f(x) = \sin 4x$		$y = C\cos 4x + D\sin 4x$
4	$f(x) = 3 - 5x^2$		$y = Cx^2 + Dx + E$
5	$f(x) = 27$		$y = C$
6	$f(x) = 5\cosh 4x$		$y = C\cosh 4x + D\sinh 4x$

All correct? If you have made a slip with any one of them, be sure that you understand where and why your result was incorrect before moving on.

Next frame

28

Let us work through a few examples. Here is the first:

Example 1

Solve $\dfrac{d^2y}{dx^2} - 5\dfrac{dy}{dx} + 6y = 24$

(1) *CF* Solve LHS = 0 $\quad \therefore m^2 - 5m + 6 = 0$

$\therefore (m-2)(m-3) = 0 \quad \therefore m = 2$ and $m = 3$

$\therefore y = Ae^{2x} + Be^{3x}$ (1)

(2) *PI* $f(x) = 24$, i.e. a constant. Assume $y = C$

Then $\dfrac{dy}{dx} = 0$ and $\dfrac{d^2y}{dx^2} = 0$

Substituting in the given equation:

$0 - 5(0) + 6C = 24 \qquad C = 24/6 = 4$

$\therefore$ PI is $y = 4$ (2)

General solution is $y = \text{CF} + \text{PI}$, i.e. $y = \underbrace{Ae^{2x} + Be^{3x}}_{\text{CF}}\underbrace{+4}_{\text{PI}}$

Now another:

Example 2

Solve $\dfrac{d^2y}{dx^2} - 5\dfrac{dy}{dx} + 6y = 2\sin 4x$

(1) *CF* This will be the same as in the previous example, since the LHS of this equation is the same

i.e. $y = Ae^{2x} + Be^{3x}$

(2) *PI* The general form of the PI in this case will be ....

29

$$\boxed{y = C\cos 4x + D\sin 4x}$$

Note: Although the RHS is $f(x) = 2\sin 4x$, it is necessary to include the full general function $y = C\cos 4x + D\sin 4x$ since, in finding the derivatives, the cosine term will also give rise to $\sin 4x$.

So we have:

$$y = C\cos 4x + D\sin 4x$$

$$\frac{dy}{dx} = -4C\sin 4x + 4D\cos 4x$$

$$\frac{d^2y}{dx^2} = -16C\cos 4x - 16D\sin 4x$$

We now substitute these expressions in the LHS of the equation and by equating coefficients, find the values of C and D.

Away you go then.

Complete the job and then move on to Frame 30

30

$$\boxed{C = \frac{2}{25} \quad D = -\frac{1}{25} \quad y = \frac{1}{25}(2\cos 4x - \sin 4x)}$$

Here is the working:

$$-16C\cos 4x - 16D\sin 4x + 20C\sin 4x - 20D\cos 4x$$
$$+ 6C\cos 4x + 6D\sin 4x = 2\sin 4x$$

$$(20C - 10D)\sin 4x - (10C + 20D)\cos 4x = 2\sin 4x$$

$$\begin{array}{lll} 20C - 10D = 2 & \left.\begin{array}{l} 40C - 20D = 4 \\ 10C + 20D = 0 \end{array}\right\} 50C = 4 & \therefore C = \dfrac{2}{25} \\ 10C + 20D = 0 & & \\ & & D = -\dfrac{1}{25} \end{array}$$

In each case the PI is $y = \dfrac{1}{25}(2\cos 4x - \sin 4x)$

The CF was $y = Ae^{2x} + Be^{3x}$

The general solution is: $y = Ae^{2x} + Be^{3x} + \dfrac{1}{25}(2\cos 4x - \sin 4x)$

31

Here is an example we can work through together:

Example 3

Solve $\dfrac{d^2y}{dx^2} + 14\dfrac{dy}{dx} + 49y = 4e^{5x}$

First we have to find the CF. To do this we solve the equation ....

32

$$\boxed{\frac{d^2y}{dx^2} + 14\frac{dy}{dx} + 49y = 0}$$

Correct. So start off by writing down the auxiliary equation, which is ....

33

$$\boxed{m^2 + 14m + 49 = 0}$$

This gives $(m+7)(m+7) = 0$, i.e. $m = -7$ (twice)

$\therefore$ The CF is $y = e^{-7x}(A + Bx)$ (1)

Now for the PI. To find this we take the general form of the RHS of the given equation, i.e. we assume $y =$ ....

34

$$\boxed{y = Ce^{5x}}$$

Right. So we now differentiate twice which gives us:

$\dfrac{dy}{dx} =$ .... and $\dfrac{d^2y}{dx^2} =$ ....

35

$$\boxed{\frac{dy}{dx} = 5Ce^{5x}, \quad \frac{d^2y}{dx^2} = 25\,Ce^{5x}}$$

The equation now becomes:

$25Ce^{5x} + 14.5Ce^{5x} + 49Ce^{5x} = 4e^{5x}$

Dividing through by e^{5x}: $25C + 70C + 49C = 4$

$$144C = 4 \quad \therefore C = \frac{1}{36}$$

$$\text{The PI is} \quad y = \frac{e^{5x}}{36} \qquad (2)$$

So there we are. The CF is $y = e^{-7x}(A + Bx)$ and the PI is $y = \dfrac{e^{5x}}{36}$

and the complete general solution is therefore ....

36

$$\boxed{y = e^{-7x}(A + Bx) + \frac{e^{5x}}{36}}$$

Correct, because in every case, the general solution is the sum of the complementary function and the particular integral.

Here is another example.

Example 4

Solve $\dfrac{d^2y}{dx^2} + 6\dfrac{dy}{dx} + 10y = 2\sin 2x$

(1) *To find CF* Solve LHS = 0 $\therefore m^2 + 6m + 10 = 0$

$$\therefore m = \frac{-6 \pm \sqrt{36-40}}{2} = \frac{-6 \pm \sqrt{-4}}{2} = -3 \pm j$$

$$y = e^{-3x}(A\cos x + B\sin x) \qquad (1)$$

(2) *To find PI* Assume the general form of the RHS

i.e. $y = \ldots\ldots\ldots\ldots$

On to Frame 37

37

$$\boxed{y = C\cos 2x + D\sin 2x}$$

Do not forget that we have to include the cosine term as well as the sine term, since that will also give $\sin 2x$ when the derivatives are found.

As usual, we now differentiate twice and substitute in the given equation $\dfrac{d^2y}{dx^2} + 6\dfrac{dy}{dx} + 10y = 2\sin 2x$ and equate coefficients of $\sin 2x$ and of $\cos 2x$.

Off you go then. Find the PI on your own.

When you have finished, check your result with that in Frame 38

38

$$\boxed{y = \frac{1}{15}(\sin 2x - 2\cos 2x)}$$

Because if:

$$y = C\cos 2x + D\sin 2x$$

$$\therefore \frac{dy}{dx} = -2C\sin 2x + 2D\cos 2x$$

$$\therefore \frac{d^2y}{dx^2} = -4C\cos 2x - 4D\sin 2x$$

Substituting in the equation gives:

$$-4C\cos 2x - 4D\sin 2x - 12C\sin 2x + 12D\cos 2x$$
$$+ 10C\cos 2x + 10D\sin 2x = 2\sin 2x$$

$$(6C + 12D)\cos 2x + (6D - 12C)\sin 2x = 2\sin 2x$$

$$6C + 12D = 0 \quad \therefore C = -2D$$

$$6D - 12C = 2 \quad \therefore 6D + 24D = 2 \quad \therefore 30D = 2 \quad \therefore D = \frac{1}{15}$$

$$\therefore C = -\frac{2}{15}$$

PI is $y = \dfrac{1}{15}(\sin 2x - 2\cos 2x)$ (2)

So the CF is $y = e^{-3x}(A\cos x + B\sin x)$

and the PI is $y = \dfrac{1}{15}(\sin 2x - 2\cos 2x)$

The complete general solution is therefore $y = \ldots\ldots\ldots\ldots$

39

$$\boxed{y = e^{-3x}(A\cos x + B\sin x) + \frac{1}{15}(\sin 2x - 2\cos 2x)}$$

Before we do another example, list what you would assume for the PI in an equation when the RHS function was:

1 $f(x) = 3\cos 4x$
2 $f(x) = 2e^{7x}$
3 $f(x) = 3\sinh x$
4 $f(x) = 2x^2 - 7$
5 $f(x) = x + 2e^x$

Jot down all five results before turning to Frame 40 to check your answers

40

1 $y = C\cos 4x + D\sin 4x$
2 $y = Ce^{7x}$
3 $y = C\cosh x + D\sinh x$
4 $y = Cx^2 + Dx + E$
5 $y = Cx + D + Ee^x$

Note that in **5** we use the general form of both the terms.

General form for x is $Cx + D$
and for e^x is Ee^x

$\therefore$ The general form of $x + e^x$ is $y = Cx + D + Ee^x$

Now do this one all on your own:

Example 5

Solve $\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} + 2y = x^2$

Do not forget: find (1) the CF and (2) the PI. Then the general solution is $y = \text{CF} + \text{PI}$.

Off you go.

When you have finished completely, move to Frame 41

41

$$\boxed{y = Ae^x + Be^{2x} + \frac{1}{4}(2x^2 + 6x + 7)}$$

Here is the solution in detail:

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = x^2$$

(1) *CF* $m^2 - 3m + 2 = 0$

$\therefore (m-1)(m-2) = 0 \quad \therefore m = 1$ or 2

$\therefore y = Ae^x + Be^{2x}$ (1)

(2) *PI* $y = Cx^2 + Dx + E \quad \therefore \dfrac{dy}{dx} = 2Cx + D \quad \therefore \dfrac{d^2y}{dx^2} = 2C$

$2C - 3(2Cx + D) + 2(Cx^2 + Dx + E) = x^2$

$2Cx^2 + (2D - 6C)x + (2C - 3D + 2E) = x^2$

$2C = 1 \quad \therefore C = \dfrac{1}{2}$

$2D - 6C = 0 \quad \therefore D = 3C \quad \therefore D = \dfrac{3}{2}$

$2C - 3D + 2E = 0 \quad \therefore 2E = 3D - 2C = \dfrac{9}{2} - 1 = \dfrac{7}{2} \quad \therefore E = \dfrac{7}{4}$

$\therefore$ PI is $y = \dfrac{x^2}{2} + \dfrac{3x}{2} + \dfrac{7}{4} = \dfrac{1}{4}(2x^2 + 6x + 7)$ (2)

General solution: $y = Ae^x + Be^{2x} + \dfrac{1}{4}(2x^2 + 6x + 7)$

Next frame

Particular solution

42 All our solutions to the equation

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x) \text{ where } f(x) \neq 0$$

have contained two unknown arbitrary constants. For instance, in the previous example the general solution to the differential equation

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = x^2 \text{ was seen to be } y = Ae^x + Be^{2x} + \frac{1}{4}(2x^2 + 6x + 7)$$

which contains the two arbitrary constants A and B. These two constants are arbitrary because whatever values are chosen for them, when inserted into the above equation for y it will still be a solution to the differential equation. This means, of course, that there is an infinite number of solutions to the differential equation, each one having specific values for A and B. To select just one solution requires additional information and this information is provided by what are called **boundary conditions** that take the form of a given specific value of y and its derivative for a particular value of x. When these boundary conditions are then imposed on the general solution we obtain a **particular solution**. For example, in the problem of Frame 40 we might have been told that at $x = 0$, $y = \dfrac{3}{4}$ and $\dfrac{dy}{dx} = \dfrac{5}{2}$.

It is important to note that the values of A and B can be found only from the complete general solution and not from the CF as soon as you obtain it. This is a common error so do not be caught out by it. Get the complete general solution before substituting to find A and B.

In this case, we are told that when $x = 0$, $y = \dfrac{3}{4}$, so inserting these values gives

. . . .

Move on to Frame 43

43

$$\boxed{A + B = -1}$$

Because

$$\frac{3}{4} = A + B + \frac{7}{4} \quad \therefore A + B = -1$$

We are also told that when $x = 0$, $\dfrac{dy}{dx} = \dfrac{5}{2}$, so we must first differentiate the general solution

$$y = Ae^x + Be^{2x} + \frac{1}{4}(2x^2 + 6x + 7)$$

to obtain an expression for $\dfrac{dy}{dx}$. So, $\dfrac{dy}{dx} = \ldots\ldots\ldots\ldots$

44

$$\boxed{\frac{dy}{dx} = Ae^x + 2Be^{2x} + \frac{1}{2}(2x + 3)}$$

Now we are given that when $x = 0$, $\dfrac{dy}{dx} = \dfrac{5}{2}$

$$\therefore \frac{5}{2} = A + 2B + \frac{3}{2} \quad \therefore A + 2B = 1$$

So we have $\left.\begin{array}{r} A + B = -1 \\ \text{and } A + 2B = 1 \end{array}\right\}$

and these simultaneous equations give:

$A = \ldots\ldots\ldots\ldots \quad B = \ldots\ldots\ldots\ldots$

Then on to Frame 45

45

$$\boxed{A = -3 \qquad B = 2}$$

Substituting these values in the general solution

$$y = Ae^x + Be^{2x} + \frac{1}{4}(2x^2 + 6x + 7)$$

gives the *particular solution*:

$$y = 2e^{2x} - 3e^x + \frac{1}{4}(2x^2 + 6x + 7)$$

And here is one for you, all on your own:

Solve the equation $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + 5y = 13e^{3x}$, given that when $x = 0$, $y = \dfrac{5}{2}$ and $\dfrac{dy}{dx} = \dfrac{1}{2}$. Remember:

(1) Find the CF.
(2) Find the PI.
(3) The general solution is $y = \text{CF} + \text{PI}$.
(4) Finally insert the given conditions to obtain the particular solution.

When you have finished, check with the solution in Frame 46

46

$$\boxed{y = e^{-2x}(2\cos x + 3\sin x) + \frac{e^{3x}}{2}}$$

Because

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = 13e^{3x}$$

(1) *CF* $m^2 + 4m + 5 = 0 \quad \therefore m = \dfrac{-4 \pm \sqrt{16-20}}{2} = \dfrac{-4 \pm j2}{2}$

$\therefore m = -2 \pm j \quad \therefore y = e^{-2x}(A\cos x + B\sin x)$ (1)

(2) *PI* $y = Ce^{3x} \quad \therefore \dfrac{dy}{dx} = 3Ce^{3x}, \quad \dfrac{d^2y}{dx^2} = 9Ce^{3x}$

$\therefore 9Ce^{3x} + 12Ce^{3x} + 5Ce^{3x} = 13e^{3x}$

$26C = 13 \quad \therefore C = \dfrac{1}{2} \quad \therefore$ PI is $y = \dfrac{e^{3x}}{2}$ (2)

General solution $y = e^{-2x}(A\cos x + B\sin x) + \dfrac{e^{3x}}{2}; \quad x = 0, \quad y = \dfrac{5}{2}$

$\therefore \dfrac{5}{2} = A + \dfrac{1}{2} \quad \therefore A = 2 \qquad y = e^{-2x}(2\cos x + B\sin x) + \dfrac{e^{3x}}{2}$

$\dfrac{dy}{dx} = e^{-2x}(-2\sin x + B\cos x) - 2e^{-2x}(2\cos x + B\sin x) + \dfrac{3e^{3x}}{2}$

$x = 0, \quad \dfrac{dy}{dx} = \dfrac{1}{2} \quad \therefore \dfrac{1}{2} = B - 4 + \dfrac{3}{2} \quad \therefore B = 3$

$\therefore$ Particular solution is $y = e^{-2x}(2\cos x + 3\sin x) + \dfrac{e^{3x}}{2}$

47

Since the CF makes the LHS = 0, it is pointless to use as a PI a term already contained in the CF. If this occurs, multiply the assumed PI by x and proceed as before. If this too is already included in the CF, multiply by a further x and proceed as usual. Here is an example:

Solve $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} - 8y = 3e^{-2x}$

(1) *CF* $m^2 - 2m - 8 = 0 \quad \therefore (m+2)(m-4) = 0 \quad \therefore m = -2$ or 4

$y = Ae^{4x} + Be^{-2x}$ (1)

(2) *PI* The general form of the RHS is Ce^{-2x}, but this term in e^{-2x} is already contained in the CF. Assume $y = Cxe^{-2x}$, and continue as usual:

$$y = Cxe^{-2x}$$

$$\frac{dy}{dx} = Cx(-2e^{-2x}) + Ce^{-2x} = Ce^{-2x}(1-2x)$$

$$\frac{d^2y}{dx^2} = Ce^{-2x}(-2) - 2Ce^{-2x}(1-2x) = Ce^{-2x}(4x-4)$$

Substituting in the given equation, we get:

$$Ce^{-2x}(4x-4) - 2.Ce^{-2x}(1-2x) - 8Cxe^{-2x} = 3e^{-2x}$$
$$(4C + 4C - 8C)x - 4C - 2C = 3$$
$$-6C = 3 \quad \therefore C = -\frac{1}{2}$$
$$\text{PI is } y = -\frac{1}{2}xe^{-2x} \qquad (2)$$

General solution: $y = Ae^{4x} + Be^{-2x} - \dfrac{xe^{-2x}}{2}$

So remember, if the general form of the RHS is already included in the CF, multiply the assumed general form of the PI by x and continue as before.

Here is one final example for you to work:

Solve $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} - 2y = e^x$

Finish it off and then move to Frame 48

48

$$\boxed{y = Ae^x + Be^{-2x} + \frac{xe^x}{3}}$$

Here is the working:

To solve $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} - 2y = e^x$

(1) *CF* $m^2 + m - 2 = 0$

$(m-1)(m+2) = 0 \quad \therefore m = 1$ or -2

$\therefore y = Ae^x + Be^{-2x}$ (1)

(2) *PI* Take $y = Ce^x$. But this is already included in the CF. Therefore, assume $y = Cxe^x$.

$$\text{Then } \frac{dy}{dx} = Cxe^x + Ce^x = Ce^x(x+1)$$
$$\frac{d^2y}{dx^2} = Ce^x + Cxe^x + Ce^x = Ce^x(x+2)$$
$$\therefore Ce^x(x+2) + Ce^x(x+1) - 2Cxe^x = e^x$$
$$C(x+2) + C(x+1) - 2Cx = 1$$
$$3C = 1 \quad \therefore C = \frac{1}{3}$$
$$\text{PI is } y = \frac{xe^x}{3} \qquad (2)$$

and so the general solution is

$$y = Ae^x + Be^{-2x} + \frac{xe^x}{3}$$

You are now almost at the end of the Programme. Before you work through the **Can You?** checklist and the **Test exercise**, however, look down the **Review summary** given in Frame 49. It lists the main points that we have established during this Programme, and you may find it very useful.

So on now to Frame 49

Review summary

49 **1** Solution of equations of the form $a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=f(x)$

(1) Auxiliary equation: $am^2+bm+c=0$

(2) Types of solutions:

(a) Real and different roots $\quad m=m_1$ and $m=m_2$

$y=Ae^{m_1x}+Be^{m_2x}$

(b) Real and equal roots $\quad m=m_1$ (twice)

$y=e^{m_1x}(A+Bx)$

(c) Complex roots $\quad m=\alpha\pm j\beta$

$y=e^{\alpha x}(A\cos\beta x+B\sin\beta x)$

2 Equations of the form $\frac{d^2y}{dx^2}+n^2y=0$

$y=A\cos nx+B\sin nx$

3 Equations of the form $\frac{d^2y}{dx^2}-n^2y=0$

$y=A\cosh nx+B\sinh nx$

4 General solution

$y=$ complementary function + particular integral

5 (1) To find CF solve $a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0$

(2) To find PI assume the general form of the RHS.
Note: If the general form of the RHS is already included in the CF, multiply by x and proceed as before, etc. Determine the complete general solution before substituting to find the values of the arbitrary constants A and B.

Now all that remains is the **Can You?** *checklist and the* **Test exercise**, *so on to Frame 50*

Can You?

50 **Checklist 26**

Check this list before and after you try the end of Programme test.

On a scale of 1 to 5 how confident are you that you can: Frames

- Use the auxiliary equation to solve certain second-order homogeneous equations? 1 to 22
  Yes ☐ ☐ ☐ ☐ ☐ *No*
- Use the complementary function and the particular integral to solve certain second-order inhomogeneous equations? 23 to 48
  Yes ☐ ☐ ☐ ☐ ☐ *No*

Test exercise 26

Here are eight differential equations for you to solve, similar to those we have dealt with in the Programme. They are quite straightforward, so you should have no difficulty with them. Set your work out neatly and take your time: this will help you to avoid making unnecessary slips. **51**

Solve the following:

1 $\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} - 2y = 8$

2 $\dfrac{d^2y}{dx^2} - 4y = 10e^{3x}$

3 $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + y = e^{-2x}$

4 $\dfrac{d^2y}{dx^2} + 25y = 5x^2 + x$

5 $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + y = 4\sin x$

6 $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + 5y = 2e^{-2x}$, given that $x = 0$, $y = 1$ and $\dfrac{dy}{dx} = -2$.

7 $3\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} - y = 2x - 3$

8 $\dfrac{d^2y}{dx^2} - 6\dfrac{dy}{dx} + 8y = 8e^{4x}$

Further problems 26

Solve the following equations: **52**

1 $2\dfrac{d^2y}{dx^2} - 7\dfrac{dy}{dx} - 4y = e^{3x}$

2 $\dfrac{d^2y}{dx^2} - 6\dfrac{dy}{dx} + 9y = 54x + 18$

3 $\dfrac{d^2y}{dx^2} - 5\dfrac{dy}{dx} + 6y = 100\sin 4x$

4 $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + y = 4\sinh x$

5 $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} - 2y = 2\cosh 2x$

6 $\dfrac{d^2y}{dx^2} - 6\dfrac{dy}{dx} + 10y = 20 - e^{2x}$

7 $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + 4y = 2\cos^2 x$

8 $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + 3y = x + e^{2x}$

9 $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + 3y = x^2 - 1$

10 $\dfrac{d^2y}{dx^2} - 9y = e^{3x} + \sin 3x$

11 For a horizontal cantilever of length l, with load w per unit length, the equation of bending is

$$EI\dfrac{d^2y}{dx^2} = \dfrac{w}{2}(l - x)^2$$

where E, I, w and l are constants. If $y = 0$ and $\dfrac{dy}{dx} = 0$ at $x = 0$, find y in terms of x. Hence find the value of y when $x = l$.

12 Solve the equation

$$\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 3x = e^{-3t}$$

given that at $t = 0$, $x = \frac{1}{2}$ and $\frac{dx}{dt} = -2$.

13 Obtain the general solution of the equation

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 5y = 6\sin t$$

and determine the amplitude and frequency of the steady-state function. [*Note*: The steady state function describes the behaviour of the solution as $t \to \infty$]

14 Solve the equation

$$\frac{d^2x}{dt^2} - 3\frac{dx}{dt} + 2x = \sin t$$

given that at $t = 0$, $x = 0$ and $\frac{dx}{dt} = 0$.

15 Solve $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 3\sin x$, given that when $x = 0$, $y = -0{\cdot}9$ and $\frac{dy}{dx} = -0{\cdot}7$.

16 Obtain the general solution of the equation

$$\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 10y = 50x$$

17 Solve the equation

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 2x = 85\sin 3t$$

given that when $t = 0$, $x = 0$ and $\frac{dx}{dt} = -20$. Show that the values of t for stationary values of the steady-state solution are the roots of $6\tan 3t = 7$ [*see* 13].

18 Solve the equation $\frac{d^2y}{dx^2} = 3\sin x - 4y$, given that $y = 0$, at $x = 0$ and that $\frac{dy}{dx} = 1$ at $x = \pi/2$. Find the maximum value of y in the interval $0 < x < \pi$.

19 A mass suspended from a spring performs vertical oscillations and the displacement x (cm) of the mass at time t (s) is given by $\frac{1}{2}\frac{d^2x}{dt^2} = -48x$. If $x = \frac{1}{6}$ and $\frac{dx}{dt} = 0$ when $t = 0$, determine the period and amplitude of the oscillations.

20 The equation of motion of a body performing damped forced vibrations is $\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 6x = \cos t$. Solve this equation, given that $x = 0{\cdot}1$ and $\frac{dx}{dt} = 0$ when $t = 0$. Write the steady-state solution in the form $K\sin(t + a)$ [*see* 13].

53 Now visit the companion website for this book at www.palgrave.com/stroud for more questions applying this mathematics to science and engineering.

Introduction to Laplace transforms

Learning outcomes

When you have completed this Programme you will be able to:

- Derive the Laplace transform of an expression by using the integral definition
- Obtain inverse Laplace transforms with the help of a Table of Laplace transforms
- Derive the Laplace transform of the derivative of an expression
- Solve first-order, constant-coefficient, inhomogeneous differential equations using the Laplace transform
- Derive further Laplace transforms from known transforms
- Use the Laplace transform to obtain the solution to linear, constant-coefficient, inhomogeneous differential equations of second and higher order

The Laplace transform

1

All the differential equations you have looked at so far have had solutions containing a number of unknown integration constants A, B, C etc. The values of these constants have then been found by applying boundary conditions to the solution, a procedure that can often prove to be tedious. Fortunately, for a certain type of differential equation there is a method of obtaining the solution where these unknown integration constants are evaluated *during the process of solution*. Furthermore, rather than employing integration as the way of unravelling the differential equation, you use straightforward algebra.

The method hinges on what is called the *Laplace transform*. If $f(t)$ represents some expression in t defined for $t \geq 0$, the *Laplace transform* of $f(t)$, denoted by $L\{f(t)\}$, is defined to be:

$$L\{f(t)\} = \int_{t=0}^{\infty} e^{-st} f(t)\,\mathrm{d}t$$

where s is a variable whose values are chosen so as to ensure that the semi-infinite integral converges. More will be said about the variable s in Frame 3. For now, what would you say is the Laplace transform $f(t) = 2$ for $t \geq 0$?

Substitute for $f(t)$ in the integral above and then perform the integration.
The answer is in the next frame

2

$$\boxed{L\{2\} = \frac{2}{s} \text{ provided } s > 0}$$

Because:

$$L\{f(t)\} = \int_{t=0}^{\infty} e^{-st} f(t)\,\mathrm{d}t$$

so

$$\begin{aligned} L\{2\} &= \int_{t=0}^{\infty} e^{-st}2\,\mathrm{d}t \\ &= 2\left[\frac{e^{-st}}{-s}\right]_{t=0}^{\infty} \\ &= 2(0 - (-1/s)) \\ &= \frac{2}{s} \end{aligned}$$

Notice that $s > 0$ is demanded because if $s < 0$ then $e^{-st} \to \infty$ as $t \to \infty$ and if $s = 0$ then $L\{2\}$ is not defined (in both of these two cases the integral diverges), so that

$$L\{2\} = \frac{2}{s} \text{ provided } s > 0$$

By the same reasoning, if k is some constant then

$$L\{k\} = \frac{k}{s} \text{ provided } s > 0$$

Now, how about the Laplace transform of $f(t) = e^{-kt}$, $t \geq 0$ where k is a constant?

Go back to the integral definition and work it out.
Again, the answer is in the next frame

3

$$\boxed{L\{e^{-kt}\} = \frac{1}{s+k} \text{ provided } s > -k}$$

Because

$$\begin{aligned} L\{e^{-kt}\} &= \int_{t=0}^{\infty} e^{-st}e^{-kt}\,\mathrm{d}t \\ &= \int_{t=0}^{\infty} e^{-(s+k)t}\,\mathrm{d}t \\ &= \left[\frac{e^{-(s+k)t}}{-(s+k)}\right]_{t=0}^{\infty} \\ &= \left(0 - \left(-\frac{1}{(s+k)}\right)\right) \\ &= \frac{1}{(s+k)} \quad \text{provided } s+k>0, \text{ that is provided } s>-k \end{aligned}$$

These two examples have demonstrated that you need to be careful about the finite existence of the Laplace transform and not just take the integral definition without some thought. For the Laplace transform to exist the integrand

$$e^{-st}f(t)$$

must converge to zero as $t \to \infty$ and this will impose some conditions on the values of s for which the integral does converge and, hence, the Laplace transform exists. In this Programme you can be assured that there are no problems concerning the existence of any of the Laplace transforms that you will meet.

Move on to the next frame

The inverse Laplace transform

4

The Laplace transform is an expression in the variable s which is denoted by $F(s)$. It is said that $f(t)$ and $F(s) = L\{f(t)\}$ form a *transform pair*. This means that if $F(s)$ is the *Laplace transform* of $f(t)$ then $f(t)$ is the *inverse Laplace transform* of $F(s)$. We write:

$$f(t) = L^{-1}\{F(s)\}$$

There is no simple integral definition of the inverse transform so you have to find it by working backwards. For example:

if $f(t) = 4$ then the Laplace transform $L\{f(t)\} = F(s) = \dfrac{4}{s}$

so

if $F(s) = \dfrac{4}{s}$ then the inverse Laplace transform $L^{-1}\{F(s)\} = f(t) = 4$

It is this ability to find the Laplace transform of an expression and then reverse it that makes the Laplace transform so useful in the solution of differential equations, as you will soon see.

For now, what is the inverse Laplace transform of $F(s) = \dfrac{1}{s-1}$?

To answer this, look at the Laplace transforms you now know.
The answer is in the next frame

5

$$\boxed{L^{-1}\{F(s)\} = f(t) = e^t}$$

Because you know that:

$L\{e^{-kt}\} = \dfrac{1}{s+k}$ you can say that $L^{-1}\left\{\dfrac{1}{s+k}\right\} = e^{-kt}$

so when $k = -1$, $L^{-1}\left\{\dfrac{1}{s-1}\right\} = e^{-(-1)t} = e^t$

To assist in the process of finding Laplace transforms and their inverses a table is used. In the next frame is a short table containing what you know to date.

6

Table of Laplace transforms

$f(t) = L^{-1}\{F(s)\}$	$F(s) = L\{f(t)\}$	
k	$\dfrac{k}{s}$	$s > 0$
e^{-kt}	$\dfrac{1}{s+k}$	$s > -k$

Reading the table from left to right gives the Laplace transform and reading the table from right to left gives the inverse Laplace transform.

Use these, where possible, to answer the questions in the **Review exercise** *that follows. Otherwise use the basic definition in Frame 1.*

Review summary

7

1 The *Laplace transform* of $f(t)$, denoted by $L\{f(t)\}$, is defined to be:

$$L\{f(t)\} = \int_{t=0}^{\infty} e^{-st} f(t)\,dt$$

where s is a variable whose values are chosen so as to ensure that the semi-infinite integral converges.

2 If $F(s)$ is the *Laplace transform* of $f(t)$ then $f(t)$ is the *inverse Laplace transform* of $F(s)$. We write:

$f(t) = L^{-1}\{F(s)\}$

There is no simple integral definition of the inverse transform so you have to find it by working backwards using a *Table of Laplace transforms*.

Review exercise

8

1 Find the Laplace transform of each of the following. In each case $f(t)$ is defined for $t \geq 0$:

(a) $f(t) = -3$ (b) $f(t) = e$ (c) $f(t) = e^{2t}$

(d) $f(t) = -5e^{-3t}$ (e) $f(t) = 2e^{7t-2}$

2 Find the inverse Laplace transform of each of the following:

(a) $F(s) = -\dfrac{1}{s}$ (b) $F(s) = \dfrac{1}{s-5}$ (c) $F(s) = \dfrac{3}{s+2}$

(d) $F(s) = -\dfrac{3}{4s}$ (e) $F(s) = \dfrac{1}{2s-3}$

Solutions in next frame

9

1 (a) $f(t) = -3$

Because $L\{k\} = \dfrac{k}{s}$ provided $s > 0$, $L\{-3\} = -\dfrac{3}{s}$ provided $s > 0$

(b) $f(t) = e$

Because $L\{k\} = \dfrac{k}{s}$ provided $s > 0$, $L\{e\} = \dfrac{e}{s}$ provided $s > 0$

(c) $f(t) = e^{2t}$

Because $L\{e^{-kt}\} = \dfrac{1}{s+k}$ provided $s > -k$, $L\{e^{2t}\} = \dfrac{1}{s-2}$ provided $s > 2$

(d) $f(t) = -5e^{-3t}$

$$L\{-5e^{-3t}\} = \int_{t=0}^{\infty} e^{-st}(-5e^{-3t})\,\mathrm{d}t = -5\int_{t=0}^{\infty} e^{-st}e^{-3t}\,\mathrm{d}t = -5L\{e^{-3t}\}$$

$L\{-5e^{-3t}\} = -\dfrac{5}{s+3}$ provided $s > -3$

(e) $f(t) = 2e^{7t-2}$

$$L\{2e^{7t-2}\} = \int_{t=0}^{\infty} e^{-st}(2e^{7t-2})\,\mathrm{d}t = 2e^{-2}\int_{t=0}^{\infty} e^{-st}e^{7t}\,\mathrm{d}t = 2e^{-2}L\{e^{7t}\}$$

$L\{2e^{7t-2}\} = \dfrac{2e^{-2}}{s-7}$ provided $s > 7$

2 (a) $F(s) = -\dfrac{1}{s}$

Because $L^{-1}\left\{\dfrac{k}{s}\right\} = k$, $L^{-1}\left\{-\dfrac{1}{s}\right\} = L^{-1}\left\{\dfrac{-1}{s}\right\} = -1$

(b) $F(s) = \dfrac{1}{s-5}$

Because $L^{-1}\left\{\dfrac{1}{s+k}\right\} = e^{-kt}$, $L^{-1}\left\{\dfrac{1}{s-5}\right\} = e^{-(-5)t} = e^{5t}$

(c) $F(s) = \dfrac{3}{s+2}$

Because $L^{-1}\left\{\dfrac{1}{s+2}\right\} = e^{-2t}$ and $L\{3e^{-2t}\} = 3L\{e^{-2t}\} = \dfrac{3}{s+2}$ so

$L^{-1}\left\{\dfrac{3}{s+2}\right\} = 3e^{-2t}$

(d) $F(s) = -\dfrac{3}{4s}$

$F(s) = -\dfrac{3}{4s} = \dfrac{(-3/4)}{s}$ so that $L^{-1}\left\{-\dfrac{3}{4s}\right\} = L^{-1}\left\{\dfrac{-3/4}{s}\right\} = -3/4$

(e) $F(s) = \dfrac{1}{2s-3}$

$$F(s) = \frac{1}{2s-3} = \frac{\frac{1}{2}}{s-\frac{3}{2}} \text{ so that } f(t) = L^{-1}\left\{\frac{1}{2s-3}\right\} = L^{-1}\left\{\frac{\frac{1}{2}}{s-\frac{3}{2}}\right\} = \frac{1}{2}e^{\frac{3}{2}t}$$

Next frame

10

Laplace transform of a derivative

Before you can use the Laplace transform to solve a differential equation you need to know the Laplace transform of a derivative. Given some expression $f(t)$ with Laplace transform $L\{f(t)\} = F(s)$, the Laplace transform of the derivative $f'(t)$ is:

$$L\{f'(t)\} = \int_{t=0}^{\infty} e^{-st} f'(t)\,\mathrm{d}t$$

This can be integrated by parts as follows:

$$\begin{aligned}L\{f'(t)\} &= \int_{t=0}^{\infty} e^{-st} f'(t)\,\mathrm{d}t \\ &= \int_{t=0}^{\infty} u(t)\mathrm{d}v(t) \\ &= \Big[u(t)v(t)\Big]_{t=0}^{\infty} - \int_{t=0}^{\infty} v(t)\mathrm{d}u(t) \quad \text{(the Parts formula – see Programme 16, Frame 21)}\end{aligned}$$

where $u(t) = e^{-st}$ so $\mathrm{d}u(t) = -se^{-st}\mathrm{d}t$ and where $\mathrm{d}v(t) = f'(t)\mathrm{d}t$ so $v(t) = f(t)$.

Therefore, substitution in the Parts formula gives:

$$\begin{aligned}L\{f'(t)\} &= \Big[e^{-st} f(t)\Big]_{t=0}^{\infty} + s\int_{t=0}^{\infty} e^{-st} f(t)\mathrm{d}t \\ &= (0 - f(0)) + sF(s) \text{ assuming } e^{-st} f(t) \to 0 \text{ as } t \to \infty\end{aligned}$$

That is:

$$L\{f'(t)\} = sF(s) - f(0)$$

So the Laplace transform of the derivative of $f(t)$ is given in terms of the Laplace transform of $f(t)$ itself and the value of $f(t)$ when $t = 0$. Before you use this fact just consider two properties of the Laplace transform in the next frame.

11

Two properties of Laplace transforms

Both the Laplace transform and its inverse are *linear transforms*, by which is meant that:

(1) *The transform of a sum (or difference) of expressions is the sum (or difference) of the individual transforms. That is:*

$$L\{f(t) \pm g(t)\} = L\{f(t)\} \pm L\{g(t)\}$$
and $$L^{-1}\{F(s) \pm G(s)\} = L^{-1}\{F(s)\} \pm L^{-1}\{G(s)\}$$

(2) *The transform of an expression that is multiplied by a constant is the constant multiplied by the transform of the expression. That is:*

$L\{kf(t)\} = kL\{f(t)\}$ and $L^{-1}\{kF(s)\} = kL^{-1}\{F(s)\}$ where k is a constant

These are easily proved using the basic definition of the Laplace transform in Frame 1.

Armed with this information let's try a simple differential equation. By using

$$L\{f'(t)\} = sF(s) - f(0)$$

take the Laplace transform of both sides of the equation

$$f'(t) + f(t) = 1 \text{ where } f(0) = 0$$

and find an expression for the Laplace transform $F(s)$.

Work through this steadily using what you know;
you will find the answer in Frame 12

12

$$\boxed{F(s) = \frac{1}{s(s+1)}}$$

Because, taking Laplace transforms of both sides of the equation you have that:

$L\{f'(t) + f(t)\} = L\{1\}$ The Laplace transform of the left-hand side equals the Laplace transform of the right-hand side

That is:

$L\{f'(t)\} + L\{f(t)\} = L\{1\}$ The transform of a sum is the sum of the transforms.

From what you know about the Laplace transform of $f(t)$ and its derivative $f'(t)$ this gives:

$$[sF(s) - f(0)] + F(s) = \frac{1}{s}$$

That is:

$(s+1)F(s) - f(0) = \frac{1}{s}$ and you are given that $f(0) = 0$ so

$(s+1)F(s) = \frac{1}{s}$, that is $F(s) = \frac{1}{s(s+1)}$

Well done. Now, separate the right-hand side into partial fractions.

You have done plenty of this before in Programme F.8; the answer is in Frame 13

13

$$\boxed{F(s) = \frac{1}{s} - \frac{1}{s+1}}$$

Because

Assume that $\frac{1}{s(s+1)} = \frac{A}{s} + \frac{B}{s+1}$ then, $1 = A(s+1) + Bs$ from which you find that $A = 1$ and $B = -1$ so that $F(s) = \frac{1}{s} - \frac{1}{s+1}$

That was straightforward enough. Now take the inverse Laplace transform and find the solution to the differential equation.

The answer is in Frame 14

14

$$\boxed{f(t) = 1 - e^{-t}}$$

Because

$f(t) = L^{-1}\{F(s)\}$

$= L^{-1}\left\{\frac{1}{s} - \frac{1}{s+1}\right\}$

$= L^{-1}\left\{\frac{1}{s}\right\} - L^{-1}\left\{\frac{1}{s+1}\right\}$ The inverse Laplace transform of a difference is the difference of the inverse transforms

$= 1 - e^{-t}$ Using the Table of Laplace transforms in Frame 6

▶

You now have a method for solving a differential equation of the form:

$af'(t) + bf(t) = g(t)$ given that $f(0) = k$

where a, b and k are known constants and $g(t)$ is a known expression in t:

(a) Take the Laplace transform of both sides of the differential equation
(b) Find the expression $F(s) = L\{f(t)\}$ in the form of an algebraic fraction
(c) Separate $F(s)$ into its partial fractions
(d) Find the inverse Laplace transform $L^{-1}\{F(s)\}$ to find the solution $f(t)$ to the differential equation.

Now you try some but before you do just look at the Table of Laplace transforms in the next frame. You will need them to solve the equations in the **Review exercise** *that follows.*

15

Table of Laplace transforms

$f(t) = L^{-1}\{F(s)\}$	$F(s) = L\{f(t)\}$	
k	$\dfrac{k}{s}$	$s > 0$
e^{-kt}	$\dfrac{1}{s+k}$	$s > -k$
te^{-kt}	$\dfrac{1}{(s+k)^2}$	$s > -k$

We will derive this third transform later in the Programme. For now, use these to answer the questions that follow the **Review summary** *in the next frame*

Review summary

16

1 If $F(s)$ is the Laplace transform of $f(t)$ then the Laplace transform of $f'(t)$ is:

$L\{f'(t)\} = sF(s) - f(0)$

2 (a) The Laplace transform of a sum (or difference) of expressions is the sum (or difference) of the individual transforms. That is:

$$L\{f(t) \pm g(t)\} = L\{f(t)\} \pm L\{(g(t)\}$$

and $$L^{-1}\{F(s) \pm G(s)\} = L^{-1}\{F(s)\} \pm L^{-1}\{G(s)\}$$

(b) The transform of an expression multiplied by a constant is the constant multiplied by the transform of the expression. That is:

$$L\{kf(t)\} = kL\{f(t)\} \text{ and } L^{-1}\{kF(s)\} = kL^{-1}\{F(s)\}$$

where k is a constant.

3 To solve a differential equation of the form:

$af'(t) + bf(t) = g(t)$ given that $f(0) = k$

where a, b and k are known constants and $g(t)$ is a known expression in t:

(a) Take the Laplace transform of both sides of the differential equation

(b) Find the expression $F(s) = L\{f(t)\}$ in the form of an algebraic fraction

(c) Separate $F(s)$ into its partial fractions

(d) Find the inverse Laplace transform $L^{-1}\{F(s)\}$ to find the solution $f(t)$ to the differential equation.

Review exercise

17

Solve each of the following differential equations:

(a) $f'(t) - f(t) = 2$ where $f(0) = 0$

(b) $f'(t) + f(t) = e^{-t}$ where $f(0) = 0$

(c) $f'(t) + f(t) = 3$ where $f(0) = -2$

(d) $f'(t) - f(t) = e^{2t}$ where $f(0) = 1$

(e) $3f'(t) - 2f(t) = 4e^{-t} + 2$ where $f(0) = 0$

Solutions in next frame

18

(a) $f'(t) - f(t) = 2$ where $f(0) = 0$

Taking Laplace transforms of both sides of this equation gives:

$sF(s) - f(0) - F(s) = \dfrac{2}{s}$ so that $F(s) = \dfrac{2}{s(s-1)} = -\dfrac{2}{s} + \dfrac{2}{s-1}$

The inverse transform then gives the solution as

$f(t) = -2 + 2e^t = 2(e^t - 1)$

(b) $f'(t) + f(t) = e^{-t}$ where $f(0) = 0$

Taking Laplace transforms of both sides of this equation gives:

$sF(s) - f(0) + F(s) = \dfrac{1}{s+1}$ so that $F(s) = \dfrac{1}{(s+1)^2}$

The Table of inverse transforms then gives the solution as $f(t) = te^{-t}$

(c) $f'(t) + f(t) = 3$ where $f(0) = -2$

Taking Laplace transforms of both sides of this equation gives:

$sF(s) - f(0) + F(s) = \dfrac{3}{s}$ so that

$F(s) = -\dfrac{2}{s+1} + \dfrac{3}{s(s+1)} = \dfrac{3-2s}{s(s+1)} = \dfrac{3}{s} - \dfrac{5}{s+1}$

The inverse transform then gives the solution as $f(t) = 3 - 5e^{-t}$

(d) $f'(t) - f(t) = e^{2t}$ where $f(0) = 1$

Taking Laplace transforms of both sides of this equation gives:

$sF(s) - f(0) - F(s) = \dfrac{1}{s-2}$ giving $(s-1)F(s) - 1 = \dfrac{1}{s-2}$

so that $F(s) = \dfrac{1}{s-1} + \dfrac{1}{(s-1)(s-2)} = \dfrac{1}{s-2}$

The inverse transform then gives the solution as $f(t) = e^{2t}$

(e) $3f'(t) - 2f(t) = 4e^{-t} + 2$ where $f(0) = 0$

Taking Laplace transforms of both sides of this equation gives:

$$3[sF(s) - f(0)] - 2F(s) = \frac{4}{s+1} + \frac{2}{s} = \frac{6s+2}{s(s+1)} \text{ so that}$$

$$F(s) = \frac{6s+2}{s(s+1)(3s-2)} = \frac{27}{5}\left(\frac{1}{3s-2}\right) - \frac{1}{s} - \frac{4}{5}\left(\frac{1}{s+1}\right)$$

$$= \frac{27}{15}\left(\frac{1}{s-\frac{2}{3}}\right) - \frac{1}{s} - \frac{4}{5}\left(\frac{1}{s+1}\right)$$

The inverse transform then gives the solution as:

$$f(t) = \frac{9}{5}e^{2t/3} - \frac{4}{5}e^{-t} - 1$$

On now to Frame 19

19

Generating new transforms

Deriving the Laplace transform of $f(t)$ often requires you to integrate by parts, sometimes repeatedly. However, because $L\{f'(t)\} = sL\{f(t)\} - f(0)$ you can sometimes avoid this involved process when you know the transform of the derivative $f'(t)$. Take as an example the problem of finding the Laplace transform of the expression $f(t) = t$. Now $f'(t) = 1$ and $f(0) = 0$ so that substituting in the equation:

$$L\{f'(t)\} = sL\{f(t)\} - f(0)$$

gives

$$L\{1\} = sL\{t\} - 0$$

that is

$$\frac{1}{s} = sL\{t\}$$

therefore

$$L\{t\} = \frac{1}{s^2}$$

That was easy enough, so what is the Laplace transform of $f(t) = t^2$?

The answer is in the next frame

20

$$\boxed{\frac{2}{s^3}}$$

Because

$$f(t) = t^2,\ f'(t) = 2t \text{ and } f(0) = 0$$

Substituting in

$$L\{f'(t)\} = sL\{f(t)\} - f(0)$$

gives

$$L\{2t\} = sL\{t^2\} - 0$$

that is

$$2L\{t\} = sL\{t^2\} \text{ so } \frac{2}{s^2} = sL\{t^2\}$$

therefore

$$L\{t^2\} = \frac{2}{s^3}$$

Just try another one. Verify the third entry in the Table of Laplace transforms in Frame 15 for $k = 1$, that is:

$$L\{te^{-t}\} = \frac{1}{(s+1)^2}$$

This is a littler harder but just follow the procedure laid out in the previous two frames and try it. The explanation is in the next frame

21

Because

$$f(t) = te^{-t},\ f'(t) = e^{-t} - te^{-t} \text{ and } f(0) = 0$$

Substituting in

$$L\{f'(t)\} = sL\{f(t)\} - f(0)$$

gives

$$L\{e^{-t} - te^{-t}\} = sL\{te^{-t}\} - 0$$

that is

$$L\{e^{-t}\} - L\{te^{-t}\} = sL\{te^{-t}\}$$

therefore

$$L\{e^{-t}\} = (s+1)L\{te^{-t}\}$$

giving

$$\frac{1}{s+1} = (s+1)L\{te^{-t}\} \text{ and so } L\{te^{-t}\} = \frac{1}{(s+1)^2}$$

On now to Frame 22

Laplace transforms of higher derivatives

22

The Laplace transforms of derivatives higher than the first are readily derived. Let $F(s)$ and $G(s)$ be the respective Laplace transforms of $f(t)$ and $g(t)$. That is

$$L\{f(t)\} = F(s) \text{ so that } L\{f'(t)\} = sF(s) - f(0)$$

and

$$L\{g(t)\} = G(s) \text{ and } L\{g'(t)\} = sG(s) - g(0)$$

Now let $g(t) = f'(t)$ so that $L\{g(t)\} = L\{f'(t)\}$ where

$$g(0) = f'(0) \text{ and } G(s) = sF(s) - f(0)$$

Now, because $g(t) = f'(t)$

$$g'(t) = f''(t)$$

This means that

$$L\{g'(t)\} = L\{f''(t)\} = sG(s) - g(0) = s[sF(s) - f(0)] - f'(0)$$

so

$$L\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$$

By a similar argument it can be shown that

$$L\{f'''(t)\} = s^3F(s) - s^2f(0) - sf'(0) - f''(0)$$

and so on. Can you see the pattern developing here?

The Laplace transform of $f^{iv}(t)$ is ....

Next frame

23

$$\boxed{L\{f^{iv}(t)\} = s^4F(s) - s^3f(0) - s^2f'(0) - sf''(0) - f'''(0)}$$

Now, using $L\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$ the Laplace transform of $f(t) = \sin kt$ where k is a constant is ....

Differentiate $f(t)$ twice and follow the procedure that you used in Frames 19 to 21. Take it carefully, the answer and working are in the following frame

24

$$\boxed{L\{\sin kt\} = \frac{k}{s^2 + k^2}}$$

Because

$f(t) = \sin kt$, $f'(t) = k\cos kt$ and $f''(t) = -k^2\sin kt$.
Also $f(0) = 0$ and $f'(0) = k$.

Substituting in

$$L\{f''(t)\} = s^2F(s) - sf(0) - f'(0) \text{ where } F(s) = L\{f(t)\}$$

gives

$$L\{-k^2\sin kt\} = s^2L\{\sin kt\} - s.0 - k$$

that is

$$-k^2L\{\sin kt\} = s^2L\{\sin kt\} - k$$

so

$$(s^2 + k^2)L\{\sin kt\} = k \text{ and } L\{\sin kt\} = \frac{k}{s^2 + k^2}$$

And $L\{\cos kt\} =$ ....

It's just the same method

25

$$\boxed{L\{\cos kt\} = \frac{s}{s^2 + k^2}}$$

Because

$f(t) = \cos kt$, $f'(t) = -k\sin kt$ and $f''(t) = -k^2\cos kt$.
Also $f(0) = 1$ and $f'(0) = 0$.

Substituting in

$$L\{f''(t)\} = s^2F(s) - sf(0) - f'(0) \text{ where } F(s) = L\{f(t)\}$$

gives

$$L\{-k^2\cos kt\} = s^2L\{\cos kt\} - s.1 - 0$$

that is

$$-k^2L\{\cos kt\} = s^2L\{\cos kt\} - s$$

so

$$(s^2+k^2)L\{\cos kt\} = s \text{ and } L\{\cos kt\} = \frac{s}{s^2+k^2}$$

The Table of transforms is now extended in the next frame.

Table of Laplace transforms

26

$f(t) = L^{-1}\{F(s)\}$	$F(s) = L\{f(t)\}$	
k	$\frac{k}{s}$	$s > 0$
e^{-kt}	$\frac{1}{s+k}$	$s > -k$
te^{-kt}	$\frac{1}{(s+k)^2}$	$s > -k$
t	$\frac{1}{s^2}$	$s > 0$
t^2	$\frac{2}{s^3}$	$s > 0$
$\sin kt$	$\frac{k}{s^2+k^2}$	$s^2+k^2 > 0$
$\cos kt$	$\frac{s}{s^2+k^2}$	$s^2+k^2 > 0$

Linear, constant-coefficient, inhomogeneous differential equations

27

The Laplace transform can be used to solve equations of the form:

$$a_nf^{(n)}(t) + a_{n-1}f^{(n-1)}(t) + \dots + a_2f''(t) + a_1f'(t) + a_0f(t) = g(t)$$

where $a_n, a_{n-1}, \dots, a_2, a_1, a_0$ are known constants, $g(t)$ is a known expression in t and the values of $f(t)$ and its derivatives are known at $t = 0$. This type of equation is called a *linear, constant-coefficient, inhomogeneous differential equation* and the values of $f(t)$ and its derivatives at $t = 0$ are called *boundary conditions*. The method of obtaining the solution follows the procedure laid down in Frame 14. For example:

To find the solution of:

$$f''(t) + 3f'(t) + 2f(t) = 4t \text{ where } f(0) = f'(0) = 0$$

(a) *Take the Laplace transform of both sides of the equation*

$$L\{f''(t)\} + 3L\{f'(t)\} + 2L\{f(t)\} = 4L\{t\}$$

to give $\left[s^2F(s) - sf(0) - f'(0)\right] + 3[sF(s) - f(0)] + 2F(s) = \frac{4}{s^2}$

(b) *Find the expression $F(s) = L\{f(t)\}$ in the form of an algebraic fraction*
Substituting the values for $f(0)$ and $f'(0)$ and then rearranging gives

$$(s^2 + 3s + 2)F(s) = \frac{4}{s^2}$$

so that

$$F(s) = \frac{4}{s^2(s+1)(s+2)}$$

(c) *Separate $F(s)$ into its partial fractions*

$$\frac{4}{s^2(s+1)(s+2)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+1} + \frac{D}{s+2}$$

Adding the right-hand side partial fractions together and then equating the left-hand side numerator with the right-hand side numerator gives

$$4 = As(s+1)(s+2) + B(s+1)(s+2) + Cs^2(s+2) + Ds^2(s+1)$$

Let $s = 0$ $\quad 4 = 2B$ therefore $B = 2$

$s = -1$ $\quad 4 = C(-1)^2(-1+2) = C$

$s = -2$ $\quad 4 = D(-2)^2(-2+1) = -4D$ therefore $D = -1$

Equate the coefficients of s:

$0 = 2A + 3B = 2A + 6$ therefore $A = -3$

Consequently:

$$F(s) = -\frac{3}{s} + \frac{2}{s^2} + \frac{4}{s+1} - \frac{1}{s+2}$$

(d) *Use the Tables to find the inverse Laplace transform $L^{-1}\{F(s)\}$ and so find the solution $f(t)$ to the differential equation*

$$f(t) = -3 + 2t + 4e^{-t} - e^{-2t}$$

So that was all very straightforward even if it was involved.
Now try your hand at the differential equations in Frame 29

Review summary

28

1 If $F(s)$ is the Laplace transform of $f(t)$ then:

$$L\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$$

and $\quad L\{f'''(t)\} = s^3F(s) - s^2f(0) - sf'(0) - f''(0)$

2 Equations of the form:

$$a_nf^{(n)}(t) + a_{n-1}f^{(n-1)}(t) + \cdots + a_2f''(t) + a_1f'(t) + a_0f(t) = g(t)$$

where $a_n, a_{n-1}, \ldots, a_2, a_1, a_0$ are constants are called linear, constant-coefficient, inhomogeneous differential equations.

3 The Laplace transform can be used to solve constant-coefficient, inhomogeneous differential equations provided $a_n, a_{n-1}, \ldots, a_2, a_1, a_0$ are known constants, $g(t)$ is a known expression in t, and the values of $f(t)$ and its derivatives are known at $t = 0$.

4 The procedure for solving these equations of second and higher order is the same as that for solving the equations of first order. Namely:

(a) Take the Laplace transform of both sides of the differential equation

(b) Find the expression $F(s) = L\{f(t)\}$ in the form of an algebraic fraction

(c) Separate $F(s)$ into its partial fractions

(d) Find the inverse Laplace transform $L^{-1}\{F(s)\}$ to find the solution $f(t)$ to the differential equation.

Review exercise

29

Use the Laplace transform to solve each of the following equations:

(a) $f'(t) + f(t) = 3$ where $f(0) = 0$

(b) $3f'(t) + 2f(t) = t$ where $f(0) = -2$

(c) $f''(t) + 5f'(t) + 6f(t) = 2e^{-t}$ where $f(0) = 0$ and $f'(0) = 0$

(d) $f''(t) - 4f(t) = \sin 2t$ where $f(0) = 1$ and $f'(0) = -2$

Answers in next frame

30

(a) $f'(t) + f(t) = 3$ where $f(0) = 0$

Taking Laplace transforms of both sides of the equation gives

$$L\{f'(t)\} + L\{f(t)\} = L\{3\} \text{ so that } [sF(s) - f(0)] + F(s) = \frac{3}{s}$$

$$\text{That is } (s+1)F(s) = \frac{3}{s} \text{ so } F(s) = \frac{3}{s(s+1)} = \frac{3}{s} - \frac{3}{s+1}$$

giving the solution as $f(t) = 3 - 3e^{-t} = 3(1 - e^{-t})$

(b) $3f'(t) + 2f(t) = t$ where $f(0) = -2$

Taking Laplace transforms of both sides of the equation gives

$$L\{3f'(t)\} + L\{2f(t)\} = L\{t\} \text{ so that } 3[sF(s) - f(0)] + 2F(s) = \frac{1}{s^2}$$

$$\text{That is } (3s+2)F(s) - (-6) = \frac{1}{s^2} \text{ so } F(s) = \frac{1 - 6s^2}{s^2(3s+2)}$$

The partial fraction breakdown gives

$$F(s) = -\frac{3}{4}.\frac{1}{s} + \frac{1}{2}.\frac{1}{s^2} - \frac{15}{4}.\frac{1}{(3s+2)} = -\frac{3}{4}.\frac{1}{s} + \frac{1}{2}.\frac{1}{s^2} - \frac{5}{4}.\frac{1}{(s+\frac{2}{3})}$$

giving the solution as

$$f(t) = -\frac{3}{4} + \frac{t}{2} - \frac{5e^{-2t/3}}{4}$$

(c) $f''(t) + 5f'(t) + 6f(t) = 2e^{-t}$ where $f(0) = 0$ and $f'(0) = 0$

Taking Laplace transforms of both sides of the equation gives

$L\{f''(t)\} + L\{5f'(t)\} + L\{6f(t)\} = L\{2e^{-t}\}$

so that $[s^2F(s) - sf(0) - f'(0)] + 5[sF(s) - f(0)] + 6F(s) = \dfrac{2}{s+1}$

That is $(s^2 + 5s + 6)F(s) = \dfrac{2}{s+1}$

so $F(s) = \dfrac{2}{(s+1)(s+2)(s+3)} = \dfrac{1}{s+1} - \dfrac{2}{s+2} + \dfrac{1}{s+3}$

giving the solution as

$f(t) = e^{-t} - 2e^{-2t} + e^{-3t}$

(d) $f''(t) - 4f(t) = \sin 2t$ where $f(0) = 1$ and $f'(0) = -2$

Taking Laplace transforms of both sides of the equation gives

$L\{f''(t)\} - L\{4f(t)\} = L\{\sin 2t\}$

so that $[s^2F(s) - sf(0) - f'(0)] - 4F(s) = \dfrac{2}{s^2+2^2}$

That is $(s^2 - 4)F(s) - s.1 - (-2) = \dfrac{2}{s^2+2^2}$

so $F(s) = \dfrac{2}{(s^2-4)(s^2+2^2)} + \dfrac{s-2}{s^2-4}$

$= \dfrac{15}{16}.\dfrac{1}{s+2} + \dfrac{1}{16}.\dfrac{1}{s-2} - \dfrac{1}{8}.\dfrac{2}{s^2+2^2}$

giving the solution as

$f(t) = \dfrac{15}{16}e^{-2t} + \dfrac{1}{16}e^{2t} - \dfrac{\sin 2t}{8}$

So, finally, the **Can You?** *checklist followed*
by the **Test exercise** *and* **Further problems**

Can You?

Checklist 27

31

Check this list before and after you try the end of Programme test.

On a scale of 1 to 5 how confident are you that you can:

Frames

- Derive the Laplace transform of an expression by using the integral definition? 1 to 3
  Yes ☐ ☐ ☐ ☐ ☐ *No*
- Obtain inverse Laplace transforms with the help of a Table of Laplace transforms? 4 to 9
  Yes ☐ ☐ ☐ ☐ ☐ *No*
- Derive the Laplace transform of the derivative of an expression? 10
  Yes ☐ ☐ ☐ ☐ ☐ *No*
- Solve first-order, constant-coefficient, inhomogeneous differential equations using the Laplace transform? 11 to 18
  Yes ☐ ☐ ☐ ☐ ☐ *No*
- Derive further Laplace transforms from known transforms? 19 to 26
  Yes ☐ ☐ ☐ ☐ ☐ *No*
- Use the Laplace transform to obtain the solution to linear, constant-coefficient, inhomogeneous differential equations of higher order than the first? 27 to 30
  Yes ☐ ☐ ☐ ☐ ☐ *No*

Test exercise 27

32

1 Using the integral definition, find the Laplace transforms for each of the following:

(a) $f(t) = 8$ (b) $f(t) = e^{5t}$ (c) $f(t) = -4e^{2t+3}$

2 Using the Table of Laplace transforms, find the inverse Laplace transforms of each of the following:

(a) $F(s) = -\dfrac{5}{(s-2)^2}$ (b) $F(s) = \dfrac{2e^3}{s^3}$

(c) $F(s) = \dfrac{3}{s^2+9}$ (d) $F(s) = -\dfrac{2s-5}{s^2+3}$

3 Given that the Laplace transform of te^{-kt} is $F(s) = \dfrac{1}{(s+k)^2}$ derive the Laplace transform of t^2e^{3t} without using the integral definition.

4 Use the Laplace transform to solve each of the following equations:

(a) $f'(t) + 2f(t) = t$ where $f(0) = 0$

(b) $f'(t) - f(t) = e^{-t}$ where $f(0) = -1$

(c) $f''(t) + 4f'(t) + 4f(t) = e^{-2t}$ where $f(0) = 0$ and $f'(0) = 0$

(d) $4f''(t) - 9f(t) = -18$ where $f(0) = 0$ and $f'(0) = 0$

Further problems 27

33

1 Find the Laplace transform of each of the following expressions (each being defined for $t \geq 0$):

(a) $f(t) = a^{kt}$, $a > 0$ (b) $f(t) = \sinh kt$ (c) $f(t) = \cosh kt$

(d) $f(t) = \begin{cases} k & \text{for } 0 \leq t \leq a \\ 0 & \text{for } t > a \end{cases}$

2 Find the inverse Laplace transform of each of the following:

(a) $F(s) = -\dfrac{2}{3s-4}$ (b) $F(s) = \dfrac{1}{s^2-8}$ (c) $F(s) = \dfrac{3s-4}{s^2+16}$

(d) $F(s) = \dfrac{7s^2+27}{s^3+9s}$ (e) $F(s) = \dfrac{4s}{(s^2-1)^2}$ (f) $F(s) = -\dfrac{s^2-6s+14}{s^3-s^2+4s-4}$

3 Show that if $F(s) = L\{f(t)\} = \displaystyle\int_{t=0}^{\infty} e^{-st} f(t)\,dt$ then:

(a) (i) $F'(s) = -L\{tf(t)\}$ (ii) $F''(s) = L\{t^2 f(t)\}$

Use part (a) to find

(b) (i) $L\{t \sin 2t\}$ (ii) $L\{t^2 \cos 3t\}$

(c) What would you say the nth derivative of $F(s)$ is equal to?

4 Show that if $L\{f(t)\} = F(s)$ then $L\{e^{kt} f(t)\} = F(s-k)$ where k is a constant. Hence find:

(a) $L\{e^{at} \sin bt\}$

(b) $L\{e^{at} \cos bt\}$ where a and b are constants in both cases.

5 Solve each of the following differential equations:

(a)	$f''(t) - 5f'(t) + 6f(t) = 0$	where $f(0) = 0$ and $f'(0) = 1$
(b)	$f''(t) - 5f'(t) + 6f(t) = 1$	where $f(0) = 0$ and $f'(0) = 0$
(c)	$f''(t) - 5f'(t) + 6f(t) = e^{2t}$	where $f(0) = 0$ and $f'(0) = 0$
(d)	$2f''(t) - f'(t) - f(t) = e^{-3t}$	where $f(0) = 2$ and $f'(0) = 1$
(e)	$f(t) + f'(t) - 2f''(t) = te^{-t}$	where $f(0) = 0$ and $f'(0) = 1$
(f)	$f''(t) + 16f(t) = 0$	where $f(0) = 1$ and $f'(0) = 4$
(g)	$2f''(t) - f'(t) - f(t) = \sin t - \cos t$	where $f(0) = 0$ and $f'(0) = 0$

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Now visit the companion website for this book at www.palgrave.com/stroud for more questions applying this mathematics to science and engineering.

"JUST THE MATHS"

UNIT NUMBER

15.1

ORDINARY DIFFERENTIAL EQUATIONS 1 (First order equations (A))

by

A.J.Hobson

UNIT 15.1 - ORDINARY DIFFERENTIAL EQUATIONS 1

FIRST ORDER EQUATIONS (A)

15.1.1 INTRODUCTION AND DEFINITIONS

1. An **ordinary differential equation** is a relationship between an independent variable (such as x), a dependent variable (such as y) and one or more ordinary derivatives of y with respect to x.

   There is no discussion, in Units 15, of **partial** differential equations, which involve partial derivatives (see Units 14). Hence, in what follows, we shall refer simply to "differential equations".

   For example,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = xe^{-2x}, \quad x\frac{\mathrm{d}y}{\mathrm{d}x} = y, \quad x^2\frac{\mathrm{d}y}{\mathrm{d}x} + y\sin x = 0 \ \text{ and } \ \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x+y}{x-y}$$

   are differential equations.

2. The **"order"** of a differential equation is the order of the highest derivative which appears in it.

3. The **"general solution"** of a differential equation is the most general algebraic relationship between the dependent and independent variables which satisfies the differential equation.

   Such a solution will not contain any derivatives; but we shall see that it will contain one or more arbitrary constants (the number of these constants being equal to the order of the equation). The solution need not be an <u>explicit</u> formula for one of the variables in terms of the other.

4. A **"boundary condition"** is a numerical condition which must be obeyed by the solution. It usually amounts to the substitution of particular values of the dependent and independent variables into the general solution.

5. An **"initial condition"** is a boundary condition in which the independent variable takes the value zero.

6. A **"particular solution"** (or **"particular integral"**) is a solution which contains no arbitrary constants.

   Particular solutions are usually the result of appplying a boundary condition to a general solution.

15.1.2 EXACT EQUATIONS

The simplest kind of differential equation of the first order is one which has the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x).$$

It is an elementary example of an **"exact differential equation"** because, to find its solution, all that it is necessary to do is integrate both sides with respect to x.

In other cases of exact differential equations, the terms which are not just functions of the independent variable only, need to be recognised as the exact derivative with respect to x of some known function (possibly involving both of the variables).

The method will be illustrated by examples.

EXAMPLES

1. Solve the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 - 6x + 5,$$

subject to the boundary condition that $y = 2$ when $x = 1$.

Solution

By direct integration, the general solution is

$$y = x^3 - 3x^2 + 5x + C,$$

where C is an arbitrary constant.

From the boundary condition,

$$2 = 1 - 3 + 5 + C, \text{ so that } C = -1.$$

Thus the particular solution obeying the given boundary condition is

$$y = x^3 - 3x^2 + 5x - 1.$$

2. Solve the differential equation

$$x\frac{\mathrm{d}y}{\mathrm{d}x} + y = x^3,$$

subject to the boundary condition that $y = 4$ when $x = 2$.

Solution

The left hand side of the differential equation may be recognised as the exact derivative with respect to x of the function xy.

Hence, we may write

$$\frac{\mathrm{d}}{\mathrm{d}x}(xy) = x^3;$$

and, by direct integration, this gives

$$xy = \frac{x^4}{4} + C,$$

where C is an arbitrary constant.

That is,

$$y = \frac{x^3}{4} + \frac{C}{x}.$$

Applying the boundary condition,

$$4 = 2 + \frac{C}{2},$$

which implies that $C = 4$ and the particular solution is

$$y = \frac{x^3}{4} + \frac{4}{x}.$$

3. Determine the general solution to the differential equation

$$\sin x + \sin y + x\cos y\frac{\mathrm{d}y}{\mathrm{d}x} = 0.$$

Solution

The second and third terms on the right hand side may be recognised as the exact derivative of the function $x\sin y$; and, hence, we may write

$$\sin x + \frac{\mathrm{d}}{\mathrm{d}x}(x \sin y) = 0.$$

By direct integration, we obtain

$$-\cos x + x \sin y = C,$$

where C is an arbitrary constant.

This result counts as the general solution without further modification; but an explicit formula for y in terms of x may, in this case, be written in the form

$$y = \mathrm{Sin}^{-1}\left[\frac{C + \cos x}{x}\right].$$

15.1.3 THE METHOD OF SEPARATION OF THE VARIABLES

The method of this section relates to differential equations of the first order which may be written in the form

$$P(y)\frac{\mathrm{d}y}{\mathrm{d}x} = Q(x).$$

Integrating both sides with respect to x gives

$$\int P(y)\frac{\mathrm{d}y}{\mathrm{d}x}\ \mathrm{d}x = \int Q(x)\ \mathrm{d}x.$$

But, from the formula for integration by substitution in Units 12.3 and 12.4, this simplifies to

$$\int P(y)\ \mathrm{d}y = \int Q(x)\ \mathrm{d}x.$$

Note:
The way to remember this result is to treat dx and dy, in the given differential equation, as if they were separate numbers; then rearrange the equation so that one side contains only y while the other side contains only x; that is, we **separate the variables**. The process is completed by putting an integral sign in front of each side.

EXAMPLES

1. Solve the differential equation

$$x\frac{\mathrm{d}y}{\mathrm{d}x} = y,$$

subject to the boundary condition that $y = 6$ when $x = 2$.

Solution

The differential equation may be rearranged as

$$\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{x};$$

and, hence,

$$\int \frac{1}{y}\,\mathrm{d}y = \int \frac{1}{x}\,\mathrm{d}x,$$

giving

$$\ln y = \ln x + C.$$

Applying the boundary condition,

$$\ln 6 = \ln 2 + C,$$

so that

$$C = \ln 6 - \ln 2 = \ln\left(\frac{6}{2}\right) = \ln 3.$$

The particular solution is therefore

$$\ln y = \ln x + \ln 3 \quad \text{or} \quad y = 3x.$$

Note:

In a general solution where most of the terms are logarithms, the calculation can be made simpler by regarding the arbitrary constant itself as a logarithm, calling it $\ln A$, for instance, rather than C. In the above example, we would then write

$$\ln y = \ln x + \ln A \quad \text{simplifying} \quad \text{to} \quad y = Ax.$$

On applying the boundary condition, $6 = 2A$, so that $A = 3$ and the particular solution is the same as before.

2. Solve the differential equation

$$x(4-x)\frac{\mathrm{d}y}{\mathrm{d}x} - y = 0,$$

subject to the boundary condition that $y = 7$ when $x = 2$.

Solution

The differential equation may be rearranged as

$$\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{x(4-x)}.$$

Hence,

$$\int \frac{1}{y}\ \mathrm{d}y = \int \frac{1}{x(4-x)}\ \mathrm{d}x;$$

or, using the theory of partial fractions,

$$\int \frac{1}{y}\ \mathrm{d}y = \int \left[\frac{\frac{1}{4}}{x} + \frac{\frac{1}{4}}{4-x}\right]\ \mathrm{d}x.$$

The general solution is therefore

$$\ln y = \frac{1}{4}\ln x - \frac{1}{4}\ln(4-x) + \ln A$$

or

$$y = A\left(\frac{x}{4-x}\right)^{\frac{1}{4}}.$$

Applying the boundary condition, $7 = A$, so that the particular solution is

$$y = 7\left(\frac{x}{4-x}\right)^{\frac{1}{4}}.$$

15.1.4 EXERCISES

1. Determine the general solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x^5 + 3e^{-2x}.$$

2. Given that differential equation

$$x^2\frac{\mathrm{d}y}{\mathrm{d}x} + 2xy = \sin x$$

is exact, determine its general solution.

3. Given that the differential equation

$$\tan x\frac{\mathrm{d}y}{\mathrm{d}x} + y\mathrm{sec}^2 x = \cos 2x$$

is exact, determine the particular solution for which $y = 1$ when $x = \frac{\pi}{4}$.

4. Use the method of separation of the variables to determine the general solution of each of the following differential equations:

(a)

$$\frac{\mathrm{d}x}{\mathrm{d}y} = (x-1)(x+2);$$

(b)

$$x(y-3)\frac{\mathrm{d}y}{\mathrm{d}x} = 4y.$$

5. Use the method of separation of the variables to solve the following differential equations subject to the given boundary condition:

(a)

$$(1+x^3)\frac{\mathrm{d}y}{\mathrm{d}x} = x^2 y,$$

where $y = 2$ when $x = 1$;

(b)

$$x^3 + (y+1)^2\frac{\mathrm{d}y}{\mathrm{d}x} = 0,$$

where $y = 0$ when $x = 0$.

15.1.5 ANSWERS TO EXERCISES

1.

$$y = \frac{x^6}{6} - \frac{3e^{-2x}}{2} + C.$$

2.

$$y = \frac{C - \cos x}{x^2}.$$

3.

$$y = \frac{3}{2}\cot x - \cos^2 x.$$

4. (a)

$$y = \ln\left[A\left(\frac{x-1}{x+2}\right)^{\frac{1}{3}}\right];$$

(b)

$$y = \ln[Ax^4y^3].$$

5. (a)

$$y^3 = 4(1 + x^3);$$

(b)

$$4[1 - (y+1)^3] = 3x^4.$$

"JUST THE MATHS"

UNIT NUMBER

15.2

ORDINARY
DIFFERENTIAL EQUATIONS 2
(First order equations (B))

by

A.J.Hobson

UNIT 15.2 - ORDINARY DIFFERENTIAL EQUATIONS 2

FIRST ORDER EQUATIONS (B)

15.2.1 HOMOGENEOUS EQUATIONS

A differential equation of the first order is said to be **"homogeneous"** if, on replacing x by λx and y by λy in all the parts of the equation except $\frac{dy}{dx}$, λ may be removed from the equation by cancelling a common factor of λ^n, for some integer n.

Note:
Some examples of homogeneous equations would be

$$(x+y)\frac{dy}{dx}+(4x-y)=0, \quad \text{and} \quad 2xy\frac{dy}{dx}+(x^2+y^2)=0,$$

where, from the first of these, a factor of λ could be cancelled and, from the second, a factor of λ^2 could be cancelled.

15.2.2 THE STANDARD METHOD

It turns out that the substitution

$$\boxed{y = vx} \quad \left(\text{giving } \frac{dy}{dx} = v + x\frac{dv}{dx}\right),$$

always converts a homogeneous differential equation into one in which the variables can be separated. The method will be illustrated by examples.

EXAMPLES

1. Solve the differential equation
$$x\frac{dy}{dx} = x + 2y,$$
subject to the condition that $y = 6$ when $x = 6$.
Solution
If $y = vx$, then $\frac{dy}{dx} = v + x\frac{dv}{dx}$, so that the differential equation becomes
$$x\left(v + x\frac{dy}{dx}\right) = x + 2vx.$$

That is,

$$v + x\frac{\mathrm{d}v}{\mathrm{d}x} = 1 + 2v$$

or

$$x\frac{\mathrm{d}v}{\mathrm{d}x} = 1 + v.$$

On separating the variables,

$$\int \frac{1}{1+v}\ \mathrm{d}v = \int \frac{1}{x}\ \mathrm{d}x,$$

giving

$$\ln(1+v) = \ln x + \ln A,$$

where A is an arbitrary constant.
An alternative form of this solution, without logarithms, is

$$Ax = 1 + v$$

and, substituting back $v = \frac{y}{x}$, the solution becomes

$$Ax = 1 + \frac{y}{x}$$

or

$$y = Ax^2 - x.$$

Finally, if $y = 6$ when $x = 1$, we have $6 = A - 1$ and, hence, $A = 7$.
The required particular solution is thus

$$y = 7x^2 - x.$$

2. Determine the general solution of the differential equation

$$(x+y)\frac{\mathrm{d}y}{\mathrm{d}x} + (4x - y) = 0.$$

Solution

If $y = vx$, then $\frac{\mathrm{d}y}{\mathrm{d}x} = v + x\frac{\mathrm{d}v}{\mathrm{d}x}$, so that the differential equation becomes

$$(x + vx)\left(v + x\frac{\mathrm{d}v}{\mathrm{d}x}\right) + (4x - vx) = 0.$$

That is,

$$(1 + v)\left(v + x\frac{\mathrm{d}v}{\mathrm{d}x}\right) + (4 - v) = 0$$

or

$$v + x\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{v - 4}{v + 1}.$$

On further rearrangement, we obtain

$$x\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{v - 4}{v + 1} - v = \frac{-4 - v^2}{v + 1};$$

and, on separating the variables,

$$\int \frac{v + 1}{4 + v^2}\ \mathrm{d}v = -\int \frac{1}{x}\ \mathrm{d}x$$

or

$$\frac{1}{2}\int \left[\frac{2v}{4 + v^2} + \frac{2}{4 + v^2}\right]\ \mathrm{d}v = -\int \frac{1}{x}\ \mathrm{d}x.$$

Hence,

$$\frac{1}{2}\left[\ln(4 + v^2) + \tan^{-1}\frac{v}{2}\right] = -\ln x + C,$$

where C is an arbitrary constant.

Substituting back $v = \frac{y}{x}$, gives the general solution

$$\frac{1}{2}\left[\ln\left(4 + \frac{y^2}{x^2}\right) + \tan^{-1}\left(\frac{y}{2x}\right)\right] = -\ln x + C.$$

3. Determine the general solution of the differential equation

$$2xy\frac{\mathrm{d}y}{\mathrm{d}x} + (x^2 + y^2) = 0.$$

Solution

If $y = vx$, then $\frac{\mathrm{d}y}{\mathrm{d}x} = v + x\frac{\mathrm{d}v}{\mathrm{d}x}$, so that the differential equation becomes

$$2vx^2\left(v + x\frac{\mathrm{d}v}{\mathrm{d}x}\right) + (x^2 + v^2x^2) = 0.$$

That is,

$$2v\left(v + x\frac{\mathrm{d}v}{\mathrm{d}x}\right) + (1 + v^2) = 0$$

or

$$2vx\frac{\mathrm{d}v}{\mathrm{d}x} = -(1 + 3v^2).$$

On separating the variables, we obtain

$$\int \frac{2v}{1 + 3v^2}\ \mathrm{d}x = -\int \frac{1}{x}\ \mathrm{d}x,$$

which gives

$$\frac{1}{3}\ln(1 + 3v^2) = -\ln x + \ln A,$$

where A is an arbitrary constant.

Hence,

$$(1 + 3v^2)^{\frac{1}{3}} = \frac{A}{x}$$

or, on substituting back $v = \frac{y}{x}$,

$$\left(\frac{x^2 + 3y^2}{x^2}\right)^{\frac{1}{3}} = Ax,$$

which can be written

$$x^2 + 3y^2 = Bx^5,$$

where $B = A^3$.

15.2.3 EXERCISES

Use the substitution $y = vx$ to solve the following differential equations subject to the given boundary condition:

1.
$$(2y - x)\frac{\mathrm{d}y}{\mathrm{d}x} = 2x + y,$$
where $y = 3$ when $x = -2$.

2.
$$(x^2 - y^2)\frac{\mathrm{d}y}{\mathrm{d}x} = xy,$$
where $y = 5$ when $x = 0$.

3.
$$x^3 + y^3 = 3xy^2\frac{\mathrm{d}y}{\mathrm{d}x},$$
where $y = 1$ when $x = 2$.

4.
$$x(x^2 + y^2)\frac{\mathrm{d}y}{\mathrm{d}x} = 2y^3,$$
where $y = 2$ when $x = 1$.

5.
$$x\frac{\mathrm{d}y}{\mathrm{d}x} - (y + \sqrt{x^2 - y^2}) = 0,$$
where $y = 0$ when $x = 1$.

15.2.4 ANSWERS TO EXERCISES

1.

$$y^2 - xy - x^2 = 11.$$

2.

$$y = 5e^{-\frac{x^2}{2y^2}}.$$

3.

$$x^3 - 2y^3 = 3x.$$

4.

$$3x^2y = 2(y^2 - x^2).$$

5.

$$e^{\sin^{-1}\frac{y}{x}} = x.$$

"JUST THE MATHS"

UNIT NUMBER

15.3

ORDINARY
DIFFERENTIAL EQUATIONS 3
(First order equations (C))

by

A.J.Hobson

UNIT 15.3 - ORDINARY DIFFERENTIAL EQUATIONS 3

FIRST ORDER EQUATIONS (C)

15.3.1 LINEAR EQUATIONS

For certain kinds of first order differential equation, it is possible to multiply the equation throughout by a suitable factor which converts it into an exact differential equation.

For instance, the equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{1}{x}y = x^2$$

may be multiplied throughout by x to give

$$x\frac{\mathrm{d}y}{\mathrm{d}x} + y = x^3.$$

It may now be written

$$\frac{\mathrm{d}}{\mathrm{d}x}(xy) = x^3$$

and, hence, it has general solution

$$xy = \frac{x^4}{4} + C,$$

where C is an arbitrary constant.

Notes:

(i) The factor, x which has multiplied both sides of the differential equation serves as an **"integrating factor"**, but such factors cannot always be found by inspection.

(ii) In the discussion which follows, we shall develop a formula for determining integrating factors, in general, for what are known as **"linear differential equations"**.

DEFINITION

A differential equation of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x)$$

is said to be **"linear"**.

RESULT

Given the linear differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x),$$

the function

$$e^{\int P(x)\ \mathrm{d}x}$$

is always an integrating factor; and, on multiplying the differential equation throughout by this factor, its left hand side becomes

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[y \times e^{\int P(x)\ \mathrm{d}x}\right].$$

Proof

Suppose that the function, $R(x)$, is an integrating factor; then, in the equation

$$R(x)\frac{\mathrm{d}y}{\mathrm{d}x} + R(x)P(x)y = R(x)Q(x),$$

the left hand side must be the exact derivative of some function of x.

Using the formula for differentiating the product of two functions of x, we can **make** it the derivative of $R(x)y$ provided we can arrange that

$$R(x)P(x) = \frac{\mathrm{d}}{\mathrm{d}x}[R(x)].$$

But this requirement can be interpreted as a differential equation in which the variables $R(x)$ and x may be separated as follows:

$$\int \frac{1}{R(x)} \, \mathrm{d}R(x) = \int P(x) \, \mathrm{d}x.$$

Hence,

$$\ln R(x) = \int P(x) \, \mathrm{d}x.$$

That is,

$$R(x) = e^{\int P(x) \, \mathrm{d}x},$$

as required.

The solution is obtained by integrating the formula

$$\frac{\mathrm{d}}{\mathrm{d}x}[y \times R(x)] = R(x)P(x).$$

Note:
There is no need to include an arbitrary constant, C, when $P(x)$ is integrated, since it would only serve to introduce a constant factor of e^C in the above result, which would then immediately cancel out on multiplying the differential equation by $R(x)$.

EXAMPLES

1. Determine the general solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{1}{x}y = x^2.$$

Solution

An integrating factor is

$$e^{\int \frac{1}{x} \, \mathrm{d}x} = e^{\ln x} = x.$$

On multiplying throughout by the integrating factor, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}[y \times x] = x^3;$$

and so,

$$yx = \frac{x^4}{4} + C,$$

where C is an arbitrary constant.

2. Determine the general solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + 2xy = 2e^{-x^2}.$$

Solution
An integrating factor is

$$e^{\int \ 2x \ \mathrm{d}x} = e^{x^2}.$$

Hence,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[y \times e^{x^2}\right] = 2,$$

giving

$$ye^{x^2} = 2x + C,$$

where C is an arbitrary constant.

15.3.2 BERNOUILLI'S EQUATION

A similar type of differential equation to that in the previous section has the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x)y^n.$$

It is called **"Bernouilli's Equation"** and may be converted to a linear differential equation by making the substitution

$$z = y^{1-n}.$$

Proof

The differential equation may be rewritten as

$$y^{-n}\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y^{1-n} = Q(x).$$

Also,

$$\frac{\mathrm{d}z}{\mathrm{d}x} = (1-n)y^{-n}\frac{\mathrm{d}y}{\mathrm{d}x}.$$

Hence the differential equation becomes

$$\frac{1}{1-n}\frac{\mathrm{d}z}{\mathrm{d}x} + P(x)z = Q(x).$$

That is,

$$\frac{\mathrm{d}z}{\mathrm{d}x} + (1-n)P(x)z = (1-n)Q(x),$$

which is a linear differential equation.

Note:
It is better not to regard this as a standard formula, but to apply the method of obtaining it in the case of particular examples.

EXAMPLES

1. Determine the general solution of the differential equation

$$xy - \frac{\mathrm{d}y}{\mathrm{d}x} = y^3 e^{-x^2}.$$

Solution

The differential equation may be rewritten

$$-y^{-3}\frac{\mathrm{d}y}{\mathrm{d}x} + x.y^{-2} = e^{-x^2}.$$

Substituting $z = y^{-2}$, we obtain $\frac{\mathrm{d}z}{\mathrm{d}x} = -2y^{-3}\frac{\mathrm{d}y}{\mathrm{d}x}$ and, hence,

$$\frac{1}{2}\frac{\mathrm{d}z}{\mathrm{d}x} + xz = e^{-x^2}$$

or

$$\frac{\mathrm{d}z}{\mathrm{d}x} + 2xz = 2e^{-x^2}.$$

An integrating factor for this equation is

$$e^{\int\ 2x\ \mathrm{d}x} = e^{x^2}.$$

Thus,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(ze^{x^2}\right) = 2,$$

giving

$$ze^{x^2} = 2x + C,$$

where C is an arbitrary constant.

Finally, replacing z by y^{-2},

$$y^2 = \frac{e^{x^2}}{2x + C}.$$

2. Determine the general solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{y}{x} = xy^2.$$

Solution

The differential equation may be rewritten

$$y^{-2}\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{1}{x}.y^{-1} = x.$$

On substituting $z = y^{-1}$ we obtain $\frac{\mathrm{d}z}{\mathrm{d}x} = -y^{-2}\frac{\mathrm{d}y}{\mathrm{d}x}$ so that

$$-\frac{\mathrm{d}z}{\mathrm{d}x} + \frac{1}{x}.z = x$$

or

$$\frac{\mathrm{d}z}{\mathrm{d}x} - \frac{1}{x}.z = -x.$$

An integrating factor for this equation is

$$e^{\int \left(-\frac{1}{x}\right)\,\mathrm{d}x} = e^{-\ln x} = \frac{1}{x}.$$

Hence,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(z \times \frac{1}{x}\right) = -1,$$

giving

$$\frac{z}{x} = -x + C,$$

where C is an arbitrary constant.

The general solution of the given differential equation is therefore

$$\frac{1}{xy} = -x + C \quad \text{or} \quad y = \frac{1}{Cx - x^2}.$$

15.3.3 EXERCISES

Use an integrating factor to solve the following differential equations subject to the given boundary condition:

1.

$$3\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = 0,$$

where $y = 10$ when $x = 0$.

2.

$$3\frac{\mathrm{d}y}{\mathrm{d}x} - 5y = 10,$$

where $y = 4$ when $x = 0$.

3.

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{y}{x} = 3x,$$

where $y = 2$ when $x = -1$.

4.

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{y}{1-x} = 1 - x^2,$$

where $y = 0$ when $x = -1$.

5.

$$\frac{\mathrm{d}y}{\mathrm{d}x} + y\cot x = \cos x,$$

where $y = \frac{5}{2}$ when $x = \frac{\pi}{2}$.

6.

$$(x^2+1)\frac{\mathrm{d}y}{\mathrm{d}x} - xy = x,$$

where $y = 0$ when $x = 1$.

7.

$$3y - 2\frac{\mathrm{d}y}{\mathrm{d}x} = y^3 e^{4x},$$

where $y = 1$ when $x = 0$.

8.

$$2y - x\frac{\mathrm{d}y}{\mathrm{d}x} = x(x-1)y^4,$$

where $y^3 = 14$ when $x = 1$.

15.3.4 ANSWERS TO EXERCISES

1.

$$y = 10e^{-\frac{2}{3}x}.$$

2.

$$y = 6e^{\frac{5}{3}x} - 2.$$

3.

$$yx = x^3 - 1.$$

4.

$$y = \frac{1}{2}(1-x)(1+x)^2.$$

5.

$$y = \frac{\sin x}{2} + \frac{2}{\sin x}.$$

6.

$$y = 1 + x^2 - \sqrt{2(1+x^2)}.$$

7.

$$y^2 = \frac{7e^{3x}}{e^{7x}+6}.$$

8.

$$y^3 = \frac{56x^6}{21x^6 - 24x^7 + 7}.$$

"JUST THE MATHS"

UNIT NUMBER

15.4

ORDINARY DIFFERENTIAL EQUATIONS 4 (Second order equations (A))

by

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UNIT 15.4 - ORDINARY DIFFERENTIAL EQUATIONS 4

SECOND ORDER EQUATIONS (A)

15.4.1 INTRODUCTION

In the discussion which follows, we shall consider a particular kind of second order ordinary differential equation which is called **"linear, with constant coefficients"**; it has the general form

$$a\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + b\frac{\mathrm{d}y}{\mathrm{d}x} + cy = f(x),$$

where a, b and c are the constant coefficients.

The various cases of solution which arise depend on the values of the coefficients, together with the type of function, $f(x)$, on the right hand side. These cases will now be dealt with in turn.

15.4.2 SECOND ORDER HOMOGENEOUS EQUATIONS

The term **"homogeneous"**, in the context of second order differential equations, is used to mean that the function, $f(x)$, on the right hand side is zero. It should not be confused with the previous use of this term in the context of first order differential equations.

We therefore consider equations of the general form

$$a\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + b\frac{\mathrm{d}y}{\mathrm{d}x} + cy = 0.$$

Note:
A very simple case of this equation is

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0,$$

which, on integration twice, gives the general solution

$$y = Ax + B,$$

where A and B are arbitrary constants. We should therefore expect <u>two</u> arbitrary constants in the solution of <u>any</u> second order linear differential equation with constant coefficients.

The Standard General Solution

The equivalent of

$$a\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + b\frac{\mathrm{d}y}{\mathrm{d}x} + cy = 0$$

in the discussion of <u>first</u> order differential equations would have been

$$b\frac{\mathrm{d}y}{\mathrm{d}x} + cy = 0; \text{ that is, } \frac{\mathrm{d}y}{\mathrm{d}x} + \frac{c}{b}y = 0$$

and this could have been solved using an integrating factor of

$$e^{\int \frac{c}{b}\,\mathrm{d}x} = e^{\frac{c}{b}x},$$

giving the general solution

$$y = Ae^{-\frac{c}{b}x},$$

where A is an arbitrary constant.

It seems reasonable, therefore, to make a trial solution of the form $y = Ae^{mx}$, where $A \neq 0$, in the second order case.

We shall need

$$\frac{\mathrm{d}y}{\mathrm{d}x} = Ame^{mx} \text{ and } \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = Am^2e^{mx}.$$

Hence, on substituting the trial solution, we require that

$$aAm^2e^{mx} + bAme^{mx} + cAe^{mx} = 0;$$

and, by cancelling Ae^{mx}, this condition reduces to

$$am^2 + bm + c = 0,$$

a quadratic equation, called the **"auxiliary equation"**, having the same (constant) coefficients as the orginal differential equation.

In general, it will have two solutions, say $m = m_1$ and $m = m_2$, giving corresponding solutions $y = Ae^{m_1 x}$ and $y = Be^{m_2 x}$ of the differential equation.

However, the linearity of the differential equation implies that the sum of any two solutions is also a solution, so that

$$y = Ae^{m_1 x} + Be^{m_2 x}$$

is another solution; and, since this contains two arbitrary constants, we shall take it to be the <u>general</u> solution.

Notes:

(i) It may be shown that there are no solutions other than those of the above form though special cases are considered later.

(ii) It will be possible to determine particular values of A and B if an appropriate number of boundary conditions for the differential equation are specified. These will usually be a set of given values for y and $\frac{\mathrm{d}y}{\mathrm{d}x}$ at a certain value of x.

EXAMPLE

Determine the general solution of the differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 5\frac{\mathrm{d}y}{\mathrm{d}x} + 6y = 0$$

and also the particular solution for which $y = 2$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = -5$ when $x = 0$.

Solution

The auxiliary equation is $m^2 + 5m + 6 = 0$,

which can be factorised as

$$(m+2)(m+3) = 0.$$

Its solutions are therefore $m = -2$ and $m = -3$.

Hence, the differential equation has general solution

$$y = Ae^{-2x} + Be^{-3x},$$

where A and B are arbitrary constants.

Applying the boundary conditions, we shall also need

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -2Ae^{-2x} - 3Be^{-3x}.$$

Hence,

$$\begin{aligned} 2 &= A + B, \\ -5 &= -2A - 3B \end{aligned}$$

giving $A = 1$, $B = 1$ and a particular solution

$$y = e^{-2x} + e^{-3x}.$$

15.4.3 SPECIAL CASES OF THE AUXILIARY EQUATION

(a) The auxilary equation has coincident solutions

Suppose that both solutions of the auxiliary equation are the same number, m_1.

In other words, the quadratic expression $am^2 + bm + c$ is a **"perfect square"**, which means that it is actually $a(m - m_1)^2$.

Apparently, the general solution of the differential equation is

$$y = Ae^{m_1x} + Be^{m_1x},$$

which does not genuinely contain two arbitrary constants since it can be rewritten as

$$y = Ce^{m_1 x} \quad \text{where} \quad C = A + B.$$

It will not, therefore, count as the general solution, though the fault seems to lie with the constants A and B rather than with m_1.

Consequently, let us now examine a new trial solution of the form

$$y = ze^{m_1 x},$$

where z denotes a function of x rather than a constant.

We shall also need

$$\frac{\mathrm{d}y}{\mathrm{d}x} = zm_1 e^{m_1 x} + e^{m_1 x}\frac{\mathrm{d}z}{\mathrm{d}x}$$

and

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = zm_1^2 e^{m_1 x} + 2m_1 e^{m_1 x}\frac{\mathrm{d}z}{\mathrm{d}x} + e^{m_1 x}\frac{\mathrm{d}^2 z}{\mathrm{d}x^2}.$$

On substituting these into the differential equation, we obtain the condition that

$$e^{m_1 x}\left[a\left(zm_1^2 + 2m_1\frac{\mathrm{d}z}{\mathrm{d}x} + \frac{\mathrm{d}^2 z}{\mathrm{d}x^2}\right) + b\left(zm_1 + \frac{\mathrm{d}z}{\mathrm{d}x}\right) + cz\right] = 0$$

or

$$z(am_1^2 + bm_1 + c) + \frac{\mathrm{d}z}{\mathrm{d}x}(2am_1 + b) + a\frac{\mathrm{d}^2 z}{\mathrm{d}x^2} = 0.$$

The first term on the left hand side of this condition is zero since m_1 is already a solution of the auxiliary equation; and the second term is also zero since the auxiliary equation, $am^2 + bm + c = 0$, is equivalent to $a(m - m_1)^2 = 0$; that is, $am^2 - 2am_1 m + am_1^2 = 0$. Thus $b = -2am_1$.

We conclude that $\frac{\mathrm{d}^2z}{\mathrm{d}x^2} = 0$ with the result that $z = Ax + B$, where A and B are arbitrary constants.

The general solution of the differential equation in the case of coincident solutions to the auxiliary equation is therefore

$$y = (Ax + B)e^{m_1 x}.$$

EXAMPLE

Determine the general solution of the differential equation

$$4\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 4\frac{\mathrm{d}y}{\mathrm{d}x} + y = 0.$$

Solution

The auxilary equation is

$$4m^2 + 4m + 1 = 0 \quad \text{or} \quad (2m + 1)^2 = 0$$

and it has coincident solutions at $m = -\frac{1}{2}$.

The general solution is therefore

$$y = (Ax + B)e^{-\frac{1}{2}x}.$$

(b) The auxiliary equation has complex solutions

If the auxiliary equation has complex solutions, they will automatically appear as a pair of **"complex conjugates"**, say $m = \alpha \pm j\beta$.

Using these two solutions instead of the previous m_1 and m_2, the general solution of the differential equation will be

$$y = Pe^{(\alpha + j\beta)x} + Qe^{(\alpha - j\beta)x},$$

where P and Q are arbitrary constants.

But, by properties of complex numbers, a neater form of this result is obtainable as follows:

$$y = e^{\alpha x}[P(\cos \beta x + j \sin \beta x) + Q(\cos \beta x - j \sin \beta x)]$$

or

$$y = e^{\alpha x}[(P + Q) \cos \beta x + j(P - Q) \sin \beta x].$$

Replacing $P+Q$ and $j(P-Q)$ (which are just arbitrary quantities) by A and B, we obtain the standard general solution for the case in which the auxiliary equation has complex solutions. It is

$$y = e^{\alpha x}[A \cos \beta x + B \sin \beta x].$$

EXAMPLE

Determine the general solution of the differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 6\frac{\mathrm{d}y}{\mathrm{d}x} + 13y = 0.$$

Solution

The auxiliary equation is

$$m^2 - 6m + 13 = 0,$$

which has solutions given by

$$m = \frac{-(-6) \pm \sqrt{(-6)^2 - 4 \times 13 \times 1}}{2 \times 1} = \frac{6 \pm j4}{2} = 3 \pm j2.$$

The general solution is therefore

$$y = e^{3x}[A\cos 2x + B\sin 2x],$$

where A and B are arbitrary constants.

15.4.4 EXERCISES

1. Determine the general solutions of the following differential equations:

   (a)

   $$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 7\frac{\mathrm{d}y}{\mathrm{d}x} + 12y = 0;$$

   (b)

   $$\frac{\mathrm{d}^2r}{\mathrm{d}\theta^2} + 6\frac{\mathrm{d}r}{\mathrm{d}\theta} + 9r = 0;$$

   (c)

   $$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + 4\frac{\mathrm{d}\theta}{\mathrm{d}t} + 5\theta = 0.$$

2. Solve the following differential equations, subject to the given boundary conditions:

   (a)

   $$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 3\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = 0,$$

   where $y = 2$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = 1$ when $x = 0$;

   (b)

   $$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} - 4\frac{\mathrm{d}x}{\mathrm{d}t} + 3x = 0,$$

   where $x = 3$ and $\frac{\mathrm{d}x}{\mathrm{d}t} = 5$ when $t = 0$;

   (c)

   $$4\frac{\mathrm{d}^2z}{\mathrm{d}s^2} - 12\frac{\mathrm{d}z}{\mathrm{d}s} + 9z = 0,$$

   where $z = 1$ and $\frac{\mathrm{d}z}{\mathrm{d}s} = \frac{5}{2}$ when $s = 0$;

(d)

$$\frac{\mathrm{d}^2r}{\mathrm{d}\theta^2} - 2\frac{\mathrm{d}r}{\mathrm{d}\theta} + 2r = 0,$$

where $r = 5$ and $\frac{\mathrm{d}r}{\mathrm{d}\theta} = 7$ when $\theta = 0$.

15.4.5 ANSWERS TO EXERCISES

1. (a)

$$y = Ae^{-3x} + Be^{-4x};$$

(b)

$$r = (A\theta + B)e^{-3\theta};$$

(c)

$$\theta = e^{-2t}[A\cos 2t + B\sin 2t].$$

2. (a)

$$y = 3e^x - e^{2x};$$

(b)

$$x = 2e^t + e^{3t};$$

(c)

$$z = (s+1)e^{\frac{3}{2}s};$$

(d)

$$r = e^{\theta}[5\cos\theta + 2\sin\theta].$$

"JUST THE MATHS"

UNIT NUMBER

15.5

ORDINARY
DIFFERENTIAL EQUATIONS 5
(Second order equations (B))

by

A.J.Hobson

UNIT 15.5 - ORDINARY DIFFERENTIAL EQUATIONS 5

SECOND ORDER EQUATIONS (B)

15.5.1 NON-HOMOGENEOUS DIFFERENTIAL EQUATIONS

The following discussion will examine the solution of the second order linear differential equation

$$a\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+b\frac{\mathrm{d}y}{\mathrm{d}x}+cy=f(x),$$

in which a, b and c are constants, but $f(x)$ is <u>not</u> identically equal to zero.

The Particular Integral and Complementary Function

(i) Suppose that $y=u(x)$ is any <u>particular</u> solution of the differential equation; that is, it contains no arbitrary constants. In the present context, we shall refer to such particular solutions as **"particular integrals"** and systematic methods of finding them will be discussed later.

It follows that

$$a\frac{\mathrm{d}^2u}{\mathrm{d}x^2}+b\frac{\mathrm{d}u}{\mathrm{d}x}+cu=f(x).$$

(ii) Suppose also that we make the substitution $y=u(x)+v(x)$ in the original differential equation to give

$$a\frac{\mathrm{d}^2(u+v)}{\mathrm{d}x^2}+b\frac{\mathrm{d}(u+v)}{\mathrm{d}x}+c(u+v)=f(x).$$

That is,

$$a\frac{\mathrm{d}^2u}{\mathrm{d}x^2}+b\frac{\mathrm{d}u}{\mathrm{d}x}+cu+a\frac{\mathrm{d}^2v}{\mathrm{d}x^2}+b\frac{\mathrm{d}v}{\mathrm{d}x}+cv=f(x);$$

and, hence,

$$a\frac{\mathrm{d}^2v}{\mathrm{d}x^2}+b\frac{\mathrm{d}v}{\mathrm{d}x}+cv=0.$$

This means that the function $v(x)$ is the general solution of the homogeneous differential equation whose auxiliary equation is

$$am^2 + bm + c = 0.$$

In future, $v(x)$ will be called the **"complementary function"** in the general solution of the original (non-homogeneous) differential equation. It complements the particular integral to provide the general solution.

Summary

$$\boxed{\textbf{General solution = particular integral + complementary function.}}$$

15.5.2 DETERMINATION OF SIMPLE PARTICULAR INTEGRALS

(a) Particular integrals, when $f(x)$ is a constant, k.

For the differential equation

$$a\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + b\frac{\mathrm{d}y}{\mathrm{d}x} + cy = k,$$

it is easy to see that a particular integral will be $y = \frac{k}{c}$, since its first and second derivatives are both zero, while $cy = k$.

EXAMPLE

Determine the general solution of the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 7\frac{\mathrm{d}y}{\mathrm{d}x} + 10y = 20.$$

Solution

(i) By inspection, we may observe that a particular integral is $y = 2$.

(ii) The auxiliary equation is

$$m^2 + 7m + 10 = 0 \quad \text{or} \quad (m+2)(m+5) = 0,$$

having solutions $m = -2$ and $m = -5$.

(iii) The complementary function is

$$Ae^{-2x} + Be^{-5x},$$

where A and B are arbitrary constants.

(iv) The general solution is

$$y = 2 + Ae^{-2x} + Be^{-5x}.$$

(b) Particular integrals, when $f(x)$ is of the form $px + q$.

For the differential equation

$$a\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + b\frac{\mathrm{d}y}{\mathrm{d}x} + cy = px + q,$$

it is possible to determine a particular integral by assuming one which has the same form as the right hand side; that is, in this case, another expression consisting of a multiple of x and constant term. The method is, again, illustrated by an example.

EXAMPLE

Determine the general solution of the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 11\frac{\mathrm{d}y}{\mathrm{d}x} + 28y = 84x - 5.$$

Solution

(i) First, we assume a particular integral of the form

$$y = \alpha x + \beta,$$

which implies that $\frac{\mathrm{d}y}{\mathrm{d}x} = \alpha$ and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0$.

Substituting into the differential equation, we require that

$$-11\alpha + 28(\alpha x + \beta) \equiv 84x - 5.$$

Hence, $28\alpha = 84$ and $-11\alpha + 28\beta = -5$, giving $\alpha = 3$ and $\beta = 1$.

Thus, the particular integral is

$$y = 3x + 1.$$

(ii) The auxiliary equation is

$$m^2 - 11m + 28 = 0 \text{ or } (m-4)(m-7) = 0,$$

having solutions $m = 4$ and $m = 7$.

(iii) The complementary function is

$$Ae^{4x} + Be^{7x},$$

where A and B are arbitrary constants.

(iv) The general solution is

$$y = 3x + 1 + Ae^{4x} + Be^{7x}.$$

Note:
In examples of the above types, the complementary function must not be prefixed by "$y =$", since the given differential equation, as a whole, is not normally satisfied by the complementary function alone.

15.5.3 EXERCISES

1. Determine the general solutions of the following differential equations:

   (a)

   $$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - \frac{\mathrm{d}y}{\mathrm{d}x} - 2y = 6;$$

   (b)

   $$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 16y = 7;$$

   (c)

   $$3\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 2\frac{\mathrm{d}y}{\mathrm{d}x} - y = x + 1;$$

   (d)

   $$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 6\frac{\mathrm{d}y}{\mathrm{d}x} + 9y = 18x + 28.$$

2. Solve, completely, the following differential equations, subject to the given boundary condidtions:

   (a)

   $$2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 7\frac{\mathrm{d}y}{\mathrm{d}x} - 4y = 100,$$

   where $y = -26$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = 5$ when $x = 0$;

   (b)

   $$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 4\frac{\mathrm{d}y}{\mathrm{d}x} + 4y = 12x + 16,$$

   where $y = 0$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = 4$ when $x = 0$;

   (c)

   $$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 6\frac{\mathrm{d}y}{\mathrm{d}x} + 10y = 10x + 14,$$

   where $y = 3$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = 2$ when $x = 0$.

15.5.4 ANSWERS TO EXERCISES

1. (a)

$$y = -3 + Ae^{x} + Be^{2x};$$

   (b)

$$y = \frac{7}{16} + A\cos 4x + B\sin 4x;$$

   (c)

$$y = 1 - x + Ae^{x} + Be^{-\frac{1}{3}x};$$

   (d)

$$y = 2x + 5 + (Ax + B)e^{3x}.$$

2. (a)

$$y = -25 + e^{4x} - 2e^{\frac{1}{2}x};$$

   (b)

$$y = 3x + 1 - (x + 1)e^{-2x};$$

   (c)

$$y = x + 2 + e^{3x}(\cos x - 2\sin x).$$

"JUST THE MATHS"

UNIT NUMBER

15.6

ORDINARY DIFFERENTIAL EQUATIONS 6
(Second order equations (C))

by

A.J.Hobson

UNIT 15.6 - ORDINARY DIFFERENTIAL EQUATIONS 6

SECOND ORDER EQUATIONS (C)

15.6.1 RECAP

For the differential equation

$$a\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + b\frac{\mathrm{d}y}{\mathrm{d}x} + cy = f(x),$$

it was seen, in Unit 15.5, that

(a) when $f(x) \equiv k$, a given **constant**, a particular integral is $y = \frac{k}{c}$;

(b) when $f(x) \equiv px+q$, a **linear** function in which p and q are given constants, it is possible to obtain a particular integral by assuming that y also has the form of a linear function; that is, we make a **"trial solution"**, $y = \alpha x + \beta$.

15.6.2 FURTHER TYPES OF PARTICULAR INTEGRAL

We now examine particular integrals for other cases of $f(x)$, the method being illustrated by examples. Also, for reasons relating to certain problematic cases discussed in Unit 15.7, we shall determine the complementary function **before** determining the particular integral.

1. $f(x) \equiv px^2 + qx + r$, a **quadratic** function in which p, q and r are given constants; $p \neq 0$.

$$\text{Trial solution :} \quad y = \alpha x^2 + \beta x + \gamma.$$

Note:
This is the trial solution even if q or r (or both) are zero.

EXAMPLE

Determine the general solution of the differential equation

$$2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 7\frac{\mathrm{d}y}{\mathrm{d}x} - 4y = 4x^2 + 10x - 23.$$

Solution

The auxilary equation is

$$2m^2 - 7m - 4 = 0 \ \text{ or } \ (2m+1)(m-4) = 0,$$

having solutions $m = 4$ and $m = -\frac{1}{2}$.
Thus, the complementary function is

$$Ae^{4x} + Be^{-\frac{1}{2}x},$$

where A and B are arbitrary constants.
To determine a particular integral, we may make a trial solution of the form $y = \alpha x^2 + \beta x + \gamma$, giving $\frac{\mathrm{d}y}{\mathrm{d}x} = 2\alpha x + \beta$ and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 2\alpha$.
We thus require that

$$4\alpha - 14\alpha x - 7\beta - 4\alpha x^2 - 4\beta x - 4\gamma \equiv 4x^2 + 10x - 23.$$

That is,

$$-4\alpha x^2 - (14\alpha + 4\beta)x + 4\alpha - 7\beta - 4\gamma \equiv 4x^2 + 10x - 23.$$

Comparing corresponding coefficents on both sides, this means that

$$-4\alpha = 4, \quad -(14\alpha + 4\beta) = 10 \text{ and } 4\alpha - 7\beta - 4\gamma = -23,$$

which give $\alpha = -1$, $\beta = 1$ and $\gamma = 3$.
Hence, the particular integral is

$$y = 3 + x - x^2.$$

Finally, the general solution is

$$y = 3 + x - x^2 + Ae^{4x} + Be^{-\frac{1}{2}x}.$$

2. $f(x) \equiv p\sin kx + q\cos kx$, a **trigonometric** function in which p, q and k are given constants.

$$\text{Trial solution : } \quad y = \alpha\sin kx + \beta\cos kx.$$

Note:
This is the trial solution even if p or q is zero.

EXAMPLE

Determine the general solution of the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 2\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = 8\cos 3x - 19\sin 3x.$$

Solution

The auxiliary equation is

$$m^2 - 2m + 2 = 0,$$

which has complex number solutions given by

$$m = \frac{2 \pm \sqrt{4-8}}{2} = 1 \pm j.$$

Hence, the complementary function is

$$e^x(A\cos x + B\sin x),$$

where A and B are arbitrary constants.

To determine a particular integral, we may make a trial solution of the form

$$y = \alpha\sin 3x + \beta\cos 3x,$$

giving $\frac{\mathrm{d}y}{\mathrm{d}x} = 3\alpha\cos 3x - 3\beta\sin 3x$ and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -9\alpha\sin 3x - 9\beta\cos 3x$.

We thus require that

$$-9\alpha\sin 3x - 9\beta\cos 3x - 6\alpha\cos 3x + 6\beta\sin 3x + 2\alpha\sin 3x + 2\beta\cos 3x \equiv 8\cos 3x - 19\sin 3x.$$

That is,

$$(-9\alpha + 6\beta + 2\alpha)\sin 3x + (-9\beta - 6\alpha + 2\beta)\cos 3x \equiv 8\cos 3x - 19\sin 3x.$$

Comparing corresponding coefficients on both sides, we have

$$\begin{aligned} -7\alpha + 6\beta &= -19, \\ -6\alpha - 7\beta &= 8. \end{aligned}$$

These equations are satisfied by $\alpha = 1$ and $\beta = -2$, so that the particular integral is

$$y = \sin 3x - 2\cos 3x.$$

Finally, the general solution is

$$y = \sin 3x - 2\cos 3x + e^x(A\cos x + B\sin x).$$

3. $f(x) \equiv pe^{kx}$, an **exponential** function in which p and k are given constants.

$$\text{Trial solution :} \quad y = \alpha e^{kx}.$$

EXAMPLE

Determine the general solution of the differential equation

$$9\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 6\frac{\mathrm{d}y}{\mathrm{d}x} + y = 50e^{3x}.$$

Solution

The auxiliary equation is

$$9m^2 + 6m + 1 = 0 \quad \text{or} \quad (3m+1)^2 = 0,$$

which has coincident solutions at $m = -\frac{1}{3}$.
The complementary function is therefore

$$(Ax + B)e^{-\frac{1}{3}x}.$$

To find a particular integral, we may make a trial solution of the form

$$y = \alpha e^{3x},$$

which gives $\frac{\mathrm{d}y}{\mathrm{d}x} = 3\alpha e^{3x}$ and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 9\alpha e^{3x}$.
Hence, on substituting into the differential equation, it is necessary that

$$81\alpha e^{3x} + 18\alpha e^{3x} + \alpha e^{3x} = 50e^{3x}.$$

That is, $100\alpha = 50$, from which we deduce that $\alpha = \frac{1}{2}$ and a particular integral is

$$y = \frac{1}{2}e^{3x}.$$

Finally, the general solution is

$$y = \frac{1}{2}e^{3x} + (Ax + B)e^{-\frac{1}{3}x}.$$

4. $f(x) \equiv p \sinh kx + q \cosh kx$, a **hyperbolic** function in which p, q and k are given constants.

$$\text{Trial solution : } \quad y = \alpha \sinh kx + \beta \cosh kx.$$

Note:
This is the trial solution even if p or q is zero.

EXAMPLE

Determine the general solution of the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 5\frac{\mathrm{d}y}{\mathrm{d}x} + 6y = 93\cosh 5x - 75\sinh 5x.$$

Solution

The auxiliary equation is

$$m^2 - 5m + 6 = 0 \ \text{ or } \ (m-2)(m-3) = 0,$$

which has solutions $m = 2$ and $m = 3$ so that the complementary function is

$$Ae^{2x} + Be^{3x},$$

where A and B are arbitrary constants.
To determine a particular integral, we may make a trial solution of the form

$$y = \alpha \sinh 5x + \beta \cosh 5x,$$

giving $\frac{\mathrm{d}y}{\mathrm{d}x} = 5\alpha \cosh 5x + 5\beta \sinh 5x$ and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 25\alpha \sinh 5x + 25\beta \cosh 5x$.
Substituting into the differential equation, the left-hand-side becomes

$$25\alpha \sinh 5x + 25\beta \cosh 5x - 25\alpha \cosh 5x - 25\beta \sinh 5x + 6\alpha \sinh 5x + 6\beta \cosh 5x.$$

This simplifies to

$$(31\alpha - 25\beta)\sinh 5x + (31\beta - 25\alpha)\cosh 5x,$$

so that we require

$$\begin{aligned} 31\alpha - 25\beta &= -75, \\ -25\alpha + 31\beta &= 93, \end{aligned}$$

and these are satisfied by $\alpha = 0$ and $\beta = 3$.

The particular integral is thus

$$y = 3\cosh 5x$$

and, hence, the general solution is

$$y = 3\cosh 5x + Ae^{2x} + Be^{3x}.$$

5. Combinations of Different Types of Function

In cases where $f(x)$ is the sum of two or more of the various types of function discussed previously, then the particular integrals for each type (determined separately) may be added together to give an overall particular integral.

15.6.3 EXERCISES

1. Determine the general solution for each of the following differential equations:

(a)

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 5\frac{\mathrm{d}y}{\mathrm{d}x} + 4y = 4x^2 + 2x - 4;$$

(b)

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 4\frac{\mathrm{d}y}{\mathrm{d}x} + 5y = 8\cos 2x - \sin 2x;$$

(c)

$$4\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 12\frac{\mathrm{d}y}{\mathrm{d}x} + 9y = 27e^{-x};$$

(d)

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 3\frac{\mathrm{d}y}{\mathrm{d}x} - 10y = \cosh 3x - \sinh 3x.$$

2. Solve completely the following differential equations subject to the given boundary conditions:

   (a)

$$\frac{d^2y}{dx^2} - y = 10 - 5x^2 - x + 16e^{-3x},$$

   where $y = 13$ and $\frac{dy}{dx} = -2$ when $x = 0$;

   (b)

$$4\frac{d^2y}{dx^2} + 8\frac{dy}{dx} + 3y = 9x + 6\cos x - 19\sin x,$$

   where $y = -2$ and $\frac{dy}{dx} = 0$ when $x = 0$.

15.6.4 ANSWERS TO EXERCISES

1. (a)

$$y = x^2 - 2x + 1 + Ae^{-x} + Be^{-4x};$$

   (b)

$$y = \sin 2x + e^{-2x}(A\cos x + B\sin x);$$

   (c)

$$y = 3e^{-x} + (Ax + B)e^{-\frac{3}{2}x};$$

   (d)

$$y = \frac{1}{8}(\cosh 3x - \sinh 3x) + Ae{-}2x + Be^{5x}.$$

2. (a)

$$y = 5x^2 + x + 2e^{-3x} + 3e^{x} - 2e^{-x};$$

   (b)

$$y = 3x - 8 + 2\cos x + \sin x + 2e^{-\frac{1}{2}x} + 2e^{-\frac{3}{2}x}.$$

"JUST THE MATHS"

UNIT NUMBER

15.7

ORDINARY
DIFFERENTIAL EQUATIONS 7
(Second order equations (D))

by

A.J.Hobson

UNIT 15.7 - ORDINARY DIFFERENTIAL EQUATIONS 7

SECOND ORDER EQUATIONS (D)

15.7.1 PROBLEMATIC CASES OF PARTICULAR INTEGRALS

Difficulties can arise if all or part of any trial solution would already be included in the complementary function. We illustrate with some examples:

EXAMPLES

1. Determine the complementary function and a particular integral for the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 3\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = e^{2x}.$$

Solution

The auxiliary equation is $m^2 - 3m + 2 = 0$, with solutions $m = 1$ and $m = 2$ and hence the complementary function is $Ae^x + Be^{2x}$, where A and B are arbitrary constants.

A trial solution of $y = \alpha e^{2x}$ gives

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2\alpha e^{2x} \quad \text{and} \quad \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 4\alpha e^{2x}$$

and, on substituting these into the differential equation, it is necessary that

$$4\alpha e^{2x} - 6\alpha e^{2x} + 2\alpha e^{2x} \equiv e^{2x}.$$

That is, $0 \equiv e^{2x}$ which is impossible.

However, if $y = \alpha e^{2x}$ has proved to be unsatisfactory, let us investigate, as an alternative, $y = F(x)e^{2x}$ (where $F(x)$ is a function of x instead of a constant).

We have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2F(x)e^{2x} + F'(x)e^{2x}$$

and, hence,

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 4F(x)e^{2x} + 2F'(x)e^{2x} + F''(x)e^{2x} + 2F'(x)e^{2x}.$$

On substituting these into the differential equation, it is necessary that

$$(4F(x) + 2F'(x) + F''(x) + 2F'(x) - 6F(x) - 3F'(x) + 2F(x))\, e^{2x} \equiv e^{2x}.$$

That is,

$$F''(x) + F'(x) = 1,$$

which is satisfied by the function $F(x) \equiv x$ and thus a suitable particular integral is

$$y = xe^{2x}.$$

Note:
It may be shown, in other cases too that, if the standard trial solution is already contained in the complementary function, then it is is necessary to multiply it by x in order to obtain a suitable particular integral.

2. Determine the complementary function and a particular integral for the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + y = \sin x.$$

Solution

The auxilary equation is $m^2 + 1 = 0$, with solutions $m = \pm j$ and, hence, the complementary function is $A \sin x + B \cos x$, where A and B are arbitrary constants.

A trial solution of $y = \alpha \sin x + \beta \cos x$ gives

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -\alpha \sin x - \beta \cos x;$$

and, on substituting into the differential equation, it is necessary that $0 \equiv \sin x$, which is impossible.

Here, we may try $y = x(\alpha \sin x + \beta \cos x)$, giving

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \alpha \sin x + \beta \cos x + x(\alpha \cos x - \beta \sin x) = (\alpha - \beta x) \sin x + (\beta + \alpha x) \cos x$$

and, therefore,

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = (\alpha - \beta x) \cos x - \beta \sin x - (\beta + \alpha x) \sin x + \alpha \cos x = (2\alpha - \beta x) \cos x - (2\beta + \alpha x) \sin x.$$

Substituting into the differential equation, we thus require that

$$(2\alpha - \beta x)\cos x - (2\beta + \alpha x)\sin x + x(\alpha \sin x + \beta \cos x) \equiv \sin x,$$

which simplifies to

$$2\alpha \cos x - 2\beta \sin x \equiv \sin x.$$

Thus $2\alpha = 0$ and $-2\beta = 1$.

An appropriate particular integral is now

$$y = -\frac{1}{2}x\cos x.$$

3. Determine the complementary function and a particular integral for the differential equation

$$9\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 6\frac{\mathrm{d}y}{\mathrm{d}x} + y == 50e^{-\frac{1}{3}x}.$$

Solution

The auxiliary equation is $9m^2 + 6m + 1 = 0$, or $(3m+1)^2 = 0$, which has coincident solutions $m = -\frac{1}{3}$ and so the complementary function is

$$(Ax + B)e^{-\frac{1}{3}x}.$$

In this example, both $e^{-\frac{1}{3}x}$ **and** $xe^{-\frac{1}{3}x}$ are contained in the complementary function. Thus, in the trial solution, it is necessary to multiply by a **further** x, giving

$$y = \alpha x^2 e^{-\frac{1}{3}x}.$$

We have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2\alpha x e^{-\frac{1}{3}x} - \frac{1}{3}x^2 e^{\frac{1}{3}x}$$

and

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 2\alpha e^{-\frac{1}{3}x} - \frac{2}{3}\alpha x e^{-\frac{1}{3}x} - \frac{2}{3}\alpha x e^{-\frac{1}{3}x} + \frac{1}{9}\alpha x^2 e^{-\frac{1}{3}x}.$$

Substituting these into the differential equation, it is necessary that

$$\left(18\alpha - 12\alpha x + \alpha x^2 + 12\alpha x - 2\alpha x^2 + \alpha x^2\right) e^{-\frac{1}{3}x} = 50e^{-\frac{1}{3}x}$$

and, hence, $18\alpha = 50$ or $\alpha = \frac{25}{9}$.
An appropriate particular integral is

$$y = \frac{25}{9}x^2 e^{-\frac{1}{3}x}.$$

4. Determine the complementary function and a particular integral for the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 5\frac{\mathrm{d}y}{\mathrm{d}x} + 6y = \sinh 2x.$$

Solution

The auxiliary equation is $m^2 - 5m + 6 = 0$ or $(m-2)(m-3) = 0$ which has solutions $m = 2$ and $m = 3$ and, hence, the complementary function is

$$Ae^{2x} + Be^{3x}.$$

However, since $\sinh 2x \equiv \frac{1}{2}(e^{2x} - e^{-2x})$, **part** of it is contained in the complementary function and we must find a particular integral for each part separately.
(a) For $\frac{1}{2}e^{2x}$, we may try

$$y = x\alpha e^{2x},$$

giving

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \alpha e^{2x} + 2x\alpha e^{2x}$$

and

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 2\alpha e^{2x} + 2\alpha e^{2x} + 4x\alpha e^{2x}.$$

Substituting these into the differential equation, it is necessary that

$$(4\alpha + 4x\alpha - 5\alpha - 10x\alpha + 6x\alpha)\, e^{2x} \equiv \frac{1}{2}e^{2x},$$

which gives $\alpha = -\frac{1}{2}$.

(b) For $-\frac{1}{2}e^{-2x}$, we may try

$$y = \beta e^{-2x},$$

giving

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -2\beta e^{-2x}$$

and

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 4\beta e^{-2x}.$$

Substituting these into the differential equation, it is necessary that

$$(4\beta + 10\beta + 6\beta)e^{-2x} \equiv -\frac{1}{2}e^{-2x},$$

which gives $\beta = -\frac{1}{40}$.

The overall particular integral is thus

$$y = -\frac{1}{2}xe^{2x} - \frac{1}{40}e^{-2x}.$$

15.7.2 EXERCISES

Solve completely the following differential equations subject to the given boundary conditions:

1.
$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 4\frac{\mathrm{d}y}{\mathrm{d}x} + 3y = e^{-x},$$

where $y = 0$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{5}{2}$ when $x = 0$.

2.
$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 9y = 2\sin 3x,$$

where $y = 2$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{8}{3}$ when $x = 0$.

3.

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 10\frac{\mathrm{d}y}{\mathrm{d}x} + 25y = 8e^{3x} + 25x^2 - 20x + 27,$$

where $y = 5$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = 13$ when $x = 0$.

4.

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 3\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = \cosh x,$$

where $y = \frac{7}{12}$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2}$ when $x = 0$.

5.

$$4\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 4\frac{\mathrm{d}y}{\mathrm{d}x} + y = 24e^{-\frac{1}{2}x}$$

where $y = 6$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = 2$ when $x = 0$.

15.7.3 ANSWERS TO EXERCISES

1.

$$y = \frac{1}{2}xe^{-x} + Ae^{-x} + Be^{-3x}.$$

2.

$$y = -\frac{1}{3}x\cos 3x + 2\cos 3x + \sin 3x.$$

3.

$$y = 2e^{3x} + x^2 + 1 + (2 - 3x)e^{5x}.$$

4.

$$y = \frac{1}{12}\left(e^{-x} - 6xe^{x} - e^{x} + 7e^{2x}\right).$$

5.

$$y = 3x^2e^{-\frac{1}{2}x} + (5x + 6)e^{-\frac{1}{2}x}.$$

"JUST THE MATHS"

UNIT NUMBER

15.8

ORDINARY DIFFERENTIAL EQUATIONS 8 (Simultaneous equations (A))

by

A.J.Hobson

UNIT 15.8 - ORDINARY DIFFERENTIAL EQUATIONS 8

SIMULTANEOUS EQUATIONS (A)

15.8.1 THE SUBSTITUTION METHOD

The methods discussed in previous Units for the solution of second order ordinary linear differential equations with constant coefficients may now be used for cases of two first order differential equations which must be satisfied simultaneously. The technique will be illustrated by the following examples:

EXAMPLES

1. Determine the general solutions for y and z in the case when

$$\begin{aligned} 5\frac{\mathrm{d}y}{\mathrm{d}x} - 2\frac{\mathrm{d}z}{\mathrm{d}x} + 4y - z &= e^{-x}, ----------- (1) \\ \frac{\mathrm{d}y}{\mathrm{d}x} + 8y - 3z &= 5e^{-x}. ---------- (2) \end{aligned}$$

Solution

First, we eliminate one of the dependent variables from the two equations; in this case, we eliminate z.

From equation (2),

$$z = \frac{1}{3}\left(\frac{\mathrm{d}y}{\mathrm{d}x} + 8y - 5e^{-x}\right)$$

and, on substituting this into equation (1), we obtain

$$5\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{2}{3}\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 8\frac{\mathrm{d}y}{\mathrm{d}x} + 5e^{-x}\right) + 4y - \frac{1}{3}\left(\frac{\mathrm{d}y}{\mathrm{d}x} + 8y - 5e^{-x}\right) = e^{-x}.$$

That is, $$-\frac{2}{3}\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - \frac{2}{3}\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{4}{3}y = \frac{8}{3}e^{-x}$$

or

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \frac{\mathrm{d}y}{\mathrm{d}x} - 2y = -4e^{-x}.$$

The auxiliary equation is

$$m^2 + m - 2 = 0 \quad \text{or} \quad (m-1)(m+2) = 0,$$

giving a complementary function of $Ae^x + Be^{-2x}$, where A and B are arbitrary constants. A particular integral will be of the form ke^{-x}, where $k - k - 2k = -4$ and hence $k = 2$. Thus,

$$y = 2e^{-x} + Ae^x + Be^{-2x}.$$

Finally, from the formula for z in terms of y,

$$z = \frac{1}{3}\left(-2e^{-x} + Ae^x - 2Be^{-2x} + 16e^{-x} + 8Ae^x + 8Be^{-2x} - 5e^{-x}\right).$$

That is,

$$z = 3e^{-x} + 3Ae^{-x} + 2Be^{-2x}.$$

Note:
The above example would have been a little more difficult if the second differential equation had contained a term in $\frac{dz}{dx}$. But, if this were the case, we could eliminate $\frac{dz}{dx}$ between the two equations in order to obtain a statement with the same form as Equation (2).

2. Solve, simultaneously, the differential equations

$$\begin{aligned} \frac{dz}{dx} + 2y &= e^x, - - - - - - - - - (1) \\ \frac{dy}{dx} - 2z &= 1 + x, - - - - - - - -(2) \end{aligned}$$

given that $y = 1$ and $z = 2$ when $x = 0$.
Solution:

From equation (2), we have

$$z = \frac{1}{2}\left[\frac{dy}{dx} - 1 - x\right].$$

Substituting into the first differential equation gives

$$\frac{1}{2}\left[\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 1\right] + 2y = e^x$$

or

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 4y = 2e^x + 1.$$

The auxiliary equation is therefore $m^2 + 4 = 0$, having solutions $m = \pm j2$, which means that the complementary function is

$$A\cos 2x + B\sin 2x,$$

where A and B are arbitrary constants.
The particular integral will be of the form $y = pe^x + q$,
where

$$pe^x + 4pe^x + 4q = 2e^x + 1.$$

We require, then, that $5p = 2$ and $4q = 1$; and so the general solution for y is

$$y = A\cos 2x + B\sin 2x + \frac{2}{5}e^x + \frac{1}{4}.$$

Using the earlier formula for z, we obtain

$$z = \frac{1}{2}\left[-2A\sin 2x + 2B\cos 2x + \frac{2}{5}e^x - 1 - x\right] = B\cos 2x - A\sin 2x + \frac{1}{5}e^x - \frac{1}{2} - \frac{x}{2}.$$

Applying the boundary conditions,

$$1 = A + \frac{2}{5} + \frac{1}{4} \quad \text{giving} \quad A = \frac{7}{20}$$

and

$$2 = B + \frac{1}{5} - \frac{1}{2} \quad \text{giving} \quad B = \frac{23}{10}.$$

The required solutions are therefore

$$y = \frac{7}{20}\cos 2x + \frac{23}{10}\sin 2x + \frac{2}{5}e^x + \frac{1}{4}$$

and

$$z = \frac{23}{10}\cos 2x - \frac{7}{20}\sin 2x + \frac{1}{5}e^x - \frac{1}{2} - \frac{x}{2}.$$

15.8.2 EXERCISES

Solve the following pairs of simultaneous differential equations, subject to the given boundary conditions:

1.

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} + 2z &= e^{-x},\\ \frac{\mathrm{d}z}{\mathrm{d}x} + 3z &= y,\end{aligned}$$

given that $y = 1$ and $z = 0$ when $x = 0$.

2.

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} - z &= \sin x,\\ \frac{\mathrm{d}z}{\mathrm{d}x} + y &= \cos x,\end{aligned}$$

given that $y = 3$ and $z = 4$ when $x = 0$.

3.

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} + 2y - 3z &= 1,\\ \frac{\mathrm{d}z}{\mathrm{d}x} - y &= e^{-2x},\end{aligned}$$

given that $y = 0$ and $z = 0$ when $x = 0$.

4.

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= 2z,\\ \frac{\mathrm{d}z}{\mathrm{d}x} &= 8y,\end{aligned}$$

given that $y = 1$ and $z = 0$ when $x = 0$.

5.

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} + 4\frac{\mathrm{d}z}{\mathrm{d}x} + 6z &= 0,\\ 5\frac{\mathrm{d}y}{\mathrm{d}x} + 2\frac{\mathrm{d}z}{\mathrm{d}x} + 6y &= 0,\end{aligned}$$

given that $y = 3$ and $z = 0$ when $x = 0$.

Hint: First eliminate the $\frac{\mathrm{d}z}{\mathrm{d}x}$ terms to obtain a formula for z in terms of y and $\frac{\mathrm{d}y}{\mathrm{d}x}$.

6.

$$\begin{aligned}10\frac{\mathrm{d}y}{\mathrm{d}x} - 3\frac{\mathrm{d}z}{\mathrm{d}x} + 6y + 5z &= 0,\\ 2\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{\mathrm{d}z}{\mathrm{d}x} + 2y + z &= 2e^{-x},\end{aligned}$$

given that $y = 2$ and $z = -1$ when $x = 0$.

Hint: First, eliminate the $\frac{\mathrm{d}y}{\mathrm{d}x}$ and z terms in one step, to obtain a formula for y in terms of $\frac{\mathrm{d}z}{\mathrm{d}x}$ and x.

15.8.3 ANSWERS TO EXERCISES

1.

$$y = (2x + 1)e^{-x} \quad \text{and} \quad z = xe^{-x}.$$

2.

$$y = (x + 4)\sin x + 3\cos x \quad \text{and} \quad z = (x + 4)\cos x - 3\sin x.$$

3.

$$y = \frac{1}{2}e^{x} + \frac{1}{2}e^{-3x} - e^{-2x} \quad \text{and} \quad z = \frac{1}{2}e^{x} - \frac{1}{6}e^{-3x} - \frac{1}{3}.$$

4.

$$y = \frac{1}{2}e^{4x} - \frac{1}{2}e^{-4x} \equiv \sinh 4x \quad \text{and} \quad z = e^{4x} + e^{-4x} \equiv 2\cosh 4x.$$

5.

$$y = 2e^{-x} + e^{-2x} \quad \text{and} \quad z = e^{-x} - e^{-2x}.$$

6.

$$y = \sin x + 2e^{-x} \quad \text{and} \quad z = e^{-x} - 2\cos x.$$

"JUST THE MATHS"

UNIT NUMBER

15.9

ORDINARY DIFFERENTIAL EQUATIONS 9 (Simultaneous equations (B))

by

A.J.Hobson

UNIT 15.9 - ORDINARY DIFFERENTIAL EQUATIONS 9

SIMULTANEOUS EQUATIONS (B)

15.9.1 INTRODUCTION

For students who have studied the principles of eigenvalues and eigenvectors (see Unit 9.6), a second method of solving two simultaneous linear differential equations is to interpret them as a single equation using matrix notation. The discussion will be limited to the simpler kinds of example, and we shall find it convenient to use t, x_1 and x_2 rather than x, y and z.

15.9.2 MATRIX METHODS FOR HOMOGENEOUS SYSTEMS

To introduce the technique, we begin by considering two simultaneous differential equations of the form

$$\begin{aligned} \frac{\mathrm{d}x_1}{\mathrm{d}t} &= ax_1 + bx_2, \\ \frac{\mathrm{d}x_2}{\mathrm{d}t} &= cx_1 + dx_2. \end{aligned}$$

which are of the "homogeneous" type, since no functions of t, other than x_1 and x_2, appear on the right hand sides.

(i) First, we write the differential equations in matrix form as

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} . \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$$

which may be interpreted as

$$\frac{\mathrm{dX}}{\mathrm{dt}} = \mathrm{MX} \quad \text{where} \quad \mathrm{X} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \text{and} \quad \mathrm{M} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

(ii) Secondly, in a similar way to the method appropriate to a single differential equation, we make a trial solution of the form

$$\mathrm{X} = \mathrm{K}e^{\lambda t},$$

where

$$\mathrm{K} = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$$

is a constant matrix of order 2×1.

This requires that

$$\lambda \mathrm{K} e^{\lambda t} = \mathrm{M} \mathrm{K} e^{\lambda t} \quad \text{or} \quad \lambda \mathrm{K} = \mathrm{M}\mathrm{K},$$

which we may recognise as the condition that λ is an eigenvalue of the matrix M, and K is an eigenvector of M.

The solutions for λ are obtained from the "characteristic equation"

$$|\mathrm{M} - \lambda \mathrm{I}| = 0.$$

In other words,

$$\begin{vmatrix} a - \lambda & b \\ c & b - \lambda \end{vmatrix} = 0,$$

leading to a quadratic equation having real and distinct solutions ($\lambda = \lambda_1$ and $\lambda = \lambda_2$), real and coincident solutions (λ only) or conjugate complex solutions ($\lambda = l \pm jm$).

(iii) The possibilities for the matrix K are obtained by solving the homogeneous linear equations

$$\begin{aligned} (a - \lambda_1 k_1 + b k_2 &= 0, \\ c k_1 + (d - \lambda_1) k_2 &= 0, \end{aligned}$$

giving $k_1 : k_2 = 1 : \alpha$ (say)

and

$$\begin{aligned}(a-\lambda_2)k_1+bk_2 &= 0,\\ ck_1+(d-\lambda_2)k_2 &= 0,\end{aligned}$$

giving $k_1 : k_2 = 1 : \beta$ (say).

Finally, it may be shown that, according to the types of solution to the auxiliary equation, the solution of the differential equation will take one of the following three forms, in which A and B are arbitrary constants:

(a)

$$A\begin{bmatrix}1\\ \alpha\end{bmatrix}e^{\lambda_1 t}+B\begin{bmatrix}1\\ \beta\end{bmatrix}e^{\lambda_2 t};$$

(b)

$$\left\{(At+B)\begin{bmatrix}1\\ \alpha\end{bmatrix}+\frac{A}{b}\begin{bmatrix}0\\ 1\end{bmatrix}\right\}e^{\lambda t};$$

or

(c)

$$e^{lt}\left\{\begin{bmatrix}A\\ pA+qB\end{bmatrix}\cos mt+\begin{bmatrix}B\\ pB-qA\end{bmatrix}\sin mt\right\},$$

where, in (c), $1:\alpha = 1:p+jq$ and $1:\beta = 1:p-jq$.

EXAMPLES

1. Determine the general solution of the simultaneous differential equations

$$\begin{aligned}\frac{\mathrm{d}x_1}{\mathrm{d}t} &= -4x_1+5x_2,\\ \frac{\mathrm{d}x_2}{\mathrm{d}t} &= -x_1+2x_2.\end{aligned}$$

Solution
The characteristic equation is

$$\begin{vmatrix} -4-\lambda & 5 \\ -1 & 2-\lambda \end{vmatrix} = 0.$$

That is,

$$\lambda^2 + 2\lambda - 3 = 0 \quad \text{or} \quad (\lambda - 1)(\lambda + 3) = 0.$$

When $\lambda = 1$, we need to solve the homogeneous equations

$$\begin{aligned} -5k_1 + 5k_2 &= 0, \\ -k_1 + k_2 &= 0, \end{aligned}$$

both of which give $k_1 : k_2 = 1 : 1$.
When $\lambda = -3$, we need to solve the homogeneous equations

$$\begin{aligned} -k_1 + 5k_2 &= 0, \\ -k_1 + 5k_2 &= 0, \end{aligned}$$

both of which give $k_1 : k_2 = 1 : \frac{1}{5}$.
The general solution is therefore

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^t + B \begin{bmatrix} 1 \\ \frac{1}{5} \end{bmatrix} e^{-3t}$$

or, alternatively,

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^t + B \begin{bmatrix} 5 \\ 1 \end{bmatrix} e^{-3t},$$

where A and B are arbitrary constants.

2. Determine the general solution of the simultaneous differential equations

$$\begin{aligned}\frac{\mathrm{d}x_1}{\mathrm{d}t} &= x_1 - x_2,\\ \frac{\mathrm{d}x_2}{\mathrm{d}t} &= x_1 + 3x_2.\end{aligned}$$

Solution

The characteristic equation is

$$\begin{vmatrix} 1-\lambda & -1 \\ 1 & 3-\lambda \end{vmatrix} = 0.$$

That is,

$$\lambda^2 - 4\lambda + 4 = 0 \text{ or } (\lambda - 2)^2 = 0.$$

When $\lambda = 2$, we need to solve the homogeneous equations

$$\begin{aligned}-k_1 - k_2 &= 0,\\ k_1 + k_2 &= 0,\end{aligned}$$

both of which give $k_1 : k_2 = 1 : -1$.

The general solution is therefore

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \left\{(At + B)\begin{bmatrix} 1 \\ -1 \end{bmatrix} - A\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\} e^{2t},$$

where A and B are arbitrary constants.

3. Determine the general solution of the simultaneous differential equations

$$\begin{aligned}\frac{\mathrm{d}x_1}{\mathrm{d}t} &= x_1 - 5x_2,\\ \frac{\mathrm{d}x_2}{\mathrm{d}t} &= 2x_1 + 3x_2.\end{aligned}$$

Solution

The characteristic equation is

$$\begin{vmatrix} 1-\lambda & -5 \\ 2 & 3-\lambda \end{vmatrix} = 0.$$

That is,

$$\lambda^2 - 4\lambda + 13 = 0,$$

which gives $\lambda = 2 \pm j3$.

When $\lambda = 2 + j3$, we need to solve the homogeneous equations

$$\begin{aligned} (-1-j3)k_1 - 5k_2 &= 0, \\ 2k_1 + (1-j3)k_2 &= 0, \end{aligned}$$

both of which give $k_1 : k_2 = 1 : \frac{-1-j3}{5}$.

When $\lambda = 2 - j3$, we need to solve the homogeneous equations

$$\begin{aligned} (-1+j3)k_1 - 5k_2 &= 0, \\ 2k_1 + (1+j3)k_2 &= 0, \end{aligned}$$

both of which give $k_1 : k_2 = 1 : \frac{-1+j3}{5}$.

The general solution is therefore

$$\frac{e^{2t}}{5}\left\{\begin{bmatrix} A \\ -A+3B \end{bmatrix}\cos 3t + \begin{bmatrix} B \\ -B-3A \end{bmatrix}\sin 3t\right\},$$

where A and B are arbitrary constants.

Note:

From any set of simultaneous differential equations of the form

$$\begin{aligned} a\frac{\mathrm{d}x_1}{\mathrm{d}t} + b\frac{\mathrm{d}x_2}{\mathrm{d}t} + cx_1 + dx_2 = 0, \\ a'\frac{\mathrm{d}x_1}{\mathrm{d}t} + b'\frac{\mathrm{d}x_2}{\mathrm{d}t} + b'x_1 + c'x_2 = 0, \end{aligned}$$

it is possible to eliminate $\frac{dx_1}{dt}$ and $\frac{dx_2}{dt}$ in turn, in order to obtain two equivalent equations of the form discussed in the above examples.

15.9.3 EXERCISES

1. Determine the general solution of the simultaneous differential equations

$$\begin{aligned}\frac{dx_1}{dt} &= x_1 + 3x_2,\\ \frac{dx_2}{dt} &= 3x_1 + x_2.\end{aligned}$$

2. Solve completely, the simultaneous differential equations

$$\begin{aligned}\frac{dx_1}{dt} &= 3x_1 + 2x_2,\\ \frac{dx_2}{dt} &= 4x_1 + x_2,\end{aligned}$$

given that $x_1 = 3$ and $x_2 = -3$ when $t = 0$.

3. Solve completely, the simultaneous differential equations

$$\begin{aligned}\frac{dx_1}{dt} &= -x_1 + 9x_2,\\ \frac{dx_2}{dt} &= 11x_1 + x_2,\end{aligned}$$

given that $x_1 = 20$ and $x_2 = 20$ when $t = 0$.

4. Solve completely, the simultaneous differential equations

$$\begin{aligned}\frac{dx_1}{dt} - \frac{dx_2}{dt} + 2x_1 - 2x_2 &= 0,\\ \frac{dx_1}{dt} + 2\frac{dx_2}{dt} - 7x_1 - 5x_2 &= 0,\end{aligned}$$

given that $x_1 = 2$ and $x_2 = 0$ when $t = 0$.

5. Determine the general solution of the simultaneous differential equations

$$\begin{aligned}\frac{\mathrm{d}x_1}{\mathrm{d}t} &= 5x_1 + 2x_2,\\ \frac{\mathrm{d}x_2}{\mathrm{d}t} &= -2x_1 + x_2.\end{aligned}$$

6. Determine the general solution of the simultaneous differential equations

$$\begin{aligned}\frac{\mathrm{d}x_1}{\mathrm{d}t} &= 8x_1 + x_2,\\ \frac{\mathrm{d}x_2}{\mathrm{d}t} &= -5x_1 + 6x_2.\end{aligned}$$

15.9.4 ANSWERS TO EXERCISES

1.
$$A\begin{bmatrix}1\\1\end{bmatrix}e^{4t} + B\begin{bmatrix}1\\-1\end{bmatrix}e^{-2t}.$$

2.
$$\begin{bmatrix}1\\1\end{bmatrix}e^{5t} + 2\begin{bmatrix}1\\-2\end{bmatrix}e^{-t}.$$

3.
$$2\begin{bmatrix}1\\-1\end{bmatrix}e^{-10t} + \begin{bmatrix}18\\22\end{bmatrix}e^{10t}.$$

4.
$$\begin{bmatrix}1\\1\end{bmatrix}e^{4t} + \begin{bmatrix}1\\-1\end{bmatrix}e^{-2t}.$$

5.
$$\left\{(At+B)\begin{bmatrix}1\\-1\end{bmatrix} + \frac{A}{2}\begin{bmatrix}0\\1\end{bmatrix}\right\}e^{3t}.$$

6.
$$e^{7t}\left\{\begin{bmatrix}A\\-A+2B\end{bmatrix}\cos 2t + \begin{bmatrix}B\\-B-2A\end{bmatrix}\sin 2t\right\}.$$

"JUST THE MATHS"

UNIT NUMBER

15.10

ORDINARY DIFFERENTIAL EQUATIONS 10 (Simultaneous equations (C))

by

A.J.Hobson

UNIT 15.10 - ORDINARY DIFFERENTIAL EQUATIONS 10

SIMULTANEOUS EQUATIONS (C)

15.10.1 MATRIX METHODS FOR NON-HOMOGENEOUS SYSTEMS

In Units 15.5, 15.6 and 15.7 , it was seen that, for a single linear differential equation with constant coefficients, the general solution is made up of a particular integral and a complementary function (the latter being the general solution of the corresponding homogeneous differential equation).

In the work which follows, a similar principle is applied to a pair of simultaneous non-homogeneous differential equations of the form

$$\begin{aligned}\frac{\mathrm{d}x_1}{\mathrm{d}t} &= ax_1 + bx_2 + f(t),\\ \frac{\mathrm{d}x_2}{\mathrm{d}t} &= cx_1 + dx_2 + g(t).\end{aligned}$$

The method will be illustrated by the following example, in which $f(t) \equiv 0$:

EXAMPLE

Determine the general solution of the simultaneous differential equations

$$\begin{aligned}\frac{\mathrm{d}x_1}{\mathrm{d}t} &= x_2, - - - - - - - - - - - - -(1)\\ \frac{\mathrm{d}x_2}{\mathrm{d}t} &= -4x_1 - 5x_2 + g(t), - - - - -(2)\end{aligned}$$

where $g(t)$ is (a) t, (b) e^{2t} (c) $\sin t$, (d) e^{-t}.

Solutions

(i) First, we write the differential equations in matrix form as

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}0 & 1\\-4 & -5\end{bmatrix}\cdot\begin{bmatrix}x_1\\x_2\end{bmatrix} + \begin{bmatrix}0\\1\end{bmatrix}g(t),$$

which may be interpreted as

$$\frac{\mathrm{dX}}{\mathrm{dt}} = \mathrm{MX} + \mathrm{N}g(t) \quad \text{where} \quad \mathrm{X} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \mathrm{M} = \begin{bmatrix} 0 & 1 \\ -4 & -5 \end{bmatrix} \quad \text{and} \quad \mathrm{N} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

(ii) Secondly, we consider the corresponding "homogeneous" system

$$\frac{\mathrm{dX}}{\mathrm{dt}} = \mathrm{MX},$$

for which the characteristic equation is

$$\begin{vmatrix} 0 - \lambda & 1 \\ -4 & -5 - \lambda \end{vmatrix} = 0,$$

and gives

$$\lambda(5 + \lambda) + 4 = 0 \quad \text{or} \quad \lambda^2 + 5\lambda + 4 = 0 \quad \text{or} \quad (\lambda + 1)(\lambda + 4) = 0.$$

(iii) The eigenvectors of M are obtained from the homogeneous equations

$$\begin{aligned} -\lambda k_1 + k_2 &= 0, \\ -4k_1 - (5 + \lambda)k_2 &= 0. \end{aligned}$$

Hence, in the case when $\lambda = -1$, we solve

$$\begin{aligned} k_1 + k_2 &= 0, \\ -4k_1 - 4k_2 &= 0, \end{aligned}$$

and these are satisfied by any two numbers in the ratio $k_1 : k_2 = 1 : -1$.

Also, when $\lambda = -4$, we solve

$$\begin{aligned} 4k_1 + k_2 &= 0, \\ -4k_1 - k_2 &= 0 \end{aligned}$$

which are satisfied by any two numbers in the ratio $k_1 : k_2 = 1 : -4$.

The complementary function may now be written in the form

$$A \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} + B \begin{bmatrix} 1 \\ -4 \end{bmatrix} e^{-4t},$$

where A and B are arbitrary constants.

(iv) In order to obtain a particular integral for the equation

$$\frac{\mathrm{dX}}{\mathrm{dt}} = \mathrm{MX} + \mathrm{N}g(t),$$

we note the second term on the right hand side and investigate a trial solution of a similar form. The three cases in this example are as follows:

(a) $g(t) \equiv t$

$$\text{Trial solution} \quad \mathrm{X} = \mathrm{P} + \mathrm{Q}t,$$

where P and Q are constant matrices of order 2×1.

We require that

$$\mathrm{Q} = \mathrm{M}(\mathrm{P} + \mathrm{Q}t) + \mathrm{N}t,$$

whereupon, equating the matrix coefficients of t and the constant matrices,

$$\mathrm{MQ} + \mathrm{N} = \mathbf{0} \quad \text{and} \quad \mathrm{Q} = \mathrm{MP},$$

giving

$$\mathrm{Q} = -\mathrm{M}^{-1}\mathrm{N} \quad \text{and} \quad \mathrm{P} = \mathrm{M}^{-1}\mathrm{Q}.$$

Thus, using

$$\mathrm{M}^{-1} = \frac{1}{4}\begin{bmatrix} -5 & -1 \\ 4 & 0 \end{bmatrix},$$

we obtain

$$\mathrm{Q} = -\frac{1}{4}\begin{bmatrix} -5 & -1 \\ 4 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.25 \\ 0 \end{bmatrix}$$

and

$$\mathrm{P} = \frac{1}{4}\begin{bmatrix} -5 & -1 \\ 4 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0.25 \\ 0 \end{bmatrix} = \begin{bmatrix} -0.3125 \\ 0.25 \end{bmatrix}.$$

The general solution, in this case, is

$$\mathrm{X} = A\begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} + B\begin{bmatrix} 1 \\ -4 \end{bmatrix} e^{-4t} + \begin{bmatrix} -0.3125 \\ 0.25 \end{bmatrix} + \begin{bmatrix} 0.25 \\ 0 \end{bmatrix} t.$$

(b) $g(t) \equiv e^{2t}$

$$\text{Trial solution} \quad \mathrm{X} = \mathrm{P}e^{2t}$$

We require that

$$2\mathrm{P}e^{2t} = \mathrm{MP}e^{2t} + \mathrm{N}e^{2t}.$$

That is,

$$2\mathrm{P} = \mathrm{MP} + \mathrm{N}.$$

The matrix, P, may now be determined from the formula

$$(2\mathrm{I} - \mathrm{M})\mathrm{P} = \mathrm{N};$$

or, in more detail,

$$\begin{bmatrix} 2 & -1 \\ 4 & 7 \end{bmatrix}.\mathrm{P} = \mathrm{N}.$$

Hence,

$$\mathrm{P} = \frac{1}{18}\begin{bmatrix} 7 & 1 \\ -4 & 2 \end{bmatrix}.\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{18}\begin{bmatrix} 7 \\ -4 \end{bmatrix}.$$

The general solution, in this case, is

$$\mathrm{X} = A\begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} + B\begin{bmatrix} 1 \\ -4 \end{bmatrix} e^{-4t} + \frac{1}{18}\begin{bmatrix} 7 \\ -4 \end{bmatrix} e^{2t}.$$

(c) $g(t) \equiv \sin t$

$$\text{Trial solution} \quad \mathrm{X} = \mathrm{P}\sin t + \mathrm{Q}\cos t.$$

We require that

$$\mathrm{P}\cos t - \mathrm{Q}\sin t = \mathrm{M}(\mathrm{P}\sin t + \mathrm{Q}\cos t) + \mathrm{N}\sin t.$$

Equating the matrix coefficients of $\cos t$ and $\sin t$,

$$\mathrm{P} = \mathrm{MQ} \quad \text{and} \quad -\mathrm{Q} = \mathrm{MP} + \mathrm{N},$$

which means that

$$-\mathrm{Q} = \mathrm{M}^2\mathrm{Q} + \mathrm{N} \quad \text{or} \quad (\mathrm{M}^2 + \mathrm{I})\mathrm{Q} = -\mathrm{N}.$$

Thus,

$$Q = -(M^2 + I)^{-1}N,$$

where

$$M^2 + I = \begin{bmatrix} 0 & 1 \\ -4 & -5 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ -4 & -5 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -3 & -5 \\ 20 & 22 \end{bmatrix}$$

and, hence,

$$Q = -\frac{1}{34} \begin{bmatrix} 22 & 5 \\ -20 & -3 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{34} \begin{bmatrix} -5 \\ 3 \end{bmatrix}.$$

Also,

$$P = MQ = \frac{1}{34} \begin{bmatrix} 0 & 1 \\ -4 & -5 \end{bmatrix} \cdot \begin{bmatrix} -5 \\ 3 \end{bmatrix} = \frac{1}{34} \begin{bmatrix} 3 \\ 5 \end{bmatrix}.$$

The general solution, in this case, is

$$X = A \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} + B \begin{bmatrix} 1 \\ -4 \end{bmatrix} e^{-4t} + \frac{1}{34} \begin{bmatrix} 3 \\ 5 \end{bmatrix} \sin t + \frac{1}{34} \begin{bmatrix} -5 \\ 3 \end{bmatrix} \cos t.$$

(d) $g(t) \equiv e^{-t}$

In this case, the function, $g(t)$, is already included in the complementary function and it becomes necessary to assume a particular integral of the form

$$X = (P + Qt)e^{-t},$$

where P and Q are constant matrices of order 2×1.

We require that

$$Qe^{-t} - (P + Qt)e^{-t} = M(P + Qt)e^{-t} + Ne^{-t},$$

whereupon, equating the matrix coefficients of te^{-t} and e^{-t}, we obtain

$$-\mathrm{Q} = \mathrm{MQ} \quad \text{and} \quad \mathrm{Q} - \mathrm{P} = \mathrm{MP} + \mathrm{N}.$$

The first of these conditions shows that Q is an eigenvector of the matrix M corresponding the eigenvalue -1 and so, from earlier work,

$$\mathrm{Q} = k \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

for any constant k.

Also,

$$(\mathrm{M} + \mathrm{I})\mathrm{P} = \mathrm{Q} - \mathrm{N};$$

or, in more detail,

$$\begin{bmatrix} 1 & 1 \\ -4 & -4 \end{bmatrix} \cdot \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = k \begin{bmatrix} 1 \\ -1 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Hence,

$$\begin{aligned} p_1 + p_2 &= k, \\ -4p_1 - 4p_2 &= -k - 1. \end{aligned}$$

Using $p_1 + p_2 = k$ and $p_1 + p_2 = \frac{k+1}{4}$, we deduce that $k = \frac{1}{3}$ and that the matrix P is given by

$$\mathrm{P} = \begin{bmatrix} l \\ \frac{1}{3} - l \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + l \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

for any number, l.

Taking $l = 0$ for simplicity, a particular integral is therefore

$$\mathrm{X} = \frac{1}{3}\left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} t\right\} e^{-t}.$$

and the general solution is

$$\mathrm{X} = A\begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} + B\begin{bmatrix} 1 \\ -4 \end{bmatrix} e^{-4t} + \frac{1}{3}\left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} t\right\} e^{-t}.$$

Note:
In examples for which neither $f(t)$ nor $g(t)$ is identically equal to zero, the particular integral may be found by adding together the separate forms of particular integral for $f(t)$ and $g(t)$ and writing the system of differential equations in the form

$$\frac{\mathrm{dX}}{\mathrm{dt}} = \mathrm{MX} + \mathrm{N}_1 f(t) + \mathrm{N}_2 g(t),$$

where

$$\mathrm{N}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } \mathrm{N}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

For instance, if $f(t) \equiv t$ and $g(t) \equiv e^{2t}$, the particular integral would take the form

$$\mathrm{X} = \mathrm{P} + \mathrm{Q}t + \mathrm{R}e^{2t},$$

where P, Q and R are matrices of order 2×1.

15.10.2 EXERCISES

1. Determine the general solutions of the following systems of simultaneous differential equations:

   (a)

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = x_1 + 3x_2 + 5t,$$
$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = 3x_1 + x_2 + e^{3t}.$$

(b)

$$\begin{aligned}\frac{\mathrm{d}x_1}{\mathrm{d}t} &= 3x_1 + 2x_2 + t^2, \\ \frac{\mathrm{d}x_2}{\mathrm{d}t} &= 4x_1 + x_2 + e^{-2t}.\end{aligned}$$

2. Determine the complete solutions of the following systems of differential equations, subject to the conditions given:

(a)

$$\begin{aligned}\frac{\mathrm{d}x_1}{\mathrm{d}t} &= -x_1 + 9x_2 + 3, \\ \frac{\mathrm{d}x_2}{\mathrm{d}t} &= 11x_1 + x_2 + e^{10t},\end{aligned}$$

given that $x_1 = \frac{1}{225}$ and $x_2 = -\frac{1}{100}$ when $t = 0$.

(b)

$$\begin{aligned}\frac{\mathrm{d}x_1}{\mathrm{d}t} &= 5x_1 + 2x_2 + 2t^2 + t, \\ \frac{\mathrm{d}x_2}{\mathrm{d}t} &= -2x_1 + x_2,\end{aligned}$$

given that $x_1 = \frac{32}{27}$ and $x_2 = -\frac{12}{27}$ when $t = 0$.

(c)

$$\begin{aligned}\frac{\mathrm{d}x_1}{\mathrm{d}t} &= 8x_1 + x_2 + \sin t, \\ \frac{\mathrm{d}x_2}{\mathrm{d}t} &= -5x_1 + 6x_2 + \cos t,\end{aligned}$$

given that $x_1 = 0$ and $x_2 = 0$ when $t = 0$.

15.10.3 ANSWERS TO EXERCISES

1. (a)

$$A\begin{bmatrix}1\\1\end{bmatrix}e^{4t}+B\begin{bmatrix}1\\-1\end{bmatrix}e^{-2t}+\frac{1}{32}\begin{bmatrix}-25\\15\end{bmatrix}+\frac{1}{8}\begin{bmatrix}5\\-15\end{bmatrix}t-\frac{1}{5}\begin{bmatrix}3\\2\end{bmatrix}e^{3t};$$

   (b)

$$A\begin{bmatrix}1\\1\end{bmatrix}e^{5t}+B\begin{bmatrix}1\\-2\end{bmatrix}e^{-t}+\frac{2}{125}\begin{bmatrix}41\\-84\end{bmatrix}+\frac{2}{25}\begin{bmatrix}-9\\16\end{bmatrix}t+\frac{1}{5}\begin{bmatrix}1\\-4\end{bmatrix}t^2+\frac{1}{7}\begin{bmatrix}2\\-5\end{bmatrix}e^{-2t}.$$

2. (a)

$$-\frac{7}{45}\begin{bmatrix}1\\-1\end{bmatrix}e^{-10t}+\frac{13}{900}\begin{bmatrix}9\\11\end{bmatrix}e^{10t}+\frac{3}{100}\begin{bmatrix}1\\-11\end{bmatrix}+\frac{1}{180}\begin{bmatrix}0\\1\end{bmatrix}e^{10t}+\frac{1}{20}\begin{bmatrix}9\\11\end{bmatrix}te^{10t};$$

   (b)

$$\left\{(2t+1)\begin{bmatrix}1\\-1\end{bmatrix}+\begin{bmatrix}0\\1\end{bmatrix}\right\}e^{3t}+\frac{1}{27}\begin{bmatrix}5\\-12\end{bmatrix}+\frac{1}{27}\begin{bmatrix}1\\-22\end{bmatrix}t-\frac{2}{9}\begin{bmatrix}1\\2\end{bmatrix}t^2;$$

   (c)

$$\frac{1}{145}\left\{e^{7t}\left(\begin{bmatrix}-1\\25\end{bmatrix}\cos 2t+\begin{bmatrix}-12\\-10\end{bmatrix}\sin 2t\right)+\begin{bmatrix}-17\\-10\end{bmatrix}\sin t+\begin{bmatrix}1\\-25\end{bmatrix}\cos t\right\}.$$

"JUST THE MATHS"

UNIT NUMBER

16.1

LAPLACE TRANSFORMS 1
(Definitions and rules)

by

A.J.Hobson

UNIT 16.1 - LAPLACE TRANSFORMS 1 - DEFINITIONS AND RULES

16.1.1 INTRODUCTION

The theory of **"Laplace Transforms"** to be discussed in the following notes will be for the purpose of solving certain kinds of **"differential equation"**; that is, an equation which involves a derivative or derivatives.

The particular differential equation problems to be encountered will be limited to the two types listed below:

(a) Given the **"first order linear differential equation with constant coefficients"**,

$$a\frac{\mathrm{d}x}{\mathrm{d}t} + bx = f(t),$$

together with the value of x when $t = 0$ (that is, $x(0)$), determine a formula for x in terms of t, which does not include any derivatives.

(b) Given the **"second order linear differential equation with constant coefficients"**,

$$a\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + b\frac{\mathrm{d}x}{\mathrm{d}t} + cx = f(t),$$

together with the values of x and $\frac{\mathrm{d}x}{\mathrm{d}t}$ when $t = 0$ (that is, $x(0)$ and $x'(0)$), determine a formula for x in terms of t which does not include any derivatives.

Roughly speaking, the method of Laplace Transforms is used to convert a calculus problem (the differential equation) into an algebra problem (frequently an exercise on partial fractions and/or completing the square).

The solution of the algebra problem is then fed backwards through what is called an **"Inverse Laplace Transform"** and the solution of the differential equation is obtained.

The background to the development of Laplace Transforms would be best explained using certain other techniques of solving differential equations which may not have been part of earlier work. This background will therefore be omitted here.

DEFINITION
The Laplace Transform of a given function $f(t)$, defined for $t > 0$, is defined by the definite integral

$$\int_0^\infty e^{-st} f(t) \ \mathrm{d}t,$$

where s is an **arbitrary positive number**.

Notes
(i) The Laplace Transform is usually denoted by $L[f(t)]$ or $F(s)$, since the result of the definite integral in the definition will be an expression involving s.

(ii) Although s is an arbitrary positive number, it is occasionally necessary to assume that it is large enough to avoid difficulties in the calculations; (see the note to the second standard result below).

16.1.2 LAPLACE TRANSFORMS OF SIMPLE FUNCTIONS

The following is a list of standard results on which other Laplace Transforms will be based:

1. $f(t) \equiv t^n$.

$$F(s) = \int_0^\infty e^{-st} t^n \ \mathrm{d}t = I_n \ \text{ say}.$$

Hence,

$$I_n = \left[\frac{t^n e^{-st}}{-s}\right]_0^\infty + \frac{n}{s}\int_0^\infty e^{-st} t^{n-1} \ \mathrm{d}t = \frac{n}{s}.I_{n-1},$$

using the fact that e^{-st} tends to zero much faster than any other function of t can tend to infinity. That is, a decaying exponential will always have the dominating effect.

We conclude that

$$I_n = \frac{n}{s}.\frac{(n-1)}{s}.\frac{(n-2)}{s}....\frac{2}{s}.\frac{1}{s}.I_0 = \frac{n!}{s^n}.I_0.$$

But,

$$I_0 = \int_0^\infty e^{-st} \ \mathrm{d}t = \left[\frac{e^{-st}}{-s}\right]_0^\infty = \frac{1}{s}.$$

Thus,

$$L[t^n] = \frac{n!}{s^{n+1}}.$$

Note:
This result also shows that

$$L[1] = \frac{1}{s},$$

since $1 = t^0$.

2. $f(t) \equiv e^{-at}$.

$$F(s) = \int_0^\infty e^{-st}e^{-at}\ \mathrm{d}t = \int_0^\infty e^{-(s+a)t}\ \mathrm{d}t$$
$$= \left[\frac{e^{-(s+a)t}}{-(s+a)}\right]_0^\infty = \frac{1}{s+a}.$$

Hence,

$$L[e^{-at}] = \frac{1}{s+a}.$$

Note:
A slightly different form of this result, less commonly used in applications to science and engineering, is

$$L[e^{bt}] = \frac{1}{s-b};$$

but, to obtain this result by integration, we would need to assume that $s > b$ to ensure that $e^{-(s-b)t}$ is genuinely a **decaying** exponential.

3. $f(t) \equiv \cos at$.

$$F(s) = \int_0^\infty e^{-st}\cos at\ \mathrm{d}t = \left[\frac{e^{-st}\sin at}{a}\right]_0^\infty + \frac{s}{a}\int_0^\infty e^{-st}\sin at\ \mathrm{d}t$$

using Integration by Parts, once.

Using Integration by Parts a second time,

$$F(s) = 0 + \frac{s}{a}\left\{\left[-\frac{e^{-st}\cos at}{a}\right]_0^\infty - \frac{s}{a}\int_0^\infty e^{-st}\cos at\ \mathrm{d}t\right\},$$

which gives

$$F(s) = \frac{s}{a^2} - \frac{s^2}{a^2}.F(s).$$

That is,

$$F(s) = \frac{s}{s^2+a^2}.$$

In other words,

$$L[\cos at] = \frac{s}{s^2+a^2}.$$

4. $f(t) \equiv \sin at$.

   The method is similar to that for $\cos at$, and we obtain

$$L[\sin at] = \frac{a}{s^2 + a^2}.$$

16.1.3 ELEMENTARY LAPLACE TRANSFORM RULES

The following list of results is of use in finding the Laplace Transform of a function which is made up of **basic** functions, such as those encountered in the previous section.

1. **LINEARITY**

   If A and B are constants, then

$$L[Af(t) + Bg(t)] = AL[f(t)] + BL[g(t)].$$

   Proof:

   This follows easily from the linearity of an integral.

   EXAMPLE

   Determine the Laplace Transform of the function,

$$2t^5 + 7\cos 4t - 1.$$

   Solution

$$L[2t^5 + 7\cos 4t - 1] = 2.\frac{5!}{s^6} + 7.\frac{s}{s^2 + 4^2} - \frac{1}{s} = \frac{240}{s^6} + \frac{7s}{s^2 + 16} - \frac{1}{s}.$$

2. **THE TRANSFORM OF A DERIVATIVE**

   The two results which follow are of special use when solving first and second order differential equations. We shall begin by discussing them in relation to an arbitrary function, $f(t)$; then we shall restate them in the form which will be needed for solving differential equations.

   (a)

$$L[f'(t)] = sL[f(t)] - f(0).$$

   Proof:

$$L[f'(t)] = \int_0^\infty e^{-st} f'(t)\ \mathrm{d}t = \left[e^{-st} f(t)\right]_0^\infty + s\int_0^\infty e^{-st} f(t)\ \mathrm{d}t$$

   using integration by parts.

   Thus,

$$L[f'(t)] = -f(0) + sL[f(t)],$$

   as required.

(b)

$$L[f''(t)] = s^2L[f(t)] - sf(0) - f'(0).$$

Proof:

Treating $f''(t)$ as the first derivative of $f'(t)$, we have

$$L[f''(t)] = sL[f'(t)] - f'(0),$$

which gives the required result on substituting from (a) the expression for $L[f'(t)]$.

Alternative Forms (Using $L[x(t)] = X(s)$)**:**

(i)

$$L\left[\frac{\mathrm{d}x}{\mathrm{d}t}\right] = sX(s) - x(0).$$

(ii)

$$L\left[\frac{\mathrm{d}^2x}{\mathrm{d}t^2}\right] = s^2X(s) - sx(0) - x'(0) \text{ or } s[sX(s) - x(0)] - x'(0).$$

3. THE (First) SHIFTING THEOREM

$$L[e^{-at}f(t)] = F(s+a).$$

Proof:

$$L[e^{-at}f(t)] = \int_0^\infty e^{-st}e^{-at}f(t)\ \mathrm{d}t = \int_0^\infty e^{-(s+a)t}f(t)\ \mathrm{d}t,$$

which can be regarded as the effect of replacing s by $s+a$ in $L[f(t)]$. In other words, $F(s+a)$.

Notes:

(i) Sometimes, this result is stated in the form

$$L[e^{bt}f(t)] = F(s-b)$$

but, in science and engineering, the exponential is more likely to be a **decaying** exponential.

(ii) There is, in fact, a Second Shifting Theorem, encountered in more advanced courses; but we do not include it in this Unit (see Unit 16.5).

EXAMPLE

Determine the Laplace Transform of the function, $e^{-2t}\sin 3t$.

Solution

First of all, we note that

$$L[\sin 3t] = \frac{3}{s^2+3^2} = \frac{3}{s^2+9}.$$

Replacing s by $(s+2)$ in this result, the First Shifting Theorem gives

$$L[e^{-2t}\sin 3t] = \frac{3}{(s+2)^2+9}.$$

4. MULTIPLICATION BY t

$$L[tf(t)] = -\frac{\mathrm{d}}{\mathrm{d}s}[F(s)].$$

Proof:

It may be shown that

$$\frac{\mathrm{d}}{\mathrm{d}s}[F(s)] = \int_0^\infty \frac{\partial}{\partial s}[e^{-st}f(t)]\mathrm{d}t = \int_0^\infty -te^{-st}f(t)\ \mathrm{d}t = -\ L[tf(t)].$$

EXAMPLE

Determine the Laplace Transform of the function,

$$t\cos 7t.$$

Solution

$$L[t\cos 7t] = -\frac{\mathrm{d}}{\mathrm{d}s}\left[\frac{s}{s^2+7^2}\right] = -\ \frac{(s^2+7^2).1 - s.2s}{(s^2+7^2)^2} = \frac{s^2-49}{(s^2+49)^2}.$$

THE USE OF A TABLE OF LAPLACE TRANSFORMS AND RULES

For the purposes of these Units, the following **brief** table may be used to determine the Laplace Transforms of functions of t without having to use integration:

$f(t)$	$L[f(t)] = F(s)$
K (a constant)	$\frac{K}{s}$
e^{-at}	$\frac{1}{s+a}$
t^n	$\frac{n!}{s^{n+1}}$
$\sin at$	$\frac{a}{s^2+a^2}$
$\sinh at$	$\frac{a}{s^2-a^2}$
$\cos at$	$\frac{s}{s^2+a^2}$
$\cosh at$	$\frac{s}{s^2-a^2}$
te^{-at}	$\frac{1}{(s+a)^2}$
$t\sin at$	$\frac{2as}{(s^2+a^2)^2}$
$t\cos at$	$\frac{(s^2-a^2)}{(s^2+a^2)^2}$
$\sin at - at\cos at$	$\frac{2a^3}{(s^2+a^2)^2}$

16.1.4 FURTHER LAPLACE TRANSFORM RULES

1.
$$L\left[\frac{\mathrm{d}x}{\mathrm{d}t}\right] = sX(s) - x(0).$$

2.
$$L\left[\frac{\mathrm{d}^2x}{\mathrm{d}t^2}\right] = s^2X(s) - sx(0) - x'(0) \quad \text{or} \quad s[sX(s) - x(0)] - x'(0).$$

3. **The Initial Value Theorem**

$$\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s),$$

provided that the indicated limits exist.

4. **The Final Value Theorem**

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s),$$

provided that the indicated limits exist.

5. **The Convolution Theorem**

$$L\left[\int_0^t f(T)g(t-T)\ \mathrm{d}T\right] = F(s)G(s).$$

16.1.5 EXERCISES

1. Use a table the table of Laplace Transforms to find $L[f(t)]$ in the following cases:

(a)

$$3t^2 + 4t - 1;$$

(b)

$$t^3 + 3t^2 + 3t + 1 \quad (\equiv (t+1)^3);$$

(c)

$$2e^{5t} - 3e^t + e^{-7t};$$

(d)

$$2\sin 3t - 3\cos 2t;$$

(e)

$$t\sin 6t;$$

(f)

$$t(e^t + e^{-2t});$$

(g)

$$\frac{1}{2}(1-\cos 2t) \quad (\equiv \sin^2 t).$$

2. Using the First Shifting Theorem, obtain the Laplace Transforms of the following functions of t:

(a)

$$e^{-3t}\cos 5t;$$

(b)

$$t^2e^{2t};$$

(c)

$$e^{-2t}\left(2t^3+3t-2\right);$$

(d)

$$\cosh 2t . \sin t;$$

(e)

$$e^{-at}f'(t),$$

where $L[f(t)] = F(s)$.

3. (a) If

$$x = t^3e^{-t},$$

determine the Laplace Transform of $\frac{\mathrm{d}^2x}{\mathrm{d}t^2}$ without differentiating x more than once with respect to t.

(b) If

$$\frac{\mathrm{d}x}{\mathrm{d}t} + x = e^t,$$

where $x(0) = 0$, show that

$$X(s) = \frac{1}{s^2-1}.$$

4. Verify the Initial and Final Value Theorems for the function

$$f(t) = te^{-3t}.$$

16.1.6 ANSWERS TO EXERCISES

1. (a)

$$\frac{6}{s^3} + \frac{4}{s^2} - \frac{1}{s};$$

(b)

$$\frac{6}{s^4} + \frac{6}{s^3} + \frac{3}{s^2} + \frac{1}{s};$$

(c)

$$\frac{2}{s-5} - \frac{3}{s-1} + \frac{1}{s+7};$$

(d)

$$\frac{6}{s^2+9} - \frac{3s}{s^2+4};$$

(e)

$$\frac{12s}{(s^2+36)^2};$$

(f)

$$\frac{1}{(s-1)^2}+\frac{1}{(s+2)^2};$$

(g)

$$\frac{1}{2}\left[\frac{1}{s}-\frac{s}{s^2+4}\right].$$

2. (a)

$$\frac{s+3}{(s+3)^2+25};$$

(b)

$$\frac{2}{(s-2)^3};$$

(c)

$$\frac{12}{(s+2)^4}+\frac{3}{(s+2)^2}-\frac{2}{s+2};$$

(d)

$$\frac{1}{2}\left[\frac{1}{(s-2)^2+1}+\frac{1}{(s+2)^2+1}\right];$$

(e)

$$(s+a)F(s+a)-f(0).$$

3. (a)

$$\frac{6s^2}{(s+1)^4};$$

(b) On the left hand side, use the formula for $L\left[\frac{\mathrm{d}x}{\mathrm{d}t}\right]$.

4.

$$\lim_{t\to 0} f(t)=0 \text{ and } \lim_{t\to\infty} f(t)=0.$$

“JUST THE MATHS”

UNIT NUMBER

16.2

LAPLACE TRANSFORMS 2
(Inverse Laplace Transforms)

by

A.J.Hobson

UNIT 16.2 - LAPLACE TRANSFORMS 2
INVERSE LAPLACE TRANSFORMS

In order to solve differential equations, we now examine how to determine a function of the variable, t, whose Laplace Transform is already known.

16.2.1 THE DEFINITION OF AN INVERSE LAPLACE TRANSFORMS

A function of t, whose Laplace Transform is the given expression, $F(s)$, is called the **"Inverse Laplace Transform"** of $f(t)$ and may be denoted by the symbol

$$L^{-1}[F(s)].$$

Notes:
(i) Since two functions which coincide for $t > 0$ will have the same Laplace Transform, we can determine the Inverse Laplace Transform of $F(s)$ only for **positive** values of t.

(ii) Inverse Laplace Transforms are **linear** since

$$L^{-1}\left[AF(s) + BG(s)\right]$$

is a function of t whose Laplace Transform is

$$AF(s) + BG(s);$$

and, by the linearity of Laplace Transforms, discussed in Unit 16.1, such a function is

$$AL^{-1}[F(s)] + BL^{-1}[G(s)].$$

16.2.2 METHODS OF DETERMINING AN INVERSE LAPLACE TRANSFORM

The type of differential equation to be encountered in simple practical problems usually lead to Laplace Transforms which are **"rational functions of s"**. We shall restrict the discussion to such cases, as illustrated in the following examples, where the table of standard Laplace Transforms is used whenever possible. The partial fractions are discussed in detail, but other, shorter, methods may be used if known (for example, the "Cover-up Rule" and "Keily's Method"; see Unit 1.9)

EXAMPLES

1. Determine the Inverse Laplace Transform of

$$F(s) = \frac{3}{s^3} + \frac{4}{s-2}.$$

Solution

$$f(t) = \frac{3}{2}t^2 + 4e^{2t} \quad t > 0$$

2. Determine the Inverse Laplace Transform of

$$F(s) = \frac{2s+3}{s^2+3s} = \frac{2s+3}{s(s+3)}.$$

Solution

Applying the principles of partial fractions,

$$\frac{2s+3}{s(s+3)} \equiv \frac{A}{s} + \frac{B}{s+3},$$

giving

$$2s+3 \equiv A(s+3) + Bs$$

Note:

Although the s of a Laplace Transform is an arbitrary **positive** number, we may temporarily ignore that in order to complete the partial fractions. Otherwise, entire partial fractions exercises would have to be carried out by equating coefficients of appropriate powers of s on both sides.

Putting $s = 0$ and $s = -3$ gives

$$3 = 3A \text{ and } -3 = -3B;$$

so that

$$A = 1 \text{ and } B = 1.$$

Hence,

$$F(s) = \frac{1}{s} + \frac{1}{s+3}$$

Finally,

$$f(t) = 1 + e^{-3t} \quad t > 0.$$

3. Determine the Inverse Laplace Transform of

$$F(s) = \frac{1}{s^2+9}.$$

Solution

$$f(t) = \frac{1}{3}\sin 3t \quad t > 0.$$

4. Determine the Inverse Laplace Transform of

$$F(s) = \frac{s+2}{s^2+5}.$$

Solution

$$f(t) = \cos t\sqrt{5} + \frac{2}{\sqrt{5}}\sin t\sqrt{5} \quad t > 0.$$

5. Determine the Inverse Laplace Transform of

$$F(s) = \frac{3s^2 + 2s + 4}{(s+1)(s^2+4)}.$$

Solution

Applying the principles of partial fractions,

$$\frac{3s^2 + 2s + 4}{(s+1)(s^2+4)} \equiv \frac{A}{s+1} + \frac{Bs+C}{s^2+4}.$$

That is,

$$3s^2 + 2s + 4 \equiv A(s^2+4) + (Bs+C)(s+1).$$

Substituting $s = -1$, we obtain

$$5 = 5A \text{ which implies that } A = 1.$$

Equating coefficients of s^2 on both sides,

$$3 = A + B \text{ so that } B = 2.$$

Equating constant terms on both sides,

$$4 = 4A + C \text{ so that } C = 0.$$

We conclude that

$$F(s) = \frac{1}{s+1} + \frac{2s}{s^2+4}.$$

Hence,

$$f(t) = e^{-t} + 2\cos 2t \quad t > 0.$$

6. Determine the Inverse Laplace Transform of

$$F(s) = \frac{1}{(s+2)^5}.$$

Solution

Using the First Shifting Theorem and the Inverse Laplace Transform of $\frac{n!}{s^{n+1}}$, we obtain

$$f(t) = \frac{1}{24}t^4 e^{-2t} \quad t > 0.$$

7. Determine the Inverse Laplace Transform of

$$F(s) = \frac{3}{(s-7)^2 + 9}.$$

Solution

Using the First Shifting Theorem and the Inverse Laplace Transform of $\frac{a}{s^2+a^2}$, we obtain

$$f(t) = e^{7t}\sin 3t \quad t > 0.$$

8. Determine the Inverse Laplace Transform of

$$F(s) = \frac{s}{s^2 + 4s + 13}.$$

Solution

The denominator will not factorise conveniently, so we **complete the square**, giving

$$F(s) = \frac{s}{(s+2)^2 + 9}.$$

In order to use the First Shifting Theorem, we must try to include $s+2$ in the numerator; so we write

$$F(s) = \frac{(s+2) - 2}{(s+2)^2 + 9} = \frac{s+2}{(s+2)^2 + 3^2} - \frac{2}{3}.\frac{3}{(s+2)^2 + 3^2}.$$

Hence,

$$f(t) = e^{-2t} \cos 3t - \frac{2}{3} e^{-2t} \sin 3t = \frac{1}{3} e^{-2t} \left[3 \cos 3t - 2 \sin 3t\right] \quad t > 0.$$

9. Determine the Inverse Laplace Transform of

$$F(s) = \frac{8(s+1)}{s(s^2 + 4s + 8)}.$$

Solution

Applying the principles of partial fractions,

$$\frac{8(s+1)}{s(s^2 + 4s + 8)} \equiv \frac{A}{s} + \frac{Bs + C}{s^2 + 4s + 8}.$$

Mutiplying up, we obtain

$$8(s+1) \equiv A(s^2 + 4s + 8) + (Bs + C)s.$$

Substituting $s = 0$ gives

$$8 = 8A \text{ so that } A = 1.$$

Equating coefficients of s^2 on both sides,

$$0 = A + B \text{ which gives } B = -1.$$

Equating coefficients of s on both sides,

$$8 = 4A + C \text{ which gives } C = 4.$$

Thus,

$$F(s) = \frac{1}{s} + \frac{-s + 4}{s^2 + 4s + 8}.$$

The quadratic denominator will not factorise conveniently, so we complete the square to give

$$F(s) = \frac{1}{s} + \frac{-s+4}{(s+2)^2+4},$$

which, on rearrangement, becomes

$$F(s) = \frac{1}{s} - \frac{s+2}{(s+2)^2+2^2} + \frac{6}{(s+2)^2+2^2}.$$

Thus, from the First Shifting Theorem,

$$f(t) = 1 - e^{-2t}\cos 2t + 3e^{-2t}\sin 2t \quad t > 0.$$

10. Determine the Inverse Laplace Transform of

$$F(s) = \frac{s+10}{s^2-4s-12}.$$

Solution

This time, the denominator **will** factorise, into $(s+2)(s-6)$, and partial fractions give

$$\frac{s+10}{(s+2)(s-6)} \equiv \frac{A}{s+2} + \frac{B}{s-6}.$$

Hence,

$$s+10 \equiv \mathrm{A}(s-6) + \mathrm{B}(s+2).$$

Putting $s = -2$,

$$8 = -8A \text{ giving } A = -1.$$

Putting $s = 6$,

$$16 = 8B \text{ giving } B = 2.$$

We conclude that

$$F(s) = \frac{-1}{s+2} + \frac{2}{s-6}.$$

Finally,

$$f(t) = -e^{-2t} + 2e^{6t} \quad t > 0.$$

However, if we did not factorise the denominator, a different form of solution could be obtained as follows:

$$F(s) = \frac{(s-2)+12}{(s-2)^2-4^2} = \frac{s-2}{(s-2)^2-4^2} + 3.\frac{4}{(s-2)^2+4^2}.$$

Hence,

$$f(t) = e^{2t}[\cosh 4t + 3\sinh 4t] \quad t > 0.$$

11. Determine the Inverse Laplace Transform of

$$F(s) = \frac{1}{(s-1)(s+2)}.$$

Solution

The Inverse Laplace Transform of this function could certainly be obtained by using partial fractions, but we note here how it could be obtained from the Convolution Theorem.

Writing

$$F(s) = \frac{1}{(s-1)}.\frac{1}{(s+2)},$$

we obtain

$$f(t) = \int_0^t e^T.e^{-2(t-T)} \, \mathrm{d}T = \int_0^t e^{(3T-2t)} \, \mathrm{d}T = \left[\frac{e^{3T-2t}}{3}\right]_0^t.$$

That is,

$$f(t) = \frac{e^t}{3} - \frac{e^{-2t}}{3} \quad t > 0.$$

16.2.3 EXERCISES

Determine the Inverse Laplace Transforms of the following rational functions of s:

1. (a)

$$\frac{1}{(s-1)^2};$$

(b)

$$\frac{1}{(s+1)^2+4};$$

(c)

$$\frac{s+2}{(s+2)^2+9};$$

(d)

$$\frac{s-2}{(s-3)^3};$$

(e)

$$\frac{1}{(s^2+4)^2};$$

(f)

$$\frac{s+1}{s^2+2s+5};$$

(g)

$$\frac{s-3}{s^2-4s+5};$$

(h)

$$\frac{s-3}{(s-1)^2(s-2)};$$

(i)

$$\frac{5}{(s+1)(s^2-2s+2)};$$

(j)

$$\frac{2s-9}{(s-3)(s+2)};$$

(k)

$$\frac{3}{s(s^2+9)};$$

(l)

$$\frac{2s-1}{(s-1)(s^2+2s+2)}.$$

2. Use the Convolution Theorem to obtain the Inverse Laplace Transform of

$$\frac{s}{(s^2+1)^2}.$$

16.2.4 ANSWERS TO EXERCISES

1. (a)

$$te^t \quad t > 0;$$

(b)

$$\frac{1}{2}e^{-t}\sin 2t \quad t > 0;$$

(c)

$$e^{-2t}\cos 3t \quad t > 0;$$

(d)

$$e^{3t}\left[t + \frac{1}{2}t^2\right] \quad t > 0;$$

(e)

$$\frac{1}{16}[\sin 2t - 2t\cos 2t] \quad t > 0;$$

(f)

$$e^{-t}\cos 2t \quad t > 0;$$

(g)

$$e^{2t}[\cos t - \sin t] \quad t > 0;$$

(h)

$$2te^{t} + e^{t} - e^{2t} \quad t > 0;$$

(i)

$$e^{-t} + e^{t}[2\sin t - \cos t] \quad t > 0;$$

(j)

$$\frac{1}{5}[13e^{-2t} - 3e^{3t}] \quad t > 0;$$

(k)

$$\frac{1}{3}[1 - \cos 3t] \quad t > 0;$$

(l)

$$\frac{1}{5}[e^{t} - e^{-t}\cos t + 8e^{-t}\sin t] \quad t > 0.$$

2.

$$\frac{1}{2}t\sin t \quad t > 0.$$

"JUST THE MATHS"

UNIT NUMBER

16.3

LAPLACE TRANSFORMS 3
(Differential equations)

by

A.J.Hobson

UNIT 16.3 - LAPLACE TRANSFORMS 3 - DIFFERENTIAL EQUATIONS

16.3.1 EXAMPLES OF SOLVING DIFFERENTIAL EQUATIONS

In the work which follows, the problems considered will usually take the form of a linear differential equation of the second order with constant coefficients.

That is,

$$a\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + b\frac{\mathrm{d}x}{\mathrm{d}t} + cx = f(t).$$

However, the method will apply equally well to the corresponding first order differential equation,

$$a\frac{\mathrm{d}x}{\mathrm{d}t} + bx = f(t).$$

The technique will be illustrated by examples.

EXAMPLES

1. Solve the differential equation

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 4\frac{\mathrm{d}x}{\mathrm{d}t} + 13x = 0,$$

given that $x = 3$ and $\frac{\mathrm{d}x}{\mathrm{d}t} = 0$ when $t = 0$.

Solution

Taking the Laplace Transform of the differential equation,

$$s[sX(s) - 3] + 4[sX(s) - 3] + 13X(s) = 0.$$

Hence,

$$(s^2 + 4s + 13)X(s) = 3s + 12,$$

giving

$$X(s) \equiv \frac{3s + 12}{s^2 + 4s + 13}.$$

The denominator does not factorise, therefore we complete the square to obtain

$$X(s) \equiv \frac{3s + 12}{(s + 2)^2 + 9} \equiv \frac{3(s + 2) + 6}{(s + 2)^2 + 9} \equiv 3.\frac{s + 2}{(s + 2)^2 + 9} + 2.\frac{3}{(s + 2)^2 + 9}.$$

Thus,

$$x(t) = 3e^{-2t}\cos 3t + 2e^{-2t}\sin 3t \quad t > 0$$

or

$$x(t) = e^{-2t}[3\cos 3t + 2\sin 3t] \quad t > 0.$$

2. Solve the differential equation

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 6\frac{\mathrm{d}x}{\mathrm{d}t} + 9x = 50\sin t,$$

given that $x = 1$ and $\frac{\mathrm{d}x}{\mathrm{d}t} = 4$ when $t = 0$.

Solution

Taking the Laplace Transform of the differential equation,

$$s[sX(s) - 1] - 4 + 6[sX(s) - 1] + 9X(s) = \frac{50}{s^2+1},$$

giving

$$(s^2 + 6s + 9)X(s) = \frac{50}{s^2+1} + s + 10.$$

Hint: Do <u>not</u> combine the terms on the right into a single fraction - it won't help !

Thus,

$$X(s) \equiv \frac{50}{(s^2+6s+9)(s^2+1)} + \frac{s+10}{s^2+6s+9}$$

or

$$X(s) \equiv \frac{50}{(s+3)^2(s^2+1)} + \frac{s+10}{(s+3)^2}.$$

Using the principles of partial fractions in the first term on the right,

$$\frac{50}{(s+3)^2(s^2+1)} \equiv \frac{A}{(s+3)^2} + \frac{B}{s+3} + \frac{Cs+D}{s^2+1}.$$

Hence,

$$50 \equiv A(s^2+1) + B(s+3)(s^2+1) + (Cs+D)(s+3)^2.$$

Substituting $s = -3$,

$$50 = 10A \text{ giving } A = 5.$$

Equating coefficients of s^3 on both sides,

$$0 = B + C. \quad (1)$$

Equating the coefficients of s on both sides (we shall not need the s^2 coefficients in this example),

$$0 = B + 9C + 6D. \quad (2)$$

Equating the constant terms on both sides,

$$50 = A + 3B + 9D = 5 + 3B + 9D. \quad (3)$$

Putting $C = -B$ into (2), we obtain

$$-8B + 6D = 0, \quad (4)$$

and we already have

$$3B + 9D = 45. \quad (3)$$

These last two solve easily to give $B = 3$ and $D = 4$ so that $C = -3$.

We conclude that

$$\frac{50}{(s+3)^2(s^2+1)} \equiv \frac{5}{(s+3)^2} + \frac{3}{s+3} + \frac{-3s+4}{s^2+1}.$$

In addition to this, we also have

$$\frac{s+10}{(s+3)^2} \equiv \frac{s+3}{(s+3)^2} + \frac{7}{(s+3)^2} \equiv \frac{1}{s+3} + \frac{7}{(s+3)^2}.$$

The total for $X(s)$ is therefore given by

$$X(s) \equiv \frac{12}{(s+3)^2} + \frac{4}{s+3} - 3.\frac{s}{s^2+1} + 4.\frac{1}{s^2+1}.$$

Finally,

$$x(t) = 12te^{-3t} + 4e^{-3t} - 3\cos t + 4\sin t \quad t > 0.$$

3. Solve the differential equation

$$\frac{d^2x}{dt^2} + 4\frac{dx}{dt} - 3x = 4e^t,$$

given that $x = 1$ and $\frac{dx}{dt} = -2$ when $t = 0$.

Solution

Taking the Laplace Transform of the differential equation,

$$s[sX(s) - 1] + 2 + 4[sX(s) - 1] - 3X(s) = \frac{4}{s-1}.$$

This gives

$$(s^2 + 4s - 3)X(s) = \frac{4}{s-1} + s + 2.$$

Therefore,

$$X(s) \equiv \frac{4}{(s-1)(s^2+4s-3)} + \frac{s+2}{s^2+4s-3}.$$

Applying the principles of partial fractions,

$$\frac{4}{(s-1)(s^2+4s-3)} \equiv \frac{A}{s-1} + \frac{Bs+C}{s^2+4s-3}.$$

Hence,

$$4 \equiv A(s^2+4s-3) + (Bs+C)(s-1).$$

Substituting $s = 1$, we obtain

$$4 = 2A; \quad \text{that is,} \quad A = 2.$$

Equating coefficients of s^2 on both sides,

$$0 = A + B, \quad \text{so that} \quad B = -2.$$

Equating constant terms on both sides,

$$4 = -3A - C, \quad \text{so that} \quad C = -10.$$

Thus, in total,

$$X(s) \equiv \frac{2}{s-1} + \frac{-s-8}{s^2+4s-3} \equiv \frac{2}{s-1} + \frac{-s-8}{(s+2)^2-7}$$

or

$$X(s) \equiv \frac{2}{s-1} - \frac{s+2}{(s+2)^2-7} - \frac{6}{(s+2)^2-7}.$$

Finally,

$$x(t) = 2e^t - e^{-2t}\text{cosh}t\sqrt{7} - \frac{6}{\sqrt{7}}e^{-2t}\text{sinh}t\sqrt{7} \quad t > 0.$$

16.3.2 THE GENERAL SOLUTION OF A DIFFERENTIAL EQUATION

On some occasions, we may either be given no boundary conditions at all; or else the boundary conditions given do not tell us the values of $x(0)$ and $x'(0)$.

In such cases, we simply let $x(0) = A$ and $x'(0) = B$ to obtain a solution in terms of A and B called the **"general solution"**.

If any non-standard boundary conditions are provided, we then substitute them into the general solution to obtain particular values of A and B.

EXAMPLE

Determine the general solution of the differential equation

$$\frac{d^2x}{dt^2} + 4x = 0$$

and, hence, determine the particular solution in the case when $x(\frac{\pi}{2}) = -3$ and $x'(\frac{\pi}{2}) = 10$.

Solution

Taking the Laplace Transform of the differential equation,

$$s(sX(s) - A) - B + 4X(s) = 0.$$

That is,

$$(s^2 + 4)X(s) = As + B.$$

Hence,

$$X(s) \equiv \frac{As + B}{s^2 + 4} \equiv A.\frac{s}{s^2 + 4} + B.\frac{1}{s^2 + 4}.$$

This gives

$$x(t) = A\cos 2t + \frac{B}{2}\sin 2t \quad t > 0;$$

but, since A and B are **arbitrary** constants, this may be written in the simpler form

$$x(t) = A\cos 2t + B\sin 2t \quad t > 0,$$

in which $\frac{B}{2}$ has been rewritten as B.

To apply the boundary conditions, we require also the formula for $x'(t)$, namely

$$x'(t) = -\ 2A\sin 2t + 2B\cos 2t.$$

Hence, $-\ 3 = -\ A$ and $10 = -\ 2B$ giving $A = 3$ and $B =$ - 5.

Therefore, the particular solution is

$$x(t) = 3\cos 2t - 5\sin 2t \quad t > 0.$$

16.3.3 EXERCISES

1. Solve the following differential equations subject to the conditions given:

   (a)
   $$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} - 2\frac{\mathrm{d}x}{\mathrm{d}t} + 5x = 0,$$
   given that $x(0) = 3$ and $x'(0) = 1$;

   (b)
   $$4\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 4\frac{\mathrm{d}x}{\mathrm{d}t} + x = 0,$$
   given that $x(0) = 4$ and $x'(0) = 1$;

   (c)
   $$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 2\frac{\mathrm{d}x}{\mathrm{d}t} - 8x = 2t,$$
   given that $x(0) = 3$ and $x'(0) = 1$;

   (d)
   $$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} - 4x = 2e^{2t},$$
   given that $x(0) = 1$ and $x'(0) = 10.5$;

   (e)
   $$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 4x = 3\cos^2 t,$$
   given that $x(0) = 1$ and $x'(0) = 2$.
   Hint: $\cos 2t \equiv 2\cos^2 t - 1$.

2. Determine the particular solution of the differential equation
   $$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} - 2\frac{\mathrm{d}x}{\mathrm{d}t} = e^t(t - 3)$$
   in the case when $x(0) = 2$ and $x(3) = -1$.

   Hint:
   Since $x(0)$ is given, just let $x'(0) = B$ to obtain a solution in terms of B; then substitute the second boundary condition at the end.

16.3.4 ANSWERS TO EXERCISES

1. (a)

$$X(s) = \frac{3s-5}{s^2-2s+5},$$

giving

$$x(t) = e^t(3\cos 2t - \sin 2t) \quad t > 0;$$

(b)

$$X(s) = \frac{4}{s+\frac{1}{2}} + \frac{3}{(s+\frac{1}{2})^2},$$

giving

$$x(t) = 4e^{-\frac{1}{2}t} + 3te^{-\frac{1}{2}t} = e^{-\frac{1}{2}t}[4+3t] \quad t > 0;$$

(c)

$$X(s) = \frac{27}{12}.\frac{1}{s-2} + \frac{39}{48}.\frac{1}{s+4} - \frac{1}{4}.\frac{1}{s^2} - \frac{1}{16}.\frac{1}{s},$$

giving

$$x(t) = \frac{27}{12}e^{2t} + \frac{39}{48}e^{-4t} - \frac{1}{4}t - \frac{1}{16} \quad t > 0;$$

(d)

$$X(s) = \frac{\frac{1}{2}}{(s-2)^2} + \frac{3}{s-2} - \frac{2}{s+2},$$

giving

$$x(t) = \frac{1}{2}te^{2t} + 3e^{2t} - 2e^{-2t} \quad t > 0;$$

(e)

$$X(s) = \frac{3}{2}.\frac{s}{(s^2+4)^2} + \frac{3}{8}.\frac{1}{s} + \frac{5}{8}.\frac{s}{s^2+4} + \frac{2}{s^2+4},$$

giving

$$x(t) = \frac{3}{8}t\sin 2t + \frac{3}{8} + \frac{5}{8}\cos 2t + \sin 2t \quad t > 0.$$

2.

$$x(t) = 3e^t - te^t - 1 \quad t > 0.$$

"JUST THE MATHS"

UNIT NUMBER

16.4

LAPLACE TRANSFORMS 4
(Simultaneous differential equations)

by

A.J.Hobson

UNIT 16.4 - LAPLACE TRANSFORMS 4
SIMULTANEOUS DIFFERENTIAL EQUATIONS

16.4.1 AN EXAMPLE OF SOLVING SIMULTANEOUS LINEAR DIFFERENTIAL EQUATIONS

In this Unit, we shall consider a pair of differential equations involving an independent variable, t, such as a time variable, and two dependent variables, x and y, such as electric currents or linear displacements.

The general format is as follows:

$$\begin{aligned} a_1\frac{\mathrm{d}x}{\mathrm{d}t} + b_1\frac{\mathrm{d}y}{\mathrm{d}t} + c_1x + d_1y &= f_1(t), \\ a_2\frac{\mathrm{d}x}{\mathrm{d}t} + b_2\frac{\mathrm{d}y}{\mathrm{d}t} + c_2x + d_2y &= f_2(t). \end{aligned}$$

To solve these equations simultaneously, we take the Laplace Transform of each equation obtaining two simultaneous algebraic equations from which we may determine $X(s)$ and $Y(s)$, the Laplace Transforms of $x(t)$ and $y(t)$ respectively.

EXAMPLE

Solve, simultaneously, the differential equations

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}t} + 2x &= e^t, \\ \frac{\mathrm{d}x}{\mathrm{d}t} - 2y &= 1 + t, \end{aligned}$$

given that $x(0) = 1$ and $y(0) = 2$.

Solution

Taking the Laplace Transforms of the differential equations,

$$sY(s) - 2 + 2X(s) = \frac{1}{s-1},$$

$$sX(s) - 1 - 2Y(s) \quad = \quad \frac{1}{s} + \frac{1}{s^2}.$$

That is,

$$\begin{aligned} 2X(s) + sY(s) &= \frac{1}{s-1} + 2, & (1) \\ sX(s) - 2Y(s) &= \frac{1}{s} + \frac{1}{s^2} + 1. & (2) \end{aligned}$$

Using $(1) \times 2 + (2) \times s$, we obtain

$$(4 + s^2)X(s) = \frac{2}{s-1} + 4 + 1 + \frac{1}{s} + s.$$

Hence,

$$X(s) = \frac{2}{(s-1)(s^2+4)} + \frac{5}{s^2+4} + \frac{1}{s(s^2+4)} + \frac{s}{s^2+4}.$$

Applying the methods of partial fractions, this gives

$$X(s) = \frac{2}{5}.\frac{1}{s-1} + \frac{7}{20}.\frac{s}{s^2+4} + \frac{23}{5}.\frac{1}{s^2+4} + \frac{1}{4}.\frac{1}{s}.$$

Thus,

$$x(t) = \frac{2}{5}e^t + \frac{1}{4} + \frac{7}{20}\cos 2t + \frac{23}{10}\sin 2t \quad t > 0.$$

We could now start again by eliminating x from equations (1) and (2) in order to calculate y, and this is often necessary; but, since

$$2y = \frac{\mathrm{d}x}{\mathrm{d}t} - 1 - t$$

in the current example,

$$y(t) = \frac{1}{5}e^t - \frac{1}{2} - \frac{7}{20}\sin 2t + \frac{23}{10}\cos 2t - \frac{t}{2} \quad t > 0.$$

16.4.2 EXERCISES

Use Laplace Transforms to solve the following pairs of simultaneous differential equations, subject to the given boundary conditions:

1.

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} + 2y &= e^{-t}, \\ \frac{\mathrm{d}y}{\mathrm{d}t} + 3y &= x,\end{aligned}$$

given that $x = 1$ and $y = 0$ when $t = 0$.

2.

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} - y &= \sin t, \\ \frac{\mathrm{d}y}{\mathrm{d}t} + x &= \cos t,\end{aligned}$$

given that $x = 3$ and $y = 4$ when $t = 0$.

3.

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} + 2x - 3y &= 1, \\ \frac{\mathrm{d}y}{\mathrm{d}t} - x + 2y &= e^{-2t},\end{aligned}$$

given that $x = 0$ and $y = 0$ when $t = 0$.

4.

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &= 2y, \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= 8x,\end{aligned}$$

given that $x = 1$ and $y = 0$ when $t = 0$.

5.

$$
\begin{aligned}
10\frac{\mathrm{d}x}{\mathrm{d}t} - 3\frac{\mathrm{d}y}{\mathrm{d}t} + 6x + 5y &= 0, \\
2\frac{\mathrm{d}x}{\mathrm{d}t} - \frac{\mathrm{d}y}{\mathrm{d}t} + 2x + y &= 2e^{-t},
\end{aligned}
$$

given that $x = 2$ and $y = -1$ when $t = 0$.

6.

$$
\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}t} + 4\frac{\mathrm{d}y}{\mathrm{d}t} + 6y &= 0, \\
5\frac{\mathrm{d}x}{\mathrm{d}t} + 2\frac{\mathrm{d}y}{\mathrm{d}t} + 6x &= 0,
\end{aligned}
$$

given that $x = 3$ and $y = 0$ when $t = 0$.

7.

$$
\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}t} &= 2y, \\
\frac{\mathrm{d}y}{\mathrm{d}t} &= 2z, \\
\frac{\mathrm{d}z}{\mathrm{d}t} &= 2x,
\end{aligned}
$$

given that $x = 1$, $y = 0$ and $z = -1$ when $t = 0$.

16.4.3 ANSWERS TO EXERCISES

1.
$$x = (2t+1)e^{-t} \quad \text{and} \quad y = te^{-t}.$$

2.
$$x = (t+4)\sin t + 3\cos t \quad \text{and} \quad y = (t+4)\cos t - 3\sin t.$$

3.
$$x = 2 - e^{-2t}[1 + \sqrt{3}\sinh t\sqrt{3} + \cosh t\sqrt{3}]$$

and
$$y = 1 - e^{-2t}\left[\cosh t\sqrt{3} + \frac{1}{\sqrt{3}}\sinh t\sqrt{3}\right].$$

4.
$$x = \sinh 4t \quad \text{and} \quad y = 2\cosh 4t.$$

5.
$$x = 4\cos t - 2e^{-t} \quad \text{and} \quad y = e^{-t} - 2\cos t.$$

6.
$$x = 2e^{-t} + e^{-2t} \quad \text{amd} \quad y = e^{-t} - e^{-2t}.$$

7.
$$x = e^{-t}\left[\frac{1}{\sqrt{3}}\sin t\sqrt{3} + \cos t\sqrt{3}\right],$$
$$y = \frac{-2}{\sqrt{3}}e^{-t}\sin t\sqrt{3}$$

and
$$z = e^{-t}\left[\frac{1}{\sqrt{3}}\sin t\sqrt{3} - \cos t\sqrt{3}\right].$$

"JUST THE MATHS"

UNIT NUMBER

16.5

LAPLACE TRANSFORMS 5
(The Heaviside step function)

by

A.J.Hobson

UNIT 16.5 - LAPLACE TRANSFORMS 5

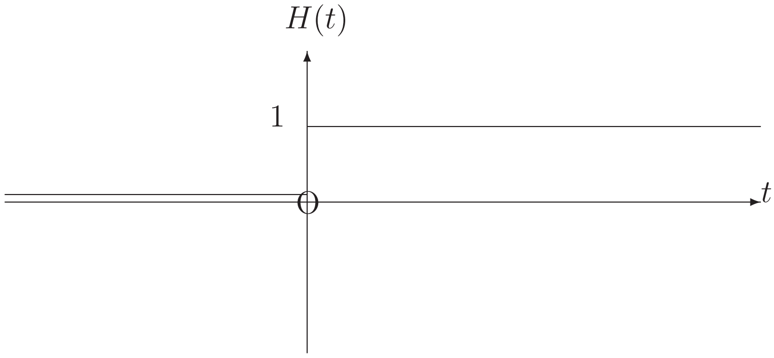
THE HEAVISIDE STEP FUNCTION

16.5.1 THE DEFINITION OF THE HEAVISIDE STEP FUNCTION

The Heaviside Step Function, $H(t)$, is defined by the statements

$$H(t) = \begin{cases} 0 & \text{for } t < 0; \\ 1 & \text{for } t > 0. \end{cases}$$

Note:
$H(t)$ is undefined when $t = 0$.

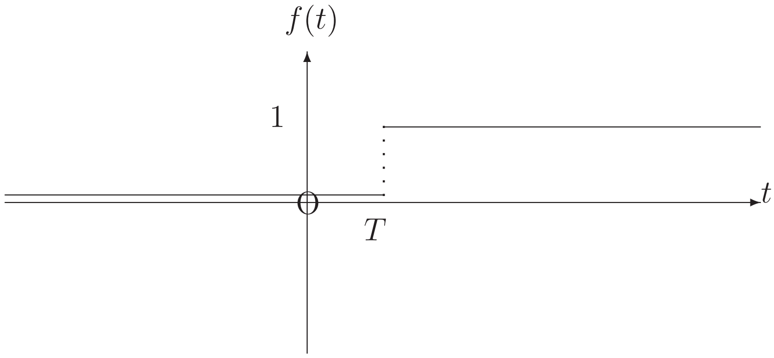


EXAMPLE

Express, in terms of $H(t)$, the function, $f(t)$, given by the statements

$$f(t) = \begin{cases} 0 & \text{for } t < T; \\ 1 & \text{for } t > T. \end{cases}$$

Solution



Clearly, $f(t)$ is the same type of function as $H(t)$, but we have effectively moved the origin to the point $(T, 0)$. Hence,

$$f(t) \equiv H(t - T).$$

Note:
The function $H(t-T)$ is of importance in constructing what are known as **"pulse functions"** (see later).

16.5.2 THE LAPLACE TRANSFORM OF H(t - T)

From the definition of a Laplace Transform,

$$L[H(t-T)] = \int_0^\infty e^{-st} H(t-T) \text{ d}t$$

$$= \int_0^T e^{-st}.0 \text{ d}t + \int_T^\infty e^{-st}.1 \text{ d}t$$

$$= \left[\frac{e^{-st}}{-s}\right]_T^\infty = \frac{e^{-sT}}{s}.$$

Note:
In the special case when $T = 0$, we have

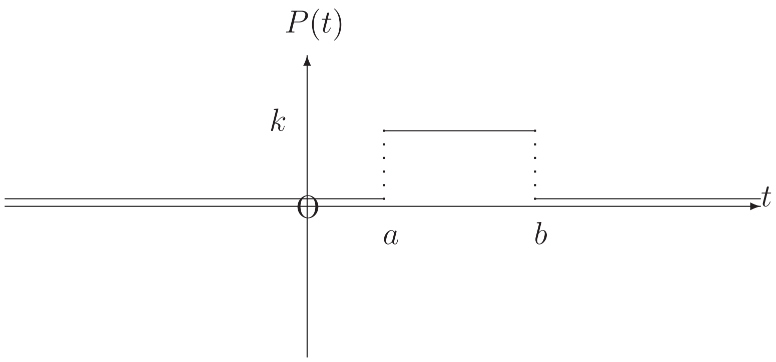
$$L[H(t)] = \frac{1}{s},$$

which can be expected since $H(t)$ and 1 are identical over the range of integration.

16.5.3 PULSE FUNCTIONS

If $a < b$, a **"rectangular pulse"**, $P(t)$, of duration, $b - a$, and magnitude, k, is defined by the statements,

$$P(t) = \begin{cases} k & \text{for } a < t < b; \\ 0 & \text{for } t < a \text{ or } t > b. \end{cases}$$



We can show that, in terms of Heaviside functions, the above pulse may be represented by

$$P(t) \equiv k[H(t-a) - H(t-b)].$$

Proof:

(i) If $t < a$, then $H(t-a) = 0$ and $H(t-b) = 0$. Hence, the above right-hand side $= 0$.

(ii) If $t > b$, then $H(t-a) = 1$ and $H(t-b) = 1$. Hence, the above right-hand side $= 0$.

(iii) If $a < t < b$, then $H(t-a) = 1$ and $H(t-b) = 0$. Hence, the above right-hand side $= k$.

EXAMPLE

Determine the Laplace Transform of a pulse, $P(t)$, of duration, $b - a$, having magnitude, k.

Solution

$$L[P(t)] = k\left[\frac{e^{-sa}}{s} - \frac{e^{-sb}}{s}\right] = k.\frac{e^{-sa} - e^{-sb}}{s}.$$

Notes:

(i) The **"strength"** of the pulse, described above, is defined as the area of the rectangle with base, $b - a$, and height, k. That is,

$$\text{strength} = k(b - a).$$

(ii) In general, the expression,

$$[H(t - a) - H(t - b)]f(t),$$

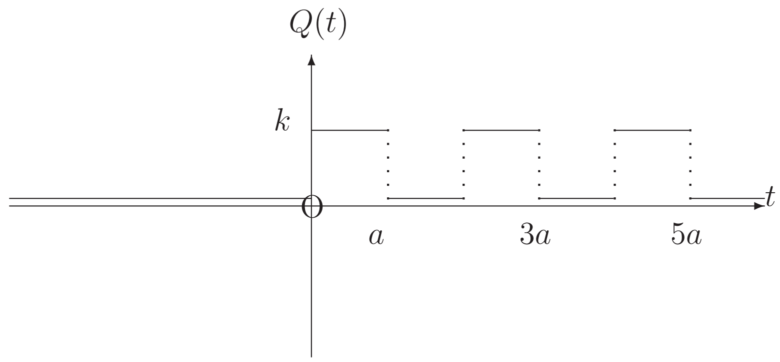
may be considered to **"switch on"** the function, $f(t)$, between $t = a$ and $t = b$ but **"switch off"** the function, $f(t)$, when $t < a$ or $t > b$.

(iii) Similarly, the expression,

$$H(t - a)f(t),$$

may be considered to **"switch on"** the function, $f(t)$, when $t > a$ but **"switch off"** the function, $f(t)$, when $t < a$.

For example, the train of rectangular pulses, $Q(t)$, in the following diagram:



may be represented by the function

$$Q(t) \equiv k\left\{[H(t) - H(t-a)] + [H(t-2a) - H(t-3a)] + [H(t-4a) - H(t-5a)] + ...\right\}.$$

16.5.4 THE SECOND SHIFTING THEOREM

THEOREM

$$L[H(t-T)f(t-T)] = e^{-sT}L[f(t)].$$

Proof:

Left-hand side =

$$\int_0^\infty e^{-st}H(t-T)f(t-T)\ \mathrm{d}t$$

$$= \int_0^T 0\ \mathrm{d}t + \int_T^\infty e^{-st}f(t-T)\ \mathrm{d}t$$

$$= \int_T^\infty e^{-st}f(t-T)\ \mathrm{d}t.$$

Making the substitution $u = t - T$, we obtain

$$\int_0^\infty e^{-s(u+T)}f(u)\ \mathrm{d}u$$

$$= e^{-sT}\int_0^\infty e^{-su}f(u)\ \mathrm{d}u = e^{-sT}L[f(t)].$$

EXAMPLES

1. Express, in terms of Heaviside functions, the function

$$f(t) = \begin{cases} (t-1)^2 & \text{for } t > 1; \\ 0 & \text{for } 0 < t < 1. \end{cases}$$

   and, hence, determine its Laplace Transform.

   Solution

   For values of $t > 0$, we may write

$$f(t) = (t-1)^2H(t-1).$$

   Therefore, using $T = 1$ in the second shifting theorem,

$$L[f(t)] = e^{-s}L[t^2] = e^{-s}.\frac{2}{s^3}.$$

2. Determine the inverse Laplace Transform of the expression

$$\frac{e^{-7s}}{s^2+4s+5}.$$

Solution

First, we find the inverse Laplace Transform of the expression,

$$\frac{1}{s^2+4s+5} \equiv \frac{1}{(s+2)^2+1}.$$

From the first shifting theorem, this will be the function

$$e^{-2t}\sin t, \quad t > 0.$$

From the second shifting theorem, the required function will be

$$H(t-7)e^{-2(t-7)}\sin(t-7), \quad t > 0.$$

16.5.5 EXERCISES

1. (a) For values of $t > 0$, express, in terms of Heaviside functions, the function,

$$f(t) = \begin{cases} e^{-t} & \text{for } 0 < t < 3; \\ 0 & \text{for } t > 3. \end{cases}$$

   (b) Determine the Laplace Transform of the function, $f(t)$, in part (a).

2. For values of $t > 0$, express, in terms of Heaviside functions, the function,

$$f(t) = \begin{cases} f_1(t) & \text{for } 0 < t < a; \\ f_2(t) & \text{for } t > a. \end{cases}$$

3. For values of $t > 0$, express the following functions in terms of Heaviside functions:

   (a)

$$f(t) = \begin{cases} t^2 & \text{for } 0 < t < 2; \\ 4t & \text{for } t > 2. \end{cases}$$

   (b)

$$f(t) = \begin{cases} \sin t & \text{for } 0 < t < \pi; \\ \sin 2t & \text{for } \pi < t < 2\pi; \\ \sin 3t & \text{for } t > 2\pi. \end{cases}$$

4. Use the second shifting theorem to determine the Laplace Transform of the function,

$$f(t) \equiv t^3 H(t-1).$$

Hint:
Write $t^3 \equiv [(t-1)+1]^3$.

5. Determine the inverse Laplace Transforms of the following:

(a)

$$\frac{e^{-2s}}{s^2};$$

(b)

$$\frac{8e^{-3s}}{s^2+4};$$

(c)

$$\frac{se^{-2s}}{s^2+3s+2};$$

(d)

$$\frac{e^{-3s}}{s^2-2s+5}.$$

6. Solve the differential equation

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 4x = H(t-2),$$

given that $x = 0$ and $\frac{\mathrm{d}x}{\mathrm{d}t} = 1$ when $t = 0$.

16.5.6 ANSWERS TO EXERCISES

1. (a)

$$e^{-t}[H(t) - H(t-3)];$$

(b)

$$L[f(t)] = \frac{1-e^{-3(s+1)}}{s+1}.$$

2.

$$f(t) \equiv f_1(t)[H(t) - H(t-a)] + f_2(t)H(t-a)].$$

3. (a)

$$f(t) \equiv t^2[H(t) - H(t-2)] + 4tH(t-2);$$

(b)

$$f(t) \equiv \sin t[H(t) - H(t-\pi)] + \sin 2t[H(t-\pi) - H(t-2\pi)] + \sin 3t[H(t-2\pi).$$

4.

$$L[f(t)] = \left[\frac{6}{s^4} + \frac{6}{s^3} + \frac{3}{s^2} + \frac{1}{s}\right] e^{-s}.$$

5. (a)

$$H(t-2)(t-2);$$

(b)

$$4H(t-3)\sin 2(t-3);$$

(c)

$$H(t-2)[2e^{-2(t-2)} - e^{-(t-2)};$$

(d)

$$\frac{1}{2}H(t-3)e^{(t-3)}\sin 2(t-3).$$

6.

$$x = \frac{1}{2}\sin 2t + \frac{1}{4}H(t-2)[1 - \cos 2(t-2)].$$

“JUST THE MATHS”

UNIT NUMBER

16.6

LAPLACE TRANSFORMS 6
(The Dirac unit impulse function)

by

A.J.Hobson

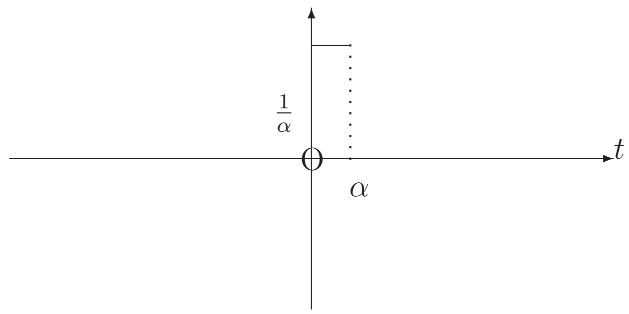
UNIT 16.6 - LAPLACE TRANSFORMS 6
THE DIRAC UNIT IMPULSE FUNCTION

16.6.1 THE DEFINITION OF THE DIRAC UNIT IMPULSE FUNCTION

A pulse of large magnitude, short duration and finite strength is called an **"impulse"**. In particular, a **"unit impulse"** is an impulse of strength 1.

ILLUSTRATION

Consider a pulse, of duration α, between $t = 0$ and $t = \alpha$, having magnitude, $\frac{1}{\alpha}$. The strength of the pulse is then 1.



From Unit 16.5, this pulse is given by

$$\frac{H(t) - H(t-\alpha)}{\alpha}.$$

If we now allow α to tend to zero, we obtain a unit impulse located at $t = 0$. This leads to the following definition:

DEFINITION 2

The **"Dirac unit impulse function"** , $\delta(t)$ is defined to be an impulse of unit strength located at $t = 0$. It is given by

$$\delta(t) = \lim_{\alpha \to 0} \frac{H(t) - H(t-\alpha)}{\alpha}.$$

Notes:

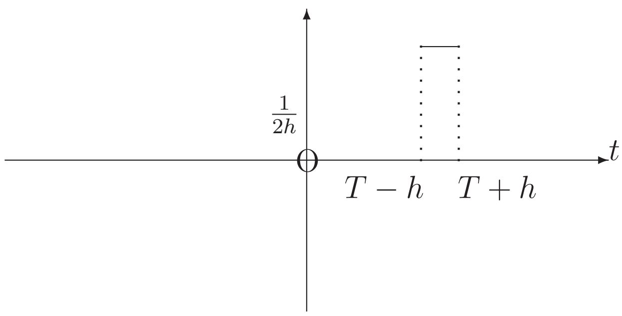
(i) An impulse of unit strength located at $t = T$ is represented by $\delta(t-T)$.

(ii) An alternative definition of the function $\delta(t-T)$ is as follows:

$$\delta(t-T) = \begin{cases} 0 & \text{for } t \neq T; \\ \infty & \text{for } t = T. \end{cases}$$

and

$$\lim_{h\to 0} \int_{T-h}^{T+h} \delta(t-T)\ \mathrm{d}t = 1.$$



THEOREM

$$\int_a^b f(t)\delta(t-T)\ \mathrm{d}t = f(T) \ \text{ if } \ a < T < b.$$

Proof:

Since $\delta(t-T)$ is equal to zero everywhere except at $t = T$, the left-hand side of the above formula reduces to

$$\lim_{h\to 0} \int_{T-h}^{T+h} f(t)\delta(t-T)\ \mathrm{d}t.$$

But, in the <u>small</u> interval from $T-h$ to $T+h$, $f(t)$ is approximately constant and equal to $f(T)$. Hence, the left-hand side may be written

$$f(T)\left[\lim_{h\to 0} \int_{T-h}^{T+h} \delta(T-T)\ \mathrm{d}t\right],$$

which reduces to $f(T)$, using note (ii) in the definition of the Dirac unit impulse function.

16.6.2 THE LAPLACE TRANSFORM OF THE DIRAC UNIT IMPULSE FUNCTION

RESULT

$$L[\delta(t-T)] = e^{-sT};$$

and, in particular,

$$L[\delta(t)] = 1.$$

Proof:

From the definition of a Laplace Transform,

$$L[\delta(t-T) = \int_0^\infty e^{-st}\delta(t-T)\ \mathrm{d}t.$$

But, from the Theorem discussed above, with $f(t) = e^{-st}$, we have

$$L[\delta(t-T)] = e^{-sT}.$$

EXAMPLES

1. Solve the differential equation,

   $$3\frac{\mathrm{d}x}{\mathrm{d}t} + 4x = \delta(t),$$

   given that $x = 0$ when $t = 0$.

   Solution

   Taking the Laplace Transform of the differential equation,

   $$3sX(s) + 4X(s) = 1.$$

   That is,

   $$X(s) = \frac{1}{3s+4} = \frac{1}{3}.\frac{1}{s+\frac{4}{3}}.$$

   Hence,

   $$x(t) = \frac{1}{3}e^{-\frac{4t}{3}}.$$

2. Show that, for any function, $f(t)$,

$$\int_0^\infty f(t)\delta'(t-a)\ \mathrm{d}t = -f'(a).$$

Solution

Using Integration by Parts, the left-hand side of the formula may be written

$$[f(t)\delta(t-a)]_0^\infty - \int_0^\infty f'(t)\delta(t-a)\ \mathrm{d}t.$$

The first term of this reduces to zero, since $\delta(t-a)$ is equal to zero except when $t=a$. The required result follows from the Theorem discussed earlier, with $T=a$.

16.6.3 TRANSFER FUNCTIONS

In scientific applications, the solution of an ordinary differential equation having the form

$$a\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + b\frac{\mathrm{d}x}{\mathrm{d}t} + cx = f(t),$$

is sometimes called the **"response of a system to the function** $f(t)$**"**.

The term **"system"** may, for example, refer to an oscillatory electrical circuit or a mechanical vibration.

It is also customary to refer to $f(t)$ as the **"input"** and $x(t)$ as the **"output"** of a system.

In the work which follows, we shall consider the special case in which $x=0$ and $\frac{\mathrm{d}x}{\mathrm{d}t}=0$ when $t=0$; that is, we shall assume zero initial conditions.

Impulse response function and transfer function

Consider, for the moment, the differential equation having the form,

$$a\frac{\mathrm{d}^2u}{\mathrm{d}t^2} + b\frac{\mathrm{d}u}{\mathrm{d}t} + cu = \delta(t).$$

Here, we refer to the function, $u(t)$, as the **"impulse response function"** of the original system.

The Laplace Transform of its differential equation is given by

$$(as^2+bs+c)U(s) = 1.$$

Hence,

$$U(s) = \frac{1}{as^2 + bs + c},$$

which is called the **"transfer function"** of the original system.

EXAMPLE

Determine the transfer function and impulse response function for the differential equation,

$$3\frac{\mathrm{d}x}{\mathrm{d}t} + 4x = f(t),$$

assuming zero initial conditions.

Solution

To find $U(s)$ and $u(t)$, we have

$$3\frac{\mathrm{d}u}{\mathrm{d}t} + 4u = \delta t,$$

so that

$$(3s + 4)U(s) = 1$$

and, hence, the transfer function is

$$U(s) = \frac{1}{3s + 4} = \frac{1}{3}.\frac{1}{s + \frac{4}{3}}.$$

Taking the inverse Laplace Transform of $U(s)$ gives the impulse response function,

$$u(t) = \frac{1}{3}e^{-\frac{4t}{3}}.$$

System response for any input

Assuming zero initial conditions, the Laplace Transform of the differential equation

$$a\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + bx + cx = f(t)$$

is given by

$$(as^2 + bs + c)X(s) = F(s),$$

which means that

$$X(s) = \frac{F(s)}{as^2 + bs + c} = F(s).U(s).$$

In order to find the response of the system to the function $f(t)$, we need the inverse Laplace Transform of $F(s).U(s)$ which may possibly be found using partial fractions but may, if necessary, be found by using the Convolution Theorem referred to in Unit 16.1

The Convolution Theorem shows, in this case, that

$$L\left[\int_0^t f(T).u(t-T)\ \mathrm{d}T\right] = F(s).U(s);$$

in other words,

$$L^{-1}[F(s).U(s)] = \int_0^t f(T).u(t-T)\ \mathrm{d}T.$$

EXAMPLE

The impulse response of a system is known to be $u(t) = \frac{10e^{-t}}{3}$.

Determine the response, $x(t)$, of the system to an input of $f(t) \equiv \sin 3t$.

Solution

First, we note that

$$U(s) = \frac{10}{3(s+1)} \quad \text{and} \quad F(s) = \frac{3}{s^2+9}.$$

Hence,

$$X(s) = \frac{10}{(s+1)(s^2+9)} = \frac{1}{s+1} + \frac{-s+1}{s^2+9},$$

using partial fractions.

Thus

$$x(t) = e^{-t} - \cos 3t + \frac{1}{3}\sin 3t \quad t > 0.$$

Alternatively, using the Convolution Theorem,

$$x(t) = \int_0^t \sin 3T.\frac{10e^{-(t-T)}}{3}\,\mathrm{d}T;$$

but the integration here can be made simpler if we replace $\sin 3T$ by e^{j3T} and use the imaginary part, only, of the result.

Hence,

$$x(t) = I_m\left(\int_0^t \frac{10}{3}e^{-t}.e^{(1+j3)T}\,\mathrm{d}T\right)$$

$$= I_m\left(\frac{10}{3}\left[e^{-t}\frac{e^{(1+j3)T}}{1+j3}\right]_0^t\right)$$

$$= I_m\left(\frac{10}{3}\left[\frac{e^{-t}.e^{(1+j3)t} - e^{-t}}{1+j3}\right]\right)$$

$$= I_m\left(\frac{10}{3}\left[\frac{[(\cos 3t - e^{-t}) + j\sin 3t](1-j3)}{10}\right]\right)$$

$$= \frac{10}{3}\left[\frac{\sin 3t - 3\cos 3t + 3e^{-t}}{10}\right]$$

$$= e^{-t} - \cos 3t + \frac{1}{3}\sin 3t \quad t > 0,$$

as before.

Note:
Clearly, in this example, the method using partial fractions is simpler.

16.6.4 STEADY-STATE RESPONSE TO A SINGLE FREQUENCY INPUT

In the differential equation,

$$a\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + b\frac{\mathrm{d}x}{\mathrm{d}t} + cx = f(t),$$

suppose that the quadratic denominator of the transfer function, $U(s)$, has negative real roots; that is, it gives rise to an impulse response, $u(t)$, involving negative powers of e and, hence, tending to zero as t tends to infinity.

Suppose also that $f(t)$ takes one of the forms, $\cos\omega t$ or $\sin\omega t$, which may be regarded, respectively, as the real and imaginary parts of the function, $e^{j\omega t}$.

It turns out that the response, $x(t)$, will consist of a **"transient"** part which tends to zero as t tends to infinity, together with a non-transient part forming the **"steady-state response"**.

We illustrate with an example:

EXAMPLE

Consider that

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2}+3\frac{\mathrm{d}x}{\mathrm{d}t}+2x=e^{j7t},$$

where $x=2$ and $\frac{\mathrm{d}x}{\mathrm{d}t}=1$ when $t=0$.

Taking the Laplace Transform of the differential equation,

$$s(sX(s)-2)-1+3(sX(s)-2)+2X(s)=\frac{1}{s-j7}.$$

That is,

$$(s^2+3s+2)X(s)=2s+7+\frac{1}{s-j7},$$

giving

$$X(s)=\frac{2s+7}{s^2+3s+2}+\frac{1}{(s-j7)(s^2+3s+2)}=\frac{2s+7}{(s+2)(s+1)}+\frac{1}{(s-j7)(s+2)(s+1)}.$$

Using the "cover-up" rule for partial fractions, we obtain

$$X(s)=\frac{5}{s+1}-\frac{3}{s+2}+\frac{1}{(-1-j7)(s+1)}+\frac{1}{(2+j7)(s+2)}+\frac{U(j7)}{(s-j7)},$$

where

$$U(s)\equiv\frac{1}{s^2+3s+2}$$

is the transfer function.

Taking inverse Laplace Transforms,

$$x(t) = 5e^{-t} - 3e^{-2t} + \frac{1}{-1-j7}e^{-t} + \frac{1}{2+j7}e^{-2t} + U(j7)e^{j7t}.$$

The first four terms on the right-hand side tend to zero as t tends to infinity, so that the final term represents the steady state response; we need its real part if $f(t) \equiv \cos 7t$ and its imaginary part if $f(t) \equiv \sin 7t$.

Summary

The above example illustrates the result that the steady-state response, $s(t)$, of a system to an input of $e^{j\omega t}$ is given by

$$s(t) = U(j\omega)e^{j\omega t}.$$

16.6.5 EXERCISES

1. Evaluate
$$\int_0^\infty e^{-4t}\delta'(t-2)\ \mathrm{d}t.$$

2. In the following cases, solve the differential equation
$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 4x = f(t),$$
where $x = 0$ and $\frac{\mathrm{d}x}{\mathrm{d}t} = 1$ when $t = 0$:

   (a)
$$f(t) \equiv \delta(t);$$

   (b)
$$f(t) \equiv \delta(t-2).$$

3. Determine the transfer function and impulse response function for the differential equation,
$$2\frac{\mathrm{d}^2x}{\mathrm{d}t^2} - 3\frac{\mathrm{d}x}{\mathrm{d}t} + x = f(t),$$
assuming zero initial conditions.

4. The impulse response function of a system is known to be $u(t) = e^{3t}$. Determine the response, $x(t)$, of the system to an input of $f(t) \equiv 6\cos 3t$.

5. Determine the steady-state response to the system

$$3\frac{\mathrm{d}x}{\mathrm{d}t} + x = f(t)$$

in the cases when

(a)

$$f(t) \equiv e^{j2t};$$

(b)

$$f(t) \equiv 3\cos 2t.$$

16.6.6 ANSWERS TO EXERCISES

1.

$$4e^{-8}.$$

2. (a)

$$x = \sin 2t \quad t > 0;$$

(b)

$$x = \sin t + H(t-2)\sin(t-2) \quad t \neq 2.$$

3.

$$U(s) = \frac{1}{2s^2 - 3s + 1} \quad \text{and} \quad u(t) = [e^t - e^{\frac{1}{2}t}].$$

4.

$$\frac{1}{13}\left[18e^{3t} - 18\cos 2t + 12\sin 2t\right] \quad t > 0.$$

5. (a)

$$\frac{(1-j6)e^{j2t}}{37} \quad t > 0;$$

(b)

$$\frac{1}{37}(\cos 2t + 6\sin 2t) \quad t > 0.$$

"JUST THE MATHS"

UNIT NUMBER

16.7

LAPLACE TRANSFORMS 7
(An appendix)

by

A.J.Hobson

One view of how Laplace Transforms might have arisen

UNIT 16.7 - LAPLACE TRANSFORMS 7 (AN APPENDIX)

ONE VIEW OF HOW LAPLACE TRANSFORMS MIGHT HAVE ARISEN.

(i) Let us consider that our main problem is to solve a second order linear differential equation with constant coefficients, the general form of which is

$$a\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + b\frac{\mathrm{d}x}{\mathrm{d}t} + cx = f(t).$$

(ii) Assuming that the solution of an equivalent first order differential equation,

$$a\frac{\mathrm{d}x}{\mathrm{d}t} + bx = f(t),$$

has already been included in previous knowledge, we examine a typical worked example as follows:

EXAMPLE

Solve the differential equation,

$$\frac{\mathrm{d}x}{\mathrm{d}t} + 3x = e^{2t},$$

given that $x = 0$ when $t = 0$.

Solution

A method called the **"integrating factor method"** uses the coefficient of x to find a function of t which multiplies both sides of the given differential equation to convert it to an **"exact"** differential equation.

The integrating factor in the current example is e^{3t} since the coefficient of x is 3.

We obtain, therefore,

$$e^{3t}\left[\frac{\mathrm{d}x}{\mathrm{d}t} + 3x\right] = e^{5t}.$$

which is equivalent to

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[xe^{3t}\right] = e^{5t}.$$

On integrating both sides with respect to t,

$$xe^{3t} = \frac{e^{5t}}{5} + C$$

or

$$x = \frac{e^{2t}}{5} + Ce^{-3t}.$$

Putting $x = 0$ and $t = 0$, we have

$$0 = \frac{1}{5} + C.$$

Hence, $C = -\frac{1}{5}$ and the complete solution becomes

$$x = \frac{e^{2t}}{5} - \frac{e^{-3t}}{5}.$$

(iii) As a lead up to what follows, we shall now examine a different way of setting out the above working in which we do not leave the substitution of the boundary condition until the very end.

We multiply both sides of the differential equation by e^{3t} as before, but we then integrate both sides of the new "exact" equation from 0 to t.

$$\int_0^t \frac{\mathrm{d}}{\mathrm{d}t}\left[xe^{3t}\right] \, \mathrm{d}t = \int_0^t e^{5t} \, \mathrm{d}t.$$

That is,

$$\left[xe^{3t}\right]_0^t = \left[\frac{e^{5t}}{5}\right]_0^t,$$

giving

$$xe^{3t} - 0 = \frac{e^{5t}}{5} - \frac{1}{5}.$$

since $x = 0$ when $t = 0$.

In other words,

$$x = \frac{e^{2t}}{5} - \frac{e^{-3t}}{5},$$

as before.

(iv) Now let us consider whether an example of a second order linear differential equation could be solved by a similar method.

EXAMPLE

Solve the differential equation,

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} - 10\frac{\mathrm{d}x}{\mathrm{d}t} + 21x = e^{9t},$$

given that $x = 0$ and $\frac{\mathrm{d}x}{\mathrm{d}t} = 0$ when $t = 0$.

Solution

Supposing that there might be an integrating factor for this equation, we shall take it to be e^{st} where s, at present, is unknown, but assumed to be positive.

Multiplying throughout by e^{st} and integrating from 0 to t, as in the previous example,

$$\int_0^t e^{st}\left[\frac{\mathrm{d}^2x}{\mathrm{d}t^2} - 10\frac{\mathrm{d}x}{\mathrm{d}t} + 21x\right]\,\mathrm{d}t = \int_0^t e^{(s+9)t}\,\mathrm{d}t = \left[\frac{e^{(s+9)t}}{s+9}\right]_0^t.$$

Now, using integration by parts, with the boundary condition,

$$\int_0^t e^{st}\frac{\mathrm{d}x}{\mathrm{d}t}\,\mathrm{d}t = e^{st}x - s\int_0^t e^{st}x\,\mathrm{d}t$$

and

$$\int_0^t e^{st}\frac{\mathrm{d}^2x}{\mathrm{d}t^2}\,\mathrm{d}t = e^{st}\frac{\mathrm{d}x}{\mathrm{d}t} - s\int_0^t e^{st}\frac{\mathrm{d}x}{\mathrm{d}t}\,\mathrm{d}t = e^{st}\frac{\mathrm{d}x}{\mathrm{d}t} - se^{st}x + s^2\int_0^t e^{st}x\,\mathrm{d}t.$$

On substituting these results into the differential equation, we may collect together (on the left hand side) terms which involve $\int_0^t e^{st}x\,\mathrm{d}t$ and e^{st} as follows:

$$(s^2 + 10s + 21)\int_0^t e^{st}x\,\mathrm{d}t + e^{st}\left[\frac{\mathrm{d}x}{\mathrm{d}t} - (s+10)x\right] = \frac{e^{(s+9)t}}{s+9} - \frac{1}{s+9}.$$

(v) OBSERVATIONS

(a) If we had used e^{-st} instead of e^{st}, the quadratic expression in s, above, would have had the same coefficients as the original differential equation; that is, $(s^2 - 10s + 21)$.

(b) Using e^{-st} with $s > 0$, if we had integrated from 0 to ∞ instead of 0 to t, the second term on the left hand side above would have been absent, since $e^{-\infty} = 0$.

(vi) Having made our observations, we start again, multiplying both sides of the differential equation by e^{-st} and integrating from 0 to ∞ to obtain

$$(s^2 - 10s + 21)\int_0^\infty e^{-st}x \text{ d}t = \left[\frac{e^{(-s+9)t}}{-s+9}\right]_0^\infty = \frac{-1}{-s+9} = \frac{1}{s-9}.$$

Of course, this works only if $s > 9$, but we can easily assume that it is so. Hence,

$$\int_0^\infty e^{-st}x \text{ d}t = \frac{1}{(s-9)(s^2-10s+21)} = \frac{1}{(s-9)(s-3)(s-7)}.$$

Applying the principles of partial fractions, we obtain

$$\int_0^\infty e^{-st}x \text{ d}t = \frac{1}{12}.\frac{1}{s-9} + \frac{1}{24}.\frac{1}{s-3} - \frac{1}{8}.\frac{1}{s-7}.$$

(vii) But, finally, it can be shown by an independent method of solution that

$$x = \frac{e^{9t}}{12} + \frac{e^{3t}}{24} - \frac{e^{7t}}{8}.$$

and we may conclude that the solution of the differential equation is closely linked to the integral

$$\int_0^\infty e^{-st}x \text{ d}t,$$

which is called the **"Laplace Transform"** of $x(t)$.