Introduction to Quantitative Trade Models

International Trade (PhD), Fall 2025

Ahmad Lashkaripour

Indiana University

Background

- The class of trade models covered in this class (*e.g.*, Armington, Krugman, Eaton-Kortum, Melitz-Pareto) deliver a common macro-level representation for general equilibrium.

- These models have two appealing features:
    1. They predict trade values consistent with a gravity equation:

    $$\text{Trade Value}_{in} \propto \frac{\text{GDP}_i \times \text{GDP}_n}{\text{Distance}_{in}^{\beta}} \qquad (\text{origin } i, \text{ destination } n)$$

    which amounts to good in-sample predictive power *w.r.t.* trade flows.

    2. They can be used to perform counterfactual analyses based on easy-to-obtain sufficient statistics: (1) trade shares, (2) national accounts data, and (3) trade elasticities.

Road Map for Today's Lecture

- *First*, we present the common representation of general equilibrium implied by quantitative trade models (*e.g.*, Armington, Krugman, Eaton-Kortum, Melitz-Pareto).

- *Second*, we overview the *ex-post* and *ex-ante* applications of these models, highlighting their merits relative to alternative research designs (*e.g.*, diff-in-diff, shift-share).

- *Third*, we discuss the structural estimation of these models and the exact hat-algebra technique for obtaining counterfactual (or out-of-sample) predictions.

Environment

- The global economy consist of $N > 1$ countries.

- We use $i, j, n \in \{1, .., N\}$ to index countries

- Labor is the only factor of production

- Country i is endowed with L_i units of labor

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[1] See Adao, Costinot, Donaldson (2017, AER) and Costinot & Rodriguez-Clare (2018, JEP).

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- **Note:** The class of trade models we study can be alternatively cast as a fictitious endowment economy in which trade values reflect the international demand for each country's labor services.[1]

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Exogenous Parameters or Variables

- L_i is country i's labor endowment

- χ_i encompasses information on country i's technological endowment

- τ_{in} is the iceberg trade cost associated with origin i's sales to destination n

- ϵ is the elasticity of trade values w.r.t. trade costs (i.e., the trade elasticity)

- D_i is country i's trade deficit vis-à-vis the rest of the world ($\sum_i D_i = 0$).

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Note: only L_i and D_i are directly observable, the remaining parameters must be estimated.

Endogenous Equilibrium Outcomes

Main independent outcome

- the vector of national-level wages $\{w_1, ..., w_N\}$

Outcomes determined by wages & exogenous parameters

- λ_{in} ~ the share of country n's expenditure on goods originating from country i

- E_n ~ country n's total expenditure (GDP + deficit)

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Note that λ_{in} and E_n are readily observable, whereas w_i is difficult to measure as it represents a national-level index of factor prices.

The General Equilibrium

Given parameters $\{\epsilon, \chi_i, L_i, D_i, \tau_{in}\}_{i,n}$, equilibrium wages, $\{w_i\}_i$, satisfy the labor market clearing condition in each country:

$$\sum_{n=1}^{N} \underbrace{\lambda_{in}(w_1,...,w_N) E_n(w_n)}_{\text{country } i\text{'s sales to country } n} = w_i L_i \quad , \forall i$$

with bilateral expenditure shares (λ_{in}) and national expenditure (E_n) given by

$$\begin{cases} \lambda_{in}(w_1,...,w_N) = \dfrac{\chi_i(\tau_{in}w_i)^{-\epsilon}}{\sum_{j=1}^{N}\chi_j(\tau_{jn}w_j)^{-\epsilon}} & \forall i,j \\[2ex] E_n(w_n) = w_n L_n + D_n & \forall n \end{cases}$$
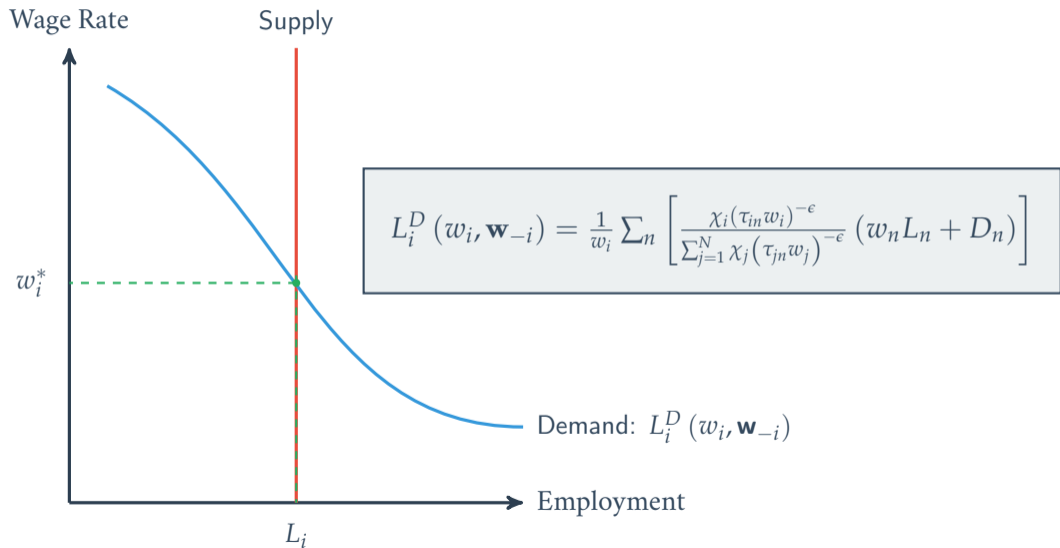
The General Equilibrium

- Given $\left\{ \epsilon, L_i, D_i, \chi_i, \tau_{ij} \right\}_{i,j}$, the vector of wages $\{w_1, ..., w_N\}$ can be computed by solving a non-linear system of *N-equations* and *N-unknowns*[2]

$$\underbrace{\frac{1}{w_i} \sum_{n=1}^{N} \left[\frac{\chi_i \left(\tau_{in} w_i \right)^{-\epsilon}}{\sum_{j=1}^{N} \chi_j \left(\tau_{jn} w_j \right)^{-\epsilon}} \left(w_n L_n + D_n \right) \right]}_{\text{demand for country } i\text{'s labor}} = \overbrace{L_i}^{\text{labor supply}}$$

- Workhorse trade models can be cast as a fictitious endowment economy in which countries directly exchange labor services *subject to* constant elasticity demand functions.

- The main equilibrium outcome is a vector of wages that equalizes the supply and demand for each country's labor.

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[2]Link to Matlab routine that solves the above system

Equilibrium in Country i given Foreign Wages ($\mathbf{w}_{-i}$)

$$L_i^D\left(w_i, \mathbf{w}_{-i}\right) = \frac{1}{w_i} \sum_n \left[\frac{\chi_i(\tau_{in}w_i)^{-\epsilon}}{\sum_{j=1}^N \chi_j(\tau_{jn}w_j)^{-\epsilon}} \left(w_n L_n + D_n\right) \right]$$

Demand: $L_i^D\left(w_i, \mathbf{w}_{-i}\right)$

The General Equilibrium

- When mapping trade models to data is useful to specify equilibrium in terms of national income or GDP ($Y_i = w_i L_i$) rather than wages.

- Given $\left\{ \epsilon, L_i, D_i, \tilde{\chi}_i, \tau_{ij} \right\}_{i,j}$, equilibrium can be alternatively defined as a vector $\{Y_1, ..., Y_N\}$ that solve the following system of equations

$$\sum_{n=1}^{N} \left[\frac{\tilde{\chi}_i \left(\tau_{in} Y_i \right)^{-\epsilon}}{\sum_{j=1}^{N} \tilde{\chi}_j \left(\tau_{jn} Y_j \right)^{-\epsilon}} \left(Y_n + D_n \right) \right] = Y_i, \quad \text{where} \quad \tilde{\chi}_i \equiv \chi_i L_i^{\epsilon}$$

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- The above formulation is also useful for deriving the gravity equation.

The Gravity Equation

– Let $X_{in} = \lambda_{in} \times E_n$ denotes trade flows from origin i to destination n

$$X_{in} = \frac{\tilde{\chi}_i \left(\tau_{in} Y_i \right)^{-\epsilon}}{\sum_{j=1}^{N} \tilde{\chi}_j \left(\tau_{jn} Y_j \right)^{-\epsilon}} \, E_n$$

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The Gravity Equation

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$$X_{in} = \tau_{in}^{-\epsilon} \; \underbrace{\tilde{\chi}_i \left(Y_i \right)^{-\epsilon}}_{\Phi_i} \; \underbrace{\frac{E_n}{\sum_{j=1}^{N} \tilde{\chi}_j \left(\tau_{jn} Y_j \right)^{-\epsilon}}}_{\Omega_n}$$

- $\tau_{in}^{-\epsilon}$ represents trade frictions relating to taste differences, transport costs, or policy.

- Φ_i is the *exporter fixed effect*, summarizing all relevant information on origin i

- Ω_n is the *importer fixed effect,* summarizing all relevant information on destination n

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The Gravity Equation

- The Labor Market Clearing condition specifies Φ_i in terms of Y_i

$$\sum_{n=1}^{N} X_{in} = \Phi_i \sum_{n=1}^{N} \left[\tau_{in}^{-\epsilon} \Omega_n \right] = Y_i \quad \Longrightarrow \quad \Phi_i = \frac{Y_i}{\sum_n \Omega_n \tau_{in}^{-\epsilon}} \quad (*)$$

- The national-level budget constraint specifies Ω_i in terms of E_i

$$\sum_{n=1}^{N} X_{ni} = \sum_{n=1}^{N} \left[\Phi_n \tau_{ni}^{-\epsilon} \right] \Omega_i = E_i \quad \Longrightarrow \quad \Omega_i = \frac{E_i}{\sum_n \Phi_n \tau_{ni}^{-\epsilon}} \quad (**)$$

- Combining equation $(*)$ and $(**)$ and noting that $\tau_{in}^{-\epsilon} \sim \text{Dist}_{in}^{-\beta}$, yields

$$X_{in} = \frac{Y_i}{\sum_n \Omega_n \text{Dist}_{in}^{-\beta}} \times \frac{E_n}{\sum_n \Phi_n \text{Dist}_{ni}^{-\beta}} \times \text{Dist}_{in}^{-\beta}$$

An Implicit Property of Quantitative Trade Models

Proposition. If trade trade costs are symmetric and there are no *aggregate* trade imbalances, then trade values are bilaterally balanced

$$\begin{cases} \tau_{ji} = \tau_{ij} & \forall i, j \\ D_i = 0 & \forall i \end{cases} \implies X_{ij} = X_{ji} \ (\forall i, j)$$

- The above proposition can be proven by appealing to Equations $(*)$ and $(**)$, and showing that $\Phi_i = \Omega_i$ if $\tau_{ji} = \tau_{ij}$ and $D_i = 0$.

- **Implication:** bilateral trade imbalances may be a mere reflection of aggregate trade imbalances rather than asymmetric trade barriers.

Applications of Quantitative Trade Models

- Quantitative trade models can be used to examine the *ex-ante* or *ex-post* impacts of shocks to the global economy.

Example of ex-ante application

- What is the impact of a eliminating aggregate trade imbalances?
- The shock we seek to examine ($D_i \to 0$) has not materialized yet, so event-based research designs like *diff-in-diff* or *shift-share* are not applicable.

Example of ex-post application

- What was the impact of NAFTA on the US economy?
- The NAFTA shock ($\Delta \tau^{\mathrm{NAFTA}} < 0$) has already materialized, but non-structural research designs (if applicable) may fail to identify the GE effects of NAFTA.

Two Approaches to Performing Counterfactual Analyses

- The noted applications require that we simulate the counterfactual equilibrium that emerges after say the NAFTA shock. This task can be accomplished in two ways.

First Approach

- Estimate the full parameters of the model

- shock the parameters and re-solve the model to obtain counterfactual outcomes

Second Approach

- Apply the exact the hat-algebra technique

- Under this approach we no longer need to estimate τ_{ni} or $\tilde{\chi}_i$, since the information on these parameters if fully embedded in expenditure shares and income levels.

Class Assignment

- Quantitative trade models predict trade flows are given by

$$X_{in} = \frac{\tilde{\chi}_i \left(\tau_{in} Y_i\right)^{-\epsilon}}{\sum_{j=1}^{N} \tilde{\chi}_j \left(\tau_{jn} Y_j\right)^{-\epsilon}} E_n$$

and satisfy the adding up constraint $\sum_n X_{in} = Y_i$ for all i.

- X_{ni}, Y_i, and E_i are observable in the data.

- **How would you estimate $\tilde{\chi}_i$, τ_{in}, and ϵ?**

Class Assignment

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- **How would you estimate $\tilde{\chi}_i$, τ_{in}, and ϵ?**

- I will create an "Announcement" on CANVAS. Submit your answer as a comment underneath the announcement before Tuesday, next week.

Estimation of Quantitative Trade Models

Estimation Setup

- Data points: $\mathbb{D} = \left\{ X_{ni}^{data}, Y_i^{data}, E_i^{data} \right\}_{i,n}$

- Unobserved parameters: $\Theta = \left\{ \tau_{in}, \tilde{\chi}_i, \epsilon \right\}_{i,n}$

- Model's prediction *w.r.t.* trade flows, given $\left\{ Y_i^{data} \right\}_i$ and $\left\{ E_i^{data} \right\}_i$

$$X_{in}\left(\Theta; \mathbb{D}\right) = \frac{\tilde{\chi}_i \left(\tau_{in} Y_i^{data}\right)^{-\epsilon}}{\sum_{j=1}^{N} \tilde{\chi}_j \left(\tau_{jn} Y_j^{data}\right)^{-\epsilon}} E_n^{data}$$

Note: ϵ cannot be separately identified from τ_{in} with information on $\mathbb{D}$

- Parameter combinations $\left\{ \tilde{\chi}_i, \tau_{in}, \epsilon \right\}_{i,n}$ and $\left\{ \tilde{\chi}_i, \tau'_{in}, \epsilon' \right\}_{i,n}$ are observationally equivalent in terms of their prediction vis-à-vis $\mathbb{D}$ *iff* $\tau_{in}^{-\epsilon} = (\tau'_{in})^{-\epsilon'}$.

Generic Estimation Strategy

- We can normalize ϵ and estimate the remaining elements of Θ by minimizing the distance between the model's predictions and data *subject to* equilibrium constraints:

$$\min_{\Theta} \sum_{n,i} \left(\log X_{in}(\Theta; \mathbb{D}) - \log X_{in}^{data} \right)^2 \qquad s.t. \qquad \sum_n X_{in}(\Theta; \mathbb{D}) = Y_i^{data} \quad (\forall i)$$

- The above problem is exactly identified, *i.e.,* there exists a Θ^* such that

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- We can use Θ^* to perform counterfactuals (*e.g.,* eliminating trade imbalances), *but* this task can be performed more efficiently with *exact hat algebra*.

Estimating the Determinants of Trade Costs

- We can use a similar strategy to estimate the determinants of τ_{in}.

- Suppose we have data on bilateral distance, FTAs, common language, common border, and conflict for many country pairs.

- We can parameterize bilateral trade costs as

$$\tau_{in} = \bar{\tau} \left(\text{Dist}_{in}\right)^{\beta_d} \cdot \beta_f^{\text{FTA}_{in}} \cdot \beta_l^{\text{Lang}_{in}} \cdot \beta_b^{\text{Border}_{in}} \cdot \beta_c^{\text{Conflict}_{in}}$$

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Interpretation of Parameters

- $\beta_f = 0.75$ implies that a typical FTA reduces trade costs by 25%

- $\beta_c = 1.5$ implies that conflict increases trade costs by 50%; *etc*.

Estimation

- Reduced set of parameters: $\tilde{\Theta} = \left\{ \tilde{\chi}_i, \beta_d, \beta_f, \beta_l, \beta_b, \beta_c, \epsilon \right\}$

- We can normalize ϵ and estimate the remaining elements of $\tilde{\Theta}$ as

$$\min_{\tilde{\Theta}} \sum_{n,i} \left(\log X_{in} \left(\tilde{\Theta}; \mathbb{D} \right) - \log X_{in}^{data} \right)^2 \qquad s.t. \qquad \sum_n X_{in} \left(\tilde{\Theta}; \mathbb{D} \right) = Y_i^{data} \qquad (\forall i)$$

- The above estimation is akin to a standard gravity estimation—though, as we'll note later in the semester, there are easier ways to perform gravity estimation (*e.g.*, PPML)

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- The estimation of β's unveils policy-relevant shocks for counterfactual analysis—*e.g.*,

$$
\text{aboloshing FTAs} \qquad \sim \qquad \Delta \ln \tau'_{in} \approx \begin{cases} \beta_f - 1 & \text{if } \text{FTA}_{in} = 1 \\ 0 & \text{if } \text{FTA}_{in} = 0 \end{cases}
$$

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- The estimation of β's unveils policy-relevant shocks for counterfactual analysis—*e.g.*,

$$\text{global conflict} \qquad \sim \qquad \Delta \ln \tau'_{in} \approx \begin{cases} 0 & \text{if } \text{Conflict}_{in} = 1 \\ \beta_c - 1 & \text{if } \text{Conflict}_{in} = 0 \end{cases}$$

The Exact Hat-Algebra Approach

Definition of Equilibrium

- For any set of exogenous parameters and variables $\{\tau_{in}, \tilde{\chi}_i, D_i, \epsilon\}$, equilibrium is a vector of national GDP levels, $\mathbf{Y} = \{Y_1, ..., Y_N\}$, that satisfy

$$Y_i = \sum_{n=1}^{N} \left[\lambda_{in}(\mathbf{Y}) \times \overbrace{(Y_n + D_n)}^{E_n} \right], \qquad (\forall i)$$

where the expenditure share $\lambda_{in}(\mathbf{Y})$ is given by

$$\lambda_{in}(\mathbf{Y}) = \frac{\tilde{\chi}_i (\tau_{in} Y_i)^{-\epsilon}}{\sum_{j=1}^{N} \tilde{\chi}_j (\tau_{jn} Y_j)^{-\epsilon}}, \qquad (\forall i, n)$$

Hat-Algebra Notation

For a generic variable (x)

- x ~ baseline value under the status quo
- x' ~ counterfactual value after some external shock
- $\widehat{x} \equiv \frac{x'}{x}$

Example: suppose countries i and n sign an FTA that lowers their bilateral trade cost by 25% and increases their bilateral trade value by 15%:

$$\hat{\tau}_{in} = \hat{\tau}_{ni} = 0.75; \qquad\qquad \widehat{X}_{in} = \widehat{X}_{ni} = 1.15$$

Counterfactual Expenditure Shares

- Consider an external shock to trade costs: $\left\{ \hat{\tau}_{in} \right\}_{i,n}$

- Considering that exogenous parameters ($\tilde{\chi}_i$ and ϵ) are unaffected by the shock, counterfactual expenditure shares are

$$\lambda'_{in} = \frac{\tilde{\chi}_i \left(\tau'_{in} Y'_i \right)^{-\epsilon}}{\sum_{j=1}^{N} \tilde{\chi}_j \left(\tau'_{jn} Y'_j \right)^{-\epsilon}}$$

- Noting that $\tau'_{in} = \hat{\tau}_{in} \tau_{in}$ and $Y'_i = \widehat{Y}_i Y_i$ we can rewrite this equation as

$$\lambda'_{in} = \frac{\tilde{\chi}_i \left(\hat{\tau}_{in} \tau_{in} \widehat{Y}_i Y_i \right)^{-\epsilon}}{\sum_{j=1}^{N} \tilde{\chi}_j \left(\hat{\tau}_{jn} \tau_{jn} \widehat{Y}_j Y_j \right)^{-\epsilon}} = \frac{\lambda_{in} \left(\hat{\tau}_{in} \widehat{Y}_i \right)^{-\epsilon}}{\sum_{j=1}^{N} \lambda_{jn} \left(\hat{\tau}_{jn} \widehat{Y}_j \right)^{-\epsilon}}$$

Counterfactual Equilibrium

- Labor-market clearing condition in the counterfactual equilibrium:

$$Y_i' = \sum_{n=1}^{N} \left[\lambda_{in}' \times \left(Y_n' + D_n \right) \right]$$

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- The above system determines $\left\{ \widehat{Y}_1, ..., \widehat{Y}_N \right\}$ with information on observables $\mathbb{D} = \left\{ Y_i, D_i, \lambda_{in} \right\}_{i,n}$ and the trade elasticity, ϵ

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- Given $\widehat{Y}_i$, we can calculate the change in trade values in response to $\left\{ \hat{\tau}_{in} \right\}_{i,n}$ as

$$\widehat{X}_{in} = \widehat{\lambda}_{in} \times \underbrace{\frac{Y_n \widehat{Y}_n + D_n}{Y_n + D_n}}_{\widehat{E}_n}, \qquad \text{where} \qquad \widehat{\lambda}_{in} = \frac{\left(\hat{\tau}_{in} \widehat{Y}_i \right)^{-\epsilon}}{\sum_{j=1}^{N} \lambda_{jn} \left(\hat{\tau}_{jn} \widehat{Y}_j \right)^{-\epsilon}}$$

Example: *the US and the Rest of the World*

- *Two countries*: US ($i = 1$) and ROW ($i = 2$)

$$\lambda = \left[\begin{array}{cc} 0.88 & 0.02 \\ 0.12 & 0.98 \end{array} \right] ; \qquad \mathbf{Y} = \left[\begin{array}{c} 1 \\ 4 \end{array} \right] ; \qquad \mathbf{D} = \left[\begin{array}{c} 0.04 \\ -0.04 \end{array} \right]$$

- Suppose international trade costs fall by 20%:

$$\hat{\boldsymbol{\tau}} = \left[\begin{array}{cc} 1 & 0.80 \\ 0.80 & 1 \end{array} \right]$$

Example: *the US and the Rest of the World*

- System of equations specifying labor-market clearing conditions:

$$Y_1 \widehat{Y_1} = \frac{\lambda_{11} \left(\hat{\tau}_{11} \widehat{Y_1} \right)^{-\epsilon} \times \left(Y_1 \widehat{Y_1} + D_1 \right)}{\lambda_{11} \left(\hat{\tau}_{11} \widehat{Y_1} \right)^{-\epsilon} + \lambda_{21} \left(\hat{\tau}_{21} \widehat{Y_2} \right)^{-\epsilon}} + \frac{\lambda_{12} \left(\hat{\tau}_{12} \widehat{Y_1} \right)^{-\epsilon} \times \left(Y_2 \widehat{Y_2} + D_2 \right)}{\lambda_{12} \left(\hat{\tau}_{12} \widehat{Y_1} \right)^{-\epsilon} + \lambda_{22} \left(\hat{\tau}_{22} \widehat{Y_2} \right)^{-\epsilon}}$$

$$Y_2 \widehat{Y_2} = \frac{\lambda_{21} \left(\hat{\tau}_{21} \widehat{Y_2} \right)^{-\epsilon} \times \left(Y_1 \widehat{Y_1} + D_1 \right)}{\lambda_{11} \left(\hat{\tau}_{11} \widehat{Y_1} \right)^{-\epsilon} + \lambda_{21} \left(\hat{\tau}_{21} \widehat{Y_2} \right)^{-\epsilon}} + \frac{\lambda_{22} \left(\hat{\tau}_{22} \widehat{Y_2} \right)^{-\epsilon} \times \left(Y_2 \widehat{Y_2} + D_2 \right)}{\lambda_{12} \left(\hat{\tau}_{12} \widehat{Y_1} \right)^{-\epsilon} + \lambda_{22} \left(\hat{\tau}_{22} \widehat{Y_2} \right)^{-\epsilon}}$$

- Assuming $\epsilon = 5$, solving the system implies[3]

$$\widehat{\mathbf{Y}} = \begin{bmatrix} 1.025 \\ 1.062 \end{bmatrix} \quad \Longrightarrow \quad \widehat{\mathbf{X}} = \begin{bmatrix} 0.86 & 3.66 \\ 2.22 & 1.01 \end{bmatrix}$$

[3] See Canvas for the Matlab code that generates these numbers.

Example: *the US and the Rest of the World*

- System of equations specifying labor-market clearing conditions:

$$\widehat{Y}_1 = \frac{0.88\left(\widehat{Y}_1\right)^{-\epsilon} \times \left(\widehat{Y}_1 + 0.04\right)}{0.88\left(\widehat{Y}_1\right)^{-\epsilon} + 0.12\left(0.80\widehat{Y}_2\right)^{-\epsilon}} + \frac{0.02\left(0.80\widehat{Y}_1\right)^{-\epsilon} \times \left(4\widehat{Y}_2 - 0.04\right)}{0.02\left(0.80\widehat{Y}_1\right)^{-\epsilon} + 0.98\left(\widehat{Y}_2\right)^{-\epsilon}}$$

$$4\widehat{Y}_2 = \frac{0.12\left(0.80\widehat{Y}_2\right)^{-\epsilon} \times \left(\widehat{Y}_1 + 0.04\right)}{0.88\left(\widehat{Y}_1\right)^{-\epsilon} + 0.12\left(0.80\widehat{Y}_2\right)^{-\epsilon}} + \frac{0.98\left(\widehat{Y}_2\right)^{-\epsilon} \times \left(4\widehat{Y}_2 - 0.04\right)}{0.02\left(0.80\widehat{Y}_1\right)^{-\epsilon} + 0.98\left(\widehat{Y}_2\right)^{-\epsilon}}$$

- Assuming $\epsilon = 5$, solving the system implies[3]

$$\widehat{\mathbf{Y}} = \begin{bmatrix} 1.025 \\ 1.062 \end{bmatrix} \implies \widehat{\mathbf{X}} = \begin{bmatrix} 0.86 & 3.66 \\ 2.22 & 1.01 \end{bmatrix}$$

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Taking Stock

- The exact hat-algebra approach enables us to perform counterfactuals without estimating the trade cost or technology parameters (τ and $\tilde{\chi}$).

- Performing counterfactuals requires two sets of sufficient statistics:
    1. Observable statistics: λ_{in}, Y_i, and E_i.
    2. Trade Elasticity: ϵ

Taking Stock

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- In the class of models we study, the *welfare change* in response to an external shock, $\{\hat{\tau}_{in}, \widehat{\chi}_i\}_{i,n}$, can also be computed as

$$\hat{W}_i = \hat{\tau}_{ii}^{-1} \times \widehat{\chi}_i^{\frac{1}{\epsilon}} \times \widehat{\lambda}_{ii}^{-\frac{1}{\epsilon}}$$

  - In the earlier example: $\hat{\tau}_{ii} = \hat{\chi}_i = 1 \longrightarrow \widehat{W}_i = \widehat{\lambda}_{ii}^{-\frac{1}{\epsilon}}$.

Key Limitation of the Exact Hat Algebra Method

The exact hat-algebra method suffers from an overfitting problem when the observed share matrix λ is sparse and contains many zeros:

- the model-predicted λ is often a choice probability derived from a discrete-choice framework with a continuum of goods or agents (*observed share = probability*)

- a sparse λ is consistent with a granular setting (*observed share $\neq$ probability*) $\longrightarrow$ calibrating probabilities to observed shares leads to overfitting (Dingel and Tintelnot, 2025).

- a manifestation of this issue is that if $\lambda_{ni} = 0$, then the method always predicts $\lambda'_{ni} = 0$

- **However:** zeros are less prevalent in aggregate trade flows, and the limitation is more applicable to calibrations involving high-dimensional product-level trade data.

Generalization of the Exact Hat Algebra Method

- A piece-wise variant of the method can be applied to setups with variable elasticity, $\epsilon = \varepsilon(\lambda, Y)$

- With a variable ϵ, the hat-algebra method can be represented locally for small shock $\hat{\tau} \approx 1$ as

$$(\hat{\lambda}, \hat{Y}) = \mathcal{F}(\lambda, Y, \epsilon; \hat{\tau})$$

- We can apply a piece-wise algorithm to obtain counterfactual outcomes under a large shock $\hat{\tau}$:
    - decompose the shock $\hat{\tau} = \prod_{t=1}^{T} \hat{\tau}_t$ into T piecemeal shocks, where $\hat{\tau}_t = \hat{\tau}^{\frac{1}{T}} \approx 1$
    - compute $(\hat{\lambda}_1, \hat{Y}_1) = \mathcal{F}(\lambda_0, Y_0, \epsilon_0; \hat{\tau}_1)$ given baseline data (λ_0, Y_0) and elasticity $\epsilon_0 = \varepsilon(\lambda_0, Y_0)$
    - update values: $\lambda_1' = \hat{\lambda}_1 \lambda_0$, $Y_1' = \hat{Y}_1 Y_0$, and $\epsilon_1' = \varepsilon(\lambda_1', Y_1')$
    - compute $(\lambda_2', Y_2') = \mathcal{F}(\lambda_1', Y_1', \epsilon_1'; \hat{\tau}_2)$ given updated inputs
    - repeat to obtain $(\lambda_T', Y_T') = \mathcal{F}(\lambda_{T-1}', Y_{T-1}', \epsilon_{T-1}'; \hat{\tau}_T)$

Testing the External Validity of Quantitative Trade Models

- Trade models perform well *in-sample*, but how credible are the *out-of-sample* predictions?
  - Adao et al. (2025, QJE) propose a goodness-of-fit measure for testing the credibility of counterfactual predictions

- **Basic Idea:** Apply hat-algebra to a past policy shock $\hat{z} \sim \hat{\tau}$ that is independent of other concurrent (non-policy) shocks and compute model predictions:

$$\hat{\mathcal{Y}} = \mathcal{F}(\mathcal{Y}, \epsilon; \hat{z}), \qquad with \qquad \mathcal{Y} = (\lambda, Y)$$

- Calculate the goodness-of-fit measure using observed changes in outcomes ($\ln \hat{\mathcal{Y}}^{(data)}$)

$$\hat{\beta}_z = (\ln \hat{\mathcal{Y}}^{(data)} - \ln \hat{\mathcal{Y}}) \cdot \ln \hat{z}$$

  - If the model is not misspecified $\implies \mathbb{E}\left[\hat{\beta}_z\right] = 0$
  - *Intuition:* if the model is not misspecified, the residual variation in $\mathcal{Y}$ that is not explained by the model should be orthogonal to $\hat{z}$

Accompanying Code

- [Link to the code and data](#) accompanying this lecture, which includes
    1. MATLAB code for MPEC estimation
    2. MATLAB code for nested fixed point estimation
    3. MATLAB code corresponding to the exact hat-algebra example

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- Class assignment: modify `HAT_ALGEBRA_EXAMPLE.m` to calculate the effect of eliminating aggregate trade imbalances ($D_i \to D_i' = 0$) on US's exports & imports.