Chapter 2

Random Variables and Expectation

1. Introduction

We have mentioned that our interest in the study of a random phenomenon is in the statements we can make concerning the events that can occur, and these statements are made based on probabilities assigned to simple outcomes. One of the immediate steps that can be taken in this unifying attempt is to require that each of the possible outcomes of a random experiment be represented by a real number. In this way, when the experiment is performed, each outcome is identified by its assigned real number rather than by its physical description. For example, when the possible outcomes of a random experiment consist of success and failure, we arbitrarily assign the number one to the event ‘success’ and the number zero to the event ‘failure’. The associated sample space has now 1, 0 as its sample points instead of success and failure, and the statement ‘the outcome is 1’ means ‘the outcome is success’.

Consequently, sample spaces associated with many random experiments of interest are already themselves sets of real numbers. The real-number assignment procedure is thus a natural unifying agent. On this basis, we may introduce a variable, which is used to represent real numbers, the values of which are determined by the outcomes of a random experiment.

2. Random variable

Consider a random experiment to which the outcomes are elements of sample space in the underlying probability space. In order to construct a model for a random variable, we assume that it is possible to assign a real number $X(s)$ for each outcome s following a certain set of rules. We see that the 'number' $X(s)$ is really a real-valued point function defined over the domain of the basic probability space

Definition1. The point function $X(s)$ is called a random variable if (a) it is a finite real-valued function defined on the sample space S of a random experiment for which the probability function is defined, and (b) for every real number X, the set $\{s : X(s) < X\}$ is an event.

To see more clearly the role a random variable plays in the study of a random phenomenon; consider again the simple example where the possible outcomes of a random experiment are success and failure. Let us again assign number one to the event success and zero to failure. If X is the random variable associated with this experiment, then X takes on two possible values: 1 and 0. Moreover, the following statements are equivalent:

1- The outcome is success.

2- The outcome is 1.

3- X=1

The random variable is called a random variable if it is defined over a sample space having a finite or a countably infinite number of sample points. In this case, random variable takes on discrete values, and it is possible to enumerate all the values it may assume. In the case of a sample space having an uncountable infinite number of sample points, the associated random variable is called a random variable, with its values distributed over one or more continuous intervals on the real line. We make this distinction because they require different probability assignment considerations. Both types of random variables are important in science and engineering.

3. Discrete Random variables

To each point in the sample space we will assign a real number denoting the value of the variable X. The value assigned to X will vary from one sample point to another, but some points may be assigned the same numerical value. Thus, we have defined a variable that is a function of the sample points in S, and {all sample points where X = a} is the numerical event assigned the number a. Indeed, the sample space S can be partitioned into subsets so that points within a subset are all assigned the same value of X. These subsets are mutually exclusive since no point is assigned two different numerical values. The partitioning of S is symbolically indicated in for a random variable that can assume values 0, 1, 2, 3, and 4.

Definition 2: A random variable *X is* said to be discrete if it can assume only a finite or countably infinite number of distinct values.

Example.1:

Let X is the random variable defined by the Head appear, in times, for toss a coin 3 times respectively. Find the value of x.

Solution: The sample space

$$S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$$

Let Head is the random variable X, then the conjugate space

$$X(s)=\{3,2,1,0\} \text{ then } x_i=0,1,2,3$$

Example.2:

Suppose a sampling plan involves sampling items from a process until a defective is observed. The evaluation of the process will depend on how many consecutive items are observed. In that regard, let X be a random variable defined by the number of items observed before a defective is found. With N a non-defective and D a defective, sample spaces are S = {D} given X = 1, S = {ND} given X = 2, S = {NND} given X = 3, and so on.

3.1. Discrete probability distribution

A discrete random variable assumes each of its values with a certain probability. For a discrete variable X can be represented by a formula, a table, or a graph that provides (i.e: example 1)

1- $P(x) = P(X = x) \ \forall \ x \in X(s)$

2- $0 \le P(x) \le 1$

3- $\sum P(x_i) = 1$

x	**0**	**1**	**2**	**3**
***P*(*x*)**	**1/8**	**3/8**	**3/8**	**1/8**

Example.3:

A shipment of 20 similar laptop computers to a retail outlet contains 3 that are defective. If a school makes a random purchase of 2 of these computers, find the probability distribution for the number of defectives. Solution: Let *X* be a random variable whose values *x* are the possible numbers of defective computers purchased by the school. Then *x* can only take the numbers 0, 1, and2.

Now we calculate, $P(0) = P(X = 0) = \frac{136}{190}$,

$P(1) = P(X = 1) = \frac{51}{190}$,

$P(3) = P(X = 3) = \frac{3}{190}$

x	**0**	**1**	**2**
***P*(*x*)**	**136/190**	**51/190**	**3/190**

Thus, the probability distribution of *X* is

Notice that the probabilities associated with all distinct values of a discrete random variable must sum to 1.

Example.4:

Suppose that a pair of fair dice is tossed and let the discrete random variable X denote the sum of the points. Obtain the range of the discrete random variable X.

Solution:

Random Variable X	**Events**
$x_1=2$	$A_1=\{(1,1)\}$
$x_2=3$	$A_2=\{(1,2),(2,1)\}$
$x_3=4$	$A_3=\{(2,2),(3,1),(1,3)\}$
$x_4=5$	$A_4=\{(1,4),(4,1),(3,2),(2,3)\}$
$x_5=6$	$A_5=\{(3,3),(2,4),(4,2),(5,1),(1,5)\}$
$x_6=7$	$A_7=\{(3,4),(4,3),(5,2),(2,5),(1,6),(6,1)\}$
$x_7=8$	$A_7=\{(4,4),(5,3),(3,5),(6,2),(2,6)\}$
$x_8=9$	$A_8=\{(4,5),(5,4),(3,6),(6,3)\}$
$x_9=10$	$A_9=\{(5,5),(4,6),(6,4)\}$
$x_{10}=11$	$A_{10}=\{(5,6),(6,5)\}$
$x_{11}=12$	$A_{11}=\{(6,6)\}$

Then $x=\{2,3,4,5,6,7,8,9,10,11,12\}$

Example.5:

Find the probability distribution corresponding to the random variable X of a coin is tossed twice. And Construct a probability graph.

Solution:

Assuming that the coin is fair we have

$$S=\{(H,H),(H,T),(T,H),(T,T)\}$$

Let $A_1=\{(T,T)\}$, $A_2=\{(H,T),(T,H)\}$, and $A_3=\{(H,H)\}$

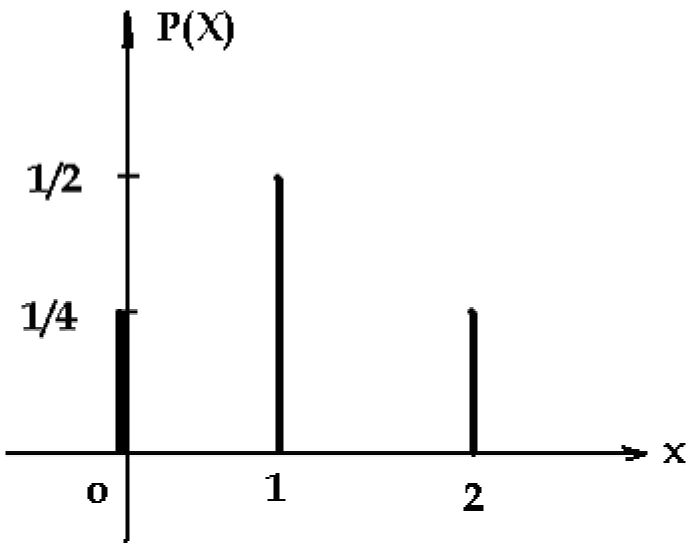
$$P(A_1)=\frac{1}{4},\quad P(A_2)=\frac{2}{4}=\frac{1}{2},\quad P(A_3)=\frac{1}{4}$$

Then $P(X=0)=P(A_1)=\frac{1}{4}$,

$P(X=1)=P(A_2)=\frac{1}{2}$ and $P(X=2)=P(A_3)=\frac{1}{4}$

X	*0*	*1*	*2*
P(X)	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

A probability graph can be obtained by use of a bar chart.



3.2. The cumulative distribution function, or distribution function for a random variable X is defined by

$$P(X \le x) = F(x).$$

Where x is any real number, i.e. $-\infty < x < \infty$. The distribution function can be obtained from the probability function by

$$F(x) = P(X \le x) = \sum_{0 \le x} P(x)$$

There are many problems where we may wish to compute the probability that the observed value of a random variable X will be less than or equal to some real number x. Writing $F(x) = P(X \le x)$ for every real number x, we define $F(x)$ to be the **cumulative distribution function** of the random variable X.

Example.6:

Consider the simple condition in which components are arriving from the production line and they are stipulated to be defective or not defective.

Solution:

Define the random variable X by

$X = 1$, if the component is defective,

0, if the component is not defective.

Clearly the assignment of 1 or 0 is arbitrary though quite convenient. The random variable for which 0 and 1 are chosen to describe the two possible values is called a Bernoulli random variable.

Example.6:

A store carries flash drives with either1 GB, 2 GB, 4 GB, 8 GB, or 16 GB of memory. The accompanying table gives the distribution of X = the amount of memory in a purchased drive:

X	1	2	4	8	16
p(X)	0.05	0.10	0.35	0.40	0.10

Let's first determine $F(x)$ for each of the five possible values of X:

$F(16) = P(X \leq 16) = 1$

$F(8) = P(X \leq 8) = p(1) + p(2) + p(4) + p(8) = .90$

$F(4) = P(X \leq 4) = P(Y = 1 \text{ or } 2 \text{ or } 4) = p(1) + p(2) + p(4) = .50$

$F(2) = P(X \leq 2) = P(Y = 1 \text{ or } 2) = p(1) + p(2) = .15$

$F(1) = P(X \leq 1) = P(Y = 1) = p(1) = .05$

Now for any other number x, $F(x)$ will equal the value of F at the closest possible value of X to the left of x.

4. Continuous Random variables

A continuous random variable has a probability of 0 of assuming exactly any of its values. Consequently, its probability distribution cannot be given in tabular form. We shall concern ourselves with computing probabilities for various intervals of continuous random variables such as:

$P(a < X < b)$, $P(W \geq c)$, and so forth.

Note that when X is continuous, for a continuous random variable, its Probability Density Function, is a continuous function of and the derivative

$$f(x)=\frac{dF(x)}{dx}.$$

The **cumulative distribution function** $F(x)$ of a continuous random variable

X with density function *f*(*x*) is

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(x)dx, \quad -\infty < x < \infty.$$

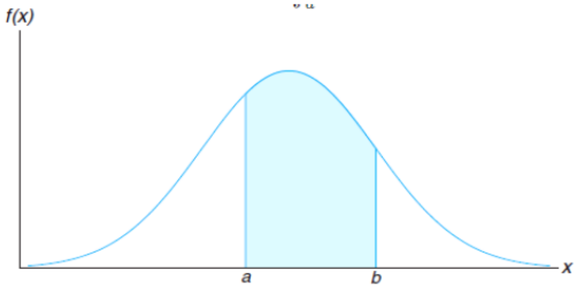
A function f(x) is called probability density function (p.d.f) if the following conditions satisfied

1- $f(x) \ge 0$

2. $\int_{-\infty}^{\infty} f(x)\,dx = 1$

3- $P(a \le X \le b) = P(a < X \le b) = P(a \le X < b) =$

$$= P(a < X < b) = \int_{a}^{b} f(x)\,dx$$



Should this range of *X* be a finite interval, it is always possible to extend the interval to include the entire set of real numbers by defining *f*(*x*) to be zero at all points in the extended portions of the interval. In the Figure, the probability that *X* assumes a value between *a* and *b* is equal to the shaded area under the density function between the ordinates at $x = a$ and $x = b$, and from integral calculus is given by

$$P(a < x < b) = \int_{a}^{b} f(x)dx$$

Example.7:

A continuous random variable X has a density function given by

$$f(x)=\begin{cases}\frac{1}{4} & 0<x\le 4\\ 0 & \text{elsewhere}\end{cases}$$

Show that $f(x)$ is a valid probability density function.

Solution

It is clear that $f(x)\ge 0$

$$\int_a^b f(x)dx=\int_0^4 \frac{1}{4}\,dx+0=1$$

Then $f(x)$ is a density function.

Example.8:

A continuous random variable X has a density function given by

$$f(x)=\begin{cases}k\,x^2 & 0<x<3\\ 0 & \text{otherwise}\end{cases}$$

(a) Find the constant k

(b) Compute $P(1<X<2)$

Solution

a) Since $f(x)$ satisfies property (1) if $k\ge 0$, it must satisfy property (2) in order to be a density function. Now:

$$\int_{-\infty}^{\infty} f(x)dx=\int_{-\infty}^{0} f(x)\,dx+\int_{0}^{3} f(x)\,dx+\int_{3}^{\infty} f(x)\,dx$$

$$=0+\int_0^3 kx^2\,dx+0=\frac{k}{3}\left[x^3\right]_0^3=1$$

$$9k=1 \quad \Rightarrow \quad k=\frac{1}{9}$$

b) $$P(1<X<2)=\int_1^2 \frac{1}{9}x^2\,dx=\frac{x^3}{27}\Big|_1^2=\frac{8}{27}-\frac{1}{27}=\frac{7}{27}$$

Example.9:

Suppose that the error in the reaction temperature, in ◦C, for a controlled laboratory experiment is a continuous random variable *X* having the probability density function

$$f(x)=\begin{cases}\frac{x^2}{3}, & -1\le x\le 2\\ 0 & elsewhere\end{cases}$$

(a) **Verify** that *f*(*x*) is a density function.

(b) **Find** $P(0 < x \le 1)$.

Solution

We use Definition.

(a) Obviously, $f(x) \ge 0$. To verify condition 2, we have

$$\int_a^b f(x)dx=\int_{-1}^{2}\frac{x^2}{3}dx=1$$

(b) $p(0<x\le 1)=\int_0^1 \frac{x^2}{3}dx=\frac{1}{9}$

5. Expectations and Moments

While a probability distribution [$F_X(x)$, $P_X(x)$, or $f_X(x)$] contains a complete description of a random variable *X*, it is often of interest to seek a set of simple numbers that gives the random variable some of its dominant features. These numbers include moments of various orders associated with *X*. Let us first provide a general definition as:

Definition. Let g(X) be a real-valued function of a random variable X. The mathematical expectation, or simply expectation, of g(X), denoted by is defined by

$$E(g(X)) = \sum_i g(x_i) P_X(x_i)$$

If X is discrete, where $x_1, x_2, \ldots$ are possible values assumed by X. When the range of i extends from 1 to infinity, the sum $\sum_i g(x_i) P_X(x_i)$ exists if it converges absolutely.

If random variable X is continuous, the expectation is defined by

$$E(g(X)) = \int_{-\infty}^{\infty} g(x) P_X(x) dx$$

Let us note some basic properties associated with the expectation operator. For any constant c and any functions g(X) and h(X) for which expectations exist, we have

1- $E(c) = c$

2- $E(cg(X)) = cE(g(X))$

3- $E(g(X) + h(X)) = E(g(X)) + E(h(X))$

5.1. Moments

Let $g(X) = X^n$, n=1, 2, …; the expectation $E(X^n)$, when it exist, is called the n^th moment of X. It is denoted by n and is given by

$$\mu_n = E(X^n) = \sum_i x_i^n P(x_i) \qquad \text{for X is discrete.}$$

$$\mu_n = E(X^n) = \int_{-\infty}^{\infty} x f(x) dx \qquad \text{for X is continuous.}$$

Example.10:

Let X is the waiting time (in minutes) of a customer waiting to be served at a ticket counter has the density function.

$$f(x)\begin{cases} 2e^{-2x} & \text{for } x \geq 0 \\ 0, & elsewhere. \end{cases}$$

Determine the average waiting time.

Solution:

$$E(X) = \int_0^{\infty} xf(x)dx$$

$$= \int_0^{\infty} 2xe^{-2x}dx = \frac{1}{2}$$

6. Exercises.2

Ex.1: In the experiment of tossing a fair die once, the range of the random variable X is {1, 3} where "1" is specified for the appearance of an odd number on the upper face and "3" is specified for the appearance of an even number. Find the probability distribution of this variable.

Ex.2: In the experiment of tossing a coin three consecutive times to observe the type of the appearing faces, if he random variable "X" is defined by "Twice the number of the appearing heads." Write the probability distribution of X.

Ex.3: Let X be a continuous random variable with probability density function given by $$f(x) = \begin{cases} 3x^2, & 0 \le x \le 1 \\ 0 & elsewhere \end{cases}$$

Find *F(x)*. Graph both *f (x)* and *F(x)*.

Ex.4: Let X be a discrete random variable taking the values x = 0, 1, 2with probabilities 1/4, 1/2, 1/4 respectively. Plot F(x).

Ex.5: Let the distribution function of random variable X given as follows

x	***1***	***2***	***3***	***4***
f(x)	***1/8***	***3/8***	***3/4***	***1***

Determine:

(a) Probability function
(b) $P(1<X\le 3)$
(c) $P(X\ge 2)$
(d) $P(X<3)$

Exe.6: Let $f(x)=\begin{cases} K(2-x) & \text{for} \quad 0 \le x \le 2 \\ 0 & \text{elsewhere} \end{cases}$

(a) Find K such that f(x).density function

(b) Plot f(x).

(c) $E(X)$

Ex.7: A continuous random variable X that can assume values between $x = 1$ and $x = 3$ has a density function given by $f(x) = 1/2$.

(a) **Show** that the area under the curve is equal to 1.

(b) **Find** $P(2 < X < 2.5)$. (c) **Find** $P(X \le 1.6)$.

(d) **Find** $F(x)$. Use it to evaluate $P(2 < X < 2.5)$.

(e) $E(X)$

Ex.8: Consider the density function

$$f(x)=\begin{cases} k\sqrt{x}, & 0 \le x \le 1 \\ 0 & elsewhere \end{cases}.$$

(a) **Evaluate** k.

(b) **Find** $F(x)$ and use it to evaluate $P(0.3 < X < 0.6)$.

(e) $E(X)$