Chapter 1

Probability Theory - Sample Space

1. Introduction

In the study of statistics we are concerned basically with the presentation and interpretation of chance outcomes that occur in a planned study or scientific investigation. For example, we may record the number of accidents that occur monthly at the intersection of Driftwood Lane and Royal Oak Drive, hoping to justify the installation of a traffic, light; we might classify items coining off an assembly line as "defective" or "non-defective"; or we may be interested in the volume of gas released in a chemical reaction when the concentration of an acid is varied. Hence, the statistician is often dealing with either experimental data, representing counts or measurements, or perhaps with categorical data that can be classified according to some criterion.

Statisticians use the word experiment to describe any process that generates a set of data. A simple example of a statistical experiment is the tossing of a coin. In this experiment there are only two possible outcomes, heads or tails. Another experiment might be the launching of a missile and observing its velocity at specified times. The opinions of voters concerning a new sales tax can also be considered as observations of an experiment. We are particularly interested in the observations obtained by repeating the experiment several times. In most cases the outcomes will depend on chance and, therefore, cannot, be predicted with certainty. If a chemist runs an analysis several times under the same conditions, he or she will obtain different measurements, indicating an element of chance in the experimental procedure. Even when a coin is tossed repeatedly, we cannot be certain that a given toss will result in a head. However, we know the entire set of possibilities for each toss, sample space and event that are important when to determine the probability of an event.

Sample space: is the set of all possible outcomes of a statistical experiment and is donated by the symbol S.

Events: is the subset of the outcomes of the sample space.

Probability of an event: is a value between 0 and 1, that describe the possibility of an event will be occur. Let A is an event subset of S, then

$$P(A)=\frac{\text{number of event}}{\text{total number of sample space}}$$

Example.1:

When we toss a coin 3 times and record the results in the sequence that they occur, then the sample space is

S = { HHH , HHT , HTH , HTT , THH , THT , TTH , TTT } .

Elements of S are sequences or ordered outcomes. We may expect each of the 8 outcomes to be equally likely.

Thus the probability of the element (HTT) is $\frac{1}{8}$.

The probability of an element that contains precisely two Heads is $\frac{3}{8}$.

Example.2:

The set of basic outcomes of rolling a die once is S = { 1 , 2 , 3 , 4 , 5 , 6 } , so the subset E = { 2 , 4 , 6 } is an example of an event. If a die is rolled once and it lands with a 2 or a 4 or a 6 up then we say that the event E has occurred.

We have already seen that the probability that E occurs is P (E) = $\frac{3}{6}$.

For sample spaces with a large event or infinite sequence of sample points are best described by a statement or rule method. For example, if the possible outcomes of an experiment are the set of cities in the world with a population over 1 million, our sample space is written as:

S = {x | x is a city with a population over 1 million},

Example.3:

Given the sample space S = {t | t > 0}, where t is the life in years of a certain electronic component, then the event A that the component fails before the end of the fifth year is the subset A = {t | 0 < t < 4}. It is conceivable that an event may be a subset that includes the entire sample space S or a subset of S called the empty set and denoted by the symbol φ, which contains no elements at all.

Example.4:

Write the sample space of each of the following random experiments

E_1 : Toss a die one time to observe the appearing on its upper face.

E_2 :Tossing a coin twice one after the other (or tossing two distinct coins simultaneously to observe the type of the appearing face in each throw.

E_3 : The number of incoming-calls on a telephone line through one day.

Solution

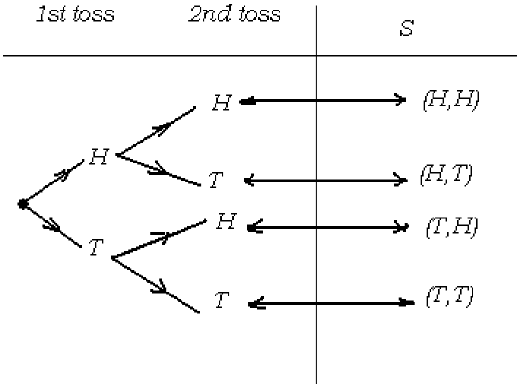
E_1 : $S=\{1,2,3,4,5,6\}$, number of elements $6^1=6$.

E_2 : Every outcome of this experiment is an ordered pair in the form (the face which appears in the first toss and the face which appears in the second toss). Let (H) denotes the head and (T) denotes the tail. Thus:

$$S=\{(H,H), (H,T), (T,H), (T,T)\}$$

Note that the number of elements of sample space is $2^2=4$ elements.

(**First**) the tree diagram



E_3: $S=\{0,1,2,\ldots\}$. Note: the elements of sample space are countably infinite.

Example.5:

If we randomly draw one character from a box containing the characters a, b, and c, then the sample space is S = {a, b, c}, and there are 8 possible events, namely, those in the set of events

$$E = \{\{\ \}, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$$

If the outcomes a, b, and c, are equally likely to occur, then

$P(\{\ \}) = 0$, $P(\{a\}) = \frac{1}{3}$, $P(\{b\}) = \frac{1}{3}$,

$P(\{c\}) = \frac{1}{3}$, $P(\{a, b\}) = \frac{2}{3}$, $P(\{a, c\}) = \frac{2}{3}$,

$P(\{b, c\}) = \frac{2}{3}$, $P(\{a, b, c\}) = 1$.

For example, P ({a, b}) is the probability the character is an a or b.

Example.6:

What is the probability of at least one head in four tosses of a coin?
The sample space S will have 16 outcomes

$$P(\text{at least one H}) = 1 - P(\text{no H}) = 1- \frac{1}{16}=\frac{15}{16}.1-\frac{15}{16}=\frac{1}{16}$$

Example.7:

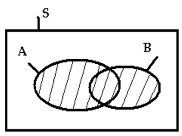
Let A = {1, 2, 3, 5}, B = {4, 6, 7} and C = {4, 3, 5}: Then it follows that

1) $A \cap B = \varnothing$, that is A and B have no elements in common and, therefore, cannot both occur simultaneously. For certain statistical experiments it is by no means unusual to define two events, A and B, which cannot both occur simultaneously. The events A and B are then said to be **mutually exclusive**.

2) $A \cap C = \{3,5\}$, that is A and C have common elements, therefore. For certain statistical experiments it is by means usual to define two events, A and B. The events A and B are then said to be **not mutually exclusive**.

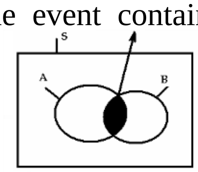
2. Operations on Events

1- The **union** of the two events A and B, is the event containing all the elements that belong to A or B or both.



$$(A \cup B) = \{x : x \in A \ or \ x \in B\}$$

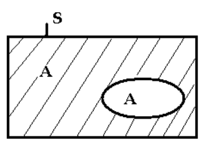
2- The **intersection** of two events A and B is the event containing all elements that are common to A and B.



$$(A \cap B) = \{x : x \in A \ and \ x \in B\}$$

3- The **complement** of an event A with respect to S is the subset of all elements of S that are not in A. We denote the complement of A by the symbol A^c.

$$A^c = \{x : x \notin A \ and \ x \in S\}$$



4- The difference between to events is all elements of that are in A and not in B. $(A-B)=\{x : x\in A \; and \; x\notin B\}$

Algebra's laws of the sets

1- $A\cup A=A,\quad A\cup S=S,\quad A\cup\varnothing=A$
$A\cap A=A,\quad A\cap S=A,\quad A\cap\varnothing=\varnothing$

2- Associative laws

$$(A\cup B)\cup C=A\cup(B\cup C),\qquad (A\cap B)\cap C=A\cap(B\cap C)$$

3- Commutative laws

$$A\cup B=B\cup A,\qquad A\cap B=B\cap A$$

4-Distributive laws

$$A\cup(B\cap C)=(A\cup B)\cap(A\cup C),\qquad A\cap(B\cup C)=(A\cap B)\cup(A\cap C)$$

5-Complement laws

$$A\cup A^c=S,\qquad A\cap A^c=\varnothing,\qquad (A^c)^c=A\quad S^c=\varnothing,\qquad \varnothing^c=S$$

6-De Morgen's law

$$(A\cup B)^c=A^c\cap B^c,\qquad (A\cap B)^c=A^c\cup B^c$$

Remarks:-

1- $If\ A\subset B,\quad B\subset C\quad\Rightarrow\quad A\subset C$

2- $If\ A\subset B,\quad B\subset A\quad\Rightarrow\quad A=B$

3- $If\ A\subset B,\quad then\quad A\cup B=B$

4- $A\subset(A\cup B),\qquad B\subset(A\cup B)$

5- $(A\cap B)\subset(A\cup B)$

6- $(A-B)\neq(B-A)$

7- $S-A=A^c,\ \ \varnothing-A=\varnothing,\ \ A-\varnothing=A,\ \ A-A=\varnothing$

8- $If\ A\subset B,\quad\Rightarrow\quad A-B=\varnothing$

3. Axioms on Probability

A probability function P assigns a real number (the probability of E) to every event E in a sample space S. P (.) must satisfy the following basic properties:

1) $0 \le P(E) \le 1$

2) $P(S) = 1$

3) For any disjoint events E_i i = 1, 2, · · · , n, we have

$$P(E_1 \cup E_2 \cup E_3 ..... \cup E_n) = P(E_1) + P(E_2) + ... + P(E_n)$$

Further Rules on probability

1) Addition Rule

Let A and B are two not mutually exclusive events then:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

2) Multiplication Rule

Let A and B are two independent events then:

$$P(A \cap B) = P(A) * P(B)$$

3) Complement Rule

Let A an event subset from sample space then:

$$P(A^C) = P(S) - P(A) = 1 - P(A)$$

Proparties:

1) $P(E \cup E^C) = P(E) + P(E^C) = 1.$

2) $P(E_1 \cap E_2 \cap E_3 ..... \cap E_n) = P(E_1) * P(E_2) * ..... * P(E_n)$

3) $P(E_1^c \cup E_2^c) = P(E_1 \cap E_2)^C = 1 - P(E_1 \cap E_2)$

4) $P(E_1^c \cap E_2^c) = P(E_1 \cup E_2)^C = 1 - P(E_1 \cup E_2)$

Example.8:

If A and B are two not mutually exclusive events, suppose $P(A)=0.5$ $P(B)=0.3$ and $P(A \cap B)=0.2$. **Find** the following probabilities:

1) $P(A \cup B)$ 2) $P(A^c \cap B^c)$ 3) $P(A \cup B^c)$

Solution:

1- $P(A \cup B) = P(A)+P(B) - P(A \cap B) = 0.5+0.3-0.2=0.6$

2- $P(A^c \cap B^c) = P(A \cup B)^c = 1-0.6=0.4$

3- $P(A \cup B^c) = P(A)+P(B^c) - P(A \cap B^c) = 0.5+0.7-(0.5-0.2)=0.9$

Example.9:

If A and B are two mutually exclusive events, suppose $P(A)=0.4$, and $P(B)=0.2$. **Find** the following probabilities

1- Probability of A or B 2- $P(A^c \cup B^c)$ 3) $P(A \cup B^c)$

Solution:

1- $P(A \text{ or } B) = P(A \cup B) = P(A)+P(B) = 0.4+0.2=0.6$

2- $P(A^c \cup B^c) = P(A \cap B)^c = 1-P(\varnothing)=1$

3- $P(A \cup B^c) = P(A)+P(B^c) - P(A \cap B^c) = 0.4+0.8-(0.4)=0.8$

Example.10:

A student is taking two courses, Math & Phys, the probability the student will pass the math is 0.6, and the probability of passing Phys is 0.7 , the probability of passing both courses is 0.5, what is the probability of passing at least one course.

Solution:

Let the probability of student pass in Math $P(A)=0.6$ and in Phys $P(B)=0.7$, also pass in both them $P(A \cap B)=0.5$

$P(\text{Math or Phys}) = P(A \cup B) = P(A)+P(B) - P(A \cap B) = 0.6+0.7-0.5=0.8$

4. Counting Sample Points

One of the problems that the statistician must consider and attempt to evaluate is the element of chance associated with the occurrence of certain events when an experiment is performed. In many cases, we shall be able to solve a probability problem by counting the number of points in the sample space without actually listing each element.

Rule1: If an operation can be performed in n_1 ways and if for each of these ways a second operation can be performed in n_2 ways, then the two operations can be performed together in n_1n_2 ways.

Example .11:

How many sample points are there in the sample space when a pair of dice is thrown once?

Solution:

The first die can land face-up in any one of $n_1 = 6$ ways. For each of these 6 ways, the second die can also land face-up in $n_2 = 6$ ways. Therefore, the pair of dice can land in $n_1n_2 = (6)\,(6) = 36$ possible.

Rule 2: If an operation can be performed in n_1 ways, and if for each of these a second operation can be performed in n_2 ways, and for each of the first two a third operation can be performed in n_3 ways, and so forth, then the sequence of k operations can be performed in $n_1n_2 \cdots n_k$ ways.

We have seen examples where the outcomes in a finite sample space S are equally likely, i.e., they have the same probability. Such sample spaces occur quite often. Computing probabilities then requires counting all outcomes and counting certain types of outcomes. The counting has to be done carefully!

We will discuss a number of representative examples in detail. Concepts that arise include **permutations** and **combinations**.

Theorem.1: The number of permutations of ***n*** distinct objects taken ***k*** at a time is

$$ {}^{n}P_{k} = \frac{n!}{(n-k)!}. $$

Theorem.2: The number of combinations of ***n*** distinct objects taken ***k*** at a time is

$$ {}^{n}C_{k} = \frac{n!}{n!(n-k)!}. $$

Example.12:

Suppose that four-letter words of lower case alphabetic characters are generated randomly with equally likely outcomes. (Assume that letters may appear repeatedly.)

1-How many four-letter words are there in the sample space S?

$$ \text{answer} = (26)^{4} = 456,976. $$

2-How many four-letter words are there are there in S that start with the letter ”s ” ?

$$ \text{answer} = (26)^{3}\ . $$

3- What is the probability of generating a four-letter word that starts with an ”s ” ?

$$ \text{answer} = \frac{(26)^{3}}{(26)^{4}} = \frac{1}{16} = 0.038. $$

Example.13:

How many re-orderings (permutations) are there of the string abc ? (Here letters may appear only once.)

Solution : Six, namely, abc , acb , bac , bca , cab , cba .

If these permutations are generated randomly with equal probability then what is the probability the word starts with the letter ”a ” ? $\frac{2}{6} = \frac{1}{3}$.

Example.14:

Three-letter words are generated randomly from the five characters a, b, c, d, e, where letters can be used at most once.

1- How many three-letter words are there in the sample space S? 60

2- How many words containing a, b are there in S?

First place the characters a , b

i.e., select the two indices of the locations to place them.

This can be done in $3 \times 2 = 6$ ways.

There remains one position to be filled with c, d or e.

Therefore the number of words is $3 \times 6 = 18$.

Suppose the 60 solutions in the sample space are equally likely.

3- What is the probability of generating a three-letter word that contains the letters a and b? $\frac{18}{60} = 0.3$

Example.15:

How many ways are there to choose a committee of 4 persons from a group of 10 persons, if order is not important?

$$^{10}C_4 = \binom{10}{4} \equiv \frac{10!}{10!(10-4)!} = 210,$$

If each of these 210 outcomes is equally likely then what is the probability that a particular person is on the committee?

$$\frac{^{9}C_3}{^{10}C_4} = \frac{84}{210} = \frac{4}{10},$$

How many ways are there to choose a committee of 4 persons from a group of 10 persons, if one is to be the chairperson?

$$^{10}C_1 *^{9}C_3 = 10\frac{9!}{3!(9\text{-}3)!} = 840,$$

Example.16:

Two balls are selected at random from a bag with four white balls and three black balls, where order is not important.

1- What would be an appropriate sample space S?

Solution: Denote the set of balls by

B = {w1, w2, w3, w4, b1, b2, b3}

The number of outcomes in S (which are sets of two balls) is then $^7C_2 = 21$

2- What is the probability that both balls are white?

$$\frac{^4C_2}{^7C_2} = \frac{6}{21} = \frac{2}{7},$$

3- What is the probability that both balls are black?

$$\frac{^3C_2}{^7C_2} = \frac{3}{21} = \frac{1}{7},$$

4- What is the probability that one is white and one is black?

$$\frac{^3C_1 *^4 C_1}{^7C_2} = \frac{4*3}{21} = \frac{4}{7},$$

Example.17:

A young boy asks his mother to get 5 Game cartridges from his collection of 10 arcade and 5 sports games. How many ways are there that his mother can get 3 arcade and 2 sports games?

Solution:

The number of ways of selecting 3 cartridges from 10 is

$$\binom{10}{3} = \frac{10!}{3!(7)!} = 120$$

The number of ways of selecting 2 cartridges from

$$\binom{5}{2} = \frac{5!}{2!(2)!} = 10$$

Using the multiplication rule with n_1 = 120 and n_2 = 10, we have (120) (10) = 1200 ways.

5. Conditional Probability

One very important concept in probability theory is conditional probability. In some applications, the practitioner is interested in the probability structure under certain restrictions. The probability of an event *A* occurring when it is known that some event *B* has occurred is called a **conditional probability** and is denoted by:

$$P(A|B) = \frac{P(A\cap B)}{P(B)}.$$

The symbol $P(A|B)$ is usually read "the probability that *B* occurs given that *A* occurs" or simply "the probability of *A*, given *B*."

Example.18:

Three coins are tossed, **find** the probability that all are heads if:

1-First coin is head.
2-At least one of the coins is head.
Solution:

The sample space consists of 8 equally probable outcomes as follows:

$$S=\left\{\begin{matrix}(H,H,H),(H,H,T),(H,T,H),(T,H,H),\\ (T,T,H),(T,H,T),(H,T,T),(T,T,T)\end{matrix}\right\}$$

$$E_1=\{\text{all outcomes are Heads}\}=\{(H,H,H)\}$$

$$E_2=\{\text{First coin is Head}\}=\{(H,H,H),(H,H,T),(H,T,H),(H,T,T)\}$$

1- $$P(E_1|E_2)=\frac{P(E_1\cap E_2)}{P(E_2)}=\frac{1/8}{4/8}=\frac{1}{4}$$

$$E_3=\{\text{at least one of the coins is Head}\}$$

$$E_3=\left\{\begin{matrix}(H,H,H),(H,H,T),(H,T,H),(T,H,H),\\ (T,T,H),(T,H,T),(H,T,T)\end{matrix}\right\}$$

2- $$P(E_1|E_3)=\frac{P(E_1\cap E_3)}{P(E_3)}=\frac{1/8}{7/8}=\frac{1}{7}$$

Example.19:

The probability that a regularly scheduled flight departs on time is $P(A) = 0.6$; the probability that it arrives on time is $P(B) = 0.8$; and the probability that it departs and arrives on time is $P(A \cap B) = 0.5$. Find the probability that

1- A plane arrives on time, given that it departed on time.

2- A plane departed on time, given that it has arrived on time.

Solution:

Using definition of conditional probability, we get

1-The probability that a plane arrives on time, given that it departed on time

$$P(B|A) = \frac{0.5}{0.6} = \frac{5}{6}$$

2-The probability that a plane departed on time, given that it has arrived on time

$$P(A|B) = \frac{0.5}{0.8} = \frac{5}{8}$$

Example.20:

If A and B are two independent events, suppose $P(A) = 0.4$ and $P(B) = 0.5$

Find the following probabilities:

1) $P(A \cup B)$ **2)** $P(A^c \cap B)$ **3)** $P(A|B)$

Solution:

1- $P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.4 + 0.5 - (0.4 * 0.5) = 0.7$

2- $P(A^c \cap B) = P(B) - P(A \cap B) = 0.5 - (0.4 * 0.5) = 0.3$

Or $P(A^c \cap B) = P(A^c) * P(B) = (1 - 0.4) * 0.5 = 0.3$

3- $P(A|B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{0.4 * 0.5}{0.5} = \dfrac{0.2}{0.5} = \dfrac{2}{5}$

6. Bayes' Formula

A useful formula that" inverts conditioning" is derived as follows: Since we have both

$$P(E\cap F) = P(E|F)\,P(F)\text{ , and }P(F\cap E) = P(F|E)\,P(E)\ .$$

If $P(E)\neq 0$, then it follows that $P(F|E)=\dfrac{P(E|F)\,P(E)}{P(E)}$

, and, using the earlier useful formula, we get

$$P(F|E)=\frac{P(E|F)\,P(E)}{P(E|F)P(F)+P(E|F)P(F)},$$

which is known as Bayes' formula for two arbitrary events E and F .

Example.21:

Suppose 1 in 1000 persons has a certain disease. A test detects the disease in 99 % of diseased persons. The test also"detects" the disease in 5 % of healthy persons. With what probability does a positive test diagnose the disease?

Solution: Let D denoted diseased, H denoted healthy, B denoted positive. We are given that

P(D) = 0.001 , P(B/D) = 0.99 , P(B/H) = 0.05 .

By Bayes' formula

$$P(D|B)=\frac{P(B|D)\,P(D)}{P(B|D)P(D)+P(B|H)P(H)}$$

$$=\frac{0.99*0.001}{0.99*0.001+0.05*0.999}=0.0194$$

Example.22:

Incidence of a rare disease. Only 1 in 1000 adults is afflicted with a rare disease for which a diagnostic test has been developed. The test is such that when an individual actually has the disease, a positive result will occur 99% of the time, whereas an individual without the disease will show a positive test result only 2% of the time. If a randomly selected individual is tested and the result is positive, what is the probability that the individual has the disease?

Solution

To use Bayes' theorem, let A_1 = individual has the disease, A_2= individual does not have the disease, and B = positive test result.

Then $P(A_1)$ =0.001, $P(A_2)$ = 0.99, , $P(B\backslash A_1)$ =0.99, and $P(B/A_2)$ = 0.02 .

$$P(A_1 | B) = 0.047$$

This result seems counterintuitive; the diagnostic test appears so accurate that we expect someone with a positive test result to be highly likely to have the disease, whereas the computed conditional probability is only .047. However, the rarity of the disease implies that most positive test results arise from errors rather than from diseased individuals.

7. Probability of Systems

Since this topic is a guaranteed part of a question on the final exam, and to make using this document simpler, I'd recommend you print this out ... It is not essential to print it, but most people will find it easier to follow in printed form ... I think! ... ;-)

There are really only two systems, and once you understand how to tackle these two then all of the others (that we will cover) are just applications of these two. The more complex ones may not look like the simple two to begin with, but some examples will show you how. We will need three probability rules:

Complementary rule: $P(A) = 1 - P(A^c)$

$$P(\text{item works}) = 1 - P(\text{item doesn't work})$$

General addition rule: $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$

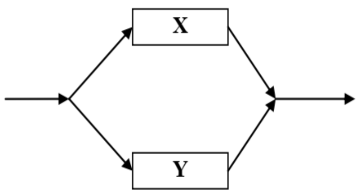
Basic multiplication rule: $P(A \text{ and } B) = P(A)*P(B)$,

at A, B are independent.

These are the basic tools for this job. Getting when to use/write AND and when to write OR is key, so to make this extra clear: You should be able to put in the word BOTH with the AND, eg BOTH A AND B; If this makes sense that use the basic multiplication rule. You should be able to put the word EITHER with OR, eg EITHER A OR B.

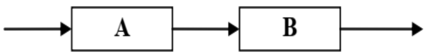
The two basic systems, the building blocks for all systems, are:

System 1:



and

System 2:

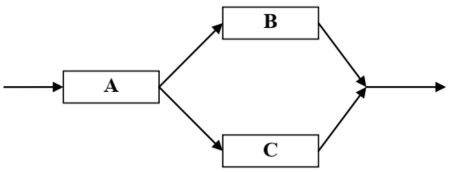


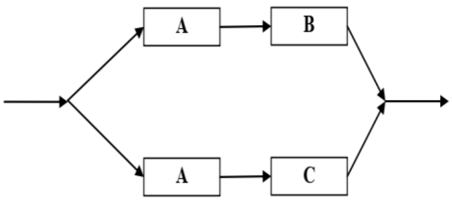
Imagine yourself on the left hand side of system 1. To get to the other side you have to follow the path and go through EITHER X **OR** Y, so this will be a general addition rule, since we have “EITHER ...**OR** ...”.

Imagine yourself on the left hand side of system 2. To get to the other side you have to follow the path and go through BOTH A **AND** B, so this will be a basic multiplication rule, since we have “BOTH ...**AND** .

Example.23:

The probability that components (or people or circuits or whatever) A, B working is 0.8, 0.9 respectively. The probability that component C failing is 0.4, and the probability of component D failing is 0.15. Each A,B,C,D is independent. Calculate probability the system works.

1- 

2- 

Solution

1- So working only now on this simple P1 system,

P(P1 works) = P((EITHER) B works **OR** C works)

= P(B works) + P(C works) – P(B works **AND** C works)

= 0.9 + 0.6 – P(B works **AND** C works)

= 1.5 – P(B works)*P(C works)

= 1.5 – 0.9*0.6

= 0.96

P(System works) = P((BOTH) A works **AND** P1works)

= P(A works)*P(P1 works)

= 0.8 * 0.96= 0.768

2- P(S1 works) = 0.72

P(S2 works) = 0.48

P(System works) = P(S1 works **OR** S2 works)

= P(S1 works) + P(S2 works) – P(S1 works **AND** S2works)

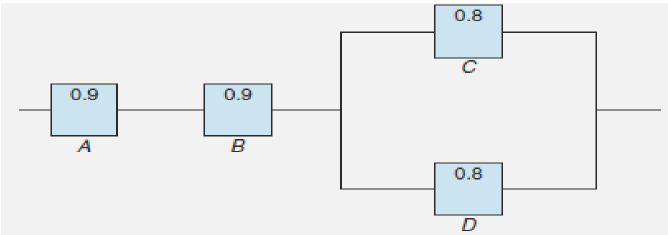
= P(S1 works) + P(S2 works) – P(S1 works)*P(S2 works)

= 0.72 + 0.48 – 0.72*0.48

= 0.8544.

Example .24:

An electrical system consists of four components as illustrated in the next Figure. The system works if components *A* and *B* work and either of the components *C* or *D* works. The reliability (probability of working) of each component is also shown In the Figure. Find the probability that (a) the entire system works and (b) the component *C* does not work, given that the entire system works. Assume that the four components work independently.



Solution: In this configuration of the system, *A*, *B*, and the subsystem *C* and *D* constitute a serial circuit system, whereas the subsystem *C* and *D* itself is a parallel circuit system.

(a) Clearly the probability that the entire system works can be calculated as follows:

$$P[A \cap B \cap (C \cup D)] = P(A)P(B)P(C \cup D)$$

$$= P(A)P(B)[1 - P(C^c \cap D^c)]$$

$$= P(A)P(B)[1 - P(C^c)P(D^c)]$$

$$= (0.9)(0.9)[1 - (1 - 0.8)(1 - 0.8)] = 0.7776.$$

The equalities above hold because of the independence among the four components.

(b) To calculate the conditional probability in this case, notice that

$$P = P(\text{the system works but } C \text{ does not work})/P(\text{the system works})$$

$$= P(A \cap \cap B \cap C^c \cap D)/P(\text{the system works})$$

$$= (0.9)(0.9)(1 - 0.8)(0.8)/0.7776 = 0.1667.$$

The multiplicative rule can be extended to more than two-event situations.

8. Exercises.1

Ex.1: Two dice are rolled as an experiment. A is the event 'sum of points is odd' and B is the event 'at least one 3 is shown' Describe in words the events:

(a) $A \bigcup B$ (b) $A \bigcap B$ (c) $\left(A \bigcap B^c\right) \cup A^c$

Ex.2: A coin is tossed twice. The sequence of heads and tails is observed. Write down the sample space for this experiment. Also list the following events.

a) A_1 : is the event of getting a tail in the first toss.

b) A_2 : is the event of getting only one tail.

c) A_3 : is the event of getting a tail in one of the two tosses.

Ex.3: A football team plays two games and in each game either he wins (w) or be equal (D) or losses (L). Write the sample space of this experiment, and then express the following events:

A_1 : Be equal (D) in one of the two games.

A_2: To win (w) in at least one of the two games.

Ex.4: A box contains 60 balls. 30 of them are white and numbered from 1 to 30, and 18 of them are yellow and numbered from 1 to 18, and 12 of them are blue and numbered from 1 to 12. One ball is drawn from the box at random, find the probability that it is:

(a) White or yellow. (b) Carries a number ≤ 6

(c) Carries a number ≤ 20 (d) Carries a number ≥ 15

(e) Carries 7 or 13 or 24.

Ex.5: If A, B are two mutually exclusive of a random. Experimental, such that $P(A)=0.35$, $P(B)=x$, $P(A^c \cup B^c)=0.1$. Find value of x then calculate:

(1) $P(A \cup B^c)$ (2) $P(A^c \cup B^c)$

Ex.6: If a multiple-choice test consists of 5 questions, each with 4 possible answers of which only 1 is correct,

(a) in how many different ways can a student check off one answer to each question?

(b) in how many ways can a student check off one answer to each question and get all the answers wrong?

Ex.7: If 3 books are picked at random from a shelf containing 5 novels, 3 books of poems, and a dictionary, what is the probability that

(a) the dictionary is selected?

(b) 2 novels and 1 book of poems are selected?

Ex.8: In a certain region of the country it is known from past experience that the probability of selecting an adult over 40 years of age with cancer is 0.05. If the probability of a doctor correctly diagnosing a person with cancer as having the disease is 0.78 and the probability of incorrectly diagnosing a person without cancer as having the disease is 0.06, what is the probability that an adult over 40 years of age is diagnosed as having cancer?

Ex.9: A particular class of 48 students, 24 of them like playing football, and 20 of them like playing handball, and 8 of them like both of the two sports. One student is chosen at random of this class, calculate the probability that:

(i)He likes at least one of the two sports.

(ii)He does not like the football or the handball.

Ex.10: If the probability that a student succeeds in math's exam. Is 0.3 and the probability that he succeeds in physics is 0.45, the probability that he succeeds in at least one of them is 0.65. Find the probability that:

(1)He succeeds in both math and phys. [0.1]

(2) He succeeds only in math. [0.2]

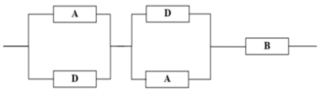
(3) He fails in both math and phys. [0.35]

Ex.11: Suppose 1 in 100 products has a certain defect. A test detects the defect in 95 % of defective products. The test also “detects” the defect in 10 % of non-defective products. With what probability does a positive test diagnose a defect?

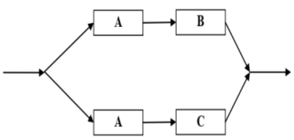
Ex.12: Suppose 1 in 2000 persons has a certain disease. A test detects the disease in 90 % of diseased persons. The test also “detects” the disease in 5 % of healthy persons. With what probability does a positive test diagnose the disease?

Ex.13: The probability that components (or people or circuits or whatever) A, B working is 0.8, 0.9 respectively. The probability that component C failing is 0.4, and the probability of component D failing is 0.15. Each A, B, C, D is independent. Calculate probability the system works.

1)



2)



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