Parabolic Cover Curve Calculation Methodology

Finding the equation of the tangent of the general equation of a curve set is a solved problem in mathematic. Then, in rock mechanics, if we consider the system equations of Mohr's circles as a general differential equation, the unusual answer of this differential equation is the equation of cover curve (or failure envelop) of Mohr circles which is known as failure criterion.

In the mathematics, the abnormal (or unusual) answer (or solution) of the first-order differential equation is a curve that is tangent to all curves generated from general equation. If assume that the $F(x, y, u) = 0$ is a general equation of a curve sets, the equation of the cover or tangent of these curve sets could be calculated by eliminating parameter u from following system of equations:

$$\begin{Bmatrix} F(x, y, u) = 0 \\ \dfrac{\partial F}{\partial u} = 0 \end{Bmatrix} \tag{1}$$

Note, in the cases that the elimination of u from system of equations is impossible, the equation of the cover curve could be parametrically demonstrated as a function of u. In addition, sometimes the equation obtained from solving the system equations demonstrated in Eq. 1 is not the tangent of the curve sets and sometimes it could result just a small part of the cover curve.

Now, the procedure will be demonstrated by an example. Assume that the Eq. 2 is a general equation of circle sets. Let's to find the equations of possible cover curves which are tangent to all circles, just in one point for each circle.

$$(x - u)^2 + y^2 = 4 \tag{2}$$

For this purpose, first derive the Eq. 2 in terms of u:

$$\frac{\partial F}{\partial u} = 0 \Rightarrow \qquad -2(x - u) = 0 \Rightarrow \qquad x = u \tag{3}$$

Then the Eq. 1 is equal to:

$$\begin{Bmatrix} (x - u)^2 + y^2 = 4 \\ x = u \end{Bmatrix} \Rightarrow \qquad y^2 = 4 \Rightarrow \qquad y = \pm 2 \tag{4}$$

As it shown in Fig. 1, lines by equations y=2 and y=-2 are tangent to all circles and just touch the circles in one point for each circle.

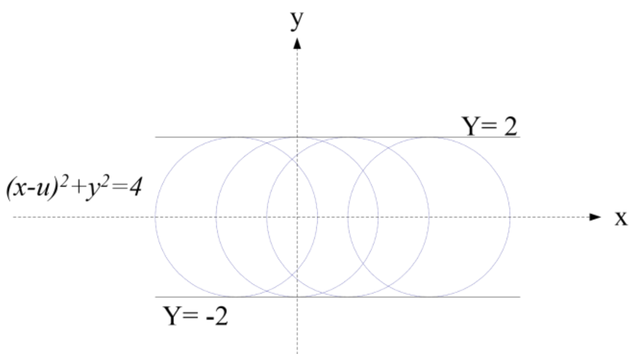

Fig. 1 Tangents of the circle sets by general equation of *(x-u)2+y2*=4.

Results obtained from triaxial experiments in rock mechanics typically show that if a rock is tested under confined condition, its strength increases with increasing confining pressure. The rate of increase in strength is high at low confining pressure. In fact, it can be said that σ_1 at failure increases at a less-than-linear rate with σ_3. So, generally, as confining pressure is increased, the rate of increase in strength decreases and non-linear increase of the sizes of circles shown in Fig. 2 illustrates this fact. Therefore, in order to provide an accurate failure criterion, Mohr (1900) suggested that it is better to use possibly nonlinear relations, such as curve form of the Fig. 2a and this curve can be determined experimentally as the envelope of all of the Mohr's circles that correspond to states of stress that cause failure (Jaeger et al. 2009, Singh M & Singh B 2012).

$$|\tau| = f\ (\sigma) \tag{5}$$

Based on this hypothesis, specifically, failure is supposed to occur if one of the Mohr's circles touches the curve defined by Eq. 5. In addition, as shown in Fig. 2b, this will necessarily occur for the circle defined by σ_1 and σ_3, and so the value of the intermediate principal stress is not expected to affect the onset of failure (Jaeger et al. 2009). In this research it is found that, aforementioned mathematical procedure could accurately provide the equation of failure envelope of Mohr circles.

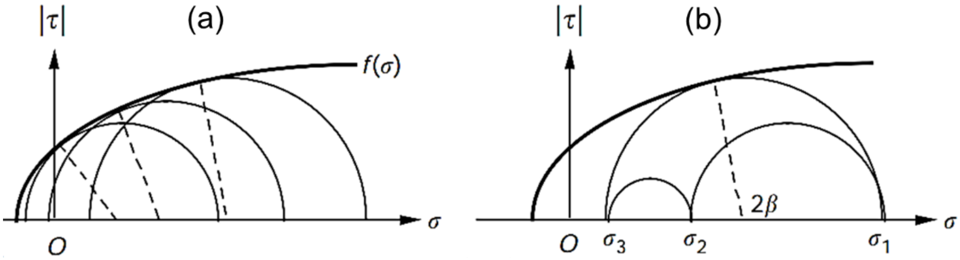

Fig. 2 (a) Nonlinear failure curve, defined as the envelope of all Mohr circles that cause failure. (b) According to the Mohr hypothesis, the intermediate principal stress does not influence the onset of failure (Jaeger et al. 2009).

Here, the failure criterion (Eq. 5) is determined by plotting the Mohr's circles for the stresses at failure, as found in a series of triaxial tests conducted under different confining stresses. As shown in Fig. 2a, each failure circle has a different centroid and radius than other circles. Then, the general equation of Mohr's circles could be written as:

$$(\sigma - C)^2 + \tau^2 = R^2 \tag{6}$$

Where C is the centroid of circle
R is the radios of circles

Now, the failure curve will be given by the envelope of these circles (Fig. 2a). For this purpose, the Eq. 6 should be written as system of equations Eq. 1, But we know that the general equation of Mohr's circles is in the form of $F\ (\sigma,\ \tau,\ C,\ R^2) = 0$. In order to find the envelope of these circles, first the general equation should be converted to the form of $F\ (\sigma,\ \tau,\ u) = 0$. In this regard, we know that the C (centroid of the failure circles) and R^2 (square of the radius of the failure circles) at failure are dependent to σ_3 and σ_1. Therefore, it is possible to find one of theme to the function of the other. For this purpose, the parameters C and R^2 for each failure state were calculated by using corresponding σ_3 and σ_1. Then, the results were plotted in the $\{C, R^2\}$ plane and equations of linear regression were derived for the determination of the relation between C and R^2. Investigations demonstrated that the correlation of these two parameters is suitably linear (Coefficient of Determination r^2>0.9). So, finally the linear correlation of C and R^2 parametrically could be written as follow:

$$R^2 = A \times C + B \tag{7}$$

Where, A and B are the coefficients of linear regression.

By substituting Eq. 7 to Eq. 6 it can be rewritten as:

$$(\sigma - C)^2 + \tau^2 = A \times C + B \tag{8}$$

Now, by renaming C to u in the Eq. 8 It can be seen that the general equation of Mohr's circles could be written in the form of $F(\sigma, \tau, u) = 0$:

$$(\sigma - u)^2 + \tau^2 = A \times u + B \tag{9}$$

As we mentioned before, the equation of the envelope or the tangent of circles could be calculated by eliminating parameter u from Eq. 1 as follow:

$$\begin{gathered}\begin{Bmatrix} F(\sigma,\tau,u) = 0 \\ \dfrac{\partial F}{\partial u} = 0 \end{Bmatrix} \Rightarrow \begin{Bmatrix} \tau^2 + (\sigma - u)^2 - (A \times u + B) = 0 \\ -2(\sigma - u) - A = 0 \end{Bmatrix} \\ \Rightarrow \quad u = \frac{A + 2\sigma}{2} \end{gathered} \tag{10}$$

By substituting value of u obtained from Eq. 10 to Eq. 9 a new non-linear Mohr's failure envelope can be expressed by a parabolic equation as follow:

$$\tau = \sqrt{A.\sigma + \left(\frac{A^2}{4} + B\right)} \tag{11}$$

Where A and B are the coefficients of linear regression between C and R^2 obtained from at least three number of triaxial test data.

All of the procedure is demonstrated by solving an example and is summarized in Fig. 3. Note that, the system of equations was solved by developing a computer code open source program which is freely available for download from https://www.researchgate.net/publication/334289138_Mohr_Circles_Non-Linear_Cover_Curve_for_results_obtained_from_Triaxial_test_on_Intact_Rocks.

Triaxial Test Data

σ_3	σ_1	C	R^2
0	60	30	900
2	70	36	1156
4	85	44.5	1640.25
8	98	53	2025
10	110	60	2500

$$C = \frac{\sigma_1 + \sigma_3}{2} \qquad R = \frac{\sigma_1 - \sigma_3}{2}$$

$$R^2 = A \times C + B$$

$$\tau = \sqrt{A.\sigma + \left(\frac{A^2}{4} + B\right)}$$

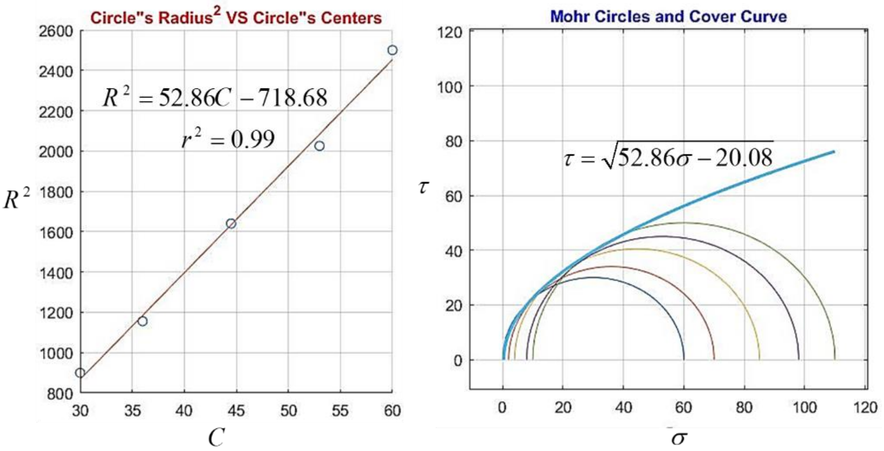

Fig. 3 Triaxial test data obtained from Limestone specimens (Mehrishal et al. 2017) and calculation of the equation of Mohr's circles envelop.