Bermudan swaption pricing

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1 Abstract

This paper focuses on the pricing of Bermudan swaptions, a type of interest rate derivative with multiple exercise dates. Unlike European swaptions, Bermudan swaptions do not have a closed-form solution, which necessitates the use of numerical techniques for accurate pricing. In this project, we employ a one-factor Hull-White model to replicate the dynamics of interest rates in a discrete lattice. Through backward induction, we calculate the option's value at each exercise point, tracing the optimal exercise strategy. This approach is essential for pricing complex derivatives like Bermudan swaptions, given the stochastic nature of interest rates and the lack of analytical solutions.

2 Introduction

Pricing derivatives like options requires a model that can accurately capture the underlying asset's behavior over time. For simpler options, such as European options, which have a single exercise date, closed-form solutions like the Black-Scholes model offer a straightforward way to determine prices. However, when it comes to more complex instruments like Bermudan swaptions, which have multiple exercise dates, pricing becomes significantly more challenging.

Bermudan swaptions are a type of option that grants the holder the flexibility to exercise the option at several predefined points before maturity, which makes their valuation complex. Closed-form solutions for such options do not exist due to the path-dependent nature of their payoff structure and the need to account for interest rate volatility. The challenge lies in modeling these interest rate dynamics in a way that reflects both their stochastic nature and the flexibility of early exercise decisions.

In this paper, we address these challenges by using a combination of the Hull-White one-factor model and a trinomial lattice. The Hull-White model, with its mean-reverting feature, provides a flexible framework for modeling interest rates, making it suitable for capturing the complex dynamics needed for Bermudan swaption pricing. The use of a trinomial lattice allows for the construction of a discrete-time representation of interest rate movements, which is crucial for handling the multiple exercise dates of Bermudan swaptions. By applying backward induction within the lattice framework, we are able to calculate the value of the swaption at each possible exercise point, ultimately determining the optimal strategy for the holder. This combination of techniques offers an effective way to tackle the complexities of Bermudan swaption pricing.

3 Relevant literature

Interest Rate Swaps and Their Derivatives: A Practitioner's Guide by Amir Sadr offers a detailed, step-by-step methodology for pricing complex instruments like Bermudan swaptions, laying out the necessary techniques for building models based on interest rate dynamics. It provides comprehensive guidance on applying models such as the Hull-White one-factor model in combination with trinomial lattice techniques for accurate valuation.

4 Methodology

4.1 The Hull-White Model

The Hull-White model is a widely used extension of the Vasicek model that allows for the calibration of interest rate models to current market conditions. This flexibility comes from the time-dependence of its parameters, which helps in fitting the initial term structure of interest rates. The stochastic differential equation (SDE) governing the one-factor Hull-White model is given by:

$$dr(t, \omega) = a(t)[b(t) - r(t, \omega)]dt + \sigma(t)dB(t, \omega)$$

where:

- $r(t, \omega)$ is the short rate at time t,
- $a(t)$ is the mean reversion speed, controlling how quickly the short rate reverts to the long-term mean,
- $b(t)$ is the long-term mean of the short rate, or the level to which the short rate reverts over time,
- $\sigma(t)$ represents the volatility of the interest rate, driving the magnitude of fluctuations due to random shocks,
- $dB(t, \omega)$ is a Wiener process, representing the stochastic or random component of the model.

The mean-reversion term $a(t)[b(t) - r(t, \omega)]dt$ ensures that the short rate drifts towards the long-term mean $b(t)$, with $a(t)$ dictating the speed of this reversion. The term $\sigma(t)dB(t, \omega)$ introduces randomness into the short rate, allowing for the uncertainty and volatility seen in real-world interest rate movements. Together, these terms capture both the deterministic mean-reversion behavior and the stochastic variability of interest rates.

4.2 Parameter Estimation

The parameters of the Hull-White model, namely $a(t)$, $b(t)$, and $\sigma(t)$, are critical for accurately describing the evolution of the short rate. However, these parameters are not directly observable and must be estimated from historical data.

To estimate the volatility parameter, $\sigma(t)$, we analyze the historical movements of interest rates. Volatility is typically derived from the variance of the log returns of the interest rate data, providing an empirical measure of how much the rates fluctuate over time.

The mean-reversion speed, $a(t)$, and the long-term mean, $b(t)$, are estimated through statistical regression methods. By analyzing how the short rate changes over time and comparing these changes to the rate's lagged values, we can determine how quickly the rate tends to revert to its long-term average and what that average level is.

Once these parameters are estimated, we can calibrate the Hull-White model to fit the historical data and market conditions. The calibrated model can then be used to generate a lattice that reflects possible future paths of interest rates, forming the basis for pricing interest rate derivatives like Bermudan swaptions.

4.3 Lattice Construction

The Bermudan swaption is priced using a trinomial lattice to model the evolution of interest rates. The lattice is constructed using the Hull-White model, which allows for mean-reverting stochastic behavior in the short rate. The lattice has $n + 1$ columns, where n is the number of time steps. The central branch evolves as the short rate reverts towards the long-term mean, with each node representing a potential future short rate. The movement in the lattice is governed by the formula:

$$dr(t) = a(b - r(t))\Delta t + \sigma\sqrt{\Delta t}dW$$

where a is the mean-reversion speed, b is the long-term mean, σ is the volatility, and dW is the Wiener process. The up, down, and central movements in the lattice are determined by the term $dr_i = \sigma\sqrt{3\Delta t}$, ensuring that the lattice captures the full range of potential interest rate changes.

The trinomial structure allows for both upward, downward, and unchanged movements, providing a robust representation of possible interest rate paths. The short rates at each node are used later for backward induction to price the swaption.

4.4 Arrow-Debreu Prices

In the backward induction algorithm, Arrow-Debreu prices are crucial for determining the continuation value of the Bermudan swaption at each node. The Arrow-Debreu price represents the price today of receiving one unit of currency

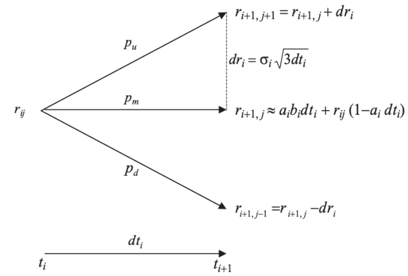

Figure 1: Construction of a lattice using HW model

at a future node, conditional on reaching that node. These prices are approximated using the discount factors computed from the short rates in the lattice.

At each step in the backward induction, we compute the Arrow-Debreu prices for upward, downward, and central movements. These prices help calculate the continuation value, which represents the expected value of holding the option and moving to the next time step. The Arrow-Debreu price at time t for a node is computed as:

$$AD(t, \omega) = \frac{1}{1 + r(t, \omega)\Delta t}$$

These prices are then used to compute the continuation value by weighting the value at the next nodes by the corresponding Arrow-Debreu prices.

4.5 Value of the Bermudan Swaption

The value of the Bermudan swaption is determined using backward induction, starting from the final time step and moving back to the present. At the last exercise date, the value of the swaption is the payoff from exercising the option, given by the difference between the swap rate K and the remaining swap value $S(t, \omega)$. The immediate exercise value at each node is computed as:

$$C(t, \omega) = \max(0, S(t, \omega))$$

At earlier time steps, the value of the option is the maximum of the immediate exercise value and the continuation value, computed using the Arrow-Debreu prices. This ensures that the holder of the option exercises optimally, either choosing to hold the option or exercise it at each time step. The Bermudan swaption value at the initial node $B(0, 0)$ is the price of the option today, given by:

$$B(0, 0) = \max(C(0, 0), H(0, 0))$$

where $H(0, 0)$ is the hold value computed using backward induction with the Arrow-Debreu prices. The final value of the Bermudan swaption reflects the flexibility of multiple exercise dates and the uncertainty in future interest rates.

5 Data

To validate the parameter estimation of the Hull-White model, we conducted a test using historical data of federal fund rates. The data was fetched from the Federal Reserve Economic Data (FRED) system, covering the period from January 2010 to the present.

6 Implementation

6.1 Parameters estimation

In this study, we opted for the estimation of unconditional parameters for the Hull-White model, given the challenges associated with calibrating a fully conditional model. The parameters a, b, and σ are not directly observable and must be inferred from historical interest rate data.

The mean-reversion speed, a, the long-term mean b, and the volatility parameter *sigma* are estimated through a maximum likelihood estimation (MLE) process. In particular, the short rate is modeled as reverting to its long-term mean over time at a rate governed by a. This estimation method focuses on fitting the model parameters so that the observed changes in interest rates over time match those predicted by the Hull-White process.

We estimate the parameters of the Vasicek, specifically the speed of reversion (a), the long-term mean rate (b), and the volatility (σ). We define the log-likelihood function for the Vasicek model as follows:

$$\mathcal{L}(\theta) = -\frac{1}{2} \sum_{i=1}^{n} \left[\log(2\pi\sigma_v^2) + \frac{(r_i - \mu_i)^2}{\sigma_v^2} \right] \tag{1}$$

where $\mu_i = b + (r_{i-1} - b)\exp(-a\Delta t)$ and $\sigma_v = \sqrt{\left(\frac{\sigma^2}{2a}\right)(1 - \exp(-2a\Delta t))}$.

6.2 Estimation testing

To assess the accuracy and significance of these estimates, standard errors and t-statistics were calculated. The Hessian matrix, which is the second derivative of the likelihood function, was numerically approximated using finite differences. This allowed for the computation of the variance-covariance matrix, from which standard errors and t-statistics were derived.

The results indicate that the parameters provide a good fit to the historical data, with reasonably low standard errors and t-statistics that confirm the significance of the estimated parameters. These results suggest that the unconditional Hull-White parameters can effectively capture the dynamics of interest rate evolution and can be reliably used for further model calibration and derivative pricing.

6.3 Backward Induction for 1-Year Bermudan Swaption with Quarterly Coupons

To price the Bermudan swaption, we will use backward induction on the interest rate lattice. We are pricing a 1-year Bermudan swaption with quarterly coupons and quarterly exercise dates starting in 3 months. The fixed rate of the swaption is $K = 5.5\%$, and we assume quarterly cash flows.

6.3.1 Step 1: Initial Setup and Interest Rate Lattice

We construct an interest rate lattice using the Hull-White model, with parameters for mean reversion (a), long-term mean (b), volatility (σ), and initial rate r_0. The lattice evolves over four time steps, corresponding to the quarterly periods in the 1-year horizon.

At each node in the lattice, we calculate the short rate $r(t, \omega)$ and the discount factor for future cash flows. The lattice is initialized with the short rate r_0, and rates evolve upward, downward, or remain unchanged at each time step.

6.3.2 Step 2: Payoff at Final Exercise Date

At the last exercise date t_4 (1 year), we compute the value of the Bermudan swaption based on the immediate exercise value. At each node, the immediate exercise value is the value of receiving $K = 5.5\%$ for the remaining swap.

The remaining swap value $S(t, \omega)$ is given by:

$$S(t_4, \omega_j) = \frac{K}{4} \left(1 - D(t_4, t_4 + 0.25, \omega_j)\right)$$

where $D(t_4, t_4+0.25, \omega_j)$ is the discount factor at the final node. The Bermudan option value at t_4 is set equal to the immediate exercise value:

$$B(t_4, \omega_j) = \max(0, S(t_4, \omega_j))$$

6.3.3 Step 3: Backward Induction for Earlier Dates

We now apply backward induction to calculate the swaption value at each earlier time step. Starting from $t_3 = 9$ months, at each node (t_3, ω_j), we calculate both the immediate exercise value and the continuation value.

The immediate exercise value at time t_3 is:

$$C(t_3, \omega_j) = \frac{K}{4} \left(1 - D(t_3, t_3 + 0.25, \omega_j)\right)$$

Next, we compute the continuation value by discounting the expected value from future nodes using Arrow-Debreu prices (approximated as discount factors):

$$H(t_3, \omega_j) = \frac{1}{1 + r(t_3, \omega_j)\Delta t} \left(\frac{B(t_4, \omega_{j-1})}{4} + \frac{B(t_4, \omega_j)}{4} + \frac{B(t_4, \omega_{j+1})}{4} \right)$$

The Bermudan option value at each node is the maximum of the immediate exercise value and the continuation value:

$$B(t_3, \omega_j) = \max(C(t_3, \omega_j), H(t_3, \omega_j))$$

6.3.4 Step 4: Final Value at Time $t = 0$

We continue this backward induction process through all time steps $t_2 = 6$ months, $t_1 = 3$ months, and finally $t_0 = 0$ months. At each node, we compute the immediate exercise value and the continuation value as shown above.

At the initial node $t_0 = 0$, the value of the Bermudan swaption is the option value at that point:

$$B(t_0, \omega_0) = \max(C(t_0, \omega_0), H(t_0, \omega_0))$$

This value represents the price of the Bermudan swaption today.

7 Results

7.1 Estimating the parameters

Using the estimation methods outlined earlier, we derived the key parameters of the Hull-White model. The mean-reversion speed (α), long-term mean (b), and volatility (σ) were estimated as follows:

7.1.1 Parameters estimation

Parameter	**Estimate**	**Std. Err.**	**T-stat**
a	0.072238	0.134898	0.53550
b	0.045204	0.064555	0.70024
σ	0.010676	0.000087	122.86434

Table 1: Estimation results for parameters a, b, and σ.

7.2 Lattice

The interest rate lattice constructed for this Bermudan swaption pricing example shows the evolution of short rates over time. Starting from a neutral rate at time t_0, the rates evolve through up, down, and neutral branches as we move forward in time. As shown in the table, the short rates remain positive throughout the lattice, reflecting a relatively stable interest rate environment.

This outcome is driven by the Hull-White model, which incorporates both upward and downward movements in interest rates based on the parameters estimated from historical data. The relatively modest volatility (σ) in the model limits extreme fluctuations, keeping the short rates positive across all time steps.

				0.081328
			0.072082	0.072098
		0.062836	0.062851	0.062867
	0.053590	0.053605	0.053621	0.053636
0.044344	0.044360	0.044375	0.044390	0.044404
	0.035098	0.035113	0.035129	0.035144
		0.025852	0.025868	0.025883
			0.016606	0.016621
				0.007360

Figure 2: Trinomial lattice

As a result, the lattice does not exhibit negative interest rates, unlike in scenarios with higher volatility where rates may dip below zero.

While negative interest rates are possible in certain economic environments, particularly in the case of high volatility or aggressive monetary policies, the current model shows a controlled range of rate movements, avoiding significant downward shifts. This stability suggests that the short rate dynamics, as captured in the model, reflect an environment with moderate economic uncertainty and controlled volatility.

7.3 Bermudan swaption value

The Bermudan swaption, resulted in a volatility price of 0.00605. This value represents the implied volatility of the option, which is a crucial metric for determining the likelihood of future rate movements that would make exercising the option profitable. In the context of interest rate options, this volatility price reflects the market's expectations of how much the underlying interest rates will fluctuate over the life of the swaption.

A low volatility value, as observed here, suggests that the market anticipates relatively small fluctuations in interest rates around the strike rate of 5.5%. This indicates that the option's intrinsic value is limited, as the opportunity to exercise the swaption profitably is expected to be low. If interest rates remain close to 5.5%, the swaption may not provide a significant benefit to the holder.

Volatility also encapsulates the uncertainty in future interest rate movements. In this case, a volatility price of 0.00605 implies that the market does not expect drastic changes in rates for a given time horizon, reducing the potential upside of the Bermudan swaption. However, the Bermudan structure still offers flexibility, allowing early exercise at multiple dates, which adds some value to the option despite the low expected volatility.

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