EMF Projet 2

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1 Static allocation

1.1 Q1.1

To solve for the optimal portfolio weights $\tilde{\alpha} = (\tilde{\alpha}_s, \tilde{\alpha}_b)'$, we need to compute the first-order derivative of the mean-variance criterion with respect to $\tilde{\alpha}$.

$$\mathcal{L} = \tilde{\alpha}'\mu + (1 - \mathbf{e}'\tilde{\alpha})R_f - \frac{\lambda}{2}\tilde{\alpha}'\Sigma\tilde{\alpha}$$

$$\frac{\partial \mathcal{L}}{\partial \tilde{\alpha}} = \mu - eRf - \lambda\Sigma\tilde{\alpha} = 0$$

$$\alpha^* = \frac{\Sigma^{-1}(\mu - eR_f)}{\lambda}$$

1.2 Q1.2

The optimal weight vector, denoted by $\tilde{\alpha}^*$, for $\lambda = 2$ and $\lambda = 10$ are:

	$\lambda = 2$	$\lambda = 10$
Stocks	1.2198	0.2440
Bonds	6.1228	1.2246
Cash	-6.3426	-0.46851

2 Estimation of a GARCH Model

2.1

By performing a Kolmogorov-Smirnov test on the simple and squared excess returns (standardized by subtracting the mean and dividing by the standard deviation), we get the following results:

	SMI	SMI (sq)	Bond	Bond (sq)
Test statistic	0.0834***	0.1370***	0.0576***	0.1045***
p-value	0.0000	0.0000	0.0007	0.0000
Critical value	-0.3199	0.4679	-0.4807	0.4918

Table 1: Kolmogorov-Smirnov Test

In Table 1, "sq." stands for the squared returns. The Kolmogorov-Smirnov test compares the actual distribution of the data to the normal distribution for excess returns and to the Chi-square distribution for squared excess returns.

The test statistic is the biggest difference between the actual and assumed distributions. If this difference is bigger than a certain critical value, we reject the idea that the data follows the assumed distribution. The results in Table 1 show that the test is always rejected at the 1% level, meaning our returns do not follow a normal distribution. This means that our data does not match what we would expect if it were normally distributed.

Concerning auto-correlations, we computed Ljung-Box tests to verify if there is significant auto-correlation in excess or squared excess returns using up to 4 lags. The results indicate that for stock excess returns, there is significant auto-correlation at all lags, with test statistics significant at the 1% alpha threshold. This strong evidence suggests predictability in stock returns.

Similarly, the squared stock excess returns show significant auto-correlation at all lags, indicating strong predictability in volatility. Since squared innovations follow a Chi-squared distribution with an expected value of 1, squared excess returns serve as a proxy for volatility, confirming the presence of volatility clustering.

For bond excess returns, the Ljung-Box test results show no significant auto-correlation at any lag, with p-values much greater than 0.05. This indicates no evidence of auto-correlation in bond returns. However, squared bond excess returns reveal significant auto-correlation at all lags, suggesting strong predictability in bond volatility.

In summary, the Ljung-Box test results demonstrate significant auto-correlation in both excess returns and squared excess returns for stocks, but only in the squared excess returns for bonds. These findings justify using an AR(1) model to capture auto-correlation in returns and a GARCH model to account for time-varying volatility.

Lag	Autocorrelation	Test statistic	p-value	Critical value
1	-0.1339	21.5552	0.0000	3.8415
2	0.0669	26.9460	0.0000	5.9915
3	-0.1036	39.8566	0.0000	7.8147
4	0.0167	40.1931	0.0000	9.4877

Table 2: Auto-correlations on excess returns - Stock

Lag	Autocorrelation	Test statistic	p-value	Critical value
1	0.2936	103.6426	0.0000	3.8415
2	0.1674	137.3752	0.0000	5.9915
3	0.0978	148.9066	0.0000	7.8147
4	0.0367	150.5340	0.0000	9.4877

Table 3: Auto-correlations on squared excess returns - Stock

Lag	Autocorrelation	Test statistic	p-value	Critical value
1	-0.0113	0.1560	0.6928	3.8415
2	0.0057	0.1961	0.9065	5.9915
3	-0.0134	0.4126	0.9376	7.8147
4	-0.0046	0.4389	0.9792	9.4877

Table 4: Auto-correlations on excess returns - Bond

Lag	Autocorrelation	Test statistic	p-value	Critical value
1	0.1517	27.6569	0.0000	3.8415
2	0.1322	48.6942	0.0000	5.9915
3	0.0795	56.3001	0.0000	7.8147
4	0.1083	70.4218	0.0000	9.4877

Table 5: Auto-correlations on squared excess returns - Bond

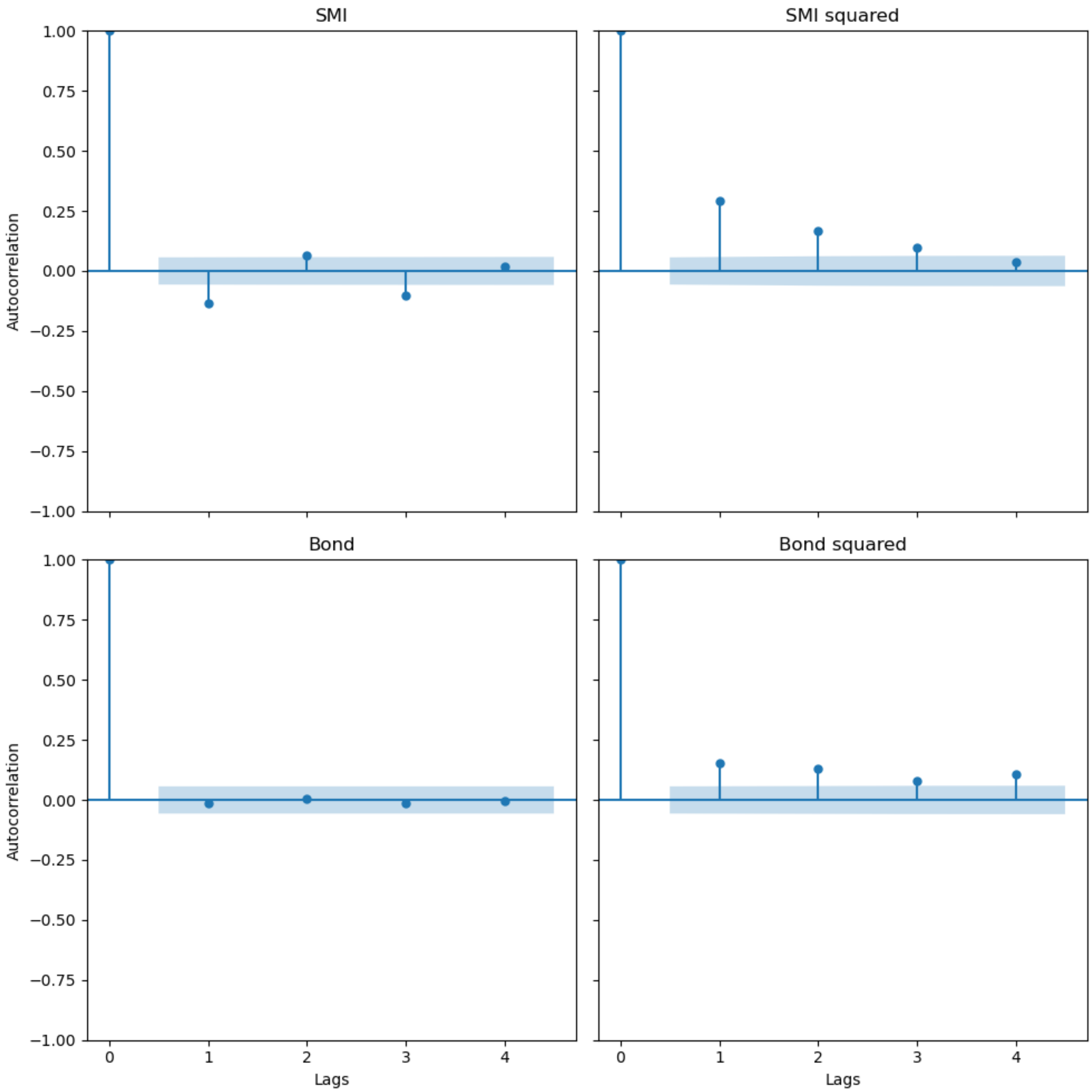

Figure 1: Auto-correlations on (squared) excess returns

2.2

As the results show, the intercept for stock returns is not significant, while the lagged return coefficient is highly significant. This indicates a mean-reverting behavior in stock returns, suggesting some degree of

	SMI	**Bond**
a_i	0.0013	0.0006**
(a_i)	(0.001)	(0.000)
ρ_i	-0.1352***	-0.0093
(ρ_i)	(0.029)	(0.029)
R^2	0.018	0.000
$\bar{R}^2$	0.017	-0.001

Table 6: AR(1) model estimation results for stock and bond returns

predictability. However, the R-squared value is very low at 0.018, implying that the AR(1) model does not explain much of the variability in stock returns. This is expected in financial return series, which are typically volatile and unpredictable.

For bond returns, the intercept is significant, but the lagged return coefficient is not. This indicates that past bond returns do not predict future bond returns. The R-squared value is effectively zero, further supporting the idea that bond returns are more random and less predictable compared to stock returns.

These results shows that stock returns can exhibit some degree of predictability due to mean reversion, while bond returns are generally less predictable. However, as shown in previous analyses, the unknown part of the return or squared excess return might be predictable.

2.3

	SMI	**Bond**
μ	0.0013	0.0000
	(0.0007)	(0.0000)
ω	0.0000***	0.0000***
	(0.0000)	(0.0000)
α_1	0.2000***	0.0500***
	(0.0531)	(0.0123)
β_1	0.7000***	0.9300***
	(0.0576)	(0.0111)
$\alpha_1 + \beta_1$	0.9000	0.9800
Wald Statistic	3.3392	21.3526
p-value	0.0675	0.0000

Table 7: GARCH(1,1) model estimation results for stock and bond residuals

The GARCH(1,1) model estimation results for the SMI and bond residuals, as shown in Table 7, show important aspects of volatility. For the SMI, the constant μ is positive but not significant, while for bonds, it is close to 0. Both ω and the parameter α_1 are significant for both SMI and bonds, showing the immediate impact of shocks on volatility. The GARCH parameter β_1 is also significant, showing the persistence of volatility.

The sum of α_1 and β_1 is 0.9000 for SMI and 0.9800 for bonds, suggesting that volatility is very persistent, especially for bonds. The Wald test for $\alpha_1 + \beta_1 = 1$ gives a p-value of 0.0675 for SMI, meaning we do not reject the null hypothesis at the 5% level. This means the volatility for SMI is very persistent. For bonds, the p-value is 0, so we reject the null hypothesis, meaning bond volatility is less persistent and tends to revert to the mean over time.

3 Dynamic allocation

3.1

The following figure shows the optimal weights for both the static approach and the new dynamic approach with time-varying volatility. For both $\lambda = 2$ and $\lambda = 10$, the optimal weights for the dynamic approach are very volatile and fluctuate around the static weights. The weights for the bond are particularly volatile especially for $\lambda = 2$ and are sometimes exceeding 20. The weights for $\lambda = 10$, despite also being very volatile, are more conservative than for $\lambda = 2$, indicating that a lower risk aversion level results in more aggressive reallocations. Overall, the dynamic weights are extreme and seem to be too risky to be applied in practice.

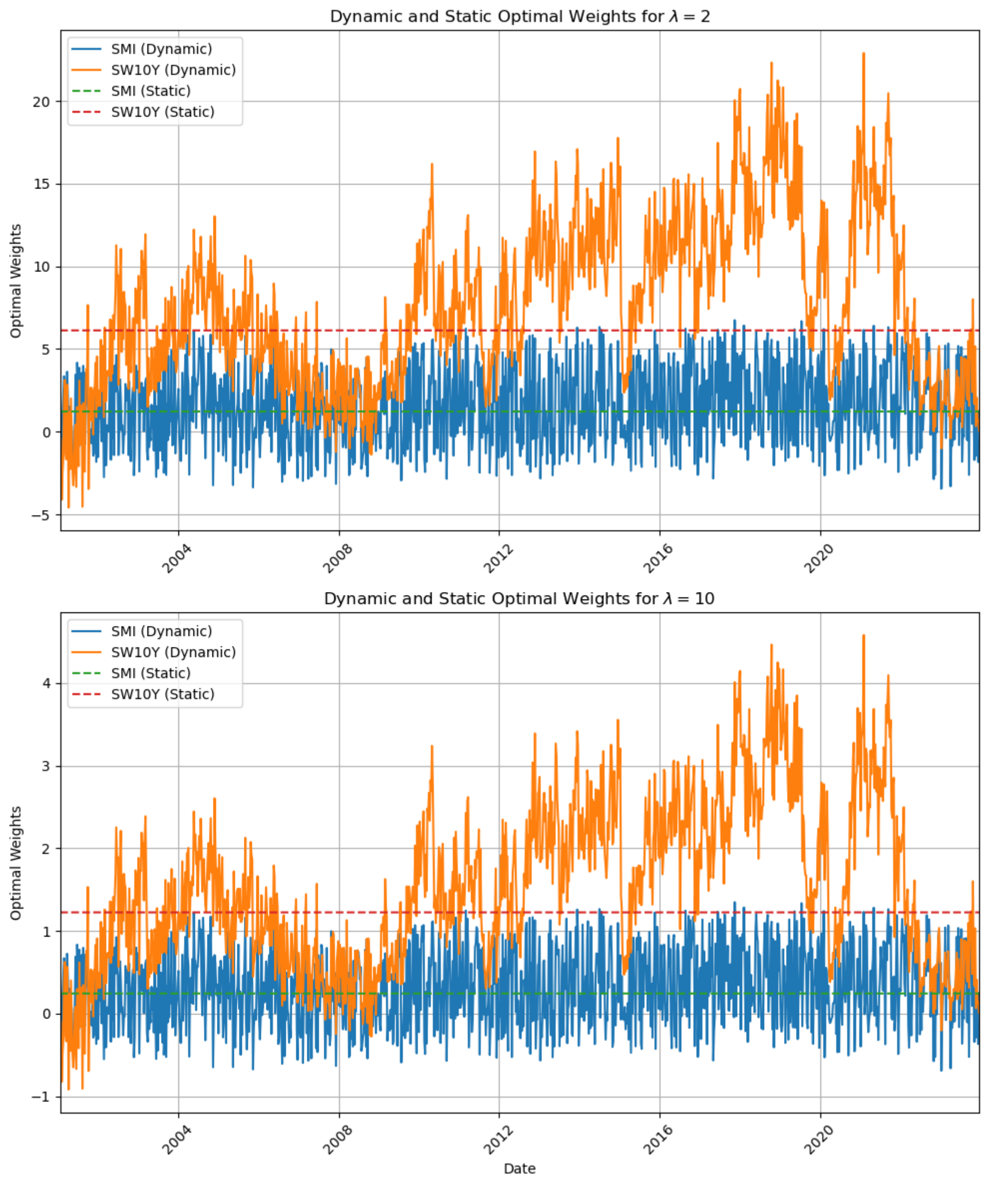

Figure 2: Dynamic and Static Optimal Weights for $\lambda = 2$ and $\lambda = 10$

3.2

Looking at the cumulative returns (here we took the log of the portfolio returns because the standard cumulative returns gave us something explosive), we can see that they are higher when the investor is less risk averse, as both the dynamic and static returns are higher for $\lambda = 10$ than for $\lambda = 2$. In general, dynamic allocations seem to have slightly higher returns then the static ones for the same risk aversion, but they come with higher volatility.

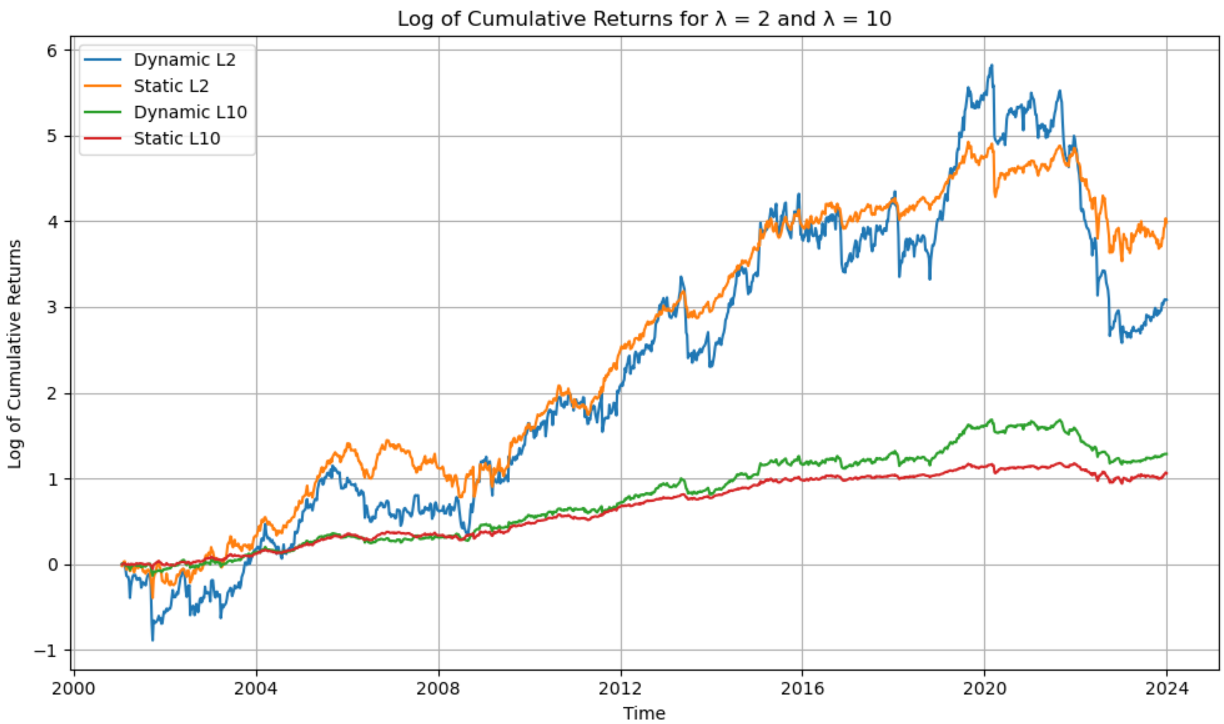

Figure 3: Log of Cumulative Returns for $\lambda = 2$ and $\lambda = 10$

3.3

In order for both strategies to break even, we need a transaction fee of 0,064%. This low transaction cost comes from the fact that we rebalance often with large changes in weights. The addition of transaction costs makes the dynamic approach less attractive because in practice getting transaction costs as low as 0,064% seems very unlikely. The dynamic strategy would benefit from adding limits to the trading activity so we wouldn't rebalance as often.

4 Computing the VaR of a portfolio

4.1

	Dyn. $\lambda = 2$	Dyn. $\lambda = 10$	St. $\lambda = 2$	St. $\lambda = 10$
Mean	-0.0012	-0.0002	-0.0009	-0.0002
Variance	0.0013	0.0000	0.0004	0.0000
Quantile$^{(Uncond)}$	0.1101	0.0220	0.0548	0.0109
VaR$_p^{(Uncond)}$	0.0839	0.0168	0.0476	0.0095

Table 8: Unconditional moments and measures

The mean and the variance of the losses give us the estimated parameters for the assumed normal distribution that the losses follow. Given the distribution, we compute the Value at Risk which is the minimum potential loss a portfolio can suffer in 1 day in the (1 - θ) % worst scenarios, over a given horizon of time. It is unconditional because we assume that the mean and the variance are constant over time.

Our unconditional VaR is higher in the dynamic portfolios. This comes from the higher variance of the losses on these portfolios due to the reallocation, while keeping the same allocation through the whole sample gives us more stable losses. The following figure and table tell us how many time the losses exceeded our metric, which should be 1% if our metric is accurate. Losses are exceeding the metric less often in the static portfolio as they are more stable, while it is much less effective for the dynamic portfolio .

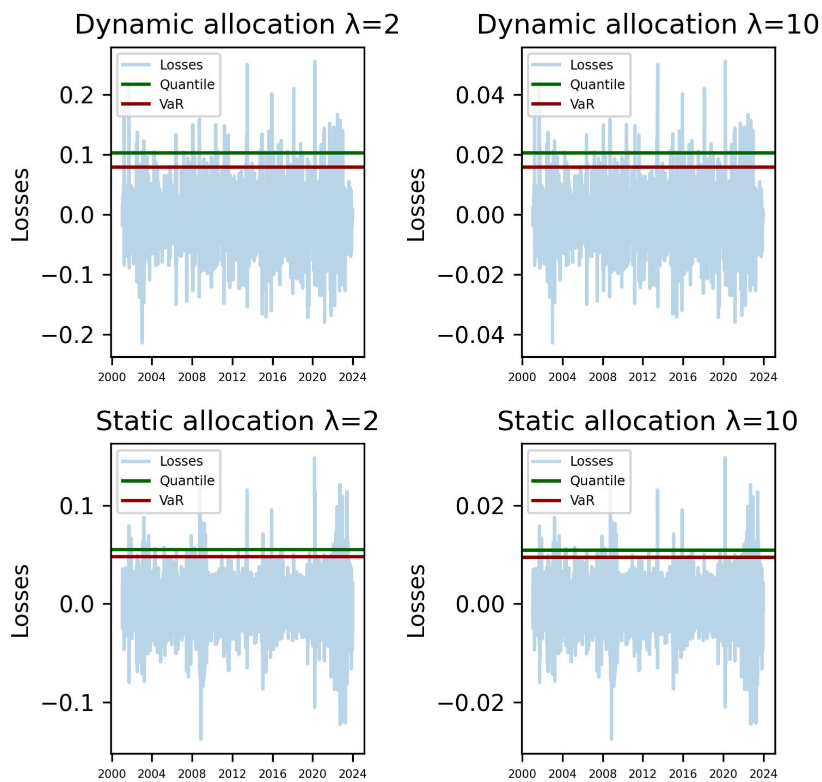

Figure 4: Losses and $VaR_p^{(Uncond)}$ comparison

	Losses over the VaR	Percentage above VaR
Dynamic allocation λ=2	121	2.0217%
Dynamic allocation λ=10	121	2.0217%
Static allocation λ=2	99	1.6541%
Static allocation λ=10	99	1.6541%

Table 9: Losses and $VaR_p^{(Uncond)}$ comparison

4.2

The estimation of AR(1)-GARCH(1,1) gave us the following results :

	Dyn. $\lambda = 2$	Dyn. $\lambda = 10$	St. $\lambda = 2$	St. $\lambda = 10$
a_i	-0.0011^{**}	-0.0002^{**}	-0.0009^{**}	-0.0002^{**}
	(0.0122)	(0.0094)	(0.0010)	(0.0006)
ρ_i	0.0273^{**}	0.0273^{**}	0.0022	0.0021
	(0.0196)	(0.0197)	(0.8634)	(0.8713)
ω_i	1.5649^{**}	0.0623^{**}	0.0741^{**}	0.0030^{**}
	(0.0000)	(0.0000)	(0.0000)	(0.0000)
α_i	0.4135^{**}	0.4132^{**}	0.0727^{**}	0.0729^{**}
	(0.0000)	(0.0000)	(0.0000)	(0.0000)
β_i	0.5864^{**}	0.5867^{**}	0.9099^{**}	0.9097^{**}
	(0.0000)	(0.0000)	(0.0000)	(0.0000)

Table 10: AR(1)-GARCH(1,1) estimators

In this case, we assume that the mean and the variance of the portfolio losses are not constant over time and estimate them using an AR(1)-GARCH(1,1) model.

Looking at the following figure, we observe that the conditional VaR captures much better the market conditions. It is still relatively stable in the static portfolios but spikes as volatility gets higher. It is even more clear when looking at the dynamic portfolio, the losses are better predicted as we are closer to the 1% threshold as shown in Table 11.

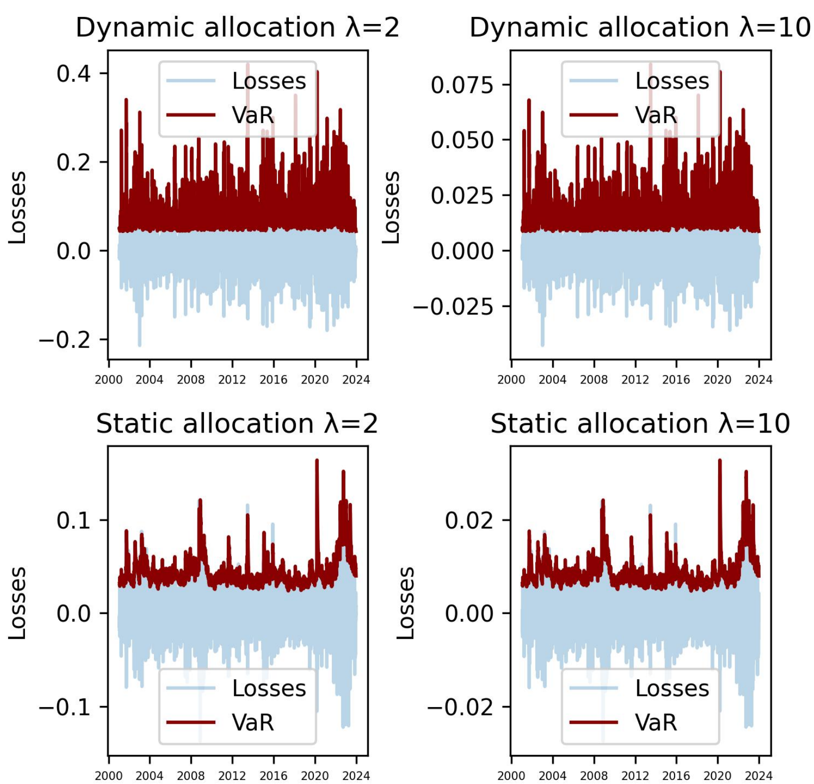

Figure 5: Losses and $VaR_{p,t+1}^{(GARCH)}$ comparison

	Losses over the VaR	Percentage above VaR
Dynamic allocation λ=2	89	1.4873%
Dynamic allocation λ=10	89	1.4873%
Static allocation λ=2	97	1.6210%
Static allocation λ=10	97	1.6210%

Table 11: Losses and $VaR_{p,t+1}^{(GARCH)}$ comparison

4.3

	Dyn. $\lambda = 2$	Dyn. $\lambda = 10$	St. $\lambda = 2$	St. $\lambda = 10$
ξ	0.0989	0.0987	-0.0204	-0.0207
μ	2.2847	2.2855	2.3203	2.3213
ψ	0.7813	0.7830	0.6010	0.6014

Table 12: GEV model parameters estimation

The shape parameter ξ determines the tail behavior of the distribution. Both dynamic portfolios have positive shape parameters ξ, indicating a higher chance of extreme returns, meaning more risk. In contrast, static portfolios have negative ξ values, showing fewer extremes and thus lower risk. The location parameter (μ) is slightly higher for static portfolios, but still very close. This is consistent with previous result shown in part 3. The scale parameter (ψ) shows that dynamic portfolios have more variability, reinforcing their riskier nature. Overall, dynamic portfolios are riskier but potentially more rewarding, while static portfolios are safer and more stable. The effect of changing λ is minimal, slightly increasing variability.

4.4

The quantiles for m_τ and $\hat{z}_{p,t+1}$ are described in the table bellow. Furthermore, we know that the quantile for the extremes m_τ is given by

$$\hat{\mu} + \frac{\hat{\psi}}{\hat{\xi}} \left[(-\log \theta)^{-\hat{\xi}} - 1 \right]$$

while the quantile for $\hat{z}_{p,t+1}$ is given by

$$\hat{\mu} + \frac{\hat{\psi}}{\hat{\xi}} \left[(-N \log \theta^*)^{-\hat{\xi}} - 1 \right]$$

with $N = 60$ in our case and $\theta^* = \theta^{1/N}$.

	Dyn. $\lambda = 2$	Dyn. $\lambda = 10$	St. $\lambda = 2$	St. $\lambda = 10$
m_τ	6.8368	6.8450	4.9597	4.9603
$\hat{z}_{p,t+1}$	2.6899	2.6915	2.6228	2.6339

Table 13: Quantiles for m_τ and $\hat{z}_{p,t+1}$

4.5

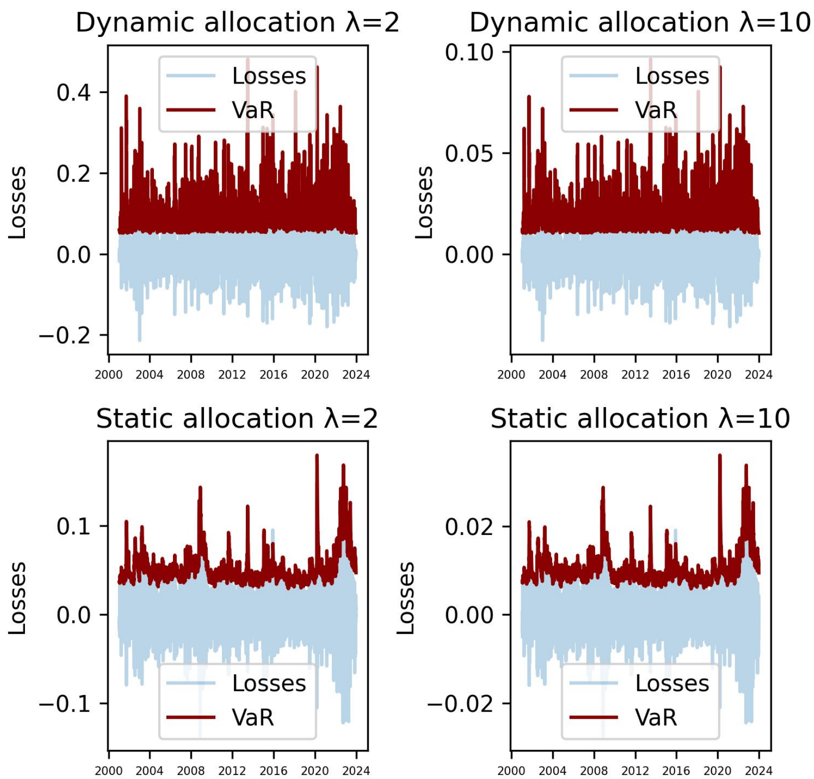

Figure 6: Losses and $VaR_{p,t+1}^{(GEV)}$ comparison

	Losses over the VaR	Percentage above VaR
Dynamic allocation λ=2	58	0.96%
Dynamic allocation λ=10	57	0.95%
Static allocation λ=2	49	0.82%
Static allocation λ=10	49	0.82%

Table 14: Losses and $VaR_{p,t+1}^{(GEV)}$ comparison

4.6

When analysing our results, we see how different measures handle risk management.. The VaR^{Uncon} is clearly the least effective. Although it is a simple model to implement, it performs poorly at capturing the risk in high volatility market conditions, thus in a dynamic allocation. Plus, in-sample testing was conducted, an out-of-sample implementation would likely yield significantly worse results. The VaR^{GARCH} performs better, reflecting more accurately market conditions and assessing risk at any time. It also measures risk better for higher volatility portfolios, as demonstrated with our dynamic portfolio.

The GEV model is the best for calculating VaR because it focuses on tail risks, where the biggest losses happen. It accurately captures extreme events, unlike normal models that often underestimate these risks. Our results have effectively shown that, the VaR^{GEV} being the close to the 1% threshold and even lower, indicating it does not underestimate risk, which is precisely what we aim to avoid in risk management.