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Pricing, assortment, and inventory decisions under nested logit demand and equal profit margins

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ABSTRACT

We consider the interdependent decisions on assortment, inventory and pricing of substitutable products that are differentiated by some primary and secondary attributes captured by a nested logit consumer choice. We examine a newsvendor-type setting with several products competing for demand over a single selling season. We assume that all products have equal profit margins. The demand has a multiplicative-additive structure where both variance and coefficient of variation depend on the common profit margin, which adds to the model applicability at the expense of tractability. Under a Taylor series-type approximation, we show that the expected profit is unimodal in the common margin of products in a given assortment. Then, we compare the optimal profit margin to the case under ample inventory, which allows understanding the effect of inventory on pricing. We also study the optimal assortment problem under exogenous pricing. We show that the classic result on the optimality of popular sets holds under tight approximations of the profit function. We finally propose a heuristic for jointly deciding on assortment, pricing, and inventory decisions, which assumes equal profit margins of products, and exploit popular sets only. Our detailed numerical study shows that this equal-margin heuristic produces high quality solutions.

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Retail management; assortment; pricing; inventory management; nested logit choice; Multi-product newsvendor

1. Introduction

Modern times have seen the rise of the service industry, adopting highly-analytical management approaches, in what is referred to as the post-industrial age, e.g., Wei (2005). Among services, retail, a large part of the US economy and a major employer, has experienced an influx of statistical and mathematical-based methods. This is referred to sometimes as "rocket science retailing" (Fisher, 2009). The emergence of data science is further contributing to the advances in this field, where for example, the funding for retail AI startups is estimated to be more than $2 billion in 2021 (CBInsights, 2021). Beyond reliance on scientific methods and being a useful vehicle for data-driven decision making, the new trend of retail management emphasizes integration of decisions across different divisions of a firm, especially marketing and operations, which has been a key characteristic in the recent literature, as reviewed, for example, in Kök *et al.* (2008), Maddah *et al.* (2011), and Mou *et al.* (2018), as well as being embraced by practitioners (Frantz, 2004; Rao and Saroj, 2021). Another key feature of the emerging retail management paradigm is being "consumer-centric" where understanding the consumer behavior in making purchasing decision is critical (Keltz and Sterneckert, 2009). This is typically captured with demand models that are based on consumer choice and utility theory, which are widely accepted in marketing and economics (Anderson and de Palma, 1992).

The current article contributes to this recent retailing research by developing mathematical models for integrated decision making on the key retail decisions of assortment, pricing, and inventory for a product line of substitutable retail products. The products we consider face uncertain demand and are sold over a single selling season with no opportunity for replenishment during the season. This applies to fashion goods, e.g., clothing and apparel products. The demand function we consider is based on the nested logit consumer choice and allows for capturing the effect of pricing and assortment decisions on demand, as well as a higher degree of heterogeneity among products than what has been reported in most recent works that typically utilize classic logit choice. Our model also allows for capturing more variety on the supply side where we consider products having different unit costs. This is a departure from a common assumption of homogeneous costs which can be justified in many settings. However, there are important cases of selling horizontally differentiated substitutable products with significantly different unit costs. For example, the difference between the costs of national brands and store brands is significant, yet these brands compete side by side for customer demand. The same observation applies to local and imported goods.

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To better understand the rational behind integrated assortment, pricing, and inventory, decision making, consider a customer who enters a store, and chooses among a set of horizontally differentiated products (similar products that serve a common purpose, e.g., wearing a knited sweater) available in the store's assortment. A major determinant that will affect her choice is the retailer's price. Price may not only affect her choice of purchasing her most favorable product, it may also shift her interest into a new product. Pricing a product does not only affect its own demand but also has an impact on the overall demand, and the allocation of demand among different products.

A common approach used in the literature is to assume a multinomial logit consumer choice model (Van Ryzin and Mahajan, 1999; Maddah and Bish, 2007; Aydin and Porteus, 2008; Maddah *et al.*, 2014). However, one downside of using logit choice is the Independence-from-Irrelevant-Alternatives (IIA) pitfall where the ratio of the probabilities of choosing any two alternatives is independent of the availability of a third alternative (McFadden *et al.*, 1977). To avoid IIA, we adopt the more general Nested MultiNomial Logit (NMNL) choice where a customer's purchasing behavior is modeled through a two-step decision process. A costumer first decides on a "nest" of products then they choose a product within this nest (Anderson and de Palma, 1992). For instance, a customer entering a store wanting to buy a knitted sweater first decides on whether to buy a cardigan, a V-neck sweater or a round neck sweater. Each of the three options can be seen as a distinct nest. Once she chooses to buy a cardigan; i.e. she chooses her nest, the costumer then decides on the color of cardigan she wants to purchase. Utilizing classic logit choice in this situation implies that adding a cardigan of a new color to this product line would decrease the purchase probability for a cardigan of a different color by the same percentage that it would decrease that of any other V-neck. This is not the case in practice, where introducing a new cardigan has the most pronounced cannibalization effect on other products (cardigans) in the same sub-set (nest). Nested logit choice closely captures this situation.

In order to simplify the pricing decision, we assume that all the products have the same profit margin. We verify that this assumption yields near-optimal results. Moreover, this pricing rule is asymptotically optimal when the demand volume is large enough, which is equivalent to assuming ample inventory levels. To clarify, we utilize a version of the nested logit model where all the nests have the same price-sensitivity parameters and dissimilarity indices. This version of the nested logit model was first introduced by Anderson and de Palma (1992) and Anderson *et al.* (1992), and continues to be used to date (Kök and Xu, 2011; Alptekinoğlu and Grasas, 2014). For this version of the nested logit model, and in the benchmark case of ample inventory (equivalently make-to-order), which is widely studied, it can be shown that the optimal profit margins are equal (Li and Huh, 2011).

The current article builds on the work done by Maddah *et al.* (2014) where the authors analyze assortment, pricing, and inventory decisions for a category of substitutable products under a "multiplicative-additive" demand model assuming: (i) logit choice; (ii) all products in the category have equal unit cost, and are accordingly priced at the same level; and (iii) a newsvendor-type inventory setting. This article relaxes Assumptions (i) and (ii) of Maddah *et al.* (2014) by adopting a nested logit instead of the classic logit choice and by considering products with heterogeneous costs, which applies in certain important settings, as aforementioned. To keep the pricing decision tractable, we assume that all products have the same profit margin (retail price minus the unit cost). It is worth noting that the structure of the expected profit as a function of the assortment and pricing is significantly more complex under heterogeneous cost than in the typical case of equal unit costs, even under the assumption of equal profit margins.

The main contributions of this article with respect to Maddah *et al.* (2014) are: (i) showing that many structural results continue to hold in our general nested logit/equal profit margins setting; and (ii) investigating how other results are altered in our more general framework. Specifically, for a given assortment, we show that the unimodularity of the profit function still holds in our case, and we derive bounds on the optimal profit margin similar to Maddah *et al.* (2014). We also analytically compare the risky and riskless prices and generalize the conditions in Maddah *et al.* (2014). On the assortment planning with exogenous pricing front, we observe that a Cartesian product of popular sets across nests constitutes an optimal assortment under tight lower and upper bound approximations of the expected profit. A corollary of this result, is that, in the special case of homogeneous unit cost in each nest, popular sets per nest are indeed optimal. This could contradict the optimality of "global popular sets" that cut across all nests (as we observe via counter examples), and deviates from recent results established under classic logit choice (Van Ryzin and Mahajan, 1999; Maddah *et al.*, 2014).

This article also contributes an extension to the work of Maddah and Bish (2007), whereby an Equal Margins Heuristic (EMH) proposed in that work under logit choice is shown to continue to perform well under nested logit choice. Our generalized EMH exploits popular sets within each nest, and assumes an equal profit margin across all nests. In a detailed numerical study, we observe a gap below 0.73% compared with the optimal solution spanning all assortments across all nests and imposing no restriction on the pricing structure.

The remainder of this article is organized as follows. In Section 2, we provide a literature review of recent approaches to study the retailing decisions on pricing, assortment and inventory. In Section 3, we introduce our model and assumptions. In Section 4, we establish structural properties of the expected profit function for a given assortment, and we compare riskless and "risky" profit margins. In Section 5, we examine the structure of the optimal assortment under exogenous pricing. In Section 6, we present the details of the EMH approach. In Section 7, we provide numerical results that illustrate the findings of Sections 4 and 5 and establish the near-optimality of EMH. Finally, in

Section 8, we present our conclusions and highlight future research directions.

2. Literature review

As it affects the effectiveness of companies' decisions on product planning, pricing and control, product substitution has recently received substantial attention in the operations management literature (Shin *et al.*, 2015). In this context, Shin *et al.* (2015) explain three substitution mechanisms: (i) assortment–based substitution, where the customer substitutes the initially preferred product by another one that is newly introduced to the retailer's assortment; (ii) inventory–based substitution, where the customer substitutes in the case of the stock out of initially preferred product; and (iii) price–based substitution, where the customer substitutes her initially preferred product due to change in relative prices of substitutable products. In this article we consider both (i) and (iii), i.e., assortment and price–based substitution, and we embed these in the demand function (via nested logit choice) and in integrated assortment, pricing, and inventory decision models. Considering these substitution mechanisms, the literature has focused on four main areas of decisions that can affect customers' substitution behavior: assortment planning, inventory decision, pricing decision, and capacity planning. In this article, we examine the first three decisions starting with a two-facet approach and then pushing the analysis further to consider the three decisions jointly. First, for a given assortment, optimal prices and inventory levels are analyzed. Next, prices are assumed to be exogenous, and the structure of optimal assortment and inventory levels are then investigated. Finally, we numerically study the joint pricing, assortment, and inventory problem based on a well-crafted heuristic. Examples of works that consider joint inventory and pricing decisions for a given assortment include Aydin and Porteus (2008), Karakul and Chan (2010), Kocabıyıkoglu and Popescu (2011), Li and Huh (2011), Akan *et al.* (2013), Roels (2013) and Maddah *et al.* (2014). Works on joint assortment and inventory decisions under exogenous pricing include Van Ryzin and Mahajan (1999), Smith and Agrawal (2000), Kök and Fisher (2007), Honhon *et al.* (2010) and Maddah et al. (2014). Finally, the literature on jointly optimizing assortment, pricing, and inventory decisions includes Maddah and Bish (2007), Tang and Yin (2010) and Ghoniem and Maddah (2015).

Recent works on retailing decisions can be further classified based on their solution methodology. Some utilize Mathematical Programming (MP), and others utilize stylized models. In the MP approach, the authors seek an exact solution (typically on two of the key pricing, assortment and inventory decisions) for a large number of products under steady (deterministic) demand over possibly a long selling season with multiple replenishment opportunities, which applies to fast-moving consumer goods. Example of works that utilize the MP approach include Dobson and Kalish (1993), Subramanian and Sherali (2010) and Gohoniem and Maddah (2015). Authors utilizing stylized models are more concerned with obtaining insight on the structure of the optimal pricing and assortment decisions and the effect of environmental parameters on these decisions over a short selling season with a highly variable (stochastic) demand, and no opportunity for replenishing inventory, in a newsvendor-type setting. The stylized approach applies mainly to fashion goods. Examples of works that utilize stylized models include Besanko *et al.* (1998), Van Ryzin and Mahajan (1999), Aydin and Ryan (2000), Hopp and Xu (2005), Aydin and Porteus (2008), Kok and Xu (2011) and Li and Huh (2011). Having fashion goods in mind, our work utilizes the stylized approach and contributes to the literature by expanding the results of Maddah and Bish (2007) and Maddah *et al.* (2014), as explained in Section 1.

Most of the works above utilize the classic logit choice for modeling customer purchase behavior over an assortment of substitutable products with few exceptions. Some authors assume exogenous demand models, where demand is predetermined for each product (Smith and Agrawal, 2000; Kök and Fisher, 2007), whereas other authors adopt locational demand models that link the attractiveness of a product to its "distance" from an ideal product location (Gaur and Honhon, 2006).

A recent stream of the literature considers pricing, assortment, and inventory decisions under nested logit choice, similar to what we do in this article. However, the focus in most of these works is on proposing efficient computational approaches.

Kalakesh (2006) analyzes the retailer's assortment, price and inventory decisions jointly under a NMNL demand model through a numerical study. He finds that popular sets and equal profit margins provide near-optimal solutions in most cases. Lee and Eun (2016) propose an approach for demand estimation under NMNL using the expectation-maximization algorithm. Farahat and Lee (2018) propose an efficient numerical search for determining the optimal inventory levels of a multi-product newsvendor under "dynamic" stock-out-based substitution, and present a detailed application under NMNL choice. Wan *et al.* (2018) numerically study another stock-out-based substitution where a customer would first select a store and then select a product within the chosen store.

The assortment problem under NMNL with exogenous pricing and ample inventory is studied in Davis *et al.* (2014) who propose efficient methods to exploit a limited number of assortments within each nest. Gallego and Topaloglu (2014) and Feldman and Topaloglu (2015) extend this work by incorporating a capacity constraint, among other requirements. The joint assortment and pricing problem under NMNL and ample inventory is studied by Kök and Xu (2011) who focus on numerically finding the optimal solution for heterogeneous products under centralized and decentralized management. Li *et al.* (2015) develop efficient algorithms for the assortment-only and the pricing-only problems (with ample inventory) under a generalization of NMML whereby the products are classified on $d \geq 2$ levels.

Few recent works also study retailing decisions under NMNL analytically. Alptekinoğlu and Grasas (2014) study the optimal assortment under consumer returns where customers make purchases following the NMNL choice. The optimal assortment was found by using a greedy algorithm under two different scenarios of make to stock and make to order. Li and Huh (2011) study the pricing problem under NMNL for a given assortment with ample inventory, and show that the expected revenue and profit are concave in the market share vector. Li and Huh (2011) show that the optimal profit margins are equal across all products under a version of the NMNL choice similar in structure to the one we consider in this article. This provides a motivation for our equal-margin assumption. Jiang *et al.* (2017) consider a similar problem to that in Li and Huh (2011), but with the multilevel NMNL, and also prove that the profit function is concave in the market share vector.

It is clear from this brief review that no work in the recent literature on NMNL establishes structural properties for the two problems of joint pricing and inventory decisions for an assortment and joint assortment and inventory decisions under exogenous pricing as do we in this article. In addition, no work considers the joint pricing, assortment and inventory decisions problem for which we propose an efficient solution via the EMH. Table 1 positions our work with respect to the recent literature in a concise manner.

3. Model and assumptions

We define Ω as the set of products, differing in secondary characteristics, from which the retailer can form her assortment S. The set Ω is partitioned into n nests, $\Omega = \cup_{i=1}^{n} N_i$, and the assortment S to be chosen would also be partitioned among nests, $S = \cup_{i=1}^{n} S_i$, where $S_i \subseteq N_i$. NMNL choice is characterized by two constants; μ_1 measures the degree of dissimilarity between nests, and μ_2 that measures the dissimilarity among products in the same nest. Products across different nests exhibit more differences than products included in the same nest, which leads to $\mu_1 \geq \mu_2$. Purchase behavior is modeled through a two-step decision process; consumers first decide on the nest of preference then they choose to purchase a product within this nest (Anderson and de Palma, 1992). The no-purchase option appears in the first stage, whereby a customer chooses between different nests and not buying. This is in-line with the random utility maximization theory, where a customer chooses the higher-utility option after observing the realized utility of all available options.

Table 1. Summary of works on NMNL choice in the recent literature.

	Decision			Approach	
Paper	*Inventory*	*Assortment*	*Pricing*	*Analytical*	*Numerical*
This paper	√	√	√	√	√
Alptekinoğlu and Grasas (2014)	×	√	×	√	×
Davis et al. (2014)	×	√	×	×	√
Farahat and Lee (2018)	√	×	×	×	√
Feldman and Topaloglu (2015)	×	√	×	×	√
Gallego and Topaloglu (2014)	×	√	×	×	√
Jiang et al. (2017)	×	×	√	√	×
Kalakesh (2006)	√	√	√	×	√
Kök and Xu (2011)	×	√	√	×	√
Li and Huh (2011)	×	×	√	√	×
Li et al. (2015)	×	√	√	×	√
Wan et al. (2018)	√	×	×	×	√

Figure 1 is an illustration of the decision process, where q_i is the probability of selecting nest i, and q_{li} is the probability of selecting product l from nest i, and $k_i = |S_i|$.

Each product $l \in S$ belongs to a nest N_i and will be denoted as product li. Variants in S are assumed to have equal profit margin, denoted by m, defined as the difference between the retail price, p_{li}, and the unit cost, c_{li} of a product $li \in S_i$. Specifically, we assume that $p_{li} - c_{li} = p_{jk} - c_{jk} = m, \forall i = 1, ..., n, j = 1, ..., n, l \in S_i, k \in S_j$.

A consumer has a mean reservation price for product li defined as α_{li}. The consumer's utility for product li is given by $U_{li} \equiv \alpha_{li} - p_{li} + \epsilon_{li}$, where the ϵ_{li} are independent and identically distributed (i.i.d.) Gumbel random variables with mean zero and shape factor μ_2. The probability of purchasing a product li given that its nest i has been selected is

$$q_{l|i}(S_i) \equiv P\{U_{li} = \max_{j\in S_i}(U_{ji})\} = \frac{e^{(\alpha_{li}-c_{li}-m)/\mu_2}}{\sum_{j\in S_i} e^{(\alpha_{ji}-c_{ji}-m)/\mu_2}}$$
$$= \frac{e^{(\alpha_{li}-c_{li})/\mu_2}}{\sum_{j\in S_i} e^{(\alpha_{ji}-c_{ji})/\mu_2}}. \tag{1}$$

It is interesting to note that the purchase probability $q_{l|i}(S_i)$ is independent of the common profit margin, m. One interpretation for this is that having all the products in a nest priced at the same level (margin) eliminates the effect of pricing on selecting a product from the nest. The utility of each nest is based on the attractiveness of nests, defined as $A_i \equiv \mu_2 \ln\left(\sum_{j\in S_i} e^{(\alpha_{lj}-c_{lj}-m)/\mu_2}\right)$ for nest S_i. Specifically, the utility of nest S_i and the no purchase option are given by $U_i \equiv A_i + \epsilon_i$ and $U_0 = u_0 + \epsilon_0$, respectively, where u_0 is the average utility of the no-purchase options and the ϵ_i, $i = 0, 1, ..., n$, are also i.i.d. Gumbel random variables with mean zero and shape factor μ_1. Hence, the probability of choosing nest i is given by

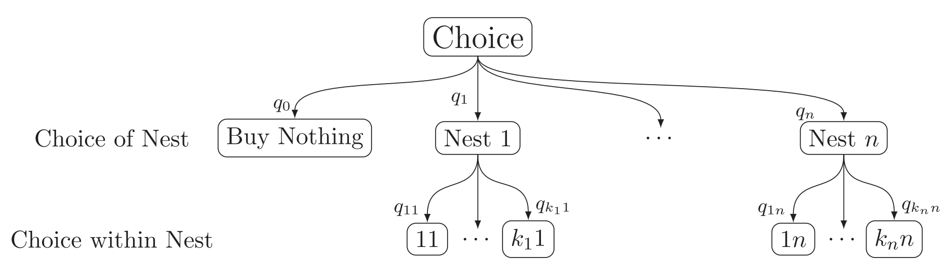

Figure 1. Two-level decision process under NMNL.

$$q_i(S,m) \equiv P\{U_i = \max_{k=0,1,\ldots,n}(U_k)\} = \frac{e^{A_i/\mu_1}}{v_0 + \sum_{k=1}^{n} e^{A_k/\mu_1}}, \tag{2}$$

$$= \frac{\left(\sum_{j\in S_i} e^{(\alpha_{ji}-c_{ji})/\mu_2}\right)^{\mu_2/\mu_1}}{v_0 e^{m/\mu_1} + \sum_{k=1}^{n}\left(\sum_{j\in S_k} e^{(\alpha_{jk}-c_{jk})/\mu_2}\right)^{\mu_2/\mu_1}}, \tag{3}$$

where $v_0 \equiv e^{u0/\mu_1}$. Finally, the probability of purchasing a product li is

$$q_{li}(S,m) \equiv q_{l|i}(S_i) \times q_i(S,m), \tag{4}$$

where $q_{l|i}(S_i)$ and $q_i(S,m)$ are given by (1) and (3), respectively.

We assume that a consumer makes her purchase decision based only on price, variety and quality (i.e., we consider assortment and price-based substitution). Assuming that the arrivals to the store follows a Poisson process with rate λ, it follows from the Poisson decomposition property that the demand for products l in nest i, X_{li}, are i.i.d. Poisson random variables with mean $\lambda q_{li}(S,m)$. We approximate this demand with a normal random variable with mean $\lambda q_{li}(S,m)$ and standard deviation $\sqrt{\lambda q_{li}(S,m)}$ i.e., $X_{li} \approx \lambda q_{li}(S,m) + \sqrt{\lambda q_{li}(S,m)} Z_{li}$, where Z_{li} are i.i.d. standard normal random variables. This represents a normal approximation to demand generated from customers arriving according to a Poisson process with rate λ per selling period. Such a normal approximation approach is common in the literature (Van Ryzin and Mahajan, 1999; Maddah and Bish, 2007; Maddah *et al.*, 2014). This demand function is classified as "multiplicative-additive," in the sense that both the demand standard deviation and coefficient of variation are dependent on the pricing reflected by the profit margin m. This demand function captures consumer behavior accurately, however it is typically hard to analyze (Petruzzi and Dada, 1999; Maddah *et al.*, 2014).

Following established results on the newsvendor model under a normal demand distribution (Silver *et al.*, 1998), the optimal inventory level, $y^*_{li}(S,m)$ for each product l in nest i and the corresponding expected profit $\Pi(S,m)$ at these optimal inventory levels can be written as

$$y^*_{li}(S,m) = \lambda q_{li}(S,m) + \Phi^{-1}\left(1 - \frac{c_{li}}{m+c_{li}}\right)\sqrt{\lambda q_{li}(S,m)}, \tag{5}$$

$$\Pi(S,m) = \sum_{i=1}^{n}\sum_{l\in S_i} m\lambda q_{li}(S,m) - (m+c_{li})\varphi\left(\Phi^{-1}\left(1-\frac{c_{li}}{m+c_{li}}\right)\right)\sqrt{\lambda q_{li}(S,m)}, \tag{6}$$

where $q_{li}(S,m)$ is given in (4), and $\varphi(.)$ and $\Phi(.)$ respectively denote the probability density function and the cumulative distribution function of the standard normal distribution. In (6), the first term is the gross profit with no inventory considerations, whereas the second term accounts for inventory cost. The aim of this article is to optimize $\Pi(S,m)$ by finding the optimal assortment $S \in \Omega$ and the profit margin m. The obtained optimal assortment and profit margin can then be replaced in (5) to determine the optimal inventory levels. Rearranging (6), the expected profit can be written as

$$\Pi(S,m) = m\lambda p(S,m)\rho(S) - \sqrt{\lambda p(S,m)}\sum_{i=1}^{n}\sum_{l\in S_i}(m+c_{li})\varphi\left(\Phi^{-1}\left(1-\frac{c_{li}}{m+c_{li}}\right)\right)\zeta_{li}, \tag{7}$$

where

$$p(S,m) \equiv \frac{e^{-m/\mu_1}}{v_0 + e^{-m/\mu_1}\rho(S)}, \tag{8}$$

$$\rho(S) \equiv \sum_{i=1}^{n}\left(\sum_{l\in S_i} e^{\frac{\alpha_{li}-c_{li}}{\mu_2}}\right)^{\frac{\mu_2}{\mu_1}}, \tag{9}$$

$$\zeta_{li} \equiv \sqrt{e^{\frac{\alpha_{li}-c_{li}}{\mu_2}}\left(\sum_{k\in S_i} e^{\frac{\alpha_{ki}-c_{ki}}{\mu_2}}\right)^{\frac{\mu_2}{\mu_1}-1}}. \tag{10}$$

To simplify the analysis of (7), we adopt a Taylor-series type approximation developed by Maddah *et al.* (2014), which is given by

$$\varphi(\Phi^{-1}(1-x)) \approx -ax(x-1), 0 \le x \le 1, \tag{11}$$

where a recommended calibration value is $a = 1.66$. Maddah *et al.* (2014) show that Approximation (11) is highly accurate with a Mean Absolute Percentage Error (MAPE) of 8.6%. Moreover, in an extensive numerical study on optimizing pricing and inventory for an assortment, the regret from utilizing Approximation (11) is observed to be below 3%, see Maddah *et al.* (2014) for more details. (In terms of deriving the approximation, Philip (1960) develops a Taylor approximation for a $B(.)$ function. Maddah *et al.* (2014) adopt this approximation and improve it by fitting a quadratic function which is error-free at the end points, $x = 0$ and $x = 1$.)

We believe that, with the similarity in the expected profit structure under the logit and nested logit demand functions, the approximation will continue to be accurate.

Using Approximation (11), the profit function in (7) is reduced to

$$\hat{\Pi}(S,m) \equiv m\lambda\rho(S)p(S,m) - am\sqrt{\lambda p(S,m)}\theta(S,m), \tag{12}$$

where $\theta(S,m) \equiv \sum_{i=1}^{n}\sum_{l\in S_i}\frac{c_{li}}{c_{li}+m}\zeta_{li}$.

4. Optimal profit margin for a given assortment

In this section, we analyze the properties of the optimal profit margin that maximizes the profit function in (12) for a given assortment S, $m^*_S \equiv \mathrm{argmax}_{m>0}\hat{\Pi}(S,m)$. In Section 4.1, we analyze the structure of the expect profit function as a function of the common profit margin, and establish its unimodularity. Then, in Section 4.2, we analyze the effect of inventory considerations on pricing, by comparing the optimal margin m_S^* to the "riskless" margin which maximizes the expected profit under ample inventory. Proofs of key analytical results are included in appendices which are available as part of the online supplement to this article.

4.1. Structure of the expected profit in function of the common margin

To ensure that the retailer is not better off by not selling anything, we make the following assumption.

Assumption 1. *The expected profit $\hat{\Pi}(S,m)$ is increasing in m at $m=0$; that is $\frac{\partial \hat{\Pi}(S,m)}{\partial m}|_{m=0} > 0$, or equivalently,*

$$\lambda > \frac{a^2(v_0+\rho(S))}{\rho(S)^2}\left(\sum_{i=1}^{n}\sum_{l\in S_i}\zeta_{li}\right)^2. \tag{13}$$

Assumption 1 ensures that there is an interval where the profit is positive, which, together with the fact that $\lim_{m\to\infty}\hat{\Pi}(S,m)\to 0^-$, implies that m_S^* is an internal point solution satisfying the first and second-order optimality conditions. The following lemma establishes this result in precise terms along with an upper bound on m_S^*.

Lemma 1. *The expected profit $\hat{\Pi}(S,m)>0$ if and only if $m\in(0,\bar{m}_S)$ where $\bar{m}_S$ is the unique solution to the equation*

$$\lambda\rho(S)^2 p(S,\bar{m}_S) - a^2\theta^2(S,\bar{m}_S) = 0. \tag{14}$$

Furthermore $m_S^* < \bar{m}_S$.

Proof. See Appendix A.1. □

Lemma 1 shows that the expected profit is only positive on the interval $[0,\bar{m}_S]$. This interval defines the feasible set of the profit margin. Furthermore, we can see numerically that $\bar{m}_S$ is decreasing in the assortment size. Hence, a retailer offering a wide variety of products cannot assign high profit margins. This finding conforms to the typical behavior of a logit choice model, where thinning of demand occurs with big assortments.

The next theorem describes the behavior of $\hat{\Pi}(S,m)$ over the feasible range defined in Lemma 1.

Theorem 1. *The expected profit $\hat{\Pi}(S,m)$ is unimodal in m on $(0,\bar{m}_S)$.*

Proof. See Appendix A.2. □

Theorem 1 generalizes a result in Maddah *et al.* (2014), where the authors prove that the expected profit is unimodal over a bounded interval in the price for products having homogeneous unit cost and price under the classic logit choice. Theorem 1 shows that there exists a single optimal solution on $[0,\bar{m}_S]$ that maximizes the profit. This, along with Lemma 1 allows the determination of the optimal profit margin via any line search technique. The insight from Theorem 1 is tied to the classic trade-off between unit margin and demand volume. Setting the profit margin, m, too low leads to a high demand volume and a low profitability per unit, whereas setting m too high leads to the opposite effects. The optimal profit margin is a "middle" value that balances these effects. It is to be noted that this trade-off is mostly inherited from the "riskless," case, which ignores inventory costs. In Section 4.2, we investigate how balancing the additional effect of the inventory cost influences the optimal profit margin.

4.2. Effect of inventory considerations on pricing

In this section, we study the properties of the optimal profit margin m_S^* by comparing it to the optimal riskless profit margin $m_S^0 \equiv \text{argmax}_m \Pi^0(S,m)$ that maximizes the profit when no inventory cost exists, i.e., for a make-to-order retailer case or when there is ample inventory. When no inventory costs are considered, the second part of (7) is dropped and the riskless expected profit is written as

$$\Pi^0(S,m) = m\lambda p(S,m)\rho(S). \tag{15}$$

For the nested logit demand we adopt in this article, where the price-sensitivity parameters and dissimilarity indices across nests are all equal (to 1 and $1/\mu_2$, respectively), Li and Huh (2011) show that having equal profit margins across all products is optimal. This provides justification for an all equal-margin pricing scheme, as it can be easily seen that the expected "risky" profit in (12) converges to the "riskless" one in (15) for a large demand volume, i.e., $\lim_{\lambda\to\infty}\hat{\Pi}(S,m)=\Pi^0(S,m)$, and accordingly, if p_{li}^* denotes the optimal price of Product li, then $\lim_{\lambda\to\infty} p_{li}^* - c_{li} = m_S^0$, $i=1,\dots n, l\in S_i$. (This formalizes our statement from Section 1 on the asymptotic optimality of equal profit margins.) In the following lemma, we develop a closed-form expression for m_S^0 in terms of the Lambert function $W(.)$. The Lambert function is the inverse of the function we_w (Corless *et al.*, 1996). Lemma 2 gives this closed-form expression of m_S^0.

Lemma 2. *$\Pi^0(S,m)$ is unimodal in m and the optimal margin m_S^0 maximizing it has a closed-form expression given by*

$$m_S^0 = \mu_1\left[1 + W\left(\frac{\rho(S)}{v_0}e^{-1}\right)\right]. \tag{16}$$

Proof. See Appendix A.3. □

Li and Huh (2011) present a similar result to Lemma 2 for the special case of the standard logit choice. Next, we compare m_S^* and m_S^0. In the literature, the optimal price p^* and the riskless price p^0 compare as follows. When the demand is additive $p^*\le p^0$, and in the case of multiplicative demand $p^* > p^0$ (Petruzzi and Dada, 1999). In additive-multiplicative demand p^* may fall above or below p^0 (Maddah *et al.*, 2014). In the following, we present a result similar to that in Maddah *et al.* (2014). Studying the inventory cost function enables us to compare m_S^* and m_S^0. From (12), the inventory cost is given by

$$\hat{C}(S,m) \equiv am\sqrt{\lambda p(S,m)}\theta(S,m). \tag{17}$$

The following lemma presents important properties of the inventory cost.

Lemma 3. *The inventory cost $\hat{C}(S,m)$ is unimodal in m.*

Proof. See Appendix A.4. □

By establishing the unimodularity of the inventory cost function, Lemma 3 implies that the inventory cost would be minimized for extreme values of m to the right of 0 and, and as m gets larger. Therefore, the optimal "risky" profit margin which attempts to hit a high value for the gross profit in (15) while minimizing the inventory cost in (17) would involve pushing the riskless profit margin in (16) up or down. This result is established formally in Theorem 2. Lemma 3 can be also used to compare m_S^* and m_S^0, which we do in Theorem 2.

Theorem 2. *If*

$$\sum_{i=1}^{n}\sum_{l\in S_i}\frac{c_{li}\zeta_{li}}{(c_{li}+m_S^0)^2}(c_{li}-m_S^0)\geq 0$$

then $m_S^*\leq m_S^0$ *otherwise* $m_S^*\geq m_S^0$.

Proof. See Appendix A.5. □

The inequality in Theorem 2 can be related to the gross profit margins of products in the assortment, as the inequality $m_S^0>c_{li}$ is equivalent to having a gross profit margin above 50%. As such, Theorem 2 implies that for assortments with high gross profit (exceeding 50%, on average), inventory costs will tend to push the pricing up, and *vice versa* when the gross profit margin is, on average, not too high. It is worth noting, that by examining the financial statements of several fashion retailers, Maddah *et al.* (2014) indicate that gross profit margins exceeding 50% are not uncommon. Theorem 2 can be also used to bound m_S^*. This is formalized in the following corollary.

Corollary 1. If

$$\sum_{i=1}^{n}\sum_{l\in S_i}\frac{c_{li}\zeta_{li}}{(c_{li}+m_S^0)^2}(c_{li}-m_S^0)\geq 0,$$

then $m_S^*\in(0,m_S^0)$. Otherwise, $m_S^*\in(m_S^0,\bar{m}_S)$.

Proof. Follows directly from Theorem 2 and Lemma 1. □

Corollary 1 simplifies the search for the optimal margin m_S^*, especially in that the riskless margin, m_S^0, has the closed-form expression in Lemma 2. One issue that may be worthy of further investigation, in the aim of sharpening Corollary 1, is comparing m_S^0 to $\bar{m}_S$. We observe numerically that the inequality $m_S^0<\bar{m}_S$ holds consistently, which suggests that not much improvement is possible over Corollary 1.

The condition in Theorem 2 can be simplified in certain special cases that we explore next. If we define $\underline{c}_S=\min_{S_i\in S,l\in S_i}c_{li}$ and $\bar{c}_S=\max_{S_i\in S,l\in S_i}c_{li}$, i.e., $\underline{c}_S$ and $\bar{c}_S$ are the cheapest and most expensive costs in the assortment, then the following corollary follows directly from Theorem 2.

Corollary 2. *If* $m_S^0\leq\underline{c}_S$ *then* $m_S^*\leq m_S^0$. *In addition, if* $m_S^0\geq\bar{c}_S$ *then* $m_S^*\geq m_S^0$.

Proof. Follows directly from Theorem 2. □

In addition, in the widely-studied special case where all products in the assortment have equal unit cost the following corollary holds.

Corollary 3. *In the special case where* $c_{li}=c$, $l\in N_i, i=1,\ldots,n$, *if* $m_S^0<c$ *then* $m_S^*\leq m_S^0$. *In addition if* $m_S^0>c$ *then* $m_S^*\geq m_S^0$.

Proof. Follows directly from Theorem 2. □

Corollary 3 extends Lemma 4 in Maddah *et al.* (2014). Furthermore, in the case of a single nest, Corollary 3 is equivalent to Lemma 4 in Maddah *et al.* (2014).

5. Structure of the optimal assortment

In this section, we study the problem of determining the products that constitute an optimal assortment assuming that the prices are exogenously determined, albeit having the same margin m. It is well known that the assortment problem exploits the following trade-offs under the classic MNL choice: Adding more products to the assortment results in higher demand for the assortment as a whole (which leads to a higher gross profit), but lowers the demand for individual products due to cannibalization (which incurs higher inventory costs). Therefore, the optimal assortment under MNL choice has been observed to be of a "moderate" size and typically composed of a set from popular products (Van Ryzin and Mahaan, 1999; Maddah *et al.*, 2014). In the following, we observe that similar trade-offs and results continue to hold under NMNL choice.

Before proceeding, we formally define the objective of the analysis in the section, which is to study the structure of the optimal assortment (under the assumptions in Section 3) that maximizes the expected profit in (12) for given a profit margin m, $S^*=\text{argmax}_{S\in\Omega}\hat{\Pi}(S,m)$.

We begin by establishing critical monotinicity and convexity properties of $\hat{\Pi}(S,m)$.

Lemma 4. *The expected profit* $\hat{\Pi}(S,m)$ *is decreasing in* c_{li} *and pseudoconvex in* α_{li}.

Proof. See Appendix B.1. □

The montonicity in the unit cost in Lemma 4 is intuitive. The interesting part of the lemma is related to the pseudoconvexity in the mean reservation price α_{li}, as it paves the way to a popular set structure of the optimal assortment once the effect of the unit costs is factored out. We point out first that pseudoconvexity implies that the profit function, $\hat{\Pi}(S,m)$, admits a unique local minimum as a function α_{li} without necessarily being convex. Specifically, the profit function starts flat for $\alpha_{li}=-\infty$ at a value equivalent to the profit from an assortment excluding product li (i.e., $\hat{\Pi}(S-\{li\},m)$), and then decreases slowly reaching a local minimum and then increasing at a fast rate approaching the profit of the assortment having product li only, i.e., $\{li\}$, as α_{li} increases. To illustrate how this helps with structuring the optimal assortment, consider the common case of homogeneous unit costs across all products, and imagine one is building the optimal assortment sequentially, starting with the product having the highest α_{li}. When it comes to adding the second product, Lemma 4 implies that it is optimal to add the product with the largest α (in nest i) or to add nothing (equivalent to $\alpha = -\infty$), as pseudoconvexity implies that the maximum values occur at the extreme points. The same thing would repeat when adding the third and fourth product leading to a popular set structure per nest.

To study the structure of the optimal assortment, we introduce two functions that serve as upper and lower bounds on the expected profit function. To define these bounds, denote $\bar{c}_i$ and $\underline{c}_i$ as the highest and lowest cost in nest N_i, respectively, i.e., ${}_{\bar{c}}i = \max_{il \in N_i} c_{il}$ and $\underline{c}_i = \min_{il \in N_i} c_{il}$. Then, it can be easily seen that the two functions in (18) and (19) respectively define upper and lower bounds on $\hat{\Pi}(S,m)$ in (12):

$$\bar{\Pi}(S,m) \equiv m\lambda\rho(S)p(S,m) - am\sqrt{\lambda p(S,m)}\bar{\theta}(S,m), \qquad (18)$$

$$\underline{\Pi}(S,m) \equiv m\lambda\rho(S)p(S,m) - am\sqrt{\lambda p(S,m)}\underline{\theta}(S,m), \qquad (19)$$

where

$$\bar{\theta}(S,m) \equiv \sum_{i=1}^{n}\sum_{l\in S_i}\frac{\underline{c}_i}{\underline{c}_i + m}\zeta_{li}$$

and

$$\underline{\theta}(S,m) \equiv \sum_{i=1}^{n}\sum_{l\in S_i}\frac{\bar{c}_i}{\bar{c}_i + m}\zeta_{li}.$$

We observe numerically in Section 7 that $\bar{\Pi}(S,m)$ and $\underline{\Pi}(S,m)$ provide tight bounds on $\hat{\Pi}(S,m)$.

We show next that when utilizing $\bar{\Pi}(S,m)$ or $\underline{\Pi}(S,m)$ as an approximate profit function instead of $\hat{\Pi}(S,m)$, an optimal assortment has a popular set structure containing products with the highest average margin $\alpha_{li} - c_{li}$ in each nest. We start by introducing a useful notion of product dominance.

Definition 1. *Consider two products l_1i and l_2i in nest N_i. If $\alpha_{l_1i} - c_{l_1i} \geq \alpha_{l_2i} - c_{l_2i}$, then product l_1i dominates product l_2i.*

Lemma 5 avails of this dominance concept and builds on Lemma 4.

Lemma 5. *Consider two products l_1i and l_2i in nest N_i where l_1i dominates l_2i. If l_2i is in an optimal assortment that maximizes the profit bound $\bar{\Pi}(S,m)$, then l_1i belongs to that optimal assortment as well. This property also holds for an optimal assortment that maximizes the profit bound $\underline{\Pi}(S,m)$.*

Proof. See Appendix B.3. □

Lemma 5 can be used to deduce the structure of the optimal assortment as highlighted in Theorem 3.

Theorem 3. *The optimal assortment that maximizes the profit bound $\bar{\Pi}(S,m)$ or $\underline{\Pi}(S,m)$ contains the κ_i products having the highest average margin $\alpha_{li} - c_{li}$ in nest N_i for some integer $\kappa_i \leq |N_i|$.*

Proof. Follows from Lemma 5. □

Theorem 3 indicates a "popular set" structure in each nest of an optimal assortment that maximizes the bounds on the expected profit in (18) and (19), in each nest. Theorem 3, along with the testing of the quality of the bounds (18) and (19) in Section 7, imply that assortments consisting of popular sets in each nest are expected to produce "good solution" and can be used to sharpen good heuristics as we do in Section 6, which are particularly useful with products having heterogeneous unit cost.

When all the products have the same unit cost, the popular set structure of the optimal assortment in Theorem 3 also holds when maximizing the actual expected profit $\hat{\Pi}(S,m)$, as indicated in Corollary 4.

Corollary 4. *If $c_{l_1i} = c_{l_2i} \forall l_1, l_2 \in N_i, i = 1, \ldots, n$ then the optimal assortment that maximizes $\hat{\Pi}(S,m)$ contains the κ_i products having the highest α_{li} in in nest N_i for some integer $\kappa_i \leq |N_i|$.*

Proof. Follows from Corollary 3 by noting that $\bar{\Pi}(S,m) = \underline{\Pi}(S,m) = \hat{\Pi}(S,m)$, when $c_{l_1i} = c_{l_2i} \forall l_1, l_2$. □

The structure of the optimal assortment in Corollary 4 is such that it contains a Cartesian product of popular sets (having the largest values of the average margin, $\alpha - c$) from different nests. This is not the same result in the previous literature (Van Ryzin *et al.*, 1999; Maddah *et al.*, 2014), where an optimal assortment is a "globally popular" set containing the products with the highest average margin across the entire product line. In fact, we observe counter examples, where globally popular sets are not optimal in our nested logit framework (See Appendix C).

Computationally, Corollary 4 simplifies the search for the optimal assortment with homogeneous unit cost in each nest, as the number of combinations considered is $|N_1||N_2|...|N_n|$ instead of $2^{|N_1||N_2|...|N_n|}$. When the unit cost in the nest is heterogeneous, Theorem 3 indicates that a "good" assortment can typically be found with a similar efficiency, as we observe in Section 7. In fact, in a detailed numerical study, we do not encounter any case where an optimal assortment is not a popular set.

6. Joint pricing, assortment, and inventory decisions via EMH

Based on the results in Sections 4 and 5, we propose, in this section, an equal margin heuristic that jointly optimizes pricing, assortment and inventory decisions under the equal profit margin requirement on the prices. The objective is to determine a profit margin m^H and assortment S^H that produce high (near-optimal) values of the the profit function in (12). The optimal inventory levels are found from (5).

- Step 1:
  Sort products in all nests $N_i \subseteq \Omega, i = 1, \ldots, n,$ in non-increasing order of $\alpha_{li} - c_{li}$. Break ties at random. Generate the $|N_1| \times |N_2| \times \cdots |N_n|$ popular sets. Denote these assortments $P_1, P_2, \ldots P_z$ where $z = |N_1| \times |N_2| \times$

$\cdots|N_n|$. Let $\boldsymbol{P} = \{\mathrm{P}_1, \mathrm{P}_2, \ldots, \mathrm{P}_z\}$ be the set containing all these assortments.

- Step 2:
  Define the contribution of product li in a given assortment S by

$$\hat{\Pi}_{li}(S,m) \equiv m\lambda q_{li}(S,m) - am\frac{c_{li}}{m+c_{li}}\sqrt{\lambda q_{li}(S,m)} \quad (20)$$

  For all the popular sets assortment generated in Step 1, identify any assortment with "loss" products, P_L, where there exists $li \in P_L$, $\frac{\partial \hat{\Pi}_{li}(P_L,m)}{\partial m}|_{m=0} \le 0$, or equivalently

$$q_{li}(P_L,0) \le \frac{a^2}{\lambda}. \quad (21)$$

  Specifically, let $\boldsymbol{L} = \{\mathrm{P_L} \in \boldsymbol{P} | \exists\ l \in \mathrm{P_L}$ such that Inequality (21) holds$\}$, be the set of assortments with loss products.
- Step 3:
  For each set $P_j \in \boldsymbol{P}\backslash\boldsymbol{L}$,
- Step 3.1:
  Find $\bar{m}_j\bar{m}j$, the upper bound on the optimal profit margin of P_j, by solving (14),
- Step 3.2:
  Find m_j^0, the riskless profit margin of P_j, from (16),
- Step 3.3:
  Calculate $\delta_j^r = \sum_{i=1}^n \sum_{l \in P_{ji}} \frac{c_{li}\zeta_{li}}{(c_{li}+m_S^0)^2}(c_{li} - m_S^0)$, where $P_{ji} = P_j \cap N_i$.
- Step 3.4:
  If $\delta_j^r \ge 0$, search for m_j^H the profit margin that maximizes $\Pi(P_j, m)$ over $(0, \bar{m}_j)$. Otherwise, search for m_j^H over $(m_j^0, \bar{m}_j)$.
- Step 4:
  Find $P_{k^h} = \mathrm{argmax}_{P_j \in \boldsymbol{P}\backslash\boldsymbol{L}} \hat{\Pi}(P_j, m_j^H)$. Choose the assortment $S^H = P_{k^h}$ with the corresponding profit margin $m^H = m_{k^h}^H$. Estimate inventory levels of products in S^H via $y_{lk^h}^*(S^H, m^H), lk^h \in S^H$ from (5).

Step 1 of EMH identifies popular sets across all nests, as this heuristic exploits such assortments only. This is motivated by the results in Section 5, on the near-optimality of popular sets, mainly Theorem 3. Step 2 eliminates from consideration any popular sets containing products that will most likely be sold at a loss. An easy way to implement Step 2 is to go over each nest and form popular sets by adding products incrementally, starting with the most popular product. Once a product satisfies Inequality (21), the resulting assortment is in $\boldsymbol{L}$. Step 3 in EMH searches for the optimal profit margin of non-trivial popular sets. The optimization in this step is guaranteed to produce an internal point solution, which is globally optimal (under the requirement of equal profit margins) due to results in Section 4, specifically Theorem 1 and Corollary 1.

The advantage of EMH is that it requires little computational effort relative to the effort required to find the optimal solution. The EMH generates at most $|N_1| \times |N_2| \times \cdots |N_n|$ assortments, each requiring a single variable search to find the corresponding optimal equal profit margin.

The computational complexity of EMH is driven mainly by the number of popular set assortments to enumerate, as defined in Steps 1 and 2. In the worst case, assuming that the set $\mathbf{L}$ defined in Step 2 is empty, this involves enumerating $\Pi_{i=1}^n |N_i|$ assortments. Assuming further that the products are split uniformly across nests, the computational complexity of the enumeration is $(|\Omega|/n)^n$. In each enumeration step, the two numerically demanding tasks are Step 3.1 requiring solving the nonlinear equation in (14), which is guaranteed to converge by Lemma 1, and the line search in Step 3.4, which is also guaranteed to converge by Theorem 1. In conclusion, with highly efficient algorithms existing for the numerical search in Steps 3.1 and 3.4, the computational complexity of EMH is, practically, in the order of $(|\Omega|/n)^n$. Although this is not polynomial, the limitation of the number of nests, n, in most applications leads to a highly efficient performance of EMH. In our Python implementation, all the EMH instances of our numerical study in Section 7 converged within few seconds, except for large instances having 96 products split over eight nests analyzed in Appendix D of the online supplement.

7. Numerical study

The objective of the numerical study is two fold. First, we show that $\overline{\Pi}(S,m)$ and $\underline{\Pi}(S,m)$ in (18) and (19), provide tight bounds on $\hat{\Pi}(S,m)$ in (12), which provides justification for adopting popular sets assortment based on Theorem 3. Second, we evaluate the performance of the EMH presented in Section 6, and demonstrate excellent results. The high-quality performance of EMH provides validation on the near-optimality of both popular sets assortments and equal profit margin pricing. This also advocates the usage of EMH as a practical assortment and pricing optimization tool for retailers who have the ability to integrate pricing, assortment and inventory decisions.

7.1. Base case

Consider a store where customers arrive according to a Poisson process with a rate $\lambda = 100$, and choose to buy at most one product from a product line organized into three nests, each having four products. The degree of dissimilarity among the nests is $\mu_1 = 2$ and within a nest $\mu_2 = 1.2$. The utility of leaving empty handed is zero, i.e., $v_0 = 1$. Other relevant demand and cost data is given in Figure 2. The fourth nest will be used in the sensitivity analysis later.

Consider a full-product offering assortment, i.e., $S_i = \{1i, 2i, 3i, 4i\}, i = 1,2,3$ and $S = \cup_{i=1}^3 S_i$, and an equal profit margin pricing. The approximate expected profit for this assortment $\hat{\Pi}(S,m)$ given in (12) is plotted in Figure 3 as a function of the profit margin m, along with the "exact" expected profit $\Pi(S,m)$ given in (7). It is obvious that $\hat{\Pi}(S,m)$ provides an excellent approximation of $\Pi(S,m)$, which is due to the accuracy of the Taylor-type approximation in (11) adapted from Maddah *et al.* (2014).

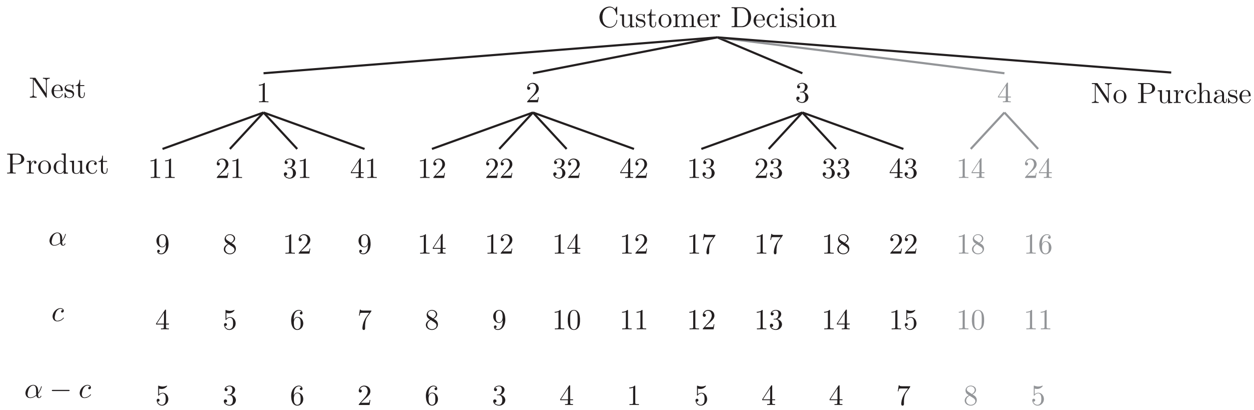

Figure 2. Base case.

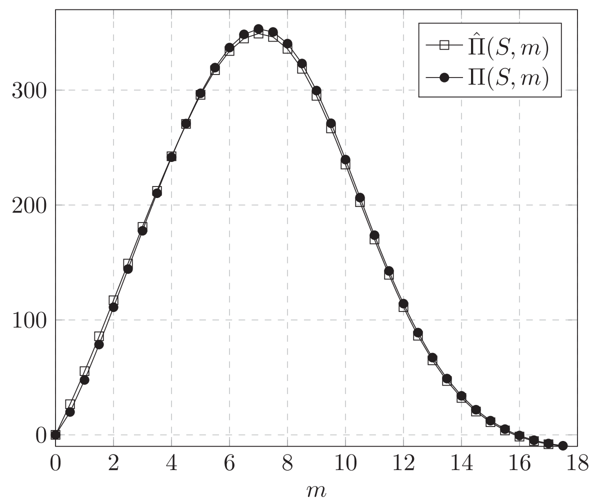

Figure 3. $\hat{\Pi}(S,m)$ and $\Pi(S,m)$ versus m for the full-offering assortment.

From Figure 3, it is obvious that $\hat{\Pi}(S,m)$ is unimodal over the interval where $\hat{\Pi}(S,m) \geq 0$, as shown in Lemma 1 and Theorem 1. This interval is $(0, {}_{\bar{m}}S)$, where ${}_{\bar{m}}S = 15.86$, from Lemma 1. To proceed with the search for the optimal profit margin, m_S^*, we determine the optimal riskless profit from Lemma 2, which we find to be $m_S^0 = 7.10$. Referring to the unit cost values in Figure 2, notice that $c_{li} \leq m_S^0, \forall li$, which implies by Theorem 2 that $m_S^* \leq m_S^0$. Then, by Corollary 1, we carry-out the search for m_S^* on $(0, m_S^0)$. This gives $m_S^* = 7.05$, which can be observed in Figure 3.

7.2. Evaluating the bounds $\bar{\Pi}(S,m)$ and $\underline{\Pi}(S,m)$

To evaluate the upper and lower bounds $\bar{\Pi}(S,m)$ and $\underline{\Pi}(S,m)$ of $\hat{\Pi}(S,m)$, we consider again the example in Section 7.1 with the full offering assortment. We perform a one-way sensitivity analysis by varying the profit margin m, the demand volume λ, the utility of the no purchase v_0, the reservation price of the first product in nest 2, α_{12}, and the unit cost of this product, c_{12}, one at a time, while keeping the other parameters at their base values given Section 7.1. The base value we use for m is the optimal value we found in Section 7.1, $m_S^* = 7.05$. We pick product 12 for illustration purpose. Our experience with varying parameters of other products are similar.

It is obvious from Figure 4 that the bounds $\bar{\Pi}(S,m)$ and $\underline{\Pi}(S,m)$ are tight enough. Figure 4 also indicates that the slope and convexity of the bounds $\bar{\Pi}(S,m)$ and $\underline{\Pi}(S,m)$ (i.e., first and second derivatives) are also close to those of $\hat{\Pi}(S,m)$. Furthermore, Tables 2 and 3 show the absolute difference between $\bar{\Pi}(S,m)$ and $\underline{\Pi}(S,m)$ on one hand, and $\hat{\Pi}(S,m)$ on the other hand. Tables 2 and 3 indicate a maximum absolute difference of around 14 (dollars) and an average difference of around 11. These are reasonable numbers given an optimal profit of around 390 in the base case. This provides a numerical justification for utilizing the popular set structure of the optimal assortment established under these bounds in Section 5 in heuristic solutions sought to maximize the actual profit $\hat{\Pi}(S,m)$.

7.3. Evaluating the equal profit margin heuristic

To evaluate the performance of the EMH presented in Section 6 in producing joint pricing, assortment and inventory decisions, we compare the expected profit produced from EMH with the optimal expected profit based on the input data defined in the base case of Section 7.1. To obtain the optimal expected profit, we enumerate all possible assortments and optimize over the expected profit at optimal inventory levels from each assortment with no restriction on prices. To clarify, based on similar arguments to those in Section 3, the expected profit from an assortment S with unrestricted prices $\boldsymbol{p} = (p_{11}, p_{12}, \ldots)$ is given by

$$\hat{\Pi}(S,\boldsymbol{p}) \equiv \sum_{i=1}^{n} \sum_{l \in S_i} (p_{li} - c_{li}) \left[\lambda q_{li}(S,\boldsymbol{p}) - a\left(\frac{c_{li}}{p_{li}}\right) \sqrt{\lambda q_{li}(S,\boldsymbol{p})} \right],$$

where

$$q_{li}(S,\boldsymbol{p}) = \left[\frac{e^{(\alpha_{li}-p_{li})/\mu_2}}{\sum_{j \in S_i} e^{(\alpha_{ji}-p_{ji})/\mu_2}} \right] \left[\frac{\left(\sum_{j \in S_i} e^{(\alpha_{ji}-p_{ji})/\mu_2} \right)^{\mu_2/\mu_1}}{v_0 + \sum_{k=1}^{n} \left(\sum_{j \in S_k} e^{(\alpha_{jk}-p_{jk})/\mu_2} \right)^{\mu_2/\mu_1}} \right].$$

The expected optimal profit is then given by $\hat{\Pi}^* = \hat{\Pi}(S^*, \boldsymbol{p}^*) = \max_{S \in \Omega} \max_{\boldsymbol{p}} \hat{\Pi}(S,\boldsymbol{p})$, where S^* and $\boldsymbol{p}^*$ are the optimal assortment and prices.

In our numerical trials, we utilize several starting solutions in an attempt to find the globally optimal prices. Since the profit function is not concave, or even pseudoconcave, in the

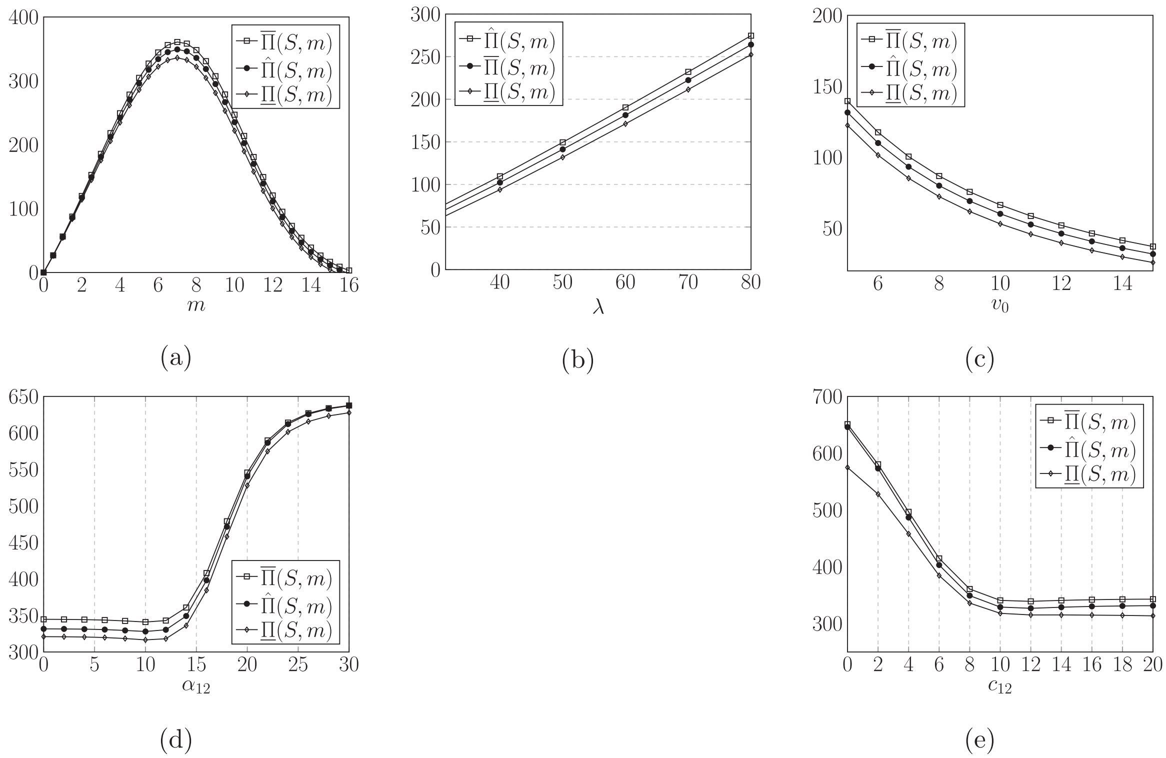

Figure 4. $\underline{\Pi}(S,m)$, $\bar{\Pi}(S,m)$ and $\hat{\Pi}(S,m)$ versus m, λ, v_0, α_{12}, and c_{12}.

Table 2. Absolute difference between the expected profit and the upper bound, $\bar{\Pi}(S,m) - \hat{\Pi}(S,m)$

	Difference from upper bound			
Varying	*Min*	*Mean*	*Median*	*Max*
m	0.25	9.26	9.14	12.31
λ	7.36	8.59	8.62	9.74
v_0	6.80	7.38	7.35	8.03
α_{12}	0.44	8.30	10.76	13.06
c_{12}	5.09	10.63	11.64	12.23

Table 3 Absolute difference between the expected profit and the lower bound, $\hat{\Pi}(S,m) - \underline{\Pi}(S,m)$.

	Difference from Lower Bound			
Varying	*Min*	*Mean*	*Median*	*Max*
m	0.27	9.34	10.42	13.98
λ	8.33	9.72	9.76	11.02
v_0	7.70	8.35	8.32	9.08
α_{12}	9.50	11.41	10.96	13.68
c_{12}	6.08	8.05	9.22	11.41

prices (see Appendix C of the online supplement), there is no guarantee that we can reach global optimality regardless of the number of the initial solutions used. However, the solutions we report appear to be globally optimal, as we failed to find better solutions despite extensive trials on select cases. This is a common issue in all non-convex optimization problems. Letting $\hat{\Pi}^H = \hat{\Pi}(S^H, m^H)$ be the expected profit produced by EMH with an assortment S^H and profit margin m^H, the optimality gap (in percentile points) is found as

$$\frac{\hat{\Pi}^* - \hat{\Pi}^H}{\hat{\Pi}^*} \times 100. \tag{22}$$

We illustrate the estimation of the optimality gap of EMH on the base case in 7.1. Enumerating over all possible assortments and running unrestricted pricing optimization gives an optimal assortment S^* partitioned as follows across the three nests $S_1^* = \{11, 31\}$, $S_2^* = \{12\}$, and $S_3^* = \{43\}$, with corresponding prices $p_{11}^* = 11.11$, $p_{31}^* = 12.89$, $p_{12}^* = 14.91$ and $p_{43}^* = 21.83$, and an optimal profit $\hat{\Pi}^* = 390.2$. Applying EMH to the base example gives an assortment which is exactly the same as the optimal assortment priced at a profit margin $m^H = 6.90$, and a corresponding profit $\Pi^H = 389.9$. This implies that EMH provides a high-quality solution with an optimality gap of 0.077%. This low gap can be understood by noting that the optimal assortment is composed of popular sets across the three nests with the two products with the two highest values of the average margin, $\alpha - c$, from nest 1, the product with the highest average margin from nest 2, and similarly with nest 3. Moreover, the optimal profit margins of the four products in the optimal assortment, $m_{11}^* = 7.11$, $m_{31}^* = 6.89$, $m_{12}^* = 6.91$, and $m_{43}^* = 6.83$ are almost equal in value and are very close to the EMH margin $m^H = 6.90$.

In Figure 5, we report on the EMH optimality gap by conducting a one-way sensitivity analysis on the parameters of the base case in Section 7.1, by changing the demand volume λ, the utility of the no purchase v_0, the reservation price, α_{12}, the unit cost, c_{12}, the $\alpha_{1i} - c_{1i}$ of all the products on nest 1, adding a new nest to the assortment and the inter- and intra-nest similarity parameters, μ_1 and μ_2, respectively, one at a time, while keeping the other parameters at their base values similar to what is done in Section 7.2. Figure 5 reveals an excellent performance of EMH with an optimality gap that is below 0.4% in all the considered cases. We again attribute this EMH performance to its exploitation of the structure of the optimal solution. In fact,

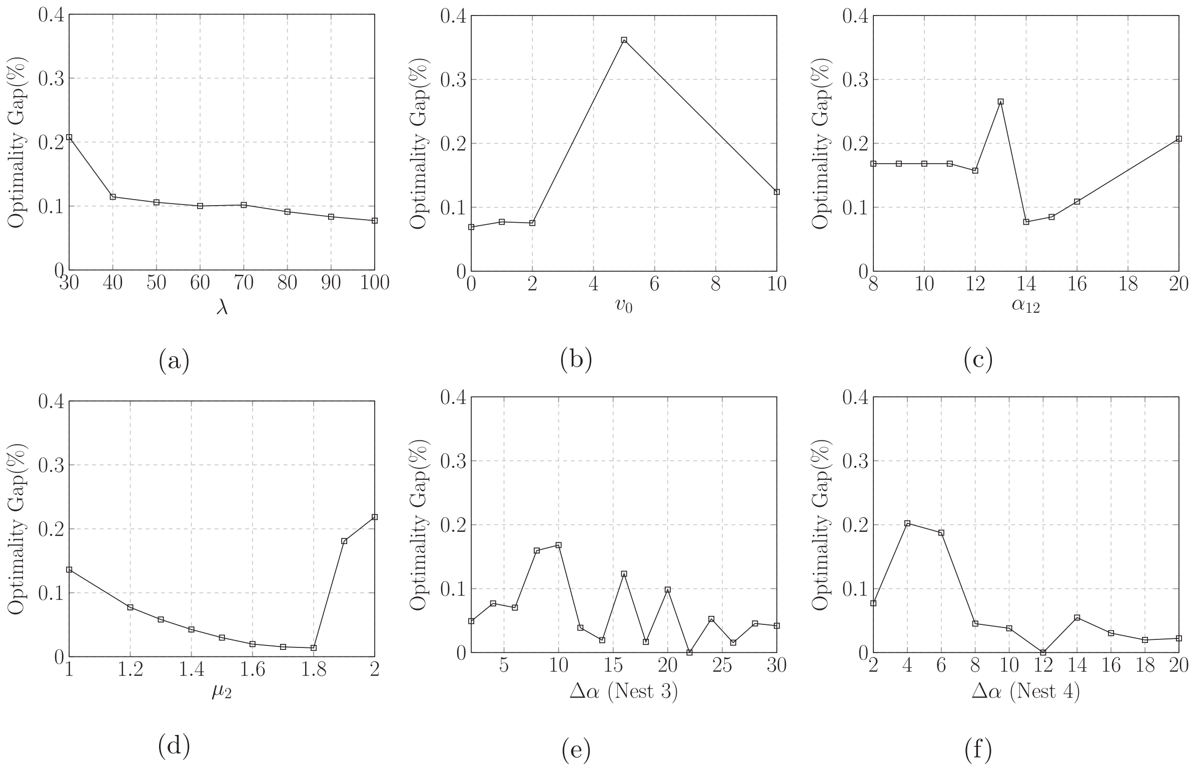

Figure 5. Performance of EMH.

in all the tested cases the optimal assortments, S^*, were composed of popular sets from the three nests, similar to the base case. Moreover, the optimal profit margins, $p^*_{li} - c_{li}$, were found to be close in almost all the test cases.

Further testing of EMH over five "extreme" scenarios is presented in Appendix D of the online supplement. In all these extreme cases, the EMH continues to exhibit an excellent performance with an optimality gap below 0.73%.

8. Conclusion

This article investigates pricing, assortment, and inventory decisions of horizontally differentiated fashion products facing a nested logit demand and sold over a single period. The pricing strategy suggested in this article is to have an equal profit margin on all the products in the offered assortment. For a given assortment, the expected profit at optimal inventory levels is found to be unimodal in the profit margin. This implies the existence of a single internal-point solution that can be easily found by a simple line search. We also compare the optimal profit margin with its "riskless" counter-part obtained under the assumption of ample inventory, which allows us to understand the effect of inventory considerations on pricing. A useful by-product of the comparison to the riskless case, are some easily obtained bounds that expedite the numerical search for the optimal profit margin. The structure of the optimal assortment under exogenous pricing and limited inventory is investigated utilizing tight bounds on the expected profit as proxy objectives. Under these approximate objective functions, the optimal assortment consists of the most popular products within each nest having the highest average margin (utility minus the cost). This suggests that popular sets lead to highly-profitable assortments in practice. In fact, for the common case of horizontally differentiate products with homogeneous unit cost in each nest, we find that an assortment consisting of popular sets within each nests is optimal under the actual objective function.

Motivated by the results on the structure of the optimal pricing and assortment, we propose an EMH that exploits popular sets to develop jointly optimal pricing, assortment, and inventory decisions. A detailed numerical study reveals an excellent performance of EMH with an optimality gap below 0.73% over all the tested cases. This suggests that EMH is a useful practical tool for integrated assortment, pricing, and inventory decision under nested logit with "no tears" due to its high numerical efficiency and simplicity.

Future work could attempt to analyze the structure of the optimal pricing and assortment without the equal profit margin requirement across all nests and products. Although we numerically demonstrate that the equal profit margin structure of pricing provides good solutions, we observe that the expected profit under general unrestricted prices is not well-behaved, in the sense of not being pseudoconcave. This complicates the unrestricted pricing problem, and suggests that efficient algorithms could be explored in the future. It is also worth noting that the recent literature (Li and Huh (2011); Li *et al.* (2015)) considers a generalized version of the nested logit choice with heterogeneous price elasticity and nest dissimilarity parameters. This literature establishes that the optimal pricing, in the riskless case with ample inventory, is characterized by distinct profit margins across nests while products in the same nest continue to have equal margins. It might be also useful in future research to investigate adapting our EMH heuristic to this more general nested logit choice.

Finally, it might be worth investigating a decentralized scenario where the products in different groups (nests) are managed by different retailers. In future work, one can analyze the equilibrium pricing and assortment strategy if the different retailers compete, as well as appropriate revenue sharing schemes when collaboration among these retailers is plausible.

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