Faculty of Computer Systems & Software Engineering

Formal methods.

Liveness and Fairness

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Liveness property

- We have developed specifications of HourClock, Async Interface and FIFO by setting restrictions on its possible states

- These are specifications of what a system *must not do* and describe so called ***safety*** *properties* of a system.
  E.g. **Len (Buf) < 3** or **hr \in (1..12)**

- Safety properties are satisfied by a **finite** behavior, which has not been violated by any step so far.

- They don't require that the system ever actually do anything.

Liveness properties

- Lets learn how to specify *that something does happen*, i.e. that the clock keeps ticking or a message is eventually read from memory;
- We will call it *liveness properties* – the ones, that cannot be violated at any particular state;
- Only by examining an entire **infinite** behavior we can tell that the clock has stopped ticking, or that a message is never sent;
- We will express liveness properties as temporal formulas.

Temporal Formulas

From the previous lectures we know, that:

- A state assigns a value to every variable of a system;
- A behavior is an infinite sequence of states;
- A temporal formula is true or false of a behavior.

Temporal formula ***F*** assigns a Boolean value to a behavior ***σ***

$$\sigma \models F$$

Note, in ASCII, we write **σ |= F**

We will say that F *is true* of behavior *σ*, or that *σ satisfies F*, iff σ |= F equals true.

Boolean combination of temporal formulas

- The formula **F /\ G** is true of a behavior σ iff both F and G are true of σ

$$\sigma \models (F \land G) \;\triangleq\; (\sigma \models F) \land (\sigma \models G)$$

- The formula **¬F** is true of σ iff F is not true of σ.

$$\sigma \models \neg F \;\triangleq\; \neg(\sigma \models F)$$

- These are the definitions of the meaning of /\ and of ¬ as operators on temporal formulas.
- The meanings of the other Boolean operators are similarly defined.

Temporal formulas

All the unquantified temporal formulas are Boolean combinations of three simple kinds of formulas:

- A *state predicate* (*an action that contains no primed variables*), is true of a behavior iff it is true *in the first state* of the behavior (e.g. HCini == hr \in (1 .. 12)).
- A formula $\Box P$ (in ASCII, []P), where **P** is a *state predicate*, is true of a behavior iff **P** is true *in every state* of the behavior (e.g. []HCini).
- A formula $\Box[N]_v$ (in ASCII, [][N]_v**)** , where **N** is an *action* and *v* is a state variable, is true of a behavior iff every successive pair of steps in the behavior is a $\Box[N]_v$ step (e.g. [][HCnxt]_hr).

Actions as temporal formulas

Let $\boldsymbol{\sigma}_{\mathbf{i}}$ – is the **(i + 1)** state of the behavior **σ** for any natural number i,

so σ is the behavior $\sigma_0 \rightarrow \sigma_1 \rightarrow \sigma_2 \rightarrow \cdots$

Arbitrary *action* **A** as a *temporal formula* iff **σ |= A** to be true for first two states of **σ** in **A** step.

That is, we define **σ |= A** to be true iff $\boldsymbol{\sigma_0 \rightarrow \sigma_1}$ is an **A** step.

In the special case, when **A** is a state predicate, $\boldsymbol{\sigma_0 \rightarrow \sigma_1}$ is an **A** step iff **A** is true in the zero state $\boldsymbol{\sigma_0}$

Actions as temporal formulas

$\Box[N]_v$ is true of a behavior iff each step is a $[N]_v$ step.

So for any natural number *n*

$\sigma \models \Box A$ to be true iff $\sigma_n \rightarrow \sigma_{n+1}$ is an A step

In other words, for any temporal formula A

$$\sigma \models \Box A \;\equiv\; \forall\, n \in Nat : \sigma^{+n} \models A$$

Where

$$\sigma^{+n} \;\triangleq\; \sigma_n \rightarrow \sigma_{n+1} \rightarrow \sigma_{n+2} \rightarrow \cdots$$

Temporal formulas - example

$$\sigma \models \Box((x = 1) \Rightarrow \Box(y > 0))$$

is true iff, for all $n \in Nat$, if $x = 1$ is true in state σ_n, then $y > 0$ is true in all states σ_{n+m} with $m \geq 0$.

Thus, it asserts that, any time **x = 1** is true, **y > 0** is true *from then on.*

We can read $\Box$ as ***always*** or *henceforth* or *from then on*.

Temporal formulas - stuttering steps

Specification should allow stuttering steps, that leave unchanged all the variables appearing in the formula.

We say that a formula **F is invariant under stuttering** iff adding or deleting a stuttering step to a behavior **σ** does not affect whether **σ** satisfies F.

A state predicate is always invariant under stuttering, i.e. since *its truth depends only on the first state of a behavior*, and adding a stuttering step doesn't change the first state.

Basic temporal formulas

Lets consider temporal formulas constructed from arbitrary temporal formulas F and G.

We read $\Diamond$ ***eventually*** (taking eventually to include now)
In ASCII, <>

$$\Diamond F \text{ is defined to equal } \neg\Box\neg F$$

$\Diamond$ asserts that F is **not always false**, which means that F is *true at some time*

Eventual action

$\Diamond\langle A \rangle_v$ is defined to equal $\neg\Box[\neg A]_v$

where A is an action and v a state function.

We define the action $\langle A \rangle_v$ by

$$\langle A \rangle_v \triangleq A \wedge (v' \neq v)$$

so $\Diamond\langle A \rangle_v$ asserts that eventually an $\langle A \rangle_v$ step occurs.

$(v' \neq v)$ is a condition that it is not stuttering step, i.e. value of v is changed

Temporal formulas

$F \leadsto G$ is defined to equal $\Box(F \Rightarrow \Diamond G)$

We read $\leadsto$ as *leads to*.

The formula $F \leadsto G$ asserts that whenever F is true, G is *eventually* true, that is, G *is true then* or *at some later time*.

Infinitely Often (always eventually)

$\Box\Diamond F$ asserts that at all times,

F is true then or at some later time.

For time 0, this implies that F is true at some time $n_0 \geq 0$.

For time n_0+1, it implies that F is true at some time $n_1 \geq n_0 + 1$.

In other words, $\Box\Diamond F$ implies that *F is infinitely often true*.

Eventually always

$\Diamond\Box F$ asserts that eventually (at some time), F becomes true and remains true thereafter. In other words, $\Diamond\Box F$ asserts that F is *eventually always* true.

Adding liveness for Hour Clock

For HourClock we can require that the clock never stops by asserting that there must be infinitely many HCnxt steps.

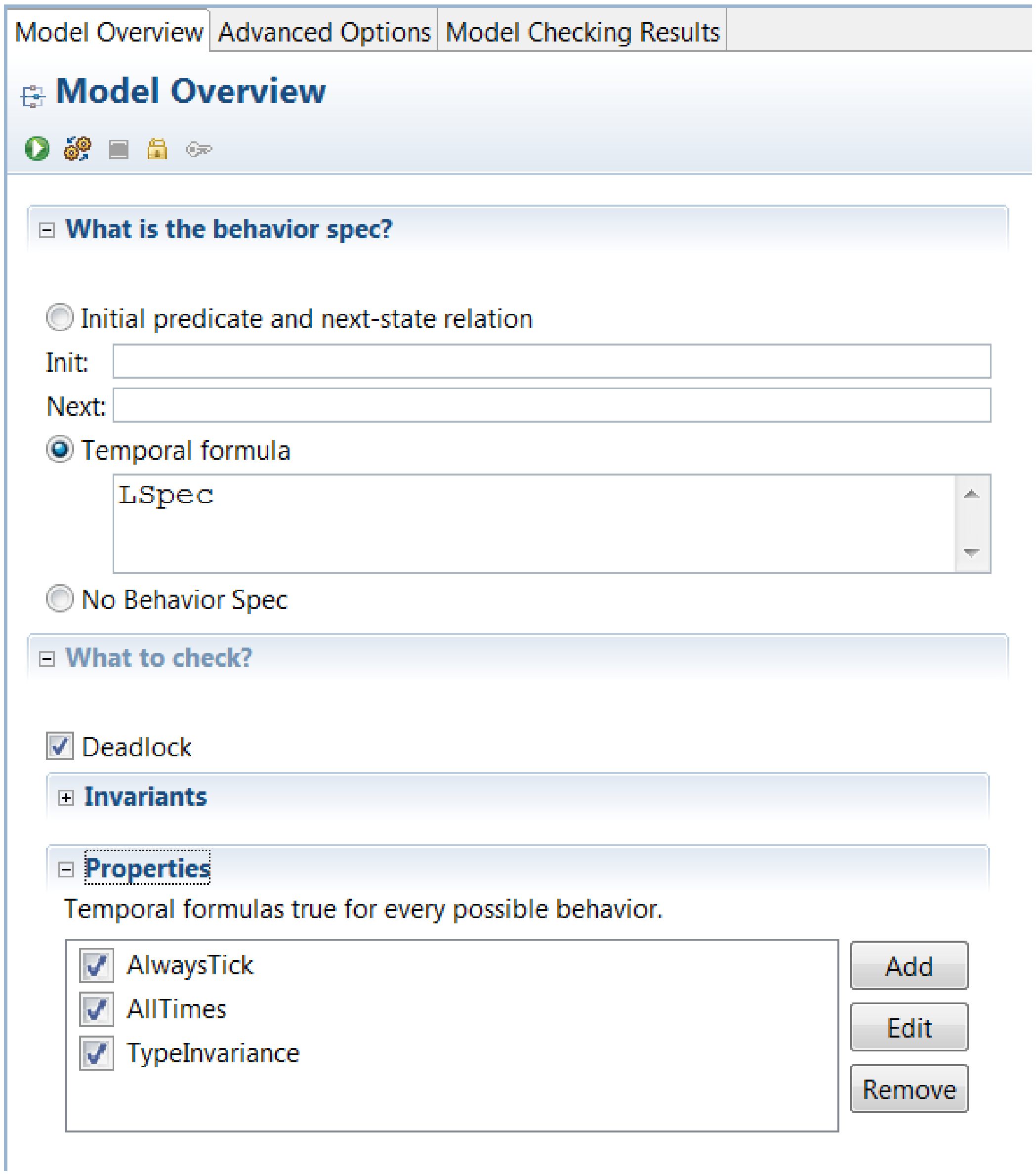
$$\Box\Diamond\langle HCnxt \rangle_{hr}$$

By conjoining the liveness condition to the safety specification HC we will have specification of a clock that never stops.

$$HC \land \Box\Diamond\langle HCnxt \rangle_{hr}$$

```
----------------------- MODULE LiveHourClock --------------------------
(* Add the liveness condition to the hour clock specification of module HourClock. *)
EXTENDS HourClock

(* Conjoin the specification with the week fairness condition *)
LSpec == HC /\ WF_hr(HCnxt)

(* Define some properties that LSpec satisfies. *)
(* Asserts that infinitely many <<HCnxt>>_hr steps occur. *)
AlwaysTick == []<><<HCnxt>>_hr

(* Asserts that, for each time n in 1..12, hr infinitely often equals n. *)
AllTimes == \A n \in 1..12 : []<>(hr = n)

TypeInvariance == []HCini
(* The temporal formula asserting that HCini is always true. *)
(* It is stated in this way to show another way of telling TLC to check an invariant. *)
------------------------------------------------------------------------------
THEOREM  LSpec => AlwaysTick /\ AllTimes /\ TypeInvariance
==========================================================
```

TLC - checking temporal properties



Temporal properties of LiveHourClock

When **hr** is equal to 1, it implies that **hr** eventually will have value 2

New == (hr = 1) => <>(hr = 2)   * True of False?

When **hr** is equal to 1, it implies that **hr** always will have value 2.

New == (hr = 1) => [](hr = 2)   * True of False?

If this property *infinitely often* true

New == (hr = 1) => []<>(hr = 2)   * True of False?

If this property *eventually always true*

New == (hr = 1) => <>[](hr = 2)   * True of False?

THEOREM LSpec => AlwaysTick /\ AllTimes /\ TypeInvariance /\ **New**

ENABLED action

In any behavior satisfying the safety specification HC, it's always possible to take an HCnxt step *that changes hr*.

Action $\langle HCnxt \rangle_{hr}$ is therefore always enabled, so $\textsc{Enabled}\ \langle HCnxt \rangle_{hr}$ is true throughout such a behavior.

Formally:

$$\Box(\textsc{Enabled}\ \langle HCnxt \rangle_{hr} \Rightarrow \Diamond\langle HCnxt \rangle_{hr})$$

In general, **ENABLED A** is a predicate that is true iff action A is enabled, meaning *that it is possible to take **A** step*, that changes state variables.

General liveness condition - Weak Fairness

The general liveness condition for an action A

$$\Box(\text{ENABLED}\ \langle A \rangle_v \Rightarrow \Diamond \langle A \rangle_v)$$

This condition asserts that, if A ever becomes enabled, then an A step will *eventually occur*

Next formula is called **Weak Fairness** $\mathrm{WF}_v(A)$

$$\Box(\Box\text{ENABLED}\ \langle A \rangle_v \Rightarrow \Diamond \langle A \rangle_v)$$

- This formula asserts that, if *A ever becomes forever enabled*, *then an A step must eventually occur*.
- **WF** stands for Weak Fairness, and the condition $\mathrm{WF}_v(A)$ is called weak fairness on A.

Strong Fairness

$\mathrm{SF}_v(A)$

If A is infinitely often enabled, then infinitely many A steps occur.

$\Box\Diamond\textsc{Enabled}\,\langle A\rangle_v \;\Rightarrow\; \Box\Diamond\langle A\rangle_v$

Temporal Tautologies

The temporal tautology $\Box F \Rightarrow F$ asserts the obvious fact that, if F is true at all times, then it's true at time 0.

$\neg\Box F \equiv \Diamond\neg F$

F is not always true iff it is eventually false.

$\Box(F \wedge G) \equiv (\Box F) \wedge (\Box G)$

F and G are both always true iff F is always true and G is always true. Another way of saying this is that $\Box$ distributes over $\wedge$.

$\Diamond(F \vee G) \equiv (\Diamond F) \vee (\Diamond G)$

F or G is eventually true iff F is eventually true or G is eventually true. Another way of saying this is that $\Diamond$ distributes over $\vee$.

Temporal Tautologies

The operator $\Box$ doesn't distribute over $\vee$, nor does $\Diamond$ distribute over $\wedge$. For example, $\Box((n \geq 0) \vee (n < 0))$ is not equivalent to $(\Box(n \geq 0) \vee \Box(n < 0))$; the first formula is true for any behavior in which n is always a number, but the second is false for a behavior in which n assumes both positive and negative values.

$\Box\Diamond$ distributes over $\vee$ and $\Diamond\Box$ distributes over $\wedge$:

$$\Box\Diamond(F \vee G) \equiv (\Box\Diamond F) \vee (\Box\Diamond G) \qquad \Diamond\Box(F \wedge G) \equiv (\Diamond\Box F) \wedge (\Diamond\Box G)$$

The first asserts that F or G is true infinitely often iff F is true infinitely often or G is true infinitely often.

Model checking example: Mutual exclusion

- The mutual exclusion problem (mutex)
  - Avoiding the simultaneous access to some kind of resources by use of the *critical sections* of concurrent processes
- The problem is to find a *protocol* for determining which process is allowed to enter its critical section
- Some expected properties for a correct protocol: Safety, Liveness, Non-blocking, No strict sequencing

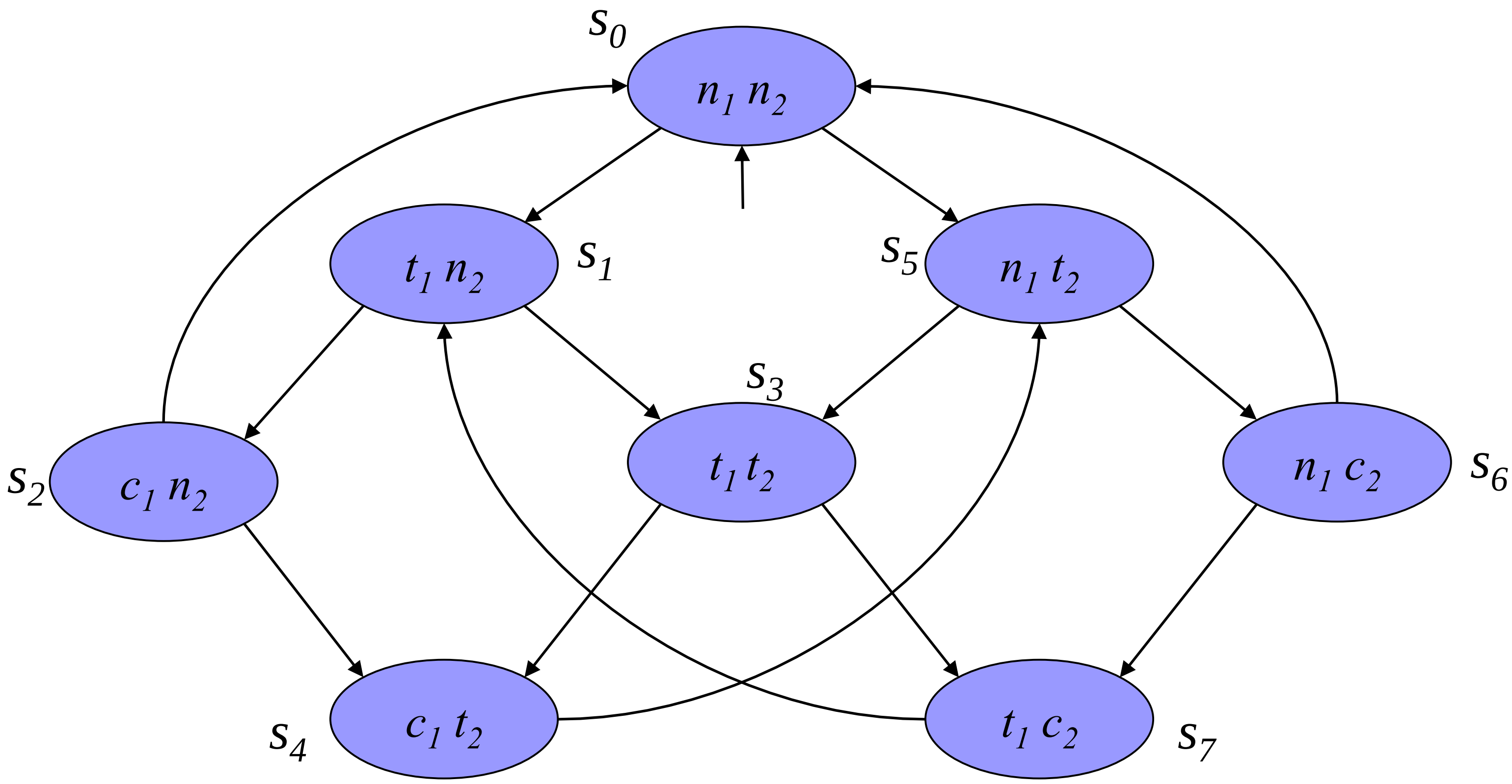
- Safety: Only one process is in its critical section at any time.
- Liveness: Whenever any process requests to enter its critical section, it will eventually be permitted to do so.
- Non-blocking: A process can always request to enter its critical section.
- No strict sequencing: Processes not need enter their critical section in a strict sequence.

Modeling mutex

- Consider each process to be either in its non-critical state *n*, trying to enter the critical section *t* or *c*
- Each individual process has this cycle:
  - $n \rightarrow t \rightarrow c \rightarrow n \rightarrow t \rightarrow c \rightarrow n \ldots$
- The processes phases are interleaved
- We will define *n, t, c* as temporal formulas

2 process mutex

- The processes are *asynchronous interleaved*
  - one of the processes makes a transition while the other remains in its current state



- Safety: Only one process is in its critical section at any time:

  $[]\ \neg(c_1 \wedge c_2)$

- Liveness: Whenever any process requests to enter its critical section, it will eventually be permitted to do so:

  $[]\ (n_1 => <>\ t_1)$

- Non-blocking: A process can always request to enter its critical section:
  - $[]\ (\backslash E\ t_1 : n_1 => t_1)$
- No strict sequencing: Processes not need enter their critical section in strict sequence:
  - $[]<>(c_1 \sim> t_1)\ /\backslash\ []<>(c_2 \sim> t_2)$

Thank you for your attention!
Please ask questions