ME 280A Fall 2014

HOMEWORK 4: ERROR ESTIMATION & ADAPTIVE MESHING

- Consider the two following problems, with their analytical solutions, defined over the domain $\Omega = (0, L)$:

$$
\begin{aligned}
&\tfrac{d}{dx}\left(A_1 \tfrac{du}{dx}\right) = -90\pi^2 \sin(3\pi x)\sin\left(36\pi x^3\right) + ... \\
&(10\sin(3\pi x) + 5)\left(216\pi x \cos\left(36\pi x^3\right) - 11664\pi^2 x^4 \sin\left(36\pi x^3\right)\right) + ... \\
&6480\pi^2 x^2 \cos(3\pi x)\cos\left(36\pi x^3\right) \\
&A_1 = 1.0 \\
&L = 1 \\
&u(0) = 0 \\
&u(L) = 0 \\
&u^{True} = (10\sin(3\pi x) + 5)\sin\left(36\pi x^3\right)
\end{aligned}
\tag{1}
$$

$$
\begin{aligned}
&\tfrac{d}{dx}\left(A_1 \tfrac{du}{dx}\right) = 256\sin\left(\tfrac{3}{4}\pi x\right)\cos(16\pi x) \\
&A_1 = \begin{cases} 0.2; x < 1/3 \\ 2.0; x \geq 1/3 \end{cases} \\
&L = 1 \\
&u(0) = 0 \\
&A_1(L)\tfrac{du}{dx}(L) = 1 \\
&u^{True} = \tfrac{512}{4087\pi A_1}\left(\frac{268\sin\left(\frac{61\pi x}{4}\right)}{61\pi} - \frac{244\sin\left(\frac{67\pi x}{4}\right)}{67\pi}\right) ... \\
&+ \begin{cases} \left(5 + \frac{7680\sqrt{2}}{4087\pi}\right)x; x < \frac{1}{3} \\ \frac{2304\sqrt{2}\left(8210 - 768\sqrt{3} + 4087\pi\right)}{16703569\pi^2} + \frac{3}{2} + \left(\frac{1}{2} + \frac{768\sqrt{2}}{4087\pi}\right)x; x \geq \frac{1}{3} \end{cases}
\end{aligned}
\tag{2}
$$

- Compute the finite element solution u^N to both problems using linear equal-sized elements. For each problem, determine how many elements are needed in order to achieve

$$
e^N \stackrel{\text{def}}{=} \frac{||u - u^N||_{A_1(\Omega)}}{||u||_{A_1(\Omega)}} \leq TOL = 0.05\,,
$$

$$
||u||_{A_1(\Omega)} \stackrel{\text{def}}{=} \sqrt{\int_\Omega \frac{du}{dx} A_1 \frac{du}{dx} dx}
$$

For the second problem, do the process twice. First, solve it with equally spaced elements. Then, place a mesh node exactly at the discontinuity, and mesh each side with equally sized elements so that all of the elements on the right side are the same size, and all of the elements on the left side are the same size. By uniformly increasing the number of elements on each side (so that $h_{left} \approx h_{right}$), determine how many total elements are required to satisfy the error criterion.

- Plot the position of the first node of each element X_I versus E_I, where

$$E_I \stackrel{\text{def}}{=} \frac{\frac{1}{h_I}||u - u^N||^2_{A_1(\Omega_I)}}{\frac{1}{L}||u||^2_{A_1(\Omega)}} .$$

  Here I is the element index, h_I is the length of element I, Ω_I is the domain of element I, and the error norm over an individual element is defined as

$$||u||^2_{A_1(\Omega_I)} \stackrel{\text{def}}{=} \int_{\Omega_I} \frac{du}{dx} A_1 \frac{du}{dx} dx .$$

- Modify the 1D FEM code from the previous assignments so that it can automatically refine the mesh using the following process:

  1. Solve the finite element proble for a given mesh.
  2. Compute E_I on each element.
  3. Subdivide every element in which $E_I \geq TOL_E$ into two equal parts.
  4. Repeat until no elements need to be subdivided.

  Refine the mesh (by dividing elements into two) until $E_I < TOL_E$ for all I. Use this criterion to refine your mesh, starting with $N = 16$ equal-sized elements. Solve both of the problems using automatic refinement.

  - By adjusting TOL_E, determine how many elements are needed to achieve $e^N \leq TOL = 0.05$ using this procedure.
  - Plot the final solution together with the exact solution. Use appropriate markers to denote how the elements are distributed in your final numerical solution.
  - Tabulate the final number of elements that fall into each of the initial 16 elements.
  - Plot X_I versus E_I for the solution.
  - For the second problem, repeat the process, but mesh around the discontinuity in A_1. Put the sixth of the initial mesh points exactly on the discontinuity and have equal sized elements on either side, so that there are 5 elements to the left and 11 to the right. Perform the refinement procedure and comment on differences from the original initial mesh.

- **Report:**

  1. Please follow the usual instructions for submission of the report. You do not need to repeat content from previous homeworks - just refer to them in your current report.
  2. For the description of procedure/implementation - please write it clearly, briefly, and in a manner such that someone else can read through your implementation details and write the program that you are describing.