# Ahmad Zareei
# Homework 2 - Higher Order Elements
# ME 280A

## Introduction

In the previous problem set we solved 1-D linear conservation problem as

$$\frac{d}{dx}\left(A_1\frac{du}{dx}\right) = f\left(x\right)$$

with finite element method and using linear elements. In this problem set, we will investigate the effect of using higher order elements (quadratic and cubic elements) in the finite element solution of this 1-D equtaion.

Increasing order of elements used, will increase the complexity of the code. However the error will decrease faster. There is a compromise between the complexity and the error. Higher order elements converge faster, however it takes more time to write the code. We will see how increasing the order of elements will affect the error and convergence to the exact solution. Note that, Increasing order of elements, assumes that the solution is smooth and deffrenciable in the elements. For the cases where the solution is not smooth higher order elements might not converge faster than linear elements. Therefore a rough understanding of the solution is needed to decide what order of elements to pick for the finite element code. In this problem set we will find the relation between error and element size for different order of elements.

## Objectives

(I) Solving a 1-D steady state conservation equation with a driving force using Finite element method with equally sized elements.

(II) Understanding the error analysis and comparing the result with exact solution.

(III) Understanding the effect of using higher order elements in the error.

(IV) Using Gauss quadrature method in computing the integrals

(V) Finding the relation between the error and the element size

## Problem, Procedure and Results

1. Solve the following boundary value problem, with domain $\Omega = (0, L)$, analytically :

$$\frac{d}{dx}\left(A_1\frac{du}{dx}\right) = 256\sin\left(\frac{3}{4}\pi x\right)\cos\left(16\pi x\right) \tag{1}$$

where $A_1 = \text{const.} = 0.2$ and $k = \text{const.}$, $L = 1$ and $u(0) = \Delta_1 = 0$ and $A_1(L)u'(L) = 1$.

**Solution:**

This equation is a inhomogenious equation, so it will have a particular and a general solution. The general solution is as

$$u(x) = Ax + B \tag{2}$$

where $A$ and $B$ are some constants to be defined. We first rewrite the right handside as

$$256 \sin\left(\frac{3}{4}\pi x\right) \cos(16\pi x) = 128 \sin(16.75\pi x) - 128 \sin(15.25\pi x) \tag{3}$$

with integrating twice of the right handisde, we will have

$$u(x) = -\frac{128}{A_1(16.75\pi)^2} \sin(16.75\pi x) + \frac{128}{A_1(15.25\pi)^2} \sin(15.25\pi x) + Ax + B \tag{4}$$

Using the boundary condition of $u(0) = 0$, results in $B = 0$. With the boundary condition of $A_1 u'(1) = 1$, we will have

$$\frac{128}{16.75\pi}\frac{\sqrt{2}}{2} - \frac{128}{15.25\pi}\frac{\sqrt{2}}{2} + A_1 A = 1 \tag{5}$$

which results in

$$u(x) = -\frac{128}{A_1(16.75\pi)^2} \sin(16.75\pi x) + \frac{128}{A_1(15.25\pi)^2} \sin(15.25\pi x) + \frac{1}{A_1}\left(\frac{3072}{4087\pi\sqrt{2}} + 1\right) x \tag{6}$$

Equation (6) gives the exact solution of the equation (1) satisfying boundary conditions.

2. Solve this with the finite element method using order $p$ equal-sized elements. In order to achieve

$$e^N = \frac{||u - u^N||_{A_1(\Omega)}}{||u||_{A_1(\Omega)}} \tag{7}$$

where

$$||u||_{A_1(\Omega)} = \sqrt{\int_\Omega \frac{du}{dx} A_1 \frac{du}{dx} dx}. \tag{8}$$

How many finite elements ($N$) are needed for $p = 1$, $p = 2$, $p = 3$.

**Solution:**

The weak form of the equation is same as what we found in the first assignment with a slight change because of the neumann boundary condition on the right hand side. According to the homework 1, we have

$$\int_\Omega \left(\frac{d}{dx}\left(A_1 \frac{du}{dx}\right) - f(x)\right) v \mathrm{d}x = 0 \tag{9}$$

where $v(x)$ is the test function and $f(x)$ is the right hand side function. With integrating by parts, we obtain

$$A_1 \frac{du}{dx} v|_{\partial\Omega} - \int_\Omega A_1 \frac{du}{dx}\frac{dv}{dx} dx - \int_\Omega f(x) v \mathrm{d}x = 0 \tag{10}$$

The first term of the above integral in the first assignment was zero, Now it has some value, since $A_1 u(1) = 1$. So we will have

$$\int_\Omega A_1 \frac{du}{dx}\frac{dv}{dx} \mathrm{d}x = -\int_\Omega f(x) v \mathrm{d}x + v(x)|_{x=1} \tag{11}$$

Here again, we chose $v(x)$, such that to be zero at $x = 0$. Now we suppose that $u = \sum a_j \phi_j$ and $v = \sum b_i \phi_i$, for some basis functions $\phi_i$. We should note that $\{b_i\}$ are arbitrary since the integral should be true for any function of $v(x)$. The goal in here is to find system of equations for $\{a_i\}$ such that $u = \sum a_j \phi_j$ be the solution. Subistituting $u(x)$ and $v(x)$ in our equation, we obtain

$$\sum_i b_i \left( \sum_j \left( \int_\Omega A_1 \frac{d\phi_i}{dx}\frac{d\phi_j}{dx} \mathrm{d}x \right) a_j + \int_\Omega f(x)\phi_i(x)\mathrm{d}x - \phi_i(x)|_{x=1} \right) = 0 \tag{12}$$

where the last term is computed on the boundary of $x = 1$. Since the equation should be true for any choice of the test function i.e. different $b_i$, we should have

$$\sum_j \left( \int_\Omega A_1 \frac{d\phi_i}{dx}\frac{d\phi_j}{dx} \mathrm{d}x \right) a_j + \int_\Omega f(x)\phi_i(x)\mathrm{d}x - \phi_i(x)|_{x=1} = 0. \tag{13}$$

Using element nodes of

$$0 = x_0 < x_1 < ... < x_i = ih < ... < x_N = 1 \tag{14}$$

and use of basis functions such that $\phi_i(x_i) = 1$ and zero at all the other nodes, we will have

$$\sum_j \left( \int_\Omega A_1 \frac{d\phi_i}{dx}\frac{d\phi_j}{dx} \mathrm{d}x \right) a_j = -\int_\Omega f(x)\phi_i(x)\mathrm{d}x + 1\delta_{i,N}. \tag{15}$$

where $\delta_{i,N}$ is 1 when $i = N$ and 0 elsewhere. So we only need to correct the loading vector for the element of last node. The other terms remain exactly the same.

For computing the integrals, we first maped the elements to the master element between $[-1, 1]$ and then used Guass quadrature method, to compute these integrals. Guass quadrature method for computing an arbitrary funcuton $g(x)$ between $[-1, 1]$ states that

$$\int_{-1}^{1} g(x)\mathrm{d}x = \sum_{i=1}^{N} w_i g(x_i) \tag{16}$$

where $N$ shows the number of points used and $w_i$'s are weighting coefficients and $x_i$'s are Guass points. Number of points needed for computing the integrals are evaluated such that it gives error of less than double precision error $10^{-16}$ compared with integrals computed with quad integral of MatLab.

We will use three different basis functions. One is linear and it will be one at the node and zero at all other elements. The key element for linear basis ($p = 1$) functions will be

$$\phi_1(\zeta) = \frac{1-\zeta}{2} \quad \phi_2(\zeta) = \frac{1+\zeta}{2}. \tag{17}$$

These basis for quadratic case ($p = 2$) are

$$\phi_1(\zeta) = \frac{\zeta(\zeta-1)}{2} \tag{18}$$

$$\phi_2(\zeta) = (1-\zeta)(1+\zeta) \tag{19}$$

$$\phi_3(\zeta) = \frac{\zeta(\zeta+1)}{2} \tag{20}$$

and for the cubic case($p = 3$), we use Legendre interpolation and we find that

$$\phi_1(\zeta) = \frac{(\zeta+1/3)(\zeta-1/3)(\zeta-1)}{(-1+1/3)(-1-1/3)(-1-1)} \tag{21}$$

$$\phi_2(\zeta) = \frac{(\zeta+1)(\zeta-1/3)(\zeta-1)}{(-1/3+1)(-1/3-1/3)(-1/3-1)} \tag{22}$$

$$\phi_3(\zeta) = \frac{(\zeta+1)(\zeta+1/3)(\zeta-1)}{(1/3+1)(1/3+1/3)(1/3-1)} \tag{23}$$

$$\phi_4(\zeta) = \frac{(\zeta+1)(\zeta+1/3)(\zeta-1/3)}{(1+1)(1+1/3)(1-1/3)} \tag{24}$$

So these will test functions will be used for elemenets.

We did a binary search similar to what we did in the first assignment to find the minimum number of elements for the error to be less than 0.01. The number of elements for needed for this error are tabulated in table 1.

| p | 1 (linear) | 2 (quadratic) | 3 (cubic) |
|---|---|---|---|
| N | 1348 | 94 | 34 |

Table 1: Minimmum number of elements to achieve relative error of 0.01

3. Plot the numerical solutions for several values of $N$, for each $p$, along with the exact solution.

The only problem in here is that, cubic and quadratic test functions, the solution will be as

$$u_{element\ i} = \sum u_j \phi_j \tag{25}$$

where $u_j$ are values at nodes found by solving the matrix and $\phi_j$'s are test functions. For the case of quadratic and cubic test functions, we need more points to plot them. We descritize the master (key) element, into 10 pieces and found the equivalent $x$ and $u$ for the points and then stored these values for each element into an array in order to plot them.

Plots of the exact solution and convergence of the nuerical solution to the exact solution are shown in figures 1, 2 for the linear case ($p = 1$), figures 4, 5 for the quadratic case ($p = 2$) and figures 7, 8 for the cubic case ($p = 3$). Errors for different runs are shown in figure 3 for the linear case ($p = 1$), figure 6 for the quadratic case ($p = 2$) and figure 9 for the cubic case ($p = 3$).

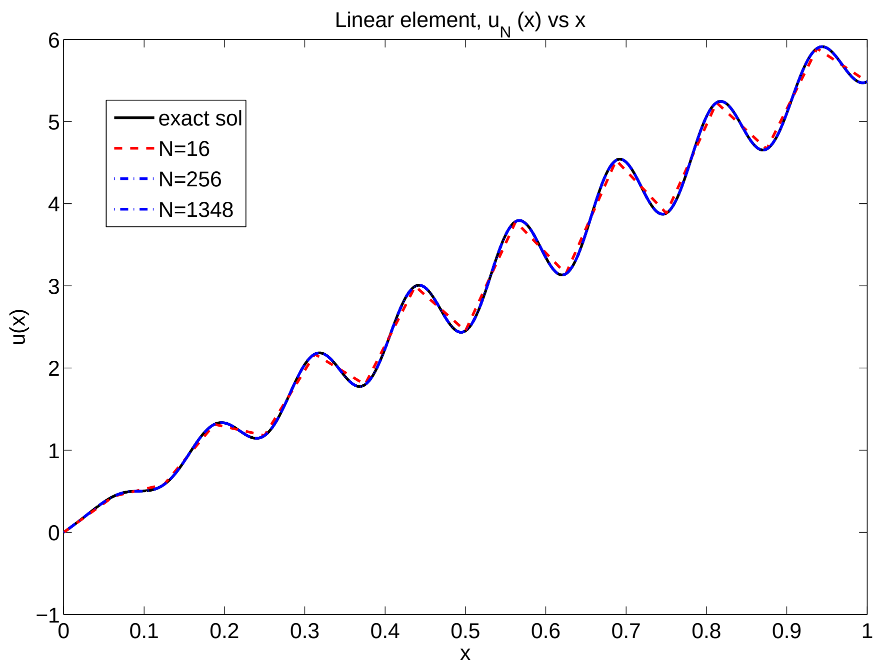


Figure 1: Exact solution of $u(x)$ with numerical solution with linear element for different $N$

4. Plot $e^N$ as a function of the element size $h$ for each $p$.

   Figures of Error vs element size are plotted in figure 10 for the linear case ($p = 1$), figure 11 for the quadratic case ($p = 2$) and figure 12 for the cubic elements ($p = 3$).

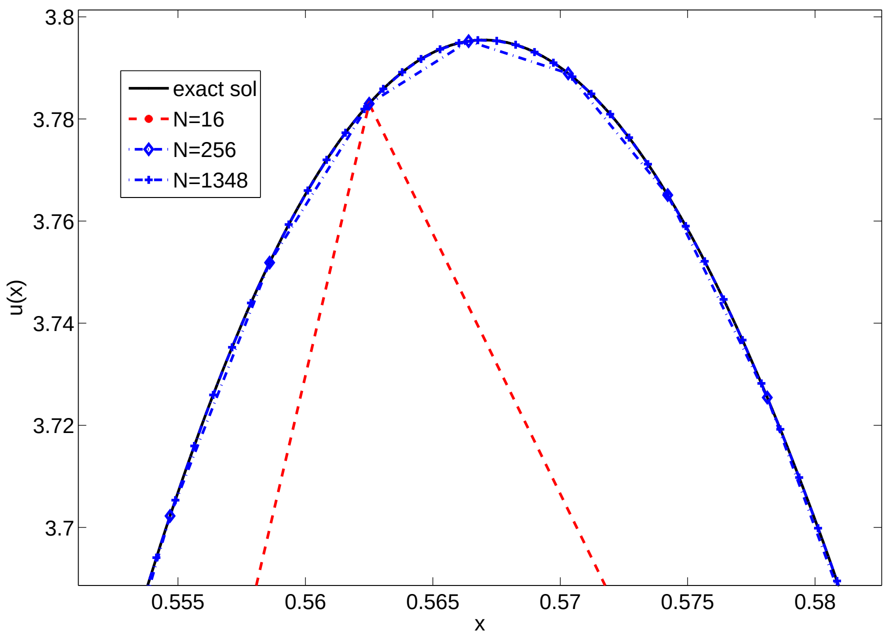


Figure 2: Exact solution of $u(x)$ with numerical solution with linear element for different $N$, zoomed

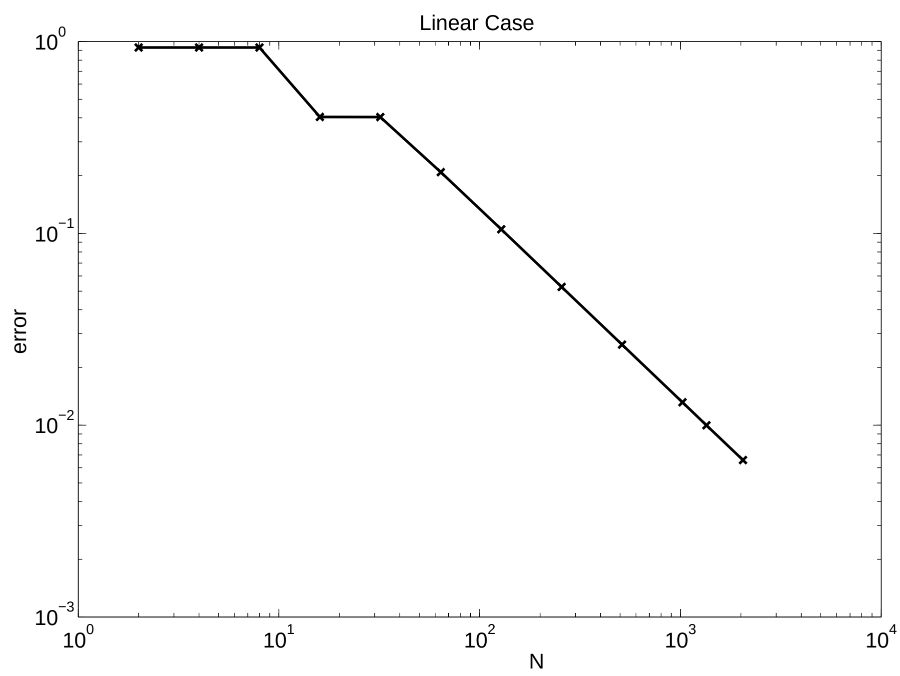


Figure 3: Error of numerical solution with linear element for different $N$

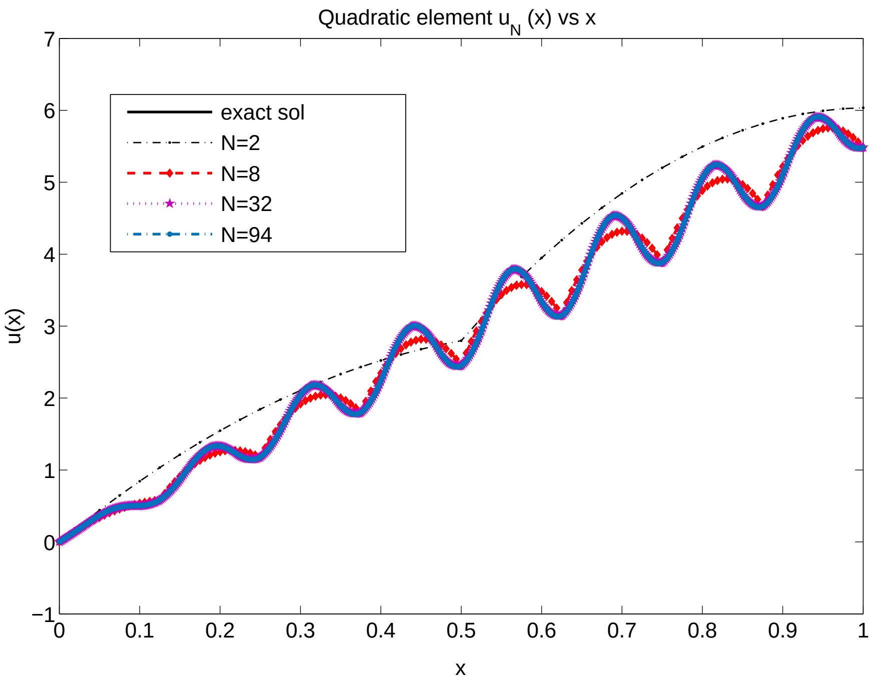


Figure 4: Exact solution of $u(x)$ with numerical solution with quadratic element for different $N$

5. Plot $e^N$ as a function of the number of degrees of freedom for each $p$.

   Figures of Error vs number of degrees of freedom are plotted in figure 13 for the linear case ($p = 1$), figure 14 for the quadratic case ($p = 2$) and figure 15 for the cubic elements ($p = 3$).

6. Determine the relationship between the error and the element size for each $p$ ($e^N = f(h)$).

   This relationship between error $e^N$ and element size $h$ is found by fitting a linear polynomial to the data with "polyfit" command in MatlLab. Since the relation is as $e^N = Ah^B$, we fitted a linear line to logarithmic values of error and step size. Result are shown in figure 10 for the linear case ($p = 1$), figure 11 for the quadratic case ($p = 2$) and figure 12 for the cubic elements ($p = 3$). Tabulated values are as table 2. Coefficients of $\log h$ in table 2, shows the power relation between error and step size.

| | |
|---|---|
| $p = 1$ | $\log e = 2.55 + 0.993 \log h$ |
| $p = 2$ | $\log e = 4.82 + 2.087 \log h$ |
| $p = 3$ | $\log e = 4.58 + 2.682 \log h$ |

Table 2: Relation between Error $e^N$ and step size $h$ for different element types $p$

This means that for linear case error decrease with first power of step size i.e. $e \sim h^1$. For quadratic case this relation is $e \sim h^2$ and for the cubic elements $e \sim h^3$.

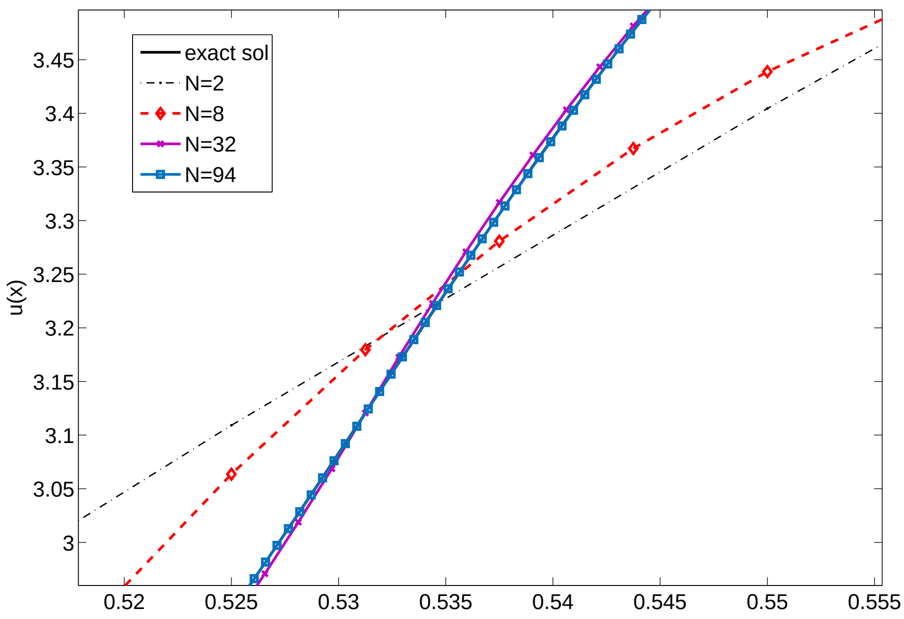


Figure 5: Exact solution of $u(x)$ with numerical solution with quadratic element for different $N$, zoomed

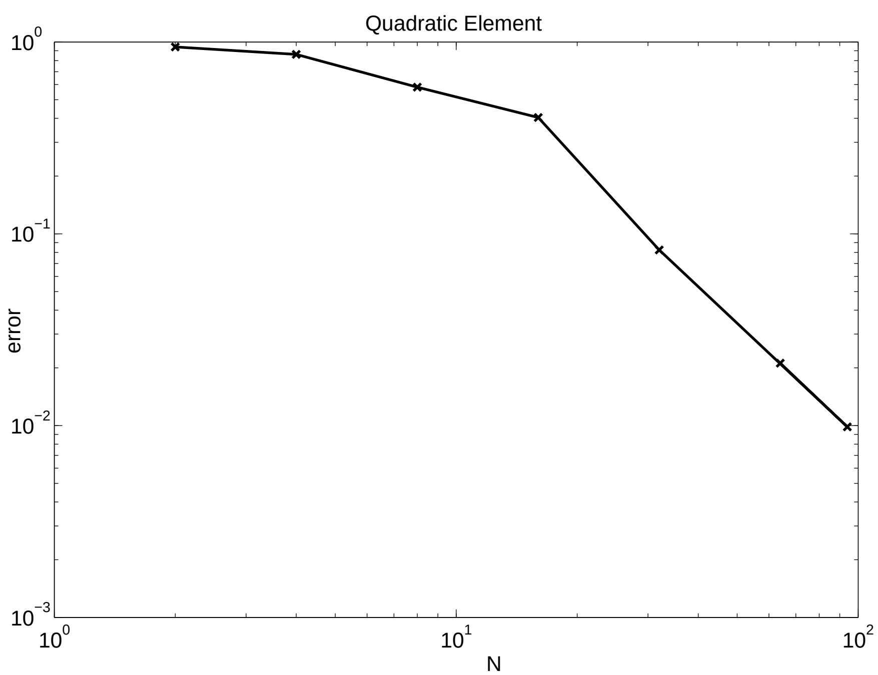


Figure 6: Error numerical solution with linear element for different $N$

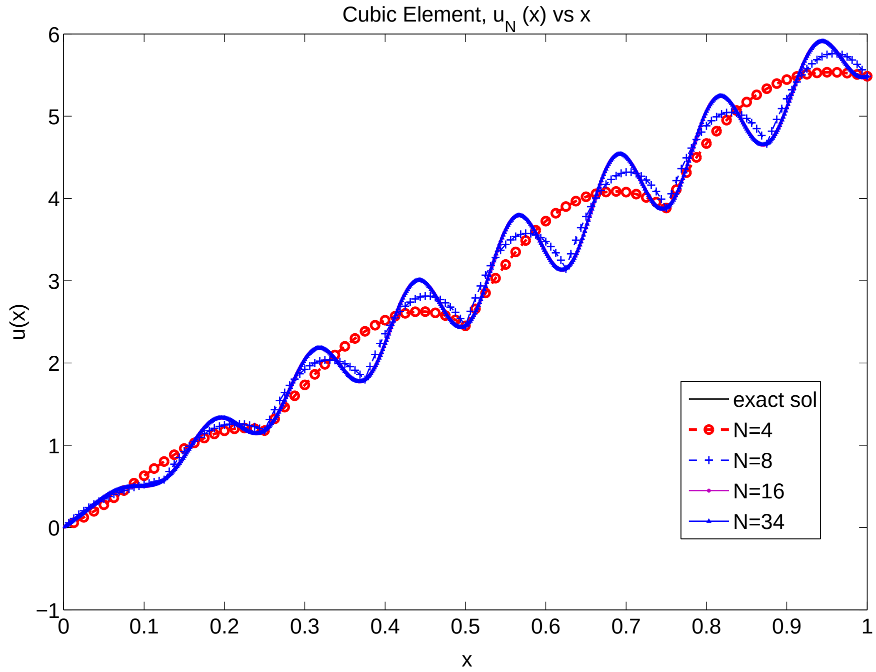


Figure 7: Exact solution of $u(x)$ with numerical solution with cubic element for different $N$

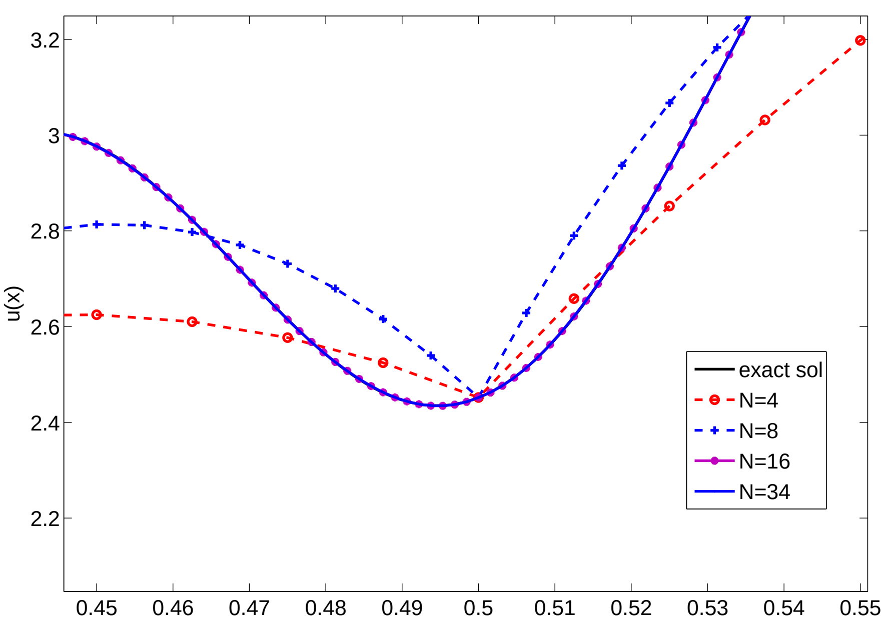


Figure 8: Exact solution of $u(x)$ with numerical solution with cubic element for different $N$, zoomed

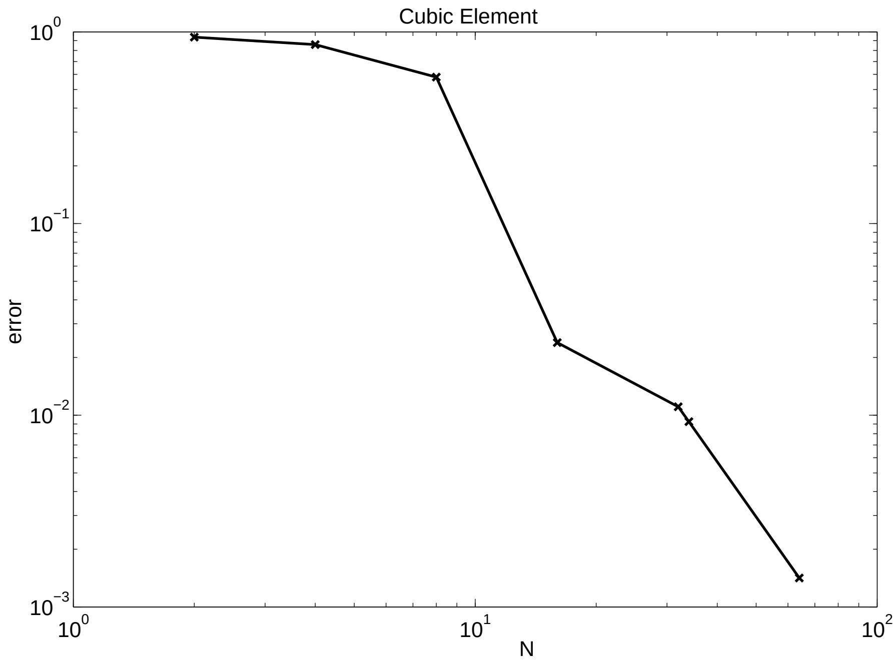


Figure 9: Error numerical solution with cubic element for different $N$

# Discussion

We found that the number of elements needed for a certain value in the error decreases dramatically by using higher order elements. Number of nodes for the error of 0.01 decreased from 1349 nodes (1348 elements) using linear elements to 103 nodes (34 elements) for the cubic elements. This is a decrease of order 10 in number of elements which is significant. I must mention that the solution is assumed to be smooth and defferenciable for using higher order elements, otherwise the linear elements will converge faster. Relation between the error $e$ and element size $h$ for different elements was found. In linear case the error will decrease with first power of the element size $e \sim h^1$, for quadratic elements ($p = 2$), this error decrease with second power of element size $e \sim h^2$ and for the cubic elements it will decrease as the third power of element size i.e. $e \sim h^3$. This power of $h$ that increases as the we use higher order elements, is the cause of the significance decrease in number of the elements needed.

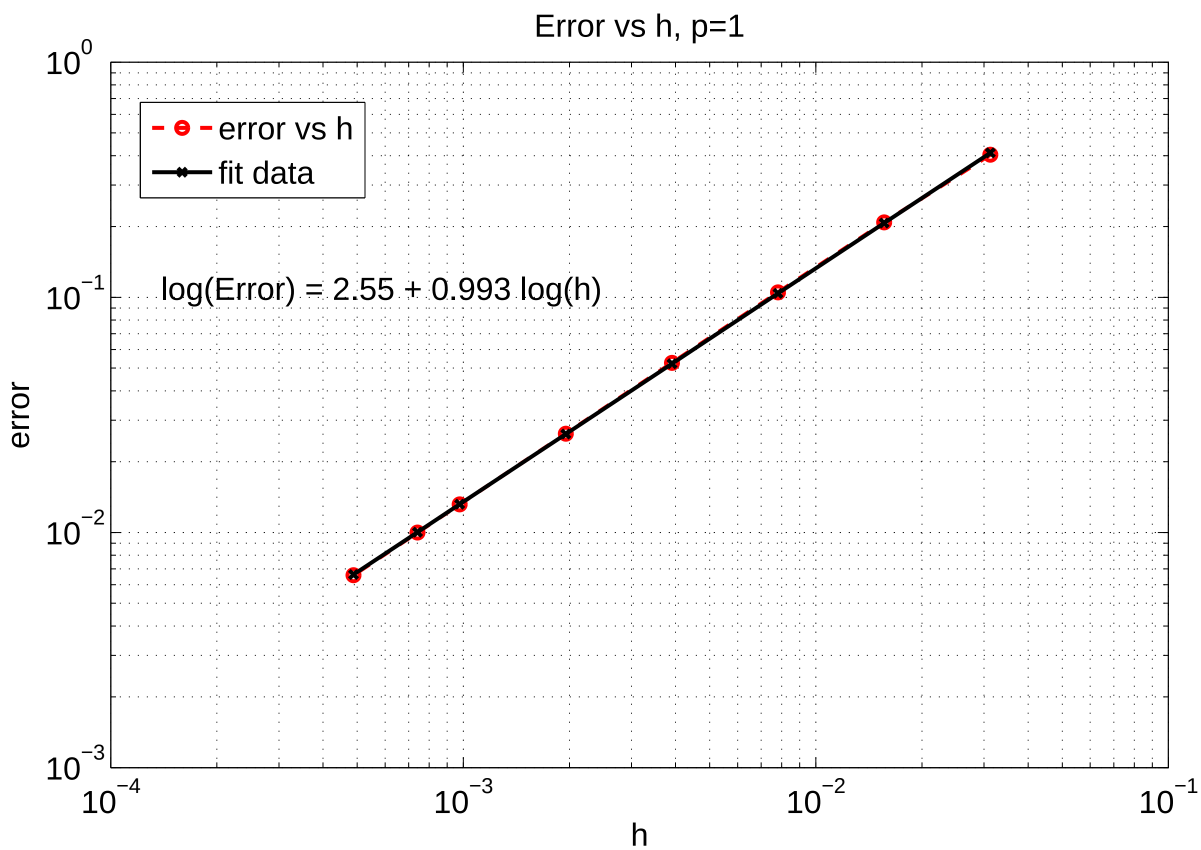


Figure 10: Error vs $h$ for p=1 with linear fitted line

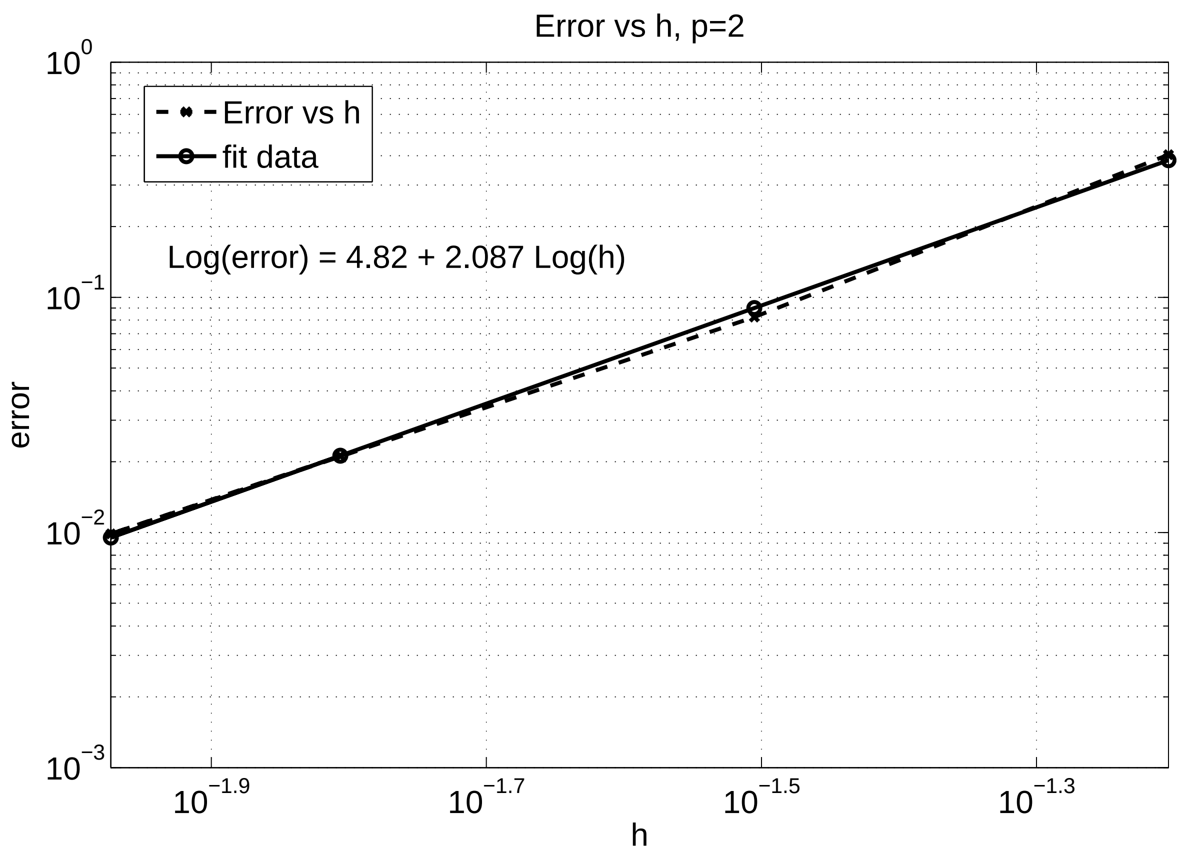


Figure 11: Error vs $h$ for p=2 with linear fitted line

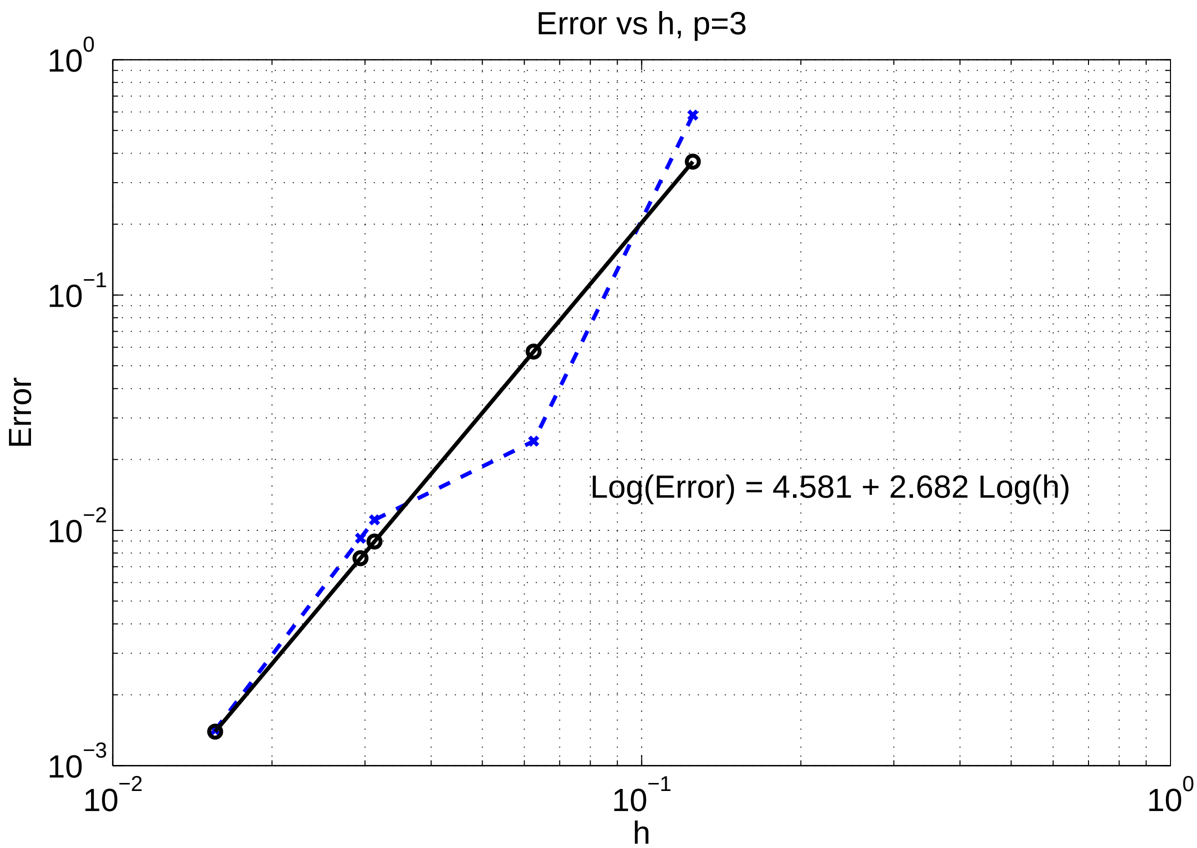


Figure 12: Error vs $h$ for p=3 with linear fitted line

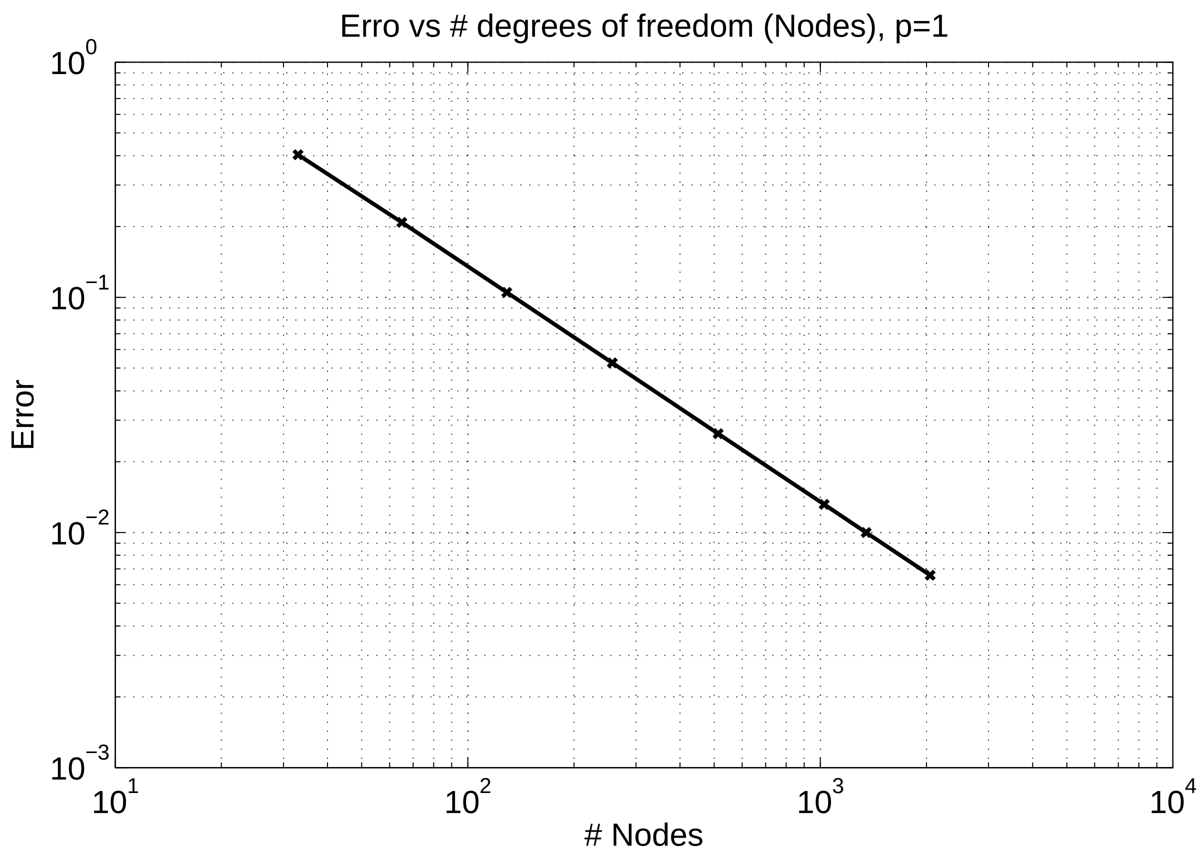


Figure 13: Error vs degrees of freedom for p=1

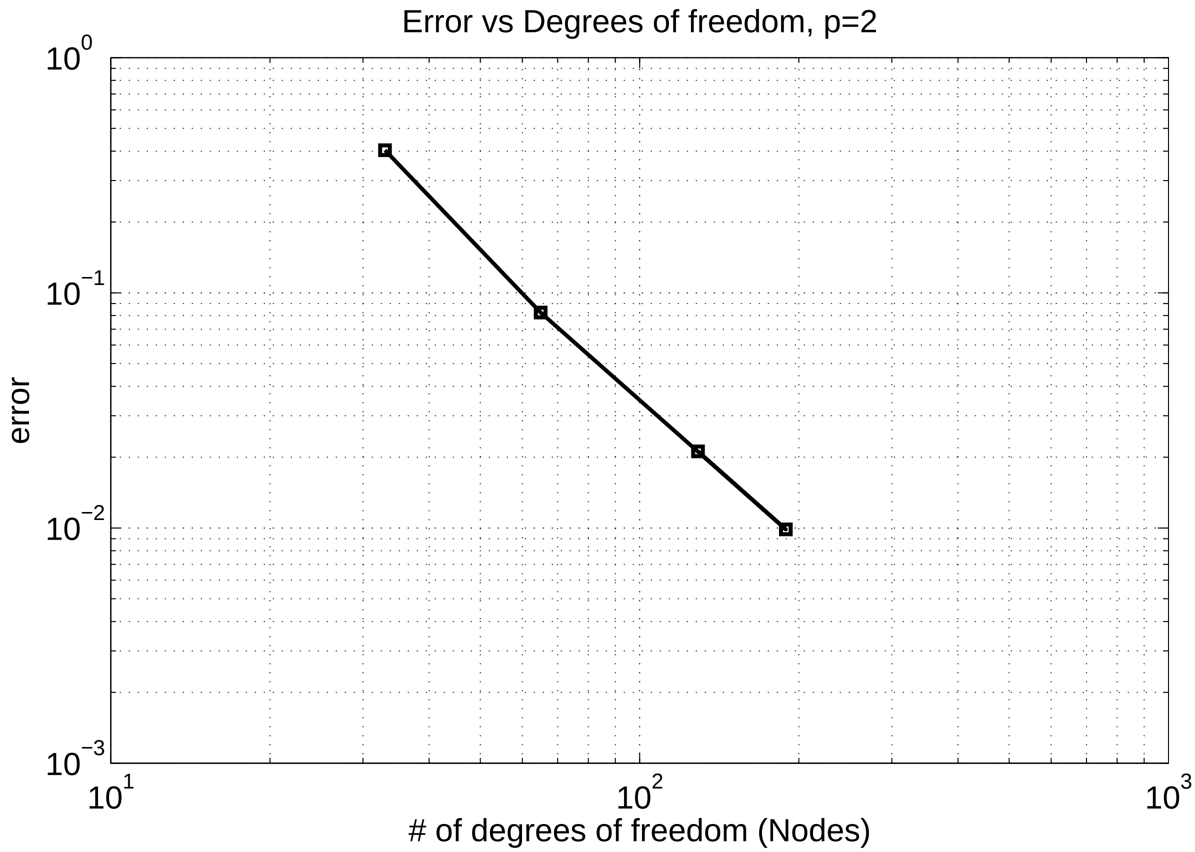


Figure 14: Error vs degrees of freedom for p=2

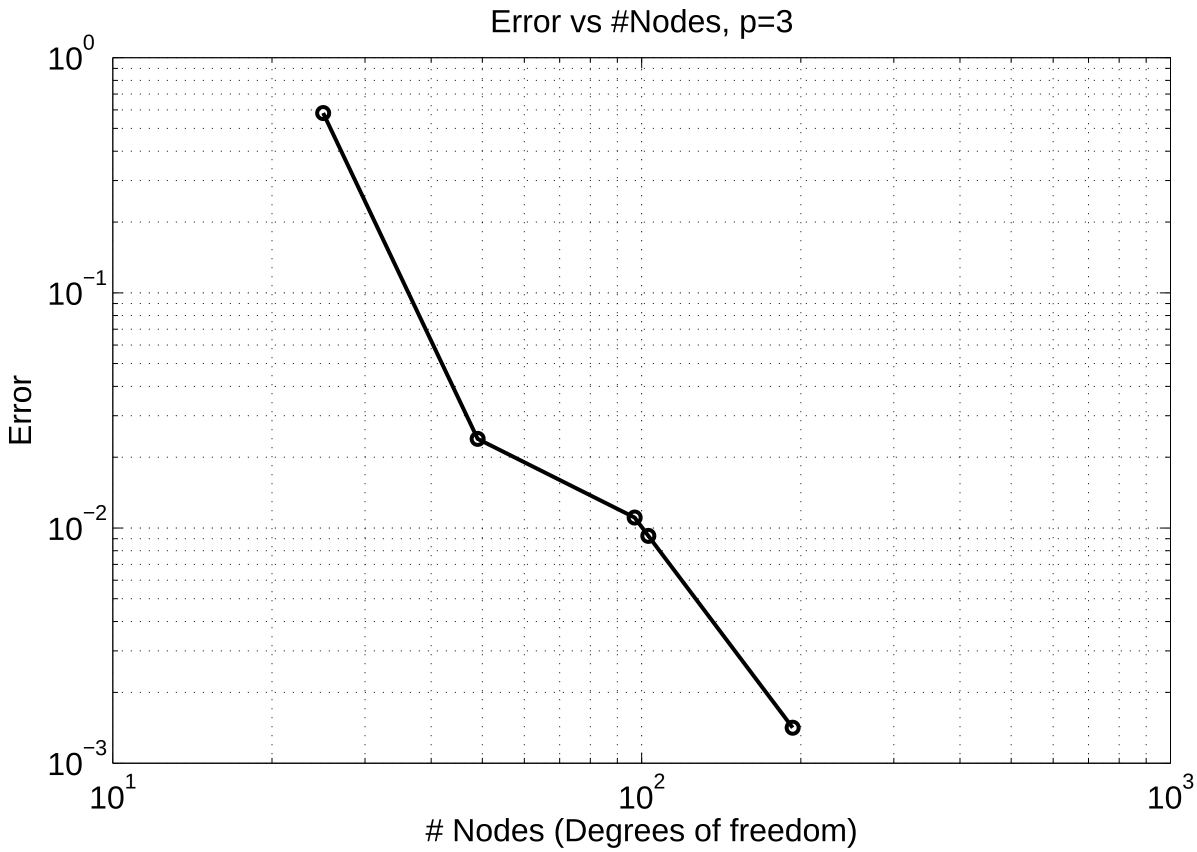


Figure 15: Error vs degrees of freedom for p=3