Math 280A
Homework 7
Ahmad Zareei

1 Introduction

In previous homeworks, we worked through finite element method in 1D and 2D. We did the simultaions and also worked out the time dependent problem. Since most real-life problems are in 3D, in this problem set we will work out throught the finite element problem in 3D. We will derive the formulations for numerical implementation. We will discuss conjugate gradient method in this case and discuss operations needed to solve using conjugate gradient method. At the end we will derive time dependent problem formulation for both cases implicit and explicit methods.

2 Objectives

- Deriving weak formulation of a 3D elasticity problem with direct calculation of Dirichelet boundary condition
- Deriving weak formulation of a 3D elasticity problem with penalty method
- Developing finite element weak form with both methods for boundary conditions
- Investigating 3D mesh for FEM and geometric error in these meshes
- Deriving time dependent 3D elasticity problem with for both explicit and implicit methods

3 Problem, Procedure

You are given a tubular multiphase structure with an elasticity of $\mathbb{E}(x, y, z)$, and with dimensions shown in the figure. It is clamped on one end and externally traction loaded everywhere else, including on the interior surface. The SMALL deformation of the body is governed by (strong form):

$$\nabla.(\mathbb{E} : \nabla \mathbf{u}) + \rho \mathbf{b} = 0 \tag{1}$$

where $\mathbb{E}$ and ρ are spacially variable and $\mathbf{b} = \mathbf{b}(x, y, z)$ is given data.

- **Develop a weak form, providing all the steps and assumptions necessary.** Carefully define the spaces of approximation.

Solution

We first summarize the main problem in the strong form as

$$\nabla(\mathbb{E} : \nabla \mathbf{u}) + \rho \mathbf{b} = 0 \tag{2}$$

$$\mathbf{u} = \mathrm{d}, \quad \mathbf{x} \in \Gamma_u \tag{3}$$

$$(\mathbb{E} : \nabla \mathbf{u}).\mathbf{n} = \mathbf{t}, \quad \mathbf{x} \in \Gamma_t \tag{4}$$

Now, In order to derive the weak form of the equation, we first multiply both sides by some test function $v(x, y, z)$, we will obtain

$$\int_\Omega \mathbf{v}.\nabla.(\mathbb{E} : \nabla \mathbf{u}) \mathrm{d}\Omega + \int_\Omega \mathbf{v}.\rho \mathbf{b} \mathrm{d}\Omega = 0 \tag{5}$$

Doing integration by parts, we will have

$$\int_\Omega \nabla \mathbf{v} : \mathbb{E} : \nabla u \mathrm{d}\Omega = \int_\Omega v.\rho \mathbf{b} \mathrm{d}\Omega + \int_{\partial\Omega} \mathbb{E} : \nabla u.\mathbf{n}\mathbf{v} \mathrm{d}A \tag{6}$$

Now assume that $\mathbf{u} \in \mathbf{H}^1_u(\Omega)$ and $\mathbf{v} \in \mathbf{H}^1_v(\Omega)$. If we denote parts of $\partial\Omega$ that the tension is known as Γ_t (Neumann boundary condition) and other parts that u is known by Γ_u (Dirichelet boundary condition), this equation would mean that:

Find $\mathbf{u} \in \mathbf{H}^1_u(\Omega)$, $\mathbf{u}_{\Gamma_u} = \mathbf{d}$ such that $\forall \mathbf{v} \in \mathbf{H}^1_v(\Omega)$ that $\mathbf{v}|_{\Gamma_u} = 0$, such that

$$\int_\Omega \nabla \mathbf{v} : \mathbb{E} : \nabla \mathbf{u} \mathrm{d}\Omega = \int_\Omega \mathbf{v}.\rho \mathbf{b} \mathrm{d}\Omega + \int_{\Gamma_t} \mathbf{t}.\mathbf{v} \mathrm{d}A \tag{7}$$

where $\mathbf{t} = \mathbb{E} : \nabla u.\mathbf{n}$ is the traction force on the surface where $\mathbf{n}$ is the normal vector to the surface. This is by implementing direct boundary condition of Dirichelet.

- **Develop a finite element weak form. Carefully define the spaces of approximation.**

  In orther to find the finite element form of the equations, we need those integrals to be finite. In other words

$$\int_\Omega \nabla \mathbf{v} : \mathbb{E} : \nabla \mathbf{u} \mathrm{d}\Omega < \infty, \quad \int_\Omega \mathbf{v}.\rho \mathbf{b} \mathrm{d}\Omega < \infty, \quad \int_{\Gamma_t} \mathbf{t}.\mathbf{v} \mathrm{d}A < \infty \tag{8}$$

  We denote $\mathbf{H}^1(\Omega)$ as the usual space of functions with generalized partial derivatives of order ≤ 1 in $\mathbf{L}^2(\Omega)$, i.e. square integrable, in other words, $\mathbf{u} \in \mathbf{H}^1(\Omega)$) if

$$||\mathbf{u}||^2_{\mathbf{H}^1(\Omega)} := \int_\Omega \frac{\partial u_i}{\partial x_j} \frac{\partial u_i}{\partial x_j} \mathrm{d}\Omega + \int_\Omega u_i u_i \mathrm{d}\Omega < \infty \tag{9}$$

  We assume that $\mathbf{b} \in \mathbf{L}^2(\Omega)$ and $\mathbf{t} \in \mathbf{L}^2(\Gamma_t)$. We require $\mathbf{u} \in \mathbf{H}^1(\Omega)$, equation of the weak form reads

  Find $\mathbf{u} \in \mathbf{H}^1(\Omega)$, $\mathbf{u}|_{\Gamma_u} = \mathbf{d}$ such that $\forall \mathbf{v} \in \mathbf{H}^1(\Omega)$ that $\mathbf{v}|_{\Gamma_u} = 0$, such that

$$\int_\Omega \nabla \mathbf{v} : \mathbb{E} : \nabla \mathbf{u} \mathrm{d}\Omega = \int_\Omega \mathbf{v}.\rho \mathbf{b} \mathrm{d}\Omega + \int_{\Gamma_t} \mathbf{t}.\mathbf{v} \mathrm{d}A \tag{10}$$

- Develop a finite element weak statement using the penalty method. Carefully define the spaces of approximation.

Solution

Equation (6) with penalty method, will give the following statement

Find $\mathbf{u} \in \mathbf{H}^1_u(\Omega)$ such $\forall \mathbf{v} \in \mathbf{H}^1_v(\Omega)$, such that

$$\int_\Omega \nabla\mathbf{v} : \mathbb{E} : \nabla\mathbf{u}\mathrm{d}\Omega = \int_\Omega \mathbf{v}.\rho\mathbf{b}\mathrm{d}\Omega + \int_{\Gamma_t} \mathbf{t}.\mathbf{v}\mathrm{d}A + P^* \int_{\Gamma_u} (\mathbf{d} - \mathbf{u}).\mathbf{v}\mathrm{d}A \tag{11}$$

This will be the weak form of our equation with both direct implementation of Dirichelet method and with penalty method . This will be the weak form of our equation with implementation of Penalty method for Dirichelet boundary condition. Again, In orther to find the finite element form of the equations, we need those integrals to be finite. In other words

$$\int_\Omega \nabla\mathbf{v} : \mathbb{E} : \nabla\mathbf{u}\mathrm{d}\Omega < \infty, \quad \int_\Omega \mathbf{v}.\rho\mathbf{b}\mathrm{d}\Omega < \infty, \quad \int_{\Gamma_t} \mathbf{t}.\mathbf{v}\mathrm{d}A < \infty, \quad \int_{\Gamma_u} (\mathbf{d} - \mathbf{u}).\mathbf{v}\mathrm{d}A < \infty \tag{12}$$

We denote $\mathbf{H}^1(\Omega)$ as the usual space of functions with generalized partial derivatives of order ≤ 1 in $\mathbf{L}^2(\Omega)$, i.e. square integrable, in other words, $\mathbf{u} \in \mathbf{H}^1(\Omega))$ if

$$||\mathbf{u}||^2_{\mathbf{H}^1(\Omega)} := \int_\Omega \frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_j}\mathrm{d}\Omega + \int_\Omega u_i u_i \mathrm{d}\Omega < \infty \tag{13}$$

We assume that $\mathbf{b} \in \mathbf{L}^2(\Omega)$ and $\mathbf{t} \in \mathbf{L}^2(\Gamma_t)$. We require $\mathbf{u} \in \mathbf{H}^1(\Omega)$, equation of the weak form reads

Find $\mathbf{u} \in \mathbf{H}^1(\Omega)$, such that $\forall \mathbf{v} \in \mathbf{H}^1(\Omega)$ that $\mathbf{v}|_{\Gamma_u} = 0$, such that

$$\int_\Omega \nabla\mathbf{v} : \mathbb{E} : \nabla\mathbf{u}\mathrm{d}\Omega = \int_\Omega \mathbf{v}.\rho\mathbf{b}\mathrm{d}\Omega + \int_{\Gamma_t} \mathbf{t}.\mathbf{v}\mathrm{d}A + \int_{\Gamma_u} (\mathbf{d} - \mathbf{u}).\mathbf{v}\mathrm{d}A \tag{14}$$

- Derive the equations for element stiffness matrices (be explicit) and load vectors. Thereafter, describe how the global stiffness matrix and load vector are generated, using the penalty method. Use trilinear subspatial approximations. There are different kinds of loading on the surfaces, so be very explicit as to what each of the individual stiffness matrices and righthandside vectors look like, as well as a generic element that is not on the surface.

Solution

Writing in the matrix form of equation (10), we have

$$\int_\Omega ([\mathbf{D}]\{v\})^T[\mathbb{E}]([\mathbf{D}]\{u\})\mathrm{d}\Omega = \int_\Omega \{v\}^T\rho\{b\}\mathrm{d}\Omega + \int_{\Gamma_t} \{v\}^T\{t\}\mathrm{d}A \tag{15}$$

where

$$[\mathbf{D}] = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \end{bmatrix} \tag{16}$$

and

$$\{u\} = \left\{ \begin{array}{c} u_1 \\ u_2 \\ u_3 \end{array} \right\}, \quad \{v\} = \left\{ \begin{array}{c} v_1 \\ v_2 \\ v_3 \end{array} \right\}, \quad \rho\{b\} = \left\{ \begin{array}{c} \rho b_1 \\ \rho b_2 \\ \rho b_3 \end{array} \right\}, \tag{17}$$

We now, expand different terms on the basis functions, we have

$$u_1(x, y, z) = \sum_{i=1}^{N} a_i \phi_i(x, y, z) \tag{18}$$

$$u_2(x, y, z) = \sum_{i=N+1}^{2N} a_i \phi_i(x, y, z) \tag{19}$$

$$u_3(x, y, z) = \sum_{i=2N+1}^{3N} a_i \phi_i(x, y, z) \tag{20}$$

or in other format, $\{u\} = [\phi]\{a\}$, where

$$\phi = \begin{bmatrix} \phi_1 & \phi_2 & ... & \phi_N & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \phi_1 & \phi_2 & ... & \phi_N & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \phi_1 & \phi_2 & ... & \phi_N \end{bmatrix} \tag{21}$$

We write all the terms of a_i in a column vector as

$$\{a\}^T = \{a_1, a_2, a_3, ..., a_{3N}\} \tag{22}$$

then we could write

$$\{u\} = \sum_{i=1}^{3N} a_i \{\phi_i\} \tag{23}$$

where

$$\{\phi_i\} = \{\phi_i, 0, 0\}^T \qquad 1 \le i \le N \tag{24}$$

$$\{\phi_i\} = \{0, \phi_i, 0\}^T \qquad N+1 \le i \le 2N \tag{25}$$

$$\{\phi_i\} = \{0, 0, \phi_i\}^T \qquad 2N+1 \le i \le 3N \tag{26}$$

Now we choose to write $\{v\}$ in the same basis functions as $\{v\} = [\phi]\{c\}$, we will obtain

$$\int_\Omega ([\mathbf{D}][\phi]\{c\})^T [\mathbb{E}]([\mathbf{D}][\phi]\{a\}) \mathrm{d}\Omega = \int_\Omega ([\phi]\{c\})^T \rho\{b\} \mathrm{d}\Omega + \int_{\Gamma_t} ([\phi]\{c\})^T \{t\} \mathrm{d}A \tag{27}$$

Since, $\{c\}$ is arbitrary, $\forall \mathbf{v} \to \forall\{c\}$, we have:

$$\{c\}^T \{[K]\{a\} - \{R\}\} = 0 \quad \to \quad [K]\{a\} - \{R\} = 0 \tag{28}$$

where

$$[K] = \int_\Omega ([\mathbf{D}][\phi])^T [\mathbb{E}]([\mathbf{D}][\phi])\mathrm{d}\Omega \tag{29}$$

$$\{R\} = \int_\Omega [\phi]^T \rho\{b\}\mathrm{d}\Omega + \int_{\Gamma_t} [\phi]^T \{t\}\mathrm{d}A \tag{30}$$

So, the system of equations to be solved is $[K]\{a\} - \{R\} = 0$.

In order to find the form of the system of equations, with penalty method, we write equation (14) in the matrix form, we have

$$\int_\Omega ([\mathbf{D}]\{v\})^T [\mathbb{E}]([\mathbf{D}]\{u\})\mathrm{d}\Omega = \int_\Omega \{v\}^T \rho\{b\}\mathrm{d}\Omega + \int_{\Gamma_t} \{v\}^T \{t\}\mathrm{d}A + P^* \int_{\Gamma_u} \{v\}^T (\{d\} - \{u\})\mathrm{d}A \tag{31}$$

All the equations, from (16) throught (26), are hold. Expanding $\{u\} = [\phi]\{a\}$ and $\{v\} = [\phi]\{c\}$, we will obtain:

$$\int_\Omega ([\mathbf{D}][\phi]\{c\})^T [\mathbb{E}]([\mathbf{D}][\phi]\{a\})\mathrm{d}\Omega = \tag{32}$$

$$\int_\Omega ([\phi]\{c\})^T \rho\{b\}\mathrm{d}\Omega + \int_{\Gamma_t} ([\phi]\{c\})^T \{t\}\mathrm{d}A + P^* \int_{\Gamma_u} ([\phi]\{c\})^T (\{d\} - [\phi]\{a\})\mathrm{d}A \tag{33}$$

Since, $\{c\}$ is arbitrary, $\forall \mathbf{v} \to \forall\{c\}$, we have:

$$\{c\}^T \{[K]\{a\} - \{R\}\} = 0 \quad \to \quad [K]\{a\} - \{R\} = 0 \tag{34}$$

where

$$[K] = \int_\Omega ([\mathbf{D}][\phi])^T [\mathbb{E}]([\mathbf{D}][\phi])\mathrm{d}\Omega + P^* \int_{\Gamma_u} [\phi]^T [\phi]\mathrm{d}A \tag{35}$$

$$\{R\} = \int_\Omega [\phi]^T \rho\{b\}\mathrm{d}\Omega + \int_{\Gamma_t} [\phi]^T \{t\}\mathrm{d}A + P^* \int_{\Gamma_u} [\phi]^T \{d\}\mathrm{d}A \tag{36}$$

So, the system of equations to be solved is $[K]\{a\} - \{R\} = 0$.

- ASSUME A TUBULAR CIRCULAR CROSS-SECTION OF OUTER RADIUS R_c, THICKNESS t AND A SEMICIRCULAR CROSS-SECTION (ALONG THE LENGTH) OF R_s . Use $N_r = 5$ elements in the r direction, $N_c = 20$ (circumferantial of the cross-section) elements in the circumferential direction and $N_\theta = 20$ (along the length) elements in the θ direction for each semicircular portion. DRAW THE MESH NOT BY

HAND. Explicitly characterize the geometric error for ageneral number (N_r, N_c, N_θ), defined as the percentage error in the solid volume as a function of the N_r elements in the r direction, the N_c elements in the circumferential direction and the N_θ elements in the θ direction

$$\text{Geometric Error} = \frac{\text{True Volume - FEM Volume}}{\text{True Volume}} \tag{37}$$

We generated the mesh, assuming $R_c = 1$, $t = 0.5$ and $Rs = 4$. The 3-D shape of the plot is shown in figure 1. Figure 2 shows the cross section of the torus at $x - z$ plane around one of the circles.

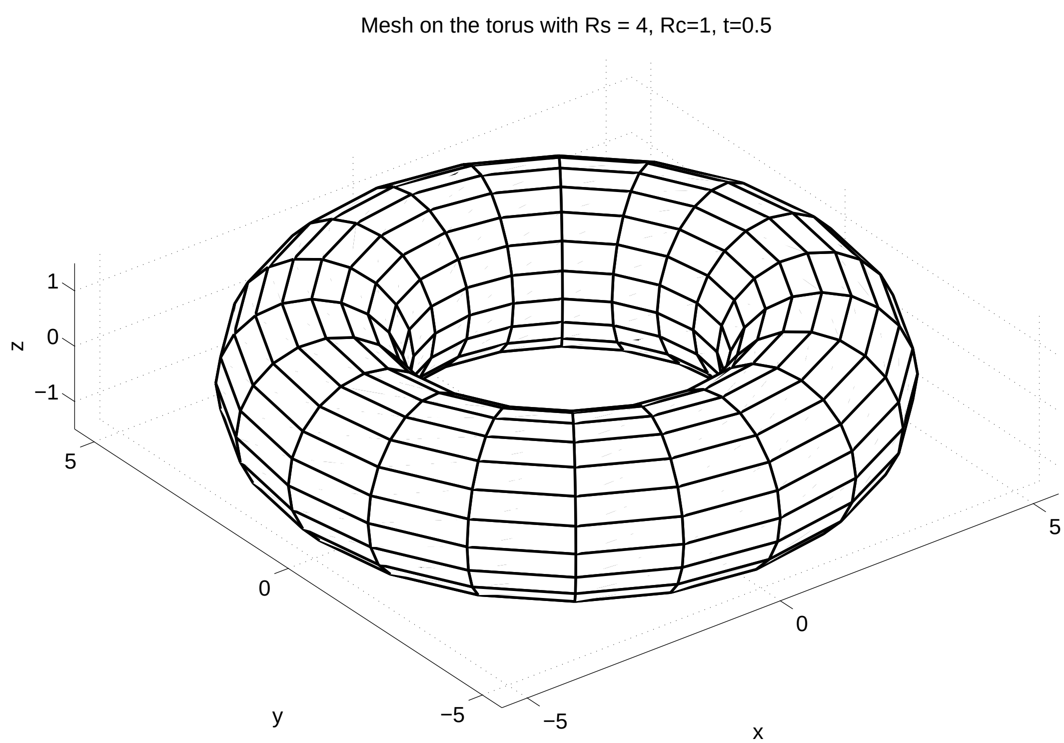

Figure 1: 3D mesh of the torus with$R_c = 1, t = 0.5, Rs = 4$ with $N_c = 20, N_\theta = 20, N_r = 5$

In order to fine the geometric error, we first need to find the true volume, which is

$$\text{True Volume} = \pi((R_c + t)^2 - R_c^2) \times 2\pi R_s \tag{38}$$

For our values of R_c, t and R_s, this true volume is

$$\text{True Volume} = 98.6960440108936 \tag{39}$$

The FEM volume, we computed by going over all the elements and computing the following integral as

$$\text{FEM Volume} = \sum_{n_e=1}^{N} \int_{\Omega_e} 1 \mathrm{d}x \mathrm{d}y \mathrm{d}z = \sum_{n_e=1}^{N} \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} 1 J \mathrm{d}\zeta_1 \mathrm{d}\zeta_2 \mathrm{d}\zeta_3 \tag{40}$$

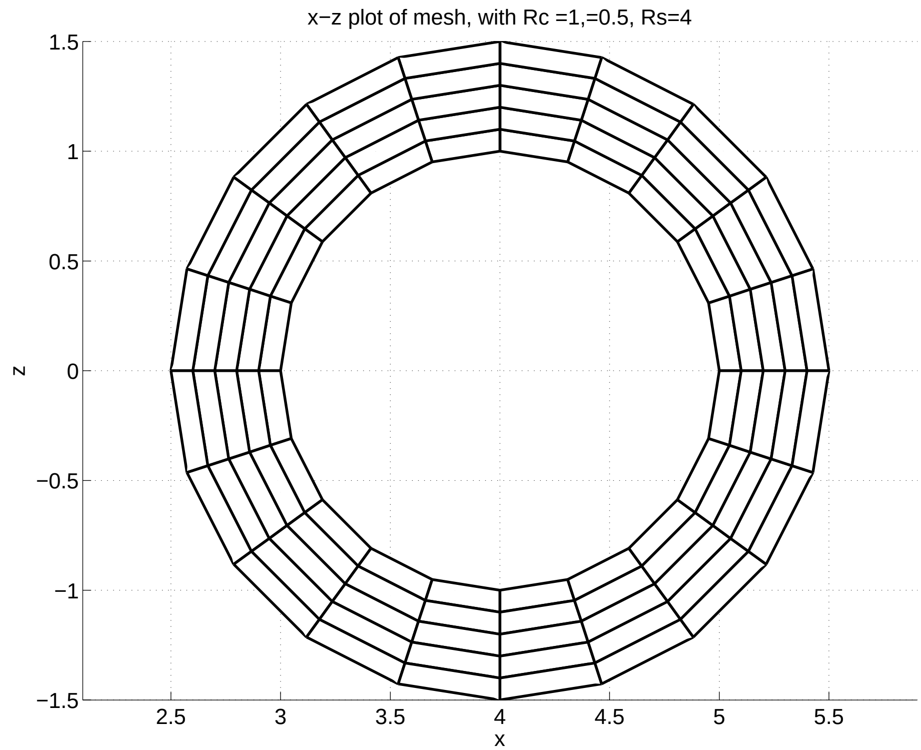

Figure 2: x-z cut of the 3D mesh of the torus with $R_c = 1$, $t = 0.5$ and $Rs = 4$ with $N_c = 20, N_\theta = 20, N_r = 5$

where $J = det|F|$ is the Jacobian of transformation determinant. The transformation is as

$$x = \sum_{i=1}^{8} X_i \phi_i \qquad y = \sum_{i=1}^{8} Y_i \phi_i \qquad z = \sum_{i=1}^{8} Z_i \phi_i \tag{41}$$

where X_i, Y_i and Z_i are the positions of the elements in the master elements. Doing so, we will find that

$$\text{FEM Volume} = 95.491502812526306 \tag{42}$$

so the geometric error would be

$$\text{Geometric Error} = -3.2469\% \tag{43}$$

- If one were to use a Conjugate-Gradient solver, theoretically how many operation counts would be needed to solve this problem for a mesh of N_r elements in the r direction, N_c elements in the circumferential direction and N_θ elements in the θ direction.

  First, we need number of degrees of freedom. To compute that, we need number of nodes, since, for each node we have 24 degrees of freedom. Numer of Nodes are $(N_r + 1)N_c N_\theta$. Note θ and circumferential directions are periodic, so the first and last nodes will collide (thats why we have N_θ nodes in θ direction not $N_\theta + 1$). Now the number of degrees of freedom will be $24(N_r+1)N_cN_\theta$. Note θ. Stiffness matrix is made of blocks with 24×24 $N_rN_\theta N_c$ elements wich are symmetric also, so in these 24×24 matrices, we have only 300 independent elements. These matrices takes $24^2N_rN_\theta N_c$ operations to multiply with a vector, and it takes I iteration to converge. So the total number of operations will be $24^2IN_rN_\theta N_c$.

- Now consider the time-transient case. The body has the same boundary conditions as before, with the initial condition that $u(t = 0, x, y, z) = u0(x, y, z)$ and $u(t = 0, x, y, z) = u_0(x, y, z)$. The governing equation is

$$\nabla.(\mathbb{E} : \nabla\mathbf{u}) + \rho\mathbf{b} = \rho\ddot{\mathbf{u}} \tag{44}$$

Develop a finite element weak statement. Carefully define the spaces of approximation. Use the

  - the IMPLICIT finite difference approximation for the time dependent term
  - the EXPLICIT finite difference approximation for the time dependent term
  - the EXPLICIT finite difference approximation for the time dependent term with a lumped mass approximation.

Explicit: First we introduce $\mathbf{w} = \rho\dot{\mathbf{u}}$, so the equation, will give two sets of equations, as

$$\rho\dot{\mathbf{u}} = \mathbf{w} \tag{45}$$

$$\dot{\mathbf{w}} = \nabla.(\mathbb{E} : \nabla\mathbf{u}) + \rho\mathbf{b} \tag{46}$$

For the explicit method, we have

$$\rho\frac{\mathbf{u}(t + \Delta t) - \mathbf{u}(t)}{\Delta t} = \mathbf{w}(t) \tag{47}$$

and also

$$\frac{\mathbf{w}(t + \Delta t) - \mathbf{w}(t)}{\Delta t} = \nabla.(\mathbb{E} : \nabla\mathbf{u}(t)) + \rho\mathbf{b}(t) \tag{48}$$

using these equation, we will obtain

$$\rho\mathbf{u}(t + \Delta t) = \rho\mathbf{u}(t) + \Delta t\mathbf{w}(t) \tag{49}$$

$$\mathbf{w}(t + \Delta t) = \mathbf{w}(t) + \Delta t\nabla.(\mathbb{E} : \nabla\mathbf{u}(t)) + \Delta t\rho\mathbf{b}(t) \tag{50}$$

Multiplying both sides of these equations with arbitrary function $\mathbf{v}$ and doing the integration by parts for the second equation, we will obtain:

$$\int_\Omega \mathbf{v}.\rho\mathbf{u}(t + \Delta t)\mathrm{d}\Omega = \int_\Omega \mathbf{v}.\rho\mathbf{u}(t) + \Delta t\int_\Omega \mathbf{v}.\mathbf{w}(t) \tag{51}$$

$$\begin{aligned}\int_\Omega \mathbf{v}.\mathbf{w}(t + \Delta t)\mathrm{d}\Omega = \int_\Omega \mathbf{v}.\mathbf{w}(t)\mathrm{d}\Omega - \Delta t\int_\Omega \nabla\mathbf{v} : \mathbb{E} : \nabla\mathbf{u}(t)\mathrm{d}\Omega + \Delta t\int_\Omega \mathbf{v}.\rho\mathbf{b}(t)\mathrm{d}\Omega \\ + \Delta t\int_{\Gamma_t} \mathbf{v}.\mathbf{t}\mathrm{d}A\end{aligned} \tag{52}$$

where $\mathbf{t} = \mathbb{E}.\nabla\mathbf{u}(t).\mathbf{n}$ is the traction force on the surface, where $\mathbf{n}$ is the normal force on the surface. In this weak form of the equation, we supposed $\mathbf{u} \in \mathbf{H}^1_u(\Omega)$, $\mathbf{u}_{\Gamma_u} = \mathbf{d}$ and $\forall\mathbf{v} \in \mathbf{H}^1_v(\Omega)$ that $\mathbf{v}|_{\Gamma_u} = 0$ and also $\mathbf{w} \in \mathbf{H}^1_w(\Omega)$, $\mathbf{w}_{\Gamma_u} = 0$, Now if we suppose that these integrals are finite. In a percise way, we denote $\mathbf{H}^1(\Omega)$ as the usual space of functions with generalized partial derivatives of order ≤ 1 in $\mathbf{L}^2(\Omega)$, i.e. square integrable, in other words, $\mathbf{u} \in \mathbf{H}^1(\Omega))$ if

$$||\mathbf{u}||^2_{\mathbf{H}^1(\Omega)} := \int_\Omega \frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_j}\mathrm{d}\Omega + \int_\Omega u_i u_i \mathrm{d}\Omega < \infty \tag{53}$$

We assume that $\mathbf{b} \in \mathbf{L}^2(\Omega)$ and $\mathbf{t} \in \mathbf{L}^2(\Gamma_t)$. We require $\mathbf{u}, \mathbf{w}, \mathbf{v} \in \mathbf{H}^1(\Omega)$.

Again, as before, equations (16) upto (26) holds, we will have

$$\int_\Omega ([\phi]\{c\})^T.\rho[\phi]\{a\}(t+\Delta t)\mathrm{d}\Omega = \int_\Omega ([\phi]\{c\})^T.[\phi]\rho\{a\}(t) + \Delta t \int_\Omega ([\phi]\{c\})^T.[\phi]\{g\}(t) \tag{54}$$

$$\begin{aligned}\int_\Omega ([\phi]\{c\})^T.[\phi]\{g\}(t+\Delta t)\mathrm{d}\Omega = & \int_\Omega ([\phi]\{c\})^T.[\phi]\{g\}(t)\mathrm{d}\Omega \\ & - \Delta t \int_\Omega ([D][\phi]\{c\})^T : \mathbb{E} : [D][\phi]\{a\}(t)\mathrm{d}\Omega \\ & + \Delta t \int_\Omega ([\phi]\{c\})^T.\rho\{b\}(t)\mathrm{d}\Omega \\ & + \Delta t \int_{\Gamma_t} ([\phi]\{c\})^T.\{t\}\mathrm{d}A \end{aligned} \tag{55}$$

where we assumed $\{u\} = [\phi]\{a\}$ and $\{v\} = [\phi]\{c\}$ and $\{w\} = [\phi]\{g\}$. Since $\{v\}$ is arbitrary (since $\forall\mathbf{v}$ this statement will be true), we will have

$$\rho[M]\{a\}(t+\Delta t) = \rho[M]\{a\}(t) + \Delta t[M]\{g\}(t) \tag{56}$$

$$[M]\{g\}(t+\Delta t) = [M]\{g\}(t) - \Delta t[A]\{a\}(t) + \Delta t\{R\} \tag{57}$$

where

$$[M] = \int_\Omega [\phi]^T[\phi]\mathrm{d}\Omega \tag{58}$$

$$[A] = \int_\Omega ([D][\phi])^T[E]([D][\phi])\mathrm{d}\Omega \tag{59}$$

$$\{R\} = \int_\Omega [\phi]^T\rho\{b\}(t)\mathrm{d}\Omega + \int_{\Gamma_t}[\phi]^T.\{t\}\mathrm{d}A \tag{60}$$

We can also combine two equations in (56) and (57) and obtain:

$$\rho[M]\{a\}(t+2\Delta t) = 2\rho[M]\{a\}(t+\Delta t) - \rho[M]\{a\}(t) + \Delta t^2(-[A]\{a\}(t) + \{R\}) \tag{61}$$

Implicit Method: For the implicit method, we have

$$\rho\mathbf{u}(t+\Delta t) = \rho\mathbf{u}(t) + \Delta t\mathbf{w}(t+\Delta t) \tag{62}$$

$$\mathbf{w}(t+\Delta t) = \mathbf{w}(t) + \Delta t\nabla.(\mathbb{E} : \nabla\mathbf{u}(t+[\mathbf{D}]t)) + \Delta t\rho\mathbf{b}(t+\Delta t) \tag{63}$$

Multiplying both sides of these equations with arbitrary function $\mathbf{v}$ and doing the integration by parts for the second equation, we will obtain:

$$\int_\Omega \mathbf{v}.\rho\mathbf{u}(t+\Delta t)\mathrm{d}\Omega = \int_\Omega \mathbf{v}.\rho\mathbf{u}(t+\Delta t) + \Delta t\int_\Omega \mathbf{v}.\mathbf{w}(t+\Delta t) \tag{64}$$

$$\begin{aligned}\int_\Omega \mathbf{v}.\mathbf{w}(t+\Delta t)\mathrm{d}\Omega = &\int_\Omega \mathbf{v}.\mathbf{w}(t+\Delta t)\mathrm{d}\Omega \\ &- \Delta t\int_\Omega \nabla\mathbf{v} : \mathbb{E} : \nabla\mathbf{u}(t+\Delta t)\mathrm{d}\Omega + \Delta t\int_\Omega \mathbf{v}.\rho\mathbf{b}(t+\Delta t)\mathrm{d}\Omega \\ &+ \Delta t\int_{\Gamma_t} \mathbf{v}.\mathbf{t}\mathrm{d}A\end{aligned} \tag{65}$$

where $\mathbf{t} = \mathbb{E}.\nabla\mathbf{u}(t+\Delta t).\mathbf{n}$ is the traction force on the surface, where $\mathbf{n}$ is the normal force on the surface. In this weak form of the equation, we supposed $\mathbf{u} \in \mathbf{H}^1_u(\Omega)$, $\mathbf{u}_{\Gamma_u} = \mathbf{d}$ and $\forall\mathbf{v} \in \mathbf{H}^1_v(\Omega)$ that $\mathbf{v}|_{\Gamma_u} = 0$ and also $\mathbf{w} \in \mathbf{H}^1_w(\Omega)$, $\mathbf{w}_{\Gamma_u} = 0$, Now if we suppose that these integrals are finite. In a percise way, we denote $\mathbf{H}^1(\Omega)$ as the usual space of functions with generalized partial derivatives of order ≤ 1 in $\mathbf{L}^2(\Omega)$, i.e. square integrable, in other words, $\mathbf{u} \in \mathbf{H}^1(\Omega))$ if

$$||\mathbf{u}||^2_{\mathbf{H}^1(\Omega)} := \int_\Omega \frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_j}\mathrm{d}\Omega + \int_\Omega u_i u_i \mathrm{d}\Omega < \infty \tag{66}$$

We assume that $\mathbf{b} \in \mathbf{L}^2(\Omega)$ and $\mathbf{t} \in \mathbf{L}^2(\Gamma_t)$. We require $\mathbf{u}, \mathbf{w}, \mathbf{v} \in \mathbf{H}^1(\Omega)$.

Again, as before, equations (16) upto (26) holds, we will have

$$\int_\Omega ([\phi]\{c\})^T.\rho[\phi]\{a\}(t+\Delta t)\mathrm{d}\Omega = \int_\Omega ([\phi]\{c\})^T.[\phi]\rho\{a\}(t) + \Delta t\int_\Omega ([\phi]\{c\})^T.[\phi]\{g\}(t+\Delta t) \tag{67}$$

$$\begin{aligned}\int_\Omega ([\phi]\{c\})^T.[\phi]\{g\}(t+\Delta t)\mathrm{d}\Omega = &\int_\Omega ([\phi]\{c\})^T.[\phi]\{g\}(t)\mathrm{d}\Omega \\ &- \Delta t\int_\Omega ([D][\phi]\{c\})^T : \mathbb{E} : [D][\phi]\{a\}(t+\Delta t)\mathrm{d}\Omega \\ &+ \Delta t\int_\Omega ([\phi]\{c\})^T.\rho\{b\}(t+\Delta t)\mathrm{d}\Omega \\ &+ \Delta t\int_{\Gamma_t} ([\phi]\{c\})^T.\{t\}(t+\Delta t)\mathrm{d}A\end{aligned} \tag{68}$$

where we assumed $\{u\} = [\phi]\{a\}$ and $\{v\} = [\phi]\{c\}$ and $\{w\} = [\phi]\{g\}$.

Since $\{v\}$ is arbitrary (since $\forall \mathbf{v}$ this statement will be true), we will have

$$\rho[M]\{a\}(t+\Delta t) = \rho[M]\{a\}(t) + \Delta t[M]\{g\}(t+\Delta t) \tag{69}$$

$$[M]\{g\}(t+\Delta t) = [M]\{g\}(t) - \Delta t[A]\{a\}(t+\Delta t) + \Delta t\{R\} \tag{70}$$

where

$$[M] = \int_\Omega [\phi]^T[\phi]\mathrm{d}\Omega \tag{71}$$

$$[A] = \int_\Omega ([D][\phi])^T[E]([D][\phi])\mathrm{d}\Omega \tag{72}$$

$$\{R\} = \int_\Omega [\phi]^T\rho\{b\}\mathrm{d}\Omega + \int_{\Gamma_t} [\phi]^T.\{t\}\mathrm{d}A \tag{73}$$

Combining equation (69) and (70), we will have

$$(\rho[M] + \Delta t[A])\{a\}(t+\Delta t) = \rho[M]\{a\}(t) + \Delta t[M]\{g\}(t) + \Delta t\{R\} \tag{74}$$

$$[M]\{g\}(t+\Delta t) = [M]\{g\}(t) - \Delta t[A]\{a\}(t+\Delta t) + \Delta t\{R\} \tag{75}$$

With the first equation, we can solve for $\{a\}(t+\Delta t)$ based on previous solutions of $\{a\}(t)$ and $\{g\}(t)$ and then with the solution we find in the first equation, i.e. $\{a\}(t+\Delta t)$, we can solve for the second equation and find $\{g\}(t+\Delta t)$.

Lumped Explicit: For the lumped mass approximation, we follow the same procedure as in explicit method, the final results would be (same as explicit)

$$\rho[M]\{a\}(t+2\Delta t) = 2\rho[M]\{a\}(t+\Delta t) - \rho[M]\{a\}(t) + \Delta t^2(-[A]\{a\}(t) + \{R\}) \tag{76}$$

where

$$[M] = \int_\Omega [\phi]^T[\phi]\mathrm{d}\Omega \tag{77}$$

$$[A] = \int_\Omega ([D][\phi])^T[E]([D][\phi])\mathrm{d}\Omega \tag{78}$$

$$\{R\} = \int_\Omega [\phi]^T\rho\{b\}(t)\mathrm{d}\Omega + \int_{\Gamma_t} [\phi]^T.\{t\}\mathrm{d}A \tag{79}$$

The only difference is in computing the mass matrix, where in here we assume that the mass is on the nodes, so that the mass matrix will become diagonal, which will be the sum of the rows of the previous mass matrix that we had.

4 Conclusion

We saw how to derive different formulation for FEM in 3D for both time dependent and time independent problems. We derived weak formulation for these problems and derived different matrices and their forms in these problems. We also see that FEM mesh will be different from true volume that we consider and we will have some geometric problem in describing the mesh for the finite element solver.