ME 280A
Homework 6
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1 Introduction

So far we solved time independent problems for steady state cases, where we have no time dependent. In many cases we want to solve for a time dependent problem. Here, we will solve a time dependent problem for heat equation, where we solved its steady state case in the previous problem set. We will derive different forms of implicit and explicit method with Euler expansion for the time derivative.

2 Objectives

- Deriving weak form of time dependent problem
- Formulation of Implicit and Explicit methods
- Convergence to the steady state solution
- Time step variation and its effect on convergence

3 Problem, Procedure

3.1 Problem Formulation

Consider the steady state heat conduction problem

$$\nabla.(\kappa\nabla T) + z = \rho c_p \dot{T} \quad \text{in } \Omega \tag{1}$$

$$T = T_0 \quad \text{on } \Omega \text{ at } t = 0 \tag{2}$$

$$T = \bar{T} \quad \text{on } \Gamma_T \text{ at } t \geq 0 \tag{3}$$

$$(\kappa\nabla T).\mathbf{n} = \bar{q}_n \quad \text{on } \Gamma_q \text{ at } t \geq 0 \tag{4}$$

for $t \geq 0$ with a constant in time conductivitt κ, heat capacitance c_p, density ρ and source term z.

- **Derive the weak form of this problem for the discrete approximation.** State the expressions of each of the matrices involved.
- **Derive the Forward Euler with lumped mass approximation, Forward Euler with a mass matrix, and Backward Euler time stepping schemes.** Comment on the differences between these formulations. Do not use a lumped mass approximation for the Backward Euler scheme.

Solution:

The strong form of the equation is as

$$\nabla.(\kappa\nabla T) + z = \rho c_p \dot{T} \quad \text{in } \Omega. \tag{5}$$

In order to find the weak form, we first descritize the term $\dot{T}$ in time. we have

$$\dot{T} = \frac{T(t+\Delta t) - T(t)}{\Delta t} \tag{6}$$

All of other expressions can be either computed at $t = t$ which is called Euler explicit or at the time $t = t + \Delta t$ which the method is known as Euler implicit method. We firt mmultiply both sides by some test function v and integrate over the whole domain of Ω. We will have

$$\int_\Omega v\nabla.(\kappa\nabla T)\mathrm{d}a + \int_\Omega zv\mathrm{d}a = \int_\Omega \frac{\rho c_p}{\Delta t}(T(t+\Delta t) - T(t))v\mathrm{d}a, \tag{7}$$

where da is the infintesimal surface element. Doing the integration by parts, for the first element we will have

$$-\int_\Omega \nabla v.\kappa\nabla T\mathrm{d}a + \int_\Gamma v\kappa\nabla T.\mathbf{n}\mathrm{d}l + \int_\Omega zv\mathrm{d}a = \int_\Omega \frac{\rho c_p}{\Delta t}(T(t+\Delta t) - T(t))v\mathrm{d}a, \tag{8}$$

where $\Gamma = \partial\Omega = \Gamma_T \cup \Gamma_q$ is the boundary of the domain Ω and we used the divergence theorem to change the integral over the domain to the integral over the surface. Now we can proceed in two ways, we can integrate over the dirichelet boundary condition (Γ_T here) with penalty method or with the or with imposing this dirichelet boundary condition directly. We will consider both methods, first we impose direct dirichelet boudnary condition:

- **Direct Dirichelet Condition**: Imposing direct dirichelet boundary condition, we will have

$$-\int_\Omega \nabla v.\kappa\nabla T\mathrm{d}a + \int_{\Gamma_q} v\bar{q}_n\mathrm{d}l + \int_{\Gamma_T} v\kappa\nabla T.\mathbf{n}\mathrm{d}l + \int_\Omega zv\mathrm{d}a = \int_\Omega \frac{\rho c_p}{\Delta t}(T(t+\Delta t) - T(t))v\mathrm{d}a, \tag{9}$$

where we set $\bar{q}_n = \kappa\nabla T.\mathbf{n}$ on the bondary Γ_q. Now, we choose test function such that to be zero on Γ_T, so we will get

$$-\int_\Omega \nabla v.\kappa\nabla T\mathrm{d}a + \int_{\Gamma_q} v\bar{q}_n\mathrm{d}l + \int_\Omega zv\mathrm{d}a = \int_\Omega \frac{\rho c_p}{\Delta t}(T(t+\Delta t) - T(t))v\mathrm{d}a. \tag{10}$$

We now suppose that test function v and the tempreture T can both be written in some basis functions ϕ_i as

$$v \approx \sum b_i(t)\phi_i, \qquad T \approx \sum a_j(t)\phi_j. \tag{11}$$

Inserting these solution forms in equation 10, we can proceed in two ways. one is to compute all the other terms in the previous time step $t = t$ (Euler Explicit) or at time $t = t + \Delta t$ (Euler Implicit).

(i) **Forward Euler, Explicit Method**: First we consider the Euler Explicit method. Inserting these solution forms in equation 10, Assuming that all terms computed at time $t = t$, we will have

$$\sum_{i=1}^{N} b_i(t) \left\{ \left(\int_\Omega \frac{\rho c_p}{\Delta t} \phi_i \phi_j \mathrm{da} - \sum_{j=1}^{N} \int_\Omega \nabla \phi_i \kappa \nabla \phi_j \mathrm{da} \right) a_j(t) \right. \tag{12}$$

$$\left. + \int_{\Gamma_q} \phi_i \bar{q}_n \mathrm{d}l + \int_\Omega z \phi_i \mathrm{da} = \left(\int_\Omega \frac{\rho c_p}{\Delta t} \phi_i \phi_j \mathrm{da} \right) a_j(t + \Delta t) \right\}. \tag{13}$$

This is true for any choice of b_i, so every multipication of b_i should be zero. This will lead to a system of equations. If we choose elements of Matrix $[K]$ and matrix $[L]$ and vector $\{R\}$ as

$$K_{ij} = \int_\Omega \nabla \phi_i \kappa \nabla \phi_j \mathrm{da} \tag{14}$$

$$R_i = \int_{\Gamma_q} \phi_i \bar{q}_n \mathrm{d}l + \int_\Omega z \phi_i \mathrm{da} \tag{15}$$

$$L_{ij} = \int_\Omega \frac{\rho c_p}{\Delta t} \phi_i \phi_j \mathrm{da} \tag{16}$$

the problem reduces to only solving

$$([L] - [K]) \{a(t)\} + \{R\} = [L]\{a(t + \Delta t)\}, \tag{17}$$

where elements of $\{a\}$ will give the solution as in equation 11. This final form equation is used to find values of $\{a(t + \Delta t)\}$ when the values of $\{a(t)\}$ is known. In other words, this equation is used as

$$[L]\{a(t + \Delta t)\} = ([L] - [K]) \{a(t)\} + \{R\} \tag{18}$$

(ii) **Backward Euler, Implicit method**: Inserting equation (11) in equation (10), Assuming that all terms computed at time $t = t$, we will have:

$$\sum_{i=1}^{N} b_i(t) \left\{ \left(\int_\Omega \frac{\rho c_p}{\Delta t} \phi_i \phi_j \mathrm{da} + \sum_{j=1}^{N} \int_\Omega \nabla \phi_i \kappa \nabla \phi_j \mathrm{da} \right) a_j(t + \Delta t) = \right. \tag{19}$$

$$\left. + \int_{\Gamma_q} \phi_i \bar{q}_n \mathrm{d}l + \int_\Omega z \phi_i \mathrm{da} + \left(\int_\Omega \frac{\rho c_p}{\Delta t} \phi_i \phi_j \mathrm{da} \right) a_j(t) \right\}. \tag{20}$$

This is true for any choice of b_i, so every multipication of b_i should be zero. This will lead to a system of equations. If we choose elements of Matrix $[K]$ and matrix $[L]$ and vector $\{R\}$ as

$$K_{ij} = \int_\Omega \nabla \phi_i \kappa \nabla \phi_j \mathrm{da} \tag{21}$$

$$R_i = \int_{\Gamma_q} \phi_i \bar{q}_n \mathrm{d}l + \int_\Omega z \phi_i \mathrm{da} \tag{22}$$

$$L_{ij} = \int_\Omega \frac{\rho c_p}{\Delta t} \phi_i \phi_j \mathrm{da} \tag{23}$$

the problem reduces to only solving

$$([L]+[K])\{a(t+\Delta t)\} = \{R\} + [L]\{a(t)\}, \tag{24}$$

and again this formulation is used to solve for $\{a(t+\Delta t)\}$ when $\{a(t)\}$ is known.

- **Penalty formulation:** In penalty formulation, we replace the dirichelet boundary condition with some penalty integral as

$$P^* \int_{\Gamma_T} (\bar{T} - T)v\mathrm{d}l, \tag{25}$$

where P^* is a large number to make sure that the values of T on the boundary are equal to $\bar{T}$. Following equation 8, and replacing the dirichelet boundary condition with the penalty, we will have

$$-\int_\Omega \nabla v.\kappa\nabla T\mathrm{d}a + \int_{\Gamma_q} v\bar{q}_n\mathrm{d}l + P^*\int_{\Gamma_T}(\bar{T}-T)v\mathrm{d}l + \int_\Omega zv\mathrm{d}a = \int_\Omega \frac{\rho c_p}{\Delta t}(T(t+\Delta t) - T(t))v\mathrm{d}a. \tag{26}$$

Again, we can proceed in two ways, either computing all the other terms of T at time $t = t$ (Forward Euler or explicit method) or computing at time $t = t + \Delta t$ (Backward Euler or implicit method). we suppose that we can write both test function v and tempreture field T as a summation over the basis functions as equation 11.

(i) **Forward Euler, Explicit Method**: First we consider the Euler Explicit method. Inserting these solution forms in equation 10, Assuming that all terms computed at time $t = t$, we will have

$$\sum_{i=1}^{N} b_i(t)\left\{\left(\int_\Omega \frac{\rho c_p}{\Delta t}\phi_i\phi_j\mathrm{d}a - \sum_{j=1}^{N}\int_\Omega \nabla\phi_i\kappa\nabla\phi_j\mathrm{d}a - P^*\int_\Omega \phi_i\phi_j\mathrm{d}a\right)a_j(t)\right. \tag{27}$$

$$\left. + \int_{\Gamma_q}\phi_i\bar{q}_n\mathrm{d}l + \int_\Omega z\phi_i\mathrm{d}a + P^*\int_\Omega \bar{T}\phi_i\mathrm{d}a = \left(\int_\Omega \frac{\rho c_p}{\Delta t}\phi_i\phi_j\mathrm{d}a\right)a_j(t+\Delta t)\right\}. \tag{28}$$

This is true for any choice of b_i, so every multipication of b_i should be zero. This will lead to a system of equations. If we choose elements of Matrix $[K]$ and matrix $[L]$ and vector $\{R\}$ as

$$K_{ij} = \int_\Omega \nabla\phi_i\kappa\nabla\phi_j\mathrm{d}a \tag{29}$$

$$R_i = \int_{\Gamma_q}\phi_i\bar{q}_n\mathrm{d}l + \int_\Omega z\phi_i\mathrm{d}a + P^*\int_\Omega \bar{T}\phi_i\mathrm{d}a \tag{30}$$

$$L_{ij} = \int_\Omega \frac{\rho c_p}{\Delta t}\phi_i\phi_j\mathrm{d}a \tag{31}$$

the problem reduces to only solving

$$([L]-[K]-P^*[L])\{a(t)\} + \{R\} = [L]\{a(t+\Delta t)\}, \tag{32}$$

where elements of $\{a\}$ will give the solution as in equation 11. This final form equation is used to find values of $\{a(t+\Delta t)\}$ when the values of $\{a(t)\}$ is known. In other words, this equation is used as

$$[L]\{a(t+\Delta t)\} = ([L] - [K] - P^*[L])\{a(t)\} + \{R\} \tag{33}$$

(ii) **Backward Euler, Implicit method**: Inserting equation (11) in equation (10), Assuming that all terms computed at time $t = t$, we will have:

$$\sum_{i=1}^{N} b_i(t) \left\{ \left(\int_\Omega \frac{\rho c_p}{\Delta t} \phi_i \phi_j \mathrm{da} + \sum_{j=1}^{N} \int_\Omega \nabla\phi_i \kappa \nabla\phi_j \mathrm{da} + P^* \int_\Omega \phi_i\phi_j \mathrm{da} \right) a_j(t+\Delta t) = \right. \tag{34}$$

$$\left. + \int_{\Gamma_q} \phi_i \bar{q}_n \mathrm{d}l + \int_\Omega z\phi_i \mathrm{da} + P^* \int_\Omega \bar{T}\phi_i \mathrm{da} + \left(\int_\Omega \frac{\rho c_p}{\Delta t} \phi_i\phi_j \mathrm{da} \right) a_j(t) \right\}. \tag{35}$$

This is true for any choice of b_i, so every multipication of b_i should be zero. This will lead to a system of equations. If we choose elements of Matrix $[K]$ and matrix $[L]$ and vector $\{R\}$ as

$$K_{ij} = \int_\Omega \nabla\phi_i \kappa \nabla\phi_j \mathrm{da} \tag{36}$$

$$R_i = \int_{\Gamma_q} \phi_i \bar{q}_n \mathrm{d}l + \int_\Omega z\phi_i \mathrm{da} + P^* \int_\Omega \bar{T} \mathrm{da} \tag{37}$$

$$L_{ij} = \int_\Omega \frac{\rho c_p}{\Delta t} \phi_i \phi_j \mathrm{da} \tag{38}$$

the problem reduces to only solving

$$([L] + [K] + P^*[L])\{a(t+\Delta t)\} = \{R\} + [L]\{a(t)\}, \tag{39}$$

and again this formulation is used to solve for $\{a(t+\Delta t)\}$ when $\{a(t)\}$ is known.

The only part remained is the lump mass matrix assumption. To do so it is just needed to change the matrix $[L]$ in the above formulation. At each row of $[L]$, we sum over all the elements in the row and then we put this sum on the diagonal. In the mathematical form, we have

$$L_{ii}^{\text{lump}} = \sum_j L_{ij} \tag{40}$$

and all of the other elements are zero. This would help us to increase the time step that we need.

3.2 Radially Symmetric Problem

Recall the simple domain from Homework 5 which is reproduced in Figure 1. Again, let $\bar{T}_i = 100$, $\kappa = 0.04$,$r_i = 0.1$, $r_o = 0.25$, $z = 500$, $\bar{q}_n = -25$. Use the initial value $T(t=0) = 100$ in Ω and $\rho = c_p = 2$. Solve the system from $t_i = 0$ to $t_f = 100$.

- **Modify your code from Homework 5 to solve the time dependent problem.** Use Forward Euler with a lumped-mass approximation for a variable timestep size δt. Use $NE_\theta = 20$ and $NEr = 10$ for your mesh. Plot the radial temperature distribution at multiple points in time to demonstrate the evolution of the problem with $\delta t = tf/10000$. Also plot the temperature at the outer surface against time for this time step. Include the steady-state analytical solution from Homework 5 on the plots (the time-dependent solution will converge to it.)

- **Provide a detailed description of your algorithm.** Detail how you enforce the boundary conditions.

- **Comment on the effect of timestep size.** Experiment with timesteps that smaller and larger than the one stated above. Make a plot of the outer surface temperature versus time with the curves for multiple timestep sizes on the same plot

Solution

We used forward Euler method with lump assumtion on the mass matrix. The formulation of our equation, in two cases of using penalty method, and using direct dirichelet method are as follows

- **Forward Euler with Direct Dirichelet method**: We need to solve

$$[L]\{a(t+\Delta t)\} = ([L]-[K])\{a(t)\} + \{R\} \tag{41}$$

where

$$K_{ij} = \int_\Omega \nabla\phi_i \kappa \nabla\phi_j \mathrm{d}a \tag{42}$$

$$R_i = \int_{\Gamma_q} \phi_i \bar{q}_n \mathrm{d}l + \int_\Omega z\phi_i \mathrm{d}a \tag{43}$$

$$L_{ij} = \int_\Omega \frac{\rho c_p}{\Delta t}\phi_i\phi_j \mathrm{d}a \tag{44}$$

The lump assumtion is that to substitue $[L]$ with $[L^{\text{lump}}]$, where $[L^{\text{lump}}]$ is as

$$L_{ii}^{\text{lump}} = \sum_j L_{ij} \tag{45}$$

- **Forward Euler with Penalty method**: In here we need to solve

$$[L]\{a(t+\Delta t)\} = ([L]-[K]-P^*[L])\{a(t)\} + \{R\} \tag{46}$$

where

$$K_{ij} = \int_\Omega \nabla\phi_i \kappa \nabla\phi_j \mathrm{d}a \tag{47}$$

$$R_i = \int_{\Gamma_q} \phi_i \bar{q}_n \mathrm{d}l + \int_\Omega z\phi_i \mathrm{d}a + P^* \int_\Omega \bar{T}\phi_i \mathrm{d}a \tag{48}$$

$$L_{ij} = \int_\Omega \frac{\rho c_p}{\Delta t}\phi_i\phi_j \mathrm{d}a \tag{49}$$

and again for lump assumption, we just need to substitue $[L]$ with $[L^{\text{lump}}]$, where $[L^{\text{lump}}]$ is as

$$L_{ii}^{\text{lump}} = \sum_j L_{ij} \tag{50}$$

For solving these systems of equations, we first mesh the annular region with the mesh generator that we wrote for the previous homework. This mesh generator will return connectivity matrix and nodes of elements. Then we go over all the elements and find their contribution to the full mass matrix $[K]$ and $[L]$ and the right hand side vector (loading vector) of $\{R\}$. For computing these integrals we will map them to master element in ζ_1, ζ_2 space where both ζ_1 and ζ_2 strecehs from $[-1, 1]$.

Mapping to Master elements

In order to compute the integrals that we have, we will map them to integrals in the ζ_1, ζ_2 space which is our master element. These elements are from $[-1, 1] \times [-1, 1]$ in both ζ_1, ζ_2 space. This mapping will have a Jacobian that is the important part of it. Jacobian of the transformation is

$$F = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \tag{51}$$

where

$$F_{11} = X_1\phi^1_{\zeta_1} + X_2\phi^2_{\zeta_1} + X_3\phi^3_{\zeta_1} + X_4\phi^4_{\zeta_1} \tag{52}$$

$$F_{12} = X_1\phi^1_{\zeta_2} + X_2\phi^2_{\zeta_2} + X_3\phi^3_{\zeta_2} + X_4\phi^4_{\zeta_2} \tag{53}$$

$$F_{21} = Y_1\phi^1_{\zeta_1} + Y_2\phi^2_{\zeta_1} + Y_3\phi^3_{\zeta_1} + Y_4\phi^4_{\zeta_1} \tag{54}$$

$$F_{22} = Y_1\phi^1_{\zeta_2} + Y_2\phi^2_{\zeta_2} + Y_3\phi^3_{\zeta_2} + Y_4\phi^4_{\zeta_2} \tag{55}$$

where (X_i, Y_i) is the position of node i in the element and ϕ^i is the shape function of the i^{th} node, which is 1 at node i and zero at other nodes. the subscripts ζ_1 and ζ_2 stands for derivatives with respect to these variables.

After mapping we will have two different types of integrals to compute: (i) integrals over the volume (area here) and (ii) integrals over the surface (line segments here). For the first type of the integral, we use the rule that volumes are related together with the relation

$$\mathrm{d}\Omega(x) = |\det F|\mathrm{d}\Omega(\zeta) \tag{56}$$

where $\mathrm{d}\Omega(x)$ is the volume (area in here) of the infinitesimal element in (x, y) space and $\mathrm{d}\Omega(\zeta)$ is the volume of the infinitesimal element in the (ζ_1, ζ_2) space. For relating the integrals on the boundary, we will use Nanson's Formula which states

$$\mathbf{n}\mathrm{d}a = |\det F| F^{-T}\mathbf{N}\mathrm{d}A \tag{57}$$

where $\mathrm{d}a$ is in the (x, y) space and $\mathrm{d}A$ is in the (ζ_1, ζ_2) space. Integrals in here are over the boundary lines, for doing so, we want the scaler part of the above relation, we will have

$$\mathrm{d}l = |\det F|\sqrt{F^{-T}\mathbf{N}.F^{-T}\mathbf{N}}\mathrm{d}A \tag{58}$$

The only remained part in computing the integrals is computing the term $\nabla T.\kappa \nabla T$. This term can be easily computed, noting that

$$\nabla_x(.) = F^{-1}\nabla_\zeta(.) \tag{59}$$

where ∇_x is the gradient operator of operator of $(\partial/\partial x, \partial/\partial y)$ and $\nabla_\zeta = (\partial/\partial\zeta_1, \partial/\partial\zeta_2)$. Using these statements integrals could be easily computed in the master element domain. I should mention that integrals in the master element are computed as before using **guass quadrature** method with 5 weighting points.

4 Results

We initialized all of the nodes with $T = 100$. The initial condition and final solution are shown in figure 1. The solution at different times, using lump assumption is shown in figure 2. In here, we used time step used $\delta t = t_f/10000 = (0.01)$. As it can be seen in the figures, the solution is converged at nearly $t = 5$, So we plotted the solution at $t < 5$ to better see the changes in the solution in time. This is shown in figures 3 and 4.

We also did the simulation with full matrix of $[L]$ and observed that using full matrix of $[L]$ we need smaller time steps for convergence. For a converging results with full matrix of $[L]$ we need $\delta t = 0.001$ instead of $\delta t = 0.1$ for lumped assumption.

We also changed the time step and observed that for $\delta t = 0.05$ the solver diverges and the results for $\delta t = 0.01, 0.001$ are plotted in figure 6. Also we observed that without lump assumption we need smaller time steps. Without lump assuption, the solver diverges for $\delta t = 0.01$.

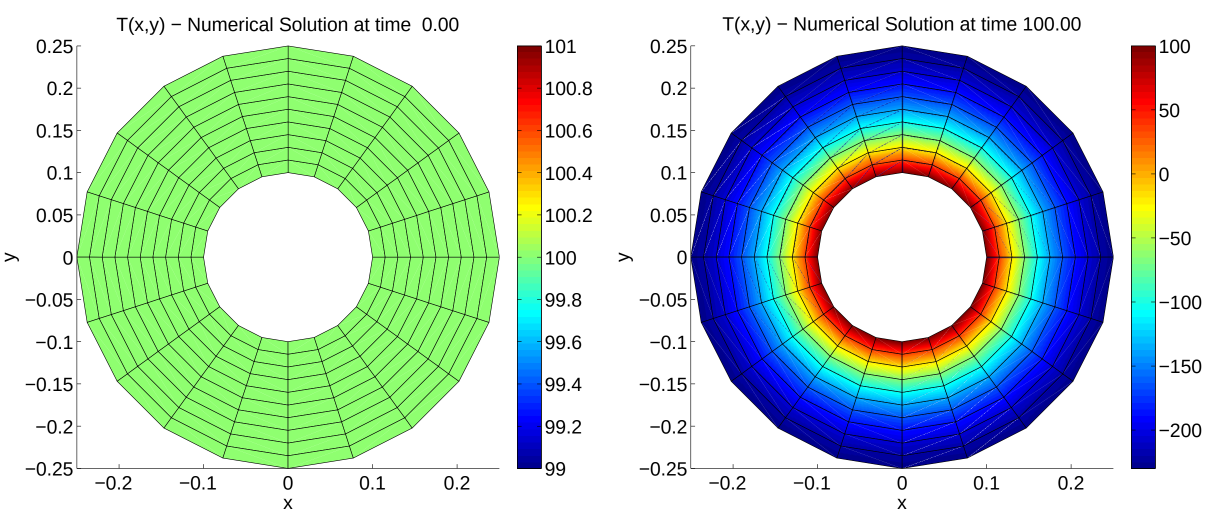

Figure 1: Solution at $t = 0$ and at $t = t_f = 100$ with $NE_r = 10$ and $NE_\theta = 20$ and $\delta t = 0.01 = t_f/10000$

5 Conclusion

In conclusion, we saw how to implement a 2D time dependent problem with different implicit or explicit methods. We used lump assumption and observed that with a higher time step we could converge to the solution.

The time dependent problem that we have here converge to the steady state solution that we solved in the previous home work.

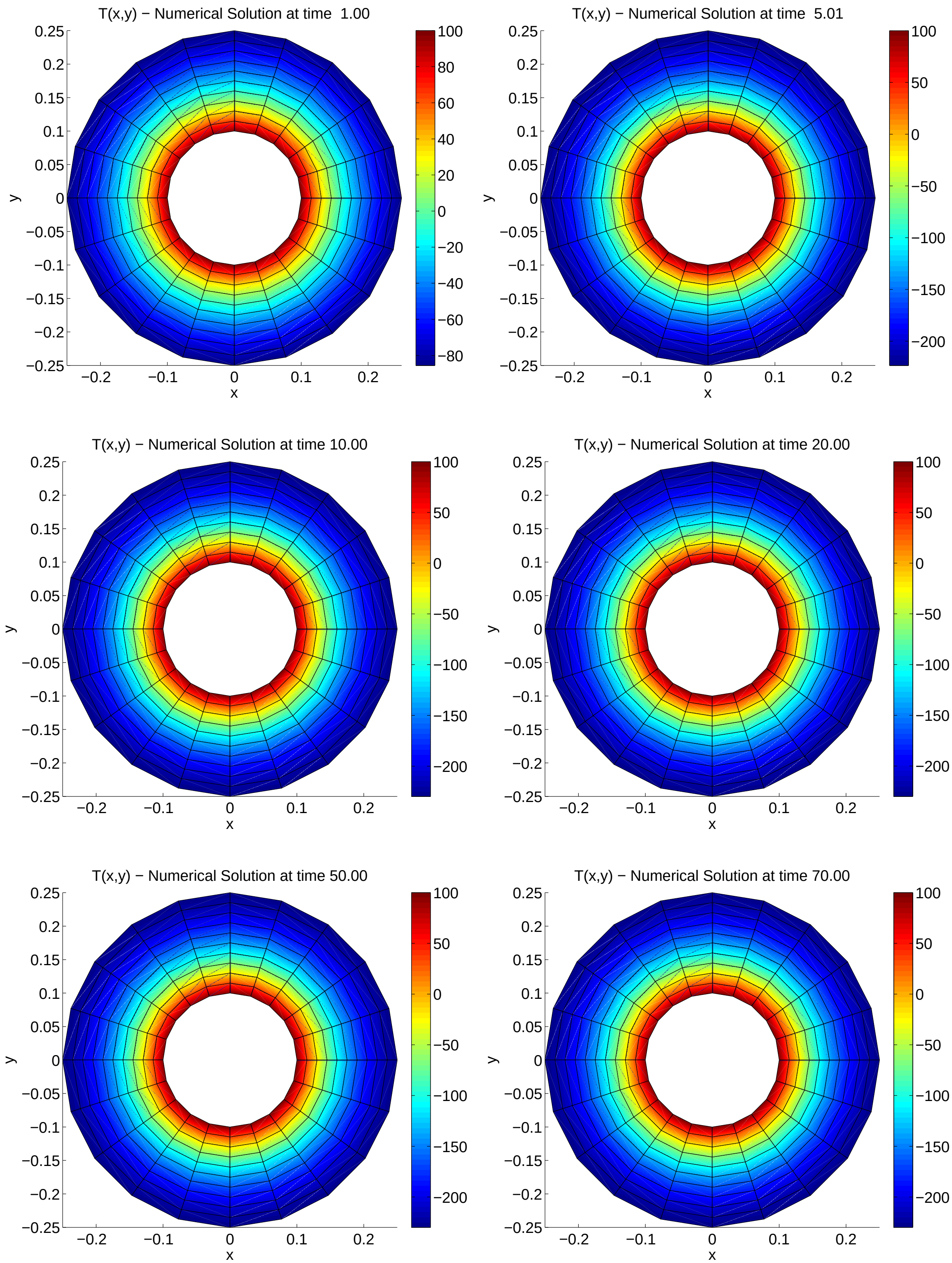

Figure 2: Solution at different times with $NE_r = 10$ and $NE_\theta = 20$ and $\delta t = 0.01 = t_f/10000$

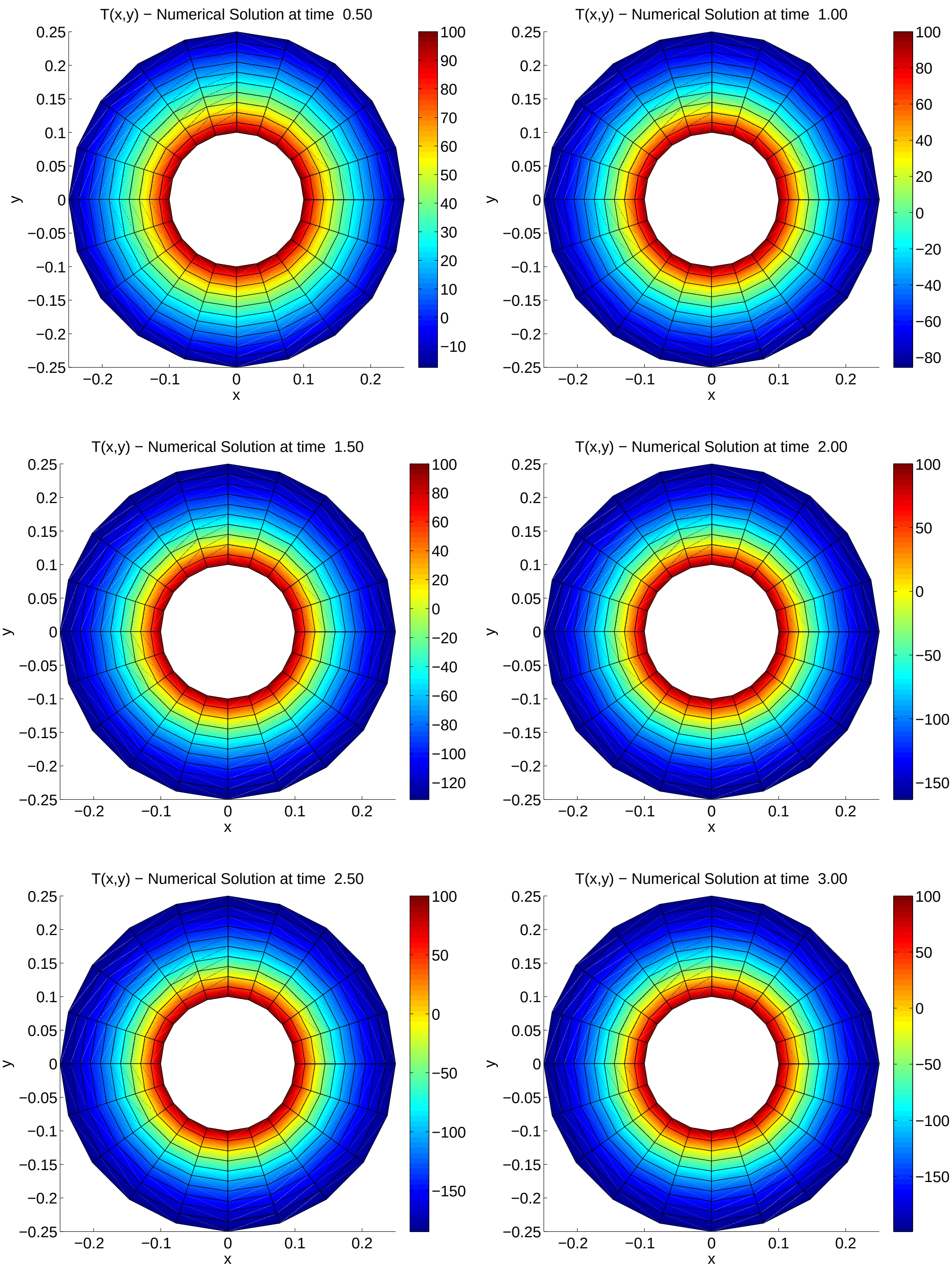

Figure 3: Solution at different times for between $t = 0$ and $t = 3$ with $NE_r = 10$ and $NE_\theta = 20$ and $\delta t = 0.01 = t_f/10000$

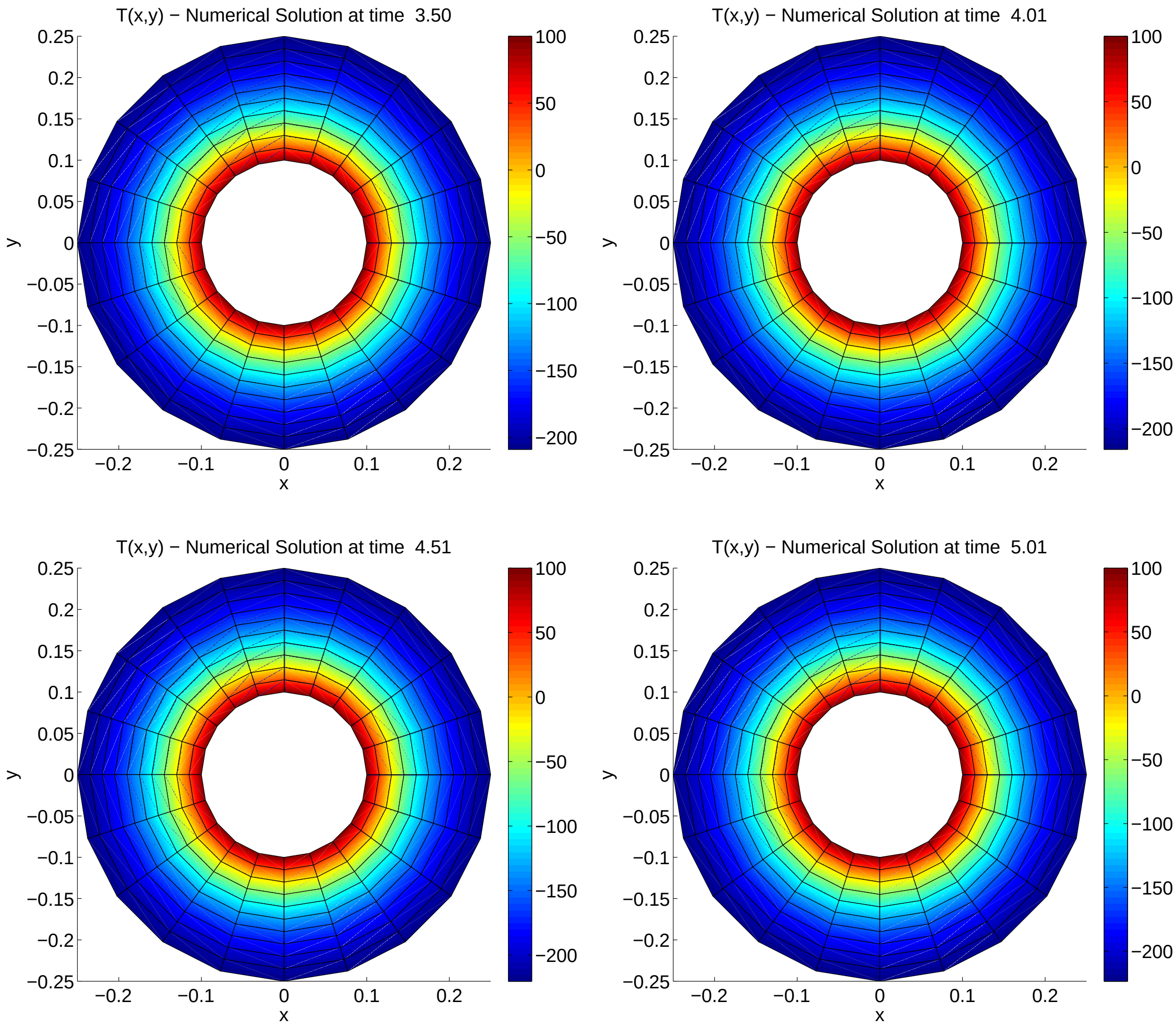

Figure 4: Solution at different times for between $t = 0$ and $t = 3$ with $NE_r = 10$ and $NE_\theta = 20$ and $\delta t = 0.01 = t_f/10000$

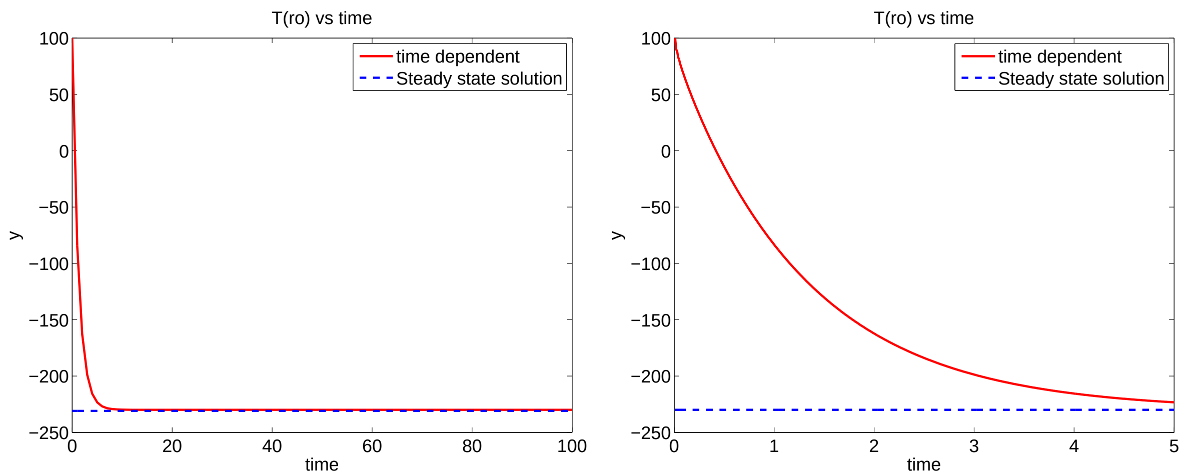

Figure 5: $T(r_o)$ vs t and the steady state solution, Numerical solution is found with lumped assumption with $NE_r = 10$ and $NE_\theta = 20$ and $\delta t = 0.01 = t_f/10000$

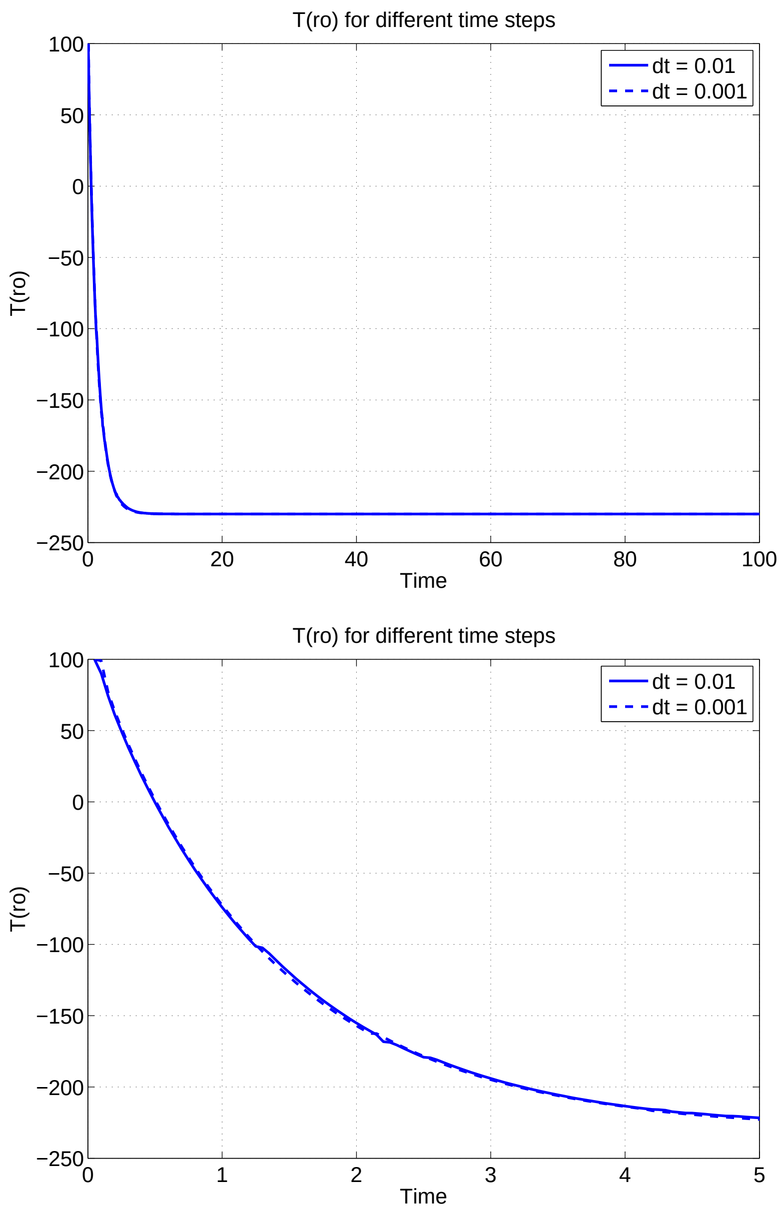

Figure 6: $T(r_o)$ vs t for time steps of $\delta t = 0.01$ and $\delta t = 0.001$, Numerical solution is found with lumped assumption with $NE_r = 10$ and $NE_\theta = 20$