ME280A Fall 2014
Homework 5: Two-Dimensional Problem

Problem Formulation

Consider the steady state heat conduction problem

$$\begin{aligned} \nabla \cdot (k \nabla T) + z &= 0 \,\mathrm{in}\, \Omega \\ T &= \bar{T} \,\mathrm{on}\, \Gamma_T \\ (k\nabla T) \cdot \mathbf{n} &= \bar{q}_n \,\mathrm{on}\, \Gamma_q \end{aligned}$$

for a conductivity k and source term z. You will mesh a domain and solve a finite element solution of the 2D problem using four node bilinear quad elements.

1. **Derive the weak form of this problem.** Derive it for both imposing Dirichlet boundary conditions directly and using the penalty formulation. Explicitly state the properties of the true solution T and the test function v for each case.

Meshing

1. **Construct a mesh of the problem domain in Figure** 1**.** Use 4-node linear quadrilateral elements. Evenly distribute the mesh in r and θ. Your method should be able to vary r_i, r_o as well as the number of elements in the radial and angular directions as input parameters. You meshes should look like the mesh in Figure 1. *MIND THE CONNECTIVITY OF THE MESH:* It does a loop, the elements need to connect in a ring.
2. **Describe your algorithm in detail.** Provide pseudo-code or a flow chart.
3. **Present images of meshes for various element numbers in either direction.** Vary the number of elements along θ by $NE_\theta = 10, 20$ and number of elements along r by $NE_r = 5, 10, 20$ to produce six different meshes.

Radially Symmetric Test Problem

To solve the problem analytically in two dimensions using polar coordinates, the above problem can be written by

$$\frac{1}{r}\frac{\partial}{\partial r}\left(kr\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial \theta}\left(k\frac{\partial T}{\partial \theta}\right) + z = 0.$$

When the problem is radially-symmetric with uniform material properties k and non-varying source term z, the temperature distribution is given by

$$T\left(r\right) = -\frac{z}{4k}r^2 + C_1 \ln r + C_2 \tag{1}$$

for some constant coefficients C_1, C_2. (To emphasize, this is just to solve the problem analytically: your finite element formulation must still be in cartesian coordinates.) Consider the domain in Figure 1. Let the radii be given by $r_i = 0.1$, $r_o = 0.25$, the internal temperature by given by $\bar{T}_i = 100$, outer heat flux $\bar{q}_n = -25$, uniform conductivity $k = 0.04$, and source term $z = -500$,

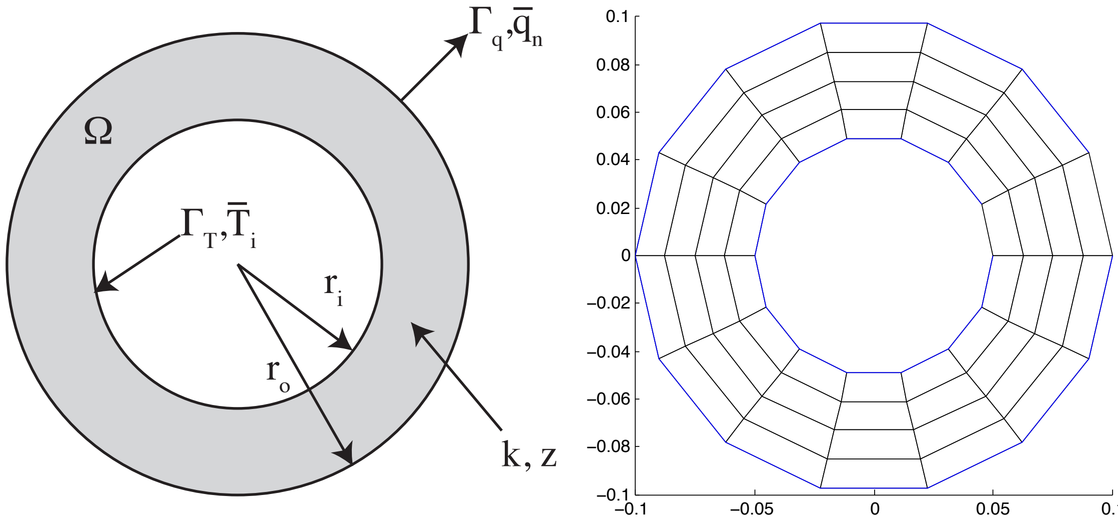

Figure 1: Radially symmetric annulus and sample mesh

1. **Solve the problem analytically.** Solve for C_1 and C_2 in Equation 1 using the boundary conditions
2. **Write a 2D finite element code that uses your meshes.** Call the mesher from within your code, or use the output of the mesher as an input to the code. As in the 1D case, make it so k and z are functions of the position. Use the penalty method to enforce the Dirichlet boundary conditions.
3. **Compare the finite element results to the analytical solution to verify your solution.** Evaluate the error of your solution. Plot your finite element solution along r against the analytical solution for the largest penalty parameter and finest mesh. Also provide a 2D color plot of the solution to verify it is actually radially symmetric.
4. **Discuss the results of refining the mesh.** Comment on the different effect on the solution between refining along r versus refining along θ. Provide a plot with the temperature distribution along r for the above mesh sizes.
5. **Discuss the effect of the different penalty parameters.** Use penalties of $10 \times \max K_{ii}$, $100 \times \max K_{ii}$, and $1000 \times \max K_{ii}$, where $\max K_{ii}$ denotes the maximum diagonal entry in K. Provide a plot of the temperature distribution along r for the most refined mesh with each penalty parameter.

Two-Phase Structure Problem

Consider the annulus in Figure 2 with the following conditions:

$$
\begin{aligned}
z\,(x, y) &= 1000 - 2000y \\
k\,(x, y) &= \begin{cases} 0.5 & y \geq 0 \text{ and } x \geq 0 \\ 1.0 & y \geq 0 \text{ and } x < 0 \\ 1.0 & y < 0 \text{ and } x \geq 0 \\ 0.25 & y < 0 \text{ and } x < 0 \end{cases}
\end{aligned}
$$

The domain dimensions and boundary conditions are the same as the previous problem.

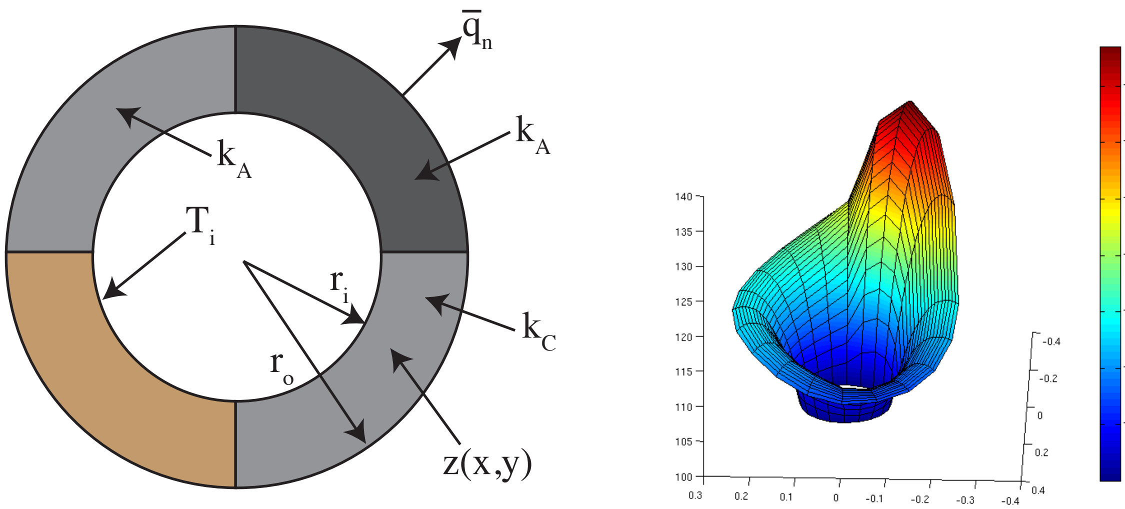

Figure 2: Annulus made from three materials and the expected solution

1. **Solve for the temperature distribution.** Use multiple values of NE_r and NE_θ. Provide a 2D plot of it. *Demonstrate that the solution converged by plotting the potential as the mesh is refined.* Your answer will be similar to the plot in Figure 2.

Hints

- I used the following Matlab command to make the mesh plot in Figure 1:

  ```
  patch('Faces',elems,'Vertices',x,'FaceColor','w')
  ```

  Where x was a vector of node positions, and elems was an array of the form [[1 2 3 4] ; [3 4 5 6] ...]. To specify scalar values to be interpolated, use this command:

  ```
  patch('Vertices',[x a],'Faces',elems,'FaceVertexCData',a,'FaceCol','interp');
  ```

  By doing [x a], I concatenate the solved values into the z-data, so that I can rotate the graph like in Figure 2. Just doing x, you get a 2D plot. That includes the edges, you may want to add

  ```
  'EdgeCol','none'
  ```

  to the options. To get a legend, call

  ```
  colorbar
  ```

  after calling patch.

- You may want to make the mesher return a list of boundaries to simplify applying boundary conditions. For example, surface_1 = [[1 2] ; [2 3] ; [3 4] ; ... [10 1]]. Standalone meshers give you this type of surface representation.