ME 280A
Homework 5
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1 Introduction

Finite element method is a powerfull method to solve for 2D and 3D problems with arbitrary geometries. In the previous problem sets we saw how to solve for 1D cases and we discussed different parts in details. In this problem set we will solve for a simple 2D case with both Newmann and Dirichelet boudnary conditions and also we will use a discontinuous material parameter and see how it easily can be implemented.

2 Objectives

- Solving a 2D problem using finite element method.
- Making the mesh and extract usefull data to pass for finite element solver
- Mapping to master element in 2D problem and constructing stiffness matrix and loading vector
- Implementing different boundary conditions as Dirichelet and Newmann in 2D problems.
- Using potential function as a criteria for convergence

3 Problem, Procedure and Results

3.1 Problem Formulation

Consider the steady state heat conduction problem

$$\nabla.(\kappa\nabla T) + z = 0 \quad \text{in } \Omega \tag{1}$$

$$T = \bar{T} \quad \text{on } \Gamma_T \tag{2}$$

$$(\kappa\nabla T).\mathbf{n} = \bar{q}_n \quad \text{on } \Gamma_q \tag{3}$$

for a conductivity κ and source term z. You will mesh a domain and solve a finite element solution of the 2D problem using four node bilinear quad elements.

- **Derive the weak form of this problem.** Derive it for both imposing Dirichlet boundary conditions directly and using the penalty formulation. Explicitly state the properties of the true solution T and the test function v for each case.

Solution:

The strong form of the equation is as

$$\nabla.(\kappa\nabla T) + z = 0 \quad \text{in } \Omega. \tag{4}$$

In order to find the weak form, we first multiply both sides by some test function v and integrate over the whole domain of Ω. We will have

$$\int_\Omega v\nabla.(\kappa\nabla T)\mathrm{d}a + \int_\Omega zv\mathrm{d}a = 0, \tag{5}$$

where da is the infintesimal surface element. Doing the integration by parts, for the first element we will have

$$-\int_\Omega \nabla v.\kappa\nabla T\mathrm{d}a + \int_\Gamma v\kappa\nabla T.\mathbf{n}\mathrm{d}l + \int_\Omega zv\mathrm{d}a = 0, \tag{6}$$

where $\Gamma = \partial\Omega = \Gamma_T \cup \Gamma_q$ is the boundary of the domain Ω and we used the divergence theorem to change the integral over the domain to the integral over the surface. Now we can proceed in two ways, we can integrate over the dirichelet boundary condition (Γ_T here) with penalty method or with imposing this dirichelet boundary condition directly. We will consider both methods, first we impose direct dirichelet boudnary condition:

- **Direct Dirichelet Condition**: Imposing direct dirichelet boundary condition, we will have

$$-\int_\Omega \nabla v.\kappa\nabla T\mathrm{d}a + \int_{\Gamma_q} v\bar{q}_n\mathrm{d}l + \int_{\Gamma_T} v\kappa\nabla T.\mathbf{n}\mathrm{d}l + \int_\Omega zv\mathrm{d}a = 0, \tag{7}$$

where we set $\bar{q}_n = \kappa\nabla T.\mathbf{n}$ on the bondary Γ_q. Now, we choose test function such that to be zero on Γ_T, so we will get

$$-\int_\Omega \nabla v.\kappa\nabla T\mathrm{d}a + \int_{\Gamma_q} v\bar{q}_n\mathrm{d}l + \int_\Omega zv\mathrm{d}a = 0. \tag{8}$$

We now suppose that test function v and the tempreture T can both be written in some basis functions ϕ_i as

$$v \approx \sum b_i\phi_i, \qquad T \approx \sum a_j\phi_j. \tag{9}$$

Inserting these equationd into equation 8, we will have

$$\sum_{i=1}^{N} b_i \left\{ \left(\sum_{j=1}^{N} \int_\Omega \nabla\phi_i\kappa\nabla\phi_j\mathrm{d}a \right) a_j = \int_{\Gamma_q} \phi_i\bar{q}_n\mathrm{d}l + \int_\Omega z\phi_i\mathrm{d}a \right\}. \tag{10}$$

This is true for any choice of b_i, so every multipication of b_i should be zero. This will lead to a system of equations. If we choose elements of Matrix $[K]$ and vector $\{R\}$ as

$$K_{ij} = \int_\Omega \nabla\phi_i\kappa\nabla\phi_j\mathrm{d}a \qquad R_i = \int_{\Gamma_q} \phi_i\bar{q}_n\mathrm{d}l + \int_\Omega z\phi_i\mathrm{d}a, \tag{11}$$

the problem reduces to only solving

$$[K]\{a\} = \{R\}, \tag{12}$$

where elements of $\{a\}$ will give the solution as in equation 9.

Mapping to Master elements

In order to compute the integrals that we have, we will map them to integrals in the ζ_1, ζ_2 space which is our master element space. These elements are from $[-1,1]\times[-1,1]$ in both ζ_1, ζ_2. This mapping will have a Jacobian that is the important part of it. Jacobian of the transformation is

$$F = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \tag{13}$$

where

$$F_{11} = X_1\phi^1_{\zeta_1} + X_2\phi^2_{\zeta_1} + X_3\phi^3_{\zeta_1} + X_4\phi^4_{\zeta_1} \tag{14}$$
$$F_{12} = X_1\phi^1_{\zeta_2} + X_2\phi^2_{\zeta_2} + X_3\phi^3_{\zeta_2} + X_4\phi^4_{\zeta_2} \tag{15}$$
$$F_{21} = Y_1\phi^1_{\zeta_1} + Y_2\phi^2_{\zeta_1} + Y_3\phi^3_{\zeta_1} + Y_4\phi^4_{\zeta_1} \tag{16}$$
$$F_{22} = Y_1\phi^1_{\zeta_2} + Y_2\phi^2_{\zeta_2} + Y_3\phi^3_{\zeta_2} + Y_4\phi^4_{\zeta_2} \tag{17}$$

where (X_i, Y_i) is the position of node i in the element and ϕ^i is the shape function of the i^{th} node, which is 1 at node i and zero at other nodes. For example, ϕ^1 which is 1 at node $\zeta_1 = -1$ and $\zeta_2 = -1$ is as

$$\phi^1(\zeta_1, \zeta_2) = \frac{-1}{4}(\zeta_1 - 1)(\zeta_2 - 1) \tag{18}$$

Note that the subscripts ζ_1 and ζ_2 in equations (14) to (17), stands for derivatives with respect to these variables.

After mapping, we will have two different types of integrals to compute: (i) integrals over the volume (area here) and (ii) integrals over the surface (line segments here). For the first type of the integral, we use the rule that volumes are related together with the relation

$$\mathrm{d}\Omega(x) = |\mathrm{det}F|\mathrm{d}\Omega(\zeta) \tag{19}$$

where $\mathrm{d}\Omega(x)$ is the volume (area in here) of the element in (x, y) space and $\mathrm{d}\Omega(\zeta)$ is the volume of the element in the (ζ_1, ζ_2) space. For relating the integrals on the boundary, we will use Nanson's Formula which states

$$\mathbf{n}\mathrm{d}a = |\mathrm{det}F|F^{-T}\mathbf{N}\mathrm{d}A \tag{20}$$

where $\mathrm{d}a$ is in the (x, y) space and $\mathrm{d}A$ is in the (ζ_1, ζ_2) space. Integrals in here are over the boundary lines, for doing so, we want the scaler part of the above relation, we will have

$$\mathrm{d}l = |\mathrm{det}F|\sqrt{F^{-T}\mathbf{N}.F^{-T}\mathbf{N}}\ \mathrm{d}L \tag{21}$$

The only remained part in computing the integrals is computing the term $\nabla T.\kappa\nabla T$. This term can be easily computed, noting that

$$\nabla_x(.) = F^{-1}\nabla_\zeta(.) \tag{22}$$

where ∇_x is the gradient operator of the operator $(\partial/\partial x, \partial/\partial y)$ and $\nabla_\zeta = (\partial/\partial\zeta_1, \partial/\partial\zeta_2)$. Using these statements integrals could be easily computed in the master element domain. I should mention that integrals in the master element are computed as before using **guass quadrature** method with 5 weighting points.

- **Penalty formulation:** In penalty formulation, we replace the dirichelet boundary condition with some penalty integral as

$$P^* \int_{\Gamma_T} (\bar{T} - T) v \mathrm{d}l, \tag{23}$$

where P^* is a large number to make sure that the values of T on the boundary are equal to $\bar{T}$. Following equation 6, and replacing the dirichelet boundary condition with the penalty, we will have

$$-\int_\Omega \nabla v.\kappa \nabla T \mathrm{d}a + \int_{\Gamma_q} v\bar{q}_n \mathrm{d}l + P^* \int_{\Gamma_T} (\bar{T} - T) v \mathrm{d}l + \int_\Omega z v \mathrm{d}a = 0. \tag{24}$$

Re organazing this equation, we will have

$$\int_\Omega \nabla v.\kappa \nabla T \mathrm{d}a + P^* \int_{\Gamma_T} T v \mathrm{d}l = \int_{\Gamma_q} v\bar{q}_n \mathrm{d}l + P^* \int_{\Gamma_T} (\bar{T}) v \mathrm{d}l + \int_\Omega z v \mathrm{d}a. \tag{25}$$

Again, we suppose that we can write both test function v and tempreture field T as a summation over the basis functions as equation 9, so we will have

$$\sum_{i=1}^{N} b_i \left\{ \sum_{j=1}^{N} \left(\int_\Omega \nabla\phi_i \kappa \nabla\phi_j \mathrm{d}a + P^* \int_{\Gamma_T} \phi_i\phi_j \mathrm{d}l \right) a_j = \int_{\Gamma_q} \phi_i \bar{q}_n \mathrm{d}l + P^* \int_{\Gamma_T} \bar{T}\phi_i \mathrm{d}l + \int_\Omega z\phi_i \mathrm{d}a \right\}. \tag{26}$$

Since this is true for any choice of b_i, every coefficient of b_i should be zero. This will lead to a system of equations. If we choose elements of matrix $[K]$ and $\{R\}$ as

$$[K]_{ij} = \int_\Omega \nabla\phi_i \kappa \nabla\phi_j \mathrm{d}a + P^* \int_{\Gamma_T} \phi_i\phi_j \mathrm{d}l, \tag{27}$$

and

$$\{R\}_i = \int_{\Gamma_q} \phi_i \bar{q}_n \mathrm{d}l + P^* \int_{\Gamma_T} \bar{T}\phi_i \mathrm{d}l + \int_\Omega z\phi_i \mathrm{d}a, \tag{28}$$

the problem reduces to only solving

$$[K]\{a\} = \{R\}, \tag{29}$$

where elements of $\{a\}$ will give the solution as in equation 9.

3.2 Meshing

- **Construct a mesh of the problem domain** in Figure 1 in the problem set. Use 4-node linear quadrilateral elements. Evenly distribute the mesh in r and θ. Your method should be able to vary r_i, r_o as well as the number of elements in the radial and angular directions as input parameters. You meshes should look like the mesh in Figure 1 in the problem set. MIND THE CONNECTIVITY OF THE MESH: It does a loop, the elements need to connect in a ring.
- **Describe your algorithm in detail**. Provide pseudo-code or a flow chart.
- **Present images of meshes for various element numbers in either direction.** Vary the number of elements along θ by $NE\theta = 10, 20$ and number of elements along r by $NEr = 5, 10, 20$ to produce six different meshes.

Solution

In order to make the mesh the annular region, we distribute the mesh in r and θ directions. We split the radial and angular directions into NEr and NEth elements with the same size. then we compute the x and y coordinates of the nodes with the following relations

$$x = r\cos\theta \qquad y = r\sin\theta. \tag{30}$$

Then we start to number the mesh points. The way the we number the mesh is that, we first start from $\theta = 0$ and start numbering the points in r direction from 0 to $\mathrm{NEr} + 1$. Then we increase the azimuthal angle with $\mathrm{d}\theta$ where $\mathrm{d}\theta = 2\pi/\mathrm{NEth}$ and start numbering the points in the radial direction again.

The pseudo algorithm is as

- Generate the mesh in r and θ directions like a cartesian coordinate
- Generate regarding points in the cartesian coordinate with polar coordinate relations
- Start numbering points in r direction first for each azimuthal angle
- Go over all the numbers and save the points
- With the knowledge of numbers generate the elements numbers in the connectivity matrix

The figures of the mesh for different values of NEr and NEth as requested are shown in figures 1.

3.3 Radially Symmetric Test problem

To solve the problem analytically in two dimensions using polar coordinates, the above problem can be written by

$$\frac{1}{r}\frac{\partial}{\partial r}\left(\kappa r\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial\theta}\left(\kappa\frac{\partial T}{\partial\theta}\right) + z = 0 \tag{31}$$

When the problem is radially-symmetric with uniform material properties κ and non-varying source term z, the temperature distribution is given by

$$T(r) = -\frac{z}{4\kappa}r^2 + C_1 \ln r + C_2 \tag{32}$$

for some constant coefficients $C1, C2$. (To emphasize, this is just to solve the problem analytically: your finite element formulation must still be in cartesian coordinates.) Consider the domain in Figure 1 in the problem set. Let the radii be given by $r_i = 0.1$, $r_o = 0.25$, the internal temperature by given by $\bar{T}_i = 100$, outer heat flux $\bar{q}_n = -25$, uniform conductivity $\kappa = 0.04$, and source term $z = -500$,

Solution

- **Solve the problem analytically**. Solve for $C1$ and $C2$ in equation 32 using the boundary conditions.

  To solve this problem analytically, we first set the boundary condition on the outer cylinder as

  $$\kappa\frac{\partial T}{\partial r} = \bar{q}_n \quad \text{on } r = r_o \tag{33}$$

  where since the normal direction on the outer cylinder is $\hat{r}$, the term $\nabla T.\mathbf{n}$ will be $\frac{\partial T}{\partial r}$. The term $\bar{q}_n$ is the normal heat flux to which is a known paramter in here. Setting this boundary condition we will get

  $$\kappa\left(-\frac{z}{2\kappa}r_o + \frac{C1}{r_o}\right) = \bar{q}_n \quad \rightarrow \quad C_1 = \frac{\bar{q}_n}{\kappa}r_o + \frac{z}{2\kappa}r_o^2 \tag{34}$$

  Now that we found C_1, we impose the boundary condition on the inner cylinder as

  $$T(r_i) = \bar{T} \quad \rightarrow \quad -\frac{z}{4\kappa}r_i^2 + C_1 \ln r_i + C_2 = \bar{T}. \tag{35}$$

  So, C_2 reads

  $$C_2 = \bar{T} + \frac{z}{4\kappa}r_i^2 - \left(\frac{\bar{q}_n}{\kappa}r_o + \frac{z}{2\kappa}r_o^2\right)\ln r_i \tag{36}$$

  The exact solution found numerically is shown in figure 2

- **Write a 2D finite element code that uses your meshes.** Call the mesher from within your code, or use the output of the mesher as an input to the code. As in the 1D case, make it so k and z are functions of the position. Use the penalty method to enforce the Dirichlet boundary conditions.

- **Compare the finite element results to the analytical solution to verify your solution.** Evaluate the error of your solution. Plot your finite element solution along r against the analytical solution for the largest penalty parameter and finest mesh. Also provide a 2D color plot of the solution to verify it is actually radially symmetric.

- **Discuss the results of refining the mesh.** Comment on the different effect on the solution between refining along r versus refining along θ. Provide a plot with the temperature distribution along r for the above mesh sizes.

Solution

We followed the penalty method as discussed earlier and used the penalty method for the dirichelet condition on the inner boundary. We used 1000 MaxK_{ii} for the P^* value in the penalty method. The results for 20 mesh points in the radial and azimuthal directions is shown in figures 3 to 5.

For the error we computed the value of e^N as follows

$$e^N = \frac{||T - T^{N_r,N_\theta}||_\Omega}{||T||_\Omega}, \tag{37}$$

where

$$||R||_\Omega = \sqrt{\int_\Omega \nabla T.\kappa \nabla T dx}. \tag{38}$$

We computed this error by going over all the elements and by mapping them to the master elements. Note that T^{N_r,N_θ} is the numerical solution with N_r and N_θ elements in r and θ directions respectively.

The results of the error analysis for different cases of the N_r and N_θ and penalty constants are tabulated in table 1. As it can be seen from the resulted values of the error, refinement in the θ direction has more impact on the error than refinement in the r direction.

N_r, N_θ	5,10	5,20	10,10	10,20	20,10	20,20
Error	0.0488	0.0284	0.0448	0.0252	0.0425	0.0230

Table 1: Elapsing and Error for different values of h for $\lambda = 1/2$

- **Discuss the effect of the different penalty parameters.** Use penalties of $10 \max K_{ii}$, $100 \max K_{ii}$, and $1000 \max K_{ii}$, where max K_{ii} denotes the maximum diagonal entry in K. Provide a plot of the temperature distribution along r for the most refined mesh with each penalty parameter.

Solution

In the code for solving using penalty method, there is an argument in the input that determines the coefficient for P^*. This constant in the input will be multiplied by the $\max K_{ii}$ and will be used as P^*. The results for the case of $N_r = 20$ and $N_\theta = 20$ and different values of this constant are plotted in figure 6.

3.4 Two-Phase Structure Problem

Consider the annulus in Figure 2 in the problem set with the following conditions:

$$z(x,y) = 1000 - 2000y \tag{39}$$

and

$$\kappa(x,y) = \begin{cases} 0.5 & y \leq 0 \text{ and } x \leq 0 \\ 1.0 & y \leq 0 \text{ and } x < 0 \\ 1.0 & y < 0 \text{ and } x \leq 0 \\ 0.25 & y < 0 \text{ and } x < 0 \end{cases} \tag{40}$$

The domain dimensions and boundary conditions are the same as the previous problem.

- **Solve for the temperature distribution.** Use multiple values of NEr and $NEth$. Provide a 2D plot of it. Demonstrate that the solution converged by plotting the potential as the mesh is refined. Your answer will be similar to the plot in Figure 2 in the problem set.

Solution

We changed the values of $\kappa(x,y)$ and $z(x,y)$ in our code and plotted the results for diferrent values of N_r and N_θ. The results are shown in figure 8 and 9.

For the potential function, we computed the following expression as:

$$\mathcal{J}(T) = \frac{1}{2}\int_\Omega \nabla T.\kappa \nabla T \mathrm{d}v - \int_\Omega zT\mathrm{d}v - \int_{\Gamma_q} T\bar{q}_n \mathrm{d}a \tag{41}$$

where Ω is the whole annular domain and Γ_q is the part where $\bar{q}_n$ is defined, which is the outer boundary. The change of the potential function for different N_θ in a fixed N_r are shown in figure 9. These values of potential nearly constant. If we change the limits of the y-axis we see that these values are pretty much constant. We plotted these results with different limits on the y-axis in the figure 10.

4 Conclusion

In conclusion, we saw how to solve 2D problem with Finite element method. We used mapping to master elements to compute the integrals. We used connectivity matrix to compute contribution of each element to stiffness matrix and loading vector.

We used the true solution to show the convergence for the case where we knew the analytical solution. For the case of two phase structure problem, we used the potential function to show how the solution is converging.

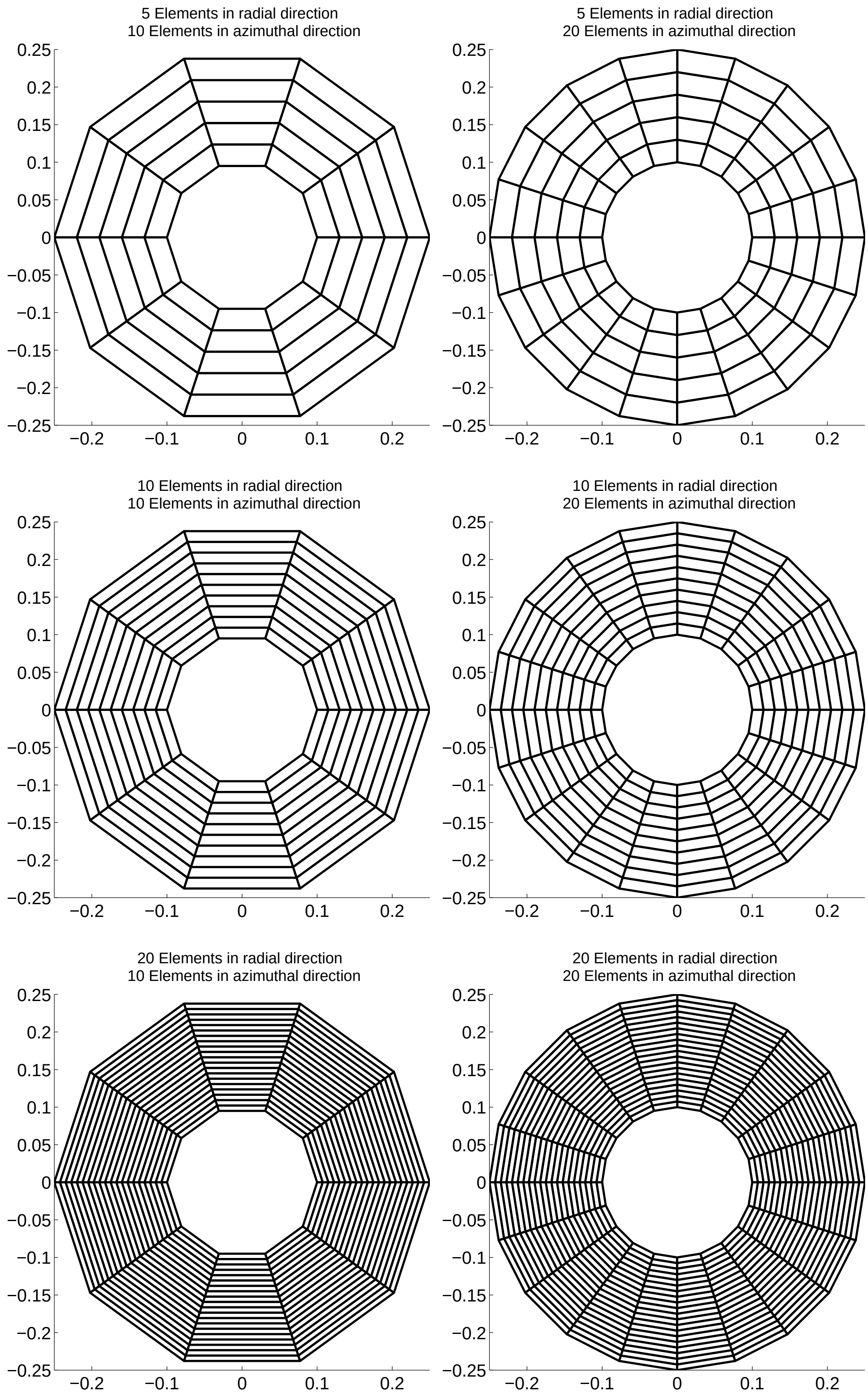

Figure 1: Mesh figures for different values of NEr and NEth

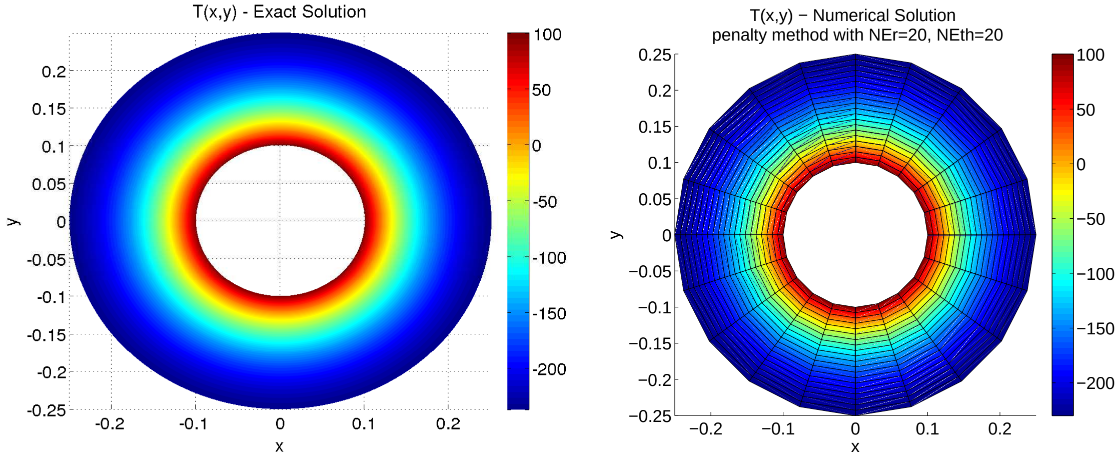

Figure 2: 2D color plot of exact solution in the domain, 2D solution of numerical result with $N_r = 20$ and $N_\theta = 20$ and with $P^* = 1000 \max K_{ii}$

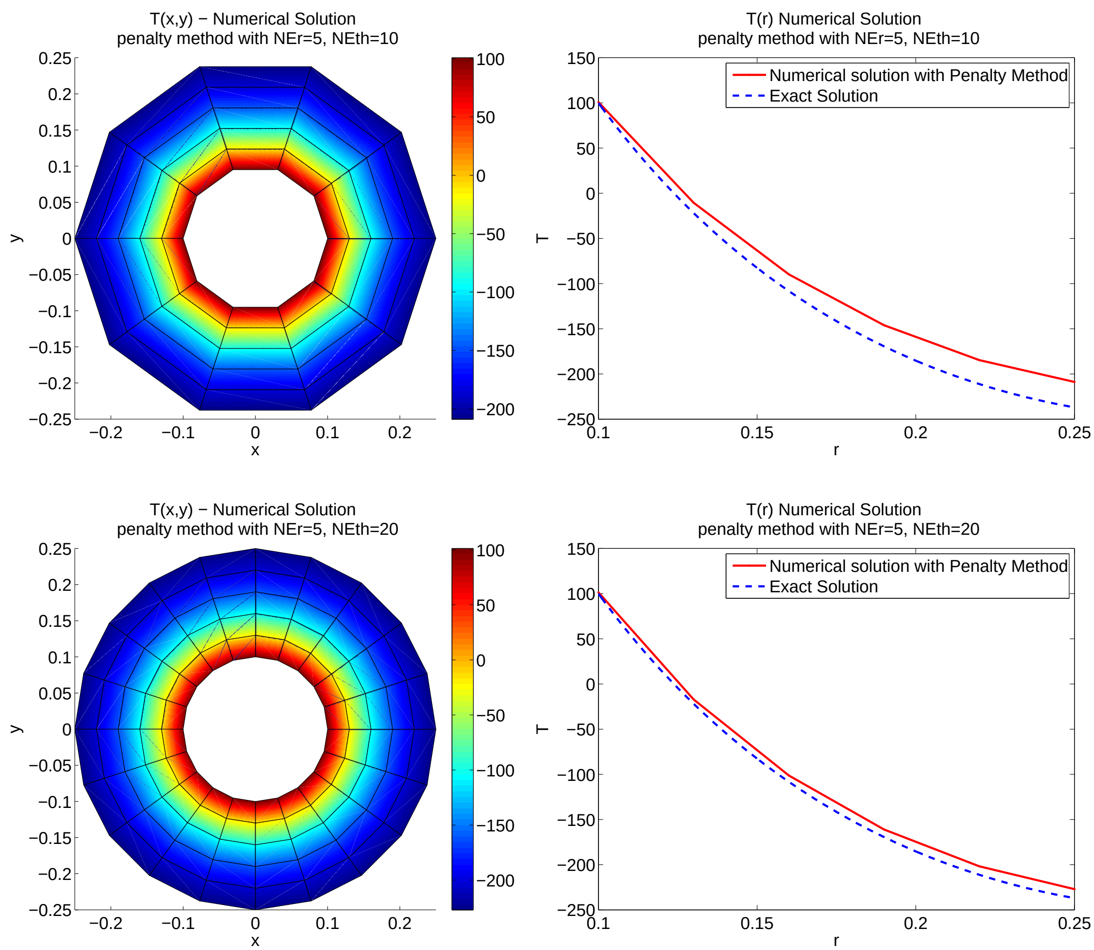

Figure 3: 2D color plot and $T(r)$ vs exact solution for $N_r = 5$ and different N_θ. $P^* = 1000 \max K_{ii}$

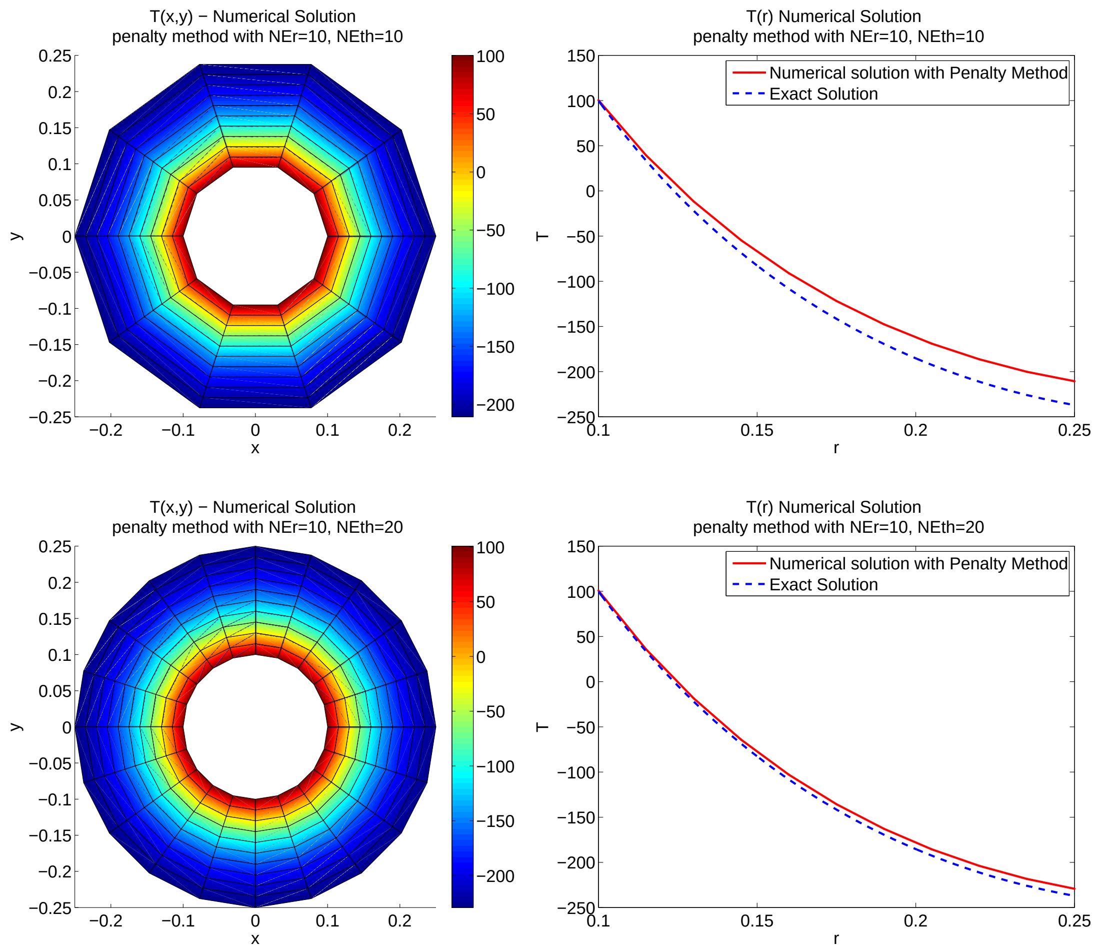

Figure 4: 2D color plot and $T(r)$ vs exact solution for $N_r = 10$ and different N_θ. $P^* = 1000 \max K_{ii}$

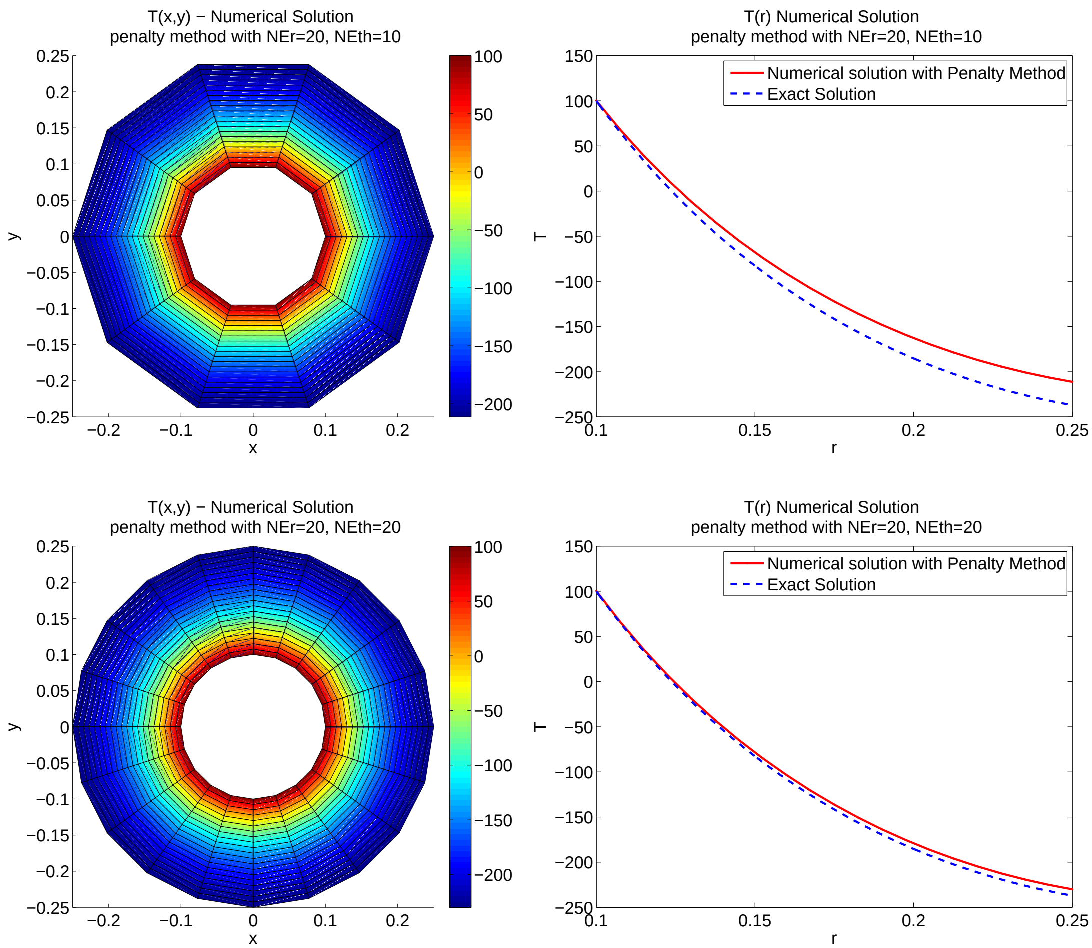

Figure 5: 2D color plot and $T(r)$ vs exact solution for $N_r = 20$ and different N_θ. $P^* = 1000 \max K_{ii}$

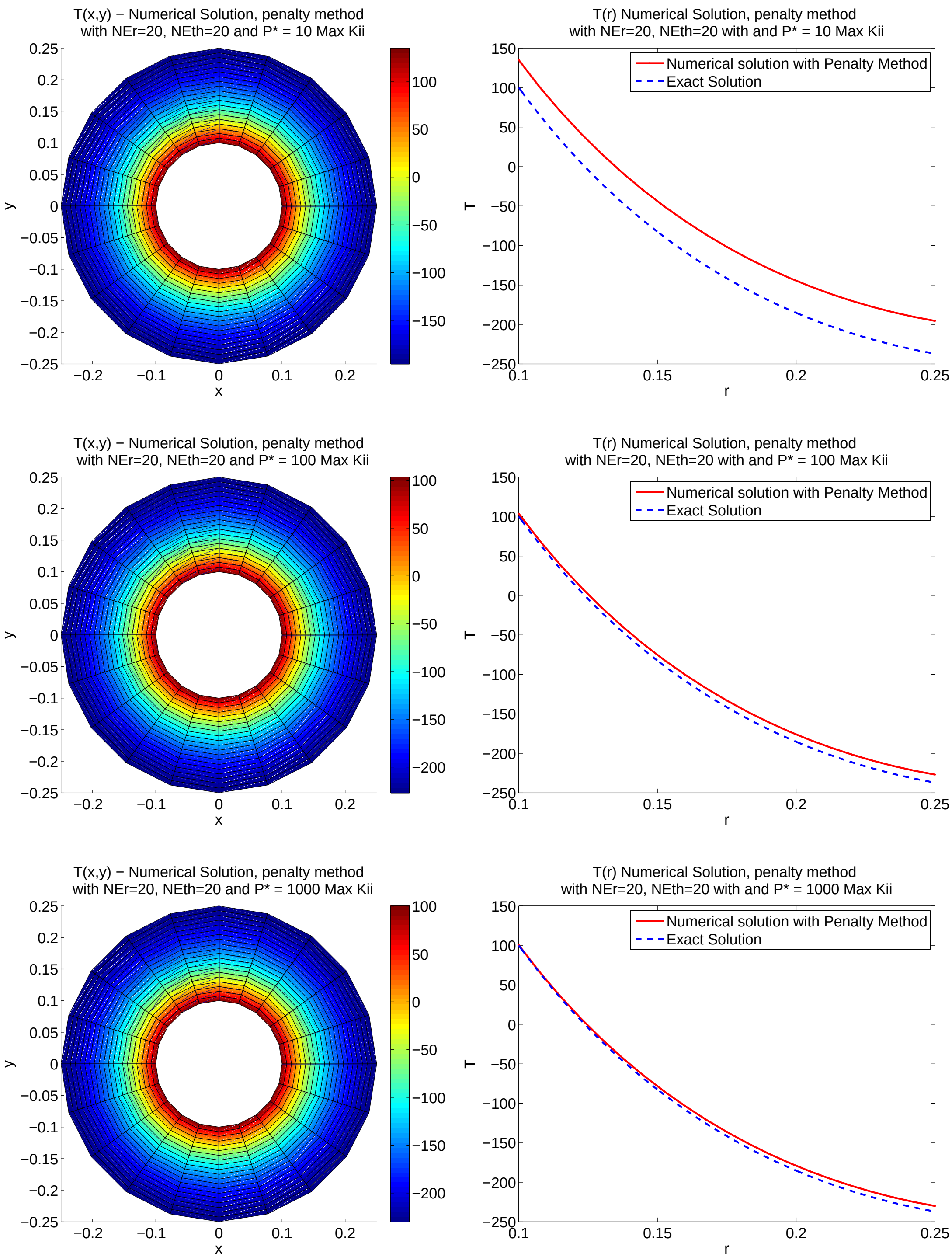

Figure 6: 2D color plot and $T(r)$ vs exact solution for $N_r = 20$ and $N_\theta = 20$ and different P^* values

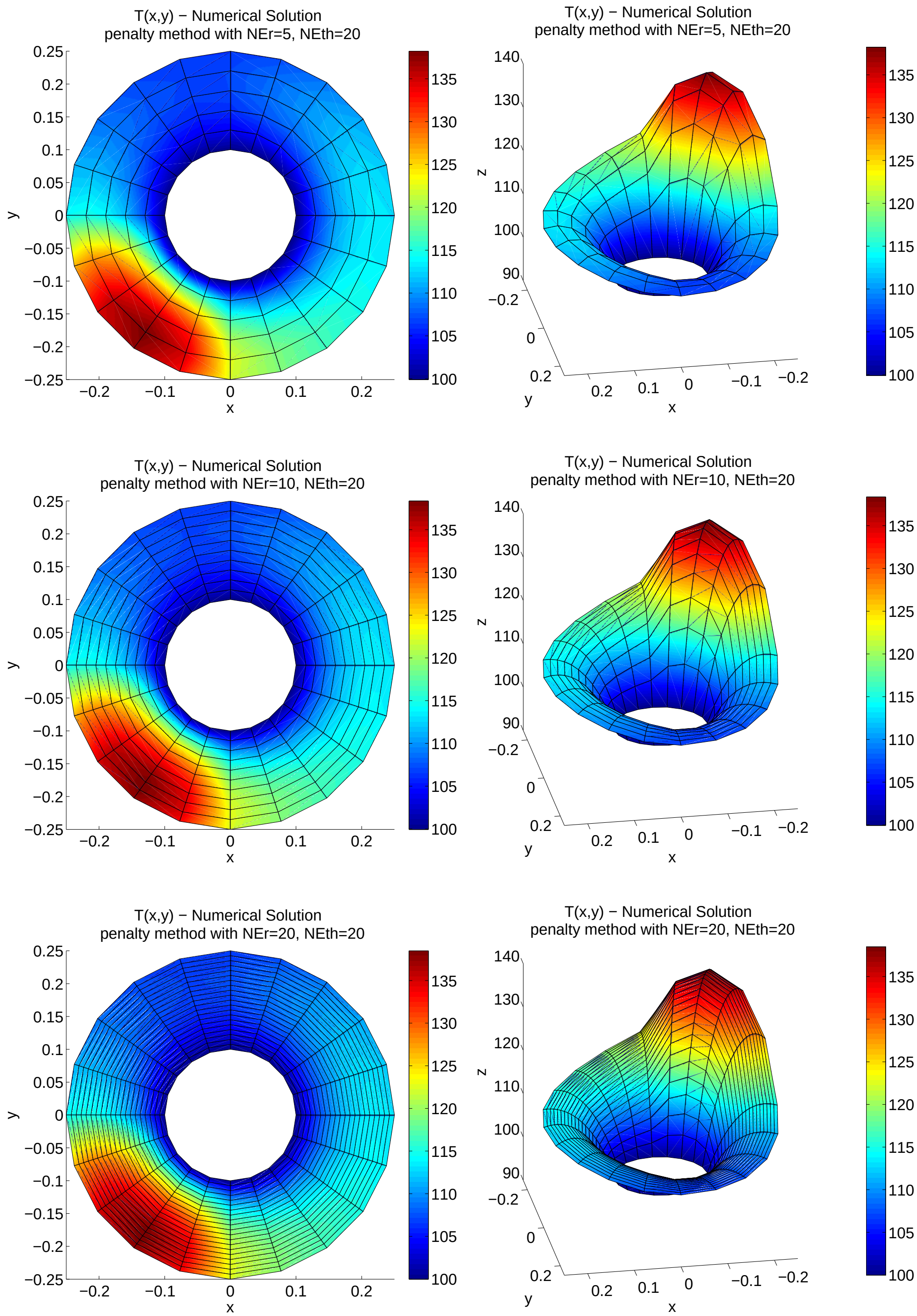

Figure 7: 2D color plot and $T(r)$ vs exact solution for $N_\theta = 20$ and different N_r, penalty parameter is kept fixed as $P^* = 1000 \max K_{ii}$

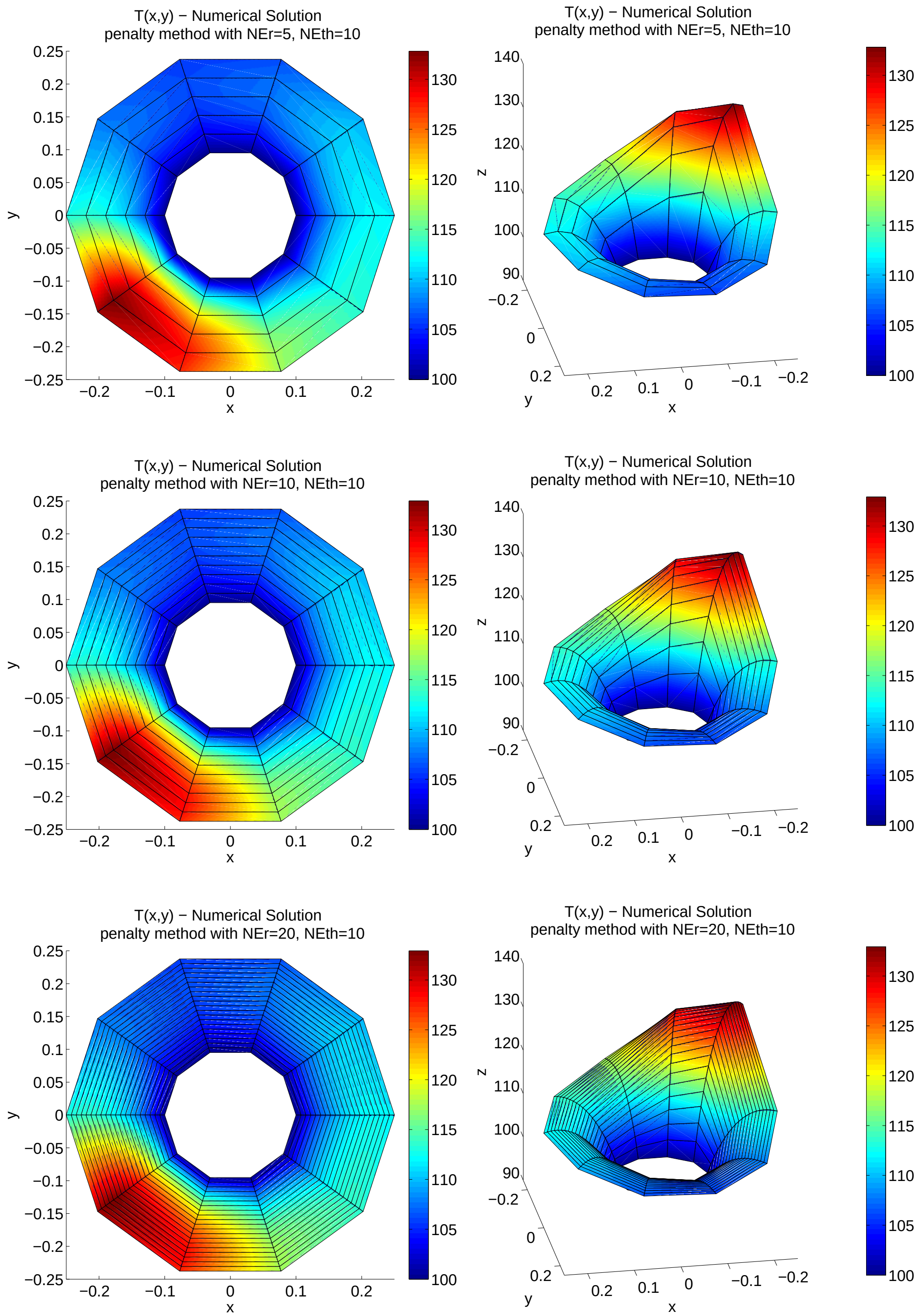

Figure 8: 2D color plot and $T(r)$ vs exact solution for $N_\theta = 20$ and different N_r, penalty parameter is kept fixed as $P^* = 1000 \max K_{ii}$

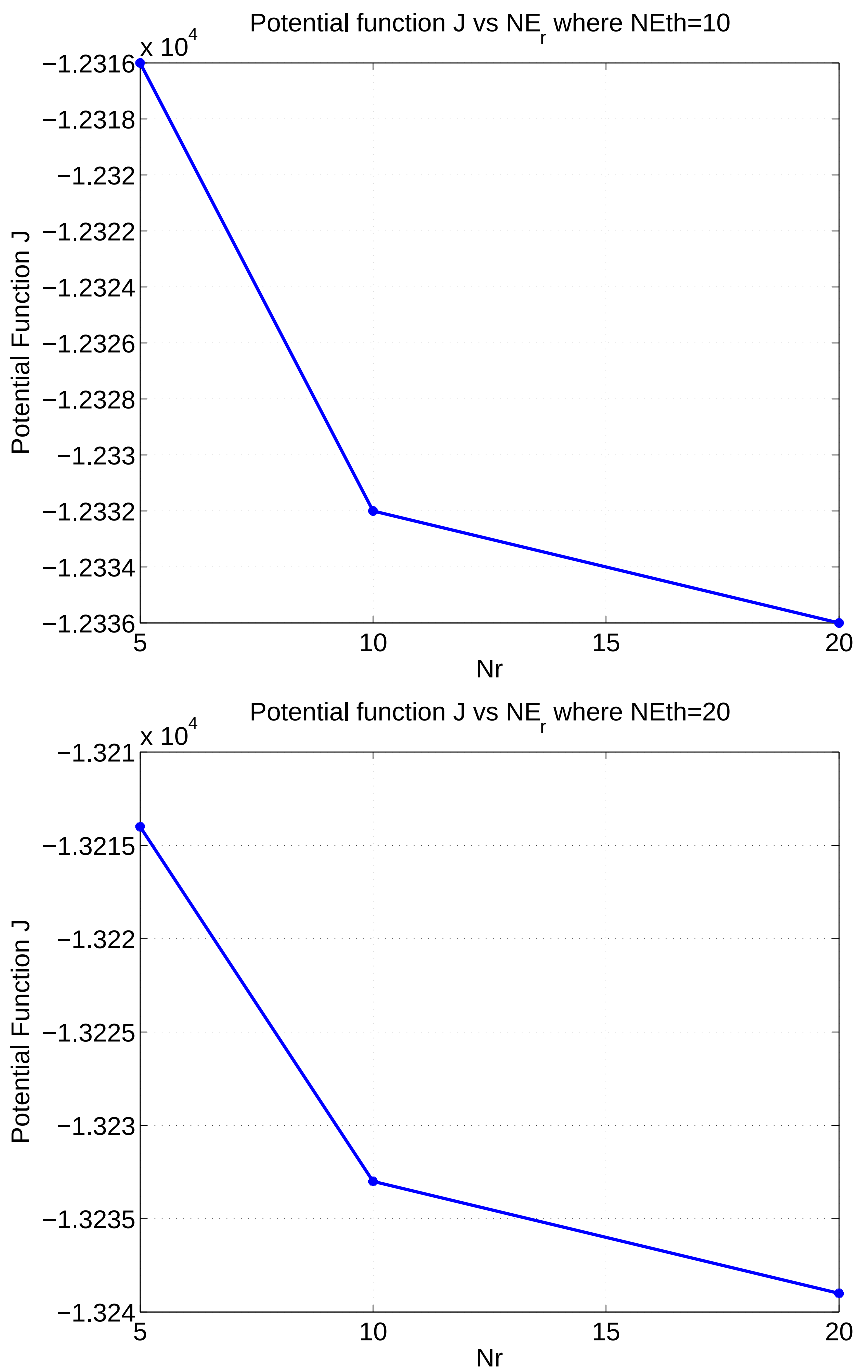

Figure 9: Potential function versus N_r for fixed $N_\theta = 10$ and $N_\theta = 20$

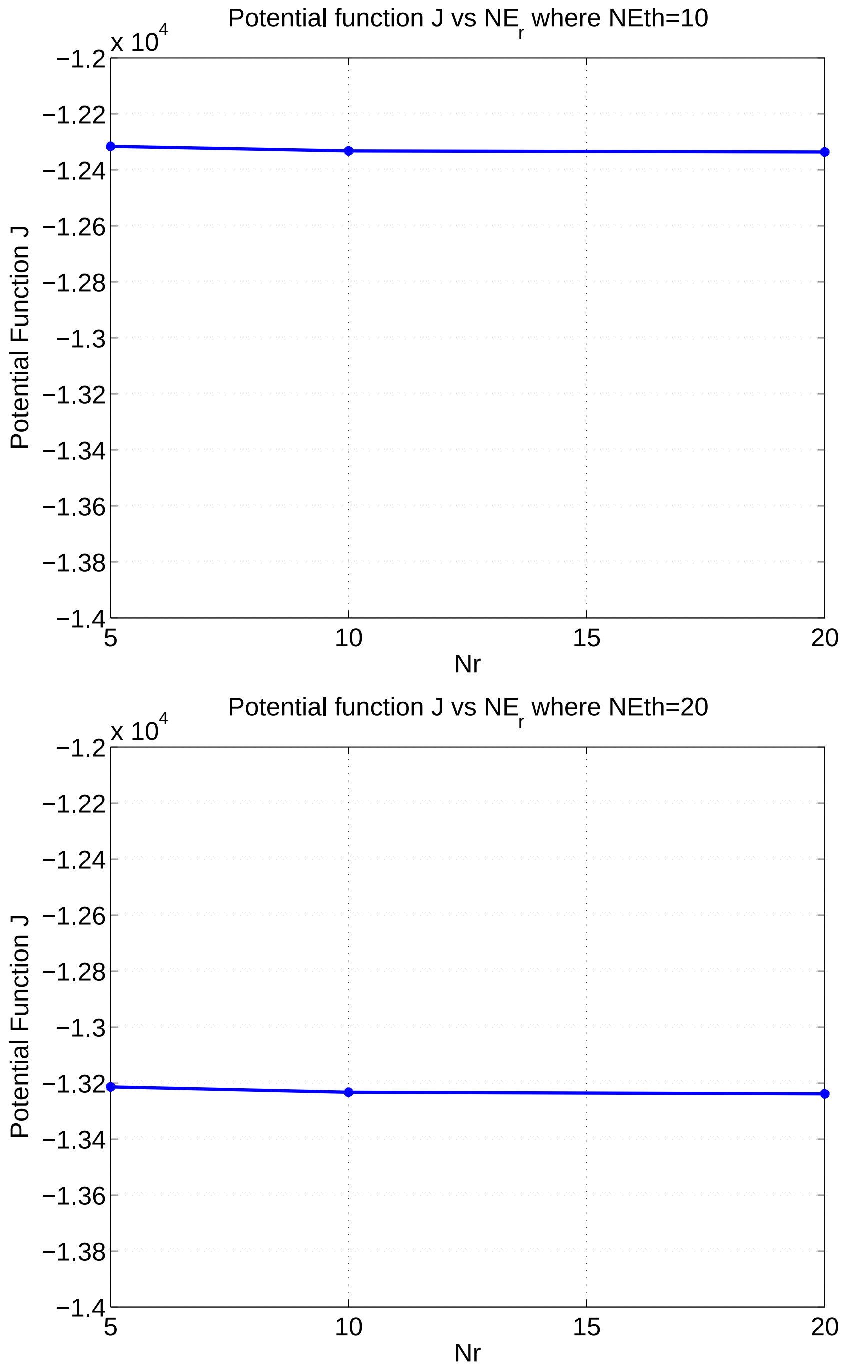

Figure 10: Potential function versus N_r for fixed $N_\theta = 10$ and $N_\theta = 20$