Math 228B
Homework 4
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Introduction

In this problem set, we want to solve burger equation which is written as

$$u_t + uu_x = 0$$

using several different methods such as Lax-Wendroff, Lax-Fredriches, Godunov, Glimm, upwind and upwind with viscosity method. We will invistigate how shocks are made in the system and how these methods keep the shocks sharp.

Problem Definition

We will be solving Burgers' equation:

$$u_t + uu_x = 0$$

This equation could be written in a conservative form as

$$u_t + [F(u)]_x = 0$$

where $F(u) = \frac{1}{2}u^2$. This form of an equation is called conservative form. We will solve this equation on $[-1, 1]$, with two different initial condition

$$1)\ u = 0 \text{ for } x < 0,\ u = 1 \text{ for } x >= 0 \tag{1}$$
$$2)\ u = 1 \text{ for } x < 0,\ u = 0 \text{ for } x >= 0 \tag{2}$$

For each set of initial data, we will solve Burgers' equation using

1. Lax-Friedrichs
2. Lax-Wendroff
3. Glimm's Method
4. Godunov's Method
5. Upwind
6. Upwind with Viscosity

We will also see how shock are formed in each method and how they will stay sharp.

Implementation

In this part, we will talk about about different methods and their numerical scheme used to simulate Burgers' equation.

The equation we are solving is

$$u_t + uu_x = 0$$

which could be written as

$$u_t + [\frac{1}{2}u^2]_x = 0 \quad \Rightarrow \quad u_t + [F(u)]_x = 0$$

where $F(u) = \frac{1}{2}u^2$. We define $a(u) := F'(u) = \frac{\mathrm{d}F(u)}{\mathrm{d}u}$, where $a(u)$ shows the speed of propagation. In burgers' equation $a(u) = u$.

Lax Wendroff

Lax Wedroff is the following finite difference following:

$$u_i^{n+1} = u_i^n - \frac{\lambda}{2}[F_{i+1} - F_{i-1}] + \frac{\lambda^2}{2}[a_{i+1/2}[F_{i+1} - Fi] - a_{i-1/2}[F_i - F_{i-1}]]$$

where

$$a_{i+1/2} = \frac{1}{2}(a_i + a_{i+1}) \qquad a_{i-1/2} = \frac{1}{2}(a_{i-1} + a_i)$$

This method can be written in conservational form, which would mean that decreasing the mesh size will result in converging to weak form solution of Burgers' equation. Lax Wendroff is easy to use and second order. However we will have dispersive errors.

Lax Fredrichs

Lax Fredriches scheme could be written as

$$u^{n+1} = \frac{1}{2}[u_{i+1} + u_{i-1}] - \frac{\lambda}{2}[F_{i+1} - F_{i-1}]$$

Lax Friedrichs method could also be written in conservational form. This would mean that this method is converging to weak form of Burgers' equation. This scheme is also easy to implement and the errors are dissipative. However the method is 1st order and it will smears shocks.

Upwind

Upwind method scheme could be written as

$$u_i^{n+1} = u_i^n - kaD_-u_i^n \qquad \text{if } a > 0 \tag{3}$$
$$= u_i^n - kaD_+u_i^n \qquad \text{if } a < 0 \tag{4}$$

Upwind with Viscosity

If we add the viscosity term to the upwind method (discussed previously), we will have upwind method with viscosity. This scheme is could be written as

$$
\begin{aligned}
u_i^{n+1} &= u_i^n - kaD_-u_i^n + k\epsilon D_-D_+u_i^n \qquad \text{if } a > 0 &\quad (5)\\
&= u_i^n - kaD_+u_i^n + k\epsilon D_-D_+u_i^n \qquad \text{if } a < 0 &\quad (6)
\end{aligned}
$$

where ϵ is the viscosity.

Godunov method

Godunov method is based on solving Reinman. The idea of Godunovs method is the following. Let U_n be a numerical solution on the n-th layer. Then we define a function $\tilde{u}(x,t)$ for $t_n \leq t \leq t_{n+1}$ as follows. At $t = t_n$

$$\tilde{u}^n(x, t_n) = U_j^n, \qquad x_j - \Delta x/2 < x < x_j + \Delta x/2, \quad j = 2, ..., n-1$$

then $\tilde{u}(x,t)$ is the solution of the collection of the Riemann problems on the interval $[t_n, t_{n+1}]$. If Δt is small enough so that the characteristics starting at the points $x_j + \Delta x/2$ do not intersect within this interval (i.e., the CFL condition is satisfied), then $\tilde{u}(x,t)$ is determined unambiguously. Then the numerical solution on the next layer, U_j^{n+1} is defined by averaging $\tilde{u}(x, t_{n+1})$ over the intervals $x_j \Delta x/2 < x < x_j + \Delta x/2$. This idea reduces to a very simple numerical procedure given by

$$U_j^{n+1} = U_j^n - \lambda\left(F\left(U_j^n, U_{j+1}^n\right) - F\left(U_j^n, U_{j-1}^n\right)\right)$$

The function $F(U_L, U_R)$ is the numerical flux defined by $F(U_L, U_R) = \frac{(u^*)^2}{2}$ where u is defined as follows.

If $U_L \geq U_R$

$$u^* = \begin{cases} U_L, & (U_L + U_R)/2 > 0 \\ U_R, & (U_L + U_R)/2 \leq 0 \end{cases} \tag{7}$$

If $U_L < U_R$

$$u^* = \begin{cases} U_L, & U_L \; > 0 \\ U_R, & U_R \; < 0 \\ 0, & U_L \; \leq 0 \leq U_R \end{cases} \tag{8}$$

Glimm's method

Like Godunovs mothod, Glimms method is also based on solving a collection of Riemann problems on the interval $[t_n, t_{n+1}]$. But instead of averaging procedure for getting the numerical solution on the next layer a random choice procedure is used. Glimms time step proceeds in two stages

$$U^{n+1/2}_{j+1/2} = \zeta\left(U^n_j, U^n_{j+1}\right) \tag{9}$$
$$U^{n+1}_j = \zeta\left(U^{n+1/2}_{j-1/2}, U^{n+1/2}_{j+1/2}\right) \tag{10}$$
$$\tag{11}$$

where $\zeta(U_L, U_R)$ is a random variable taking values either U_L or U_R with probabilities proportional to lengths of the corresponding intervals

$$\zeta(U_L, U_R) = \begin{cases} U_L, & p = \frac{1}{2}(1 + s\lambda) \\ U_R, & p = \frac{1}{2}(1 - s\lambda) \end{cases} \tag{12}$$

where $s = (U_L + U_R)/2$ is the shock speed and $\lambda = \frac{\Delta t}{\Delta x}$.

Problem

1. Take the one-dimensional Burgers' equation:

$$u_t + uu_x = 0$$

Take two sets of initial Riemann data:

$$1)\ u = 0 \text{ for } x < 0,\ u = 1 \text{ for } x >= 0 \tag{13}$$
$$2)\ u = 1 \text{ for } x < 0,\ u = 0 \text{ for } x >= 0 \tag{14}$$

For each set of initial data, solve Burgers' equation using

(a) Lax-Friedrichs

(b) Lax-Wendroff

(c) Glimm's Method

(d) Godunov's Method

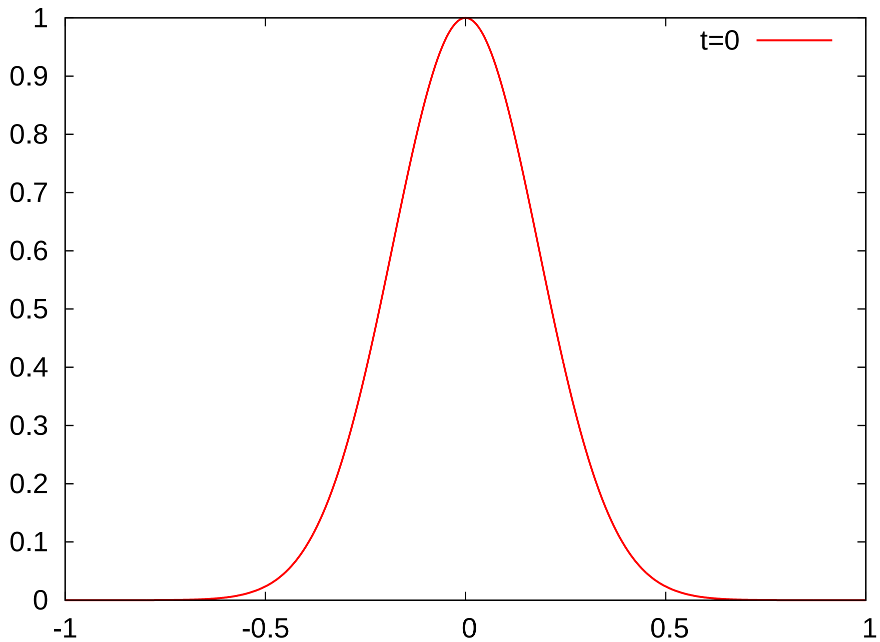
(e) Upwind

(f) Upwind with Viscosity

Resolve the mesh, see what happens to oscillations, sharpness, etc. Figure out a way to assess your error.

Solution

We have implemented different schemes numerically. To show the difference between different schemes, I picked a guassian bell shape initial condition. With this condition we will a a shock will form and and after that, we will see how this shock is moving in our space. Initial conditon is as in Figure 1



[h]

Figure 1: Initial Bell shape figure

Here we will plot what we see in each different scheme at different times (Figures 2 to 7).

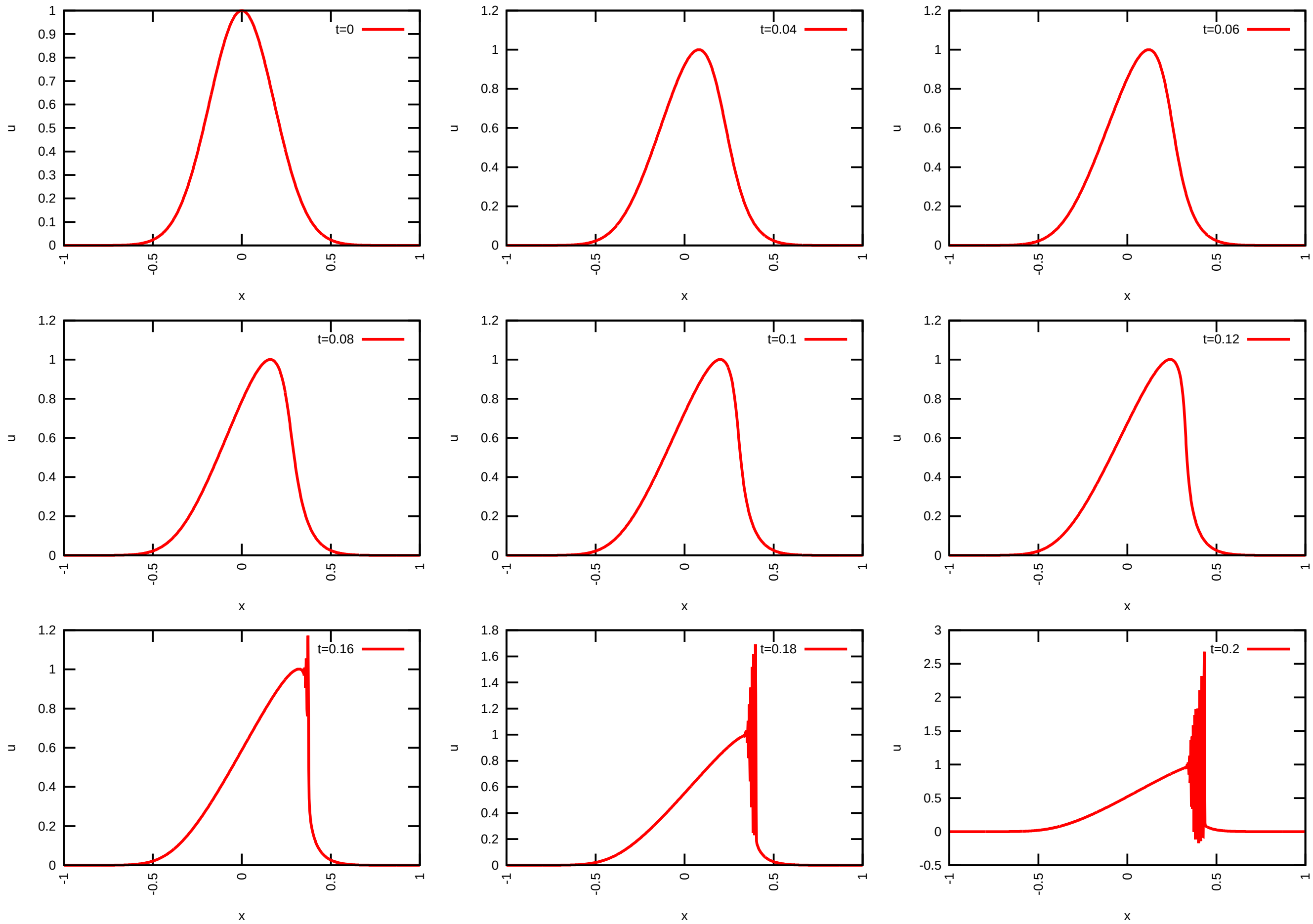


Figure 2: Lax Wendroff method

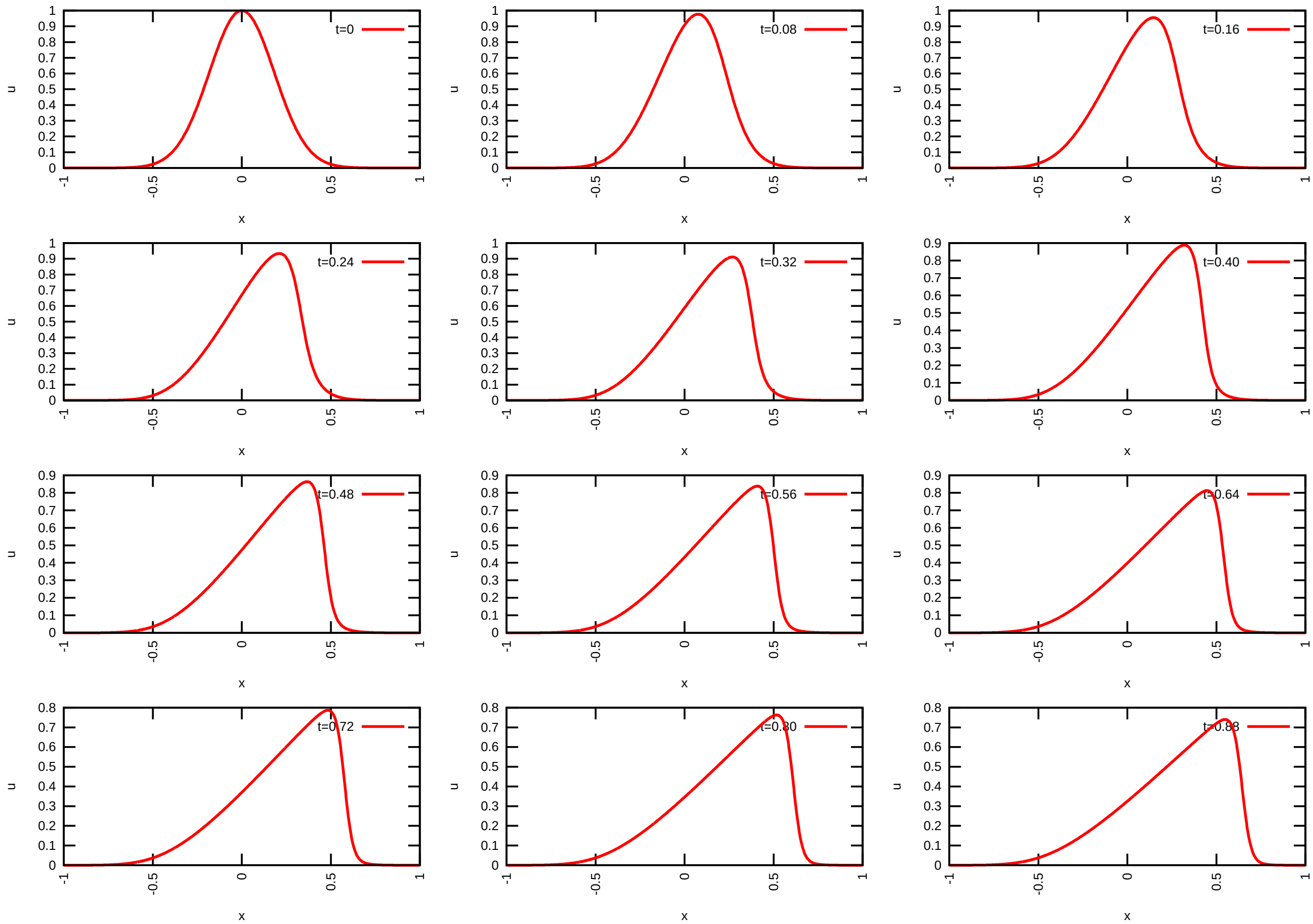


Figure 3: Lax Fredriech method

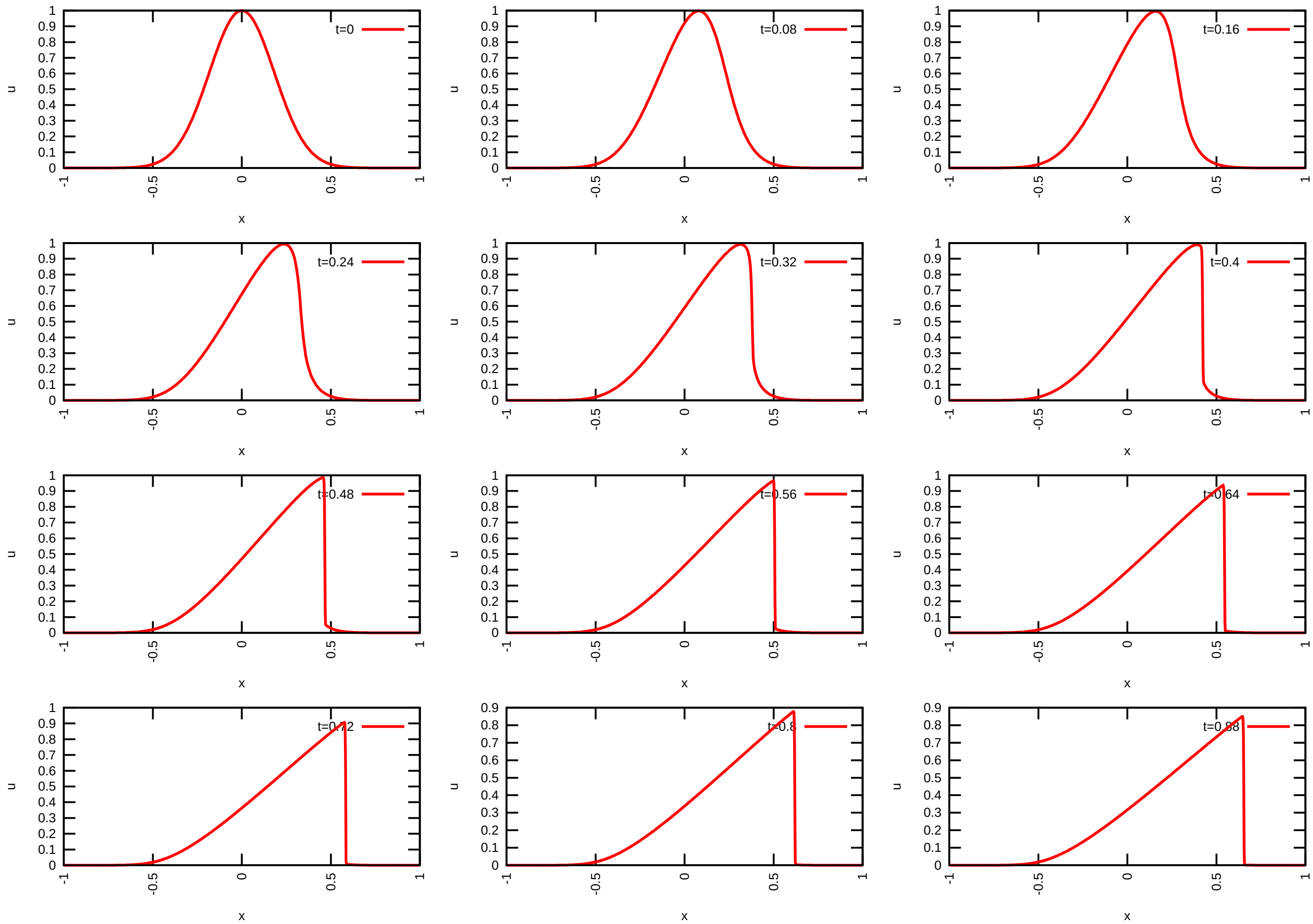


Figure 4: Godunov method

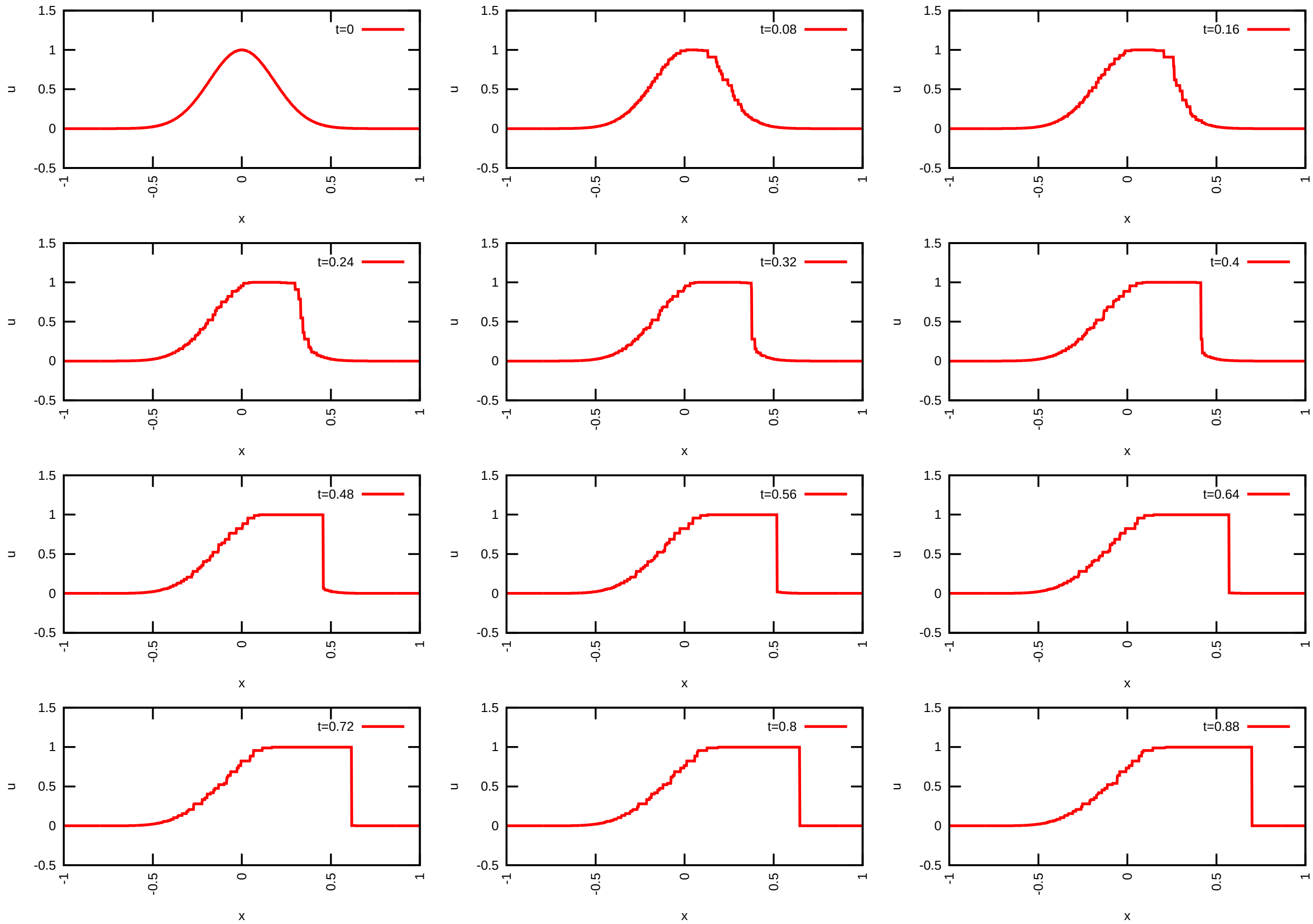


Figure 5: Glimms method

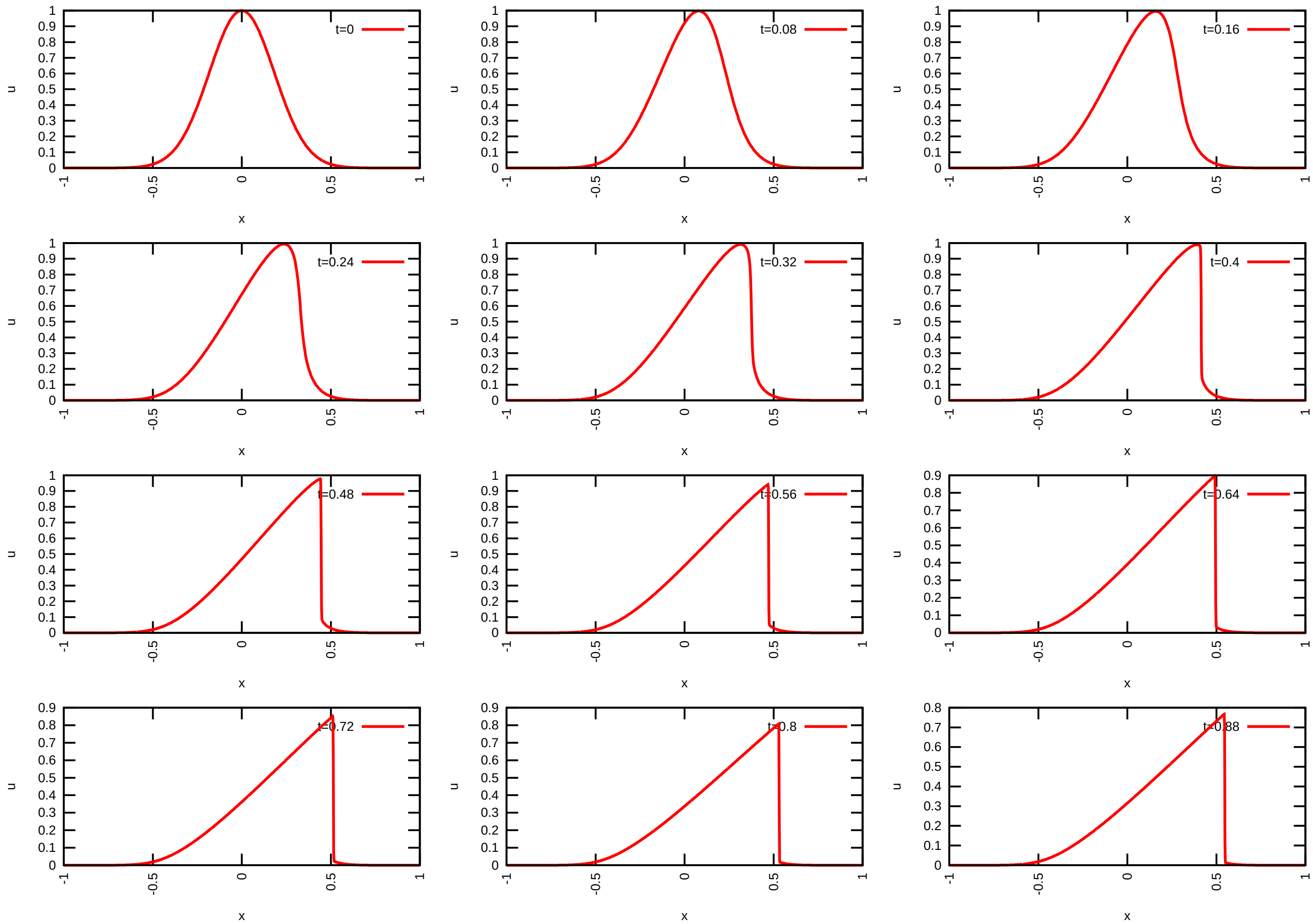


Figure 6: Upwind method

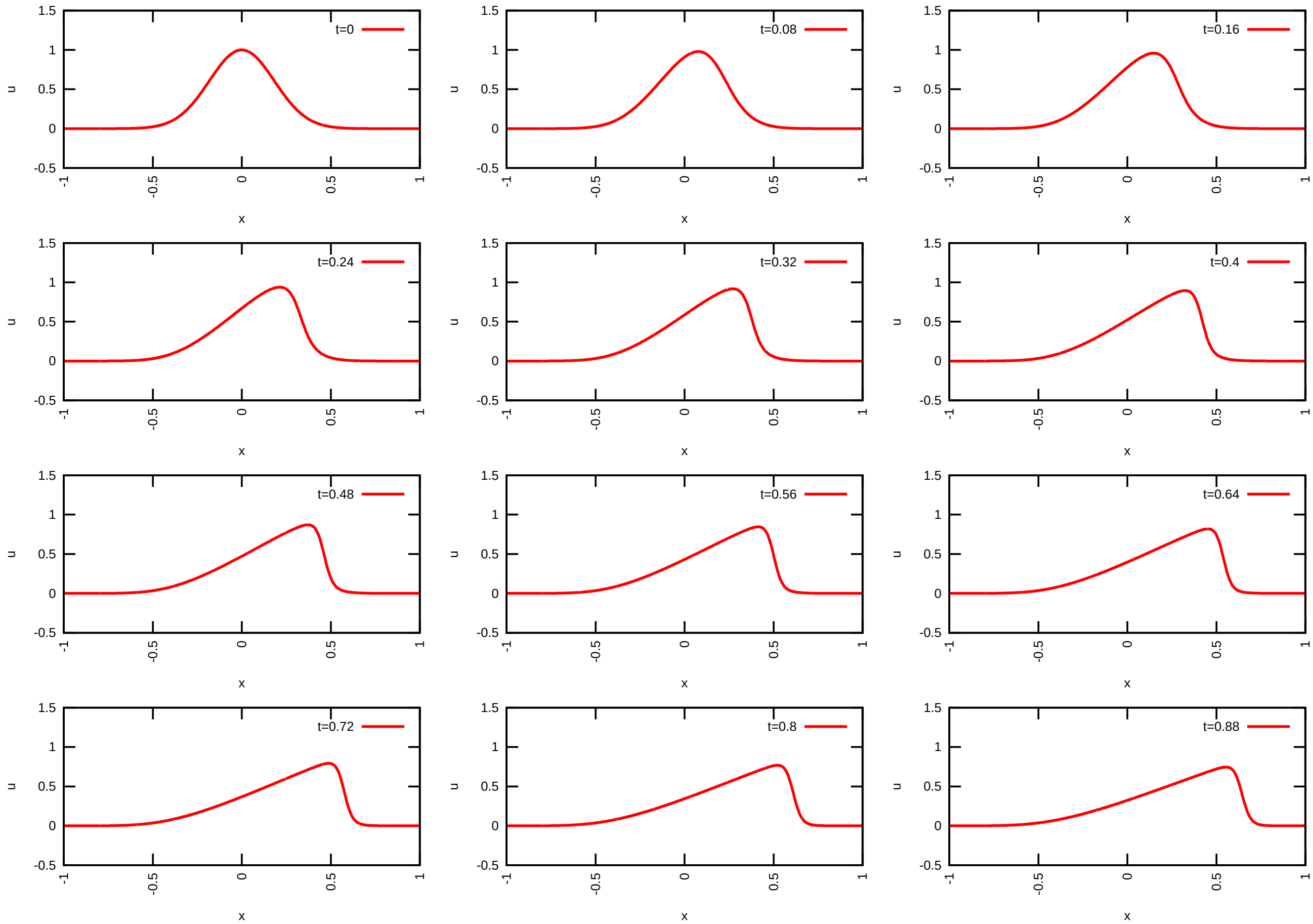


Figure 7: Upwind with viscosity $\epsilon = 0.01$

We also set our initial condition to

$$u = 1 \text{ for } x < 0,\, u = 0 \text{ for } x >= 0$$

The initial conditon looks like 8 .

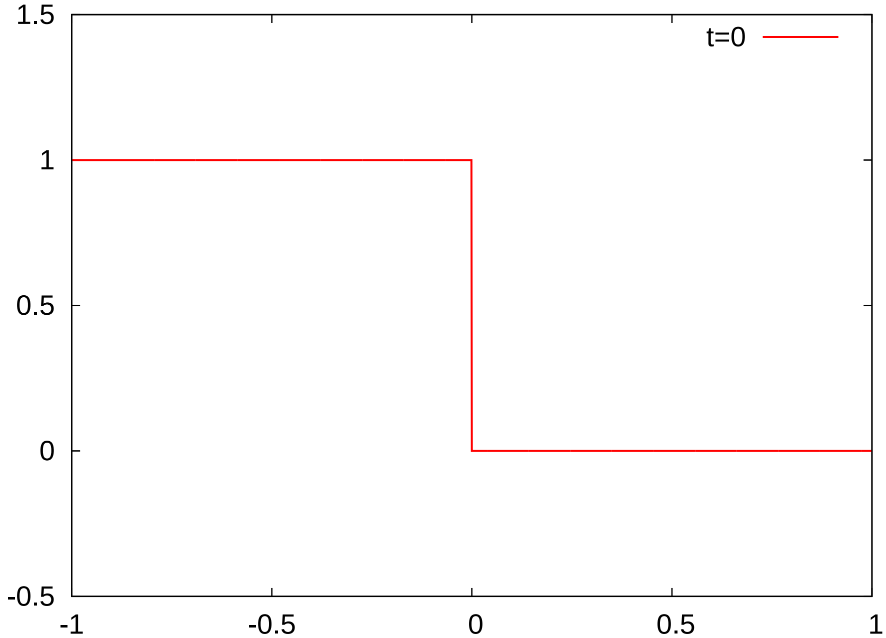


Figure 8: Initial step condition

The results for different method is as in Figures 9 to 14. For error estimation, we find the distance from the exact solution at the final time step. The exact solution in this case is:

$$u_{exact} = \begin{cases} 1.0 & x \leq 0.5 \\ 0.0 & x > 0.5 \end{cases} \tag{15}$$

The error for different schemes is as in following table

Method	Glimm	Godunov	Lax Fredriech	upwind	viscos upwind
Error	0.02098	0.00063	0.013828	0.00063	0.01312

Table 1: Error in different methods, for the shock case

Error is founded using infinity norm. The error is:

$$\text{Error} = \frac{\sum_{i=1}^{N} |u_i - u_i^{exact}|}{N}$$

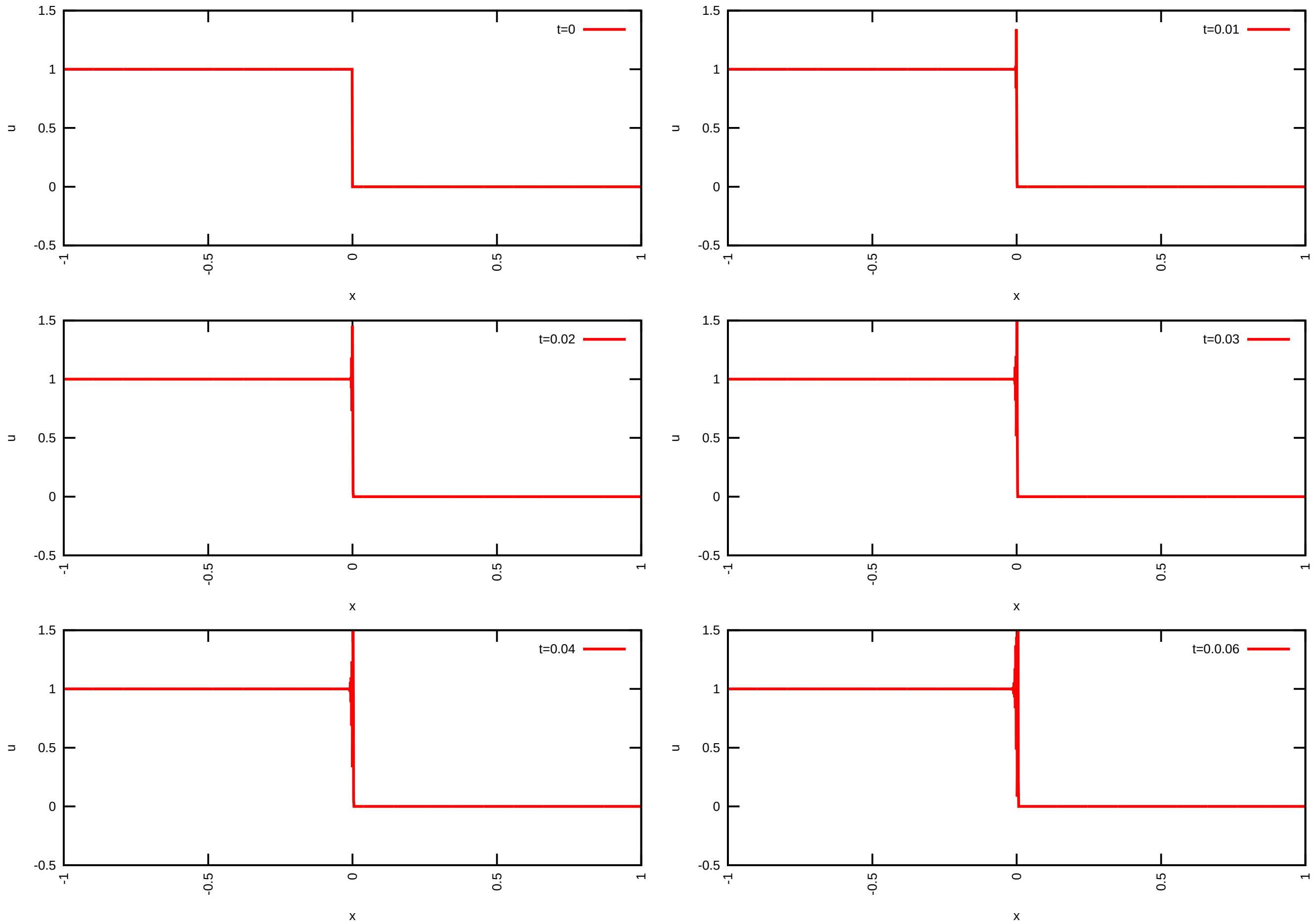


Figure 9: Lax Wendroff method

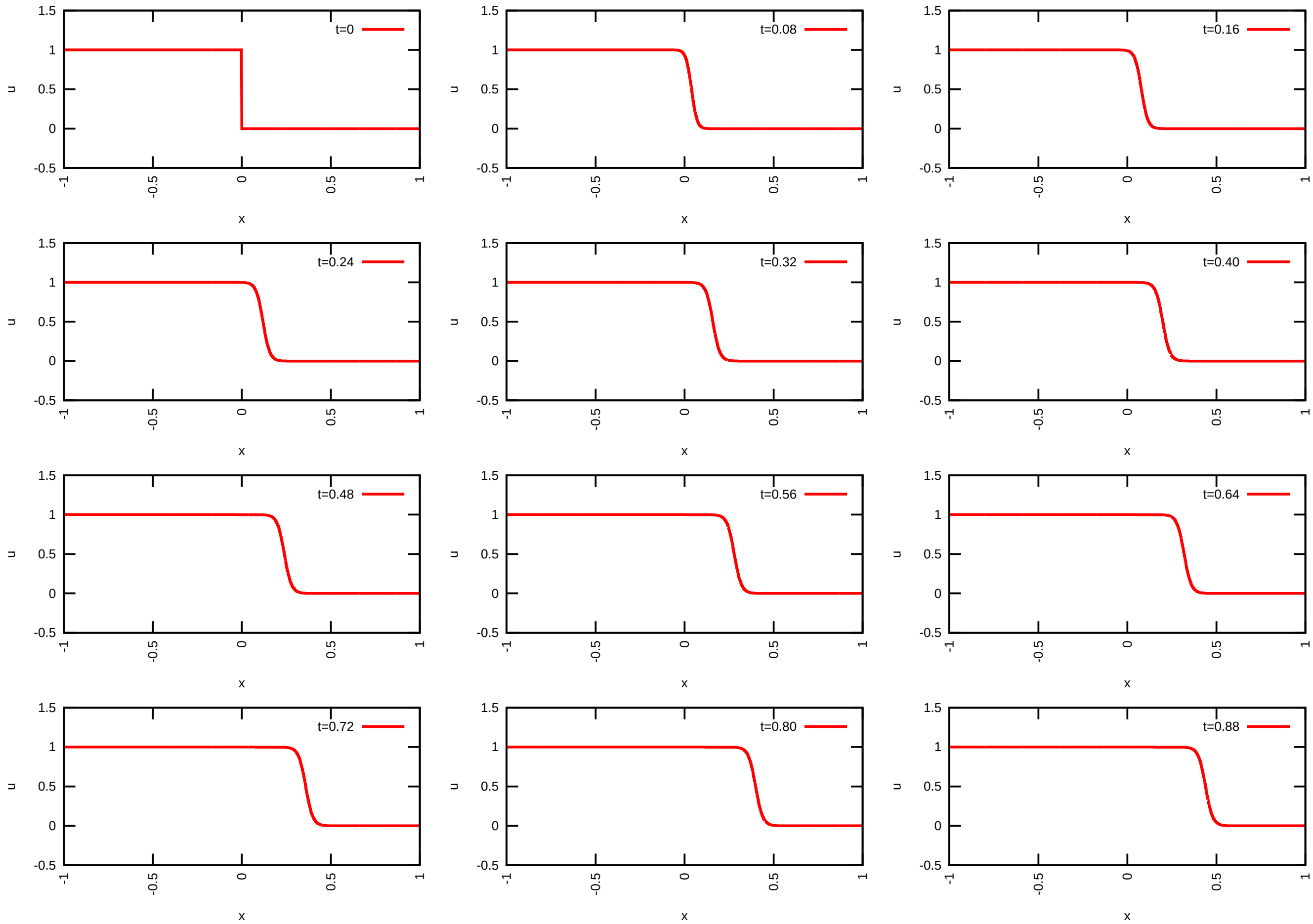


Figure 10: Lax Fredriech method

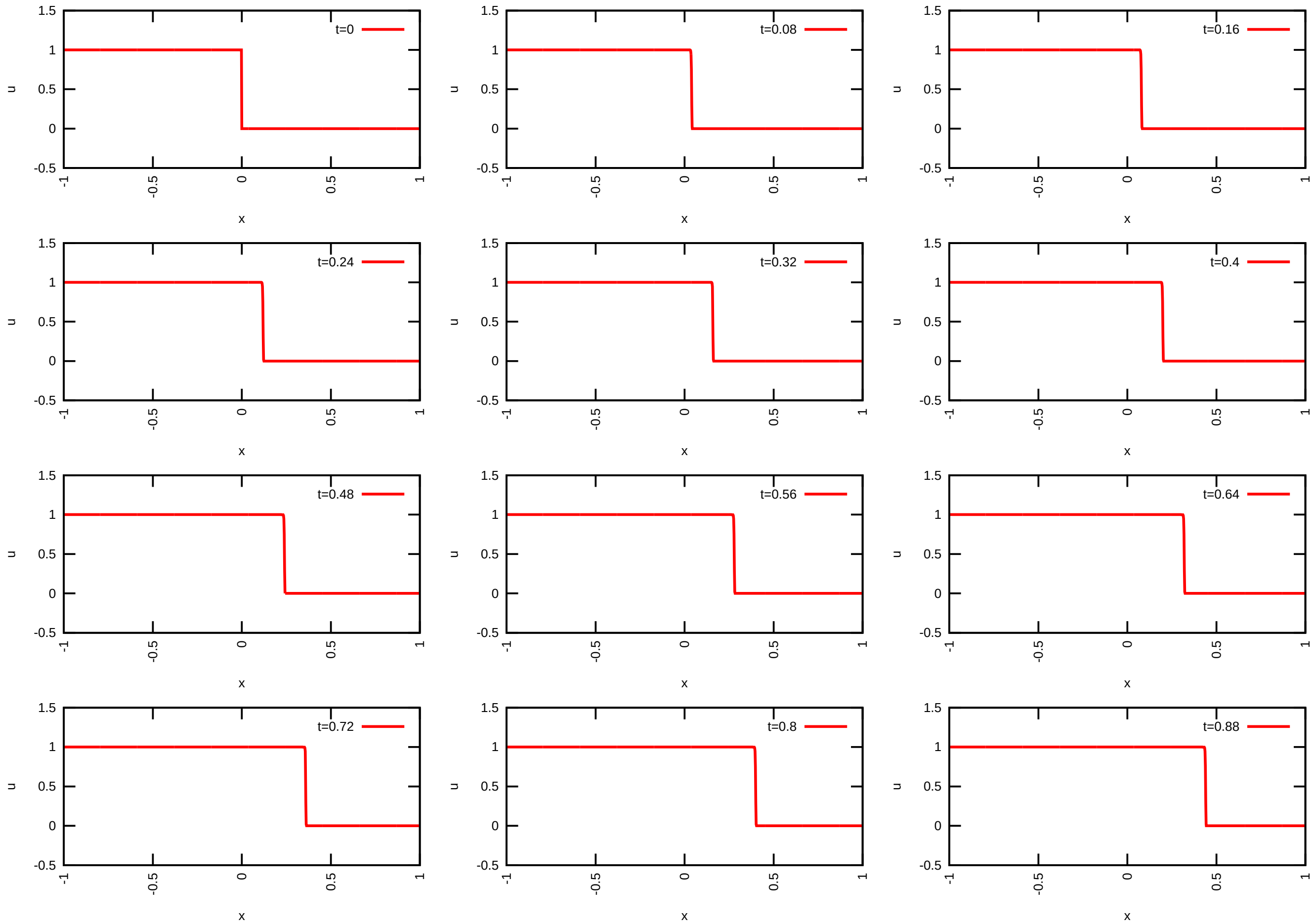


Figure 11: Godunov method

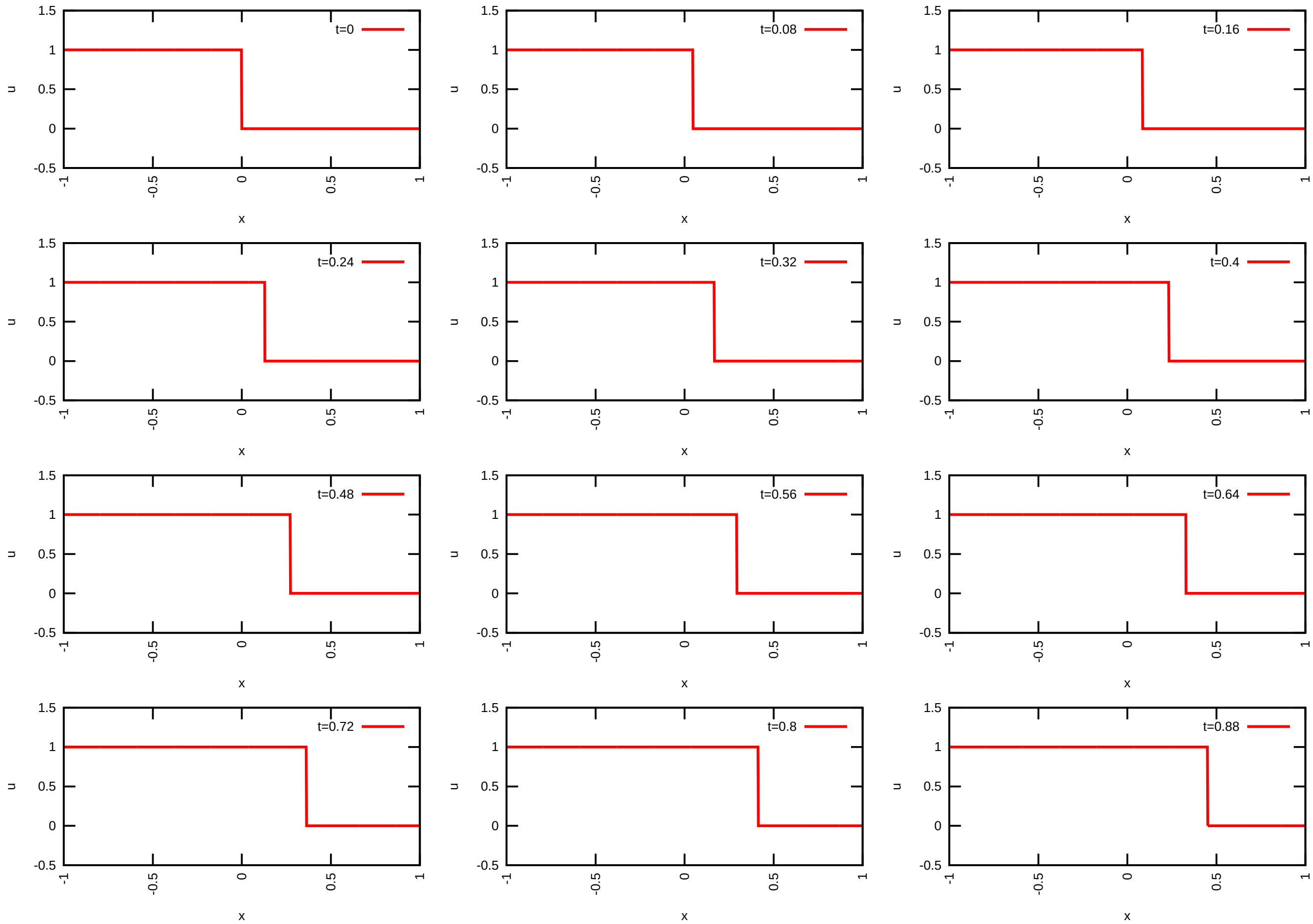


Figure 12: Glimms method

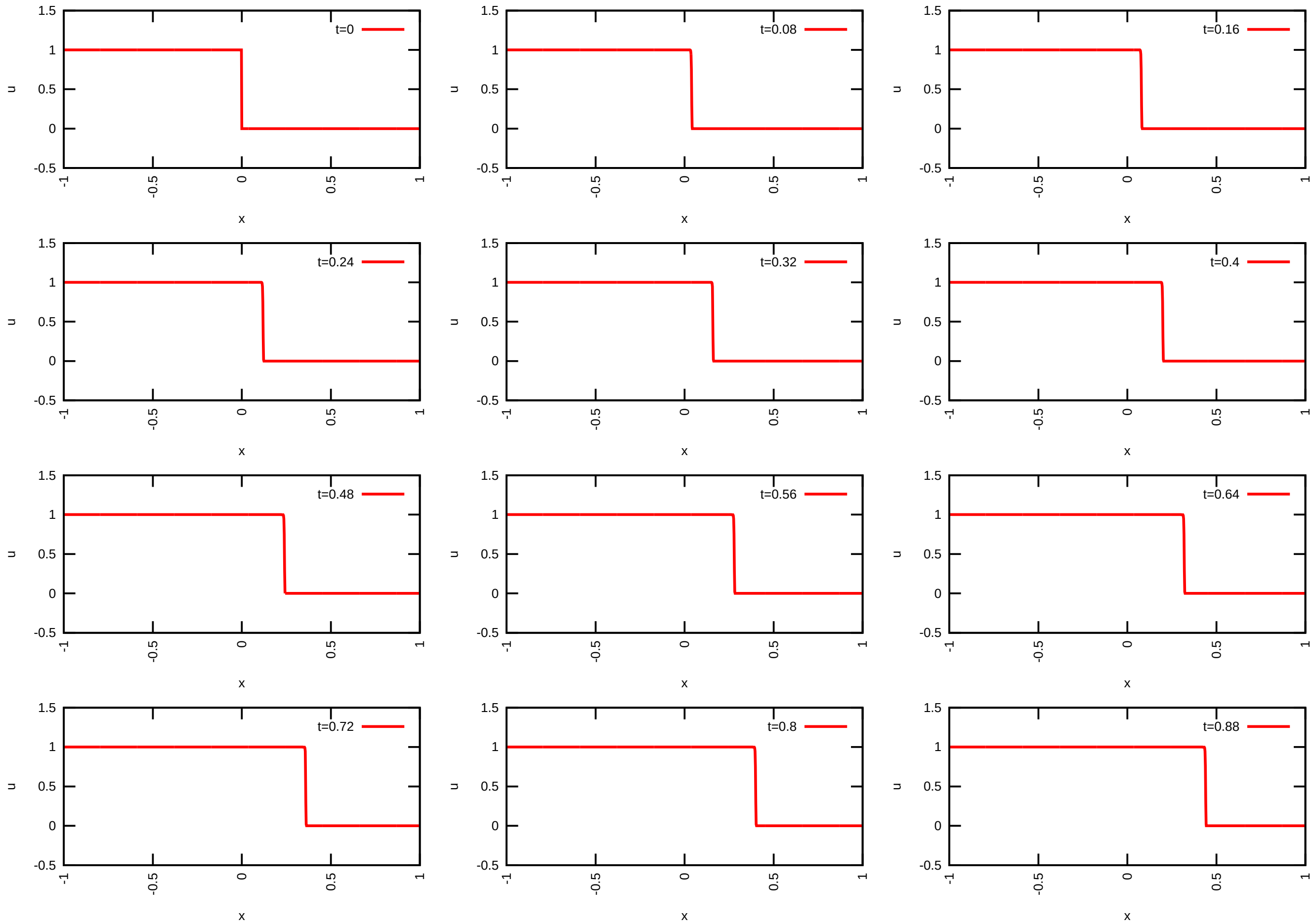


Figure 13: Upwind method

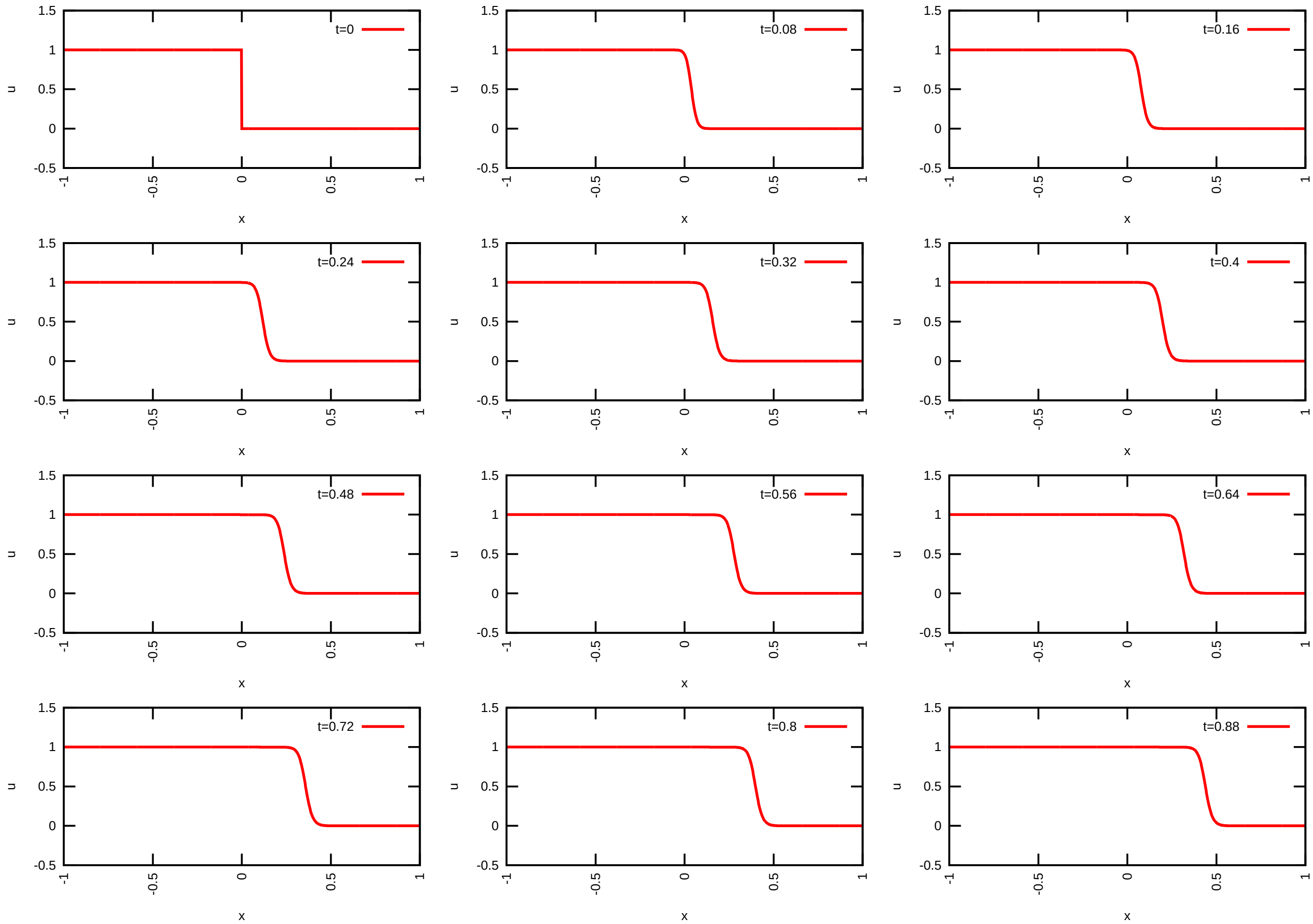


Figure 14: Upwind with viscosity $\epsilon = 0.01$

We also set our initial condition to

$$u = 0 \text{ for } x < 0,\ u = 1 \text{ for } x >= 0$$

The initial conditon looks like **??**. The shape of the wave for each method could be find in the following figures. The exact solution in this case is:

$$u_{exact} = \begin{cases} 1.0 & x/t > 1 \\ 0.0 & x/t < 0 \\ x/t & else \end{cases} \tag{16}$$

The error for different schemes is as in following table

Error is founded using infinity norm. The error is:

$$\text{Error} = \frac{\sum_{i=1}^{N} |u_i - u_i^{exact}|}{N}$$

Method	Glimm	Godunov	Lax Fredriech	upwind	viscos upwind
Error	0.19724	0.00249	0.02467	0.00249	0.00241

Table 2: Error in different methods, for the shock case

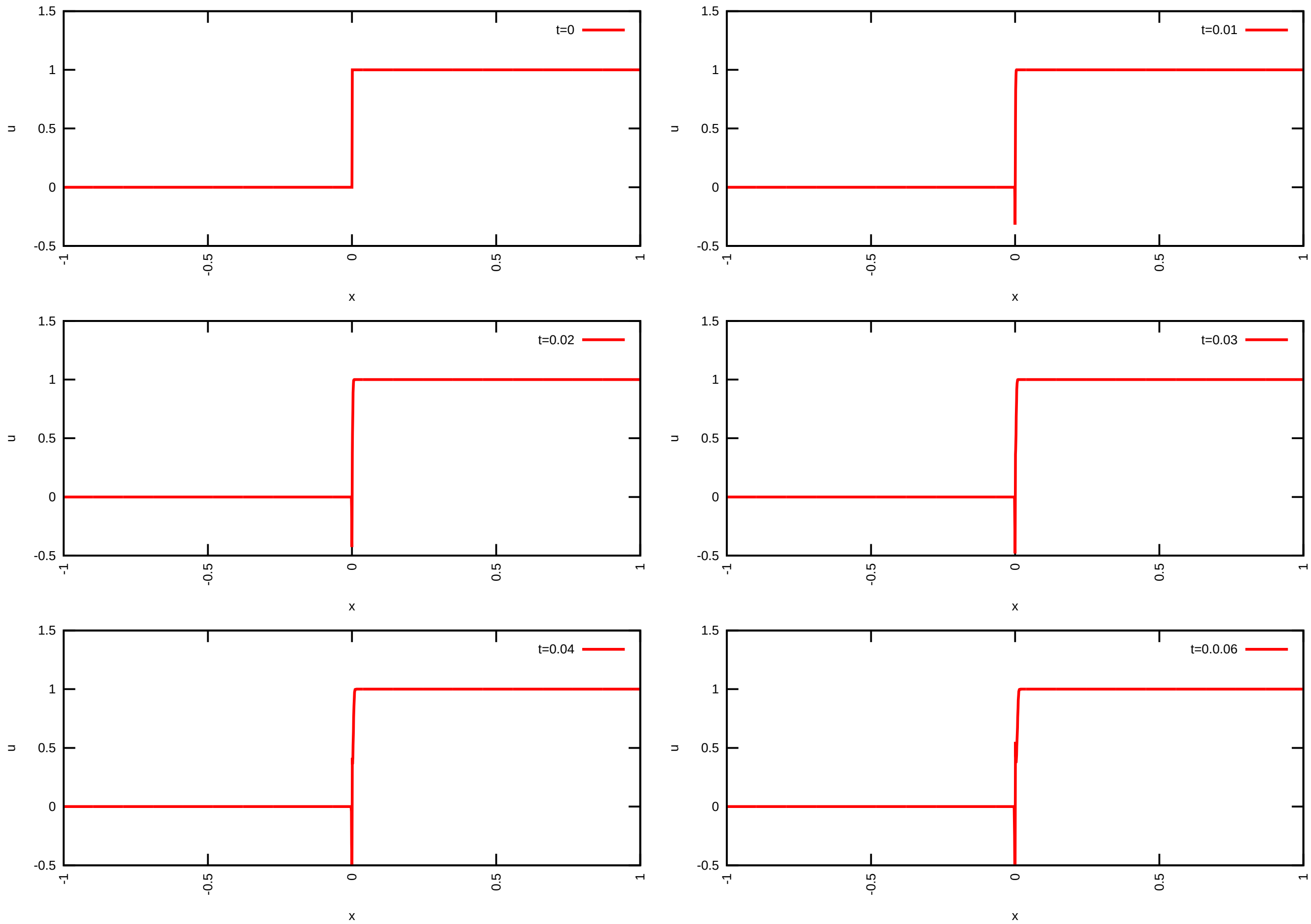


Figure 15: Lax Wendroff method

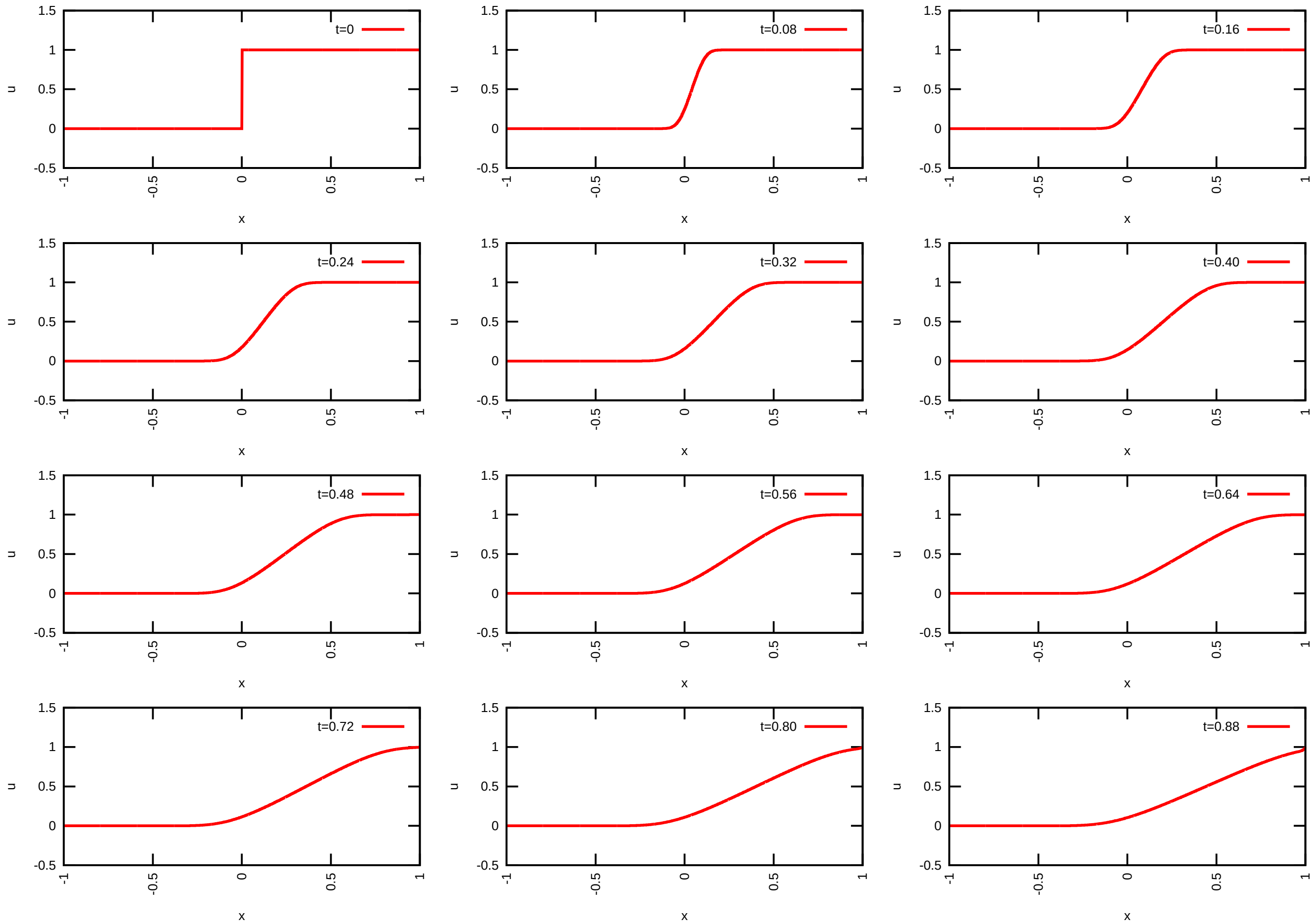


Figure 16: Lax Fredriech method

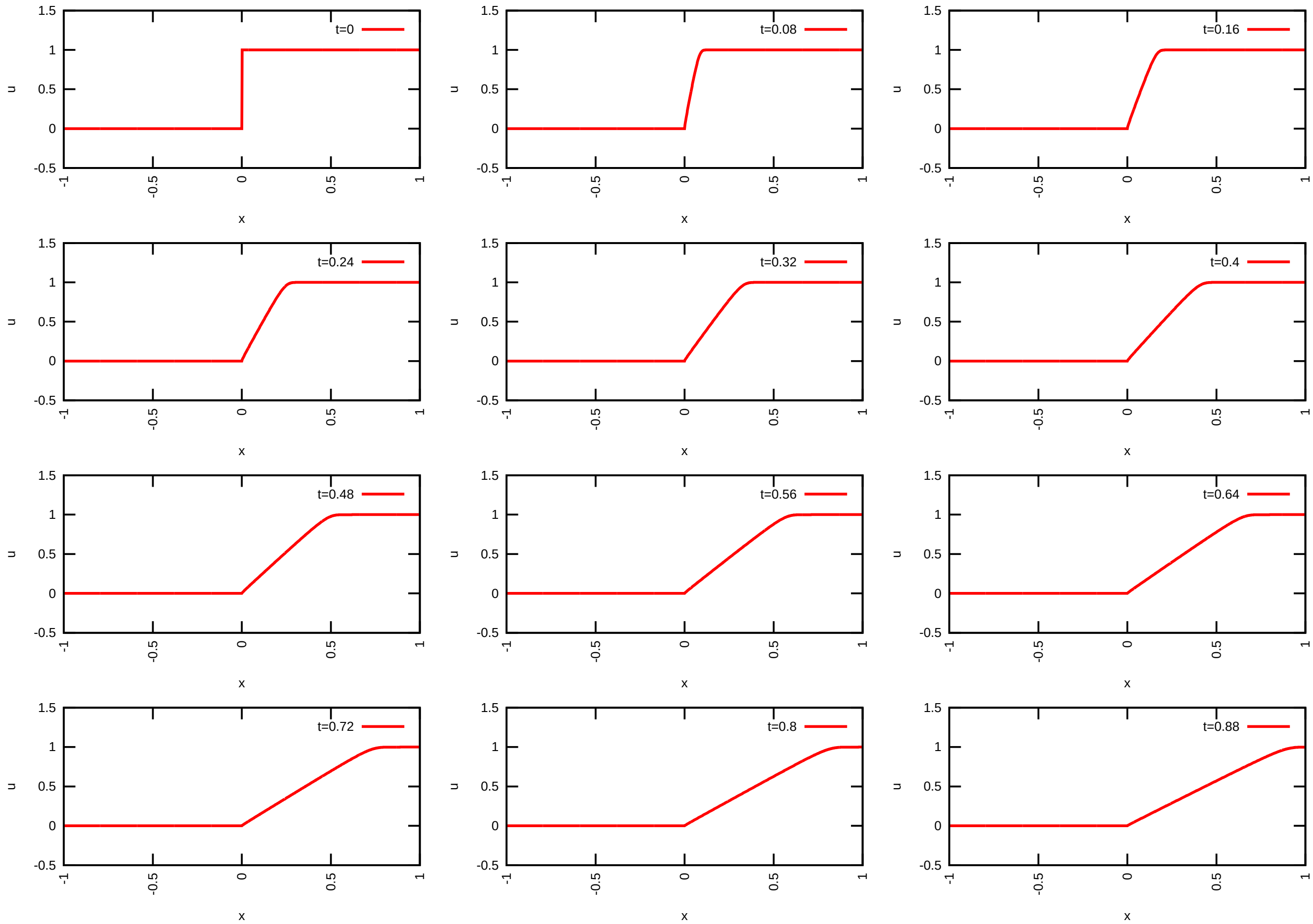


Figure 17: Godunov method

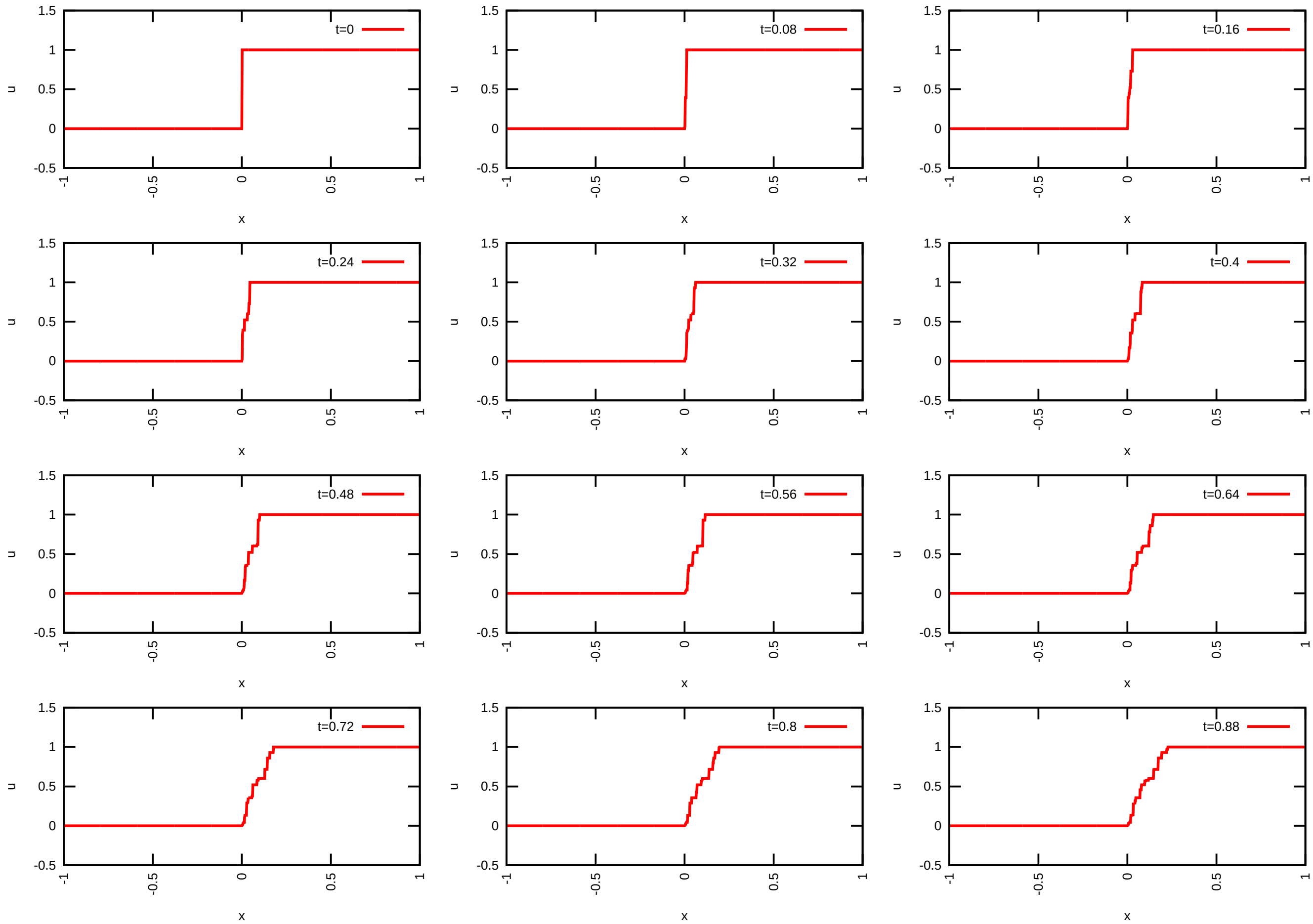


Figure 18: Glimms method

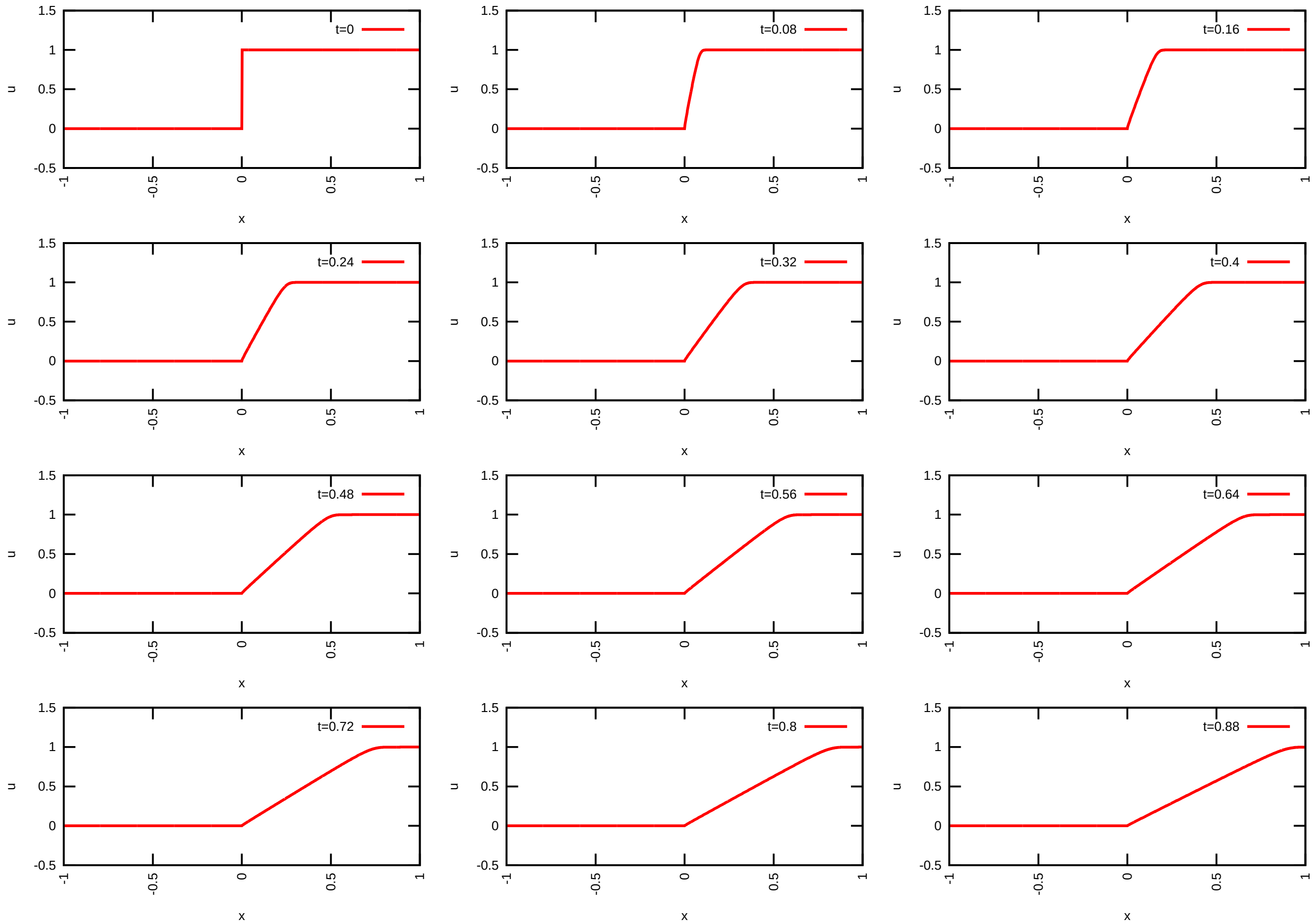


Figure 19: Upwind method

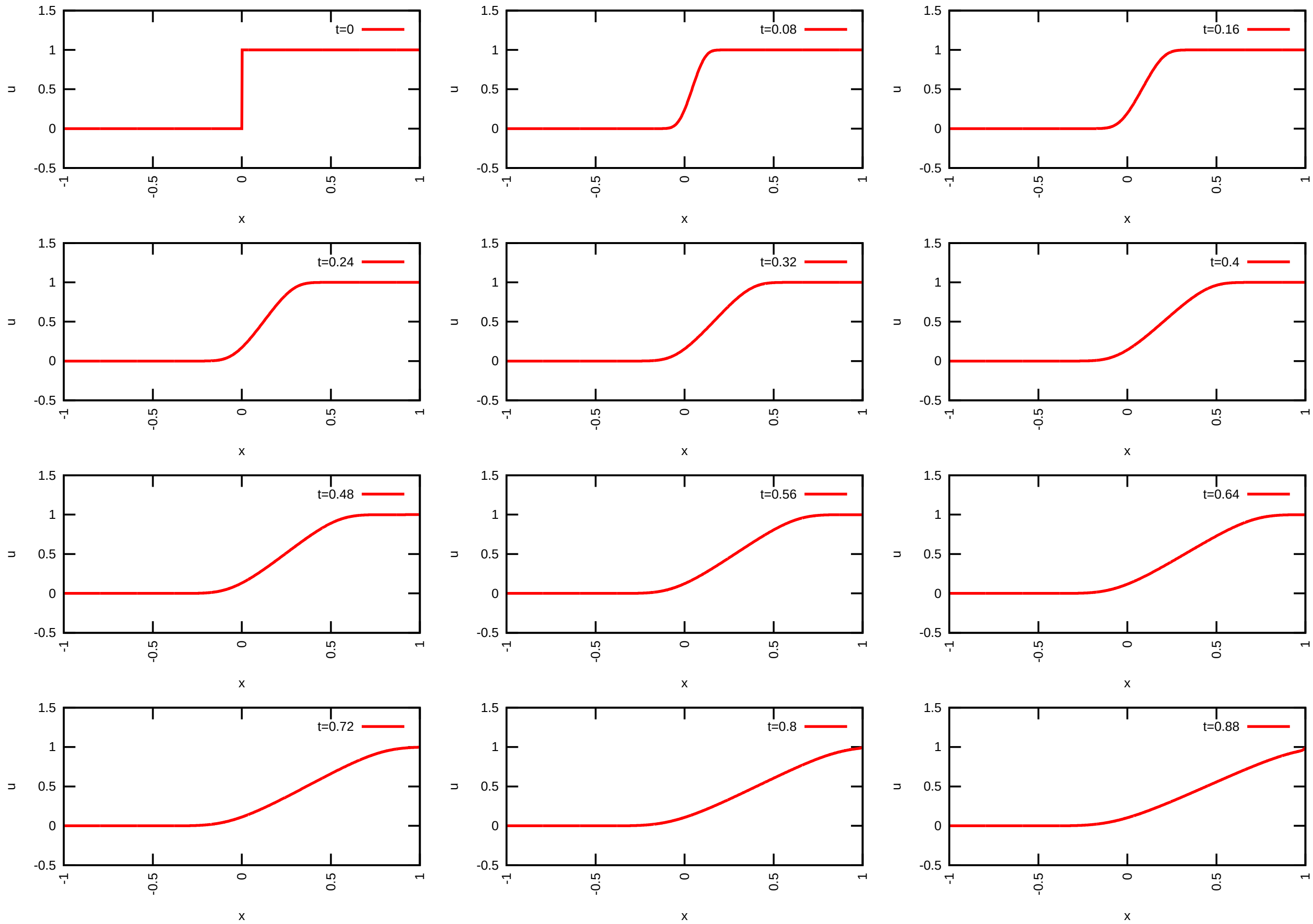


Figure 20: Upwind with viscosity $\epsilon = 0.01$

Conclusion

In this problem set we saw that how different methods can be used to solve Burgers' equation. Burgers' equation could cause shocks or fans in the system. We will see how different schemes respond to these shocks and fans. In this paper, infinity norm is used to compare the results with the exact solution. We saw how exact these schemes give the solution of burgers' equation. The worst scheme is Lax-Wendroff. This scheme could not handle shocks and fans. It will have an overshoot at place of shocks or fans. Godunov method, is the best scheme in solving burgurs' equation. However, in our case, the solution of Reimann problem was easy. In general cases, the solution of Reimann problem would need some Newton-Rophson schemes, persay, to find the solution of the inverse function. So this method gets harder to implement. Glimm's method is another good method to solve burgur's equation. In the case of Glimms method, the solution is not smooth. It will have steps, however with finer mesh one would get nicer results. Upwind method, is an easy and accurate method. This method is super easy to code and the results has a great accordance with exact solution. Upwind with viscosity is also a good scheme, however it can not capture the shocks. It will smoothes the sharpness of the shocks. Lax-Fredriech is way better than Lax-Wendroff, however it still cannot compete with upwind, Godunov or Glimm methods in accuracy. All in all, in case of accuracy Godunov scheme is the best, in case of efficiency and accuracy both at the same time, upwind method is the best. Not to mention, the worst scheme was Lax-Wendroff.