Math 228B
Homework 3
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Introduction

In this problem set, we want to solve 3-D heat equation which is written as

$$u_t = \alpha \nabla^2 u$$

This equation describes heat (or temperature variation) in a region (S) over time. α is a positive constant and is called thermal diffusivity of the medium. Actually this constant physically imply that how fast the temperature is moving in the region.For solving heat equation we need boundary conditions on ∂S and one initial condition in S.

This equation is important since volumetric concentration, heat diffusion or even wave function of a single particle are all described with this equation. In this problem set, we will solve a 3-D model of this equation.

Problem Definition

Heat equation or diffusion equation in 3-D is as

$$u_t = \alpha \nabla^2 u$$

$$u_t = \alpha \nabla^2 u = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) u \tag{1}$$

In this problem set we will solve a this equation in 3-D using forward Euler for time and centeral difference for space. Question in here is solving this equation using finite difference method and getting the results.

For the boundary condition we have that normal derivative of u along the normal vector of the boundary surface is zero i.e.

$$\frac{\partial u}{\partial n} = 0 \tag{2}$$

This is called Neumann boundary condition (since the derivative of u is known on the boudary).

Implementation

Equation in here we are solving is

$$u_t = \alpha \nabla^2 u = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) u \tag{3}$$

Which is 3-D diffusion equation. We are using forward Euler for time and centeral difference for spatial coordinates.

$$D_+^t u = D_+^x D_-^x u + D_+^y D_-^y u + D_+^z D_-^z u \tag{4}$$
$$\tag{5}$$

So we will get

$$u_{i,j,k}^{n+1} = (1 - 6\lambda)\, u_j^n + \lambda \left(u_{i,j+1,k}^n + u_{i,j-1,k}^n + u_{i+1,j,k}^n + u_{i-1,j,k}^n + u_{i,j,k+1}^n + u_{i,j,k-1}^n \right) \tag{6}$$

Also for the boundary condition, after we solve for the whole inside domain, we just set the values on the boundary to it's nearest normal to surfacee neighour. This way we make sure that the value of normal derivative of u is zero on the boundary.

Problem

1. Downstairs, on the ground floor of Evans Hall, an immense bag of potato chips explodes in room 9. Model the diffusion of "potato chip smell", and compute when the smell in room 3 is 1% of the initial smell. Make whatever assumptions you want about the layout of the ground floor, open doors, windows, etc.

   The simplified sketch of the rooms and the hall that we used for numerical solution is as follows.

   Rooms are $10m$ by $7m$ by $3m$. Both rooms are the same size. The hall is size $3m$ by $26m$ by $3m$.

   We used 3-D finite difference method as

$$D_+^t u = D_+^x D_-^x u + D_+^y D_-^y u + D_+^z D_-^z u \tag{7}$$
$$\tag{8}$$

   Writing this method by as what we used in the code, we have

$$u_{i,j,k}^{n+1} = (1 - 6\lambda)\, u_j^n + \lambda \left(u_{i,j+1,k}^n + u_{i,j-1,k}^n + u_{i+1,j,k}^n + u_{i-1,j,k}^n + u_{i,j,k+1}^n + u_{i,j,k-1}^n \right) \tag{9}$$

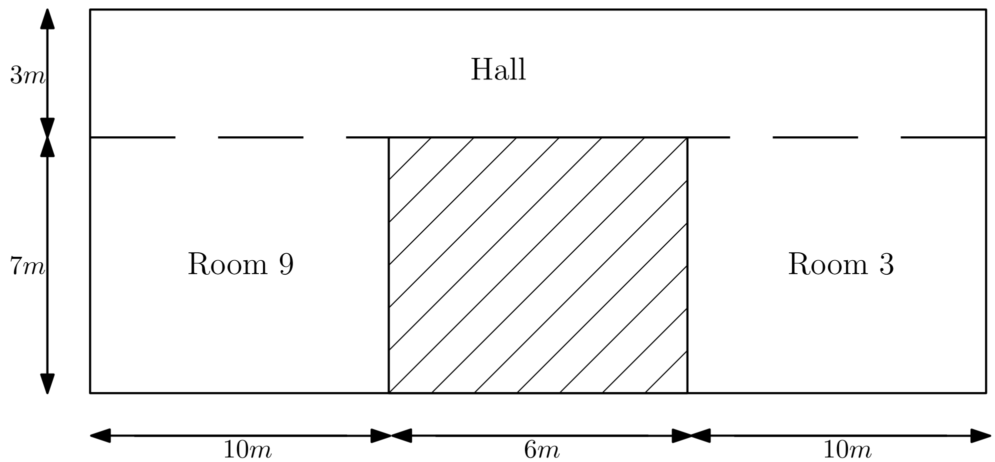

Figure 1: Plan of the rooms, extrude 3 meters for z direction

where $\lambda := h/k^2$. h is difference in time and k is difference between elements in space. We have used same spacing for different coordinates. We have seperately meshed the rooms and the hall.

For the boundary conditions of the wall we have, that $\frac{\partial u}{\partial n} = 0$ where n is normal vector. It is the same as what we did in homework 2 problem 3. After solving inside of the rooms, we set values of u on the wall to it's neighbour node inside so making normal derivative of u equal to zero. Values of nodes on the opened door is computed simoltaneously with other room or hall that is connected to the region, where we are computing our values.

For the stability condition we have done Von-Neumann analysis. Our scheme is

$$\begin{aligned} \frac{u_{i,j,k}^{n+1} - u_{i,j,k}^{n}}{\Delta t} = \alpha \Bigg(& \frac{u_{i+1,j,k}^{n} - 2u_{i,j,k}^{n} + u_{i-1,j,k}^{n}}{\Delta x^2} \\ & + \frac{u_{i,j+1,k}^{n} - 2u_{i,j,k}^{n} + u_{i,j-1,k}^{n}}{\Delta y^2} \\ & + \frac{u_{i,j,k+1}^{n} - 2u_{i,j,k}^{n} + u_{i,j,k-1}^{n}}{\Delta z^2} \Bigg) \end{aligned} \tag{10}$$

Von-Neumann stability analysis for finding amplification factor, considering fourier transform on x,y and z we will get

$$u_{i,j,k}^{n}(x,y,z) = \frac{1}{2\pi} \int \hat{u}_{i,j,k}^{n}(l,m,n)\, e^{ilx} e^{imy} e^{inz} dl\ dm\ dn \tag{11}$$

if we put this equation into our scheme and suppose that $\Delta x = \Delta y = \Delta z = h$ and $\Delta t = k$, we will have

$$\frac{1}{2\pi}\int \hat{u}^{n+1}_{i,j,k} e^{i(lx+my+nz)} dl\ dm\ dn =$$
$$\left(\frac{\alpha k}{h^2}\right)\frac{1}{2\pi}\int \hat{u}^{n}_{i,j,k} e^{i(lx+my+nz)}\left(e^{ilh}+e^{-ilh}+e^{imh}+e^{-imh}+e^{inh}+e^{-inh}-8\right) \quad (12)$$

Since $e^{ilh}+e^{-ilh}-2=-2\sin(lh/2)$, It can be easily seen that

$$\hat{u}^{n+1}=\hat{u}^{n}\left(1-\frac{4\alpha k}{h^2}\left(\sin^2\frac{lh}{2}+\sin^2\frac{mh}{2}+\sin^2\frac{nh}{2}\right)\right) \quad (13)$$

So the amplification factor $\hat{G}$ becomes

$$\hat{G}(l,m,n)=1-\frac{4\alpha k}{h^2}\left(\sin^2\frac{lh}{2}+\sin^2\frac{mh}{2}+\sin^2\frac{nh}{2}\right)$$

In the above equation we define $\lambda := \frac{k}{h^2}$. For stability, we demand that $|\hat{G}|\le 1$, so this would mean that

$$-1\le 1-4\alpha\lambda(1+1+1)\le 1$$

This would easily imply that

$$\boxed{\alpha\lambda\le\frac{1}{6}}$$

So our method, is stable if $\lambda\le\frac{1}{6\alpha}$. Here $\alpha=1$, So we just make sure that λ to be less equal to 1/6. For different values of h we have computed the solution to see if our method is convergence.

I should also mention that instead of value of 1 for u as the source, I make this value 100 and tried to see when the value in the other room at the center of the room gets to be 1. So it's the same as making source value 1 and see when the other room gets the value of 0.01.

For the results we have computed concentrecity of the middle point of the other room, and plotted it's concentricity vs time. The results is as follows

$h(m)$	0.5	0.25	0.2	0.16
Time(sec)	162.88	196.01	212.5	223.6

Table 1: Time elapsed vs space size h for $\lambda=1/6$ for $\alpha=1m^2/s$

We have also plotted the vlaue of u along the center of the hall along the x axis which goes from one room to another. The value of u looks as what follows

In this figure one can easily see that the normal derivative of u at the boundaries are zero.

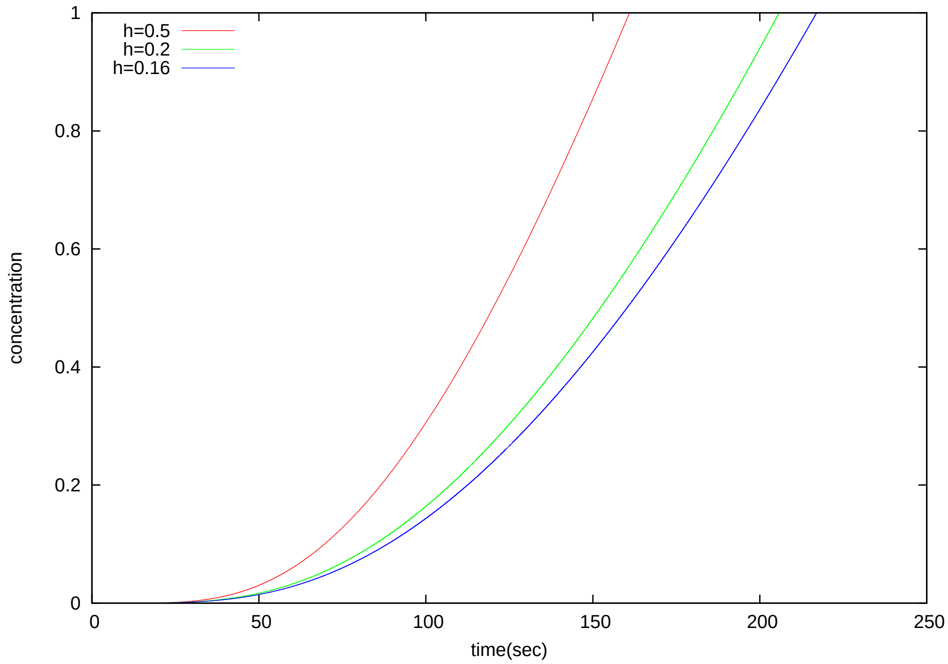

Figure 2: Concentration over time for different h, for $\lambda = 1/6$

Value of u in both rooms along the x-axis (the axis along the hall way) is shown in following figures.

As it can be seen from the figure 4 the value at the center of this room is always one, as the source is right there.

As it is shown in figure 5, values of u at this center line is different along the axis and it is higher near the first door. This is an obvious result, since we are not in an equilibrium, the values are getting higher, and since this door in more near to where the source is (values of u is higher near this door) so the values along the center of the plane x-axis is higher.

Values of u are shown in figures 6, 7 and 8. As it can be seen in these figures, both values match together, and the difference can not be seen in these figures.

Conclusion

In this problem set we solved 3-D heat equation with an explicit method, using forward Euler for time and centeral difference for second derivative of spatial coordinates. We saw that stability condition changes and λ gets smaller values for our method to be stable. We also dealt with normal derivative boundary condition for boundary surfaces. We also saw that it will take a pretty long time for the smell to diffuse to the other room (around 200 sec) if $\alpha = 1$.

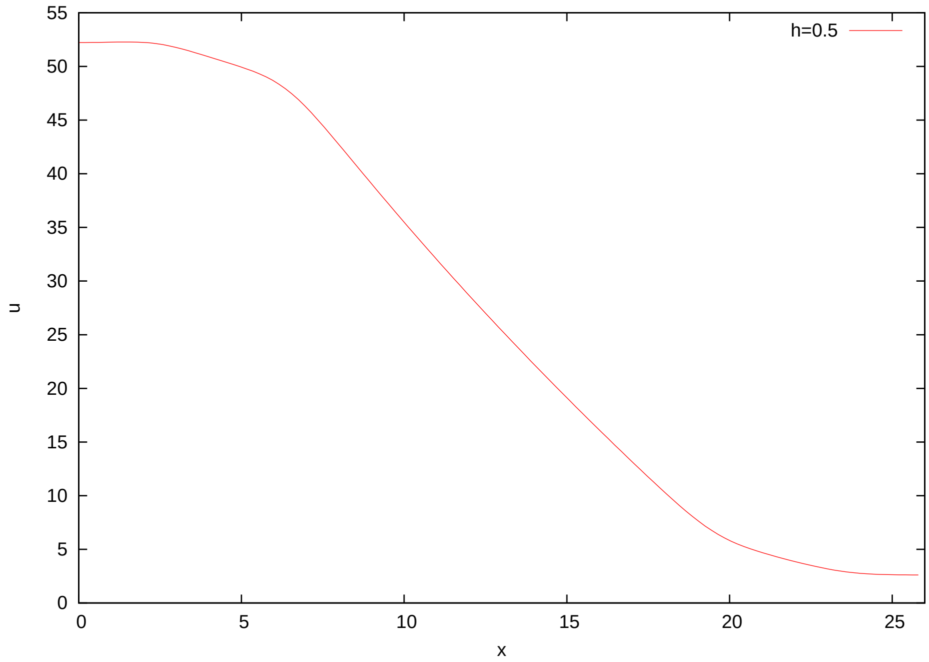

Figure 3: value of u along x axis, for line at the center of $y - z$ plane at the final time, $\lambda = 1/6$

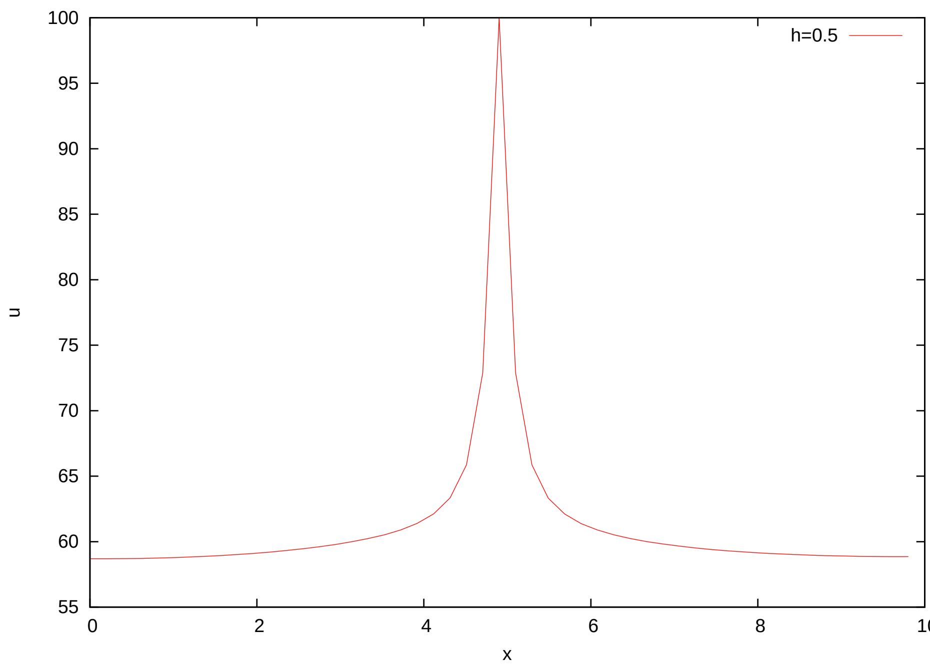

Figure 4: value of u along x-axis at the center of $y - z$ plane for source room,$\lambda = 1/6$

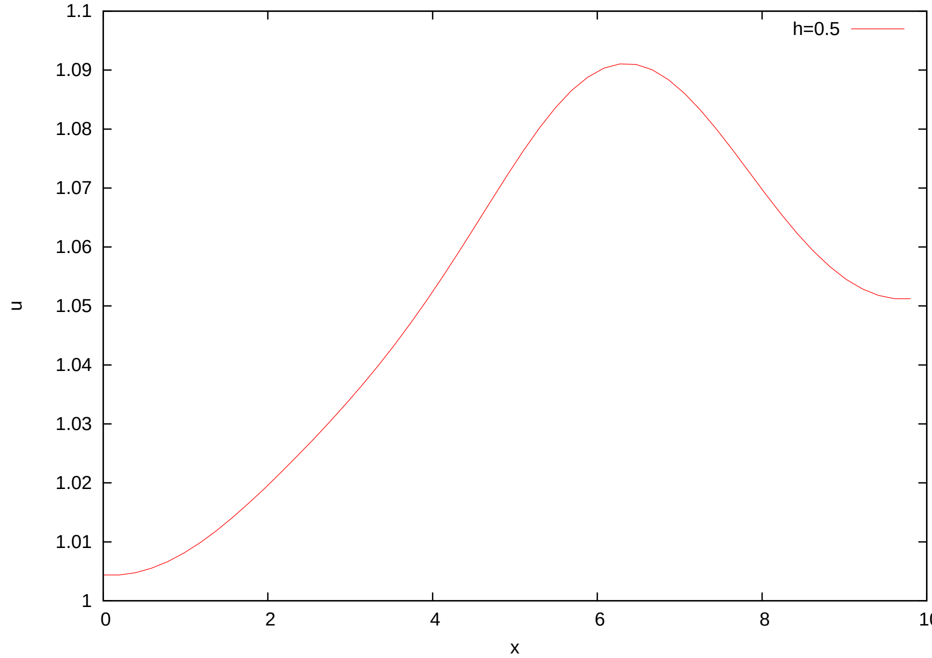

Figure 5: value of u along x axis, for line at the center of $y-z$ plane at the final time for the second room,$\lambda = 1/6$

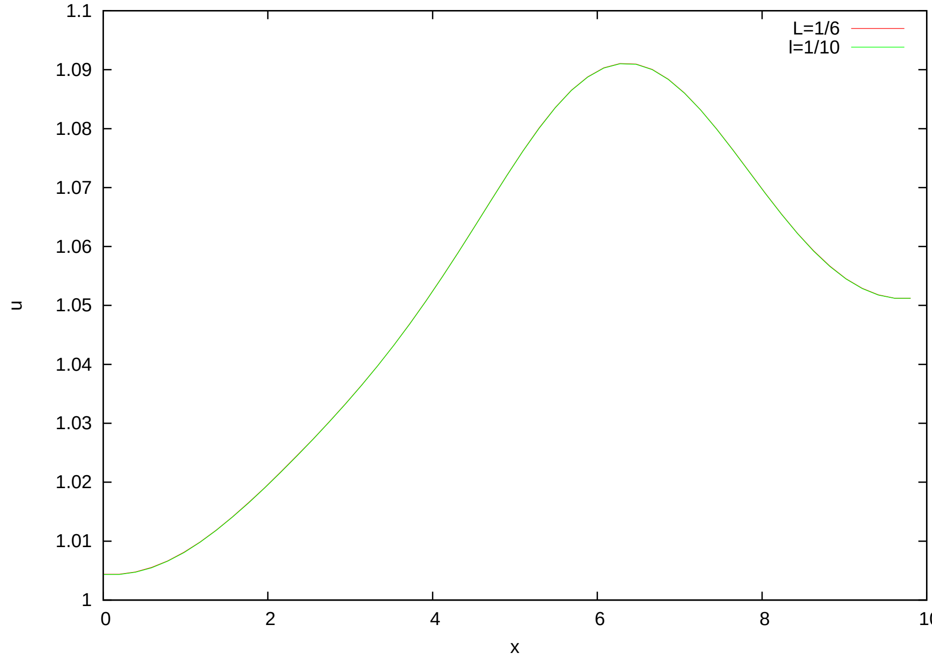

Figure 6: value of u along x axis, at the center of $y-z$ plane for the second room for different lamda

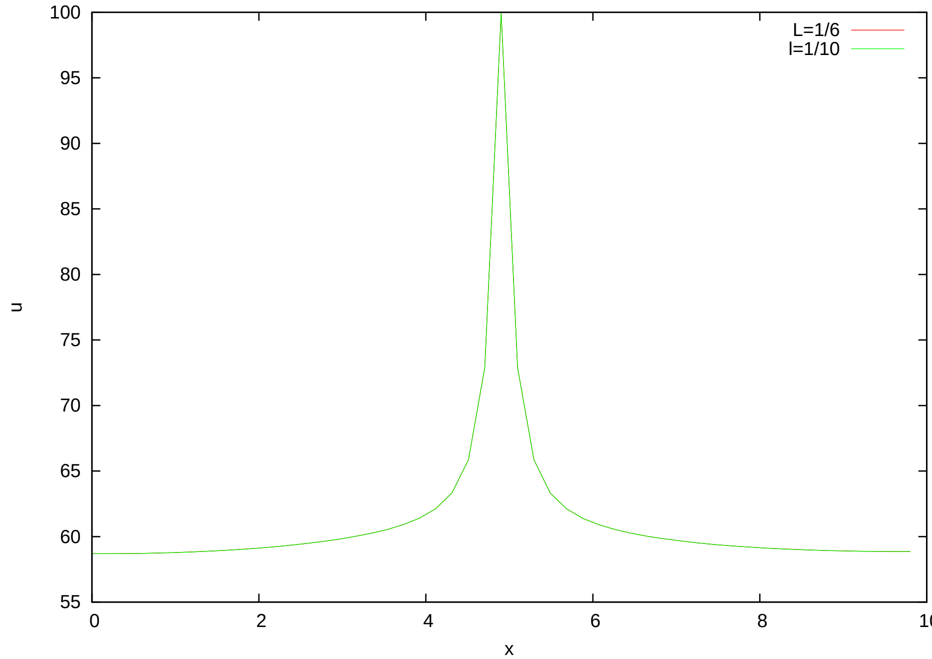

Figure 7: value of u along x axis, for line at the center of $y - z$ plane at the final time for the source room for different lamda

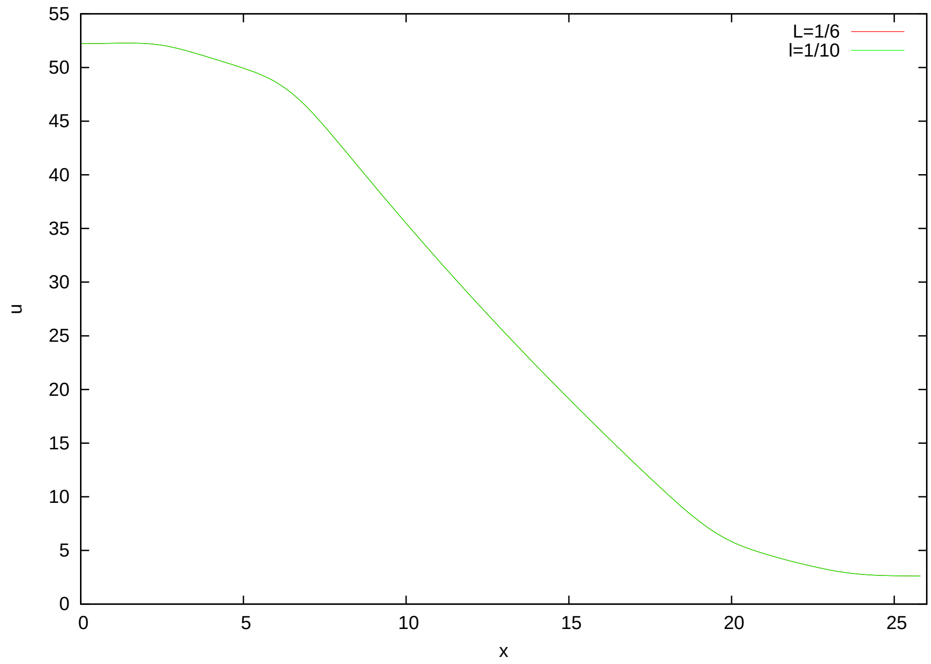

Figure 8: value of u along x axis, for line at the center of $y - z$ plane at the final time for the hall for different lamda