Math 228B
Homework 4
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Introduction

In this problem set, we want to solve wave equation which is written as

$$u_{tt} = \alpha(x)u_{xx}$$

using Lax-Wendroff and Leap-Frog method. We will invistigate energy conservation using these two methods. We will do our computations for two case, one when α is constant and the other case is when the α has a discontinuity in the domain.

Problem Definition

The equation that we are solving is 1-D wave equation as

$$u_{tt} = \alpha(x)u_{xx}$$

on domain $[0, 1]$. Initial condition on u is as

$$\begin{cases} u = 0 & 0 \le x \le 1/3 \ \&\ 2/3 \le x \le 1 \\ u = 3(x - 1/3) & 1/3 \le x \le 1/2, \\ u = 3(2/3 - x) & 1/2 \le x \le 2/3, \end{cases} \tag{1}$$

The other initial condition is $u_t(x, t = 0) = 0$. Our boundary conditions are

$$u(x = 0, t) = 0 \tag{2}$$

$$u_x(x = 1, t) = 0 \tag{3}$$

We will solve two cases. One α is constant and 1 in the whole domain. The second case α is variable and is defined as

$$\alpha = \begin{cases} 1 \text{ if } x < 1/2 \\ 2 \text{ if } x > 1/2 \end{cases} \tag{4}$$

Implementation

We will use two methods: Lax-Wendroff and Leap-Frog method for solving this wave equation. First we will write our equation in a system form as

$$\begin{pmatrix} u_t \\ u_x \end{pmatrix}_t = \begin{pmatrix} 0 & \alpha \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u_t \\ u_x \end{pmatrix}_x \tag{5}$$

Then we will change our basis such that we get two decoupled baby-wave equations ($u_t = \lambda u_x$). Using transformation

$$w := \mathbf{T}^{-1} \begin{pmatrix} u_t \\ u_x \end{pmatrix} \tag{6}$$

in which

$$\mathbf{T}^{-1} = \frac{\sqrt{\alpha+1}}{2\sqrt{\alpha}} \begin{pmatrix} 1 & \sqrt{\alpha} \\ 1 & -\sqrt{\alpha} \end{pmatrix} \tag{7}$$

we see that our equations, become

$$w_t = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} w_x \tag{8}$$

where λ_1 and λ_2 are eigenvalues of matrix $\begin{pmatrix} 0 & a \\ 1 & 0 \end{pmatrix}$. Now that we find easy wave equations, we will solve these equations using Lax-Wendroff and Leap-Frog equations. For simple wave equation $u_t = cu_x$ Lax-Wendroff is as

$$u_j^{n+1} = u_j^n + k.c\frac{u_{j+1}^n - u_{j-1}^n}{2h} + \frac{k^2c^2}{2}\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{h^2} \tag{9}$$

and Leap-Frog method is

$$u_j^{n+1} = u_j^{n-1} + \frac{k}{h}.c\left(u_{j+1}^n - u_{j-1}^n\right) \tag{10}$$

$$\tag{11}$$

Problem

1. You want to solve the one-dimensional wave equation $u_{tt} = \alpha(x)u_{xx}$ on the interval from $x = 0$ to $x = 1$.
   Numerically solve the following problem. Set $u = 0$ for all time at $x = 0$, and $u_x = 0$ for all time at $x = 1$. Set $\alpha(x) = 1$
   Start with an initial condition of a triangle wave between $x = 1/3$ and $x = 2/3$. That is,

   $$u = 0 \text{ for } 0 \leq x \leq 1/3 \text{ and } 2/3 \leq x \leq 1 \tag{12}$$
   $$u = 3(x - 1/3) \text{ for } 1/3 \leq x \leq 1/2, \tag{13}$$
   $$u = 3(2/3 - x) \text{ for } 1/2 \leq x \leq 2/3, \tag{14}$$

   Set the initial velocity $u_t = 0.0$
   Check what happens to the energy, the height, etc. How badly does it diffuse? After how many pulses is it gone?
   Repeat, only this time, set $\alpha(x) = 1$ for $x < 1/2$, and $\alpha(x) = 2$ for $x >= 1/2$. Check what happens to the energy, the height, etc. How badly does it diffuse? After how many pulses is it gone?

Solution

For solving the wave equation as in

$$u_{tt} = \alpha(x)u_{xx}$$

First, we transform this wave equation into two 1-D baby wave equation (i.e. $u_t = \lambda u_x$). For doing this transformation, we write, our equation in the following format:

$$\begin{pmatrix} u_t \\ u_x \end{pmatrix}_t = \begin{pmatrix} 0 & a \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u_t \\ u_x \end{pmatrix}_x \tag{15}$$

This way, we have two sets of baby wave equation which are coupled together. We find the eigen values of matrix $\mathbf{A} := \begin{pmatrix} 0 & a \\ 1 & 0 \end{pmatrix}$. We solve the equation $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$. Eigenvalues and eigenvectors are

$$\lambda_1 = \sqrt{\alpha} \quad u_1 = \tfrac{1}{\sqrt{\alpha+1}} \begin{pmatrix} \sqrt{\alpha} \\ 1 \end{pmatrix} \tag{16}$$

$$\lambda_2 = -\sqrt{\alpha} \quad u_2 = \tfrac{1}{\sqrt{\alpha+1}} \begin{pmatrix} \sqrt{\alpha} \\ -1 \end{pmatrix} \tag{17}$$

So the transformation matrix becomes

$$\mathbf{T} = \frac{1}{\sqrt{\alpha+1}} \begin{pmatrix} \sqrt{\alpha} & \sqrt{\alpha} \\ 1 & -1 \end{pmatrix} \tag{18}$$

For the inverse we get:

$$\mathbf{T}^{-1} = \frac{\sqrt{\alpha+1}}{2\sqrt{\alpha}} \begin{pmatrix} 1 & \sqrt{\alpha} \\ 1 & -\sqrt{\alpha} \end{pmatrix} \tag{19}$$

we define

$$w := \mathbf{T}^{-1} \begin{pmatrix} u_t \\ u_x \end{pmatrix} \tag{20}$$

So this w, will satisfy the 1-D baby wave equations, as:

$$w_t = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} w_x \tag{21}$$

where $\lambda_1 = \sqrt{\alpha}$ and $\lambda_2 = -\sqrt{\alpha}$ and $w = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$, where w_1 is a wave moving to left and w_2 is a wave moving to the right. For boundart conditions we have

$$u(x=0,t) = 0 \Rightarrow u_t(x=0,t) = 0 \tag{22}$$

$$u_x(x=1,t) = 0 \tag{23}$$

Using the relation between w and u, we have

$$\begin{pmatrix} u_t \\ u_x \end{pmatrix} = \mathbf{T} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \frac{1}{\sqrt{\alpha+1}} \begin{pmatrix} \sqrt{\alpha} & \sqrt{\alpha} \\ 1 & -1 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \tag{24}$$

Setting the boundary conditions that we found for u_t and u_x at the boundaries, we have:

$$u_t(x=0,t) = 0 \Rightarrow w_1 + w_2 = 0 \tag{25}$$
$$u_x(x=1,t) = 0 \Rightarrow w_1 - w_2 = 0 \tag{26}$$

Since w_1 is going left, at $x = 0$ it is going to be give its value to w_2 with negative sign(reflection at node), and w_2 will give its value to w_1 with a positive sign (reflection at open boundary).

The methods that we are using is Lax-Wendroff and Leap-Frog method.

Lax-Wendroff

We transformed our solution to w_1 and w_2 in which, we should only solve the simple baby wave equation (i.e. $u_t = cu_x$). We have

$$w_t^1 = \sqrt{\alpha} w_x^1 \tag{27}$$
$$w_t^2 = -\sqrt{\alpha} w_x^2 \tag{28}$$

In Lax-Wendroff, we do forward difference on time and central difference, and by adding second term in taylor series, we try to increase the accuracy. Actually, This extra term is a viscous term, which will damp the oscilations. Deriving Lax-Wendroff method, we have

$$u_t = cu_x \tag{29}$$
$$u^{n+1} = u^n + ku_t + \frac{k^2}{2} u_{tt} + O(k^3) \tag{30}$$
$$u^n + k.cu_x + \frac{k^2}{2} .c^2 u_{xx} + O(k^3) \tag{31}$$

then we use central difference for u_x (i.e. $D_0 u$) and central difference for u_{xx} (i.e. $D_+ D_- u$). The method becomes

$$u_j^{n+1} = u_j^n + k.c \frac{u_{j+1}^n - u_{j-1}^n}{2h} + \frac{k^2 c^2}{2} \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{h^2} \tag{32}$$

where h is space difference and k is time difference and c is the speed of wave (in our case is $\sqrt{\alpha}$ for w_1 and $-\sqrt{\alpha}$ for w_2). This would be our method, which also could be written as

$$u_j^{n+1} = u_j^n + \frac{\lambda c}{2} \left(u_{j+1}^n - u_{j-1}^n \right) + \frac{\lambda^2 c^2}{2} \left(u_{j+1}^n - 2u_j^n + u_{j-1}^n \right) \tag{33}$$

Lax-Wendroff Stability Analysis

For stability analysis of Lax-Wendroff method, we put $e^{ix\xi}$ into equation for finding the $|\rho(\xi)|$. This is called Von-Neumann Stability analysis.

The idea is that, we take Fourier transform of the scheme and then we say that for each frequency, the amplitude should remain finite over time (i.e. $|\rho(\xi)| < 1 + ck$).

$$\rho(\xi) = \frac{\tilde{u}^{n+1}}{\tilde{u}^n} = 1 - \frac{c\lambda}{2}\left(e^{i\xi} - e^{-i\xi}\right) + \frac{c^2\lambda^2}{2}\left(e^{i\xi} - 2 + e^{-i\xi}\right) \tag{34}$$

$$= 1 - ic\lambda\sin(\xi) - c^2\lambda^2(1 - \cos(\xi)) \tag{35}$$

$$= 1 - 2c^2\lambda^2\sin^2(\frac{\xi}{2}) - ic\lambda\sin(\xi) \tag{36}$$

So

$$|\rho(\xi)|^2 = \left(1 - 2c^2\lambda^2\sin^2\left(\frac{\xi}{2}\right)\right)^2 + (c\lambda\sin(\xi))^2 \tag{37}$$

$$= \left(1 - 2c^2\lambda^2\sin^2\left(\frac{\xi}{2}\right)\right)^2 + \left(c\lambda 2\sin(\frac{\xi}{2})\cos(\frac{\xi}{2})\right)^2 \tag{38}$$

$$= 1 - 4c^2\lambda^2\left(1 - c^2\lambda^2\right)\sin^4(\frac{\xi}{2}) \tag{39}$$

where $\lambda = k/h$. This result imply that if $c\lambda < 1$, then our method is stable. So we choose our value of h and k such that our method be stable.

Simulation

We have simulated the wave equation, using Lax Wendroff method. For the convergence test, We ploted the shape of the wave at different times with different N, to see if they converge.

Energy in the Lax-Wendroff method is decreasing with time because of the second order term that we added to increase the accuracy. This will cause energy dissipation in our system. Changes of energy vs time is shown in the figure 3.

In the second part of the simulation we change the value of α and make it variable over the domain. We have

$$\alpha = \begin{cases} 1 \text{ if } x < 1/2 \\ 2 \text{ if } x > 1/2 \end{cases} \tag{40}$$

For this variable α, we again checked if the scheme is converging. As is shown in figure 5. Wave will propagate as shown in figure 6. Energy of the wave is different than constant α. This energy is shown in figure 7.

Leap-Frog

In this part, we will solve the baby wave equation ($u_t = cu_x$) using Leap-Frog method. In Leap-Frog method, we will use past time step in forward time and central difference for

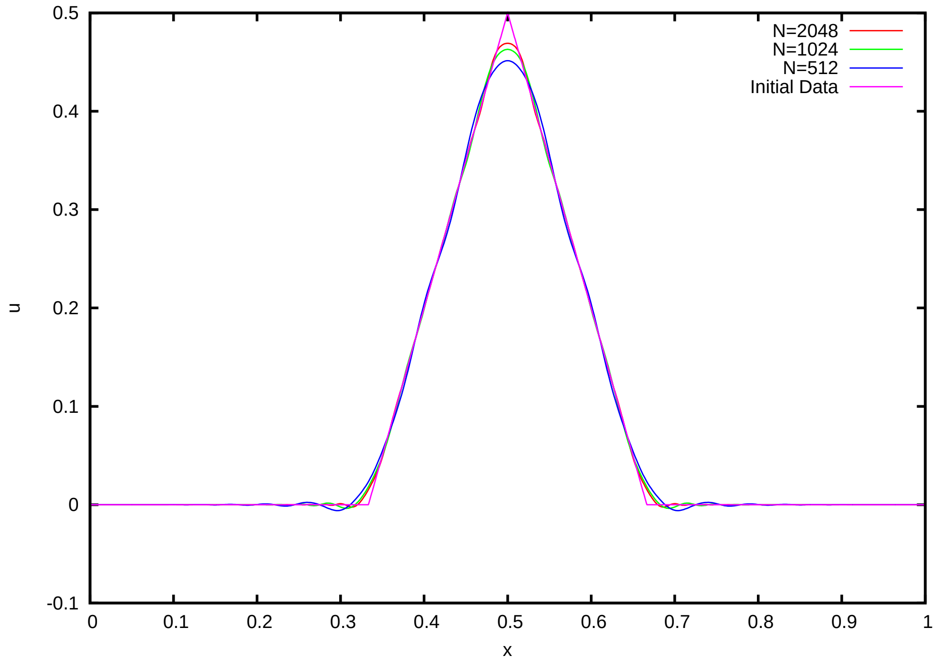

Figure 1: u vs x at $t = 3s$ for different grid point N with comparison to initial data

spatial derivative. We have

$$u_t = cu_x \tag{41}$$

$$\frac{u_j^{n+1} - u_j^{n-1}}{2k} = c\frac{u_{j+1}^n - u_{j-1}^n}{2h} \tag{42}$$

or

$$u_j^{n+1} = u_j^{n-1} + \frac{k}{h}.c\left(u_{j+1}^n - u_{j-1}^n\right) \tag{43}$$

$$u_j^{n+1} = u_j^{n-1} + \lambda.c\left(u_{j+1}^n - u_{j-1}^n\right) \tag{44}$$

This would be the Leap-Frog scheme for our method.

Simulation

We have simulated the wave equation, using Leap-Frog method. For the convergence test, we ploted the shape of the wave at different times with different N, to see if they converge.

Energy in the Lax-Wendroff method is decreasing with time because of the second order term that we added to increase the accuracy. This will cause energy dissipation in our system. Changes of energy vs time is shown in the figure 9.

For comparison between the Leap frog and Lax wendroff and their energy preservation, we have plotted both energies in the same figure (Figure11).

For this variable α, we again checked if the scheme is converging. As is shown in figure 12. Wave will propagate as shown in figure 13. Energy of the wave is different than constant α. This energy is shown in figure 14.

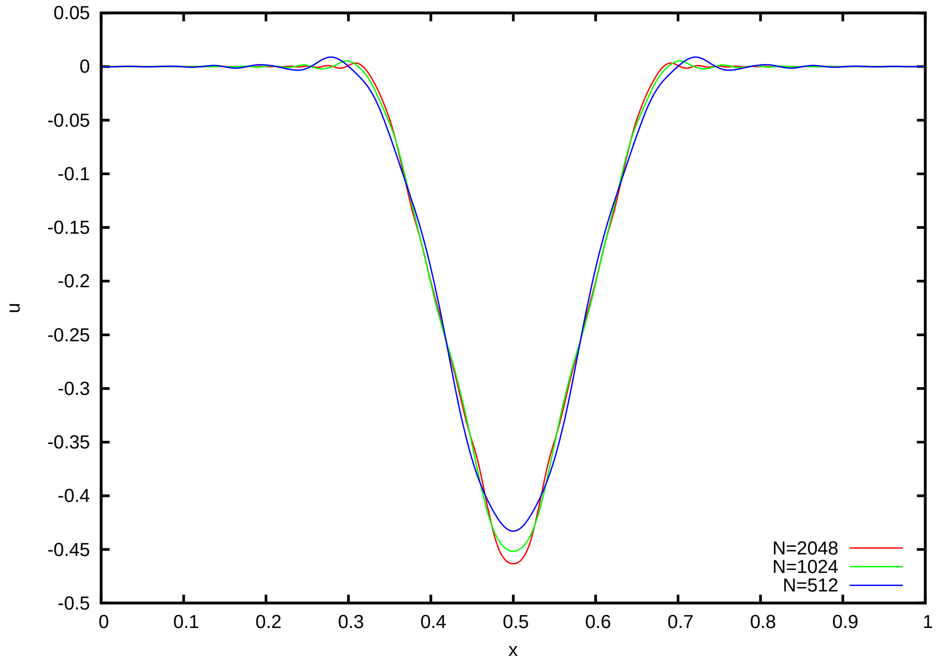

Figure 2: u vs x at $t = 10s$ for different grid point N as convergence test

Conclusion

We saw that how we can solve 1-D wave equation by change of variable. We used two methods to simulate the propagation of waves. We also discussed about boundary conditions and what will happen to them in the transformed coordinate. Both methods are of order Δt^2 and Δx^2. The main difference between these two methods is that Leap-Frog is energy conserving and Lax-Wendroff is not. In Lax-Wendroff we added an extra term using taylor series to attain more accuracy, however this extra term will dissipate energy from the system. Both methods smooth the tip of our wave. These two methods will smooth the first order discontinuities in the initial condition. We saw that this sharpness is all gone after the first reflection and we could not keep it sharp. To sum up, Leap-Frog method is better thatn Lax-Wendroff since this scheme is the same order as Lax-Wendroff and is also energy conserving. Impelimentaion is easier and Computationaly it is faster!

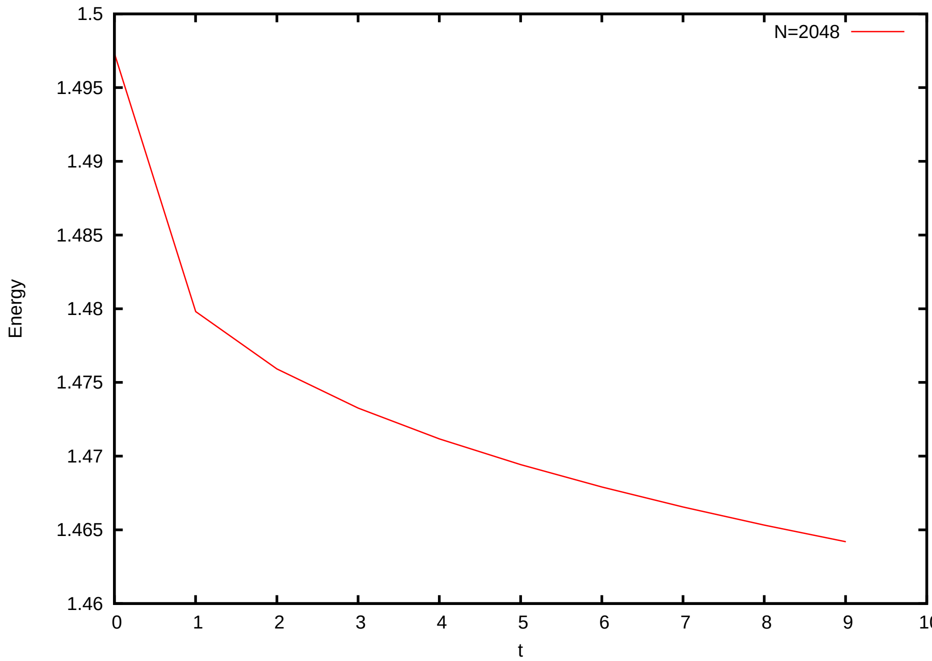

Figure 3: Lax-Wendroff Energy vs time

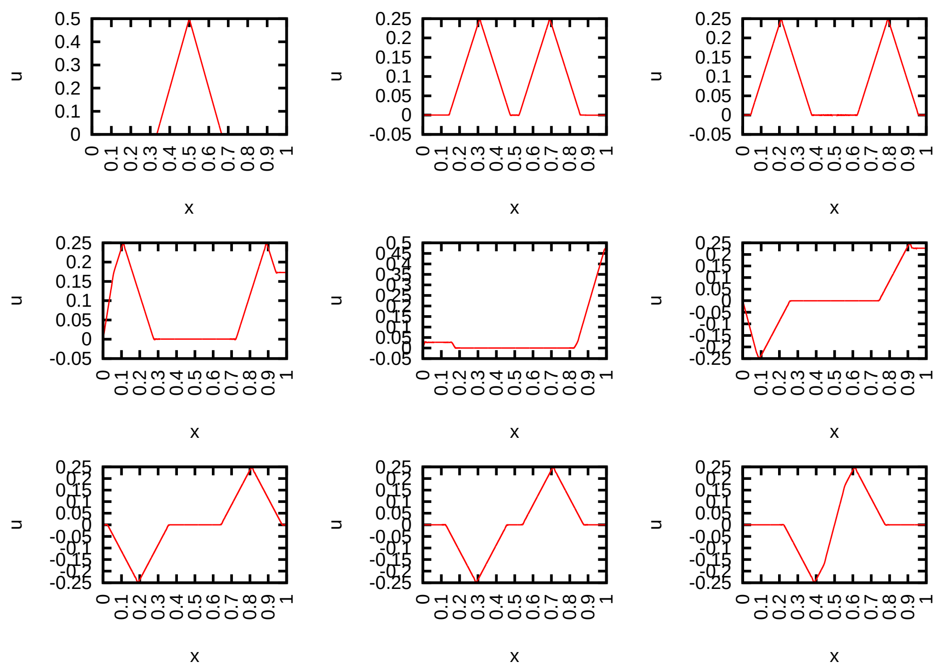

Figure 4: Shape of the Wave

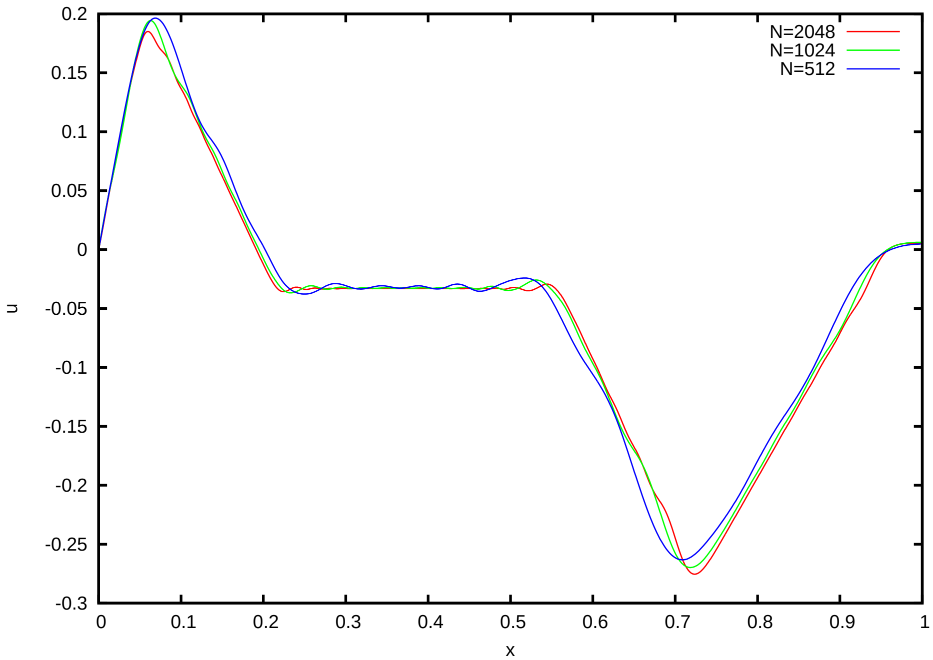

Figure 5: Shape of the Wave at time $t = 8$ for different N in case of changing α

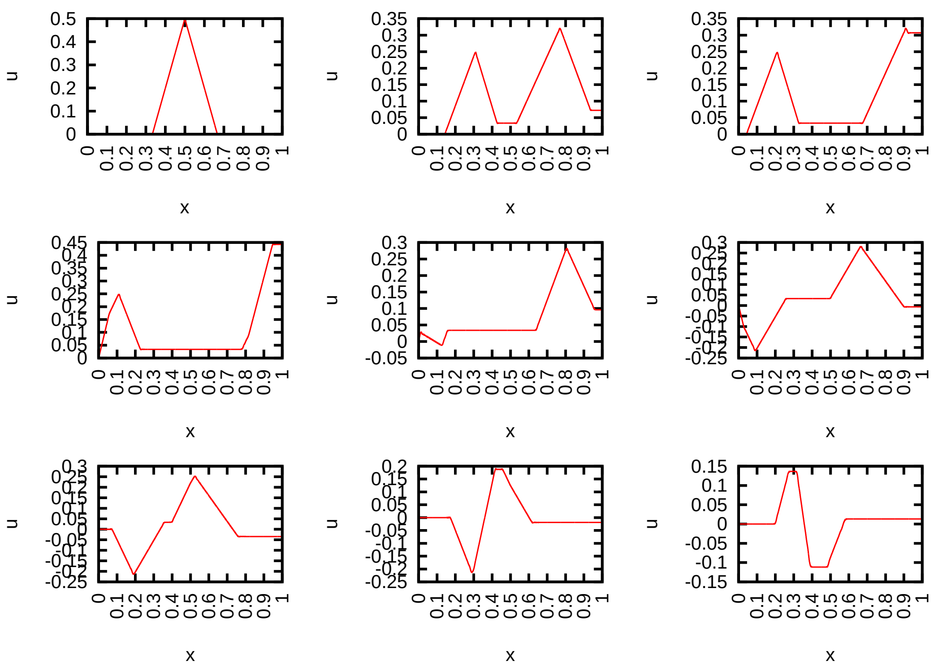

Figure 6: Propagating wave in case of variable α using Lax-Wendroff

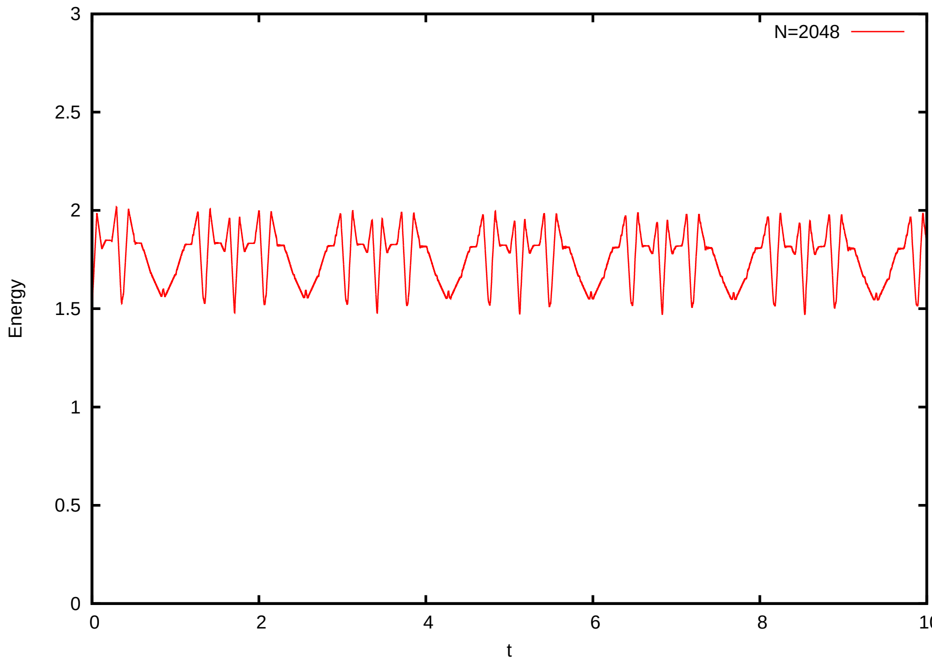

Figure 7: Wave Energy over time, variable α using Lax-Wendroff

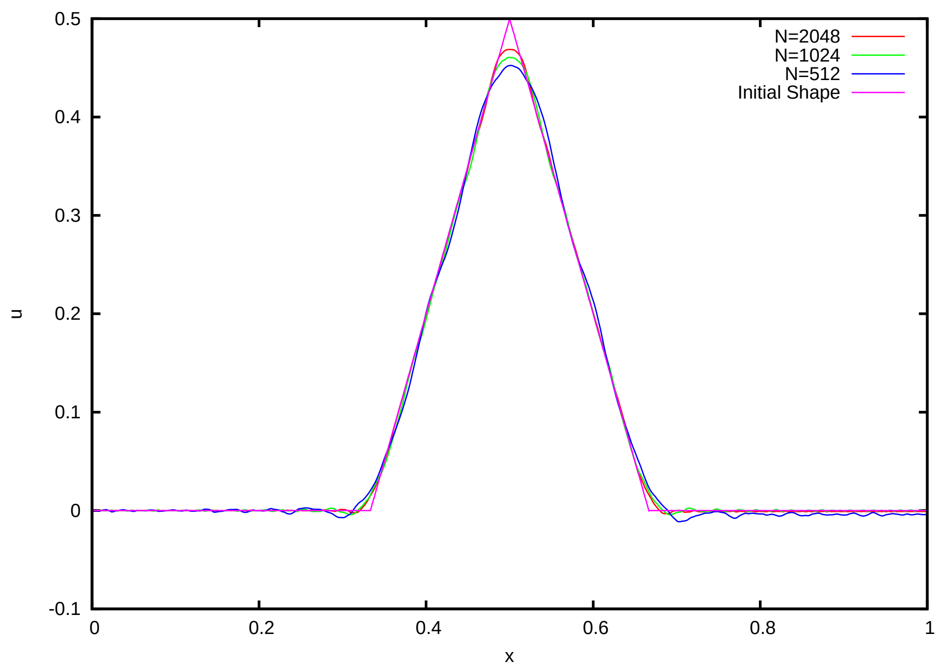

Figure 8: u vs x at $t = 3s$ for different grid point N with comparison to initial data

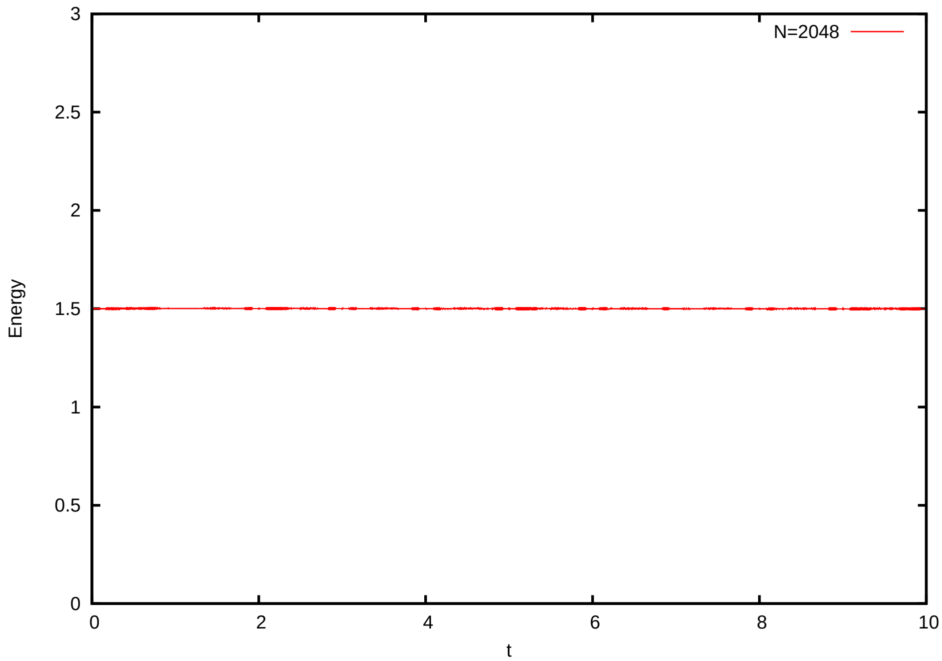

Figure 9: Energy vs time for Leap Frog Scheme

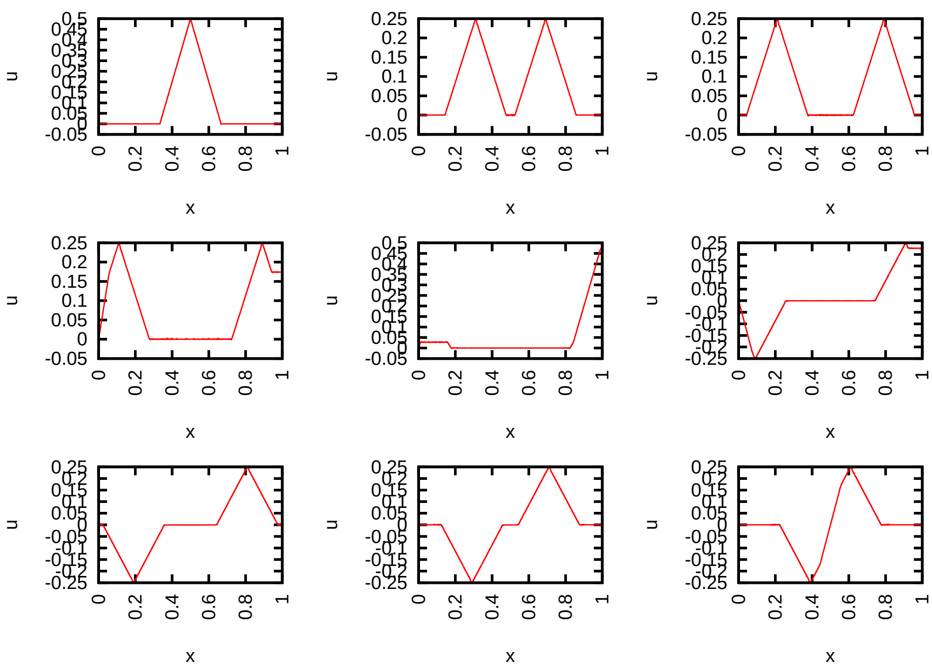

Figure 10: Shape of the Wave Over time

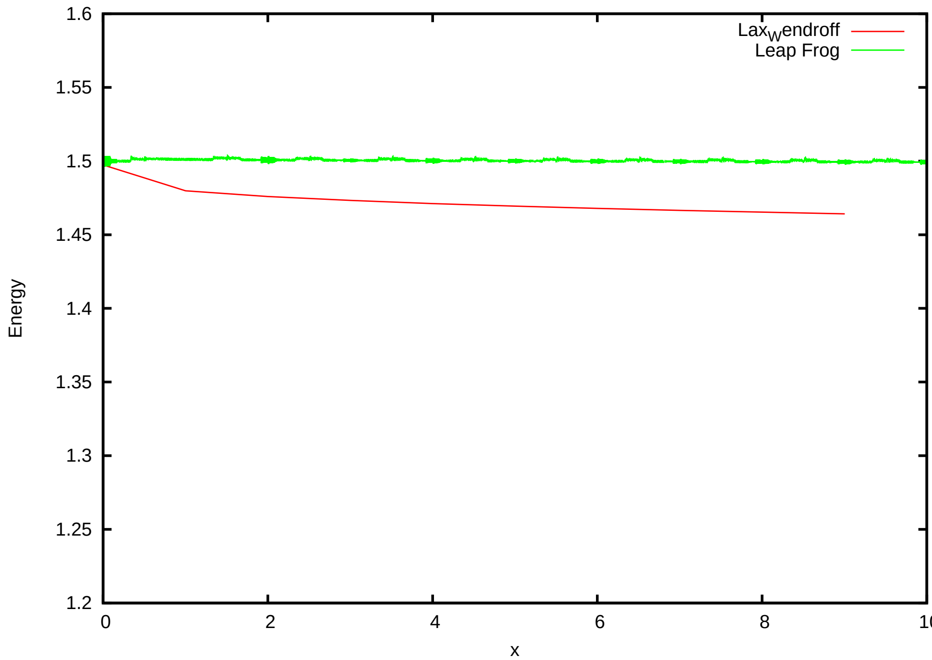

Figure 11: Energy over domain

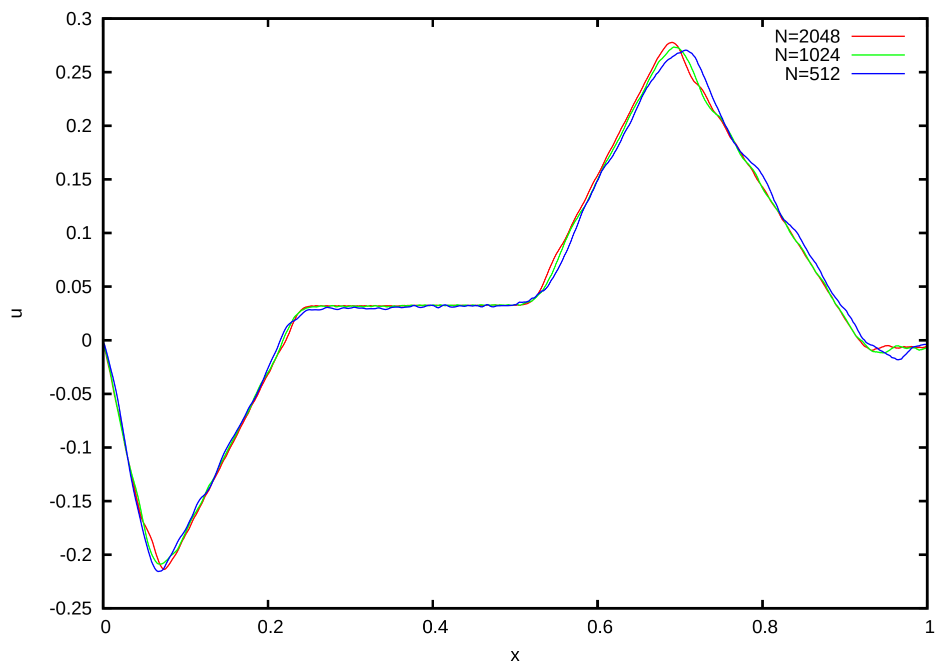

Figure 12: Shape of the Wave at time $t = 8$ for different N in case of changing α using Leap Frog

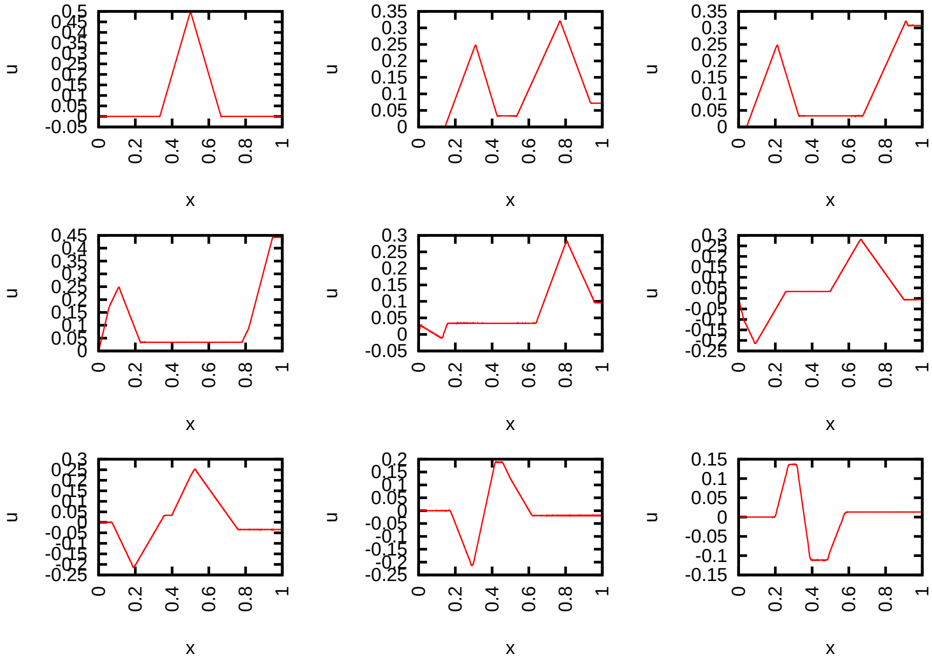

Figure 13: Propagating wave in case of variable α using Leap-Frog

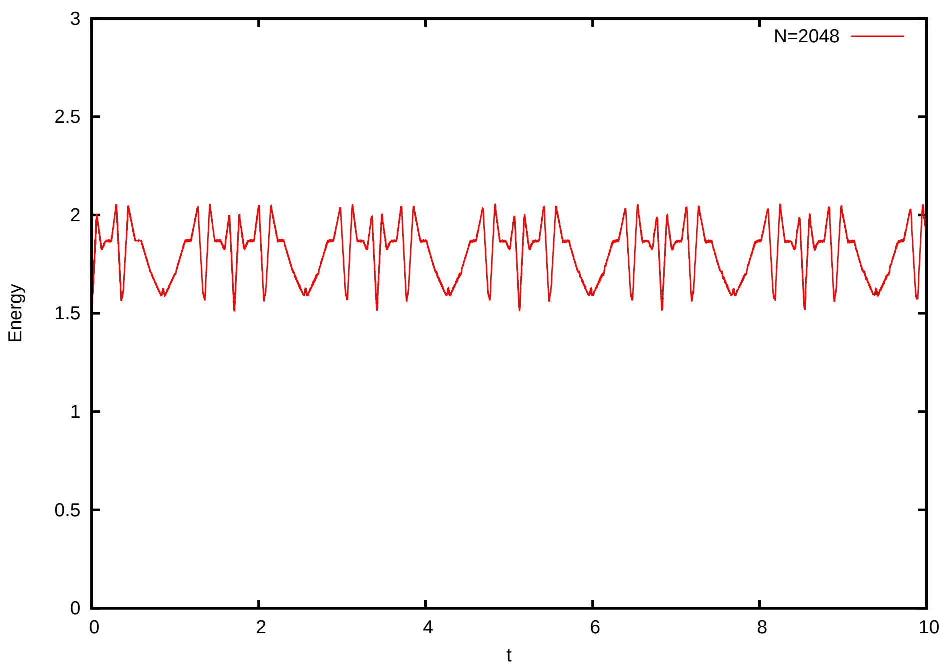

Figure 14: Wave Energy over time, variable α using Leap-Frog