# EE 226A Notes

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# Contents

# 1 Elements of Probability Theory

## 1.1 Probability Spaces and Events

**Definition 1.1.1: Kolmogorov's axioms**

For any **probability space** $(\Omega, \mathcal{F}, P)$, the function $P$ is called a **probability measure**. It is assumed to satisfy Kolmogorov's axioms:

*i.)* $P(A) \geqslant 0$ for all $A \in \mathcal{F}$;

*ii.)* $P(\Omega) = 1$;

*iii.)* if $A_1, A_2, \dots \in \mathcal{F}$ are disjoint events, then $P(\cup_{i\geqslant 1} A_i) = \sum_{i\geqslant 1} P(A_i)$.

The probability space we are working in encodes the model of our experiment, with the **measurable space** $(\Omega, \mathcal{F})$ being the most fine-grained representation of outcomes we can hope to observe.

**Theorem 1.1.2**

For a probability space $(\Omega, \mathcal{F}, P)$, the probability measure $P$ enjoys the following properties:

*i.)* Monotonicity: If $A \subset B$ are events, then $P(A) \leqslant P(B)$.

*ii.)* Subadditivity (Union bound): If $(A_i)_{i\geqslant 1}$ is a sequence of events in $\mathcal{F}$ and $A = \bigcup_{i\geqslant 1} A_i$, then $P(A) \leqslant \sum_{i\geqslant 1} P(A_i)$.

*iii.)* Continuity from below: If $A_1 \subset A_2 \subset \dots$ are events in $\mathcal{F}$ and $A = \bigcup_{i\geqslant 1} A_i$, then $P(A_i) \to P(A)$.

*iv.)* Continuity from above: If $A_1 \supset A_2 \supset \dots$ are events in $\mathcal{F}$ and $A = \bigcap_{i\geqslant 1} A_i$, then $P(A_i) \to P(A)$.

*Proof.*

*i.)* Monotonicity follows from the following: $P(A) \leqslant P(A) + P(A^C \cap B) = P(B)$.

*ii.)* Define $E_1 = A_1$ and $A_i = A_i \cap (\cup_{j<i} A_i)^C$ for $i \geqslant 2$. Then the $E_i$'s are disjoint, $E_i \subseteq A_i$ and $A = \cup_{i\geqslant 1} A_i = \cup_{i\geqslant 1} E_i$. Now we have $P(A) = P(\cup_{i\geqslant 1} E_i) = \sum_{i\geqslant 1} P(E_i) \leqslant \sum_{i\geqslant 1} P(A_i)$.

*iii.)* Define $E_i$'s as above, and note that

$$P(A) = \sum_{i\geqslant 1} P(E_i) = \lim_{n\to\infty} \sum_{i\geqslant 1}^{n} P(E_i) = \lim_{n\to\infty} P(A_n).$$

*iv.)* We now apply the previous part to $A_1^C \subset A_2^C \subset \cdots$.

$$\lim_{n\to\infty} P(A_n) = 1 - \lim_{n\to\infty} P(A_n^C) = 1 - P(A^C) = P(A).$$

$\square$

**Theorem 1.1.3: Law of total probability**

If events $A_1, A_2, \ldots$ partition $\Omega$, then

$$P(B) = \sum_{i \geqslant 1} P(A_i \cap B), \quad B \in \mathcal{F}.$$

### 1.1.1 Infinitely often and Borel-Cantelli lemmas

**Definition 1.1.4: Infinitely often**

$$\{A_n \text{ infinitely often}\} = \bigcap_{n \geqslant 1} \bigcup_{i \geqslant n} A_i.$$

We should understand $\{A_n \text{ i.o.}\}$ to be the set of samples $\omega \in \Omega$ such that $\omega \in A_i$ for infinitely many $i \geqslant 1$.

**Lemma 1.1.5: Borel-Cantelli**

Let $A_1, A_2, \ldots$ be a sequence of events. If

$$\sum_{i \geqslant 1} P(A_i) < \infty$$

then $P(\{A_i \text{ infinitely often}\}) = 0$.

*Proof.* Observe that $(\cup_{i \geqslant n} A_i)_{n \geqslant 1}$ is a decreasing sequence of events. Therefore, continuity from above and subadditivity together imply

$$P\left(\bigcap_{n \geqslant 1} \bigcup_{i \geqslant n} A_i\right) = \lim_{n \to \infty} P\left(\bigcup_{i \geqslant n} A_i\right) \leqslant \lim_{n \to \infty} \sum_{i \geqslant n} P(A_i) \to 0.$$

□

**Definition 1.1.6: Independent events**

A collection of events $A_1, A_2, \ldots$ are **independent** if

$$P\left(\bigcap_{i \in S} A_i\right) = \prod_{i \in S} P(A_i)$$

for every finite subset $S \subset \{1, 2, 3, \ldots\}$. If $A_1, A_2, \ldots$ are independent, then $A_1^C, A_2, \ldots$ are independent. By induction, the complements $A_1^C, A_2^C, \ldots$ are also independent.

**Lemma 1.1.7: Converse to Borel-Cantelli**

Let $A_1, A_2, \ldots$ be independent events. If

$$\sum_{i \geqslant 1} P(A_i) = \infty,$$

then $P(\{A_i \text{ infinitely often}\}) = 1$.

*Proof.* By definitions and continuity from above,

$$P\left(\{A_i \text{ i.o.}\}\right) = \lim_{n\to\infty} P\left(\bigcup_{i\geqslant n} A_i\right) = 1 - \lim_{n\to\infty} P\left(\bigcap_{i\geqslant n} A_i^C\right).$$

By independence, we have for any $m \geqslant n$

$$P\left(\bigcap_{i=n}^{m} A_i^C\right) = \prod_{i=n}^{m} P(A_i^C) = \prod_{i=n}^{m}(1 - P(A_i)) \leqslant \exp\left(-\sum_{i=n}^{m} P(A_i)\right),$$

where we made use of the inequality $1 - x \leqslant e^{-x}$ for all $x \in \mathbb{R}$. Since $\sum_{i\geqslant 1} P(A_i)$ doesn't converge, we must have that

$$P\left(\bigcap_{i\geqslant n} A_i^C\right) = \lim_{m\to\infty} P\left(\bigcap_{i=n}^{m} A_i^C\right) \leqslant \exp\left(-\sum_{i\geqslant n} P(A_i)\right) = 0.$$

□

### 1.1.2 Existence of probability measures

Some sets are simply not measurable, well call such sets non-measurable. A set which is non-measurable cannot be assigned a probability. In general, we tend to stick with $\sigma-$algebras, since they have nice properties. A deep theorem from probability theory called **Carathéodory's extension theorem** basically tells us that as long as we assign probabilities to a small collection of events in a consistent way, then there exists a natural and unique assignment of probabilities to the $\sigma-$algebra generated by the original collection.

**Theorem 1.1.8: Carathéodory's extension theorem**

Suppose $\mathcal{G}$ is a family of subsets of $\Omega$ that satisfies the following (relatively modest) properties:

*i.)* $\varnothing, \Omega \in \mathcal{G}$;

*ii.)* if $A, B \in \mathcal{G}$, then $A \cap B \in \mathcal{G}$;

*iii.)* if $A, B \in \mathcal{G}$, then there is a *finite* number of *disjoint* sets $C_1, \dots, C_n \in \mathcal{G}$ such that $A \setminus B = \bigcup_{i=1}^{n} C_i$.

(Note: (*iii*) is weaker than imposing the assumption $A \in \mathcal{G} \implies A^C \in \mathcal{G}$. )

The extension theorem says that if we assign numbers (i.e., probabilities) $p(A)$ to the sets $A \in \mathcal{G}$ so that

A. $p(A) \geqslant 0$ for $A \in \mathcal{G}$;

B. $p(\Omega) = 1$;

C. if $B \in \mathcal{G}$ and $A_1, A_2, \dots \in \mathcal{G}$ are disjoint with $B = \cup_{i\geqslant 1} A_i$, then $p(B) = \sum_{i\geqslant 1} p(A_i)$,

then there exists a unique probability measure $P$ on $\sigma(\mathcal{G})$ that satisfies A-C and has the property that $P(A) = p(A)$ for all $A \in \mathcal{G}$.

## 1.2 Random Variables and Expectation

### 1.2.1 Random variables and algebraic properties

**Definition 1.2.1: Random Variable**

We define a random variable to be a function $X : \Omega \to \overline{\mathbb{R}}$ that satisfies

$$\{\omega \in \Omega : X(\omega) \leqslant \alpha\} \in \mathcal{F} \text{ for each } \alpha \in \overline{\mathbb{R}}.$$

Note that $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$.

A function $X : \Omega \to \overline{\mathbb{R}}$ satisfying the definition above is said to be $\mathcal{F}$−measurable. If $X$ does not take values $\pm\infty$ (say, with probability one), then we say it is a **real-valued random variable**.

**Proposition 1.2.2**

If $X$ is a random variable, then $pX$ and $|X|^p$ are random variables for $p \in \mathbb{R}$. Moreover, if $X, Y$ are real-valued random variables, then $X + Y$, and $XY$ are also random variables.

*Proof.* We leave the first claim as an exercise. For the second, note that we can write

$$\{\omega : X(\omega) + Y(\omega) > \alpha\} = \bigcup_{q \in \mathbb{Q}} \{\omega : X(\omega) > q\} \cap \{\omega : Y(\omega) > \alpha - q\}.$$

Since $\mathcal{F}$ is closed under countable unions, intersections, and complements, it follows from the assumption that $X, Y$ are random variables that $\{\omega : X(\omega) + Y(\omega) \leqslant \alpha\} \in \mathcal{F}$. The third claim now follows easily by writing $XY = \left[(X + Y)^2 - (X - Y)^2\right]/4$. $\square$

**Proposition 1.2.3**

If $(X_n)_{n \geqslant 1}$ is a sequence of random variables defined on a common probability space $(\Omega, \mathcal{F}, P)$, then

- $\sup_{n \geqslant 1} X_n$ and $\inf_{n \geqslant 1} X_n$ are random variables; and
- $\limsup_{n \to \infty} X_n$ and $\liminf_{n \to \infty} X_n$ are random variables; and
- if $\lim_{n \to \infty} X_n$ exists point wise, it is also a random variable.

*Proof.* Note that $\sup_{n \geqslant 1} X_n(\omega) \leqslant a$ if and only if $X_n(\omega) \leqslant a$ for each $n \geqslant 1$. It follows that

$$\left\{\omega : \sup_{n \geqslant 1} X_n(\omega) \leqslant a\right\} = \bigcap_{n \geqslant 1} \{\omega : X_n(\omega) \leqslant \alpha\} \in \mathcal{F},$$

where we used the fact that $\mathcal{F}$ is closed under countable intersections. Hence, $\sup_{n \geqslant 1} X_n$ is a random variable. The rest of the claims are consequences of this. Indeed, since $\inf_{n \geqslant 1} X_n = -\sup_{n \geqslant 1}(-X_n)$, it holds that $\inf_{n \geqslant 1} X_n$ is also a random variable. So are $\limsup_{n \to \infty} X_n = \inf_{m \geqslant 1} \sup_{n \geqslant m} X_n$, and $\liminf_{n \to \infty} X_n$ by similar logic. As a result, if $\lim_{n \to \infty} X_n$ exists pointwise, then it is equal to $\limsup_{n \to \infty} X_n$ and is therefore also a random variable. $\square$

**Definition 1.2.4: Almost sure equivalence of random variables**

If $X, Y$ are random variables and $P(\{\omega : X(\omega) \neq Y(\omega)\}) = 0$, then we say $X = Y$ almost surely (abbreviated a.s.), or $X = Y$ with probability one.

### 1.2.2 Distribution functions and distributions

**Definition 1.2.5: Distribution function**

A random variable $X$ on a probability space $(\Omega, \mathcal{F}, P)$ is described in part by its distribution function $F_X : \mathbb{R} \to [0,1]$, defined as

$$F_X(x) := P\{X \leqslant x\}, \quad x \in \mathbb{R}.$$

**Theorem 1.2.6: Properties of the distrbution function**

A function $F : \mathbb{R} \to [0,1]$ is the distribution function of a random variable if and only if

*i.)* $F$ is nondecreasing

*ii.)* $F$ is right-continuous, that is $\lim_{y \downarrow x} F(y) = F(x)$, for all $x \in \mathbb{R}$.

Moreover, $F$ is the distribution function of a real-valued random variable if and only if it further holds that

$$\lim_{x \to -\infty} F(x) = 0 \text{ and } \lim_{x \to \infty} F(x) = 1.$$

*Proof.* If $F$ is a distribution function for $X$, then *(i)* follows by monotonicity of $P$ since

$$\{\omega : X(\omega) \leqslant x\} \subset \{\omega : X(\omega) \leqslant x'\}, \quad x \leqslant x'.$$

Next, for $x \in \mathbb{R}$, note that

$$\{\omega : X(\omega) \leqslant x\} = \bigcap_{n \geqslant 1} \left\{\omega : X(\omega) \leqslant x + \frac{1}{n}\right\}.$$

Continuity from above again gives $\lim_{n\to\infty} F(x + \frac{1}{n}) = F(x)$. This, together with *(i)*, gives *(ii)*.

If we have $X$ real-valued, then we can write $\Omega = \cup_{n \geqslant 1}\{\omega : X(\omega) \leqslant n\}$. Hence, we find that $\lim_{n\to\infty} F(n) = 1$ follows by continuity from below and $P(\Omega) = 1$; together with *(i)*, this implies $\lim_{x\to\infty} F(x) = 1$. Similarly, since $\varnothing = \cap_{n\geqslant 1}\{\omega : X(\omega) \leqslant -n\}$, continuity from above together with $P(\varnothing) = 0$ implies $\lim_{n\to\infty} F(-n) = 0$. As before, with $(i)$ this yields $\lim_{x\to-\infty} F(x) = 0$.

For the other direction of the proof we refer to the textbook. □