Midterm Review
EE 226A

Ahmed Shakil

Contents

1 Tools and Tricks

2 Time Series Analysis

2.1 Second-Order Processes

Definition 2.1: Second-Order Process

Let $X = (X_n)_{n\in\mathbb{Z}}$ be a stochastic process on a probability space $(\Omega, \mathcal{F}, P)$. The process X is said to be a (discrete-time) second-order process if it has finite second moments $\mathbb{E}|X_n|^2 < \infty$ for all $n \in \mathbb{Z}$. Since all X_n are elements of $L^2(\Omega, \mathcal{F}, P)$, it follows that second-order processes also form a vector space.

Example 2.2

Gaussian processes are second-order processes, and the collection of Gaussian processes is a subspace of second-order processes.

Definition 2.3: Second-Order Statistics

For second-order processes X and Y the second-order statistics are summarized by the mean function $\mu_X(n) := \mathbb{E}[X_n]$ and covariance function

$$R_{XY}(m, n) := \mathrm{Cov}(X_m, Y_n), \quad m, n \in \mathbb{Z}.$$

For second-order processes, the mean and covariance functions are finite everywhere.

Example 2.4

If X is a Gaussian process, then all finite-dimensional marginals are characterized by the functions μ_X and R_{XX}.

Definition 2.5: Wide Sense Stationary

A second-order process $X = (X_n)_{n\in\mathbb{Z}}$ is wide sense stationary (WSS), if $\mu_x(n) = \mu_X(0)$ for all n, and $R_{XX}(m, n)$ is a function of only the difference $(m - n)$. In this case, we often abbreviate $R_{XX}(m, n)$ as $R_{XX}(m - n)$ to denote parametrization of the covariance function by the difference $(m - n)$.

Remark 2.6

For WSS processes the covariance enjoys the follow symmetries:

$$R_{XX}(n, n + k) = R_{XX}(0, k) = R_{XX}(k, 0) = R_{XX}(0, -k).$$

In our compact notation, $R_{XX}(k) = R_{XX}(-k)$, so that R_{XX} is a symmetric function of k.

Definition 2.7: Jointly Wide Senese Stationary

Processes $X = (X_n)_{n\in\mathbb{Z}}$ and $Y = (Y_n)_{n\in\mathbb{Z}}$ are jointly wide sense stationary (JWSS) if each are WSS and the covariance function $R_{XY}(m, n) = \mathrm{Cov}(X_m, Y_n)$ depends only on the difference $m - n$. In this case, we abbreviate $R_{XY}(m, n)$ as $R_{XY}(m - n)$.

Remark 2.8

Unlike R_{XX}, the function R_{XY} is not symmetric in its argument. However, if X and Y are JWSS, then we do have the following identities

$$R_{XY}(n+k,n) = \mathrm{Cov}(X_k, Y_0) = \mathrm{Cov}(X_0, Y_{-k}) = R_{XY}(k,0) = R_{XY}(0,-k).$$

In particular, noting the order of subscripts, we have $R_{XY}(k) = R_{YX}(-k)$.

2.2 Spectral Theory of Second-Order Processes

2.2.1 Fourier transform speedrun

To start, we note some info and results about Fourier transforms.

Definition 2.9: Energy Spectral Density

Given $x \in l^1(\mathbb{Z})$, we can define a sequence $a \in l^1(\mathbb{Z})$ via the self-convolution

$$a(n) = \sum_k x(k)x(n-k), \quad n \in \mathbb{Z}.$$

By the convolution theorem and time-reversal property of Fourier transforms, the discrete-time Fourier transform of a is equal to

$$\hat{a}(\omega) = \hat{x}(\omega)\hat{x}^*(\omega) = |\hat{x}(\omega)|^2 \geqslant 0.$$

The function $\hat{a}$ is called the energy spectral density of x, since it is a nonnegative function with the property that its integral over any subset of frequencies in $[-\pi, \pi)$ is equal to the energy of the sequence x restricted to those frequencies.