Second Course in Analysis

Ahmed Shakil

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Contents

Part I:

Analysis in $\mathbb{R}^n$

1 Linear Algebra

1.1 The Basics

Definition 1.1.1: Vector Space

A set V over a field $\mathbb{F}$ ($\mathbb{R}$ or $\mathbb{C}$), is a vector space if there is an operation $+ : V \times V \to V$ such that for $u, v, w \in V$ the following properties hold

1. Commutativity: $u + v = v + u$
2. Associativity: $(u + v) + w = u + (v + w)$
3. 0 element: There exists $0 \in V$ such that $u + 0 = u \ \forall u \in V$
4. Additive inverse: For each $u \in V$, $\exists - u \in V$ such that $u + (-u) = 0$.

Furthermore, scalar multiplication must be supported: there must be an operation $\times : \mathbb{F} \times V \to V$ such that for all $a, b \in \mathbb{F}$ and $u, v \in V$ the following properties hold:

1. Compatibility with field multiplication: $a \times (b \times u) = (a \cdot b) \times u$
2. Distributivity in the following senses:

$$a \times (u + v) = a \times u + a \times v$$
$$(a + b) \times u = a \times u + b \times u$$

3. Identity element of scalar multiplication: $1 \times u = u$ where $1 \in \mathbb{F}$ is the identity element.

Definition 1.1.2: Inner Product

An inner product is a function over a vector space V. The function $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{F}$ has the following properties

1. Conjugate symmetry: $\langle u, v \rangle = \overline{\langle v, u \rangle}$
2. Linearity in the first argument: $\langle au + bv, u \rangle = a \langle u, w \rangle + b \langle v, w \rangle$
3. Positive-definiteness: $\langle \cdot, \cdot \rangle \geqslant 0$ where $\langle u, u \rangle = 0$ if and only if $u = 0$.

A vector space equipped with an inner product is called an inner product space.

Definition 1.1.3: Norm

A norm $\|\cdot\|$ one a vector space V is a function such that for all $u, v \in V$ and for all $c \in \mathbb{F}$ the following properties hold:

1. Non-negativity: $\|v\| \geqslant 0$, $\forall\, v \in V$ and $\|v\| = 0 \iff v = 0$
2. Absolute homogeneity: $\|cv\| = |c|\,\|v\|$, $\forall\, c \in \mathbb{F}$ and $v \in V$
3. Triangle inequality: $\|u + v\| \leqslant \|u\| + \|v\|$, $\forall\, u, v \in V$

A vector space equipped with a norm forms a normed space.

Remark 1.1.4

A norm induces a metric or distance d :

$$d(u, v) := \|u - v\|$$

where d satisfies the following properties:

1. $d(u, v) \geqslant 0$
2. $d(u, v) = 0 \iff u = v$
3. $d(u, v) = d(v, u)$
4. $d(u, w) \leqslant d(u, v) + d(v, w)$

Definition 1.1.5: Linear Transformation

A linear transformation between two vector spaces V and W (with the same field $\mathbb{F}$) is a function:

$$T : V \to W$$

such that $\forall\ u, v \in V$ and $c \in \mathbb{F}$ the following holds

1. Additivity: $T(u + v) = T(u) + T(v)$
2. Homogeneity: $T(cu) = cT(u)$

Some properties as a result of the definition above is the following:

1. Linear transformation between finite-dimensional vector spaces can always be represented as a matrix
2. Rank-Nullity theorem: $\dim(V) = \dim(\ker(T)) + \operatorname{rank}(T)$

Definition 1.1.6: Isomorphism

Two vector spaces V and W are said to be isomorphic if there exists a bijective linear transformation $T : V \to W$. Here, T is known as an isomorphism between V and W, and when V and W are isomorphic we denote this as

$$V \cong W.$$

Definition 1.1.7: Metric Space

A metric space is a set X with a function $d : X \times X \to \mathbb{R}$ (known as metric) that satisfies the following properties

1. Nonnegativity: $d(x, y) \geqslant 0$ and $d(x, y) = 0 \iff x = y$
2. Symmetry: $d(x, y) = d(y, x)$
3. Triangle inequality: $d(x, z) \leqslant d(x, y) + d(y, z)$

Definition 1.1.8: Homeomorphism

A homeomorphism is a function $f : X \to Y$ between two metric spaces X and Y that satisfies:

1. Bijectivity: f is a bijection

2. Continuity: f is continuous
3. Inverse continuity: The inverse function $f^{-1} : Y \to X$ is also continuous

The difference between isomorphism and homeomorphism is that an isomorphism preserves the algebraic structure such as addition and multiplication whereas a homeomorphism preserves the topological structure such as continuity and compactness.

1.2 Some Results From Linear Algebra

An *m*-by-*n* matrix represents a linear transformation $T_A : \mathbb{R}^n \to \mathbb{R}^m$ defined as

$$T_A(v) = Av \text{ for } v \in \mathbb{R}^n.$$

The set of linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$ denoted as $\mathcal{L} = \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ also forms a vector space.

Theorem 1.2.1

Given a matrix A (*m*-by-*k*) and a matrix B (*k*-by-*n*), we have $T_A \circ T_B = T_{AB}$.

Proof. Let $\{e_j\}_{j=1}^n$ be the basis vectors of $\mathbb{R}^n$. We will show that both linear transformations are the same on all basis vectors. Consider any basis vector of $\mathbb{R}^n$, say e_j, observe the following:

$$\begin{aligned}(T_A \circ T_B)(e_j) &= T_{AB}(e_j)\\ T_A(T_B(e_j)) &= AB(e_j)\\ T_A(Be_j) &= Ab_j\\ T_A(b_j) &= Ab_j\\ Ab_j &= Ab_j\end{aligned}$$

Since both sides are equal on all basis vectors we are done. □

Definition 1.2.2: Operator Norm

Given to normed spaces V and W we define the operator norm as

$$\|T\| := \sup\left\{ \frac{\|Tv\|_W}{\|v\|_V} \,\middle|\, v \neq 0 \right\}.$$

The operator norm as defined above is a norm over the vector space $\mathcal{L}(V, W)$.

Theorem 1.2.3

Let $T : V \to W$ be a linear transformation between two normed spaces. Then the following are equivalent:

1. $\|T\| < \infty$
2. T is uniformly continuous
3. T is continuous
4. $\|T\|$ is continuous at the origin.

Theorem 1.2.4

Every linear transformation $T : \mathbb{R}^m \to W$ is continuous, and every isomorphism is a homeomorphism.

2 Derivatives

Definition 2.0.1: Derivative

Let $f : U \to \mathbb{R}^m$ be a function defined on an open set $U \subseteq \mathbb{R}^n$. We say f is differentiable at $p \in U$ with derivative $(Df)_p = T$ if $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation such that the remainder $R : \mathbb{R}^n \to \mathbb{R}^m$ defined through

$$f(p+v) = f(p) + T(v) + R(v) \implies \lim_{\|v\| \to 0} \frac{R(v)}{\|v\|} = 0.$$

Lemma 2.0.2

Suppose $(Df)_p$ exists then it is unique.