# Midterm 2 Review
## Math H185

Ahmed Shakil

## Contents

# 1 Tools and Tricks

**Theorem 1.1: Weierstrass's Approximation Theorem**

Any continuous function $f(x)$ on $[0, 1]$ can be uniformly approximated by polynomials, meaning that one can find a sequence of polynomials $p_n(x)$ converging uniformly to $f(x)$ on $[0, 1]$.

**Theorem 1.2**

If $f : \mathbb{C} \to \mathbb{C}$ is holomorphic and non-constant, then $f(\mathbb{C})$ is dense in $\mathbb{C}$.

## 2 Analytic Functions

**Theorem 2.1: Dominated Convergence**

Suppose there exist a function $G(x)$ such that $|F_n(x)| < |G(x)| \ \forall n, x$ and $\int |G(x)|\, dx < \infty$, then

$$\lim_{n\to\infty} \int F_n(x) dx = \int \lim_{n\to\infty} F_n(x) dx.$$

Many times a constant can serve in the place of such $G(x)$.
As a reminder we note the following results again.

**Theorem 2.2: Cauchy's formula**

Suppose we have $f : \Omega \subset \mathbb{C} \to \mathbb{C}$ which is defined on an open set as the domain, and is holomorphic. Let $z_0 \in \Omega$ and $r > 0$ such that $B_r(z_0) \subset \Omega$. Then for all $z \in B_r(z_0)$, we have

$$f(z) = \frac{1}{2\pi i} \int_{\partial B_r(z_0)} \frac{f(w)}{w - z} dw.$$

**Corollary 2.3: Infinite differentiability**

As a result of Cauchy's formula we have that wherever $f$ is holomorphic it is infinitely differentiable.
We can differentiate Cauchy's formula to derive the following more general formula for higher order derivatives:

$$\boxed{f^{(n)}(z) = \frac{n!}{2\pi i} \int_{\partial B_r(z_0)} \frac{f(w)}{(w - z)^{n+1}} dw}$$

We use this formula's to derive a bound on the $n^{\text{th}}$ derivative of $f$.

**Theorem 2.4: Cauchy's Inequality**

Let $f : U \to \mathbb{C}$ be a holomorphic function and take some $\overline{B_r(z)} \subset U$, then

$$\left|f^{(n)}(z)\right| \leqslant \frac{n!}{r^n} \sup_{w \in \partial B_r(z)} |f(w)|.$$

Apply the integral triangle inequality to Cauchy's Integral formula and the result follows.

*Proof.*

$$\begin{aligned}
\left|f^{(n)}(z)\right| &\leqslant \frac{n!}{2\pi i} \int_{\partial B_r(z)} \left|\frac{f(w)}{(w - z)^{n+1}}\right| |dw| \\
&\leqslant \frac{n!}{2\pi i} \frac{2\pi i r}{r^{n+1}} \sup_{w \in \partial B_r(z)} f(w) \\
&= \frac{n!}{r^n} \sup_{w \in \partial B_r(z)} f(w)
\end{aligned}$$

□

**Definition 2.5: Analytic function**

A function $f : U \to \mathbb{C}$ is analytic if $\forall z_0 \in U$, $\exists r > 0$ such that

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \quad \forall z \in B_r(z_0).$$

It is important to note that being analytic implies being holomorphic and vice-versa.

**Theorem 2.6: Taylor representation**

If $f$ is holomorphic on $B_r(z_0) \subset U$, and we have that $f(z) = f(z_0) + f'(z_0)(z - z_0) + \frac{1}{2!} f''(z_0)(z - z_0)^2 + \dots$ Then this series converges for $z \in B_r(z_0)$.

*Proof.* Follows from using Cauchy's inequality to bound the terms in the Taylor series in order to calculate a radius of convergence which ends up being $\geqslant r$. □

**Theorem 2.7: Taylor Expansion of a Holomorphic Function**

If $f$ is holomorphic $B_r(z_0) \subset U$, then

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \frac{f''(z_0)}{2!}(z - z_0)^2 + \dots$$

*Proof.* We start with Cauchy's formula:

$$f(z) = \frac{1}{2\pi i} \int_{\partial B_r(z)} \frac{f(w)}{w - z} dz.$$

$$\frac{1}{w - z} = \frac{1}{(w - z_0) - (z - z_0)} = \frac{1}{w - z_0} \left( \frac{1}{1 - \frac{z - z_0}{w - z_0}} \right)$$

Note that since $|z - z_0| < |w - z_0|$ we have $\frac{|z - z_0|}{|w - z_0|} < 1$.
So

$$\frac{1}{1 - \frac{z - z_0}{w - z_0}} = 1 + \frac{z - z_0}{w - z_0} + \left( \frac{z - z_0}{w - z_0} \right)^2 + \dots$$

Hence, we now have:

$$\frac{1}{w - z} = \frac{1}{w - z_0} \left( 1 + \frac{z - z_0}{w - z_0} + \left( \frac{z - z_0}{w - z_0} \right)^2 + \dots \right)$$

$$f(z) = \int_{\partial B_r(z_0)} \frac{f(w)}{w - z_0} + \frac{f(w)}{(w - z_0)^2}(z - z_0) + \dots dw$$

Using the Dominated Convergence Theorem, we can interchange the sum and integral, from there we can leverage Cauchy's integral formula:

$$f(z) = \frac{1}{2\pi i}\int_{B_r(z_0)} \frac{f(w)}{w - z_0} dw + \int_{B_r(z_0)} \frac{f(w)}{(w - z_0)^2}(z - z_0) dw + \dots$$
$$= f(z_0) + f'(z_0)(z - z_0) + \frac{f''(z_0)}{2!}(z - z_0)^2 + \dots$$

This is the Taylor expansion, hence we are done. □

An important slogan to remember is that holomorphic functions are automatically analytic.

**Definition 2.8: Entire function**

A priori we know what it means for a function to be bounded. We say a function is entire if it is defined and holomorphic on all the complex numbers.

**Theorem 2.9: Liouville's Theorem**

If $f : \mathbb{C} \to \mathbb{C}$ is entire, and bounded, then $f$ is constant.

*Proof.* Apply Cauchy's inequality to $B_r(z_0)$,

$$|f'(z_0)| \leqslant \frac{1}{r} \sup_{z \in \partial B_r(z_0)} |f(z)| \leqslant A.$$

Taking $r \to \infty$, we get $f'(z_0) = 0$. □

**Theorem 2.10: Fundamental Theorem of Algebra**

Every nonconstant polynomial

$$f(z) = a_n z^n + \dots + a_1 z + a_0$$

has a root.

*Proof.* Suppose not for the sake of contradiction. Since $f(z)$ is entire and non-zero, $\frac{1}{f(z)}$ is entire. Indeed,

$$\frac{1}{f(z)} = \frac{1}{z^n}\left(\frac{1}{\frac{a_0}{z^n} + \dots + a_n}\right)$$

As $z \to \infty$, then $\frac{1}{f(z)} \to 0$. This means for every $\epsilon$ there exists an $R$ such that $\forall\ |z| > R : \left|\frac{1}{f(z)}\right| < \epsilon$. Now for the $|z| \leqslant R$ we have the following:

$$\left|\frac{1}{f(z)}\right| \leqslant \sup_{w \in \partial B_R(0)} \left|\frac{1}{f(w)}\right| < A \neq \infty.$$

Now since $\frac{1}{f(z)}$ is bounded and entire, by Liouville's theorem we have that $\frac{1}{f(z)}$ is constant. However, this is a contradiction to our initial assumption, hence, we are done. □

**Corollary 2.11: Number of Roots**

As a direct consequence of the previous theorem we have that a polynomial of degree $n$ has $n$ roots when counted with multiplicity. The proof goes by induction, where you first factor out a root and then continuously proceed to factor roots.

## 2.1 Analytic Continuation

**Theorem 2.12: About the sparsity of zeroes.**

Suppose $f : U \to \mathbb{C}$ is holomorphic and $U$ is connected and open, and $f$ vanishes on $\{w_n\} \subset U$ such that $\lim_{n\to\infty} w_n$ exists in $U$. Given that $\{w_n \in U : f(w_n) = 0\}$ has a limit point in $U$, then $f \equiv 0$.

*Proof.* Let $z_0 = \lim_{n\to\infty} w_n \in U$. Since $f$ is holomorphic at $z_0$ we have that $f$ is analytic at $z_0$.

$$\implies f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \quad \forall z \in B_r(z_0)$$

For some $r \in \mathbb{R}_{>0}$. Suppose $f \neq 0$ on $B_r(z_0)$. Then $\exists$ some smallest $m$ such that $a_m \neq 0$,

$$f(z) = a_m (z - z_0)^m (1 + g(z)),$$

where $\lim_{z\to z_0} g(z) = 0$. For $|z - z_0|$ small enough, say $< \delta$, then $|1 + g(z)| > \frac{1}{2}$, but then $f(z) \neq 0$ if $z \in B_\delta(z_0) \setminus \{z_0\}$. However, this is a contradiction based on the assumptions we laid out initially about the sequence $\{w_n\}$. Thus, $f(z) = 0 \;\; \forall z \in B_r(z_0)$.
We are not done: we have to use that $U$ is connected.

**Definition 2.13: Connected Set**

A subset $S \subset \mathbb{C}$ is connected if whenever $S = V_1 \cup V_2$ then $V_1 = \varnothing$ or $V_2 = \varnothing$, where $V_1$ and $V_2$ are open subsets of $S$.

Back to the Proof:
Let $\Omega = \{w_n \in U : f(w_n) = 0\}^-$, i.e. the interior of the set. Now based on our previous work we know that $\exists B_r(z_0) \subset \{z \in U : f(z) = 0\}$, hence the interior is not empty. By definition the interior is open. Now we show that it is closed. Suppose we have a sequence $\{z_n\} \subset \Omega$ such that $\lim_{n\to\infty} z_n = z$. Since (by our previous work) $f$ vanishes in a ball around $z$, we have $z \in \Omega$. Hence, $\Omega$ is closed.

$$\text{So } U = \underbrace{\Omega}_{\text{open}} \cup \underbrace{(U \setminus \Omega)}_{\Omega \text{ closed } \implies \text{ open}}.$$

But by the definition of a connected set it follows that $U = \Omega$. $\square$

**Corollary 2.14**

If $f, g$ are holomorphic on connected $U \subset \mathbb{C}$ and $f(z) = g(z)$ on a convergent sequence, then $f = g$ on all of $U$.

**Definition 2.15: Analytic Continuation**

If $f$ is holomorphic on $U \subset V \subset \mathbb{C}$ and $g$ is holomorphic on $V \subset \mathbb{C}$, and $g = f$ on $U$, then $g$ is an analytic continuation of $f$.

**Example 2.16: Riemann $\zeta$ function**

$$\zeta(s) = \frac{1}{1^s} + \frac{1}{2^s} + \dots$$

We have that $n^s = e^{s\log(n)}$ makes sense for $s \in \mathbb{C}$ (note we are using the principal branch).

$$\left|n^{x+iy}\right| = n^x$$

This implies that the zeta function converges for $\mathrm{Re}(s) > 1$.
Riemann proved that there is an analytic continuation of $\zeta(s)$ to $\mathbb{C} \setminus \{1\}$.

# 3 Analytic Continuation of $\zeta(s)$

**3.1: Proposition**

For $\mathrm{Re}(s) > 0$,

$$\zeta(s) = \frac{s}{s-1} - s\int_1^\infty \frac{\{x\}}{x^{s+1}}dx.$$

# 4 Sequences of Holomorphic Functions

**Theorem 4.1: Morera's Theorem**

If $f$ is continuous on $B_r(z_0) \subset U$ (where $U$ is open) and $\int_T f(z)dz = 0$ for every $T \subset B_r(z_0)$, then $f$ is holomorphic on $B_r(z_0)$. Here $T$ is a triangle.

**Theorem 4.2**

If $\{f_n\}_{n=1}^\infty$ sequence of holomorphic functions converges to $f$ uniformly on compact subsets, then $f$ is holomorphic.

*Proof.* Suffices to show $0 = \int_T f(z)dz$ for all triangles $T \subset U$.

$$\int_T \lim_{n\to\infty} f_n(z)dz = \lim_{n\to\infty}\int_T f_n(z)dz = 0$$

We can do this since we can exchange limits and integrals for uniformly converging sequences of functions. □

**Theorem 4.3**

If a sequence $\{f_n\}_{n=1}^{\infty}$ of holomorphic functions converges to $f$ uniformly on compact subsets, then $f_n'$ converges uniformly to $f'$ on compact subsets.

*Proof.* Let $z \in U$ and $r > 0$ such that $\overline{B_r(z_0)} \subset U$. Now based on Cauchy's Integral formula

$$f_n'(z_0) = \frac{1}{2\pi i} \int_{\partial B_r(z_0)} \frac{f_n(z)}{(z - z_0)^2} dz.$$

So now:

$$\lim_{n\to\infty} f_n'(z_0) = \lim_{n\to\infty} \frac{1}{2\pi i} \int_{\partial B_r(z_0)} \frac{f_n(z)}{(z-z_0)^2} dz = \frac{1}{2\pi i} \int_{\partial B_r(z_0)} \lim_{n\to\infty} \frac{f_n(z_0)}{(z-z_0)^2} dz = \frac{1}{2\pi i} \int_{\partial B_r(z_0)} \frac{f(z)}{(z-z_0)^2} dz.$$

□

## 4.1 Schwartz Reflection Principal

The goal is to establish criterion for analytic continuation in a specific situation. Take a symmetric set $U$ which is symmetric about the real axis.

**Theorem 4.4: Glueing holomorphic functions along a boundary**

If $f^+$ is holomorphic on $U^+$ and $f^-$ is holomorphic on $U^-$ and extend continuously to $I$, then

$$f(z) = \begin{cases} f^+(z) & z \in U^+ \\ f^-(z) & z \in U^- \\ f^+(z) = f^-(z) & z \in I \end{cases}$$

is holomorphic.

*Proof.* Using morera's theorem we can split triangles up into cases, and then we are done. □

**Theorem 4.5: Schwarz reflection principle**

If $f^+$ is holomorphic on $U^+$ and extends continuously to $I$ and $f(z) \in \mathbb{R}$ $\forall z \in I$ then $\exists$ analytic continuation of $f$ to $U$.

# 5 Singularities

**Definition 5.1: Singularity**

An isolated singularity of a function $f$ is $z_0 \in \mathbb{C}$ such that $f$ is defined on $B_r(z_0) \setminus \{z_0\}$ for some $r > 0$.

**Definition 5.2: Zero**

A zero of a function $f$ is a $z_0 \in \mathbb{C}$ such that $f(z_0) = 0$, we say $f$ "vanishes" at $z_0$.

**Theorem 5.3**

If $f$ holomorphic near $z_0$ and $f$ is not identically zero near $z_0$, then $z_0$ is an isolated zero.

**Theorem 5.4**

If $f$ holomorphic near some root $z_0$ and $f$ is not identically 0 near $z_0$, then $z_0$ is an isolated zero of $f$, i.e., $\exists r > 0$ such that $f$ does not vanish at any $z \in B_r(z_0) \setminus \{z_0\}$. This is the contrapositive of analytic continuation.

**Theorem 5.5**

Let $f$ be holomorphic near some root $z_0$, then $\exists n \geqslant 1$, and $g$ holomorphic and nonvanishing near $z_0$ such that $f(z) = (z - z_0)^n g(z) \quad \forall z \in B_r(z_0),$ for some $r > 0$.

*Proof.* Since $f$ is holomorphic near $z_0$, then we have

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k.$$

Let $n$ be the minimal $n$ such that $a_n \neq 0$. Then

$$f(z) = (z - z_0)^n \underbrace{\sum_{k \geqslant n}^{\infty} a_k (z - z_0)^{k-n}}_{g(z)}.$$

By assumption we see the $g(z)$ is non vanishing on $B_r(z_0)$ and still has a radius of convergence. □

**Definition 5.6: Merorphic function**

Suppose $f$ has an isolated singularity at $z_0$. We say $f$ is meromorphic at $z_0$ if $\frac{1}{f}$ is holomorphic near $z_0$.
Another way to say it:
$f$ is mermorphic at isolated singularity $z_0$ if $\exists r > 0$ such that $\frac{1}{f}$ or $f$ extends to holomorphic function on $B_r(z_0)$.
(this is the old definition we used)

**Theorem 5.7**

If $f$ is meromorphic near some pole $z_0$, then $\exists r > 0$ such that $\forall z \in B_r(z_0)$ then

$$f(z) = \frac{h(z)}{(z - z_0)^n} \quad n \geqslant 0$$

where $h(z)$ is holomorphic and non-vanishing on $B_r(z_0)$.

*Proof.* By definition $f$ is holomorphic near $z_0$. By the previous theorem we have that $\frac{1}{f(z)} = (z - z_0)^n g(z)$ near $z_0$.

$$f(z) = \frac{\frac{1}{g(z)}}{(z-z_0)^n} = \frac{h(z)}{(z-z_0)^n}.$$

□

**Definition 5.8: Singularity types**

If $f$ is meromorphic, and we are given some singularity $z_0$, if $n = 0$ then the singularity is removable, if $n > 0$ then the singularity is a pole. If $f$ is not meromorphic then the singularity is essential.

**Theorem 5.9: Laurent Series**

If $f$ is meromorphic of order $n$ at $z_0 \in \mathbb{C}$, then $\exists r > 0$ such that on $B_r(z_0)^* := B_r(z_0) \setminus \{z_0\}$,

$$f(z) = \frac{a_{-n}}{(z-z_0)^n} + \frac{a_{-n+1}}{(z-z_0)^{n-1}} + \cdots + \frac{a_{-1}}{(z-z_0)} + \underbrace{\sum_{n\geqslant 0}^{\infty} a_n (z-z_0)^n}_{G(z) \text{ holomorphic on } B_r(z_0)}$$

*Proof.*

$$\begin{aligned} f(z) &= \frac{h(z)}{(z-z_0)^n} \\ &= \frac{A_0}{(z-z_0)^n} + \frac{A_1}{(z-z_0)^{n-1}} + \cdots + \frac{A_{n-1}}{(z-z_0)} + \sum_{k\geqslant 0}^{\infty} A_{n+k}(z-z_0)^n \end{aligned}$$

The radius of convergence is the same after the substitution, so we have that $f$ is holomorphic on the $B_r(z_0)$. □

**Definition 5.10: Principal Part and Residue**

The principal part of $f$ at $z_0$ is $a_{-1}$. The residue of $f$ at $z_0$ is $a_{-1} = \text{Res}_{z_0}(f)$. It is important to note that a removable singularity has residue 0.

**Theorem 5.11: The Residue Formula**

Suppose $f$ holomorphic in a neighborhood of $\overline{U}$ except for a finite set of isolated singularities, $\{z_i\} \subset U$. Then

$$\int_{\partial U} f(z)dz = 2\pi i \sum_j \text{Res}_{z_j}(f).$$

*Proof.* Apply Cauchy's theorem.

$$\int_{\partial U} f(z)dz = \sum_j \int_{B_\epsilon(z_j)} f(z)dz$$

On a neighborhood near the singularities the Laurent series of $f$ converges uniformly.

$$\begin{aligned}\int_{B_\epsilon(z_j)} f(z)dz &= \int_{B_\epsilon(z_j)} \sum_{k=-n}^{\infty} a_k(z-z_j)^k dz \\ &= \sum_{k=-n}^{\infty} \int_{B_\epsilon(z_j)} a_k(z-z_j)^k dz \\ &= 2\pi i a_{-1} = 2\pi i \mathrm{Res}_{z_j}(f)\end{aligned}$$

It is important to note that the series uniformly converges, hence why we can exchange the sum and integral. There we have,

$$\int_{\partial U} f(z)dz = 2\pi i \sum_j \mathrm{Res}_{z_j}(f).$$

□

**Definition 5.12: Meromorphic Function**

$f$ is meromorphic at a singularity $z_0$ if $\frac{1}{f}$ or $f$ extends to a holomorphic function on $B_r(z_0)$. Meromorphic functions are preserved by addition, multiplication, and division.

**Theorem 5.13**

Let $f$ be holomorphic except at a singularity, and let the limit at that singularity exist. Suppose $f$ is bounded on $U \setminus \{z_0\}$. Then $z_0$ is a removable singularity of $f$.

*Proof.* Idea is the if $f$ extends to $z_0$ as a holomorphic function, then we could have its value by Cauchy's integral formula.
Try extending $g(z)$ on $B_r(z_0)$ by

$$g(z) = \frac{1}{2\pi i}\int_{\partial B_r(z_0)} \frac{f(w)}{w-z_0}dw.$$

We need to show this extension equals the original function and is holomorphic.

$$\begin{aligned}g'(z) &= \frac{d}{dz}\frac{1}{2\pi i}\int_{\partial B_r(z_0)} \frac{f(w)}{(z-z_0)}dw \\ &= \frac{1}{2\pi i}\int_{\partial B_r(z_0)} \frac{f(w)}{(w-z_0)^2}dw\end{aligned}$$

Since the derivative exists $g$ is holomorphic. Since the function on the inside is continuous on the boundary as a function of $z$ we can exchange operators. □

**Corollary 5.14**

Let $f$ be holomorphic in a punctured neighborhood of $z_0$. Then $z_0$ is a pole of $f$ if and only if the limit as the function approaches that point is infinity.

**Theorem 5.15: Casorati - Weierstrass**

Suppose $f$ is holomorphic on punctured $B_r(z_0)$, and has essential singularity at $z_0$. Then $f(B_r(z_0)^\star)$ is dense in $\mathbb{C}$.

*Proof.* Suppose for the sake of contradiction that $f(B_r(z_0)^\star)$ is not dense in $\mathbb{C}$. Then $z_0$ cannot be an essential singularity. We now prove this:
Since $f(B_r(z_0)^\star)$ is not dense, we can find some $w_0 \in \mathbb{C}$ and $\epsilon > 0$ such that $B\epsilon(w_0) \cap f(B_r(z_0)^\star) = \varnothing$. Observe the following

$$|f(z) - w_0| \geqslant \epsilon \implies \frac{1}{|f(z) - w_0|} \leqslant \frac{1}{\epsilon}.$$

$\square$