A cubic curve has the parametric equation:

$$x(t) = \alpha_0 + \alpha_1 t + \alpha_2 t^2 + \alpha_3 t^3$$

$$y(t) = \beta_0 + \beta_1 t + \beta_2 t^2 + \beta_3 t^3$$

If we restrict t to $(0,1)$, we can let the parametric equation define our curve if we set up for points to various different t values.

For instance, if we set $t = 0$ to (x_{st}, y_{st}) and $t = 1$ to (x_{en}, y_{en}). We are sure that the endpoints of the curve are correct. For a complete cubic curve definition, we need four points. The last two points we need we will define as the *derivative* at those end points instead.

We will define the *derivative* or to better call it the *speed* of the endpoints as (u_{st}, v_{st}) and (u_{en}, v_{en}). We prefer the term *speed* as *derivative* lack direction.

$x(t) = \alpha_0 + \alpha_1 t + \alpha_2 t^2 + \alpha_3 t^3$ $x'(t) = \alpha_1 + 2\alpha_2 t + 3\alpha_3 t^2$ $x(0) = \alpha_0 = x_{st}$ $x(1) = \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 = x_{en}$ $x'(0) = \alpha_1 = u_{st}$ $x'(1) = \alpha_1 + 2\alpha_2 + 3\alpha_3 = u_{en}$	$y(t) = \beta_0 + \beta_1 t + \beta_2 t^2 + \beta_3 t^3$ $y'(t) = \beta_1 + 2\beta_2 t + 3\beta_3 t^2$ $y(0) = \beta_0 = y_{st}$ $y(1) = \beta_0 + \beta_1 + \beta_2 + \beta_3 = y_{en}$ $y'(0) = \beta_1 = v_{st}$ $y'(1) = \beta_1 + 2\beta_2 + 3\beta_3 = v_{en}$

Note that we will be interested in equations for the $x - axis$, equally equivalent equations can be easily found on the $y - axis$.

Using $\alpha_0 = x_{st}$ and $\alpha_1 = u_{st}$:

$$\#1: x_{st} + u_{st} + \alpha_2 + \alpha_3 = x_{en}$$

$$\#2: u_{st} + 2\alpha_2 + 3\alpha_3 = u_{en}$$

By $3 * \#1 - \#2$

$$3\, x_{st} + 2u_{st} + \alpha_2 = 3x_{en} - u_{en}$$

$$\alpha_2 = -3x_{st} - 2u_{st} + 3x_{en} - u_{en}$$

By $2 * \#1 - \#2$

$$2\, x_{st} + u_{st} - \alpha_3 = 2x_{en} - u_{en}$$

$$\alpha_3 = 2x_{st} + u_{st} - 2x_{en} + u_{en}$$

By matrices we get:

$MA = V$ $M^{-1}MA = M^{-1}V$ $A = M^{-1}V$	$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} x_{st} \\ u_{st} \\ x_{en} \\ u_{en} \end{bmatrix}$ $\begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -3 & -2 & 3 & -1 \\ 2 & 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} x_{st} \\ u_{st} \\ x_{en} \\ u_{en} \end{bmatrix}$

The downside to the previous method is that (u_{st}, v_{st}) and (u_{en}, v_{en}) are not representative of a solid concept in a computer graphics setting, it would be better to define the curve using four points, instead of two points and two speeds.

We will start by renaming the starting point and the ending point to (x_0, y_0) and (x_3, y_3) respectively, and define two new points (x_1, y_1) and (x_2, y_2).

We will set the four points as key points when $t = 0$, $t = \frac{1}{3}$, $t = \frac{2}{3}$, and $t = 1$ respectively.

$$u_{st} = \frac{dx}{dt}_{st} \cong \frac{x_1 - x_0}{t_1 - t_0} = \frac{x_1 - x_0}{\frac{1}{3} - 0} = 3(x_1 - x_0)$$

$$v_{st} = 3(y_1 - y_0)$$

$$u_{en} = \frac{dy}{dt}_{en} \cong \frac{x_3 - x_2}{1 - \frac{2}{3}} = 3(x_2 - x_3)$$

$$v_{en} = 3(y_2 - y_3)$$