DrawCircle1

The equation of a circle residing at point (x_{st}, y_{st}) with radius r is $(x - x_{st})^2 + (y - y_{st})^2 = r^2$

Using $\sin^2\theta + \cos^2\theta = 1$. We get:

$r^2\cos^2\theta + r^2\sin^2\theta = r^2$

As such:

$x = x_{st} + r\cos\theta$

$y = y_{st} + r\sin\theta$

Where $0 \le \theta < 2\pi$. Simply put, we will loop over all possible values of θ in the range, and calculate the corresponding x and y values and draw at this point. However, $\theta \in R$, and as such, we need some sort of $\Delta\theta$. What is the optimal value for $\Delta\theta$?

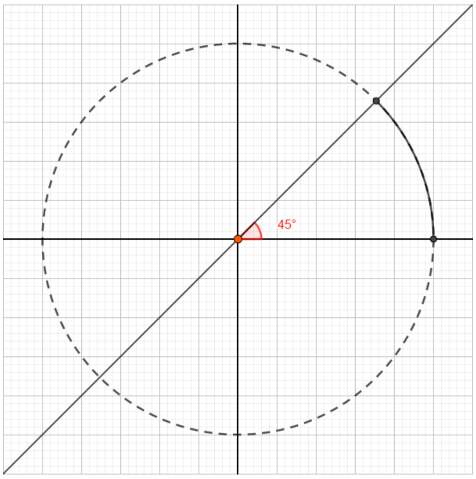
$$\theta_{n+1} = \theta_n + \Delta\theta$$

For now, we set $\Delta\theta = \frac{1}{r}$, we will prove this later.

From this point on, we will consider the center point of the circle at $(0, 0)$ We can simply translate the point before drawing.

Optimizing loop

A circle is highly symmetric, so instead of finding all points from $0 \le \theta < 2\pi$, we will only be concerned with $0 \le \theta < \frac{\pi}{4}$ and use the cos and sin value of this to interpolate all other points:



Reflect around	x comp	y comp
$-$	x	y
$y - axis$	$-x$	y
$x - axis$	x	$-y$
$x - axis\ \&\ y - axis$	$-x$	$-y$
$x = y$	y	x
$x = y\ \&\ y - axis$	$-y$	x
$x = y\ \&\ x - axis$	y	$-x$
$x = y\ \&\ x - axis\ \&\ y - axis$	$-y$	$-x$

Simply put, for each x and y we calculate (using cos and sin or any other method), we will SetPixel the following:

```
SetPixel(hdc, xc + x, xc + y, color);
SetPixel(hdc, xc - x, xc + y, color);
SetPixel(hdc, xc + x, xc - y, color);
SetPixel(hdc, xc - x, xc - y, color);
SetPixel(hdc, xc + y, xc + x, color);
SetPixel(hdc, xc - y, xc + x, color);
SetPixel(hdc, xc + y, xc - x, color);
SetPixel(hdc, xc - y, xc - x, color);
```

DrawCircle2

As a revision:

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$\sin(\alpha + \beta) = \cos\alpha\sin\beta + \sin\alpha\cos\beta$$

x_{n+1}	y_{n+1}
$x_{n+1} = r\cos\theta_{n+1}$ $x_{n+1} = r\cos(\theta_n + \Delta\theta)$ $x_{n+1} = r\cos\theta_n\cos\Delta\theta - r\sin\theta_n\sin\Delta\theta$ $x_{n+1} = x\cos\Delta\theta - y\sin\Delta\theta$ $x_0 = r$	$y_{n+1} = r\sin\theta_{n+1}$ $y_{n+1} = r\sin(\theta_n + \Delta\theta)$ $y_{n+1} = r\cos\theta_n\sin\Delta\theta + r\sin\theta_n\cos\Delta\theta$ $y_{n+1} = x\sin\Delta\theta + y\cos\Delta\theta$ $y_0 = 0$

Optimal step:

For a smooth circle, we want the distance between two consecutive points to be at most one pixel.

$$(x_{n+1} - x_n)^2 + (y_{n+1} - y_n)^2 \leq 1$$

$$(r\cos\theta_{n+1} - r\cos\theta_n)^2 + (r\sin\theta_{n+1} - r\sin\theta_n)^2 \leq 1$$

$$(\cos\theta_{n+1} - \cos\theta_n)^2 + (\sin\theta_{n+1} - \sin\theta_n)^2 \leq {}^{1}/_{r^2}$$

$$\cos^2\theta_{n+1} + \cos^2\theta_n - 2\cos\theta_{n+1}\cos\theta_n + \sin^2\theta_{n+1} + \sin^2\theta_n - 2\sin\theta_{n+1}\sin\theta_n \leq {}^{1}/_{r^2}$$

$$2 - 2\cos\theta_{n+1}\cos\theta_n - 2\sin\theta_{n+1}\sin\theta_n \leq {}^{1}/_{r^2}$$

$$2 - 2\cos(\theta_{n+1} - \theta_n) \leq {}^{1}/_{r^2}$$

$$2 - 2\cos(\Delta\theta) \leq {}^{1}/_{r^2}$$

$$1 - {}^{1}/_{2r^2} \leq \cos(\Delta\theta)$$

Using Taylor series: $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$

We will use the approximation $\cos x \cong 1 - \frac{x^2}{2}$

$$1 - {}^{1}/_{2r^2} \leq 1 - \frac{(\Delta\theta)^2}{2}$$

$$(\Delta\theta)^2 \leq {}^{1}/_{r^2}$$

$\Delta\theta \leq {}^{1}/_{r}$

We will use the upper limit of this equation as a value of $\Delta\theta$.

DrawCircle3

As a revision, to figure out if a point (x, y) is inside a circle residing at $(0,0)$, we calculate its distance from the center point. If this distance is greater than r, the point is outside the circle, otherwise it is inside the circle. To skip the root, we can compare the square of the distance with the square of the radius, or in another word: $f(x, y) = x^2 + y^2 - r^2$

If $f > 0$, point is outside the circle, otherwise it is inside the circle.

Out of all 8 arcs, we are only concerned with calculating x and y values for the first arc (highlighted in the earlier figure). Observe that starting from $(r, 0)$ towards $(r\cos\frac{\pi}{4}, r\sin\frac{\pi}{4})$, y always increases, while x sometimes decreases. The arc stops at the point where $x = y$. We can use the mid-point method to estimate the value of x.

For a point (x_n, y_n) that we have drawn, the next point is either going to be $(x_n, y_n + 1)$ or $(x_n - 1, y_n + 1)$. The midpoint is $\left(x_n - \frac{1}{2}, y_n + 1\right)$, if this point is inside the circle, then the outer point $(x_n, y_n + 1)$ is closer, otherwise $(x_n - 1, y_n + 1)$ is closer.

$$d = \left(x - \frac{1}{2}\right)^2 + (y + 1)^2 - r^2$$

$$d = x^2 - x + \frac{1}{4} + y^2 + 2y + 1 - r^2$$

$$d = x^2 + y^2 - x + 2y - r^2 + \frac{5}{4}$$

$$d' = 4x^2 + 4y^2 - 4x + 8y - 4r^2 + 5$$

Similar to previous procedures, we will find Δd:

$$d_0 = 4r^2 - 4r - 4r^2 + 5 = 5 - 4r$$

$d_n - d_{n-1} = (4x_n^2 + 4y_n^2 - 4x + 8y_n - 4r^2 + 5) - (4x_{n-1}^2 + 4y_{n-1}^2 - 4x + 8y_{n-1} - 4r^2 + 5)$

$= (4x_n^2 + 4y_n^2 - 4x_n + 8y_n) - (4x_{n-1}^2 + 4y_{n-1}^2 - 4x_{n-1} + 8y_{n-1})$

$= 4\left((x_n^2 + y_n^2 - x_n + 2y_n) - (x_{n-1}^2 + y_{n-1}^2 - x_{n-1} + 2y_{n-1})\right)$

$= 4\left((x_n^2 - x_{n-1}^2) + (y_n^2 - y_{n-1}^2) - (x_n - x_{n-1}) + 2(y_n - y_{n-1})\right)$

$= 4\big((x_n - x_{n-1})(x_n + x_{n-1}) + (y_n - y_{n-1})(y_n + y_{n-1}) - (x_n - x_{n-1}) + 2(y_n - y_{n-1})\big)$

$= 4(\Delta x(2x_{n-1} + \Delta x) + (2y_{n-1} + 1) - \Delta x + 2)$

If Δx is 0

$\Delta d = 4\big((2y_{n-1} + 1) + 2\big) = 8y_{n-1} + 12$

If Δx is -1

$\Delta d = 4(-(2x_{n-1} - 1) + (2y_{n-1} + 1) - (-1) + 2)$

$= -8x_{n-1} + 4 + 8y_{n-1} + 4 + 4 + 8$

$= -8x_{n-1} + 8y_{n-1} + 20$

In another word, always add $8y_{n-1} + 12$, and add $-8x_{n-1} + 8$ if the condition is true