DrawLine

We will loop over one axis and draw the corresponding points on the other axis. We loop over the longer axis.

Formally: define $dx = x_1 - x_0, dy = y_1 - y_0$. Loop over y if $|dy| > |dx|$ otherwise loop over x

We will always assume that if we loop over x then $x_{st} < x_{en}$ and if we loop over y then $y_{st} < y_{en}$. This simplifies the equations and can be forced simply by swapping the two points whenever the condition is violated.

Line equation:

$$(y - y_0) = m(x - x_0)$$

Slope equation:

$$m = \frac{y_1 - y_0}{x_1 - x_0}$$

DrawLine v1:

From Line equation:

$y = m\,(x - x_0) + y_0$

$x = m^{-1}(y - y_0) + x_0$

Loop over x	Loop over y
$\boldsymbol{x_n = x_{n-1} + 1}$ $\boldsymbol{y_n = m\,(x_n - x_0) + y_0}$	$y_n = y_{n-1} + 1$ $x = m^{-1}(y - y_0) + x_0$

DrawLine v2:

Given:

#1	$y_n = m\,(x_n - x_0) + y_0$
#2	$x_n = x_{n-1} + 1$

$y_n - y_{n-1} = m\,(x_n - x_0) + y_0 - (m\,(x_{n-1} - x_0) + y_0)$	Using #1
$y_n - y_{n-1} = m\,x_n - m\,x_{n-1}$	By Cancelation
$y_n - y_{n-1} = m\,(x_{n-1} + 1) - m\,x_{n-1}$	Using #2
$y_n - y_{n-1} = m$	By Cancelation
$y_n = y_{n-1} + m$	Reordering

Using the similar reasoning for x, we arrive at $x_n = x_{n-1} + m^{-1}$

Loop over x	Loop over y
$\boldsymbol{x_n = x_{n-1} + 1}$ $\boldsymbol{y_n = y_{n-1} + m}$	$y_n = y_{n-1} + 1$ $x_n = x_{n-1} + m^{-1}$

DrawLine v3:

Notice that during this analysis we assume we are looping over x, the proof for looping over y is very similar but not written extensively here, however the results are written in the tables.

Subproblem: you have point (x, y) and you need to deduce whether the point is above/below the line and whether it is to the right/left of the line

$y = m\,(x - x_0) + y_0$	Line equation
$y = mx - mx_0 + y_0$	Expansion
$y = \frac{\Delta y}{\Delta x} x - \frac{\Delta y}{\Delta x} x_0 + y_0$	Substituting m using slope equation
$\Delta x \cdot y = \Delta y \cdot x - \Delta y \cdot x_0 + \Delta x \cdot y_0$	Multiplying by Δx
$\Delta y \cdot x - \Delta x \cdot y + (\Delta x \cdot y_0 - \Delta y \cdot x_0) = 0$	Reordering

This is now in the form $ax + by + c = 0$, where $a = \Delta y,\ b = -\Delta x$, and $c = (\Delta x \cdot y_0 - \Delta y \cdot x_0)$.

$f(x, y) = \Delta y \cdot x - \Delta x \cdot y + (\Delta x \cdot y_0 - \Delta y \cdot x_0)$

$f(x, y) = a\,x + by + c$

So, if we set $f(x, y) = ax + by + c$ and plug in a value x_q, y_q we can deduce the relation between the point and the line as follows:

$f(x, y)$ and b are the same sign iff (x, y) is above the line, otherwise it is below the line.

Similarly, $f(x, y)$ and a are the same sign iff (x, y) is to the right of the line, otherwise it is to the left of the line.

If f and b are the same sign, it means that increasing y increases $|f|$, so the point becomes farther from the line as it moves upwards, therefore it is above the line in the first place.

If f and b are of a different sign, it means that decreasing y increases $|f|$, so the point becomes farther from the line as it moves downwards, therefore it is below the line in the first place.

Similar reasoning can be deduced on f and a for the rightwards and leftwards relation between the point and the line.

Let's assume we loop over x and $y_{st} < y_{en}$

If we drew a point (x, y) the next point we draw should either be $(x + 1, y)$ or $(x + 1, y + 1)$. We will draw the point that is closer to the line. To know which point is closer, we will check the midpoint between them, if the midpoint is above the line, the first is closer, if the midpoint is below the line, the second is closer. The midpoint is $\left(x + 1, y + \frac{1}{2}\right)$, we will use $f(x, y)$ as we demonstrated to check the relation between the point and the line.

Since we only care about the sign of $f(x, y)$, we can multiply it by 2 so we don't have to use a double to store the midpoint.

Set $c = \ 2(\Delta x \cdot y_0 - \Delta y \cdot x_0)$

$$f(x, y) = 2\Delta y \cdot x - 2\,\Delta x \cdot y + c$$

So, set $d_n = 2\Delta y \cdot (x_{n-1} + 1) - 2\,\Delta x \cdot (y_{n-1} + 0.5) + c$

$d_n = 2\Delta y \cdot (x_{n-1} + 1) - \Delta x \cdot (2\,y_{n-1} + 1) + c$

To know if d_n and b are the same sign we check whether $d_n * b$ is > 0 or not. $b = -\Delta x$ so we check $-d_n \Delta x$

We will simply set:

$$x_n = x_{n-1} + 1$$

$$y_n = \begin{cases} y_{n-1} & -d_n \Delta x \geq 0 \\ y_{n-1} + 1 & -d_n \Delta x < 0 \end{cases}$$

To remove the constraint $y_{st} < y_{en}$. We assign:

$$diff = \begin{cases} 1 & y_{st} \leq y_{en} \\ -1 & y_{st} > y_{en} \end{cases}$$

the next point we draw should either be $(x + 1, y)$ or $(x + 1, y + diff)$. To know which point is closer, we will check the midpoint between them, midpoint being $\left(x + 1, y + \frac{diff}{2}\right)$.

This changes the equation of d_n to:

$d_n = 2\Delta y \cdot (x_{n-1} + 1) - \Delta x \cdot (2\,y_{n-1} + diff) + c$

However, which point is closer now also depends on $diff$. If diff is positive, it is as stated before, if it is negative, then the opposite is true: the second point is closer if the midpoint is above the line. In another word, the equation we set for y_n need to be reversed if diff is negative. To do that, we simply multiply the condition by diff so the sign is flipped if it needs to be flipped:

$$x_n = x_{n-1} + 1$$

$$y_n = \begin{cases} y_{n-1} & -d_n \Delta x\, diff \geq 0 \\ y_{n-1} + diff & -d_n \Delta x\, diff < 0 \end{cases}$$

The exact same procedure will be done for looping over y:

$$diff = \begin{cases} 1 & x_{st} \leq x_{en} \\ -1 & x_{st} > x_{en} \end{cases}$$

$$y_n = y_{n-1} + 1$$

$$x_n = \begin{cases} x_{n-1} & d_n \Delta y\, diff \geq 0 \\ x_{n-1} + diff & d_n \Delta y\, diff < 0 \end{cases}$$

Notice that the condition doesn't have a negative sign because we compare a instead of b, and a is just Δy not $-\Delta x$.

Loop over x	Loop over y
$\boldsymbol{diff = \begin{cases} 1 & y_{st} \leq y_{en} \\ -1 & y_{st} > y_{en} \end{cases}}$ $\boldsymbol{x_n = x_{n-1} + 1}$ $\boldsymbol{y_n = \begin{cases} y_{n-1} & -d_n \Delta x\, diff \geq 0 \\ y_{n-1} + diff & -d_n \Delta x\, diff < 0 \end{cases}}$	$diff = \begin{cases} 1 & x_{st} \leq x_{en} \\ -1 & x_{st} > x_{en} \end{cases}$ $y_n = y_{n-1} + 1$ $x_n = \begin{cases} x_{n-1} & d_n \Delta y\, diff \geq 0 \\ x_{n-1} + diff & d_n \Delta y\, diff < 0 \end{cases}$

Our final optimization is as follows: instead of calculating d_n and each step, we will calculate it from the previous step as follows: $d_n = d_{n-1} + \Delta d$

$$\Delta d = d_n - d_{n-1} =$$

$$2\Delta y \cdot (x_{n-1} + 1) - \Delta x \cdot (2\,y_{n-1} + 1) + c - (2\Delta y \cdot (x_{n-2} + 1) - \Delta x \cdot (2\,y_{n-2} + 1) + c)$$

$$2\Delta y \cdot \big(x_{n-1} + 1 - (x_{n-2} + 1)\big) - \Delta x \cdot \big(2\,y_{n-1} + 1 - (2\,y_{n-2} + 1)\big)$$

$$2\Delta y \cdot (x_{n-1} - x_{n-2}) - \Delta x \cdot (2\,y_{n-1} - 2\,y_{n-2})$$

$$2\Delta y \cdot (x_{n-1} - x_{n-2}) - 2\,\Delta x \cdot (y_{n-1} - y_{n-2})$$

Recall that $(x_{n-1} - x_{n-2})$ and $(y_{n-1} - y_{n-2})$ is the difference between the current step and the previous step. In our analysis, we loop over x, so $(x_{n-1} - x_{n-2}) = 1$ while $(y_{n-1} - y_{n-2})$ is either 0 or 1.

$$\boldsymbol{\Delta d = 2\Delta y - 2\Delta x \cdot (y_{n-1} - y_{n-2})}$$

At the start, we need to find d_0 in the initial step:

$$d_0 = 2\Delta y \cdot (x_0 + 1) - \Delta x \cdot (2\,y_0 + diff) + c$$

$$d_0 = 2\Delta y \cdot (x_0 + 1) - \Delta x \cdot (2\,y_0 + diff) + 2(\Delta x \cdot y_0 - \Delta y \cdot x_0)$$

$$d_0 = 2\Delta y \cdot x_0 + 2\Delta y - \Delta x \cdot 2\,y_0 - \Delta x \cdot diff + 2\Delta x \cdot y_0 - 2\Delta y \cdot x_0$$

$$d_0 = 2\Delta y - \Delta x \cdot diff$$

All the above reasoning can be done for looping over y too, doing both will result in the following formulas:

Loop over x		Loop over y	
Using $\boldsymbol{x_n = x_{n-1} + 1}$ $\boldsymbol{y_n = \begin{cases} y_{n-1} & -d_n \Delta x\, diff \geq 0 \\ y_{n-1} + diff & -d_n \Delta x\, diff < 0 \end{cases}}$ **We get:** $\Delta d = 2\Delta y - 2\Delta x \cdot (y_{n-1} - y_{n-2})$		**Using** $y_n = y_{n-1} + 1$ $x_n = \begin{cases} x_{n-1} & d_n \Delta y\, diff \geq 0 \\ x_{n-1} + diff & d_n \Delta y\, diff < 0 \end{cases}$ **We get:** $\Delta d = 2\Delta y (x_{n-1} - x_{n-2}) - 2\Delta x$	
If diff is added	**If diff is not added**	**If diff is added**	**If diff is not added**
$2\Delta y - 2\Delta x \cdot diff$	$2\Delta y$	$-2\Delta x$	$2\Delta y \cdot diff - 2\Delta x$