Comparison of different types of controllers for stabilizing Double Pendulum System

I. Introduction

This report presents a comparison of the time response between two different types of controller for stabilizing the double pendulum system hanged from a cart during transferring it from one position to another in x direction. This problem is very important because if a crane that transfers a cargo from the port to the ship is considered, the transfer task should be done with minimum oscillations and minimum energy consumption. In this report different types of controller are used which are state feedback controller with pole placement and state feedback controller with linear quadratic regulator (LQR). All approaches are simulated using MATLAB, Simulink and Simscape.

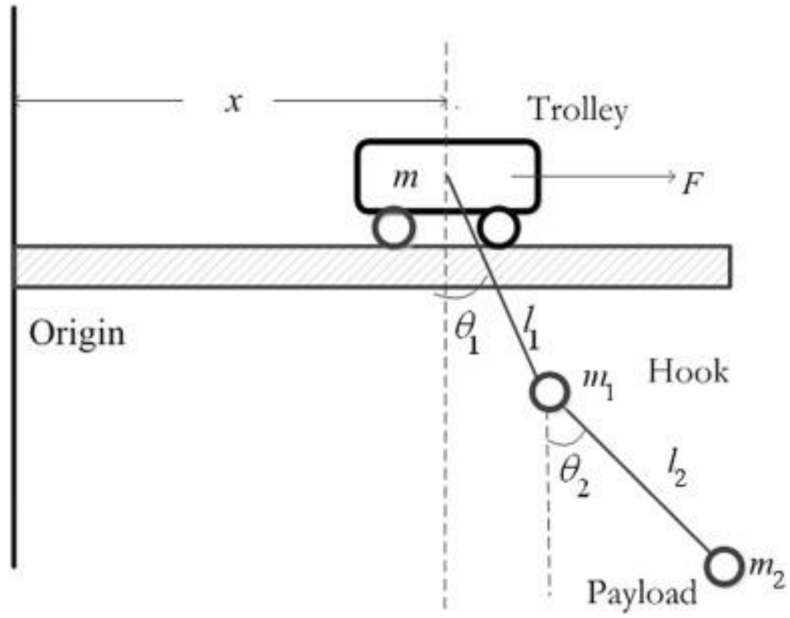
Pole Placement is the basic design approach for systems described in state-space form. Its job is to reallocate the poles of the plant using the state feedback scheme to more desirable locations. The poles characterize the response of the system. For example, reallocation the poles of the system to the left hand side of the S plane guarantees the stability of the system.

Another interesting approach is the Linear Quadratic Regulator (LQR). This approach uses the state feedback scheme and determines the desired closed-loop poles such that it balances between the acceptable response and the amount of control energy required by minimizing a standard cost function.

All mentioned control schemes works on linear systems. Hence, we have to linearize the nonlinear model of the double pendulum on a cart system.

All files used can be found on the link https://goo.gl/14anCx

II. Mathematical Model



In this section, the model of the double pendulum is established and the derivations of dynamical equations of the system are shown. The above figure shows the schematic drawing of the double pendulum hanged from the cart. Certain assumptions are made that the initial conditions are zero so that the system will start in a state of equilibrium and the pendulum will move only a few degrees away from its initial state to meet the requirement of a linear model.

Symbol	Parameter	Value	Unit
m0	Mass of cart	0.8	kg
m1	Mass of 1st pendulum	0.5	kg
m2	Mass of 2nd pendulum	0.3	kg
b	Friction on cart	0.1	N/m/sec
l1	Length of the 1st pendulum	0.3	m
l2	Length of the 2nd pendulum	0.2	m
J	Inertia of the pendulum	0.006	kg.m^2
g	Center of gravity	9.8	m/s^2

Assume that the parameters of double pendulum system be as shown in the above table.

To derive its equations of motion, one of the possible ways is to use Lagrange equations, which are given as follows.

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0\,; \qquad i = 0,1,2, \ldots .n \qquad (1)$$

where L = T - V is a Lagrangian, Q is a vector of generalized forces acting in the direction of generalized coordinates q. T is the kinetic energy, and V is the potential energy.

The kinetic energy of the system is composed of the cart, the first pendulum and the second pendulum kinetic energies. The cart kinetic energy is:

$$T_0 = \frac{1}{2} m_0 \dot{x}^2 \qquad (2)$$

The first pendulum kinetic energy is equal to the sum of three horizontal, vertical and rotational energy of pendulum.

$$T_1 = \frac{1}{2} m_1 \left(\frac{d}{dt} (x + l_1 \sin \theta_1) \right)^2 + \frac{1}{2} m_1 \left(\frac{d}{dt} (l_1 \cos \theta_1) \right)^2 + \frac{1}{2} J \dot{\theta}_1{}^2$$

$$T_1 = \frac{1}{2} m_1 \left[\dot{x} + l_1 \dot{\theta}_1 \cos \theta_1 \right]^2 + \frac{1}{2} m_1 l_1{}^2 \dot{\theta}_1{}^2 \sin^2 \theta_1 + \frac{1}{2} J \dot{\theta}_1{}^2 \qquad (3)$$

The kinetic energy of the second pendulum is also identical to the first pendulum.

$$T_2 = \frac{1}{2} m_2 \left[\dot{x} + l_1 \dot{\theta}_1 \cos \theta_1 + l_2 \dot{\theta}_2 \cos \theta_2 \right]^2 + \frac{1}{2} m_2 \left[l_1 \dot{\theta}_1 \sin \theta_1 + l_2 \dot{\theta}_2 \sin \theta_2 \right]^2 + \frac{1}{2} J \dot{\theta}_2{}^2 \qquad (4)$$

The system kinetic energy is gained from the sum of three equations.

$$T = T_0 + T_1 + T_2$$

$$T = \frac{1}{2} m_0 \dot{x}^2 + \frac{1}{2} m_1 \left[\dot{x} + l_1 \dot{\theta}_1 \cos \theta_1 \right]^2 + \frac{1}{2} m_1 l_1{}^2 \dot{\theta}_1{}^2 \sin^2 \theta_1 + \frac{1}{2} J \dot{\theta}_1{}^2 + \frac{1}{2} m_2 \left[\dot{x} + l_1 \dot{\theta}_1 \cos \theta_1 + l_2 \dot{\theta}_2 \cos \theta_2 \right]^2 + \frac{1}{2} m_2 \left[l_1 \dot{\theta}_1 \sin \theta_1 + l_2 \dot{\theta}_2 \sin \theta_2 \right]^2 + \frac{1}{2} J \dot{\theta}_2{}^2$$

$$T = \frac{1}{2} m_0 \dot{x}^2 + \frac{1}{2} m_1 \dot{x}^2 + \frac{1}{2} m_1 l_1{}^2 \dot{\theta}_1{}^2 \cos^2 \theta_1 + m_1 l_1 \dot{x} \dot{\theta}_1 \cos \theta_1 + \frac{1}{2} m_1 l_1{}^2 \dot{\theta}_1{}^2 \sin^2 \theta_1 + \frac{1}{2} J \dot{\theta}_1{}^2 + \frac{1}{2} m_2 \dot{x}^2 + \frac{1}{2} m_2 l_1{}^2 \dot{\theta}_1{}^2 \cos^2 \theta_1 + \frac{1}{2} m_2 l_2{}^2 \dot{\theta}_2{}^2 \cos^2 \theta_2 + m_2 l_1 \dot{x} \dot{\theta}_1 \cos \theta_1 + m_2 l_2 \dot{x} \dot{\theta}_2 \cos \theta_2 + m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos \theta_1 \cos \theta_2 + \frac{1}{2} m_2 l_1{}^2 \dot{\theta}_1{}^2 \sin^2 \theta_1 + \frac{1}{2} m_2 l_2{}^2 \dot{\theta}_2{}^2 \sin^2 \theta_2 + m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin \theta_1 \sin \theta_2 + \frac{1}{2} J \dot{\theta}_2{}^2$$

Then:

$$T = \frac{1}{2}m_0\dot{x}^2 + \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_1{l_1}^2{\dot{\theta}_1}^2(\cos^2\theta_1 + \sin^2\theta_1)$$
$$+ m_1 l_1\dot{x}\dot{\theta}_1\cos\theta_1 + \frac{1}{2}J{\dot{\theta}_1}^2 + \frac{1}{2}m_2\dot{x}^2$$
$$+ \frac{1}{2}m_2{l_1}^2{\dot{\theta}_1}^2(\cos^2\theta_1 + \sin^2\theta_1)$$
$$+ \frac{1}{2}m_2{l_2}^2{\dot{\theta}_2}^2(\cos^2\theta_2 + \sin^2\theta_2) + m_2 l_1\dot{x}\dot{\theta}_1\cos\theta_1$$
$$+ m_2 l_2\dot{x}\dot{\theta}_2\cos\theta_2$$
$$+ m_2 l_1 l_2\dot{\theta}_1\dot{\theta}_2(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2)$$
$$+ \frac{1}{2}J{\dot{\theta}_2}^2 \qquad (5)$$

By simplification of the relationship (5) we have:

$$T = \frac{1}{2}(m_0 + m_1 + m_2)\dot{x}^2 + \frac{1}{2}\left(m_1{l_1}^2 + m_2{l_1}^2 + J\right){\dot{\theta}_1}^2$$
$$+ \frac{1}{2}\left(m_2{l_2}^2 + J\right){\dot{\theta}_2}^2 + (m_1 l_1 + m_2 l_1)\dot{x}\dot{\theta}_1\cos\theta_1$$
$$+ m_2 l_2\dot{x}\dot{\theta}_2\cos\theta_2 + m_2 l_1 l_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2) \qquad (6)$$

Now we calculate the potential energy of the system separately for three parts of the system. The cart potential energy is zero.

$$V_0 = 0 \qquad (7)$$

The potential energy of the two pendulums will be as the following respectively:

$$V_1 = -m_1 g l_1\cos\theta_1 \qquad (8)$$
$$V_2 = -m_2 g(l_1\cos\theta_1 + l_2\cos\theta_2) \qquad (9)$$

The system potential energy is gained from the sum of three (7), (8) and (9) equations.

$$V = V_0 + V_1 + V_2$$
$$V = -(m_1 l_1 + m_2 l_1)g\cos\theta_1 - m_2 l_2 g\cos\theta_2 \qquad (10)$$

Putting the (6) and the (10) equations in the Lagrange equation we have:

$$L = \frac{1}{2}(m_0 + m_1 + m_2)\dot{x}^2 + \frac{1}{2}\left(m_1{l_1}^2 + m_2{l_1}^2 + J\right){\dot{\theta}_1}^2$$
$$+ \frac{1}{2}\left(m_2{l_2}^2 + J\right){\dot{\theta}_2}^2 + (m_1 l_1 + m_2 l_1)\dot{x}\dot{\theta}_1\cos\theta_1$$
$$+ m_2 l_2\dot{x}\dot{\theta}_2\cos\theta_2 + m_2 l_1 l_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2)$$
$$+ (m_1 l_1 + m_2 l_1)g\cos\theta_1 + m_2 l_2 g\cos\theta_2 \qquad (11)$$

Differentiating the Lagrangian by q and q yields Lagrange equation (1) as:

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = u - b\dot{x}\,; \qquad (12)$$

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) - \frac{\partial L}{\partial \theta_1} = 0\ ; \qquad (13)$$

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) - \frac{\partial L}{\partial \theta_2} = 0\ ; \qquad (14)$$

According to the equation (12), there is an external force u and a friction force only in X direction. Now, we apply the equation (12) on the equation (11).

$$\begin{aligned}(m_0 + m_1 + m_2)\ddot{x} + (m_1 l_1 + m_2 l_1)\ddot{\theta}_1 \cos\theta_1 \\ - (m_1 l_1 + m_2 l_1){\dot{\theta}_1}^2 \sin\theta_1 + m_2 l_2 \ddot{\theta}_2 \cos\theta_2 \\ - m_2 l_2 {\dot{\theta}_2}^2 \sin\theta_2 = u - b\dot{x} \qquad (15)\end{aligned}$$

Then we apply the equation (13) on the equation (11).

$$\begin{aligned}\left(m_1 {l_1}^2 + m_2 {l_1}^2 + J\right)\ddot{\theta}_1 + (m_1 l_1 + m_2 l_1)\ddot{x}\cos\theta_1 \\ - (m_1 l_1 + m_2 l_1)\dot{x}\dot{\theta}_1 \sin\theta_1 + m_2 l_1 l_2 \ddot{\theta}_2 \cos(\theta_1 - \theta_2) \\ - m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) + m_2 l_1 l_2 {\dot{\theta}_2}^2 \sin(\theta_1 - \theta_2) \\ - \big(-(m_1 l_1 + m_2 l_1)\dot{x}\dot{\theta}_1 \sin\theta_1 - m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) \\ - (m_1 l_1 + m_2 l_1) g \sin\theta_1\big) = 0\end{aligned}$$

$$\begin{aligned}\left(m_1 {l_1}^2 + m_2 {l_1}^2 + J\right)\ddot{\theta}_1 + (m_1 l_1 + m_2 l_1)\ddot{x}\cos\theta_1 \\ + m_2 l_1 l_2 \ddot{\theta}_2 \cos(\theta_1 - \theta_2) + m_2 l_1 l_2 {\dot{\theta}_2}^2 \sin(\theta_1 - \theta_2) \\ + (m_1 l_1 + m_2 l_1) g \sin\theta_1 = 0 \qquad (16)\end{aligned}$$

Then we apply the equation (14) on the equation (11).

$$\begin{aligned}\left(m_2 {l_2}^2 + J\right)\ddot{\theta}_2 + m_2 l_2 \ddot{x}\cos\theta_2 - m_2 l_2 \dot{x}\dot{\theta}_2 \sin\theta_2 \\ + m_2 l_1 l_2 \ddot{\theta}_1 \cos(\theta_1 - \theta_2) - m_2 l_1 l_2 {\dot{\theta}_1}^2 \sin(\theta_1 - \theta_2) \\ + m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) \\ - \big(-m_2 l_2 \dot{x}\dot{\theta}_2 \sin\theta_2 + m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) \\ - m_2 l_2 g \sin\theta_2\big) = 0\end{aligned}$$

$$\begin{aligned}\left(m_2 {l_2}^2 + J\right)\ddot{\theta}_2 + m_2 l_2 \ddot{x}\cos\theta_2 + m_2 l_1 l_2 \ddot{\theta}_1 \cos(\theta_1 - \theta_2) \\ - m_2 l_1 l_2 {\dot{\theta}_1}^2 \sin(\theta_1 - \theta_2) + m_2 l_2 g \sin\theta_2 = 0 \qquad (17)\end{aligned}$$

Let we assume that

$$a_0 = m_0 + m_1 + m_2$$

$$a_1 = m_1 l_1 + m_2 l_1$$

$$a_2 = m_1 {l_1}^2 + m_2 {l_1}^2 + J$$

$$a_3 = m_2 l_2$$

$$a_4 = m_2 l_1 l_2$$

$$a_5 = m_2 {l_2}^2 + J$$

Then we get

$$a_0\ddot{x} + a_1\ddot{\theta}_1 \cos\theta_1 - a_1\dot{\theta}_1^{\;2} \sin\theta_1 + a_3\ddot{\theta}_2 \cos\theta_2 - a_3\dot{\theta}_2^{\;2} \sin\theta_2 = u - b\dot{x} \qquad (15)$$

$$a_2\ddot{\theta}_1 + a_1\ddot{x}\cos\theta_1 + a_4\ddot{\theta}_2 \cos(\theta_1 - \theta_2) + a_4\dot{\theta}_2^{\;2} \sin(\theta_1 - \theta_2) + a_1 g \sin\theta_1 = 0 \qquad (16)$$

$$a_5\ddot{\theta}_2 + a_3\ddot{x}\cos\theta_2 + a_4\ddot{\theta}_1 \cos(\theta_1 - \theta_2) - a_4\dot{\theta}_1^{\;2} \sin(\theta_1 - \theta_2) + a_3 g \sin\theta_2 = 0 \qquad (17)$$

The controller can only work with the linear function so these set of equation should be linearized about $\theta_1 = \theta_2 = 0$. Then we get $\sin\theta_1 = \varphi_1$, $\sin\theta_2 = \varphi_2$, $\sin(\theta_1 - \theta_2) = \varphi_1 - \varphi_2$, $\cos\theta_1 = 1$, $\cos\theta_2 = 1$, $\cos(\theta_1 - \theta_2) = 1$ and $\dot{\theta}_1 = \dot{\theta}_2 = 0$. The linearized version of equations (15), (16) and (17) is

$$a_0\ddot{x} + a_1\ddot{\theta}_1 + a_3\ddot{\theta}_2 = u - b\dot{x}$$

$$a_1\ddot{x} + a_2\ddot{\theta}_1 + a_4\ddot{\theta}_2 = -a_1 g \sin\theta_1$$

$$a_3\ddot{x} + a_4\ddot{\theta}_1 + a_5\ddot{\theta}_2 = -a_3 g \sin\theta_2$$

We got a system of linear equations

$$\begin{bmatrix} a_0 & a_1 & a_3 \\ a_1 & a_2 & a_4 \\ a_3 & a_4 & a_5 \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} = \begin{bmatrix} u - b\dot{x} \\ -a_1 g \sin\theta_1 \\ -a_3 g \sin\theta_2 \end{bmatrix} \qquad (18)$$

First to be selected the state variable, we assume the x as the cart displacement, $\dot{x}$ as the displacement velocity, θ_1 the first pendulum angle and $\dot{\theta}_1$ its angular velocity, θ_2 the second pendulum angle and $\dot{\theta}_2$ its angular velocity, all as the state variables of the double pendulum system. We obtain the following state space equation.

$$\dot{X} = AX + Bu$$

$$Y = CX + Du$$

We can get numerical values For A, B matrices directly by solving the system of linear equations for $\ddot{x}$, $\ddot{\theta}_1$ and $\ddot{\theta}_2$ using Matlab as following

```matlab
syms u x_dot phi1 phi2;

m0 = 0.8; m1 = 0.5; m2 = 0.3;
l1 = 0.3; l2 = 0.2;
b = 0.1; g = 9.8; J = 0.006;

a0 = m0 +m1 + m2;
a1 = m1*l1 + m2*l1;
a2 = m1*l1^2 + m2*l1^2 + J;
a3 = m2*l2;
a4 = m2*l1*l2;
a5 = m2*l2^2 + J;

Z = [a0 a1 a3;
     a1 a2 a4;
     a3 a4 a5];
N = [u - b*x_dot;
     -a1*g*phi1;
     -a3*g*phi2];

% numerical_answer is a vector of
% x_d_dot, phi1_d_dot, phi2_d_dot
numerical_answer = Z\N;

A = zeros(6,6);
B = zeros(6,1);
C = eye(6);
D = zeros(6,1);

A(1,2) = 1; A(3,4) = 1 ; A(5,6) = 1;

v = [x_dot phi1 phi2 u];
r = [2 4 6];
c = [2 3 5];

for i = 1:4
    v_subs = subs(v,v(:),zeros(4,1));
    v_subs(i) = 1;
    col = double(subs(numerical_answer,v,v_subs));
    for j = 1:3
        if i<4
            A(r(j),c(i)) = col(j);
        else
            B(r(j)) = col(j);
        end
    end
end

disp(A); disp(B); disp(C); disp(D);
```

We get A, B, C and D as following

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -0.116 & 8.205 & 0 & 0.228 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0.349 & -63.8 & 0 & 9.116 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0.039 & 36.47 & 0 & -42.5 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 1.163 \\ 0 \\ -3.488 \\ 0 \\ -0.388 \end{bmatrix}$$

C matrix is chosen to be the identity matrix since we have measurements to all 6 state variables.

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

III. Controller

The system has been expressed in state space form as $\dot{X} = AX + Bu$. If $u = -K * x$, then $\dot{X} = (A - B * K)X$. If the eigenvalues of $(A - B * K)$ are at the left side of S plane, then the linearized system is asymptotic stable. If system is completely controllable, we can reassign the eigenvalues wherever we want. Hence, we need to check the controllability of the system. Matlab has an instruction to do it which is $r = rank(ctrb(A, B))$). If r equals the dimension of the system, the system is fully controllable. We got $r = 6$ which equals the dimension of the system. Hence, the system is fully controllable. Moreover, Matlab has an instruction that gives you the value of K vector required to reassign the eigenvalues on the desired location. This instruction is $K = \text{place}(A, B, P)$ where A, B are given from system state space form, and P is a vector of the desired eigenvalues. We will try two different choices for the eigenvalues. The first one is $(-5, -5.5, -6, -6.5, -7, -7.5)$. Using the Matlab instruction $K = \text{place}(A, B, P)$; K obtained equals $(37.8, 36.9, -126.3, 2.86, 18.3, -11.5)$. The second choice is $(-3, -3.5, -5, -5.5, -1.5, -2)$. Using the same Matlab instruction; K obtained equals $(0.582, 1.16, -20.58, -5.65, 25.97, 1.75)$.

Another approach is the Linear Quadratic Regulator (lqr) which is an optimization technique with following cost function

$$J = \int_0^\infty \left(x^T Q x + u^T R u\right) dt$$

The feedback control law that minimizes the value of the cost is

$$u = -R^{-1} B^T P x$$

Where P is found by solving the Algebraic Riccati equation

$$A^T P + PA - PBR^{-1} B^T P + Q = 0$$

Fortunately, there is a Matlab instruction called lqr that takes A, B, Q, R as input and gives the value of K vector required for optimality which equals $K = R^{-1}B^{T}P$. This is a better approach because choosing the desired values of new eigenvalues in Pole Placement approach needs some experience, but choosing Q and R matrices in lqr is straight forward since they affect the cost function directly. Therefore, we will choose Q and R as following

$$Q = \begin{bmatrix} 10 & 0 & 0 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 & 0 & 0 \\ 0 & 0 & 50 & 0 & 0 & 0 \\ 0 & 0 & 0 & 10 & 0 & 0 \\ 0 & 0 & 0 & 0 & 50 & 0 \\ 0 & 0 & 0 & 0 & 0 & 10 \end{bmatrix} \qquad R = 1$$

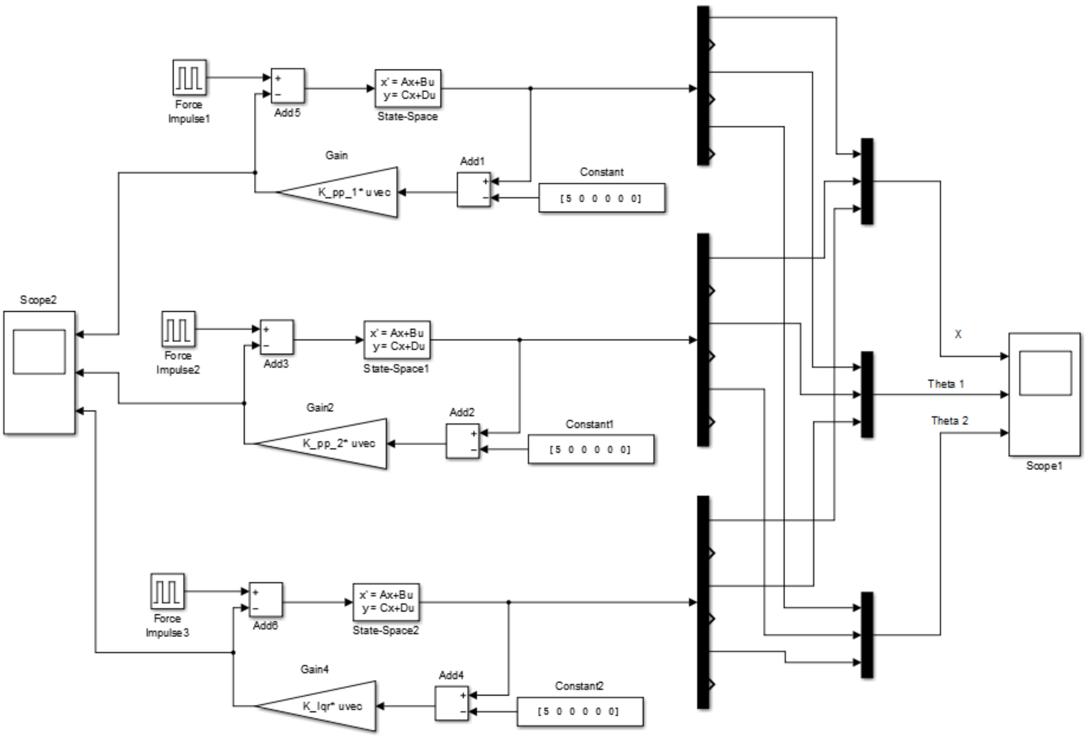
They reflect that we are interested in minimizing φ_1 and φ_2 more than all other state variables. Using the Matlab instruction $K = \mathrm{lqr}(A, B, Q, R)$; K obtained equals $(3.16, 5.37, -27.32, -3.11, 12.18, -2.14)$.

IV. Simulation / the Linearized Model

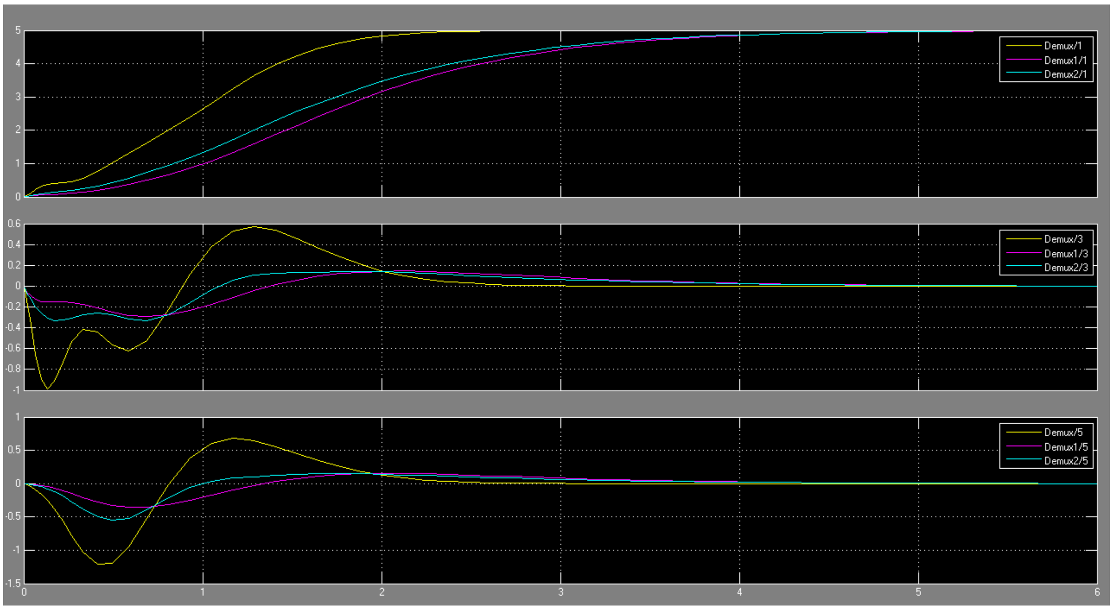
The code of the previous few steps as following

```
P1 = [-5 -5.5 -6 -6.5 -7 -7.5];
K_pp_1 = place(A,B,P1);

P2 = [-3 -3.5 -5 -5.5 -1.5 -2];
K_pp_2 = place(A,B,P2);

Q = diag([10 10 50 10 50 10]);
R = 1;
K_lqr = lqr(A,B,Q,R);
```

Now, we construct the Simulink for the two cases of the Pole Placement approach and the LQR case and compare the results. The simulation will be for move the cart 5 meters to the right with keeping the oscillations of φ_1 and φ_2 to the minimum.



And the Scope 1 simulation results which show the measurements of x, φ_1, φ_2 as following

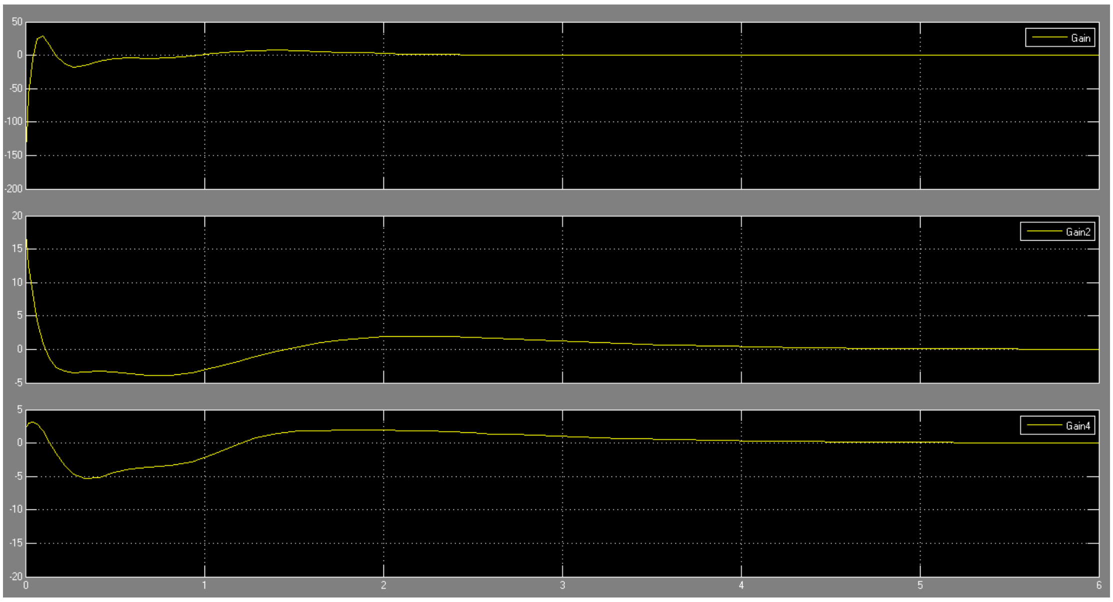


The yellow curves are for the Pole Placement first case, the purple curves are for the Pole Placement second case, and the light blue curves are for the LQR case. The above graph is for x state variable, the middle graph is for φ_1 state variable, and the below graph is for φ_2 state variable.

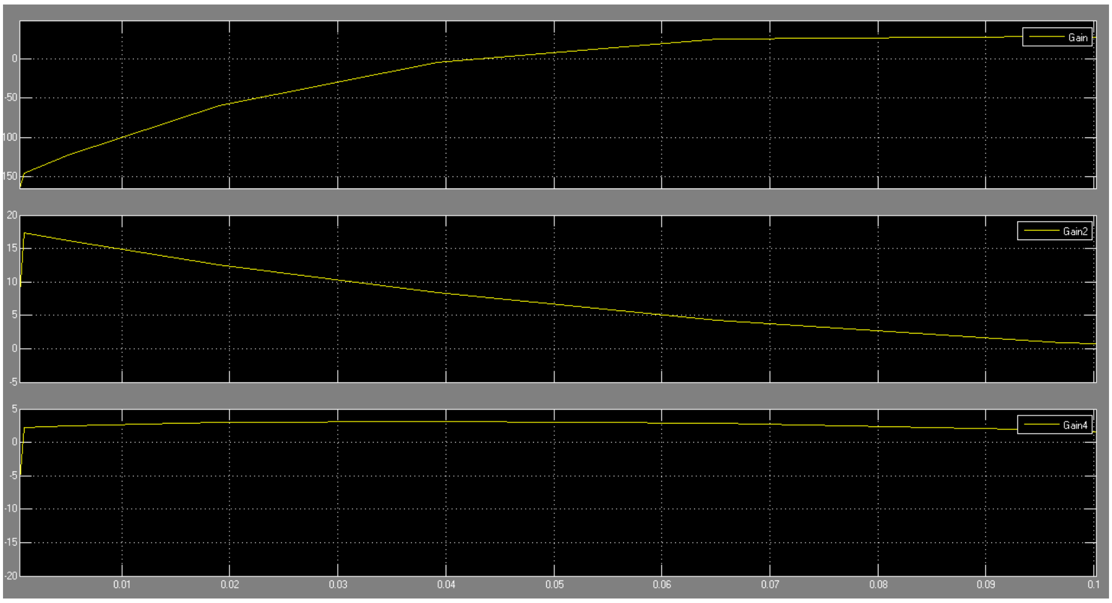
It can be seen that the Pole Placement first case reaches the final value of x faster than the others but at the expense of the more oscillations in φ_1

and φ_2. However, the Pole Placement second case and the LQR case are very much close to each other.

The Scope 2 simulation results which show the control efforts as following



And a closer look to the first 0.1 second



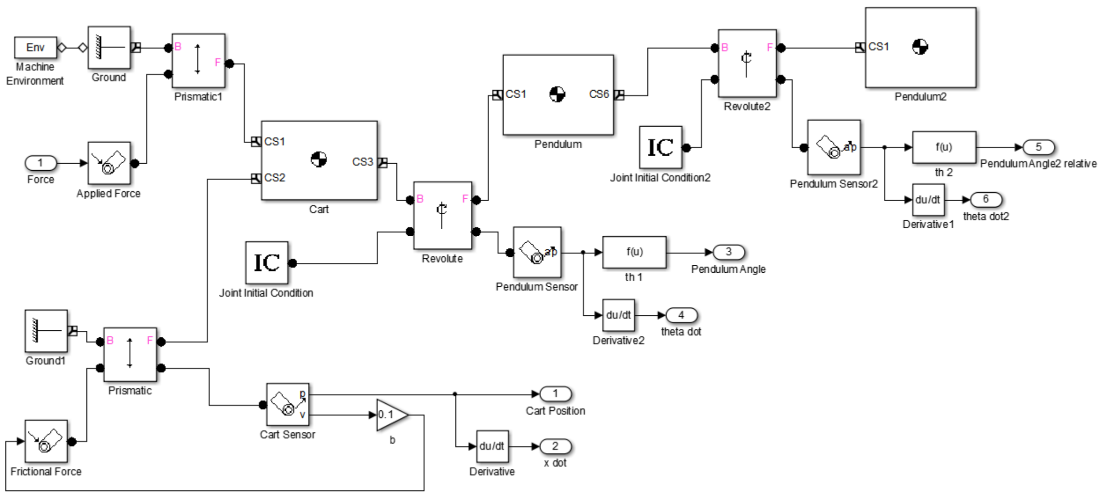
The above graph is for the Pole Placement first case, the middle graph is for the Pole Placement second case, and the below graph is the LQR case.

It can be seen that the Pole Placement first case achieves very high control effort at the beginning which reaches -150, the Pole Placement second case achieves lower control effort which reaches 17, and the LQR case achieves the lowest control effort which doesn't exceed -5. Although Pole

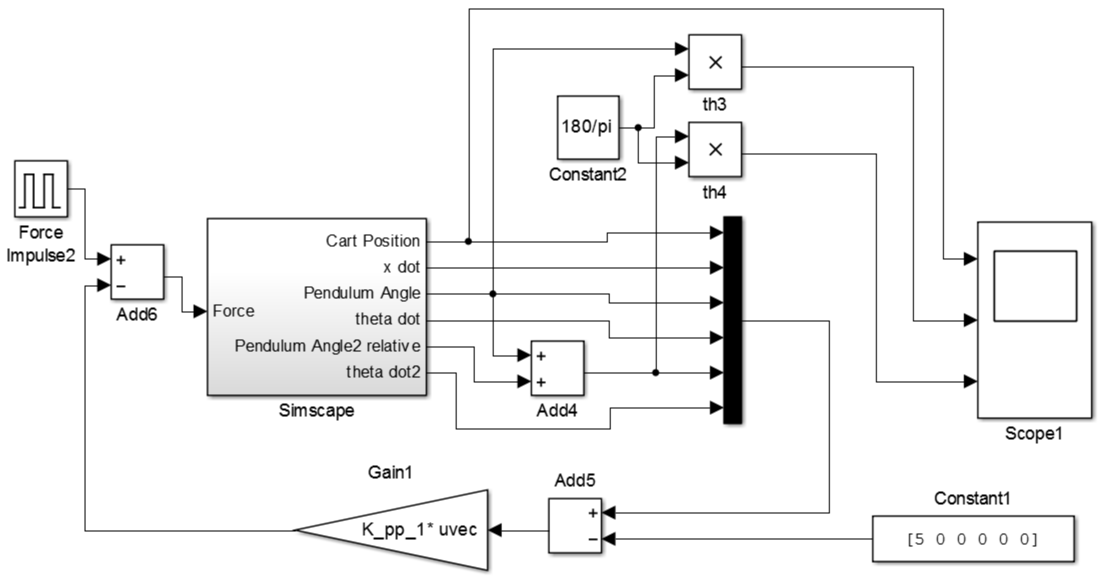
Placement second case and LQR case give close results of the output, the LQR case give better result regarding the control effort required to achieve these good results.

V. Simulation / the Nonlinear Model

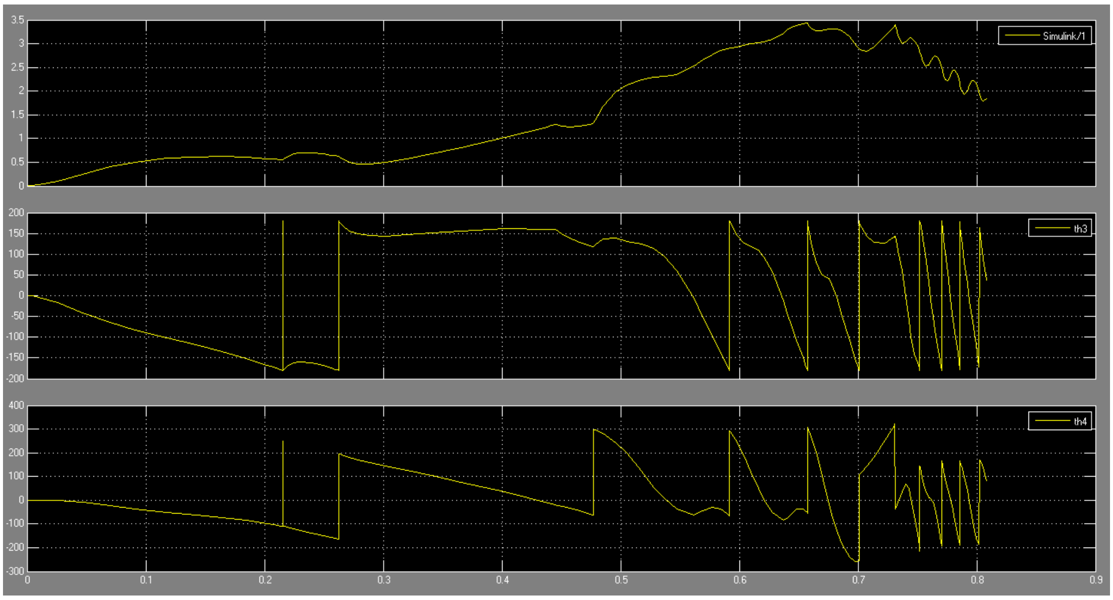
Now, we try these controllers on the nonlinear model. First, we have to construct the Simscape model. Simscape enables us to create models of physical systems within the Simulink environment. It enables us to build physical component models based on physical connections that directly integrate with block diagrams. The double pendulum model using Simscape is as following



Now, we put it as a subsystem in a feedback loop using the gains of the Pole Placement first case as following, and then run it.

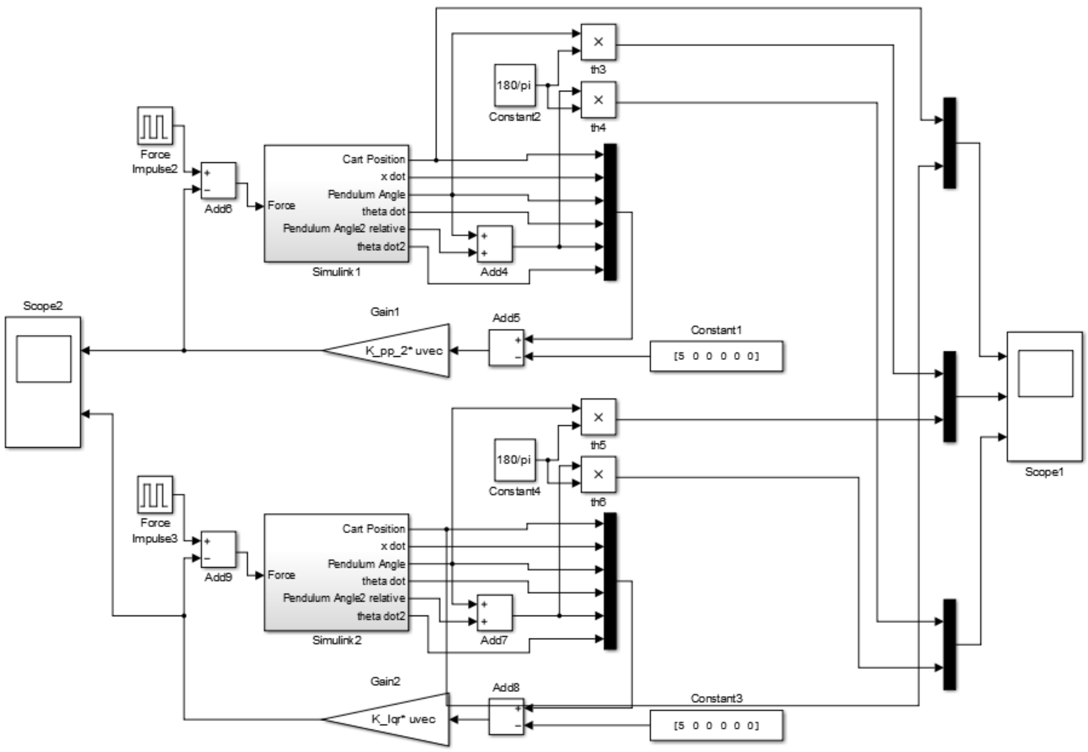


The Scope 1 simulation results which show the measurements of x, φ_1, φ_2 as following

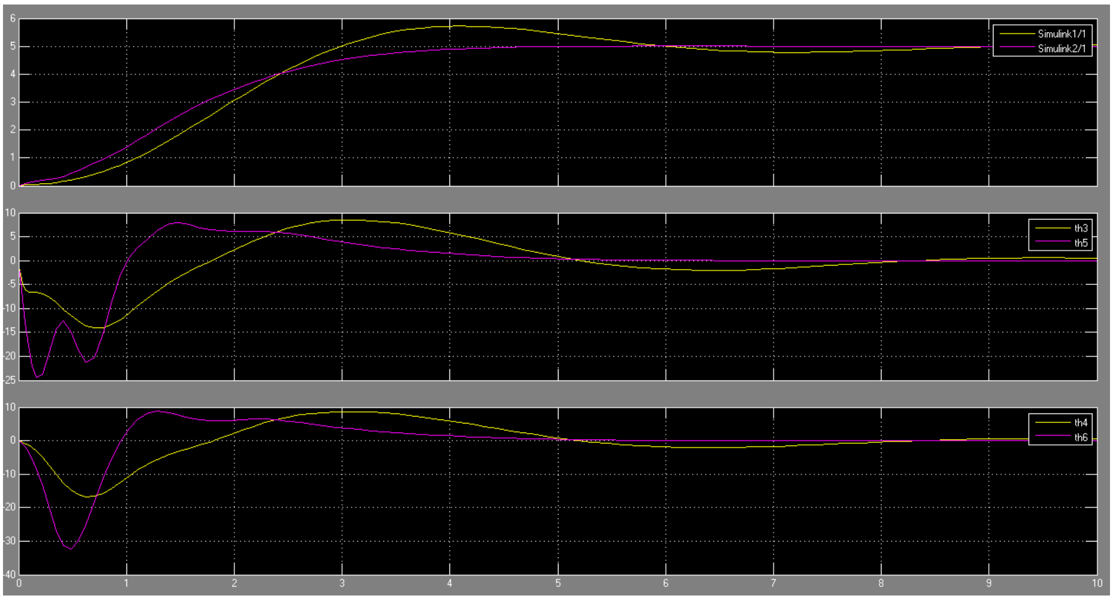


It can be seen easily that the Pole Placement first case delivers unstable behavior for the nonlinear model of the double pendulum system.

Now, we do the same for the Pole Placement second case and the LQR case and compare the results.



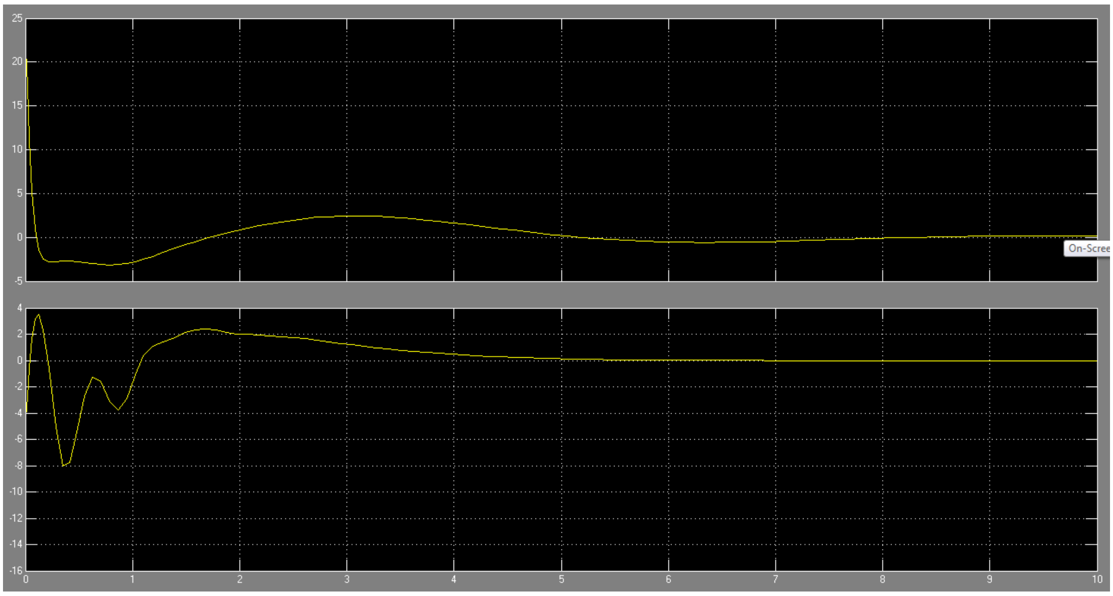
And the Scope 1 simulation results which show the measurements of x, φ_1, φ_2 as following



The yellow curves are for the Pole Placement second case, and the purple curves are for the LQR case. The above graph is for x state variable, the middle graph is for φ_1 state variable, and the below graph is for φ_2 state variable.

It can be seen that the Pole Placement second case has fewer oscillations in φ_1 and φ_2 than the LQR case during the first second; however, the LQR case reaches its steady state faster than the Pole Placement second case. Moreover, the LQR case reaches the steady state of x state variable with no oscillations at all; on the contrary of the Pole Placement second case.

The Scope 2 simulation results which show the control efforts as following



The above graph is for the Pole Placement second case, and the below graph is the LQR case.

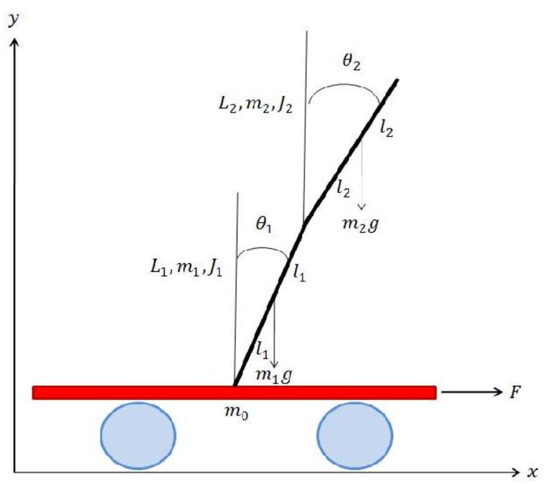
It can be seen that the Pole Placement second case achieves a high control effort at the beginning which reaches 23, and the LQR case achieves a lower control effort which doesn't exceed abs(8). Although Pole Placement second case and the LQR case give close results of the output, the LQR case give better result regarding the control effort required to achieve these good results.

VI. Conclusion

LQR is an essential method to combine between the asymptotic stability of a given system and reduce the total control effort to achieve this asymptotic stability. Pole Placement can achieve asymptotic stability, but it has no effect on the control effort. Hence, optimal control is a necessity if we want to optimize the energy consumption and maintain the stability.

VII. Extra Work / the Double Inverted Pendulum

This part extends the previous work to the double inverted pendulum on a cart problem. It is a challenging control problem which is used by many researchers for designing and testing the control techniques. The objective is to determine a control strategy that delivers better performance for the two angles of the pendulum and the cart position besides decreasing the control effort required. The nonlinear model of the double inverted pendulum needs to be linearized too. All work is simulated using MATLAB and Simulink.



Regarding the system model, the only difference between the double pendulum system and the inverted double pendulum system is the potential energy since in the inverted double pendulum system we got positive potential energy as following

$$V_0 = 0$$

$$V_1 = m_1 g l_1 \cos\theta_1$$

$$V_2 = m_2 g (l_1 \cos\theta_1 + l_2 \cos\theta_2)$$

$$V = V_0 + V_1 + V_2$$

$$V = (m_1 l_1 + m_2 l_1) g \cos\theta_1 + m_2 l_2 g \cos\theta_2$$

Repeat all the modeling work at section II, and we get the linearized version of the double inverted pendulum system as following

$$a_0 \ddot{x} + a_1 \ddot{\theta}_1 + a_3 \ddot{\theta}_2 = u - b\dot{x}$$

$$a_1 \ddot{x} + a_2 \ddot{\theta}_1 + a_4 \ddot{\theta}_2 = a_1 g \sin\theta_1$$

$$a_3 \ddot{x} + a_4 \ddot{\theta}_1 + a_5 \ddot{\theta}_2 = a_3 g \sin\theta_2$$

Then we get a system of linear equations as following

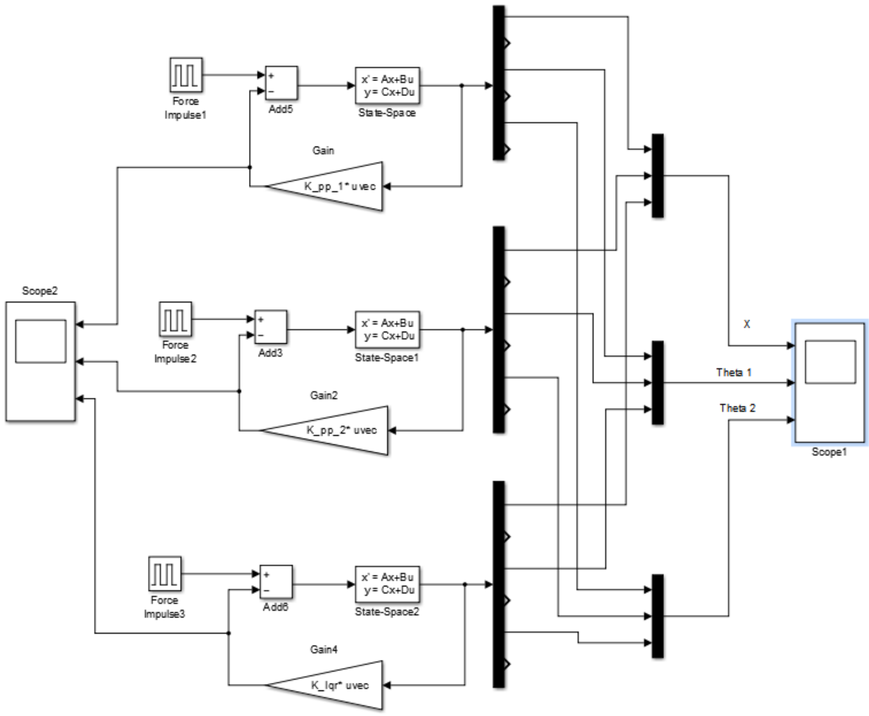
$$\begin{bmatrix} a_0 & a_1 & a_3 \\ a_1 & a_2 & a_4 \\ a_3 & a_4 & a_5 \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} = \begin{bmatrix} u - b\dot{x} \\ a_1 g \sin\theta_1 \\ a_3 g \sin\theta_2 \end{bmatrix}$$

Then we get A, B as following

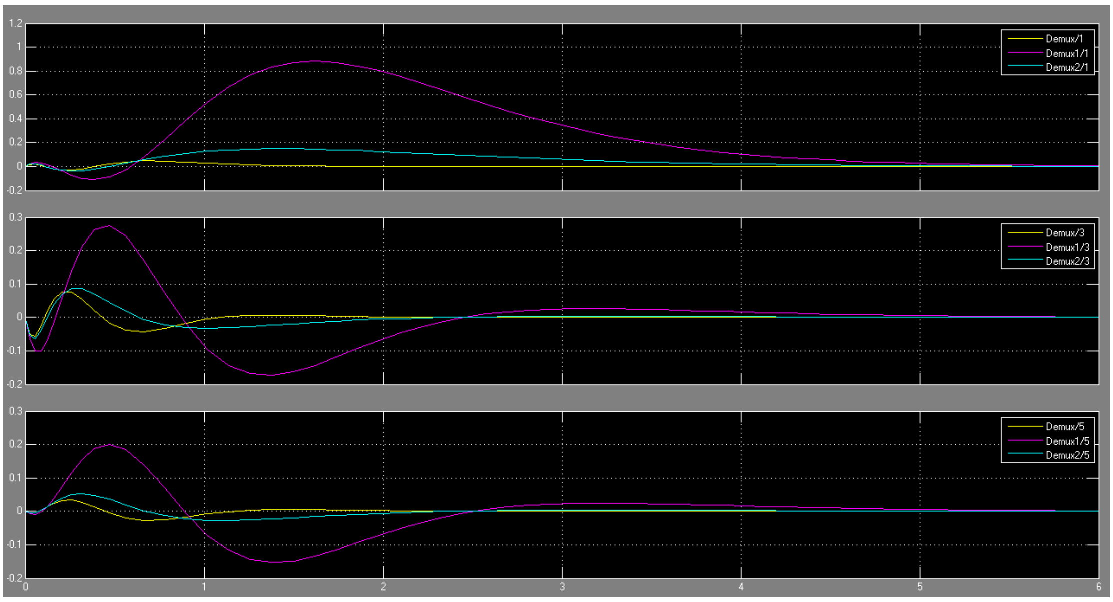
$$\mathrm{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -0.116 & -8.205 & 0 & -0.228 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0.349 & 63.8 & 0 & -9.116 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0.039 & -36.47 & 0 & 42.5 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 1.163 \\ 0 \\ -3.488 \\ 0 \\ -0.388 \end{bmatrix}$$

C matrix and D vector are the same.

Now, we generate gains K_pp_1, K_pp_2 and K_lqr such that the first two are cases of the Pole Placement method, and the third one is a case of the LQR method the same way as before. Then, apply these gains to the same Simulink model but using A, B of the double inverted pendulum.

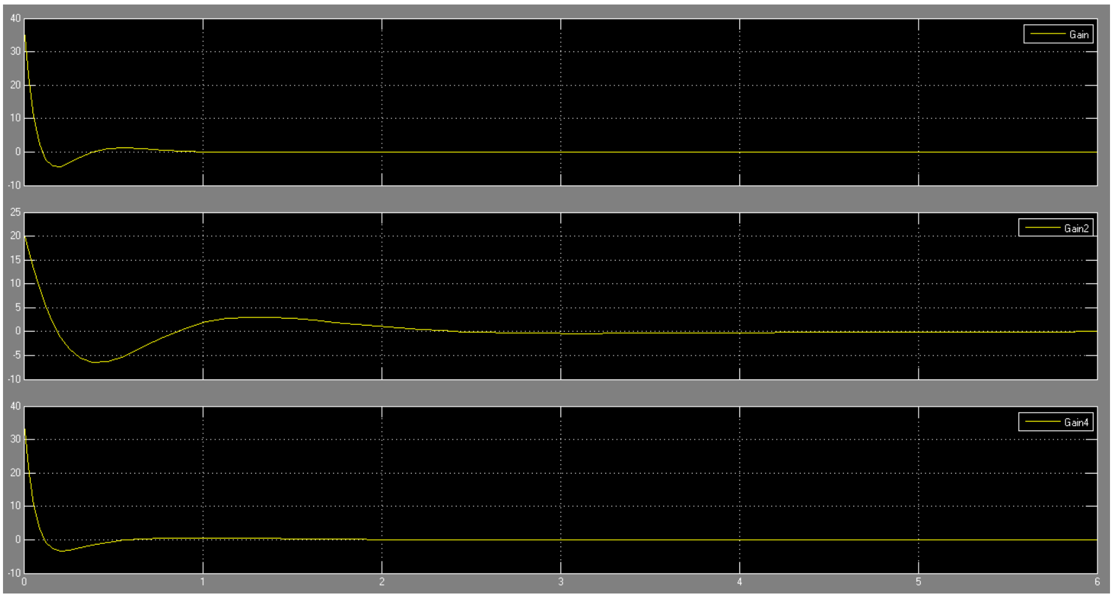


The Scope 1 simulation results which show the measurements of x, φ_1, φ_2 as following



The yellow curves are for the Pole Placement first case, the purple curves are for the Pole Placement second case, and the green curves are for the LQR case. The above graph is for x state variable, the middle graph is for φ_1 state variable, and the below graph is for φ_2 state variable. It can be seen that the Pole Placement first case and the LQR case give better results than the Pole Placement second case.

The Scope 2 simulation results which show the control efforts as following



The above graph is for the Pole Placement first case, the middle graph is for the Pole Placement second case, and the below graph is the LQR case.

It can be seen that the Pole Placement second case achieves the lowest control effort at the beginning, but it suffers from some oscillations after that. The Pole Placement first case and the LQR case are smoother, but they suffer from high control effort at the beginning. We can decrease the control effort for LQR by increasing the R variable a little bit.

VIII. Future Work

The future work can be testing the Pole Placement approach and the LQR approach by the nonlinear model of the double inverted pendulum. We can do this using Simscape. Also, we can try a nonlinear controller to compensate some of the nonlinearity of the double inverted pendulum.

IX. References

[1] Sandeep Kumar Yadav, Sachin Sharma, Narinder Singh, "Optimal Control of double inverted pendulum using LQR controller", International J. of advanced Research in computer science and software engineering, 2(2),2012, 189-192.

[2] T. Ashish, "Modern Control Design with MATLAB and SIMULINK" Indian Institute of Technology, Kanpur, India, John Wiley & Sons, 2002.

[3] K. Zhou, J. C. Doyle, and K. Glover, "Robust and optimal control", Prentice Hall Englewood Cliffs, NJ, 1996.

[4]http://ctms.engin.umich.edu/CTMS/index.php?example=InvertedPendulum&section=SimulinkModeling.