

Department : Mathematics and Computer Science  
 Level : Third Level  
 Course Code : 040101309  
 Course Title : Integral Equations  
 Semester : Fall 2023  
 Time Allowed : 30 minutes  
 Lecturer : Dr.Hanna R. Ebead  
 Total Marks : 30 Points



## Mid-Term Examination-Model Answer

[10 marks]

### Question 1: (a)

Using *Successive Approximation Method*

Let initial guess  $y_0(t) = x$  Then

$$y_1(t) = x - \int_0^x (x-t)t \, dt = x - \frac{x^3}{3!}$$

$$y_2(t) = x - \int_0^x (x-t)\left(t - \frac{t^3}{3!}\right) dt = x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

$$y_3(t) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$$

$\vdots$

$$y_n(t) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + \frac{(-1)^{n-1}x^{2n+1}}{(2n+1)!} = \sum_{k=1}^n \frac{(-1)^{k-1}x^{2k+1}}{(2k+1)!}$$

Since  $y(x) = \lim_{n \rightarrow \infty} y_n(t)$  we deduce that

$$\begin{aligned}
 y(t) &= \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{(-1)^{k-1}x^{2k+1}}{(2k+1)!} \\
 &= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}x^{2k+1}}{(2k+1)!} = \sin(x)
 \end{aligned}$$

### Question 1: (b)

Since  $k(t, s) = ts$  separable, we can use *Direct Computational Method*

$$y(t) = \frac{7}{8}t + \frac{1}{2}t \int_0^1 tsy^2(s) \, ds$$

Let  $C := \int_0^1 sy^2(s) \, ds$

$$y(t) = \frac{7}{8}t + \frac{1}{2}tC$$

$$C = \int_0^1 s \left[ \frac{7}{8}s + \frac{1}{2}sC \right]^2 ds$$

$$C = \int_0^1 s^3 \left[ \frac{7}{8} + \frac{1}{2}C \right]^2 ds = \frac{1}{4} \left[ \frac{7}{8} + \frac{1}{2}C \right]^2$$

$$16C^2 - 200C + 49 = 0$$

We get two solutions

$$\begin{cases} y_1(t) = 7t & \text{At } C_1 = \frac{49}{4} \\ y_2(t) = t & \text{At } C_2 = \frac{1}{4} \end{cases}$$

[10 marks]

**Question 2: (a)**

Since  $k$  is continuous on a closed domain  $[a, b] \times [a, b]$ . so it guarantees the existence of a finite  $L > 0$

From *Existence and uniqueness theorem* we are able to prove that the integral equation has a unique continuous solution  $y \in C[a, b]$

Since

$$f, k(t, \cdot) \in C[a, b] \subset L_2[a, b] \quad , \quad t \in [a, b]$$

Then

$$M = \int_a^b \int_a^b |k(t, s)|^2 ds dt \leq \int_a^b \int_a^b L^2 ds dt = L^2(b-a)^2$$

$$\sqrt{M} \leq L(b-a)$$

Thus, we deduce that

$$|\lambda| < \frac{1}{\sqrt{M}} \rightarrow |\lambda| < \frac{1}{L(b-a)}$$

**Question 2: (b)**

Consider the Integral Equation

$$y(t) = -4 + \int_0^1 (2t + 3s)y(s) ds \quad , \quad t \in [0, 1]$$

We have  $\lambda = 1$  ,  $L = \max_{t, s \in [a, b] \times [a, b]} |2t + 3s| = 5$  ,  $b - a = 1$  which means that it

doesn't satisfy the theorem because  $|\lambda| < \frac{1}{L(b-a)}$  doesn't hold

But the equation has an exact solution given by  $y(t) = 4t$

[10 marks]

**Question 3:**

Let  $y$  solves the IVP.

Integrate the equation from  $0 \rightarrow x$

$$\int_0^x y''(t) dt - 2 \underbrace{\int_0^x ty'(t) dt}_J - 3 \int_0^x y(t) dt = 0 \quad (1)$$

Using integration by parts on  $J$  we get

$$J := \int_0^x ty'(t) dt = [ty(t)]_0^x - \int_0^x y(t) dt = xy(x) - \int_0^x y(t) dt$$

Substitute in (1)

$$\begin{aligned}
 y'(x) - y'(0) - 2xy(x) + 2 \int_0^x y(t) dt - 3 \int_0^x y(t) dt &= 0 \\
 y'(x) - 2xy(x) - \int_0^x y(t) dt &= 0 \quad (2)
 \end{aligned}$$

Integrate (2) from  $0 \rightarrow x$

$$\begin{aligned}
 y(x) - y(0) - 2 \int_0^x ty(t) dt - \int_0^x \int_0^\xi y(t) dt d\xi &= 0 \\
 y(x) &= 1 + 2 \int_0^x ty(t) dt + \int_0^x \int_t^x y(t) d\xi dt \\
 y(x) &= 1 + 2 \int_0^x ty(t) dt + \int_0^x (x-t)y(t) dt \\
 y(x) &= 1 + \int_0^x (x+t)y(t) dt
 \end{aligned}$$

