Department : Mathematics and Computer Science
Level : Third Level
Course Code : 040101309
Course Title : Integral Equations
Semester : Fall 2023
Time Allowed : 30 minutes
Lecturer : Dr.Hanna R. Ebead
Total Marks : 30 Points

Mid-Term Examination-Model Answer

Question 1: (a)

Using *Successive Approximation Method*
Let initial guess $y_o(t) = x$ Then

$$y_1(t) = x - \int_0^x (x-t)t\,dt = x - \frac{x^3}{3!}$$
$$y_2(t) = x - \int_0^x (x-t)(t - \frac{t^3}{3!})\,dt = x - \frac{x^3}{3!} + \frac{x^5}{5!}$$
$$y_3(t) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$$
$$\vdots$$
$$y_n(t) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + \frac{(-1)^{n-1}x^{2n+1}}{(2n+1)!} = \sum_{k=1}^{n} \frac{(-1)^{k-1}x^{2k+1}}{(2k+1)!}$$

Since $y(x) = \lim_{n\to\infty} y_n(t)$ we deduce that

$$\begin{aligned} y(t) &= \lim_{n\to\infty} \sum_{k=1}^{n} \frac{(-1)^{n-1}x^{2n+1}}{(2n+1)!} \\ &= \sum_{k=1}^{\infty} \frac{(-1)^{n-1}x^{2n+1}}{(2n+1)!} = \sin(x) \end{aligned}$$

Question 1: (b)

Since $k(t,s) = ts$ separable, we can use *Direct Computational Medthod*

$$y(t) = \frac{7}{8}t + \frac{1}{2}t \int_0^1 tsy^2(s)\,ds$$

Let $C := \int_0^1 sy^2(s)\,ds$

$$\begin{aligned} y(t) &= \frac{7}{8}t + \frac{1}{2}tC \\ C &= \int_0^1 s\left[\frac{7}{8}s + \frac{1}{2}sC\right]^2 ds \\ C &= \int_0^1 s^3\left[\frac{7}{8} + \frac{1}{2}C\right]^2 ds = \frac{1}{4}\left[\frac{7}{8} + \frac{1}{2}C\right]^2 \end{aligned}$$

$$16C^2 - 200C + 49 = 0$$

We get two solutions

$$\begin{cases} y_1(t) = 7t \quad \text{At} \quad C_1 = \dfrac{49}{4} \\ \\ y_2(t) = t \quad \text{At} \quad C_2 = \dfrac{1}{4} \end{cases}$$

Question 2: (a)

Since k is continuous on a closed domain $[a,b] \times [a,b]$. so it guarantees the existence of a finite $L > 0$
From *Existence and uniqueness theorem* we are able to prove that the integral equation has a unique continuous solution $y \in C[a,b]$
Since

$$f, k(t,.) \in C[a,b] \subset L_2[a,b] \quad , \quad t \in [a,b]$$

Then

$$M = \int_a^b \int_a^b |k(t,s)|^2 \, dsdt \leq \int_a^b \int_a^b L^2 \, dsdt = L^2(b-a)^2$$
$$\sqrt{M} \leq L(b-a)$$

Thus, we deduce that

$$|\lambda| < \frac{1}{\sqrt{M}} \to |\lambda| < \frac{1}{L(b-a)}$$

Question 2: (b)

Consider the Integral Equation

$$y(t) = -4 + \int_0^1 (2t+3s)y(s) \, ds \quad , \quad t \in [0,1]$$

We have $\lambda = 1$, $L = \max\limits_{t,s \in [a,b] \times [a,b]} |2t+3s| = 5$, $b - a = 1$ which means that it doesn't satisfy the theorem because $|\lambda| < \dfrac{1}{L(b-a)}$ doesn't hold
But the equation has an exact solution given by $y(t) = 4t$

Question 3:

Let y solves the IVP.
Integerate the equation from $0 \to x$

$$\int_0^x y''(t) \, dt - 2 \underbrace{\int_0^x ty'(t) \, dt}_{J} - 3 \int_0^x y(t) \, dt = 0 \tag{1}$$

Using integeration by parts on J we get

$$J := \int_0^x ty'(t) \, dt = [ty(t)]_0^x - \int_0^x y(t) \, dt = xy(x) - \int_0^x y(t) \, dt$$

Subtitute in (1)

$$y'(x) - y'(0) - 2xy(x) + 2 \int_0^x y(t) \, dt - 3 \int_0^x y(t) \, dt = 0$$

$$y'(x) - 2xy(x) - \int_0^x y(t)\,dt = 0 \tag{2}$$

Integerate (2) from $0 \to x$

$$\begin{aligned}
y(x) - y(0) - 2\int_0^x ty(t)\,dt - \int_0^x \int_0^\xi y(t)\,dt d\xi = 0 \\
y(x) = 1 + 2\int_0^x ty(t)\,dt + \int_0^x \int_t^x y(t)\,d\xi dt \\
y(x) = 1 + 2\int_0^x ty(t)\,dt + \int_0^x (x-t)y(t)dt \\
y(x) = 1 + \int_0^x (x+t)y(t)dt
\end{aligned}$$