


Partial Differential Equations

01356 Spring 2023

Lecture Notes

Lecturer: Prof. Mahmoud M. El-Borai
Prepared by: Ossama Abdelwahed And
Ahmed.M.Habib



Contents

1 Introduction 3

1 Introduction

The main goal of many scientific disciplines can be summarized to the following:

1. Formulate a set of mathematical equations to model a phenomena of interest
2. Analyze solutions to these equations in order to extract information and make predictions.

The result of 1 is often a system of partial differential equations, thus the second becomes solving those partial differential equations.

A partial differential equation (PDE) is a differential equation containing partial derivatives of the dependent variable with respect to more than one independent variable.

Order of PDE

The order of a PDE is determined by the highest derivative in the equation.

$$\frac{\partial u}{\partial t} + \left(\frac{\partial u}{\partial x}\right)^2 = 0 \quad \text{first order}$$

$$\frac{\partial^4 u}{\partial y^4} + \frac{\partial u}{\partial x} = c \quad \text{fourth order}$$

do not mistake the order of the PDE with its degree, the degree of the PDE is the highest exponent appearing in the equation.

Linearity

A linear PDE is one that is of first degree in all of its field variables and partial derivatives.

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0 \quad \text{linear}$$

$$\frac{\partial^4 u}{\partial y^4} + \frac{\partial u}{\partial x} = y \quad \text{linear}$$

$$\frac{\partial u}{\partial t} + \left(\frac{\partial u}{\partial x}\right)^2 = 0 \quad \text{nonlinear}$$

$$\frac{\partial^3 u}{\partial x^3} + \left(\frac{\partial^2 u}{\partial y^2}\right)^5 = \sin(x) \quad \text{nonlinear}$$

a linear operator can be defined for any linear equation, taking the first equation in the previous list, the linear operator L can be defined as.

$$L = \frac{\partial}{\partial t} + \frac{\partial u}{\partial x}$$

and the equation can be written as.

$$L(u) = 0$$

Homogeneity

Let L be a linear operator. Then a linear partial differential equation can be written in the form.

$$L(u) = f(x_1, x_2, \dots, t)$$

if $f = 0$ then the equation is homogeneous, otherwise it is inhomogeneous.

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0 \quad \text{homogeneous}$$

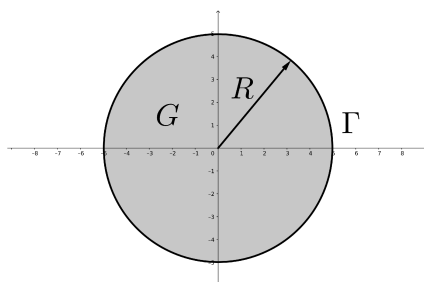
$$\frac{\partial^4 u}{\partial y^4} + \frac{\partial u}{\partial x} = y \quad \text{inhomogeneous}$$

Boundary Conditions

Boundary conditions are constraints necessary for the solution of a boundary value problem. A boundary value problem is a differential equation to be solved in a domain on whose boundary the function is known. We will be interested in one type of boundary conditions in this course which is the Dirichlet Conditions, specifies the value that the unknown function needs to take on along the boundary of the domain. For example, the Laplace equation on a circle with Dirichlet condition will be.

$$\nabla^2 u(x) = 0 \quad \forall x \in G$$

$$u(x) = f(x) \quad \forall x \in \Gamma$$



$$G = \{(x, y) : x^2 + y^2 < R^2\} \quad \Gamma = \{(x, y) : x^2 + y^2 = R^2\}$$

Equations involving such conditions are classified as Dirichlet problems.

Initial Condition

The initial condition is a condition that a solution must have at only one instant of time, which is the starting time as it can be found experimentally. An example is the heat equation with initial condition.

$$\frac{\partial u(x, t)}{\partial t} = c^2 \frac{\partial^2 u(x, t)}{\partial x^2}$$

$$u(x, 0) = f(x)$$

Equations involving such conditions are classified as Cauchy problems.

Equations of Mathematical Physics

The most frequently encountered equations in physics are the following

1. Heat Equation

$$\frac{\partial u(x, t)}{\partial t} = c^2 \frac{\partial^2 u(x, t)}{\partial x^2}$$

2. Wave Equation

$$\frac{\partial^2 u(x, t)}{\partial t^2} = c^2 \frac{\partial^2 u(x, t)}{\partial x^2}$$

3. Laplace's Equation

$$\nabla^2 u(x) = \frac{\partial^2 u(x)}{\partial x_1^2} + \frac{\partial^2 u(x)}{\partial x_2^2} + \frac{\partial^2 u(x)}{\partial x_3^2} + \dots = 0$$

those will be our main focus in this course.