Partial Differential Equations

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Lecture Notes

Lecturer: Prof. Mahmoud M. El-Borai
Prepared by: Ossama Abdelwahed And
Ahmed.M.Habib

Contents

1 Introduction 3
- 1.1 Order of PDE 3
- 1.2 Linearity 3
- 1.3 Homogeneity 3
- 1.4 Boundry Conditions 4
- 1.5 Intial Condition 4
- 1.6 Equations of Mathematical Physics 4

2 Canonical Form 5
- 2.1 Examples 6

3 Heat Equation 9
- 3.1 Fourier Transform 9
- 3.2 Cauchy Problem 10
- 3.3 Mixed Homogeneous Problem 12
- 3.4 Mixed Inhomogeneous Problem 14

4 Wave Equation 15
- 4.1 Cauchy Problem (1D) 15
- 4.2 Mixed Problem (1D) 17
- 4.3 Spherical Mean and Darboux's Equation 19
- 4.4 Kirchhoff's Formula 22

5 Laplace's Equation 25
- 5.1 Harmonic Functions 25

1 Introduction

The main goal of many scientfic displines can be summerized to the following:

1. Formulate a set of mathematical equations to model a phenomena of intrest
2. Analyze solutions to these equations in order to extract information and make predictions.

The result of 1 is often a system of partial differential equations, thus the second becomes solving those partial differential equations.

A partial differential equation (PDE) is is a differential equation containing partial derivatives of the dependent variable with repect to more than one independent variable.

1.1 Order of PDE

The order of a PDE is determined by the highest derivative in the equation.

$$\frac{\partial u}{\partial t} + (\frac{\partial u}{\partial x})^2 = 0 \qquad \implies \text{First order}$$
$$\frac{\partial^4 u}{\partial y^4} + \frac{\partial u}{\partial x} = c \qquad \implies \text{Fourth order}$$

do not mistake the order of the PDE with its degree, the degree of the PDE is the highest exponent appearing in the equation.

1.2 Linearity

A linear PDE is one that is of first degree in all of its field variables and partial derivatives.

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0 \qquad \text{linear}$$
$$\frac{\partial^4 u}{\partial y^4} + \frac{\partial u}{\partial x} = y \qquad \text{linear}$$
$$\frac{\partial u}{\partial t} + (\frac{\partial u}{\partial x})^2 = 0 \qquad \text{nonlinear}$$
$$\frac{\partial^3 u}{\partial x^3} + (\frac{\partial^2 u}{\partial y^2})^5 = \sin(x) \qquad \text{nonlinear}$$

a linear operator can be defined for any linear equation, taking the first equation in the previous list, the linear operator L can be defined as.

$$L = \frac{\partial}{\partial t} + \frac{\partial u}{\partial x}$$

and the equation can be written as.

$$L(u) = 0$$

1.3 Homogeneity

Let L be a linear operator. Then a linear partial differential equation can be written in the form.

$$L(u) = f(x_1, x_2, \dots, t)$$

if $f = 0$ then the equation is homogeneous, otherwise it is inhomogeneous.

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0 \qquad \text{homogeneous}$$
$$\frac{\partial^4 u}{\partial y^4} + \frac{\partial u}{\partial x} = y \qquad \text{inhomogeneous}$$

1.4 Boundry Conditions

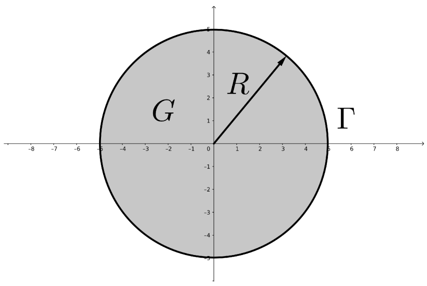
Definition 1.1 — Boundary Conditions. Boundary conditions are constraints necessary for the solution of a boundary value problem.

Definition 1.2 — Boundary Value Problem. BVP is a differential equation to be solved in a domain on whose boundary the function is known.

We will be intrested in one type of boundry conditions in this course which is the Drichlet Conditions, specifies the value that the unknown function needs to take on along the boundary of the domain. For example, the Laplace equation on a circle with drichlet condition will be.

$$\nabla^2 u(x) = 0 \quad \forall x \in G$$

$$u(x) = f(x) \quad \forall x \in G$$



$$G = \{(x, y) : x^2 + y^2 < R^2\} \qquad \Gamma = \{(x, y) : x^2 + y^2 = R^2\}$$

Definition 1.3 — Drichlet problems. are problems that has only boundry conditions

1.5 Intial Condition

Definition 1.4 — The initial condition. is a condition that a solution must have at only on instant of time, which is the starting time as it can be found exprementally.

An example is the heat equation with intial condition.

$$\frac{\partial u(x,t)}{\partial t} = c^2 \frac{\partial u(x,t)}{\partial x}$$

$$u(x, 0) = f(x)$$

Definition 1.5 — Cauchy problems. are problems that has only intial conditions

1.6 Equations of Mathematical Physics

The most frequently encountered equations in physics are the following

1. Heat Equation

$$\frac{\partial u(x,t)}{\partial t} = c^2 \frac{\partial^2 u(x,t)}{\partial x^2}$$

2. Wave Equation

$$\frac{\partial^2 u(x,t)}{\partial t^2} = c^2 \frac{\partial^2 u(x,t)}{\partial x^2}$$

3. Laplace's Equation

$$\nabla^2 u(x) = \frac{\partial^2 u(x)}{\partial x_1^2} + \frac{\partial^2 u(x)}{\partial x_2^2} + \frac{\partial^2 u(x)}{\partial x_3^2} + \cdots = 0$$

2 Canonical Form

Consider the following PDE with variable coefficients. We are aiming to transform this equation into its canonical form

$$A\left(x,y\right)\frac{\partial^2 u\left(x,y\right)}{\partial x^2}+2B\left(x,y\right)\frac{\partial^2 u\left(x,y\right)}{\partial x\partial y}+C\left(x,y\right)\frac{\partial^2 u\left(x,y\right)}{\partial y^2}+F\left(x,y,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}\right)=0 \qquad (1)$$

The first three terms are called the principle terms and the last term is called the Yound term (Y.T) which does not contain second order derivatives of u.

We start by preforming a change of variables such that.

$$\xi=\xi\left(x,y\right),\qquad \eta=\eta\left(x,y\right)$$

taking into consideration the Jacobian of the transformation

$$J=\begin{vmatrix}\frac{\partial\xi}{\partial x} & \frac{\partial\eta}{\partial x}\\ \frac{\partial\xi}{\partial y} & \frac{\partial\eta}{\partial y}\end{vmatrix}\neq 0$$

Then we find our derivatives.

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial\xi}\frac{\partial\xi}{\partial x}+\frac{\partial u}{\partial\eta}\frac{\partial\eta}{\partial x}$$
$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial u}{\partial\xi}\frac{\partial^2\xi}{\partial x^2}+\frac{\partial\xi}{\partial x}\left[\frac{\partial^2 u}{\partial\xi^2}\frac{\partial\xi}{\partial x}+\frac{\partial^2 u}{\partial\eta\partial\xi}\frac{\partial\eta}{\partial x}\right]+\frac{\partial u}{\partial\eta}\frac{\partial^2\eta}{\partial x^2}+\frac{\partial\eta}{\partial x}\left[\frac{\partial^2 u}{\partial\eta^2}\frac{\partial\eta}{\partial x}+\frac{\partial^2 u}{\partial\eta\partial\xi}\frac{\partial\xi}{\partial x}\right]$$

adding similar terms and simplifying

$$\frac{\partial^2 u}{\partial x^2}=(\frac{\partial\xi}{\partial x})^2\frac{\partial^2 u}{\partial\xi^2}+2\frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial x}\frac{\partial^2 u}{\partial\eta\partial\xi}+(\frac{\partial\eta}{\partial x})^2\frac{\partial^2 u}{\partial\eta^2}+Y.T \qquad (2)$$

and in similar fashion we can get.

$$\frac{\partial^2 u}{\partial y^2}=(\frac{\partial\xi}{\partial y})^2\frac{\partial^2 u}{\partial\xi^2}+2\frac{\partial\xi}{\partial y}\frac{\partial\eta}{\partial y}\frac{\partial^2 u}{\partial\eta\partial\xi}+(\frac{\partial\eta}{\partial y})^2\frac{\partial^2 u}{\partial\eta^2}+Y.T \qquad (3)$$

$$\frac{\partial^2 u}{\partial y\partial x}=\frac{\partial\xi}{\partial x}\frac{\partial\xi}{\partial y}\frac{\partial^2 u}{\partial\xi^2}+\left[\frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial y}+\frac{\partial\xi}{\partial y}\frac{\partial\eta}{\partial x}\right]\frac{\partial^2 u}{\partial\xi\partial\eta}+\frac{\partial\eta}{\partial x}\frac{\partial\eta}{\partial y}\frac{\partial^2 u}{\partial\eta^2}+Y.T \qquad (4)$$

substituting (2), (3), and (4) in (1) we get.

$$\begin{aligned}A&\left[(\frac{\partial\xi}{\partial x})^2\frac{\partial^2 u}{\partial\xi^2}+2\frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial x}\frac{\partial^2 u}{\partial\eta\partial\xi}+(\frac{\partial\eta}{\partial x})^2\frac{\partial^2 u}{\partial\eta^2}\right]\\
+2B&\left[\frac{\partial\xi}{\partial x}\frac{\partial\xi}{\partial y}\frac{\partial^2 u}{\partial\xi^2}+\left[\frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial y}+\frac{\partial\xi}{\partial y}\frac{\partial\eta}{\partial x}\right]\frac{\partial^2 u}{\partial\xi\partial\eta}+\frac{\partial\eta}{\partial x}\frac{\partial\eta}{\partial y}\frac{\partial^2 u}{\partial\eta^2}\right]\\
+C&\left[(\frac{\partial\xi}{\partial y})^2\frac{\partial^2 u}{\partial\xi^2}+2\frac{\partial\xi}{\partial y}\frac{\partial\eta}{\partial y}\frac{\partial^2 u}{\partial\eta\partial\xi}+(\frac{\partial\eta}{\partial y})^2\frac{\partial^2 u}{\partial\eta^2}\right]+Y.T=0\end{aligned}$$

rearranging terms.

$$\begin{aligned}&\left[A(\frac{\partial\xi}{\partial x})^2+2B\frac{\partial\xi}{\partial x}\frac{\partial\xi}{\partial y}+C(\frac{\partial\xi}{\partial y})^2\right]\frac{\partial^2 u}{\partial\xi^2}+\left[A(\frac{\partial\eta}{\partial x})^2+2B\frac{\partial\eta}{\partial x}\frac{\partial\eta}{\partial y}+C(\frac{\partial\eta}{\partial y})^2\right]\frac{\partial^2 u}{\partial\eta^2}\\
&+\left[2A\frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial x}+2B\left[\frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial y}+\frac{\partial\xi}{\partial y}\frac{\partial\eta}{\partial x}\right]+2C\frac{\partial\xi}{\partial y}\frac{\partial\eta}{\partial y}\right]\frac{\partial^2 u}{\partial\eta\partial\xi}+Y.T=0\end{aligned}$$

we now try to find ξ and η such that.

$$\left[A(\frac{\partial\xi}{\partial x})^2+2B\frac{\partial\xi}{\partial x}\frac{\partial\xi}{\partial y}+C(\frac{\partial\xi}{\partial y})^2\right]=0$$

$$\left[A(\frac{\partial\eta}{\partial x})^2+2B\frac{\partial\eta}{\partial x}\frac{\partial\eta}{\partial y}+C(\frac{\partial\eta}{\partial y})^2\right]=0$$

we notice that both equations are the same quadratic equation, thus we solve for one of them to find both ξ and η, we choose the first one and start by dividing the equation by $(\frac{\partial\xi}{\partial y})^2$.

$$A\frac{(\frac{\partial\xi}{\partial x})^2}{(\frac{\partial\xi}{\partial y})^2}+2B\frac{\left(\frac{\partial\xi}{\partial x}\right)}{\left(\frac{\partial\xi}{\partial y}\right)}+C=0$$

$$A(\frac{\partial y}{\partial x})^2-2B\frac{\partial y}{\partial x}+C=0$$

now using the quadratic formula to solve for $\frac{\partial y}{\partial x}$.

$$\frac{\partial y}{\partial x}=\frac{-(-2B)\pm\sqrt{(-2B)^2-4AC}}{2A}=\frac{B\pm\sqrt{B^2-AC}}{A}$$

this is called the charctaristic equation.

Equations Classification

Equations are classified based on the value of the expression under the root
$B^2>AC$ $\quad\forall x,y\in G$, Hyperbolic PDE (the general case of the wave equation)
$B^2<AC$ $\quad\forall x,y\in G$, Elliptic PDE (the general case of the Laplace equation)
$B^2=AC$ $\quad\forall x,y\in G$, Parabolic PDE (the general case of the Heat equation)

2.1 Examples

■ **Example 2.1.1** Transform to the canonical form.

$$4y^2\frac{\partial^2u}{\partial x^2}-e^{2x}\frac{\partial^2u}{\partial y^2}+\underbrace{6y^3}_{\text{Y.T}}=0$$

—— Solution ——

we start by determining the functions A,B, and C.

$$A(x,y)=4y^2\quad,\quad B(x,y)=0\quad,\quad C(x,y)=-e^{2x}$$

we conclude from this that it has the form of a Hyperbolic PDE.

$$B^2=0\quad>\quad AC=-4y^2e^{2x},\quad\forall y\neq0,\quad\forall x$$

now we use the charchtarstic equation to determine the value of ξ and η.

$$\frac{\partial y}{\partial x}=\frac{B\pm\sqrt{B^2-AC}}{A}$$

$$= \frac{\pm\sqrt{4y^2e^{2x}}}{4y^2} = \pm\frac{e^x}{2y}$$

rearranging and intgrating.

$$\begin{aligned}
2ydy &= \pm e^x dx \\
\int 2ydy &= \pm \int e^x dx \\
y^2 &= \pm e^x + constant \quad \implies \quad y^2 \pm e^x = constant
\end{aligned}$$

we now set ξ and η.

$$\xi = e^x + y^2 \quad , \quad \eta = e^x - y^2$$

we now work out the derivatives.

$$\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x} \\
&= \frac{\partial u}{\partial \xi}e^x + \frac{\partial u}{\partial \eta}e^x \\
\frac{\partial^2 u}{\partial x^2} &= e^x\left[\frac{\partial^2 u}{\partial \xi^2}e^x + \frac{\partial^2 u}{\partial \eta\partial \xi}e^x\right] + e^x\left[\frac{\partial^2 u}{\partial \eta^2}e^x + \frac{\partial^2 u}{\partial \xi\partial \eta}e^x\right] + Y.T \\
&= e^{2x}\frac{\partial^2 u}{\partial \xi^2} + 2e^{2x}\frac{\partial^2 u}{\partial \xi\partial \eta} + e^{2x}\frac{\partial^2 u}{\partial \eta^2} + Y.T \\
\frac{\partial^2 u}{\partial y^2} &= 4y^2\frac{\partial^2 u}{\partial \xi^2} - 8y^2\frac{\partial^2 u}{\partial \xi\partial \eta} + 4y^2\frac{\partial^2 u}{\partial \eta^2} + Y.T
\end{aligned}$$

substituting in our original equation.

$$4y^2\left[e^{2x}\frac{\partial^2 u}{\partial \xi^2} + 2e^{2x}\frac{\partial^2 u}{\partial \xi\partial \eta} + e^{2x}\frac{\partial^2 u}{\partial \eta^2}\right] - e^{2x}\left[4y^2\frac{\partial^2 u}{\partial \xi^2} - 8y^2\frac{\partial^2 u}{\partial \xi\eta} + 4y^2\frac{\partial^2 u}{\partial \eta^2}\right] + Y.T = 0$$

$$\begin{aligned}
16y^2e^{2x}\frac{\partial^2 u}{\partial \xi\partial \eta} + Y.T = 0 \\
\frac{\partial^2 u}{\partial \xi\partial \eta} + Y.T = 0
\end{aligned}$$

■

■ **Example 2.1.2** Transform to the canonical form

$$x^2\frac{\partial^2 u}{\partial x^2} + y^2\frac{\partial^2 u}{\partial y^2} = 0$$

—— Solution ——

determining the functions A,B, and C.

$$A(x,y) = x^2 \quad , \quad B(x,y) = 0 \quad , \quad C(x,y) = y^2$$

it has the form of a Elliptic PDE.

$$B^2 = 0 \quad < \quad AC = x^2y^2, \quad \forall y, x$$

using the charchtarstic equation.

$$\frac{\partial y}{\partial x} = \frac{B \pm \sqrt{B^2 - AC}}{A}$$

$$
\begin{aligned}
&= \frac{\pm\sqrt{-x^2y^2}}{x^2} = \pm i\frac{y}{x} \\
\int \frac{dy}{y} &= \pm i \int \frac{dx}{x} \\
\ln(y) &= \pm i \ln(x) + \text{constant}
\end{aligned}
$$

we will choose ξ to be the imaginary part and η to be the real part.

$$
\xi = \ln(x) \quad , \quad \eta = \ln(y)
$$

working out the derivatives.

$$
\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x} \\
&= \frac{\partial u}{\partial \xi}\frac{1}{x} + \frac{\partial u}{\partial \eta}(0) \\
\frac{\partial^2 u}{\partial x^2} &= \frac{1}{x}\left[\frac{\partial^2 u}{\partial \xi^2}\frac{1}{x} + \frac{\partial^2 u}{\partial \eta \partial \xi}(0)\right] + Y.T \\
\frac{\partial^2 u}{\partial x^2} &= \frac{1}{x^2}\frac{\partial^2 u}{\partial \xi^2} + Y.T \\
\frac{\partial^2 u}{\partial y^2} &= \frac{1}{y^2}\frac{\partial^2 u}{\partial \eta^2} + Y.T
\end{aligned}
$$

substituting in our original equation.

$$
\begin{aligned}
x^2\left[\frac{1}{x^2}\frac{\partial^2 u}{\partial \xi^2}\right] + y^2\left[\frac{1}{y^2}\frac{\partial^2 u}{\partial \eta^2}\right] + Y.T = 0 \\
\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} + Y.T = 0
\end{aligned}
$$

■

■ **Example 2.1.3** Transform to the canonical form

$$
y^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + x^2\frac{\partial^2 u}{\partial y^2} = 0
$$

—— Solution ——

determining the functions A,B, and C.

$$
A(x,y) = y^2 \quad , \quad B(x,y) = xy \quad , \quad C(x,y) = x^2
$$

it has the form of a Parabolic PDE.

$$
B^2 = x^2y^2 \quad = \quad AC = x^2y^2, \quad \forall y, x
$$

using the charchtarstic equation.

$$
\begin{aligned}
\frac{\partial y}{\partial x} &= \frac{B \pm \sqrt{B^2 - AC}}{A} \\
&= \frac{xy}{y^2} = \frac{x}{y} \\
\int y dy &= \int x dx \\
y^2 &= x^2 + \text{constant}
\end{aligned}
$$

we will assign ξ to be this function

$$\xi = y^2 - x^2$$

and for η it's Optional but to make the solution easier we will assign the previous function with different sign

$$\eta = y^2 + x^2 \quad \text{or} \quad \eta = -y^2 - x^2$$

rest of the solution same as Hyperbolic and elliptic PDEs the Canonical form in the end will be

$$\frac{\partial^2 u}{\partial \xi^2} + Y.T = 0$$

$$\text{or}$$

$$\frac{\partial^2 u}{\partial \eta^2} + Y.T = 0$$

■

Observation 2.1 the Canonical form of all Hyperbolic equations is

$$\frac{\partial^2 u}{\partial \xi \partial \eta} + Y.T = 0$$

Observation 2.2 the Canonical form of all elliptic equations is

$$\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} + Y.T = 0$$

Observation 2.3 the Canonical form of all Parabolic equations is

$$\frac{\partial^2 u}{\partial \xi^2} + Y.T = 0$$

$$\text{or}$$

$$\frac{\partial^2 u}{\partial \eta^2} + Y.T = 0$$

3 Heat Equation

The heat equation is a prototypical example for a prabolic equation. The general form of the heat equation is the following.

$$\frac{\partial u(x,t)}{\partial t} = c^2 \nabla^2 u(x,t) = C^2 \left[\frac{\partial^2 u(x)}{\partial x_1^2} + \frac{\partial^2 u(x)}{\partial x_2^2} + \frac{\partial^2 u(x)}{\partial x_3^2} + \dots \right]$$

we will be studying the heat equation only in one dimension thus this reduces to.

$$\frac{\partial u(x,t)}{\partial t} = c^2 \frac{\partial^2 u(x,t)}{\partial x^2}$$

3.1 Fourier Transform

The fourier transform of the function $f(x)$ is defined as.

$$\mathscr{F}\left[f\left(x\right)\right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ixs} f\left(x\right) dx = g\left(s\right)$$

the inverse fourier transform is

$$\mathscr{F}^{-1}\left[g\left(s\right)\right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ixs} g\left(s\right) ds = f\left(x\right)$$

the fourier transform of a derivative

$$\mathscr{F}\left[\frac{df(x)}{dx}\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ixs}\frac{df(x)}{dx}dx \tag{1}$$

notice that

$$\frac{d}{dx}\left[e^{ixs}f(x)\right] = e^{ixs}\frac{df(x)}{dx} + ise^{ixs}f(x) \tag{2}$$

$$e^{ixs}\frac{df(x)}{dx} = \frac{d}{dx}\left[e^{ixs}f(x)\right] - ise^{ixs}f(x) \tag{3}$$

substitute (3) in (2)

$$\mathscr{F}\left[\frac{df(x)}{dx}\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{d}{dx}\left[e^{ixs}f(x)\right]dx - is\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{ixs}f(x)\,dx$$

$$\mathscr{F}\left[\frac{df(x)}{dx}\right] = \frac{1}{\sqrt{2\pi}}\left[e^{ixs}f(x)\right]_{-\infty}^{\infty} - is\mathscr{F}\left[f(x)\right]$$

the first term must vanish as we assume f is absolutely intgrable on$\mathbb{R}$

$$\mathscr{F}\left[\frac{df(x)}{dx}\right] = -is\mathscr{F}\left[f(x)\right]$$

in the same way the fourier transform for the second derivative will yield

$$\begin{aligned}\mathscr{F}\left[\frac{d^2f(x)}{dx^2}\right] &= \mathscr{F}\left[\frac{d}{dx}\frac{df(x)}{dx}\right]\\ &= -is\mathscr{F}\left[\frac{df(x)}{dx}\right] = -s^2\mathscr{F}\left[f(x)\right]\end{aligned}$$

and in general

$$\mathscr{F}\left[\frac{d^nf(x)}{dx^n}\right] = (-is)^n\mathscr{F}\left[f(x)\right]$$

3.2 Cauchy Problem

Consider the following one dimensional heat equation cauchy problem

$$\frac{\partial u(x,t)}{\partial t} = c^2\frac{\partial^2 u(x,t)}{\partial x^2}, \qquad c \neq 0, \qquad -\infty < x < \infty \tag{1}$$

$$u(x,0) = \phi(x) \tag{2}$$

the fourier transform of u is

$$\mathscr{F}[u(x,t)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{ix\xi}u(x,t)\,dx = \nu(\xi,t)$$

preforming the transform on both sides of equation (1)

$$\frac{\partial \nu(\xi,t)}{\partial t} = -c^2\xi^2\nu(\xi,t) \tag{3}$$

Augustin-Louis Cauchy

Baron Augustin-Louis Cauchy (1789-1857) was a renowned French mathematician and physicist who made significant contributions to various fields of mathematics, including partial differential equations (PDEs). His work in PDEs laid the groundwork for modern analysis and helped establish the rigorous theoretical foundation for the study of these equations.

and the new intial condition.

$$\mathscr{F}[u(x,0)] = \nu(\xi,0) = \mathscr{F}[\phi(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ix\xi} \phi(x)\, dx = \psi(\xi) \tag{4}$$

the solution of (3) is.

$$\nu(\xi,t) = Ae^{-c^2\xi^2 t}$$

and A can be found from (4).

$$\nu(\xi,t) = \psi(\xi)e^{-c^2\xi^2 t} \tag{5}$$

to find u we preform the inverse fourier transform on (5).

$$\begin{aligned} u(x,t) = \mathscr{F}^{-1}[\nu(\xi,t)] &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ix\xi} \nu(\xi,t) d\xi \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ix\xi} e^{-c^2\xi^2 t} \psi(\xi) d\xi \end{aligned}$$

substituting the value of ψ from (4) and renamming the varable of intgration to y.

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ix\xi} e^{-c^2\xi^2 t} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iy\xi} \phi(y)\, dy \right] d\xi$$

simplify.

$$u(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} e^{iy\xi - ix\xi - c^2\xi^2 t} d\xi \right] \phi(y) dy \tag{6}$$

considering the inner intgral.

$$J = \int_{-\infty}^{\infty} e^{iy\xi - ix\xi - c^2\xi^2 t} d\xi \tag{7}$$

simplifying the power by completing the square.

$$\begin{aligned} iy\xi - ix\xi - c^2\xi^2 t &= -tc^2 \left(\xi^2 - \frac{i\xi}{tc^2}(y-x) \right) \\ &= -tc^2 \left(\left(\xi - \frac{i}{tc^2}(y-x)\right)^2 + \frac{(y-x)^2}{4t^2c^4} \right) \end{aligned}$$

substituting in (7)

$$\begin{aligned} J &= \int_{-\infty}^{\infty} e^{-tc^2(\xi - \frac{i}{tc^2}(y-x))^2} e^{-tc^2 \frac{(y-x)^2}{4t^2c^4}} d\xi \\ &= e^{-\frac{(y-x)^2}{4tc^2}} \int_{-\infty}^{\infty} e^{-tc^2(\xi - \frac{i}{tc^2}(y-x))^2} d\xi \end{aligned}$$

shift the function by $\frac{i}{tc^2}(y-x)$ to the left and since the limits of intgration are infinity the value of the intgral is the same.

$$J == e^{-\frac{(y-x)^2}{4tc^2}} \int_{-\infty}^{\infty} e^{-tc^2\xi^2} d\xi$$

it is now clear that J is a gaussian intgral that we can eaisly find its value.

$$J = e^{-\frac{(y-x)^2}{4tc^2}} \sqrt{\frac{\pi}{tc^2}}$$

substituing in (6) we finally get u.

$$u(x,t) = \frac{1}{2\sqrt{\pi tc^2}} \int_{-\infty}^{\infty} e^{-\frac{(y-x)^2}{4tc^2}} \phi(y) dy$$

this equation is called Poisson's Formula.

3.3 Mixed Homogeneous Problem

Definition 3.1 — Mixed Problem. is a problem that has both Drichlet boundry condition and Cauchy intial condition

Consider the following one dimensional heat equation

$$\frac{\partial u(x,t)}{\partial t} = c^2 \frac{\partial^2 u(x,t)}{\partial x^2}, \qquad c \neq 0, \qquad 0 < x < L \tag{1}$$

$$\text{I.C} \quad \Longrightarrow \quad u(x,0) = \phi(x) \tag{2}$$

$$\text{B.C} \quad \Longrightarrow \quad u(0,t) = u(L,t) = 0 \tag{3}$$

using the sepration of variables method.

$$u(x,t) = X(x)T(t) \tag{4}$$

substituting in (1)

$$X(x)\dot{T}(t) = c^2 X''(x)T(t)$$
$$\frac{\dot{T}(t)}{c^2 T(t)} = \frac{X''(x)}{X(x)} = \lambda$$

now there are 3 cases for λ

1. if $\lambda > 0$ then the solution of $X(x)$ will be

   $$X(x) = A\cosh(\sqrt{\lambda}x) + B\sinh(\sqrt{\lambda}x)$$

   and by using the boundary condition $u(0,t) = 0$ we get that $X(0) = 0$ then

   $$0 = A\cosh(\sqrt{\lambda}0) + B\sinh(\sqrt{\lambda}0)$$
   $$\therefore 0 = A$$

   and by using the boundary condition $u(L,t) = 0$ we get that $X(L) = 0$ then

   $$0 = B\sinh(\sqrt{\lambda}L)$$

   $$\text{either B} = 0 \text{ or } \sinh(\sqrt{\lambda}L) = 0$$

   $\sinh(\sqrt{\lambda}L)$ cannot equal to zero because it's roots are $i\pi n, n \in \mathbb{N}$ they are imaginary except when n $= 0$ but that cannot happen because we choose $\lambda > 0$ and $L > 0$

   $$\therefore B = 0$$

   then the $X(x) = 0$ then $u(x,t) = 0$ which is a trivial solution
2. if $\lambda = 0$ then $X(x) = 0$ then $u(x,t) = 0$ which is also a trivial solution
3. if $\lambda < 0$ then the solution of $X(x)$ will be

   $$X(x) = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x)$$

   and by using the boundary condition $u(0,t) = 0$ we get that $X(0) = 0$ then

   $$0 = A\cos(\sqrt{\lambda}0) + B\sin(\sqrt{\lambda}0)$$
   $$\therefore 0 = A$$

   and by using the boundary condition $u(L,t) = 0$ we get that $X(L) = 0$ then

   $$0 = B\sin(\sqrt{\lambda}L)$$

now if B $= 0$ it gives trivial solution then $\sin(\sqrt{\lambda}L) = 0$

$$\because \sin(\sqrt{\lambda}L) = 0$$
$$\therefore \sqrt{\lambda}L = n\pi$$
$$\lambda = \frac{n^2\pi^2}{L^2}$$

and to make sure it is a negative number we will add a negative sign to it

$$-\lambda = -\frac{n^2\pi^2}{L^2}$$

therefore $X(x)$ will be.

$$\frac{X''(x)}{X(x)} = -\lambda = -\frac{n^2\pi^2}{L^2}$$
$$X(x) = B\sin\left(\frac{n\pi x}{L}\right)$$

now solving for $T(t)$ using our constant.

$$\dot{T}(t) = -\frac{c^2n^2\pi^2}{L^2}T(t)$$
$$T(t) = Ae^{-\frac{c^2n^2\pi^2}{L^2}}$$

substituting in (4).

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-\frac{c^2n^2\pi^2}{L^2}} \sin\left(\frac{n\pi x}{L}\right)$$

using the intial condition.

$$u(x,0) = \phi(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

we found fourier coefficient which is our final step to determine u.

Fourier Coefficient

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{n\pi x}{L}\right) + b_n\sin\left(\frac{n\pi x}{L}\right)\right]$$
$$a_0 = \frac{1}{L}\int_0^L f(x)dx$$
$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$
$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

3.4 Mixed Inhomogeneous Problem

Consider a more general heat equation than the previous one. Notice that the boundry conditions are functions not zeros as before.

$$\frac{\partial u(x,t)}{\partial t} = c^2 \frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t), \qquad c \neq 0, \quad 0 < x < L \tag{1}$$

$$\text{I.C} \quad \Longrightarrow \quad u(x,0) = \phi(x) \tag{2}$$

$$\text{B.C} \quad \Longrightarrow \quad \begin{cases} u(0,t) = A(t) \\ u(L,t) = B(t) \end{cases}$$

we start by defining a new function v as.

$$v(x,t) = u(x,t) - u_E(x,t)$$

we find the equilibrium temperature $u_E(x,t)$ by solving

$$\frac{\partial^2 u_E(x,t)}{\partial x^2} = 0$$

$$\text{B.C} \quad \Longrightarrow \quad \begin{cases} u_E(0,t) = A(t) \\ u_E(L,t) = B(t) \end{cases}$$

we get that $u_E(x,t) = c_1 x + c_2$ and by using the B.C we can find c_1, c_2 as following

$$u_E(x,t) = \frac{x}{L}[B(t) - A(t)] + A(t)$$

thus we have.

$$v(x,t) = u(x,t) - \frac{x}{L}[B(t) - A(t)] - A(t) \tag{3}$$

thus we have the new equation.

$$\begin{aligned} \frac{\partial v(x,t)}{\partial t} &= \frac{\partial u(x,t)}{\partial t} - \frac{\partial R(x,t)}{\partial t} \\ \frac{\partial^2 v(x,t)}{\partial x^2} &= \frac{\partial^2 u(x,t)}{\partial x^2} \\ \frac{\partial v(x,t)}{\partial t} - c^2 \frac{\partial^2 v(x,t)}{\partial x^2} &= \left[\frac{\partial u(x,t)}{\partial t} - c^2 \frac{\partial^2 u(x,t)}{\partial x^2}\right] - \frac{\partial R(x,t)}{\partial t} \\ &= f(x,t) - \frac{\partial R(x,t)}{\partial t} = g(x,t) \end{aligned}$$

we finally have the new equation.

$$\frac{\partial v(x,t)}{\partial t} = c^2 \frac{\partial^2 v(x,t)}{\partial x^2} + g(x,t) \tag{4}$$

$$\text{I.C} \quad \Longrightarrow \quad v(x,0) = \phi(x) - \frac{x}{L}[B(0) - A(0)] - A(0) = \psi(x) \tag{5}$$

$$\text{B.C} \quad \Longrightarrow \quad \begin{cases} v(0,t) = u(0,t) - A(t) = 0 \\ v(L,t) = u(L,t) - B(t) = 0 \end{cases} \tag{6}$$

we have solved a similer problem before and know that the solution for the homogenous equation in v will be of the form.

$$v(x,t) = \sum_{n=0}^{\infty} T_n(t) \sin\left(\frac{n\pi x}{L}\right) \tag{7}$$

similerly

$$g(x,t) = \sum_{n=0}^{\infty} g_n(t) \sin\left(\frac{n\pi x}{L}\right) \tag{8}$$

substitute (7) , (8) in (4)

$$\sum_{n=1}^{\infty} \frac{dT_n(t)}{dt} \sin\left(\frac{n\pi x}{L}\right) = c^2 \sum_{n=1}^{\infty} \left(-\frac{n^2\pi^2}{L^2}\right) T_n(t) \sin\left(\frac{n\pi x}{L}\right) + \sum_{n=0}^{\infty} g_n(t) \sin\left(\frac{n\pi x}{L}\right)$$

by comparing the coefficients of $\sin\left(\frac{n\pi x}{L}\right)$ in both sides we get that

$$\frac{dT_n(t)}{dt} + c^2 \frac{n^2\pi^2}{L^2} T_n(t) = g_n(t)$$

this is an linear ODE and it's solution like

$$y' + f(x)y = g(x)$$

it's solution is

$$\mu y = \int \mu g(x)dx + c \qquad \text{where} \qquad \mu = e^{\int f(x)dx}$$

then in our case

$$\mu = e^{\int c^2 \frac{n^2\pi^2}{L^2} dt} = e^{c^2 \frac{n^2\pi^2}{L^2} t}$$

$$\therefore T_n(t) = e^{-\frac{n^2c^2\pi^2}{L^2} t} c + e^{-\frac{n^2c^2\pi^2}{L^2} t} \int_0^t e^{\frac{n^2c^2\pi^2}{L^2} s} G(s)ds$$

and to get c put $t = 0$

$$T_n(0) = c$$

$$\therefore T_n(t) = e^{-\frac{n^2c^2\pi^2}{L^2} t} T_n(0) + \int_0^t e^{\frac{n^2c^2\pi^2}{L^2}(s-t)} G(s)ds$$

now by substitute $T_n(t)$ in (7) we get v then we can obtain u by $u = v + u_E$.

4 Wave Equation

The wave equation is the prototypical example for hyperpolic PDE's, it is a second order linear PDE that is used extensively in physics in studying the propagation of waves and wave fields. The general form of the wave equation is.

$$\frac{\partial^2 u(x,t)}{\partial t^2} = c^2 \nabla^2 u(x,t) = C^2 \left[\frac{\partial^2 u(x)}{\partial x_1^2} + \frac{\partial^2 u(x)}{\partial x_2^2} + \frac{\partial^2 u(x)}{\partial x_3^2} + \dots\right]$$

we will be studying one dimensional and three dimensional wave equations.

4.1 Cauchy Problem (1D)

Consider the following one dimensional wave equation Cauchy problem.

$$\frac{\partial^2 u(x,t)}{\partial t^2} = c^2 \frac{\partial^2 u(x,t)}{\partial x^2}, \quad c \neq 0, \quad t > 0 \tag{1}$$

$$\text{I.C} \quad \Longrightarrow \quad \begin{cases} u(x,0) = \phi(x) \\ \frac{\partial u(x,0)}{\partial t} = \psi(x) \end{cases} \tag{2}$$

we start by preforming a change of variables such that.

$$\xi = x + ct \quad , \quad \eta = x - ct$$

$$u(\xi, \eta)$$

$$
\begin{aligned}
\frac{\partial u}{\partial t} &= \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial t} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial t} \\
\frac{\partial^2 u}{\partial t^2} &= \frac{\partial}{\partial t}\left[\frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial t} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial t}\right] \\
&= \frac{\partial}{\partial t}\left[\frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial t}\right] + \frac{\partial}{\partial t}\left[\frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial t}\right] \\
&= \frac{\partial u}{\partial \xi}\frac{\partial^2 \xi}{\partial t^2} + \frac{\partial \xi}{\partial t}\left[\frac{\partial^2 u}{\partial \xi^2}\frac{\partial \xi}{\partial t} + \frac{\partial^2 u}{\partial \eta \partial \xi}\frac{\partial \eta}{\partial t}\right] + \frac{\partial u}{\partial \eta}\frac{\partial^2 \eta}{\partial t^2} + \frac{\partial \eta}{\partial t}\left[\frac{\partial^2 u}{\partial \eta^2}\frac{\partial \eta}{\partial t} + \frac{\partial^2 u}{\partial \xi \partial \eta}\frac{\partial \xi}{\partial t}\right]
\end{aligned}
$$

adding similar terms and simlifying.

$$
\because \frac{\partial \xi}{\partial t} = c \quad , \quad \frac{\partial \eta}{\partial t} = -c \quad , \quad \frac{\partial^2 \xi}{\partial t^2} = 0 \quad , \quad \frac{\partial^2 \eta}{\partial t^2} = 0
$$

$$
\therefore \frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial \xi^2} - 2c^2\frac{\partial^2 u}{\partial \xi \partial \eta} + c^2\frac{\partial^2 u}{\partial \eta^2} \tag{3}
$$

in similar manner.

$$
\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial \xi^2} + 2\frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\partial^2 u}{\partial \eta^2} \tag{4}
$$

substituting (3) and (4) in (1).

$$
c^2\frac{\partial^2 u}{\partial \xi^2} - 2c^2\frac{\partial^2 u}{\partial \xi \partial \eta} + c^2\frac{\partial^2 u}{\partial \eta^2} = c^2\left[\frac{\partial^2 u}{\partial \xi^2} + 2\frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\partial^2 u}{\partial \eta^2}\right]
$$

simplify.

$$
\frac{\partial^2 u}{\partial \xi \partial \eta} = \frac{\partial}{\partial \eta}\left(\frac{\partial u}{\partial \xi}\right) = 0
$$

we can deduce that $\frac{\partial u}{\partial \xi}$ is only a function in ξ thus.

$$
\begin{aligned}
u(\xi, \eta) &= f(\xi) + g(\eta) \\
u(x,t) &= f(x+ct) + g(x-ct) \qquad (5) \\
\frac{\partial u(x,t)}{\partial t} &= cf'(x+ct) - cg'(x-ct)
\end{aligned}
$$

from the intial conditions (2).

$$
\begin{aligned}
\phi(x) &= u(x,0) = f(x) + g(x) \\
\phi'(x) &= f'(x) + g'(x) \qquad (6) \\
\psi(x) &= \frac{\partial u(x,0)}{\partial t} = cf'(x) - cg'(x) \qquad (7)
\end{aligned}
$$

multiplaying (6) by c and adding to (7).

$$
\begin{aligned}
f'(x) &= \frac{1}{2c}\left(c\phi'(x) + \psi(x)\right) \\
f(x) &= \frac{1}{2c}\left[c\phi(x) + \int_0^x \psi(s)\,ds\right] + A \qquad (8)
\end{aligned}
$$

multiplaying (6) by c and subtracting (7) from it.

$$
\begin{aligned}
g'(x) &= \frac{1}{2c}\left(c\phi'(x) - \psi(x)\right) \\
g(x) &= \frac{1}{2c}\left[c\phi(x) - \int_0^x \psi(s)\,ds\right] + B \qquad (9)
\end{aligned}
$$

substituting (8) and (9) in (5)

$$u(x,t) = \frac{1}{2c}\left[c\phi(x+ct) + \int_0^{x+ct} \psi(s)\,ds\right] + \frac{1}{2c}\left[c\phi(x-ct) - \int_0^{x-ct} \psi(s)\,ds\right] + K$$

$$u(x,t) = \frac{1}{2}\left[\phi(x+ct) + \phi(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,ds + K$$

from the intial condition $u(x,0)$, we find that K=0 and finally we are left with.

$$u(x,t) = \frac{1}{2}\left[\phi(x+ct) + \phi(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,ds$$

this Equation is called d'Alembert formula.

4.2 Mixed Problem (1D)

Consider the following one dimensional wave equation mixed problem.

$$\frac{\partial^2 u(x,t)}{\partial t^2} = \frac{\partial^2 u(x,t)}{\partial x^2} \tag{1}$$

$$\text{I.C} \quad \Longrightarrow \quad \begin{cases} u(x,0) = \phi(x) \\ \frac{\partial u(x,0)}{\partial t} = \psi(x) \end{cases} \tag{2}$$

$$\text{B.C} \quad \Longrightarrow \quad \begin{cases} u(0,t) = 0 \\ u(\pi,t) = 0 \end{cases} \tag{3}$$

we start our solution by seprating the variables.

$$u(x,t) = X(x)\,T(t) \tag{4}$$

substituting in (1).

$$\ddot{T}(t)\,X(x) = T(t)\,X''(x)$$

$$\frac{\ddot{T}(t)}{T(t)} = \frac{X''(x)}{X(x)} = \text{constant} = -n^2, \quad \forall n \in \mathbb{N}$$

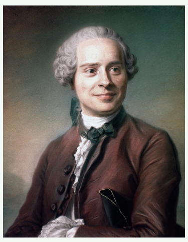
and like what happened in the heat equation we will follow the case that the constant is negative

$$X''(x) = -n^2 X(x)$$

$$\ddot{T}(t) = -n^2 T(t)$$

Jean le Rond d'Alembert

Jean le Rond d'Alembert (1717-1783) was a renowned French mathematician and physicist who made significant contributions to partial differential equations. His groundbreaking work on wave equations and the d'Alembertian operator laid the foundation for understanding wave-like behavior in various physical systems, influencing modern engineering, acoustics, and mathematical physics. D'Alembert's profound insights continue to be integral to the study of PDEs, serving as a lasting legacy in the field of mathematical analysis.



this is an equation in the form of simple harmonic motion thus the solution is.

$$X(x) = Asin(nx) + Bcos(nx)$$
$$T(t) = Csin(nt) + Dcos(nt)$$

substituting in (4).

$$u(x,t) = [Asin(nx) + Bcos(nx)][Csin(nt) + Dcos(nt)]$$

using the boundry condition (3).

$$B[Csin(nt) + Dcos(nt)] = 0$$

and we choose $B = 0$, the general solution for all n will then be.

$$u(x,t) = \sum_{n=0}^{\infty} [a_n \sin(nt) + b_n \cos(nt)] \sin(nx)$$

using the intial condition $u(x,0)$ (2) to find the constants.

$$u(x,0) = \phi(x) = \sum_{n=0}^{\infty} b_n \sin(nx)$$

and from fourier coefficients we can find b_n

$$b_n = \frac{2}{\pi} \int_0^{\pi} \phi(x) \sin(nx)\, dx$$

to find the other constant a_n.

$$\frac{\partial u(x,t)}{\partial t} = \sum_{n=0}^{\infty} [na_n \cos(nt) - nb_n \sin(nt)] \sin(nx)$$

using the intial condition for the dierivative.

$$\frac{\partial u(x,0)}{\partial t} = \psi(x) = \sum_{n=0}^{\infty} na_n \sin(nx)$$

$$nb_n = \frac{2}{\pi} \int_0^{\pi} \psi(x) \sin(nx)$$
$$b_n = \frac{2}{n\pi} \int_0^{\pi} \psi(x) \sin(nx)$$

substitute a_n and b_n to get $u(x,t)$

4.3 Spherical Mean and Darboux's Equation

The spherical mean of a function f is defined as.

$$M_f(x,r) = \frac{1}{\omega_n r^{n-1}} \int_{|y-x|=r} f(y) dS_y \tag{1}$$

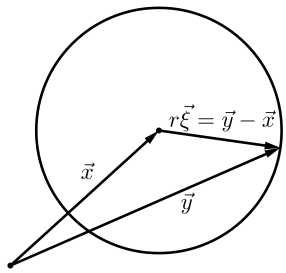
where ω_n is the surface area of an n-diminsional unit sphere[1], thus the amount $\omega_n r^{n-1}$ is the surface area of an n-diminsional sphere of raduis r, dS_y is the element of intgration on the surface of the sphere, $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ are vectors in n-diminsions, where x is the center of the sphere and y is any point of the surface on the sphere, thus we have $|y-x|^2 = (y_1 - x_1)^2 + (y_2 - x_2)^2 + \cdots + (y_n - x_n)^2 = r^2$, and the intgral is over the whole surface of the sphere.

we will show that this function satisfy an important partial differential equation, which is the Euler-Poisson-Darboux equation or just Darboux's equation.

$$\frac{\partial^2 V}{\partial r^2} + \frac{n-1}{r}\frac{\partial V}{\partial r} = \nabla_x^2 V \tag{2}$$

where ∇_x^2 is the Laplace-Beltrami operator, a generaliation of the laplace operator in n-diminsional spherical coordinates.

we will start by defining the unit vector ξ such that the vector is normal to the surface.



$$\begin{aligned} y - x &= \xi r \\ y &= x + \xi r \\ |y-x| = |\xi| r &= r, \qquad |\xi| = 1 \end{aligned} \tag{3}$$

thus the surface elements are related as.

$$dS_y = r^{n-1} dS_\xi$$

substituting in (1).

$$M_f(x,r) = \frac{1}{\omega_n} \int_{|\xi|=1} f(x + \xi r) dS_\xi \tag{4}$$

at $r = 0$.

$$M_f(x,0) = \frac{f(x)}{\omega_n} \int_{|\xi|=1} dS_\xi = f(x) \tag{5}$$

the intgral $\int_{|\xi|=1} dS_\xi$ is the surface area of a unit sphere in n diminsions which is equale to ω_n

from (3) and remembering that y, x, and ξ are all vectors, then we can write their components as.

$$y_i = x_i + \xi_i r$$

[1] for n =3 or the three diminsional case $\omega_3 = 4\pi$

using this vector notation to differntiate (4) with respect to r.

$$\begin{aligned}
\frac{\partial M_f(x,r)}{\partial r} &= \frac{1}{\omega_n}\int_{|\xi|=1}\frac{\partial f(y)}{\partial r}dS_\xi \\
&= \frac{1}{\omega_n}\int_{|\xi|=1}\sum_{i=1}^{n}\frac{\partial f(y)}{\partial y_i}\frac{\partial y_i}{\partial r}dS_\xi \\
&= \frac{1}{\omega_n}\int_{|\xi|=1}\sum_{i=1}^{n}\frac{\partial f(y)}{\partial y_i}\xi_i dS_\xi
\end{aligned}$$

using the chain rule we find. $\frac{\partial f(y)}{\partial y_i}\frac{\partial x_i}{\partial x_i} = \frac{\partial f(y)}{\partial y_i}\frac{\partial y_i}{\partial x_i}\frac{\partial x_i}{\partial y_i} = \frac{\partial f(y)}{\partial x_i}$, since $\frac{\partial x_i}{\partial y_i} = 1$

$$\frac{\partial M_f(x,r)}{\partial r} = \frac{1}{\omega_n}\int_{|\xi|=1}\sum_{i=1}^{n}\frac{\partial f(y)}{\partial x_i}\xi_i dS_\xi$$

using the chain rule one more time. $\frac{\partial f(y)}{\partial x_i}\frac{\partial \xi_i}{\partial \xi_i} = \frac{\partial f(y)}{\partial x_i}\frac{\partial x_i}{\partial \xi_i}\frac{\partial \xi_i}{\partial x_i} = \frac{\partial f(y)}{\partial \xi_i}\frac{1}{r}$, since $\frac{\partial \xi_i}{\partial x_i} = \frac{1}{r}$

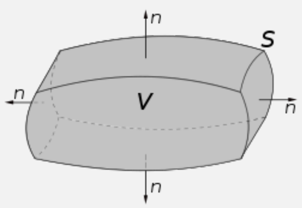
$$\frac{\partial M_f(x,r)}{\partial r} = \frac{1}{\omega_n r}\int_{|\xi|=1}\sum_{i=1}^{n}\frac{\partial f(y)}{\partial \xi_i}\xi_i dS_\xi \tag{6}$$

Theorem 4.1 — Divergence Theorem. Also called Green-Gauss-Ostrogradsky theorem realtes the volume intgral to the surface intgral under some conditions, more precisely, the surface integral of a vector field over a closed surface (flux) is equal to the volume integral of the divergence over the region. We will use this later in our treatment.

$$\int\int\int_{Volume}\frac{\partial \mathbf{F}(x,y,z)}{\partial x}dV = \int\int_{Surface}\mathbf{F}(x,y,z)\cos(\angle x,\vec{n})dS$$

or in general.

$$\int\int\int_{Volume}\nabla\cdot\mathbf{F}dV = \int\int_{Surface}\mathbf{F}\cdot\vec{n}dS$$



comparing the form we have with the divergence theorem we see that the sum we have is the dot product of our vector function with ξ[2] which is the unit vector always normal to the surface of the sphere by our definition, and we can transform our integral to a volume integral.

$$\frac{\partial M_f(x,r)}{\partial r} = \frac{1}{\omega_n r}\int_{|\xi|\le 1}\sum_{i=1}^{n}\frac{\partial^2 f(y)}{\partial \xi_i^2}d\xi$$

using the chain rule we have.

$$\frac{\partial M_f(x,r)}{\partial r} = \frac{r^2}{\omega_n r}\int_{|\xi|\le 1}\sum_{i=1}^{n}\frac{\partial^2 f(y)}{\partial x_i^2}d\xi$$

[2]the components ξ_i are dirction cosines, i.e $\xi_1^2+\xi_2^2+...+\xi_n^2=1$

$$= \frac{r}{\omega_n} \nabla_x^2 \int_{|\xi| \leq 1} f(y) d\xi$$

notice that .

$$\frac{\partial M_f(x,0)}{\partial r} = 0$$

now we will change our volume element from the relation (3).[3]

$$d\xi = \frac{1}{r^n} dy$$

thus we have

$$\begin{aligned}
\frac{\partial M_f(x,r)}{\partial r} &= \frac{r}{\omega_n r^n} \nabla_x^2 \int_{|y-x| \leq r} f(y) dy \\
&= \frac{1}{\omega_n r^{n-1}} \nabla_x^2 \int_{|y-x| \leq r} f(y) dy \\
&= \frac{1}{\omega_n r^{n-1}} \nabla_x^2 \int_{\rho=0}^{r} \int_{|y-x| = \rho} f(y) dS_y d\rho
\end{aligned}$$

rearranging and multiply and divide by ρ^{n-1}.

$$r^{n-1} \frac{\partial M_f(x,r)}{\partial r} = \nabla_x^2 \int_{\rho=0}^{r} \frac{\rho^{n-1}}{\omega_n \rho^{n-1}} \int_{|y-x| = \rho} f(y) dS_y d\rho$$

notice that on the right hand side an expression the same as the spherical mean (4) is attained thus we substitute .

$$r^{n-1} \frac{\partial M_f(x,r)}{\partial r} = \nabla_x^2 \int_{\rho=0}^{r} \rho^{n-1} M_f(x,r) d\rho$$

diffrentiating the expression with respect to r.

$$\frac{\partial}{\partial r} \left(r^{n-1} \frac{\partial M_f(x,r)}{\partial r} \right) = \nabla_x^2 \frac{\partial}{\partial r} \int_{\rho=0}^{r} \rho^{n-1} M_f(x,r) d\rho$$

$$r^{n-1} \frac{\partial^2 M_f(x,r)}{\partial r^2} + (n-1) r^{n-2} r^{n-1} \frac{\partial M_f(x,r)}{\partial r} = \nabla_x^2 r^{n-1} M_f(x,r)$$

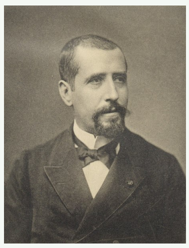
rearranging and simplifying we finally arive at Darboux's equation (2) where $V = M_f(x,r)$.

$$\frac{\partial^2 M_f(x,r)}{\partial r^2} + \frac{n-1}{r} \frac{\partial M_f(x,r)}{\partial r} = \nabla_x^2 M_f(x,r)$$

[3] remember that y is a vector thus $dy = dy_1 dy_2 dy_3 ... dy_n$ and since $dy_i = r d\xi_i$ then $dy = rrr ... d\xi_1 d\xi_2 d\xi_3 ...$

Jean Gaston Darboux

Born in Nimes Jean Gaston Darboux (1842-1917) was a renowned French mathematician who made significant contributions to Partial Differential Equations (PDEs). Darboux's work centered on the theory of linear and nonlinear partial differential equations, which are fundamental to understanding various physical phenomena and mathematical models. He tackled important questions related to the existence, uniqueness, and regularity of solutions to these equations, laying the groundwork for future investigations in the field.



4.4 Kirchhoff's Formula

We have seen how formula d'Almbert can solve the one dimensional wave equation cauchy problem but the solution of the wave equation in dimensions higher than one shows to be more complicated. Our attempt to solve the three dimensional wave equation we will use the results of the spherical mean and Darboux's equation to develop an effective formula to solve three diminsional wave equation Cahuchy problems called Kircchoff's formula.

Consider the following three dimensional wave equation Cauchy problem.

$$\frac{\partial^2 u(x,t)}{\partial t^2} = c^2 \nabla_x^2 u(x,t), \qquad c \neq 0, \quad t > 0 \tag{1}$$

$$\text{I.C} \quad \Longrightarrow \quad \begin{cases} u\,(x,0) = \phi\,(x) \\ \\ \dfrac{\partial u\,(x,0)}{\partial t} = \psi\,(x) \end{cases} \tag{2}$$

where $x = (x_1, x_2, x_3)$ is a three diminsional vector.

we start by considering Darboux's equation (54) in n = 3.

$$\begin{aligned} \frac{\partial^2 M_f(x,r)}{\partial r^2} + \frac{2}{r}\frac{\partial M_f(x,r)}{\partial r} &= \nabla_x^2 M_f(x,r) \\ r\frac{\partial^2 M_f(x,r)}{\partial r^2} + 2\frac{\partial M_f(x,r)}{\partial r} &= r\nabla_x^2 M_f(x,r) \end{aligned} \tag{3}$$

we notice that.

$$\begin{aligned} \frac{\partial}{\partial r}\left(rM_f(x,r)\right) &= r\frac{\partial M_f(x,r)}{\partial r} + M_f(x,r) \\ \frac{\partial^2}{\partial r^2}\left(rM_f(x,r)\right) &= r\frac{\partial^2 M_f(x,r)}{\partial r^2} + \frac{\partial M_f(x,r)}{\partial r} + \frac{\partial M_f(x,r)}{\partial r} \end{aligned}$$

$$\frac{\partial^2}{\partial r^2}\left(rM_f(x,r)\right) = r\frac{\partial^2 M_f(x,r)}{\partial r^2} + 2\frac{\partial M_f(x,r)}{\partial r} \tag{4}$$

substituting (4) in (3).

$$\frac{\partial^2}{\partial r^2}\left(rM_f(x,r)\right) = r\nabla_x^2 M_f(x,r) \tag{5}$$

we now consider the spherical mean of the function u where n = 3.[1]

$$M_u(x,t,r) = \frac{1}{4\pi r^2}\int_{|y-x|=r} f(y)\,dS_y$$

calculating the spherical mean of the whole equation (1) we get.

$$\frac{\partial^2}{\partial t^2}M_u(x,t,r) = c^2\nabla_x^2 M_u(x,t,r)$$

multiply by r.

$$\frac{\partial^2}{\partial t^2} rM_u(x,t,r) = c^2 r\nabla_x^2 M_u(x,t,r)$$

from (5) we get.

$$\frac{\partial^2}{\partial t^2}\left[rM_u(x,t,r)\right] = c^2\frac{\partial^2}{\partial r^2}\left[rM_u(x,t,r)\right] \tag{6}$$

[1] when n = 3 then $\omega_3 = 4\pi$ the surface area of a 3d sphere

the intial conditions will be.

$$rM_u(x,0,r) = rM_\phi(x,r) = \frac{r}{4\pi r^2}\int_{|y-x|=r} \phi(y)dS_y \tag{7}$$

$$\frac{\partial}{\partial t}\left(rM_u(x,0,r)\right) = rM_\psi(x,r) = \frac{r}{4\pi r^2}\int_{|y-x|=r} \psi(y)dS_y \tag{8}$$

we notice that this is a one diminsional wave equation that can be solved using the method developed earlier d'Alembert formula.

$$rM_u(x,t,r) = \frac{1}{2}\left[(r+ct)M_\phi(x,r+ct) + (r-ct)M_\phi(x,r-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} sM_\psi(x,s)ds$$

$$M_u(x,t,r) = \frac{1}{2r}\left[(r+ct)M_\phi(x,r+ct) + (r-ct)M_\phi(x,r-ct)\right] + \frac{1}{2cr}\int_{x-ct}^{x+ct} sM_\psi(x,s)ds \tag{9}$$

we note that.

$$\lim_{r\to 0} M_u(x,t,r) = u(x,t)$$

now we start studying the limit to find our target function $u(x,t)$.

$$\begin{aligned}\lim_{r\to 0} M_u(x,t,r) = \lim_{r\to 0}\frac{1}{2r}&\left[(r+ct)M_\phi(x,r+ct) + (r-ct)M_\phi(x,r-ct)\right]\\ &+ \lim_{r\to 0}\frac{1}{2cr}\int_{r-ct}^{r+ct} sM_\psi(x,s)ds\end{aligned} \tag{10}$$

the first limit can be rearranged to.

$$\lim_{r\to 0}\frac{1}{2r}\left[rM_\phi(x,r+ct) + ctM_\phi(x,r+ct) + rM_\phi(x,r-ct) - ctM_\phi(x,r-ct)\right]$$

rearranging one more time and noting that the spherical mean function is an even function $M_f(r) = M_f(-r)$ thus we can multiplay the r varialble by negative one.

$$\begin{gathered}\lim_{r\to 0}\frac{1}{2r}\left[rM_\phi(x,r+ct) + rM_\phi(x,r-ct)\right] + \lim_{r\to 0}\frac{1}{2r}ct\left[M_\phi(x,r+ct) - M_\phi(x,ct-r)\right]\\ \frac{1}{2}\lim_{r\to 0}\left[M_\phi(x,r+ct) + M_\phi(x,r-ct)\right] + ct\lim_{r\to 0}\frac{\left[M_\phi(x,r+ct) - M_\phi(x,ct-r)\right]}{2r}\\ \frac{1}{2}\left[M_\phi(x,ct) + M_\phi(x,-ct)\right] + ct\frac{\partial M_\phi(x,ct)}{\partial ct}\end{gathered}$$

the first limit is equale to $M_\phi(x,ct)$ due to the fnction being even and we notice that the second limit is the defenition of the derivative$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x-h)}{2h}$ with respect to ct thus the limit is finally evaluted to.

$$\lim_{r\to 0}\frac{1}{2r}\left[(r+ct)M_\phi(x,r+ct) + (r-ct)M_\phi(x,r-ct)\right] = M_\phi(x,ct) + ct\frac{\partial}{\partial ct}(M_\phi(x,ct))$$

consider .

$$\frac{\partial}{\partial t}[tM_\phi(x,ct)] = M_\phi(x,ct) + t\frac{\partial M_\phi(x,ct)}{\partial ct}\frac{\partial ct}{\partial t} = M_\phi(x,ct) + ct\frac{\partial M_\phi(x,ct)}{\partial ct}$$

$$\therefore \lim_{r\to 0}\frac{1}{2r}\left[(r+ct)M_\phi(x,r+ct) + (r-ct)M_\phi(x,r-ct)\right] = \frac{\partial}{\partial t}[tM_\phi(x,ct)]$$

now considering the second limit.

$$\lim_{r\to 0}\frac{1}{2cr}\int_{r-ct}^{r+ct} sM_\psi(x,s)ds$$

direct substitution will result in an indetermined form $\frac{0}{0}$ since the function inside the intgral is an odd function thus the symmetrical intgral will vanish thus we will utalize l'Hopital's rule to find the limit.

l'Hopital's rule

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$$

$$\lim_{r \to 0} \frac{1}{2cr} \int_{r-ct}^{r+ct} sM_{\psi}(x,s)ds = \lim_{r \to 0} \frac{1}{2c} \frac{\partial}{\partial r} \left[\int_{r-ct}^{r+ct} sM_{\psi}(x,s)ds \right]$$

to find the derivative of the intgral we use Libenz rule.

Libenz rule

$$\frac{d}{dx} \int_{a(x)}^{b(x)} f(x,y)dy = \frac{db(x)}{dx} f(x,b(x)) - \frac{da(x)}{dx} f(x,a(x)) + \int_{a(x)}^{b(x)} \frac{\partial f(x,y)}{\partial x} dy$$

in our case

$$\begin{aligned}
\frac{\partial}{\partial r} \left[\int_{r-ct}^{r+ct} sM_{\psi}(x,s)ds \right] &= (r+ct)M_{\psi}(x,r+ct) - (r-ct)M_{\psi}(x,r-ct) + \int_{r-ct}^{r+ct} \frac{\partial}{\partial r} sM_{\psi}(x,s)ds \\
&= ct[M_{\psi}(x,r+ct) + M_{\psi}(x,r-ct)]
\end{aligned}$$

thus the limit will become.

$$\lim_{r \to 0} \frac{1}{2cr} \int_{r-ct}^{r+ct} sM_{\psi}(x,s)ds = \lim_{r \to 0} \frac{ct[M_{\psi}(x,r+ct) + M_{\psi}(x,r-ct)]}{2c}$$

which is finally evaluated to.

$$\lim_{r \to 0} \frac{1}{2cr} \int_{r-ct}^{r+ct} sM_{\psi}(x,s)ds = tM_{\psi}(x,ct)$$

we now finally substitute the first and the second limit in (10).

$$u(x,t) = \lim_{r \to 0} M_u(x,t,r) = \frac{\partial}{\partial t}[tM_{\phi}(x,ct)] + tM_{\psi}(x,ct)$$

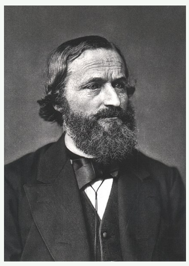
using the defenition of spherical mean with $r = ct$ our final formula becomes.

$$u(x,t) = \frac{\partial}{\partial t} \left[\frac{1}{4\pi c^2 t} \int_{|y-x|=ct} \phi(y)dS_y \right] + \frac{1}{4\pi c^2 t} \int_{|y-x|=ct} \psi(y)dS_y$$

which is called Kirchhoff's Formula.

Gustav Kirchhoff

Gustav Kirchhoff (1824-1887) was a prominent German physicist Kirchhoff made significant advancements in the understanding of wave phenomena and the mathematical modeling of physical systems using PDEs. One of Kirchhoff's most notable contributions in PDEs is his formulation of the Kirchhoff equations, which describe the propagation of waves in elastic materials, such as solids and thin plates. These equations are fundamental to the study of vibrations, acoustics, and the behavior of mechanical systems subjected to dynamic loads.



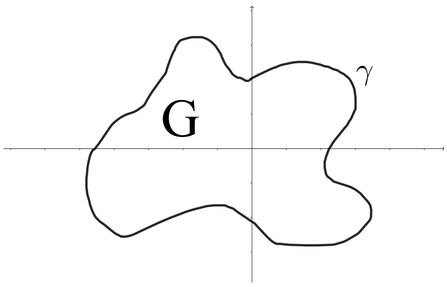
5 Laplace's Equation

Laplace's equation is one of the most important PDE in matheatical physics most notably in the study of electromagnetism. Our first stem is get accustomed to some termnology. Consider the following problem.

$$\nabla^2 u(x,y) = \frac{\partial^2 u(x,y,z)}{\partial x^2} + \frac{\partial^2 u(x,y,z)}{\partial y^2} = 0 \quad , (x,y) \in G$$

where G is a simply bounded connected domain whose boundry γ is a contor.
the boundry condition for this problem is.

$$u(x,y)|_{\gamma} = f(x,y)$$



we can see that this is a Dirchlet problem.

if the function u that satisfy the laplace eqution and is in the set of two dimensional contineous functions on the domain G $u \in C^2(G)$, i.e. all of u, $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial^2 u}{\partial x^2}$, $\frac{\partial^2 u}{\partial y^2}$, are contineous finctions on G, then the function u is called a harmonic function on G.

5.1 Harmonic Functions

we will discuss some properties or theorems of the harmonic function

Observation 5.1 u(x,y) at the center of the circle where r=0 that is when x=0 and y=0 is

$$u(0,0) = \frac{1}{2\pi R}\int_0^{2\pi R} H(\beta)\frac{R^2-(0)^2}{R^2-2R(0)cos(\alpha-\theta)+(0)^2}d\beta = \frac{1}{2\pi R}\int_0^{2\pi R} H(\beta)d\beta$$

Theorem 5.2 — Liouville Theorem. A harmonic function of the whole plane, i.e. $(x,y) \in \mathbb{R}^2$, can not be bounded from above or bounded from below unless it is constant.

Proof. assume the harmonic function is bounded from below, i.e.

$$u(x,y) \geq m \;\; \forall (x,y) \in \mathbb{R}^2$$

without loss of generality $m > 0$ is assumed.
we know that $-1 \leq cos(\theta) \leq 1$

$$\begin{aligned}
\therefore \frac{R^2-r^2}{R^2+2Rr+r^2} &\leq \frac{R^2-r^2}{R^2-2Rrcos(\alpha-\theta)+r^2} \leq \frac{R^2-r^2}{R^2-2Rr+r^2} \\
\frac{(R-r)(R+r)}{(R+r)^2} &\leq \frac{R^2-r^2}{R^2-2Rrcos(\alpha-\theta)+r^2} \leq \frac{(R-r)(R+r)}{(R-r)^2} \\
\frac{R-r}{R+r} &\leq \frac{R^2-r^2}{R^2-2Rrcos(\alpha-\theta)+r^2} \leq \frac{R+r}{R-r}
\end{aligned}$$

therefore

$$\frac{1}{2\pi R}\int_0^{2\pi R} H(\beta)\frac{R-r}{R+r}d\beta \leq \frac{1}{2\pi R}\int_0^{2\pi R} H(\beta)\frac{R^2-(0)^2}{R^2-2R(0)cos(\alpha-\theta)+(0)^2}d\beta \leq \frac{1}{2\pi R}\int_0^{2\pi R} H(\beta)\frac{R+r}{R-r}d\beta$$

and from observation 5.1

$$\frac{R-r}{R+r}u(0,0) \leq u(x,y) \leq \frac{R+r}{R-r}u(0,0)$$

taking the limit as $R \to \infty$ to include the whole plane we get.

$$u(0,0) \leq u(x,y) \leq u(0,0) \quad \Rightarrow \quad u(x,y) = u(0,0)$$

therefore the function is constant when it is bounded from below, the same argument can be used for bounded from above case. ■

Theorem 5.3 — Regularity Theorem. If u is a harmonic and contineous function on G and its boundry γ, then u is analytic at every point on G, i.e. can be represented as as a power series and can be diffrentiated infinit times.

Proof. we know that.

$$u(x,y) = \frac{a_o}{2} + \sum_{n=1}^{\infty} r^n [a_n \ \cos(n\theta) + b_n \ \sin(n\theta)]$$

and we know from de Moivre's formula that.

$$\cos(n\theta) = \sum_{k=0}^{n} A_k (\cos\theta)^k (\sin\theta)^{n-k}$$

and

$$\sin(n\theta) = \sum_{k=0}^{n} B_k (\cos\theta)^k (\sin\theta)^{n-k}$$

substituting in $u(x,y)$ we get.

$$u(x,y) = \frac{a_o}{2} + \sum_{n=1}^{\infty} r^n \left[a_n \sum_{k=0}^{n} A_k (cos\theta)^k (sin\theta)^{n-k} \right] + \sum_{n=1}^{\infty} r^n \left[b_n \sum_{k=0}^{n} B_k (cos\theta)^k (sin\theta)^{n-k} \right]$$

put $r^n = r^{n-k} r^k$ and we know that $x = rcos\theta$ and $y = rsin\theta$.

$$u(x,y) = \frac{a_o}{2} + \sum_{n=1}^{\infty} \sum_{k=0}^{n} a_n A_k x^k y^{n-k} + \sum_{n=1}^{\infty} \sum_{k=0}^{n} b_n B_k x^k y^{n-k}$$

$$u(x,y) = \frac{a_o}{2} + \sum_{n=1}^{\infty} \sum_{k=0}^{n} C_{n,k} x^k y^{n-k}$$

therefore the function can be represented as a power series at any point, i.e. is analytical. ■

Analytic Function

is a Function that can be represented by a convergent power series in a region around each point within its domain. It is a function that is differentiable at every point within this region, and its derivative exists everywhere in that domain. and it can be differntiated nth time for $n \in \mathbb{N}$
example (any Polynomial functions , Exponential function , Trigonometric functions ,etc.)

Theorem 5.4 — **Uniqueness of Drichlet Problem.** if there exist a solution of Laplace's equation with a boundery condition (Drichlet problem)

$$\exists u \in C^2(G), u \in C(G \cup \gamma), \quad \text{such that} \quad \nabla^2 u = 0 \quad \text{and} \quad u|_\gamma = f$$

then the solution is unique.

Proof. assuming the two function u_1 and u_2 are harmonic functions on G with boundry γ, i.e.

$$u_1, u_2 \in C^2(G), u_1, u_2 \in C(G \cup \gamma), \nabla^2 u_1 = 0, \nabla^2 u_2 = 0, u_1|_\gamma = f, u_2|_\gamma = f$$

we define the function w as follow

$$w = u_1 - u_2 \Rightarrow w \in C^2(G), w \in C(G \cup \gamma), \nabla^2 w = 0, w|_\gamma = f - f = 0$$

the value of the function attains its maximum or minimum value on its boundry according to the maximum principle thus we have

$$\min(w) = 0 \leq w(x) \leq 0 = \max(w) \forall x \in G \quad thus \quad w = 0 \Rightarrow u_1 = u_2$$

■

Theorem 5.5 — **Stability Theorem.** supposing the two functions u_1, u_2, are harmonic on domain G with boundry γ, and have the boundry conditions,
$u_1|_\gamma = f_1$ and $u_2|_\gamma = f_2$, if $|f_1 - f_2| \leq \epsilon \quad \forall \epsilon > 0$ then $|u_1(x) - u_2(x)| \leq \epsilon \quad \forall x \in G \cup \gamma$

Proof. according to he maximum principle

$$-\epsilon \leq u_1 - u_2 \leq \epsilon \quad \Rightarrow \quad |u_1 - u_2| \leq \epsilon$$

■

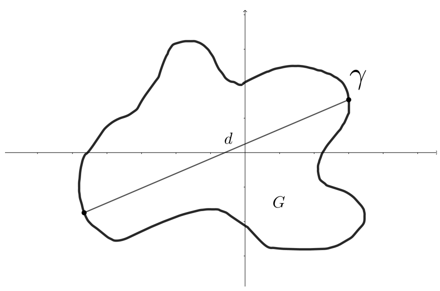
Theorem 5.6 — **Oleinik's Theorem (maximum principle).** if u is a harmonic function on G and u is contineous on $G \cup \gamma$, then u attains its maximum or minimum value on γ

Proof. suppose that the maximum value of u on $G \cup \gamma$ is M, and the minimum value of u on $G \cup \gamma$ is m, then it is clear that $M \geq m$.

suppose that $M > m$

we first define

$$v(x, y) = u(x, y) + \frac{M - m}{2d^2} \left[(x - x_1)^2 + (y - y_1)^2\right]$$



where d is the diameter of the domain G, and (x_1, y_1) is a point inside G such that, $u(x_1, y_1) = M$, thus we have.

$$v(x_1, y_1) = u(x_1, y_1) = M$$

and on γ we find

$$\begin{aligned} v(x,y) &\leq m + \frac{M-m}{2} \\ v(x,y) &\leq \frac{M+m}{2} < \frac{M+M}{2} = M \\ v(x,y) &< M \end{aligned}$$

this means that v attains a maximum value at a point call it (x_2, y_2) inside G thus we have.

$$\frac{\partial^2 v(x_2,y_2)}{\partial x^2} \leq 0 \quad \text{and} \quad \frac{\partial^2 v(x_2,y_2)}{\partial y^2} \leq 0$$

$$\therefore \nabla^2 v(x_2,y_2) = \frac{\partial^2 v(x_2,y_2)}{\partial x^2} + \frac{\partial^2 v(x_2,y_2)}{\partial y^2} \leq 0$$

from the defenition of v e obtain

$$\begin{aligned} \frac{\partial^2 v}{\partial x^2} &= \frac{\partial^2 u}{\partial x^2} + \frac{M-m}{d^2} \\ \frac{\partial^2 v}{\partial y^2} &= \frac{\partial^2 u}{\partial y^2} + \frac{M-m}{d^2} \\ \nabla^2 v = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial x^2} &= \nabla^2 u + \frac{2(M-m)}{d^2} = \frac{2(M-m)}{d^2} \\ \nabla^2 v &= \frac{2(M-m)}{d^2} > 0 \end{aligned}$$

which is a contradiction thus M must equale m ■