Partial Differential Equations

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Lecture Notes

Lecturer: Prof. Mahmoud M. El-Borai
Prepared by: Ossama Abdelwahed And
Ahmed.M.Habib

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1 Introduction

The main goal of many scientfic displines can be summerized to the following:

1. Formulate a set of mathematical equations to model a phenomena of intrest
2. Analyze solutions to these equations in order to extract information and make predictions.

The result of 1 is often a system of partial differential equations, thus the second becomes solving those partial differential equations.

A partial differential equation (PDE) is is a differential equation containing partial derivatives of the dependent variable with repect to more than one independent variable.

1.1 Order of PDE

The order of a PDE is determined by the highest derivative in the equation.

$$\begin{aligned}
\frac{\partial u}{\partial t} + (\frac{\partial u}{\partial x})^2 = 0 \qquad &\Longrightarrow \text{First order} \\
\frac{\partial^4 u}{\partial y^4} + \frac{\partial u}{\partial x} = c \qquad &\Longrightarrow \text{Fourth order}
\end{aligned}$$

do not mistake the order of the PDE with its degree, the degree of the PDE is the highest exponent appearing in the equation.

1.2 Linearity

A linear PDE is one that is of first degree in all of its field variables and partial derivatives.

$$\begin{aligned}
\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0 \qquad & \text{linear} \\
\frac{\partial^4 u}{\partial y^4} + \frac{\partial u}{\partial x} = y \qquad & \text{linear} \\
\frac{\partial u}{\partial t} + (\frac{\partial u}{\partial x})^2 = 0 \qquad & \text{nonlinear} \\
\frac{\partial^3 u}{\partial x^3} + (\frac{\partial^2 u}{\partial y^2})^5 = \sin(x) \qquad & \text{nonlinear}
\end{aligned}$$

a linear operator can be defined for any linear equation, taking the first equation in the previous list, the linear operator L can be defined as.

$$L = \frac{\partial}{\partial t} + \frac{\partial u}{\partial x}$$

and the equation can be written as.

$$L(u) = 0$$

1.3 Homogeneity

Let L be a linear operator. Then a linear partial differential equation can be written in the form.

$$L(u) = f(x_1, x_2, \ldots, t)$$

if $f = 0$ then the equation is homogeneous, otherwise it is inhomogeneous.

$$\begin{aligned}
\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0 \qquad & \text{homogeneous} \\
\frac{\partial^4 u}{\partial y^4} + \frac{\partial u}{\partial x} = y \qquad & \text{inhomogeneous}
\end{aligned}$$

1.4 Boundry Conditions

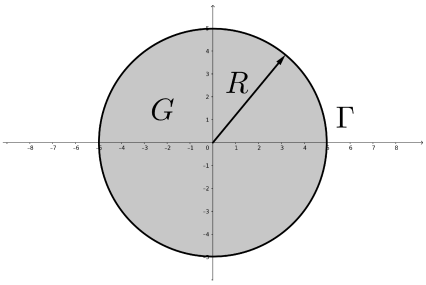
Definition 1.1 Boundary conditions are constraints necessary for the solution of a boundary value problem.

Definition 1.2 boundary value problem is a differential equation to be solved in a domain on whose boundary the function is known.

We will be intrested in one type of boundry conditions in this course which is the Drichlet Conditions, specifies the value that the unknown function needs to take on along the boundary of the domain. For example, the Laplace equation on a circle with drichlet condition will be.

$$\nabla^2 u(x) = 0 \quad \forall x \in G$$

$$u(x) = f(x) \quad \forall x \in G$$



$$G = \left\{(x, y) : x^2 + y^2 < R^2\right\} \qquad \Gamma = \left\{(x, y) : x^2 + y^2 = R^2\right\}$$

Equations involving such conditions are classified as Drichlet problems.

1.5 Intial Condition

Definition 1.3 The initial condition is a condition that a solution must have at only on instant of time, which is the starting time as it can be found expremintally.

An example is the heat equation with intial condition.

$$\frac{\partial u(x,t)}{\partial t} = c^2 \frac{\partial u(x,t)}{\partial x}$$
$$u(x,0) = f(x)$$

Equations involving such conditions are classified as Cauchy problems.

1.6 Equations of Mathematical Physics

The most frequently encountered equations in physics are the following

1. Heat Equation
$$\frac{\partial u(x,t)}{\partial t} = c^2 \frac{\partial^2 u(x,t)}{\partial x^2}$$
2. Wave Equation
$$\frac{\partial^2 u(x,t)}{\partial t^2} = c^2 \frac{\partial^2 u(x,t)}{\partial x^2}$$
3. Laplace's Equation
$$\nabla^2 u(x) = \frac{\partial^2 u(x)}{\partial x_1^2} + \frac{\partial^2 u(x)}{\partial x_2^2} + \frac{\partial^2 u(x)}{\partial x_3^2} + \cdots = 0$$

2 Canonical Form

Consider the following PDE with variable coefficients. We are aiming to transform this equation into its canonical form

$$A\left(x,y\right)\frac{\partial^2 u\left(x,y\right)}{\partial x^2}+2B\left(x,y\right)\frac{\partial^2 u\left(x,y\right)}{\partial x\partial y}+C\left(x,y\right)\frac{\partial^2 u\left(x,y\right)}{\partial y^2}+F\left(x,y,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}\right)=0 \tag{1}$$

The first three terms are called the principle terms and the last term is called the Yound term (Y.T) which does not contain second order derivatives of u.

We start by preforming a change of variables such that.

$$\xi=\xi\left(x,y\right),\qquad \eta=\eta\left(x,y\right)$$

taking into consideration the Jacobian of the transformation

$$J=\begin{vmatrix}\frac{\partial\xi}{\partial x} & \frac{\partial\eta}{\partial x}\\ \frac{\partial\xi}{\partial y} & \frac{\partial\eta}{\partial y}\end{vmatrix}\neq 0$$

Then we find our derivatives.

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial\xi}\frac{\partial\xi}{\partial x}+\frac{\partial u}{\partial\eta}\frac{\partial\eta}{\partial x}$$
$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial u}{\partial\xi}\frac{\partial^2\xi}{\partial x^2}+\frac{\partial\xi}{\partial x}\left[\frac{\partial^2 u}{\partial\xi^2}\frac{\partial\xi}{\partial x}+\frac{\partial^2 u}{\partial\eta\partial\xi}\frac{\partial\eta}{\partial x}\right]+\frac{\partial u}{\partial\eta}\frac{\partial^2\eta}{\partial x^2}+\frac{\partial\eta}{\partial x}\left[\frac{\partial^2 u}{\partial\eta^2}\frac{\partial\eta}{\partial x}+\frac{\partial^2 u}{\partial\eta\partial\xi}\frac{\partial\xi}{\partial x}\right]$$

adding similar terms and simplifying

$$\frac{\partial^2 u}{\partial x^2}=(\frac{\partial\xi}{\partial x})^2\frac{\partial^2 u}{\partial\xi^2}+2\frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial x}\frac{\partial^2 u}{\partial\eta\partial\xi}+(\frac{\partial\eta}{\partial x})^2\frac{\partial^2 u}{\partial\eta^2}+Y.T \tag{2}$$

and in similar fashion we can get.

$$\frac{\partial^2 u}{\partial y^2}=(\frac{\partial\xi}{\partial y})^2\frac{\partial^2 u}{\partial\xi^2}+2\frac{\partial\xi}{\partial y}\frac{\partial\eta}{\partial y}\frac{\partial^2 u}{\partial\eta\partial\xi}+(\frac{\partial\eta}{\partial y})^2\frac{\partial^2 u}{\partial\eta^2}+Y.T \tag{3}$$

$$\frac{\partial^2 u}{\partial y\partial x}=\frac{\partial\xi}{\partial x}\frac{\partial\xi}{\partial y}\frac{\partial^2 u}{\partial\xi^2}+\left[\frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial y}+\frac{\partial\xi}{\partial y}\frac{\partial\eta}{\partial x}\right]\frac{\partial^2 u}{\partial\xi\partial\eta}+\frac{\partial\eta}{\partial x}\frac{\partial\eta}{\partial y}\frac{\partial^2 u}{\partial\eta^2}+Y.T \tag{4}$$

substituting (2), (3), and (4) in (1) we get.

$$\begin{aligned}&A\left[(\frac{\partial\xi}{\partial x})^2\frac{\partial^2 u}{\partial\xi^2}+2\frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial x}\frac{\partial^2 u}{\partial\eta\partial\xi}+(\frac{\partial\eta}{\partial x})^2\frac{\partial^2 u}{\partial\eta^2}\right]\\ &+2B\left[\frac{\partial\xi}{\partial x}\frac{\partial\xi}{\partial y}\frac{\partial^2 u}{\partial\xi^2}+\left[\frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial y}+\frac{\partial\xi}{\partial y}\frac{\partial\eta}{\partial x}\right]\frac{\partial^2 u}{\partial\xi\partial\eta}+\frac{\partial\eta}{\partial x}\frac{\partial\eta}{\partial y}\frac{\partial^2 u}{\partial\eta^2}\right]\\ &+C\left[(\frac{\partial\xi}{\partial y})^2\frac{\partial^2 u}{\partial\xi^2}+2\frac{\partial\xi}{\partial y}\frac{\partial\eta}{\partial y}\frac{\partial^2 u}{\partial\eta\partial\xi}+(\frac{\partial\eta}{\partial y})^2\frac{\partial^2 u}{\partial\eta^2}\right]+Y.T=0\end{aligned}$$

rearranging terms.

$$\begin{aligned}&\left[A(\frac{\partial\xi}{\partial x})^2+2B\frac{\partial\xi}{\partial x}\frac{\partial\xi}{\partial y}+C(\frac{\partial\xi}{\partial y})^2\right]\frac{\partial^2 u}{\partial\xi^2}+\left[A(\frac{\partial\eta}{\partial x})^2+2B\frac{\partial\eta}{\partial x}\frac{\partial\eta}{\partial y}+C(\frac{\partial\eta}{\partial y})^2\right]\frac{\partial^2 u}{\partial\eta^2}\\ &+\left[2A\frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial x}+2B\left[\frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial y}+\frac{\partial\xi}{\partial y}\frac{\partial\eta}{\partial x}\right]+2C\frac{\partial\xi}{\partial y}\frac{\partial\eta}{\partial y}\right]\frac{\partial^2 u}{\partial\eta\partial\xi}+Y.T=0\end{aligned}$$

we now try to find ξ and η such that.

$$\left[A(\frac{\partial \xi}{\partial x})^2 + 2B\frac{\partial \xi}{\partial x}\frac{\partial \xi}{\partial y} + C(\frac{\partial \xi}{\partial y})^2\right] = 0$$

$$\left[A(\frac{\partial \eta}{\partial x})^2 + 2B\frac{\partial \eta}{\partial x}\frac{\partial \eta}{\partial y} + C(\frac{\partial \eta}{\partial y})^2\right] = 0$$

we notice that both equations are the same quadratic equation, thus we solve for one of them to find both ξ and η, we choose the first one and start by dividing the equation by $(\frac{\partial \xi}{\partial y})^2$.

$$A\frac{(\frac{\partial \xi}{\partial x})^2}{(\frac{\partial \xi}{\partial y})^2} + 2B\frac{\left(\frac{\partial \xi}{\partial x}\right)}{\left(\frac{\partial \xi}{\partial y}\right)} + C = 0$$

$$A(\frac{\partial y}{\partial x})^2 - 2B\frac{\partial y}{\partial x} + C = 0$$

now using the quadratic formula to solve for $\frac{\partial y}{\partial x}$.

$$\frac{\partial y}{\partial x} = \frac{-(-2B) \pm \sqrt{(-2B)^2 - 4AC}}{2A} = \frac{B \pm \sqrt{B^2 - AC}}{A}$$

this is called the charctaristic equation.

Equations Classification

Equations are classified based on the value of the expression under the root
$B^2 > AC$ $\quad \forall x, y \in G$, Hyperbolic PDE (the general case of the wave equation)
$B^2 < AC$ $\quad \forall x, y \in G$, Elliptic PDE (the general case of the Laplace equation)
$B^2 = AC$ $\quad \forall x, y \in G$, Parabolic PDE (the general case of the Heat equation)

2.1 Examples

■ **Example 2.1.1** Transform to the canonical form.

$$4y^2\frac{\partial^2 u}{\partial x^2} - e^{2x}\frac{\partial^2 u}{\partial y^2} + \underbrace{6y^3}_{\text{Y.T}} = 0$$

—— Solution ——

we start by determining the functions A,B, and C.

$$A(x,y) = 4y^2 \quad , \quad B(x,y) = 0 \quad , \quad C(x,y) = -e^{2x}$$

we conclude from this that it has the form of a Hyperbolic PDE.

$$B^2 = 0 \quad > \quad AC = -4y^2e^{2x}, \quad \forall y \neq 0, \quad \forall x$$

now we use the charchtarstic equation to determine the value of ξ and η.

$$\frac{\partial y}{\partial x} = \frac{B \pm \sqrt{B^2 - AC}}{A}$$

$$= \frac{\pm\sqrt{4y^2 e^{2x}}}{4y^2} = \pm\frac{e^x}{2y}$$

rearranging and intgrating.

$$\begin{aligned} 2ydy &= \pm e^x dx \\ \int 2ydy &= \pm \int e^x dx \\ y^2 &= \pm e^x + constant \quad \Longrightarrow \quad y^2 \pm e^x = constant \end{aligned}$$

we now set ξ and η.

$$\xi = e^x + y^2 \quad , \quad \eta = e^x - y^2$$

we now work out the derivatives.

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x} \\ &= \frac{\partial u}{\partial \xi}e^x + \frac{\partial u}{\partial \eta}e^x \\ \frac{\partial^2 u}{\partial x^2} &= e^x\left[\frac{\partial^2 u}{\partial \xi^2}e^x + \frac{\partial^2 u}{\partial \eta \partial \xi}e^x\right] + e^x\left[\frac{\partial^2 u}{\partial \eta^2}e^x + \frac{\partial^2 u}{\partial \xi \partial \eta}e^x\right] + Y.T \\ &= e^{2x}\frac{\partial^2 u}{\partial \xi^2} + 2e^{2x}\frac{\partial^2 u}{\partial \xi \partial \eta} + e^{2x}\frac{\partial^2 u}{\partial \eta^2} + Y.T \\ \frac{\partial^2 u}{\partial y^2} &= 4y^2\frac{\partial^2 u}{\partial \xi^2} - 8y^2\frac{\partial^2 u}{\partial \xi \partial \eta} + 4y^2\frac{\partial^2 u}{\partial \eta^2} + Y.T \end{aligned}$$

substituting in our original equation.

$$4y^2\left[e^{2x}\frac{\partial^2 u}{\partial \xi^2} + 2e^{2x}\frac{\partial^2 u}{\partial \xi \partial \eta} + e^{2x}\frac{\partial^2 u}{\partial \eta^2}\right] - e^{2x}\left[4y^2\frac{\partial^2 u}{\partial \xi^2} - 8y^2\frac{\partial^2 u}{\partial \xi \eta} + 4y^2\frac{\partial^2 u}{\partial \eta^2}\right] + Y.T = 0$$

$$\begin{aligned} 16y^2 e^{2x}\frac{\partial^2 u}{\partial \xi \partial \eta} + Y.T = 0 \\ \frac{\partial^2 u}{\partial \xi \partial \eta} + Y.T = 0 \end{aligned}$$

■

■ **Example 2.1.2** Transform to the canonical form

$$x^2\frac{\partial^2 u}{\partial x^2} + y^2\frac{\partial^2 u}{\partial y^2} = 0$$

—— Solution ——

determining the functions A,B, and C.

$$A(x,y) = x^2 \quad , \quad B(x,y) = 0 \quad , \quad C(x,y) = y^2$$

it has the form of a Elliptic PDE.

$$B^2 = 0 \quad < \quad AC = x^2y^2, \quad \forall y, x$$

using the charchtarstic equation.

$$\frac{\partial y}{\partial x} = \frac{B \pm \sqrt{B^2 - AC}}{A}$$

$$
\begin{aligned}
&= \frac{\pm\sqrt{-x^2y^2}}{x^2} = \pm i\frac{y}{x} \\
\int \frac{dy}{y} &= \pm i \int \frac{dx}{x} \\
\ln(y) &= \pm i \ln(x) + \text{constant}
\end{aligned}
$$

we will choose ξ to be the imaginary part and η to be the real part.

$$
\xi = \ln(x) \quad , \quad \eta = \ln(y)
$$

working out the derivatives.

$$
\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x} \\
&= \frac{\partial u}{\partial \xi}\frac{1}{x} + \frac{\partial u}{\partial \eta}(0) \\
\frac{\partial^2 u}{\partial x^2} &= \frac{1}{x}\left[\frac{\partial^2 u}{\partial \xi^2}\frac{1}{x} + \frac{\partial^2 u}{\partial \eta \partial \xi}(0)\right] + Y.T \\
\frac{\partial^2 u}{\partial x^2} &= \frac{1}{x^2}\frac{\partial^2 u}{\partial \xi^2} + Y.T \\
\frac{\partial^2 u}{\partial y^2} &= \frac{1}{y^2}\frac{\partial^2 u}{\partial \eta^2} + Y.T
\end{aligned}
$$

substituting in our original equation.

$$
\begin{aligned}
x^2\left[\frac{1}{x^2}\frac{\partial^2 u}{\partial \xi^2}\right] + y^2\left[\frac{1}{y^2}\frac{\partial^2 u}{\partial \eta^2}\right] + Y.T = 0 \\
\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} + Y.T = 0
\end{aligned}
$$

■

■ **Example 2.1.3** Transform to the canonical form

$$
y^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + x^2\frac{\partial^2 u}{\partial y^2} = 0
$$

—— Solution ——

determining the functions A,B, and C.

$$
A(x,y) = y^2 \quad , \quad B(x,y) = xy \quad , \quad C(x,y) = x^2
$$

it has the form of a Parabolic PDE.

$$
B^2 = x^2y^2 \quad = \quad AC = x^2y^2, \quad \forall y, x
$$

using the charchtarstic equation.

$$
\begin{aligned}
\frac{\partial y}{\partial x} &= \frac{B \pm \sqrt{B^2 - AC}}{A} \\
&= \frac{xy}{y^2} = \frac{x}{y} \\
\int y dy &= \int x dx \\
y^2 &= x^2 + \text{constant}
\end{aligned}
$$

we will assign ξ to be this function

$$\xi = y^2 - x^2$$

and for η it's Optional but to make the solution easier we will assign the previous function with different sign

$$\eta = y^2 + x^2 \quad \text{or} \quad \eta = -y^2 - x^2$$

rest of the solution same as Hyperbolic and elliptic PDEs the Canonical form in the end will be

$$\frac{\partial^2 u}{\partial \xi^2} + Y.T = 0$$

or

$$\frac{\partial^2 u}{\partial \eta^2} + Y.T = 0$$

∎

Observation 2.1 the Canonical form of all Hyperbolic equations is

$$\frac{\partial^2 u}{\partial \xi \partial \eta} + Y.T = 0$$

Observation 2.2 the Canonical form of all elliptic equations is

$$\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} + Y.T = 0$$

Observation 2.3 the Canonical form of all Parabolic equations is

$$\frac{\partial^2 u}{\partial \xi^2} + Y.T = 0$$

or

$$\frac{\partial^2 u}{\partial \eta^2} + Y.T = 0$$