Lecture 6
Linear System Of ODEs

1 Linear ODE System

consider the system

$$\begin{cases} \dfrac{dX(t)}{dt} = A(t)X(t) \\ X(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} \quad , \quad A(t) = \begin{bmatrix} a_{11}(t) & a_{12}(t) & \dots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \dots & a_{2n}(t) \\ \vdots & \vdots & & \vdots \\ a_{n1}(t) & a_{n2}(t) & \dots & a_{nn}(t) \end{bmatrix} \end{cases} \tag{1}$$

where $a_{ij}(t)$ are continuous functions on $[\alpha, \beta]$

suppose that $\mathbf{S}(t)$ is a square matrix of order n such that

$$\frac{d\mathbf{S}(t)}{dt} = A(t)\mathbf{S}(t) \quad , \quad \mathbf{S}(t) = \begin{bmatrix} s_{11}(t) & s_{12}(t) & \dots & s_{1n}(t) \\ s_{21}(t) & s_{22}(t) & \dots & s_{2n}(t) \\ \vdots & \vdots & & \vdots \\ s_{n1}(t) & s_{n2}(t) & \dots & s_{nn}(t) \end{bmatrix}$$

if the columns of $\begin{bmatrix} s_{11}(t) \\ s_{21}(t) \\ \vdots \\ s_{n1}(t) \end{bmatrix}, \begin{bmatrix} s_{12}(t) \\ s_{22}(t) \\ \vdots \\ s_{n2}(t) \end{bmatrix}, \dots, \begin{bmatrix} s_{1n}(t) \\ s_{2n}(t) \\ \vdots \\ s_{nn}(t) \end{bmatrix}$ are Linearly independent then $\mathbf{S}(t)$ is called a fundamental solution of the system

Theorem 1.1 The solution $\mathbf{S}(t)$ is fundamental iff $\det(\mathbf{S}(t)) \neq 0$ on the interval $[\alpha, \beta]$
i.e Linearly independent $\iff \det(\mathbf{S}(t)) \neq 0$

Theorem 1.2 if $\mathbf{S}(t)$ is a fundamental solution of system (1) and if $\mathbf{C}$ is constant matrix such that $\det(\mathbf{C}) \neq 0$ then $\mathbf{S}(t)\mathbf{C}$ is a fundamental solution of system (1)

Proof. set

$$f(t) = \mathbf{S}(t)\mathbf{C}$$

where $\det(\mathbf{C}) \neq 0$

$$\begin{aligned} \det f(t) &= \det[\mathbf{S}(t)\mathbf{C}] \\ &= \det \mathbf{S}(t) \det \mathbf{C} \\ &\neq 0 \end{aligned}$$

now we proof that it's a solution

$$\frac{df(t)}{dt} = \frac{d\mathbf{S}(t)}{dt}\mathbf{C} = A(t)\mathbf{S}(t)\mathbf{C} = A(t)f(t)$$

then $\mathbf{S}(t)\mathbf{C}$ is a fundamental solution ■

Theorem 1.3 if $\mathbf{S}_1(t)$ and $\mathbf{S}_2(t)$ are fundamental solution of system (1) then $\mathbf{S}_2(t) = \mathbf{S}_1(t)\mathbf{C}$ where $\mathbf{C}$ is constant matrix such that $\det(\mathbf{C}) \neq 0$

Proof. let $\mathbf{S}_1(t)$ and $\mathbf{S}_2(t)$ be fundamental solutions
set

$$\mathbf{S}_2(t) = \mathbf{S}_1(t)\mathbf{G}(t)$$

then

$$\begin{aligned}
\frac{d\mathbf{S}_2(t)}{dt} &= \mathbf{S}_1(t)\frac{d\mathbf{G}(t)}{dt} + \frac{d\mathbf{S}_1(t)}{dt}\mathbf{G}(t) \\
A(t)\mathbf{S}_2(t) &= \mathbf{S}_1(t)\frac{d\mathbf{G}(t)}{dt} + A(t)\mathbf{S}_1(t)\mathbf{G}(t) \\
A(t)\mathbf{S}_1(t)\mathbf{G}(t) &= \mathbf{S}_1(t)\frac{d\mathbf{G}(t)}{dt} + A(t)\mathbf{S}_1(t)\mathbf{G}(t) \\
\therefore \mathbf{S}_1(t)\frac{d\mathbf{G}(t)}{dt} &= 0
\end{aligned}$$

because $\det(\mathbf{S}_1(t)) \neq 0$ then it has inverse
then multiply by this inverse from the left

$$\begin{aligned}
\mathbf{S}_1^{-1}(t)\mathbf{S}_1(t)\frac{d\mathbf{G}(t)}{dt} &= 0 \\
\frac{d\mathbf{G}(t)}{dt} &= 0 \\
\therefore \mathbf{G}(t) &= \mathbf{C}
\end{aligned}$$

then $\mathbf{G}(t)$ is a constant matrix ■

2 Matrix and Determinant Differentiation

if a differentiation applied on matrix then the result will be as following

$$\frac{d}{dt}\begin{bmatrix} a_{11}(t) & a_{12}(t) & \dots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \dots & a_{2n}(t) \\ \vdots & \vdots & & \vdots \\ a_{n1}(t) & a_{n2}(t) & \dots & a_{nn}(t) \end{bmatrix} = \begin{bmatrix} \dot{a}_{11}(t) & \dot{a}_{12}(t) & \dots & \dot{a}_{1n}(t) \\ \dot{a}_{21}(t) & \dot{a}_{22}(t) & \dots & \dot{a}_{2n}(t) \\ \vdots & \vdots & & \vdots \\ \dot{a}_{n1}(t) & \dot{a}_{n2}(t) & \dots & \dot{a}_{nn}(t) \end{bmatrix}$$

on the other hand if it's applied on Determinant

$$\begin{aligned}
\frac{d}{dt}\begin{vmatrix} a_{11}(t) & a_{12}(t) & \dots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \dots & a_{2n}(t) \\ \vdots & \vdots & & \vdots \\ a_{n1}(t) & a_{n2}(t) & \dots & a_{nn}(t) \end{vmatrix} &= \begin{vmatrix} \dot{a}_{11}(t) & \dot{a}_{12}(t) & \dots & \dot{a}_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \dots & a_{2n}(t) \\ \vdots & \vdots & & \vdots \\ a_{n1}(t) & a_{n2}(t) & \dots & a_{nn}(t) \end{vmatrix} + \begin{vmatrix} a_{11}(t) & a_{12}(t) & \dots & a_{1n}(t) \\ \dot{a}_{21}(t) & \dot{a}_{22}(t) & \dots & \dot{a}_{2n}(t) \\ \vdots & \vdots & & \vdots \\ a_{n1}(t) & a_{n2}(t) & \dots & a_{nn}(t) \end{vmatrix} + \dots \\
&+ \begin{vmatrix} a_{11}(t) & a_{12}(t) & \dots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \dots & a_{2n}(t) \\ \vdots & \vdots & & \vdots \\ \dot{a}_{n1}(t) & \dot{a}_{n2}(t) & \dots & \dot{a}_{nn}(t) \end{vmatrix}
\end{aligned}$$

property :
let $\mathbf{S}(t) = e^{\mathbf{A}t}$ where $\mathbf{A}$ is a constant matrix then

$$\frac{d\mathbf{S}(t)}{dt} = \mathbf{A}e^{\mathbf{A}t} = \mathbf{A}\mathbf{S}(t)$$

where $\det\left(e^{\mathbf{A}t}\right) \neq 0$ then $e^{\mathbf{A}t}$ is a fundamental solution of $\dfrac{dX(t)}{dt} = AX(t)$

Theorem 2.1 if $\mathbf{S}(t)$ is fundamental solution of system (1) i.e

$$\frac{d\mathbf{S}(t)}{dt} = \mathbf{A}(t)\mathbf{S}(t)$$

then $\dfrac{d}{dt}\det(\mathbf{S}(t)) = \operatorname{tr}[\mathbf{A}(t)]\det(\mathbf{S}(t))$
and $\operatorname{Det}(\mathbf{S}(t)) = \det(\mathbf{S}(t_0))e^{\int_{t_0}^{t} tr[\mathbf{A}(\theta)]d\theta}$
where $\operatorname{tr}[\mathbf{A}(t)] = a_{11}(t) + a_{22}(t) + \cdots + a_{nn}(t)$

Proof. because

$$\frac{d\mathbf{S}(t)}{dt} = \mathbf{A}(t)\mathbf{S}(t)$$

let

$$\mathbf{A}(t)\mathbf{S}(t) = b_{ij}(t) = \sum_{k=1}^{n} a_{ik}(t)s_{kj}(t)$$

then

$$\frac{d}{dt}\det(\mathbf{S}(t)) = \begin{vmatrix} \sum_{k=1}^{n} a_{1k}(t)s_{k1}(t) & \sum_{k=1}^{n} a_{1k}(t)s_{k2}(t) & \dots & \sum_{k=1}^{n} a_{1k}(t)s_{kn}(t) \\ s_{21}(t) & s_{22}(t) & \dots & s_{2n}(t) \\ \vdots & \vdots & & \vdots \\ s_{n1}(t) & s_{n2}(t) & \dots & s_{nn}(t) \end{vmatrix} + \dots$$

$$+ \begin{vmatrix} s_{11}(t) & s_{12}(t) & \dots & a_{1n}(t) \\ s_{21}(t) & s_{22}(t) & \dots & a_{2n}(t) \\ \vdots & \vdots & & \vdots \\ \sum_{k=1}^{n} a_{nk}(t)s_{k1}(t) & \sum_{k=1}^{n} a_{nk}(t)s_{k2}(t) & \dots & \sum_{k=1}^{n} a_{nk}(t)s_{kn}(t) \end{vmatrix}$$

using the properties of determinants we get

$$\frac{d}{dt}\det(\mathbf{S}(t)) = a_{11}\det(\mathbf{S}(t)) + a_{22}\det(\mathbf{S}(t)) + \cdots + a_{nn}\det(\mathbf{S}(t)) = \operatorname{tr}(\mathbf{A}(t))\det(\mathbf{S}(t))$$

then we can write that

$$\begin{aligned} \frac{d}{dt}\det(\mathbf{S}(t)) &= \operatorname{tr}(\mathbf{A}(t))\det(\mathbf{S}(t)) \\ \frac{d\det(\mathbf{S}(t))}{\det(\mathbf{S}(t))} &= \operatorname{tr}(\mathbf{A}(t))dt \\ \ln\left(\frac{\det(\mathbf{S}(t))}{\det(\mathbf{S}(t_0))}\right) &= \int_{t_0}^{t} \operatorname{tr}[\mathbf{A}(\theta)]d\theta \\ \det(\mathbf{S}(t)) &= \det(\mathbf{S}(t_0))e^{\int_{t_0}^{t} \operatorname{tr}[\mathbf{A}(\theta)]d\theta} \end{aligned}$$

∎

special case : If $\mathbf{A}$ is a constant matrix and $t_0 = 0$ then

$$\det(\mathbf{S}(t)) = \det(\mathbf{S}(0))e^{t.\operatorname{tr}[\mathbf{A}]}$$

Consider the linear homogeneous system

$$\begin{cases} \dfrac{dX(t)}{dt} = \mathbf{A}X(t) \end{cases} \tag{2}$$

We try to find a solution $X(t)$ of the form

$$X(t) = e^{\lambda t}\mathbf{v}$$

Where $\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$

We notice that

$$\frac{\mathrm{d}X(t)}{\mathrm{d}t} = \lambda e^{\lambda t}\mathbf{v} = e^{\lambda t}\mathbf{A}\mathbf{v}$$

Hence $X(t) = e^{\lambda t}$ is a solution of system $\iff$ $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$

A non zero vector $\mathbf{v}$ satisfying the last equation is called **eigenvector** of matrix $\mathbf{A}$ corresponding to **eigenvalue** λ

■ **Example 2.0.1** if $\mathbf{A} = \begin{bmatrix} 8 & 5 & 2 \\ 9 & 1 & 3 \\ 4 & -5 & 6 \end{bmatrix}$ then

$$\det e^{\mathbf{A}} = \operatorname{tr}(\mathbf{A}) = 8 + 1 + 6 = 15$$

■

Remark

Now, to find $e^{\mathbf{A}}$, $\mathbf{A}$ is of order n

$$\frac{dX(t)}{dt} = \mathbf{A}X(t)$$

we find the n^{th} solution $\begin{bmatrix} x_{11}(t) \\ x_{21}(t) \\ \vdots \\ x_{n1}(t) \end{bmatrix}, \begin{bmatrix} x_{12}(t) \\ x_{22}(t) \\ \vdots \\ x_{n2}(t) \end{bmatrix}, \dots, \begin{bmatrix} x_{1n}(t) \\ x_{2n}(t) \\ \vdots \\ x_{nn}(t) \end{bmatrix}$ and by compining them in one matrix we get the solution

$$X(t) = e^{\mathbf{A}} = \begin{bmatrix} x_{11}(t) & x_{12}(t) & \dots & x_{1n}(t) \\ x_{21}(t) & x_{22}(t) & \dots & x_{2n}(t) \\ \vdots & \vdots & & \vdots \\ x_{n1}(t) & x_{n2}(t) & \dots & x_{nn}(t) \end{bmatrix}$$

■ **Example 2.0.2** Solve the IVP

$$\begin{cases} \dfrac{dx(t)}{dt} = \mathbf{A}X(t) \\ A = \begin{bmatrix} 1 & 12 \\ 3 & 1 \end{bmatrix} \\ x(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{cases} \tag{3}$$

Characteristic polynomial of $\mathbf{A}$ is given by

$$\begin{vmatrix} 1-\lambda & 12 \\ 3 & 1-\lambda \end{vmatrix} = 0$$

$$\begin{aligned} (1-\lambda)^2 &= 36 \\ 1-\lambda &= \pm 6 \\ \lambda = -5 \quad &, \quad \lambda = 7 \end{aligned}$$

At $\lambda = 7$, we seek a non zero vector $\mathbf{v}$ such that

$$(\mathbf{A} - 7I)\mathbf{v} = \begin{bmatrix} -6 & 12 \\ 3 & -6 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

By solving this system we get

$$v_1 = 2v_2$$

Consequently, every vector $\mathbf{v} = \mathbf{c} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is an eigen vector of $\mathbf{A}$ corresponding to eigenvector $\lambda = 7$

Thus,

$$X(t) = e^{7t} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

is a solution of our system

Similarly, At $\lambda = -5$

$$(\mathbf{A} + 5I)\mathbf{v} = \begin{bmatrix} 6 & 12 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

By solving this system we get

$$v_1 = -2v_2$$

Consequently, every vector $\mathbf{v} = \mathbf{c} \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ is an eigen vector of $\mathbf{A}$ corresponding to eigenvector $\lambda = 7$

Thus,

$$X(t) = e^{-5t} \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Then the general solution is given by

$$X(t) = \mathbf{c_1} e^{7t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \mathbf{c_2} e^{-5t} \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

The constants $\mathbf{c_1}$ and $\mathbf{c_2}$ are determined from initial condition

$$X(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \mathbf{c_1} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \mathbf{c_2} \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

We get

$$\begin{aligned} 0 &= 2c_1 - 2c_2 \\ 1 &= c_1 + c_2 \end{aligned}$$

$c_1 = c_2 = \frac{1}{2}$

Thus, The general solution is given by

$$X(t) = \frac{1}{2} e^{7t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \frac{1}{2} e^{-5t} \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

■

Moreover, Evaluate $e^{\mathbf{A}}$

Remember we considered that the solution of the form $X(t) = e^{\lambda t}\mathbf{v}$ and we know that $\mathbf{Av} = \lambda \mathbf{v}$

$X(t) = e^{\mathbf{A}t}\mathbf{v}$

We can get $\mathbf{v}$ easily from initial condition $\mathbf{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$e^{\mathbf{A}t} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} e^{7t} - e^{-5t} \\ \frac{1}{2}e^{7t} + \frac{1}{2}e^{-5t} \end{bmatrix}$$

Assume $e^{\mathbf{A}t} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$ We can deduce that

$$\begin{aligned} b_{12} &= e^{7t} - e^{-5t} \\ b_{22} &= \frac{1}{2}e^{7t} + \frac{1}{2}e^{-5t} \end{aligned}$$

If we consider a new initial condition. It is easy to obtain that

$$e^{\mathbf{A}t}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}e^{7t} + \frac{1}{2}e^{7t} \\ \frac{1}{4}e^{7t} - \frac{1}{4}e^{-5t} \end{bmatrix}$$

Thus, we get

$$\begin{aligned} b_{11} &= \frac{1}{2}e^{7t} + \frac{1}{2}e^{7t} \\ b_{21} &= \frac{1}{4}e^{7t} - \frac{1}{4}e^{-5t} \end{aligned}$$

$$e^{\mathbf{A}t} = \begin{bmatrix} \frac{1}{2}e^{7t} + \frac{1}{2}e^{7t} & e^{7t} - e^{-5t} \\ \frac{1}{4}e^{7t} - \frac{1}{4}e^{-5t} & \frac{1}{2}e^{7t} + \frac{1}{2}e^{-5t} \end{bmatrix}$$

At t=1

$$e^{\mathbf{A}} = \begin{bmatrix} \frac{1}{2}e^{7} + \frac{1}{2}e^{7} & e^{7} - e^{-5} \\ \frac{1}{4}e^{7} - \frac{1}{4}e^{-5} & \frac{1}{2}e^{7} + \frac{1}{2}e^{-5} \end{bmatrix}$$