Lecture 6
Linear System Of ODEs

1 Linear ODE System

consider the system

$$\begin{cases} \dfrac{dX(t)}{dt} = A(t)X(t) \\ X(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} \quad , \quad A(t) = \begin{bmatrix} a_{11}(t) & a_{12}(t) & \dots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \dots & a_{2n}(t) \\ \vdots & \vdots & & \vdots \\ a_{n1}(t) & a_{n2}(t) & \dots & a_{nn}(t) \end{bmatrix} \end{cases} \tag{1}$$

where $a_{ij}(t)$ are continuous functions on $[\alpha, \beta]$

suppose that $\mathbf{S}(t)$ is a square matrix of order n such that

$$\frac{d\mathbf{S}(t)}{dt} = A(t)\mathbf{S}(t) \quad , \quad \mathbf{S}(t) = \begin{bmatrix} s_{11}(t) & s_{12}(t) & \dots & s_{1n}(t) \\ s_{21}(t) & s_{22}(t) & \dots & s_{2n}(t) \\ \vdots & \vdots & & \vdots \\ s_{n1}(t) & s_{n2}(t) & \dots & s_{nn}(t) \end{bmatrix}$$

if the columns of $\begin{bmatrix} s_{11}(t) \\ s_{21}(t) \\ \vdots \\ s_{n1}(t) \end{bmatrix}, \begin{bmatrix} s_{12}(t) \\ s_{22}(t) \\ \vdots \\ s_{n2}(t) \end{bmatrix}, \dots, \begin{bmatrix} s_{1n}(t) \\ s_{2n}(t) \\ \vdots \\ s_{nn}(t) \end{bmatrix}$ are Linearly independent then $\mathbf{S}(t)$ is called a fundamental solution of the system

Theorem 1.1 The solution $\mathbf{S}(t)$ is fundamental iff $\det(\mathbf{S}(t)) \neq 0$ on the interval $[\alpha, \beta]$
i.e Linearly independent $\iff \det(\mathbf{S}(t)) \neq 0$

Theorem 1.2 if $\mathbf{S}(t)$ is a fundamental solution of system (1) and if $\mathbf{C}$ is constant matrix such that $\det(\mathbf{C}) \neq 0$ then $\mathbf{S}(t)\mathbf{C}$ is a fundamental solution of system (1)

Proof. set

$$f(t) = \mathbf{S}(t)\mathbf{C}$$

where $\det(\mathbf{C}) \neq 0$

$$\begin{aligned} \det f(t) &= \det[\mathbf{S}(t)\mathbf{C}] \\ &= \det \mathbf{S}(t) \det \mathbf{C} \\ &\neq 0 \end{aligned}$$

now we proof that it's a solution

$$\frac{df(t)}{dt} = \frac{d\mathbf{S}(t)}{dt}\mathbf{C} = A(t)\mathbf{S}(t)\mathbf{C} = A(t)f(t)$$

then $\mathbf{S}(t)\mathbf{C}$ is a fundamental solution ∎

Theorem 1.3 if $\mathbf{S}_1(t)$ and $\mathbf{S}_2(t)$ are fundamental solution of system (1) then $\mathbf{S}_2(t) = \mathbf{S}_1(t)\mathbf{C}$ where $\mathbf{C}$ is constant matrix such that $\det(\mathbf{C}) \neq 0$

Proof. let $\mathbf{S}_1(t)$ and $\mathbf{S}_2(t)$ be fundamental solutions
set

$$\mathbf{S}_2(t) = \mathbf{S}_1(t)\mathbf{G}(t)$$

then

$$\begin{aligned}
\frac{d\mathbf{S}_2(t)}{dt} &= \mathbf{S}_1(t)\frac{d\mathbf{G}(t)}{dt} + \frac{d\mathbf{S}_1(t)}{dt}\mathbf{G}(t) \\
A(t)\mathbf{S}_2(t) &= \mathbf{S}_1(t)\frac{d\mathbf{G}(t)}{dt} + A(t)\mathbf{S}_1(t)\mathbf{G}(t) \\
A(t)\mathbf{S}_1(t)\mathbf{G}(t) &= \mathbf{S}_1(t)\frac{d\mathbf{G}(t)}{dt} + A(t)\mathbf{S}_1(t)\mathbf{G}(t) \\
\therefore \mathbf{S}_1(t)\frac{d\mathbf{G}(t)}{dt} &= 0
\end{aligned}$$

because $\det(\mathbf{S}_1(t)) \neq 0$ then it has inverse
then multiply by this inverse from the left

$$\begin{aligned}
\mathbf{S}_1^{-1}(t)\mathbf{S}_1(t)\frac{d\mathbf{G}(t)}{dt} &= 0 \\
\frac{d\mathbf{G}(t)}{dt} &= 0 \\
\therefore \mathbf{G}(t) &= \mathbf{C}
\end{aligned}$$

then $\mathbf{G}(t)$ is a constant matrix ■

2 Matrix and Determinant Differentiation

if a differentiation applied on matrix then the result will be as following

$$\frac{d}{dt}\begin{bmatrix} a_{11}(t) & a_{12}(t) & \dots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \dots & a_{2n}(t) \\ \vdots & \vdots & & \vdots \\ a_{n1}(t) & a_{n2}(t) & \dots & a_{nn}(t) \end{bmatrix} = \begin{bmatrix} \dot{a}_{11}(t) & \dot{a}_{12}(t) & \dots & \dot{a}_{1n}(t) \\ \dot{a}_{21}(t) & \dot{a}_{22}(t) & \dots & \dot{a}_{2n}(t) \\ \vdots & \vdots & & \vdots \\ \dot{a}_{n1}(t) & \dot{a}_{n2}(t) & \dots & \dot{a}_{nn}(t) \end{bmatrix}$$

on the other hand if it's applied on Determinant

$$\begin{aligned}
\frac{d}{dt}\begin{vmatrix} a_{11}(t) & a_{12}(t) & \dots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \dots & a_{2n}(t) \\ \vdots & \vdots & & \vdots \\ a_{n1}(t) & a_{n2}(t) & \dots & a_{nn}(t) \end{vmatrix} &= \begin{vmatrix} \dot{a}_{11}(t) & \dot{a}_{12}(t) & \dots & \dot{a}_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \dots & a_{2n}(t) \\ \vdots & \vdots & & \vdots \\ a_{n1}(t) & a_{n2}(t) & \dots & a_{nn}(t) \end{vmatrix} + \begin{vmatrix} a_{11}(t) & a_{12}(t) & \dots & a_{1n}(t) \\ \dot{a}_{21}(t) & \dot{a}_{22}(t) & \dots & \dot{a}_{2n}(t) \\ \vdots & \vdots & & \vdots \\ a_{n1}(t) & a_{n2}(t) & \dots & a_{nn}(t) \end{vmatrix} + \dots \\
&+ \begin{vmatrix} a_{11}(t) & a_{12}(t) & \dots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \dots & a_{2n}(t) \\ \vdots & \vdots & & \vdots \\ \dot{a}_{n1}(t) & \dot{a}_{n2}(t) & \dots & \dot{a}_{nn}(t) \end{vmatrix}
\end{aligned}$$

property :
let $\mathbf{S}(t) = e^{\mathbf{A}t}$ where $\mathbf{A}$ is a constant matrix then

$$\frac{d\mathbf{S}(t)}{dt} = \mathbf{A}e^{\mathbf{A}t} = \mathbf{A}\mathbf{S}(t)$$

where $\det\big(e^{\mathbf{A}t}\big) \neq 0$ then $e^{\mathbf{A}t}$ is a fundamental solution of $\dfrac{dX(t)}{dt} = AX(t)$

Theorem 2.1 if $\mathbf{S}(t)$ is fundamental solution of system (1) i.e

$$\frac{d\mathbf{S}(t)}{dt} = \mathbf{A}(t)\mathbf{S}(t)$$

then $\dfrac{d}{dt}\det(\mathbf{S}(t)) = \operatorname{tr}[\mathbf{A}(t)]\det(\mathbf{S}(t))$
and $\operatorname{Det}(\mathbf{S}(t)) = \det(\mathbf{S}(t_0))e^{\int_{t_0}^{t} tr[\mathbf{A}(\theta)]d\theta}$
where $\operatorname{tr}[\mathbf{A}(t)] = a_{11}(t) + a_{22}(t) + \cdots + a_{nn}(t)$

Proof. because

$$\frac{d\mathbf{S}(t)}{dt} = \mathbf{A}(t)\mathbf{S}(t)$$

let

$$\mathbf{A}(t)\mathbf{S}(t) = b_{ij}(t) = \sum_{k=1}^{n} a_{ik}(t)s_{kj}(t)$$

then

$$\frac{d}{dt}\det(\mathbf{S}(t)) = \begin{vmatrix} \sum_{k=1}^{n} a_{1k}(t)s_{k1}(t) & \sum_{k=1}^{n} a_{1k}(t)s_{k2}(t) & \dots & \sum_{k=1}^{n} a_{1k}(t)s_{kn}(t) \\ a_{21}(t) & a_{22}(t) & \dots & a_{2n}(t) \\ \vdots & \vdots & & \vdots \\ a_{n1}(t) & a_{n2}(t) & \dots & a_{nn}(t) \end{vmatrix} + \dots$$

$$+ \begin{vmatrix} a_{11}(t) & a_{12}(t) & \dots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \dots & a_{2n}(t) \\ \vdots & \vdots & & \vdots \\ \sum_{k=1}^{n} a_{nk}(t)s_{k1}(t) & \sum_{k=1}^{n} a_{nk}(t)s_{k2}(t) & \dots & \sum_{k=1}^{n} a_{nk}(t)s_{kn}(t) \end{vmatrix}$$

using the properties of determinants we get

$$\frac{d}{dt}\det(\mathbf{S}(t)) = a_{11}\det(\mathbf{S}(t)) + a_{22}\det(\mathbf{S}(t)) + \cdots + a_{nn}\det(\mathbf{S}(t)) = \operatorname{tr}(\mathbf{A}(t))\det(\mathbf{S}(t))$$

then we can write that

$$\begin{aligned} \frac{d}{dt}\det(\mathbf{S}(t)) &= \operatorname{tr}(\mathbf{A}(t))\det(\mathbf{S}(t)) \\ \frac{d\det(\mathbf{S}(t))}{\det(\mathbf{S}(t))} &= \operatorname{tr}(\mathbf{A}(t))dt \\ \ln\left(\frac{\det(\mathbf{S}(t))}{\det(\mathbf{S}(t_0))}\right) &= \int_{t_0}^{t} \operatorname{tr}[\mathbf{A}(\theta)]d\theta \\ \det(\mathbf{S}(t)) &= \det(\mathbf{S}(t_0))e^{\int_{t_0}^{t} \operatorname{tr}[\mathbf{A}(\theta)]d\theta} \end{aligned}$$

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special case : If $\mathbf{A}$ is a constant matrix and $t_0 = 0$ then

$$\det(\mathbf{S}(t)) = \det(\mathbf{S}(0))e^{\operatorname{tr}[\mathbf{A}]}$$

■ **Example 2.0.1** if $\mathbf{A} = \begin{bmatrix} 8 & 5 & 2 \\ 9 & 1 & 3 \\ 4 & -5 & 6 \end{bmatrix}$ then

$$\det e^{\mathbf{A}} = \operatorname{tr}(\mathbf{A}) = 8 + 1 + 6 = 15$$

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now to find $e^{\mathbf{A}}$, $\mathbf{A}$ is of order n

$$\frac{dX(t)}{dt} = \mathbf{A}X(t)$$

we find the n^{th} solution $\begin{bmatrix} x_{11}(t) \\ x_{21}(t) \\ \vdots \\ x_{n1}(t) \end{bmatrix}, \begin{bmatrix} x_{12}(t) \\ x_{22}(t) \\ \vdots \\ x_{n2}(t) \end{bmatrix}, \dots, \begin{bmatrix} x_{1n}(t) \\ x_{2n}(t) \\ \vdots \\ x_{nn}(t) \end{bmatrix}$ and by compining them in one matrix we get the solution

$$X(t) = e^{\mathbf{A}} = \begin{bmatrix} x_{11}(t) & x_{12}(t) & \dots & x_{1n}(t) \\ x_{21}(t) & x_{22}(t) & \dots & x_{2n}(t) \\ \vdots & \vdots & & \vdots \\ x_{n1}(t) & x_{n2}(t) & \dots & x_{nn}(t) \end{bmatrix}$$

■ **Example 2.0.2** solve

$$\begin{cases} \dfrac{dx(t)}{dt} = \mathbf{A}X(t) \\ A = \begin{bmatrix} 1 & 12 \\ 3 & 1 \end{bmatrix} \\ x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{cases} \tag{2}$$

let

$$X(t) = e^{\lambda t}\mathbf{V} \quad , \quad \mathbf{V} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$\dot{X}(t) = \lambda e^{\lambda t}\mathbf{V} = \mathbf{A}e^{\lambda t}\mathbf{V}$$

$$\therefore e^{\lambda t}(\mathbf{A} - \lambda\mathbf{I})\mathbf{V} = 0 \quad , \quad \mathbf{AV} = \lambda\mathbf{V}$$

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} 1-\lambda & 12 \\ 3 & 1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)^2 = 36$$

$$1 - \lambda = \pm 6$$

$$\lambda = -5 \quad , \quad \lambda = 7$$

because

$$\mathbf{AV} = \lambda\mathbf{V}$$

then at $\lambda = 7$

$$\begin{bmatrix} 1 & 12 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 + 12v_2 \\ 3v_1 + v_2 \end{bmatrix} = 7\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 7v_1 \\ 7v_2 \end{bmatrix}$$

solving this system we get

$$v_1 = 2v_2$$

Next at $\lambda = -5$

$$\begin{bmatrix} 1 & 12 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 + 12v_2 \\ 3v_1 + v_2 \end{bmatrix} = -5\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -5v_1 \\ -5v_2 \end{bmatrix}$$

solving this system we get

$$v_1 = -2v_2$$

then the solution of (2) is

$$X(t) = e^{7t}v\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

or

$$X(t) = e^{-5t}v\begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

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