Lecture 9
Integro – PDE

1 Integro-Partial Differential Equations

Consider the equation

$$\frac{\partial u(x,t)}{\partial t} = \sum_{|q|\le m} a_q(x,y,t) D_x^q u(x,t) + \int_G \sum_{|q|\le m} b_q(x,y,t) D_y^q u(x,t)\, dy$$

Where

- $G \subset \mathbb{R}^n$ is bounded region with smooth surface
- $D_x^q = \dfrac{\partial^{|q|}}{\partial x_1^{q_1} \partial x_2^{q_2} \partial x_3^{q_3} \dots \partial x_n^{q_n}}$
- $|q| = q_1 + q_2 + \cdots + q_n$
- $x = (x_1, x_2, \dots, x_n)$
- $y = (y_1, y_2, \dots, y_n)$
- $dy = dy_1 dy_2 \dots dy_n$
- $\int_G = \underbrace{\int\int \dots \int}_{n}$

■ **Example 1.0.1** Consider Integro-Partial differential equation

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = \dfrac{\partial^2 u(x,t)}{\partial x^2} + \displaystyle\int_{a_1}^{b_1} K_1(x,y,t) \frac{\partial u(y,t)}{\partial y} dy + \int_{a_2}^{b_2} K_2(x,y,t) u(y,t)\, dy \\ u(x,0) = \phi(x) \end{cases} \tag{1}$$

And $\phi(x)$, K_1 , K_2 are bounded and continuous known functions on $(-\infty, \infty)$
Consider the Cauchy problem

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = \dfrac{\partial^2 u(x,t)}{\partial x^2} + V(x,t) \\ u(x,0) = \phi(x) \end{cases} \tag{2}$$

The solution of this problem is given by

$$u(x,t) = \underbrace{\frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-\xi)^2}{4t}} \phi(\xi) d\xi}_{=\psi(x,t)} + \int_0^t \int_{-\infty}^{\infty} \frac{1}{2\sqrt{\pi (t-\theta)}} e^{-\frac{(x-\xi)^2}{4(t-\theta)}} V(\xi,\theta) d\xi d\theta \tag{3}$$

(More information about this problem in the end of the lecture)
Comparing problem (1) with (2) we get that

$$V(x,t) = \int_{a_1}^{b_1} K_1(x,y,t) \frac{\partial u(y,t)}{\partial y} dy + \int_{a_2}^{b_2} K_2(x,y,t) u(y,t)\, dy$$

Now substitute for u from (3)

$$\begin{aligned} V(x,t) = &\int_{a_1}^{b_1} K_1(x,y,t) \frac{\partial \psi(y,t)}{\partial y} dy \\ &+ \int_{a_1}^{b_1} K_1(x,y,t) \frac{\partial}{\partial y} \int_0^t \int_{-\infty}^{\infty} \frac{1}{2\sqrt{\pi(t-\theta)}} e^{-\frac{(y-\xi)^2}{4(t-\theta)}} V(\xi,\theta) d\xi d\theta dy \\ &+ \int_{a_2}^{b_2} K_2(x,y,t) \psi(y,t)\, dy \\ &+ \int_{a_2}^{b_2} K_2(x,y,t) \int_0^t \int_{-\infty}^{\infty} \frac{1}{2\sqrt{\pi(t-\theta)}} e^{-\frac{(y-\xi)^2}{4(t-\theta)}} V(\xi,\theta) d\xi d\theta\, dy \end{aligned}$$

Put $K_1 = 0$ to make it a little simpler

$$V(x,t) = \psi^*(y,t) + \int_{a_2}^{b_2} K_2(x,y,t) \int_0^t \int_{-\infty}^{\infty} \frac{1}{2\sqrt{\pi(t-\theta)}} e^{-\frac{(y-\xi)^2}{4(t-\theta)}} V(\xi,\theta) d\xi d\theta \, dy$$

Now we have a volttera integral equation that can be solved by the integral equation methods like successive approximation ∎

Remember From PDE Course(Extra Information)

1.1 Cauchy In-Homogeneous Problem

Also known as Heat with a source Cauchy problem
Consider the in-homogeneous heat equation on the whole line

$$\begin{cases} \dfrac{\partial u\,(x,t)}{\partial t} = c^2 \dfrac{\partial^2 u(x,t)}{\partial x^2} + f(x,t), \quad c \neq 0, \quad -\infty < x < \infty, \quad t > 0 \\ u\,(x,0) = \phi\,(x) \end{cases} \tag{1}$$

Where $f(x,t)$ and $\phi(x)$ are arbitrary given functions.
$f(x,t)$ is called the source term, and it measures the physical effect of an external heat source.

From the superposition principle, we know that the solution of the in-homogeneous equation can be written as the sum of the solution of the homogeneous equation, and a particular solution of the in-homogeneous equation.
Thus we can break problem (1) into the following two problems

$$\begin{cases} \dfrac{\partial u_h\,(x,t)}{\partial t} = c^2 \dfrac{\partial^2 u_h(x,t)}{\partial x^2} \\ u_h\,(x,0) = \phi\,(x) \end{cases} \tag{2}$$

$$\begin{cases} \dfrac{\partial u_p\,(x,t)}{\partial t} = c^2 \dfrac{\partial^2 u_p(x,t)}{\partial x^2} + f(x,t) \\ u_p\,(x,0) = 0 \end{cases} \tag{3}$$

Obviously, $u = u_h + u_p$ will solve the original problem (1).
We have solved problem (2) using Poisson formula which is

$$u_h(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\phi(y)dy \tag{4}$$

Where $S(x,t)$ is the heat kernel and it's equal to $\dfrac{e^{-\frac{x^2}{4tc^2}}}{2\sqrt{\pi tc^2}}$.
Notice that the physical meaning of expression (4) is that the heat kernel averages out the initial temperature distribution along the entire rod.

Since $f(x,t)$ plays the role of an external heat source, it is clear that this heat contribution must be averaged out too. But in this case one needs to average not only over the entire rod, but over time as well, since the heat contribution at an earlier time will effect the temperatures at all later times. We claim that the solution to (3) is given by

$$u_p(x,t) = \int_0^t \int_{-\infty}^{\infty} S(x-y,t-s)f(y,s)dyds \tag{5}$$

Combining (4) and (5) we obtain the solution to the IVP (1)

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\phi(y)dy + \int_0^t \int_{-\infty}^{\infty} S(x-y,t-s)f(y,s)dyds \tag{6}$$

Now substitute the heat kernel

$$u(x,t) = \frac{1}{2\sqrt{\pi t c^2}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4tc^2}} \phi(y)dy + \frac{1}{2\sqrt{\pi(t-s)c^2}} \int_0^t \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4(t-s)c^2}} f(y)dyds$$

We now need to show that (6) indeed solves problem (1) by direct substitution.
Since we have solved the homogeneous equation before, it suffices to show that u_p solves problem (3).

By differentiating (5) with respect to t gives

$$\frac{\partial u_p}{\partial t} = \int_{-\infty}^{\infty} S(x-y,0)f(y,t)dy + \int_0^t \int_{-\infty}^{\infty} \frac{\partial}{\partial t} S(x-y,t-s)f(y,s)dyds$$

Note that the heat kernel solves the heat equation and has the Dirac delta function as its initial means that $S_t = c^2 S_{xx}$ and $S(x-y,0) = \delta(x-y)$
When integrating the Dirac Delta function we would get

$$\int_{-\infty}^{\infty} \delta(x-y)dy = 1$$

If we have another function $f(y,t)$ multiplied to the Dirac Delta function and integrating them we would get

$$\int_{-\infty}^{\infty} \delta(x-y)f(y,t)dy = f(x,t) \int_{-\infty}^{\infty} \delta(x-y)dy = f(x,t)$$

$$\begin{aligned}
\frac{\partial u_p}{\partial t} &= \int_{-\infty}^{\infty} \delta(x-y)f(y,t)dy + \int_0^t \int_{-\infty}^{\infty} c^2 \frac{\partial^2}{\partial x^2} S(x-y,t-s)f(y,s)dyds \\
&= f(x,t) + c^2 \frac{\partial^2}{\partial x^2} \int_0^t \int_{-\infty}^{\infty} S(x-y,t-s)f(y,s)dyds \\
&= f(x,t) + c^2 \frac{\partial^2 u_p}{\partial x^2}
\end{aligned}$$

Which shows that $u_p(x,t)$ solves the in-homogeneous heat equation. It is also clear that

$$\lim_{t\to 0} u_p(x,t) = \lim_{t\to 0} \int_0^t \int_{-\infty}^{\infty} S(x-y,t-s)f(y,s)dyds = 0$$

Therefore $u_p(x,t)$ indeed solves problem (3) which finishes the proof that (6) solves the original IVP (1).

Duhamel's principle

If one can solve an initial value problem for a homogeneous linear differential equation then an in-homogeneous linear differential equation can be solved as well.