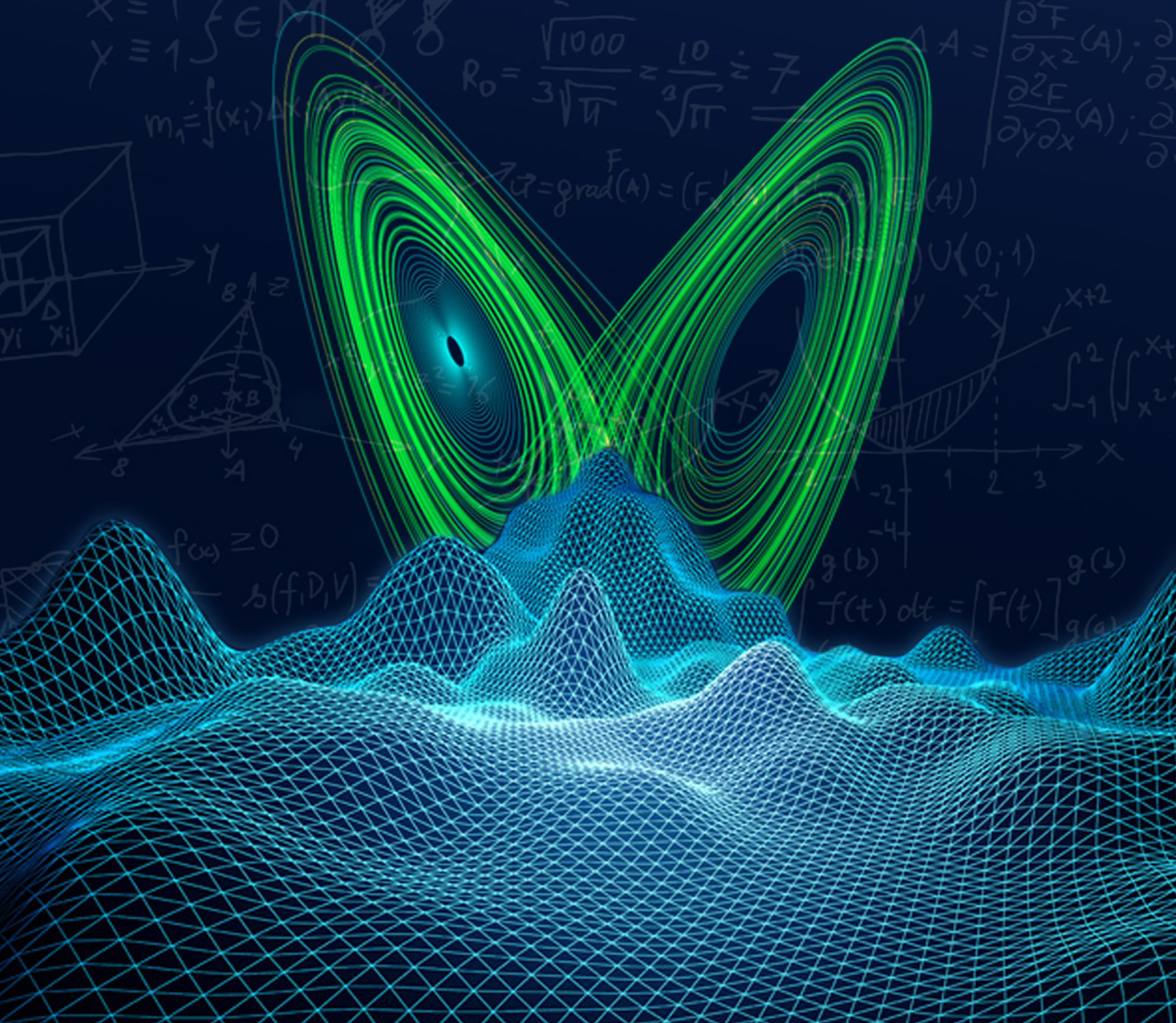
Selected Topics In Pure Mathematics



Selected Topics In Pure Mathematics (040101401)

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Contents

1 Fractional Calculus

1.1 Fractional Integral

let f be a continuous function on $[a, b]$ and let I donate the integral operator

$$If(t) = \int_0^t f(s)ds = g(t) \quad , \quad t \in [a, b]$$

and if we apply it again

$$\begin{aligned} I^2 f(t) &= \int_0^t g(s)ds \\ &= \int_0^t \int_0^s f(\theta)d\theta \\ &= \int_0^t \left(\int_s^t d\theta \right) f(s)ds \\ &= \int_0^t (t-s)f(s)ds \end{aligned}$$

we can get the general formula for integrating n time by

$$\begin{aligned} I^n f(t) &= \frac{1}{(n-1)!} \int_0^t (t-s)^{n-1} f(s)ds \\ &= \frac{1}{\Gamma(n)} \int_0^t (t-s)^{n-1} f(s)ds \end{aligned}$$

then we can say that the fractional integral of order α is defined as

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds \tag{1}$$

$$0 < \alpha \leq 1$$

1.2 Fractional Derivative

the fractional derivative is defined by

$$D^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \frac{df(s)}{ds} ds \tag{2}$$

or

$$D^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-s)^{-\alpha} f(s)ds \tag{3}$$

$$0 \leq \alpha \leq 1$$

where α is the order of differentiation
the need of having 2 formulas that each has a problem that the other solves like that formula (2) need the 1^{st} derivative to exist to get the fractional derivative and formula (3) the derivative of the constant not equal zero

$$\begin{aligned} D^\alpha 1 &= \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-s)^{-\alpha} ds \\ &= \frac{-1}{\Gamma(1-\alpha)} \frac{d}{dt} \left[\frac{(t-s)^{-\alpha+1}}{-\alpha+1} \right]_0^t \\ &= \frac{-1}{\Gamma(1-\alpha)} \frac{t^{1-\alpha}}{1-\alpha} \neq 0 \end{aligned}$$

1.3 Laplace transform for fractional integral

we know that

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)dt = F(s)$$

then we can do the following

$$\begin{aligned}\mathcal{L}\{I^\alpha f(t)\} &= \int_0^\infty e^{-st} \frac{1}{\Gamma(\alpha)} \int_0^t (t-\theta)^{\alpha-1} f(\theta) d\theta dt \\ &= \frac{1}{\Gamma(\alpha)} \mathcal{L}\{t^{\alpha-1} \times f(t)\}\end{aligned}$$

from the convolution property

$$\mathcal{L}\{I^\alpha f\} = \frac{1}{\Gamma(\alpha)} \mathcal{L}\{t^{\alpha-1}\} \times \mathcal{L}\{f(t)\} \tag{4}$$

let's handle the first transformation

$$\begin{aligned}\mathcal{L}\{t^{\alpha-1}\} &= \int_0^\infty e^{-st} t^{\alpha-1} dt \\ &\text{put } st = \eta \Longrightarrow dt = \frac{1}{s} d\eta \\ \mathcal{L}\{t^{\alpha-1}\} &= \int_0^\infty e^{-\eta} \eta^{\alpha-1} s^{1-\alpha} \frac{1}{s} d\eta \\ &= \int_0^\infty e^{-\eta} \eta^{\alpha-1} s^{-\alpha} \frac{1}{s} d\eta = s^{-\alpha}\Gamma(\alpha)\end{aligned}$$

now substitute in equation (4)

$$\mathcal{L}\{I^\alpha f\} = s^{-\alpha} \mathcal{L}\{f(t)\} = s^{-\alpha} F(s)$$

1.4 The Integral of Derivative

Now that we defined the integral and the differential operator logically they suppose to cancel each other we need to proof that

$$I^\alpha D^\alpha f(t) = f(t)$$

using the formula (2)

$$\begin{aligned}I^\alpha D^\alpha f(t) &= I^\alpha \left[\frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \frac{df(s)}{ds} ds\right] \qquad 0 < \alpha < 1 \\ &= \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \int_0^t (t-s)^{\alpha-1} \int_0^s (s-\theta)^{-\alpha} \frac{df(\theta)}{d\theta} d\theta ds \\ &= \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \int_0^t \underbrace{\int_\theta^t (t-s)^{\alpha-1}(s-\theta)^{-\alpha} ds}_{J} \frac{df(\theta)}{d\theta} d\theta \end{aligned} \tag{5}$$

let's handle the inner integral first

$$\begin{aligned}J &= \int_\theta^t (t-s)^{\alpha-1}(s-\theta)^{-\alpha} ds \\ &\text{put } s-\theta = \eta \Longrightarrow ds = d\eta \\ &= \int_0^{t-\theta} (t-\theta-\eta)^{\alpha-1} (\eta)^{-\alpha} d\eta \\ &= (t-\theta)^{\alpha-1} \int_0^{t-\theta} (1 - \frac{\eta}{t-\theta})^{\alpha-1} (\eta)^{-\alpha} d\eta\end{aligned}$$

$$
\begin{aligned}
&\text{put } \eta = (t-\theta)\xi \Longrightarrow d\eta = (t-\theta)d\xi \\
&= (t-\theta)^{\alpha-1}\int_0^1 (1-\xi)^{\alpha-1}(t-\theta)^{1-\alpha}\xi^{-\alpha}d\xi \\
&= \int_0^1 (1-\xi)^{\alpha-1}\xi^{-\alpha}d\xi = \beta(\alpha, 1-\alpha)
\end{aligned}
$$

substitute in (5) we get that

$$
\begin{aligned}
I^\alpha D^\alpha f &= \frac{\beta(\alpha, 1-\alpha)}{\Gamma(\alpha)\Gamma(1-\alpha)}\int_0^t \frac{df(\theta)}{d\theta}d\theta \\
&= \frac{\Gamma(\alpha)\Gamma(1-\alpha)}{\Gamma(\alpha+1-\alpha)\Gamma(\alpha)\Gamma(1-\alpha)}[f(t)-f(0)] \\
&= f(t)-f(0)
\end{aligned}
$$

2 Stability

consider

$$\begin{cases} \frac{dx(t)}{dt} = & f(x(t), t) \qquad t > 0 \\ x(0) = a \end{cases} \tag{1}$$

we say that the solutions of equation (1) are stable if and only if

$$\forall \epsilon > 0 \ , \ \exists \delta > 0 \text{ s.t } x(0) = a, x^*(0) = b$$

$$|a - b| < \delta \Longrightarrow |x(t) - x^*(t)| \leq \epsilon$$

where $x(t)$ and $x^*(t)$ are solutions of equation (1) moreover we say that the solutions of equation (1) are asymptotically stable if and only if they satisfies the previous Conditions and $\lim_{t \to \infty} (x(t) - x^*(t)) = 0$

2.1 Lipschitz Condition

we say that $f(x)$ satisfies lipschitz condition with lipschitz constant N if and only if

$$|f(x) - f(x^*)| \leq N |x - x^*|$$

$$f(x) \ \text{ defined on } \ [a, b] \ , \ x, x^* \in [a, b]$$

if $f(x)$ is differentiable and $f'(x)$ is bounded i.e. $|f'(x)| \leq M$

$$f(x) - f(x^*) = (x - x^*) f'(x^{**})$$

$$x \leq x^{**} \leq x^*$$

$$|f(x) - f(x^*)| \leq M |x - x^*|$$

Theorem 2.1 let $f(x, t)$ be a continues function on $G := \{(x, t) \mid a \leq x \leq b \ , \ 0 \leq t \leq T\}$ and satisfies lipschitz condition with respect to x and with lipschitz constant N , i.e.

$$|f(x, t) - f(x^*, t)| \leq N |x - x^*|$$

suppose that $\begin{cases} \frac{ds_1(t)}{dt} = f(s_1(t), t) \ , \ s_1(0) = \beta_1 \\ \frac{ds_2(t)}{dt} = f(s_2(t), t) \ , \ s_2(0) = \beta_2 \end{cases}$

if $|\beta_1 - \beta_2| \leq \delta$ then $|s_1(t) - s_2(t)| \leq \delta e^{Nt}$

proof.

$$s_1(t) = \beta_1 + \int_0^t f(s_1(\theta), \theta) d\theta \tag{2}$$

$$s_2(t) = \beta_2 + \int_0^t f(s_2(\theta), \theta) d\theta \tag{3}$$

subtract equation (3) from (2)

$$s_1(t) - s_2(t) = \beta_1 - \beta_2 + \int_0^t [f(s_1(\theta), \theta) - f(s_2(\theta), \theta)] \, d\theta$$

taking the absolute value to both sides
and because $|\beta_1 - \beta_2| \leq \delta$ and f satisfies lipschitz condition $|f(x, t) - f(x^*, t)| \leq N |x - x^*|$ then

$$|s_1(t) - s_2(t)| \leq \delta + N \int_0^t |s_1(\theta) - s_2(\theta)| \, d\theta$$

put $|s_1(t) - s_2(t)| = r(t)$

$$r(t) \leq \delta + N \int_0^t r(\theta) d\theta \tag{4}$$

set $R(t) = \displaystyle\int_0^t r(\theta) d\theta$ i.e. $r(t) = \dfrac{dR(t)}{dt}$
and Substitute in (4)

$$\frac{dR(t)}{dt} - NR(t) \leq \delta \tag{5}$$

multiply both sides by e^{-Nt}

$$e^{-Nt}\left[\frac{dR(t)}{dt} - NR(t)\right] \leq \delta e^{-Nt}$$
$$\frac{d}{dt}\left[e^{-Nt}R(t)\right] \leq \delta e^{-Nt}$$

integrating both sides from $0 \to t$ we get that

$$e^{-Nt}R(t) - R(0) \leq \frac{\delta}{N}\left[1 - e^{-Nt}\right]$$

and we know that

$$R(t) = \int_0^t r(\theta) d\theta$$

then

$$R(0) = \int_0^0 r(\theta) d\theta = 0$$

therefore we get

$$R(t) \leq \frac{\delta}{N}\left[e^{Nt} - 1\right]$$

Substitute in (5) to get the following

$$r(t) \leq \delta + \delta\left[e^{Nt} - 1\right]$$
$$|s_1(t) - s_2(t)| \leq \delta e^{Nt}$$

■

Theorem 2.2 let $f(x,t)$ be a continues function on $G := \{(x,t) \mid a \le x \le b \ , \ 0 \le t \le T\}$ and satisfies lipschitz condition with respect to x and with lipschitz constant N , i.e.

$$|f(x,t) - f(x^*,t)| \le N\,|x - x^*|$$

suppose that

$$\frac{dx(t)}{dt} = -\gamma x(t) + f(x(t),t) \tag{6}$$

and let $s_1(t)$ and $s_2(t)$ be solutions for equation (6) corresponding to $\begin{cases} s_1(0) = \beta_1 \\ s_2(0) = \beta_2 \end{cases}$

if $\gamma > N$ and $|\beta_1 - \beta_2| \le \delta$ then $\lim_{t\to\infty} |s_1(t) - s_2(t)| = 0$ and $|s_1(t) - s_2(t)| \le \delta e^{-(\gamma - N)t}$

proof. let $y(t) = e^{\gamma t} x(t)$

$$\frac{dy(t)}{dt} = e^{\gamma t}\frac{dx(t)}{dt} + \gamma e^{\gamma t} x(t)$$

Substitute $\dfrac{dx(t)}{dt}$ from equation (6)

$$\begin{aligned} \frac{dy(t)}{dt} &= e^{\gamma t}\left[-\gamma x(t) + f(x(t),t)\right] + \gamma e^{\gamma t} x(t) \\ &= e^{\gamma t} f(x(t),t) \\ &\because y(t) = e^{\gamma t} x(t) \\ &\therefore x(t) = e^{-\gamma t} y(t) \end{aligned}$$

therefore

$$\frac{dy(t)}{dt} = e^{\gamma t} f(e^{-\gamma t} y(t), t) \tag{7}$$

let $\mathbf{S}_1(t)$ and $\mathbf{S}_2(t)$ be solution of equation (7)

$$\begin{cases} \mathbf{S}_1(t) = e^{\gamma t} s_1(t) \ , \ s_1(0) = \beta_1 \\ \mathbf{S}_2(t) = e^{\gamma t} s_2(t) \ , \ s_2(0) = \beta_2 \end{cases} \tag{8}$$

$y(0) = x(0)$ i.e. $\mathbf{S}_1(t) = \beta_1$ and $\mathbf{S}_2(t) = \beta_2$

$$\mathbf{S}_1(t) = \beta_1 + \int_0^t e^{\gamma\theta} f(e^{-\gamma\theta}\mathbf{S}_1(\theta),\theta)d\theta \tag{9}$$

$$\mathbf{S}_2(t) = \beta_2 + \int_0^t e^{\gamma\theta} f(e^{-\gamma\theta}\mathbf{S}_2(\theta),\theta)d\theta \tag{10}$$

subtract equation (10) from (9)

$$\mathbf{S}_1(t) - \mathbf{S}_2(t) = \beta_1 - \beta_2 + \int_0^t e^{\gamma\theta}\left[f(e^{-\gamma\theta}\mathbf{S}_1(\theta),\theta) - f(e^{-\gamma\theta}\mathbf{S}_2(\theta),\theta)\right]d\theta$$

taking the absolute value to both sides
and because $|\beta_1 - \beta_2| \le \delta$ and f satisfies lipschitz condition $|f(x,t) - f(x^*,t)| \le N\,|x - x^*|$
then

$$\begin{aligned} |\mathbf{S}_1(t) - \mathbf{S}_2(t)| &\le \delta + N\int_0^t e^{\gamma\theta}\left|e^{-\gamma\theta}\mathbf{S}_1(\theta) - e^{-\gamma\theta}\mathbf{S}_2(\theta)\right|d\theta \\ &\le \delta + N\int_0^t |\mathbf{S}_1(\theta) - \mathbf{S}_2(\theta)|\,d\theta \end{aligned}$$

put $|\mathbf{S}_1(t) - \mathbf{S}_2(t)|\, d\theta = r(t)$

$$r(t) \leq \delta + N \int_0^t r(\theta) d\theta \tag{11}$$

set $R(t) = \displaystyle\int_0^t r(\theta) d\theta$ i.e. $r(t) = \dfrac{dR(t)}{dt}$
and Substitute in (11)

$$\frac{dR(t)}{dt} - NR(t) \leq \delta \tag{12}$$

multiply both sides by e^{-Nt}

$$e^{-Nt}\left[\frac{dR(t)}{dt} - NR(t)\right] \leq \delta e^{-Nt}$$
$$\frac{d}{dt}\left[e^{-Nt}R(t)\right] \leq \delta e^{-Nt}$$

integrating both sides from $0 \to t$ we get that

$$R(t) \leq \frac{\delta}{N}\left[e^{Nt} - 1\right]$$

Substitute in (12) to get the following

$$r(t) \leq \delta + \delta\left[e^{Nt} - 1\right]$$
$$|\mathbf{S}_1(t) - \mathbf{S}_2(t)| \leq \delta e^{Nt}$$

multiply both sides by $e^{-\gamma t}$

$$e^{-\gamma t}|\mathbf{S}_1(t) - \mathbf{S}_2(t)| \leq \delta e^{-(\gamma - N)t}$$

from equations (8) we get that

$$e^{-\gamma t}e^{\gamma t}|s_1(t) - s_2(t)| \leq \delta e^{-(\gamma - N)t}$$

$$|s_1(t) - s_2(t)| \leq \delta e^{-(\gamma - N)t}$$

because $\gamma > N$ is given in the theorem then the power of R.H.S is negative therefore when $t \to \infty$ then $e^{-(\gamma - N)} \to 0$ then

$$\lim_{t\to\infty} |s_1(t) - s_2(t)| \leq 0$$
$$\therefore \lim_{t\to\infty} |s_1(t) - s_2(t)| = 0$$

■

Rudolf Lipschitz

Rudolf Otto Sigismund Lipschitz (14 May 1832 – 7 October 1903) was a German mathematician who made contributions to mathematical analysis (where he gave his name to the Lipschitz continuity condition) and differential geometry, as well as number theory, algebras with involution and classical mechanics.

Theorem 2.3 Let A be a constant matrix suppose that all the characteristic roots of A with negative real part
Now consider the equation
$$\frac{dx(t)}{dt} = Ax(t) + f(t, x(t))$$
if $||f(t, x(t))||= o(||x(t)||)$ and $f(t, 0) = 0$, then the rest point is asymptotically stable

we can define A as $\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$ and x as $\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

and the Norm as $||x(t)||= \sum_{i=1}^{n} |x_i(t)|$ or $||x(t)||= \left(\sum_{i=1}^{n} |x_i(t)|^2 \right)^{\frac{1}{2}}$

and $||A||= \sum_{i,j=1}^{n} |a_{ij}|$ or $||A||= \left(\sum_{i,j=1}^{n} |a_{ij}|^2 \right)^{\frac{1}{2}}$
and rest point is the zero solution of the equation (1)
the rest point is stable if
$$\forall \epsilon > 0 \ , \ \exists \delta > 0 \text{ s.t } ||x(0)|| \leq \delta \Longrightarrow ||x(t)|| \leq \epsilon$$
and it is asymptotically stable if it satisfies the last condition and
$$\lim_{t \to \infty} ||x(t)||= 0$$

proof. we have
$$\frac{dx(t)}{dt} = Ax(t) + f(t, x(t)) \tag{1}$$
we can write it as
$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\theta)} f(x(\theta), \theta) d\theta \tag{2}$$
this is a representation for equation (1) to see that they are the same take the derivative of it with respect to t
$$\frac{dx(t)}{dt} = Ae^{At}x(0) + \frac{d}{dt} \int_0^t e^{A(t-\theta)} f(x(\theta), \theta) d\theta$$
using Leibniz rule
$$\begin{aligned} \frac{dx(t)}{dt} &= Ae^{At}x(0) + A \int_0^t e^{A(t-\theta)} f(x(\theta), \theta) d\theta + f(x(t), t) \\ &= A \left(e^{At}x(0) + \int_0^t e^{A(t-\theta)} f(x(\theta), \theta) d\theta \right) + f(x(t), t) \\ &= Ax(t) + f(x(t), t) \end{aligned}$$

Leibniz rule
$$\frac{d}{dt} \int_{\alpha(t)}^{\beta(t)} f(t, \theta) d\theta = \frac{d\beta(t)}{dt} f(t, \beta(t)) - \frac{d\alpha(t)}{dt} f(t, \alpha(t)) + \int_{\alpha(t)}^{\beta(t)} \frac{\partial f(t, \theta)}{\partial t} d\theta$$

we can find $K > 0, \sigma > 0$ such that $||e^{At}||< Ke^{-\sigma t}$
Take the Norm for equation (2)

$$||x(t)||\leq Ke^{-\sigma t}||x(0)||+K\int_0^t e^{-\sigma(t-\theta)}||f(x(\theta),\theta)||d\theta$$

and we know that $||f(t,x(t))||= o(||x(t)||)$ or in other word $\lim\limits_{||x(t)||\to 0} \dfrac{||f(x(t),t)||}{||x(t)||} = 0$ i.e

$$\forall \epsilon > 0\ ,\ \exists \delta > 0 \text{ such that } ||x(t)||\leq \delta \Longrightarrow ||f(x(t),t)||\leq \epsilon||x(t)||$$

thus

$$||x(t)||\leq Ke^{-\sigma t}||x(0)||+K\epsilon\int_0^t e^{-\sigma(t-\theta)}||x(\theta)||d\theta$$

put $\epsilon = \dfrac{\epsilon}{K}$ and multiply by $e^{\sigma t}$

$$e^{\sigma t}||x(t)||\leq K||x(0)||+\epsilon\int_0^t e^{\sigma\theta}||x(\theta)||d\theta$$

as long as $||x(t)||\leq \delta$

set $R(t) = \displaystyle\int_0^t e^{\sigma\theta}||x(\theta)||d\theta$ i.e. $\dfrac{dR(t)}{dt} = e^{\sigma t}||x(t)||$

$$\frac{dR(t)}{dt} \leq K||x(0)||+\epsilon R(t) \tag{3}$$

$$\frac{dR(t)}{dt} - \epsilon R(t) \leq K||x(0)||$$

multiply by $e^{-\epsilon t}$

$$e^{-\epsilon t}\left[\frac{dR(t)}{dt} - \epsilon R(t)\right] \leq K||x(0)||e^{-\epsilon t}$$
$$\frac{d}{dt}\left[e^{-\epsilon t}R(t)\right] \leq K||x(0)||e^{-\epsilon t}$$

integrate with respect to t

$$e^{-\epsilon t}R(t) \leq \frac{K\sigma}{\epsilon}(1-e^{-\epsilon t})$$

multiply by $e^{\epsilon t}$

$$R(t) \leq \frac{K\sigma}{\epsilon}(e^{\epsilon t}-1)$$

Substitute $\dfrac{dR(t)}{dt} = e^{\sigma t}||x(t)||$ and $R(t) \leq \dfrac{K\sigma}{\epsilon}(e^{\epsilon t}-1)$ in equation (3)

$$\begin{aligned} e^{\sigma t}||x(t)|| &\leq K||x(0)||+\epsilon R(t) \\ &\leq K\delta + K\delta e^{\epsilon t} - K\delta \\ &\leq K\delta e^{\epsilon t} \end{aligned}$$

$$||x(t)||\leq K\delta e^{(\epsilon-\sigma)t}$$

put $\epsilon < \sigma$ and take the limit as $t \to \infty$

$$\lim_{t\to\infty}||x(t)||\leq 0$$

$$\lim_{t\to\infty}||x(t)||= 0$$

■

2.2 Lyapunov Function

consider the Dynamical System or the Autonomous ODE

$$\frac{dx(t)}{dt} = f(x(t))$$

$$x(t) = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad f(x(t)) = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix}$$

Theorem 2.4 — Lyapunov's theorem. suppose that there exist a function $V(x)$, such that $||V(x)|| \geq 0, \forall x$ and $||V(x)|| = 0$ only at $x = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$

$$\frac{dV}{dt} = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} \frac{dx_i}{dt} = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} f_i(x,t) \leq 0$$

in some neighborhood of $\begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$ it's supposed that f(0) $= \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$ then the rest point is stable

if also $\frac{dV}{dt} \leq -\beta, \beta > 0$ outside $||x(t)|| \leq \delta$

then the rest point is asymptotically stable

■ **Example 2.2.1** check the stability of the system

$$\begin{cases} \dfrac{dx}{dt} = -y - x^3 \\ \\ \dfrac{dy}{dt} = x - y^3 \\ \\ V = x^2 + y^2 \end{cases}$$

for sure $V \geq 0, \forall x, y$ and $V = 0$ only at $x = y = 0$

$$\begin{aligned} \frac{dV}{dt} &= 2x\frac{dx}{dt} + 2y\frac{dy}{dt} \\ &= 2x(-y - x^3) + 2y(x - y^3) \\ &= -2xy - 2x^4 + 2xy - 2y^4 \\ &= -2(x^4 + y^4) \end{aligned}$$

$\frac{dV}{dt}$ is negative and $< \beta$ then the system is asymptotically stable ■

■ **Example 2.2.2** check the stability of the system

$$\begin{cases} \dfrac{dx}{dt} = -xy^4 \\ \\ \dfrac{dy}{dt} = yx^4 \\ \\ V = x^4 + y^4 \end{cases}$$

for sure $V \geq 0, \forall x, y$ and $V = 0$ only at $x = y = 0$

$$\begin{aligned} \frac{dV}{dt} &= 4x^3\frac{dx}{dt} + 4y^3\frac{dy}{dt} \\ &= -4x^4y^4 + 4x^4y^4 \\ &= 0 \end{aligned}$$

then the system is stable but not asymptotically stable ■

■ **Example 2.2.3** check the stability of the system

$$\begin{cases} \dfrac{dx(t)}{dt} = -y(t) - x^3(t) + z(t) \\ \\ \dfrac{dy(t)}{dt} = x(t) - y^3(t) - z(t) \\ \\ \dfrac{dz(t)}{dt} = y(t) - x(t) - z^3(t) \end{cases}$$

set Lyapunov function as following

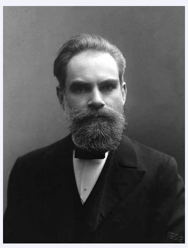
$$V = x^2 + y^2 + z^2$$

for sure $V \geq 0, \forall x, y, z$ and $V = 0$ only at $x = y = z = 0$

$$\begin{aligned} \frac{\partial V}{\partial t} &= 2x\dot{x} + 2y\dot{y} + 2z\dot{z} \\ &= 2x[-y - x^3 + z] + 2y[x - y^3 - z] + 2z[y - x - z^3] \\ &= -2xy - 2x^4 + 2xz + 2xy - 2y^4 - 2yz + 2zy - 2zx - 2z^4 \\ &= -2x^4 - 2y^4 - 2z^4 \\ \therefore \dot{V} &= -2(x^4 + y^4 + z^4) \leq 0 \end{aligned}$$

i.e. $\dot{V}$ outside $(0, 0, 0)$ is < 0 then the system is asymptotically stable ■

Aleksandr Lyapunov



Aleksandr Mikhailovich Lyapunov was a Russian mathematician, mechanician and physicist. Lyapunov contributed to several fields, including differential equations, potential theory, dynamical systems and probability theory. His main preoccupations were the stability of equilibria and the motion of mechanical systems, especially rotating fluid masses, and the study of particles under the influence of gravity. Lyapunov's impact was significant, and the following mathematical concepts are named after him: Lyapunov equation , Lyapunov exponent , Lyapunov function , Lyapunov fractal , Lyapunov stability , Lyapunov's central limit theorem , Lyapunov vector

3 Dynamical System And Climate Change Models

consider the following Autonomous

$$R\frac{dT(t)}{dt} = a - bT(t)$$
$$a = (1-\alpha)Q - A \qquad , \qquad b = B$$

This equation represents a simple energy balance model used in climate science. This type of model is often used to study the Earth's energy budget, taking into account various factors that influence the planet's temperature over time. In this equation:

- $\frac{dT(t)}{dt}$ represents the rate of change of temperature with respect to time.
- Q represents the incoming solar radiation.
- α is the albedo, which represents the fraction of incoming solar radiation that is reflected back to space.
- R is the averaged heat capacity of the Earth/atmosphere system (heat capacity is the amount of heat required to raise the temperature of an object or substance 1 kelvin(= 1 C))
- A and B are empirically determined parameters.

now let's try to solve it

$$\frac{dT(t)}{a - bT(t)} = \frac{1}{R}dt$$

multiply both sides with $-b$ and integrating with respect to t

$$\int_0^t \frac{-bdT(t)}{a - bT(t)} = \frac{-bt}{R}$$
$$ln(a - bT(t)) - ln(a - bT(0)) = \frac{-bt}{R}$$
$$ln\left(\frac{a - bT(t)}{a - bT(0)}\right) = \frac{-bt}{R}$$
$$a - bT(t) = (a - bT(0))e^{\frac{-bt}{R}}$$
$$T(t) = \frac{a}{b} + \frac{1}{b}(bT(0) - a)e^{\frac{-bt}{R}}$$

when taking the limit of $T(t)$ as t goes to ∞

$$\lim_{t\to\infty} T(t) = \frac{a}{b}$$

this is called the equilibrium point (or the zero solution that makes $T(t)$ constant)

3.1 Kaper and Engler Climate Model

Consider the next model

$$R\frac{dT(t)}{dt} = (1-\alpha)Q - \sigma T^4(t) \qquad 0 < \alpha < 1$$

The Kaper and Engler climate model is a simplified mathematical representation of the Earth's climate system. The model describes the rate of change of the Earth's temperature $T(t)$ over time t where :

- σ is the Stefan-Boltzmann constant, which relates the temperature of a black body (in this case, the Earth) to the amount of radiation it emits.

This equation captures two main factors influencing the Earth's temperature change:

1. Solar Radiation (First Term): The term $(1-\alpha)Q$ represents the solar radiation absorbed by the Earth. $(1-\alpha)$ is the fraction of incoming solar radiation that is absorbed (since α is the albedo, the fraction that is reflected), and Q represents the total incoming solar radiation.
2. Radioactive Cooling (Second Term): The term $-\sigma T^4(t)$ represents the Earth's radioactive cooling. This term describes how the Earth emits thermal radiation into space as a function of its temperature $T(t)$. According to the Stefan-Boltzmann law, the rate at which a black body radiates energy is proportional to the fourth power of its temperature.

the equilibrium point of this model is

$$(1-\alpha)Q - \sigma T^4(t) = 0$$
$$T^4(t) = \frac{(1-\alpha)Q}{\sigma}$$

4 Adomian Decomposition Method(A.D.M)

consider the nonlinear differential equation

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = x^2 - \dfrac{1}{4}(\dfrac{\partial u(x,t)}{\partial x})^2 \\ u(x,0) = 0 \end{cases} \tag{1}$$

this equation can be solved by successive approximation or the method that we will discuss which is A.D.M

set

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t)$$
$$N(u) = \sum_{n=0}^{\infty} A_n(x,t)$$

where $N(u)$ represents the nonlinear form of u in our case in equation (1) $N(u) = \left(\frac{\partial u}{\partial x}\right)^2$

$$A_n(x,t) = \left[\frac{1}{n!}\frac{d^n}{d\lambda^n} N\left(\sum_{i=0}^{n} \lambda^i u_i(x,t)\right)\right]_{\lambda=0}$$
$$A_n(x,t) = \left[\frac{1}{n!}\frac{d^n}{d\lambda^n} \left(\sum_{i=0}^{n} \lambda^i \frac{\partial u_i(x,t)}{\partial x}\right)^2\right]_{\lambda=0}$$

integrating equation (1) from $0 \to t$

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) = x^2 t - \frac{1}{4}\int_0^t \sum_{n=0}^{\infty} A_n(x,\theta) d\theta$$

now we get A_0,A_1,A_2...

$$A_0(x,\theta) = \left[\sum_{i=0}^{0} \lambda^i \frac{\partial u_i(x,\theta)}{\partial x}^2\right]_{\lambda=0} = \left(\frac{\partial u_0(x,\theta)}{\partial x}\right)^2$$
$$A_1(x,\theta) = \left[\frac{d}{d\lambda}\left(\sum_{i=0}^{1} \lambda^i \frac{\partial u_i(x,\theta)}{\partial x}\right)^2\right]_{\lambda=0}$$
$$= \left[\frac{d}{d\lambda}\left(\frac{\partial u_0(x,\theta)}{\partial x} + \lambda \frac{\partial u_1(x,\theta)}{\partial x}\right)^2\right]_{\lambda=0}$$

$$= 2\left[\left(\frac{\partial u_0(x,\theta)}{\partial x}+\lambda\frac{\partial u_1(x,\theta)}{\partial x}\right)\frac{\partial u_1(x,\theta)}{\partial x}\right]_{\lambda=0} = 2\frac{\partial u_0(x,\theta)}{\partial x}\frac{\partial u_1(x,\theta)}{\partial x}$$

$$A_2(x,\theta) = \left[\frac{1}{2!}\frac{d^2}{d\lambda^2}\left(\sum_{i=0}^{2}\lambda^i\frac{\partial u_i(x,\theta)}{\partial x}\right)^2\right]_{\lambda=0}$$

$$= \left[\frac{1}{2}\frac{d^2}{d\lambda^2}\left(\frac{\partial u_0(x,\theta)}{\partial x}+\lambda\frac{\partial u_1(x,\theta)}{\partial x}+\lambda^2\frac{\partial u_2(x,\theta)}{\partial x}\right)^2\right]_{\lambda=0}$$

$$= \left(\frac{\partial u_1(x,\theta)}{\partial x}\right)^2 + 2\left(\frac{\partial u_0(x,\theta)}{\partial x}\frac{\partial u_2(x,\theta)}{\partial x}\right)$$

$$A_3(x,\theta) = 2\frac{\partial u_1(x,\theta)}{\partial x}\frac{\partial u_2(x,\theta)}{\partial x} + 2\frac{\partial u_0(x,\theta)}{\partial x}\frac{\partial u_2(x,\theta)}{\partial x}$$

now because

$$u_0 + u_1 + u_2 + \cdots = u(x,t) = x^2t - \frac{1}{4}[A_0 + A_1 + A_2 + \dots]$$

then

$$u_0 = x^2t$$

$$u_1 = -\frac{1}{4}\int_0^t A_0 d\theta = -\frac{1}{4}\int_0^t \left(\frac{\partial u_0(x,\theta)}{\partial x}\right)^2 = -\int_0^t x^2\theta^2 d\theta = \frac{-1}{3}x^2t^3$$

$$u_2 = \frac{2}{15}x^2t^5 \quad , \quad u_3 = \frac{-17}{315}x^2t^7 \quad , \quad \dots$$

$$u(x,t) = x^2\left[t - \frac{1}{3}t^3 + \frac{2}{15}t^5 - \frac{17}{315}t^7\dots\right] = x^2\tanh(t)$$

consider the nonlinear differential equation

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = x\dfrac{\partial u(x,t)}{\partial x} + u(x,t)\dfrac{\partial u(x,t)}{\partial x} - xt - xt^2 + x \\ u(x,0) = 0 \end{cases} \tag{2}$$

the solution of equation (2) is given by $u(x,t) = xt$ by Substitute in (2)

$$L.H.S = x \qquad R.H.S = xt + xt^2 - xt - xt^2 + x = x$$

now let's use A.D.M to solve it
integrating (2) with respect to t

$$u(x,t) = \int_0^t x\frac{\partial u(x,s)}{\partial x}ds + \int_0^t u(x,s)\frac{\partial u(x,s)}{\partial x}ds - \frac{xt^2}{2} - \frac{xt^3}{3} + xt$$

$$= \int_0^t L(u)ds + \int_0^t N(u)ds + g(x,t)$$

$L(u)$ represents the linear part and $N(u)$ represents the nonlinear part

Now set

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t)$$

$$N(u) = \sum_{n=0}^{\infty} A_n(x,t)$$

in this case $N(u) = u\frac{\partial u}{\partial x}$

$$A_n(x,t) = \left[\frac{1}{n!}\frac{d^n}{d\lambda^n}N\left(\sum_{i=0}^{n}\lambda^i u_i(x,t)\right)\right]_{\lambda=0}$$

$$A_n(x,t) = \left[\frac{1}{n!}\frac{d^n}{d\lambda^n}\left(\sum_{i=0}^{n}\lambda^i u_i(x,t)\right)\left(\sum_{i=0}^{n}\lambda^i \frac{\partial u_i(x,t)}{\partial x}\right)\right]_{\lambda=0}$$

and as before we get $A_0, A_1, A_2 \ldots$

$$\begin{aligned}
A_0 &= u_0 \frac{\partial u_0}{\partial x} \\
A_1 &= \left[\frac{d}{d\lambda}\left(\sum_{i=0}^{1}\lambda^i u_i(x,t)\right)\left(\sum_{i=0}^{1}\lambda^i \frac{\partial u_i(x,t)}{\partial x}\right)\right]_{\lambda=0} \\
&= \left[\frac{d}{d\lambda}\left(u_0 + \lambda u_1\right)\left(\frac{\partial u_0}{\partial x} + \lambda \frac{\partial u_1}{\partial x}\right)\right]_{\lambda=0} \\
&= u_1 \frac{\partial u_0}{\partial x} + u_0 \frac{\partial u_1}{\partial x} \\
A_2 &= u_0 \frac{\partial u_2}{\partial x} + u_1 \frac{\partial u_1}{\partial x} + u_2 \frac{\partial u_0}{\partial x}
\end{aligned}$$

now

$$u_0 + u_1 + u_2 + \cdots = \int_0^t x \frac{\partial u(x,s)}{\partial x} ds + \int_0^t u(x,s) \frac{\partial u(x,s)}{\partial x} ds - \frac{xt^2}{2} - \frac{xt^3}{3} + xt$$

put

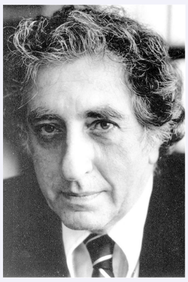
$$\begin{aligned}
u_0 &= -\frac{xt^2}{2} - \frac{xt^3}{3} + xt \\
u_1 &= \int_0^t x \frac{\partial u_0(x,s)}{\partial x} ds + \int_0^t A_0(x,s) ds \\
u_2 &= \int_0^t x \frac{\partial u_1(x,s)}{\partial x} ds + \int_0^t A_1(x,s) ds \\
&\vdots \\
u_n &= \int_0^t x \frac{\partial u_{n-1}(x,s)}{\partial x} ds + \int_0^t A_{n-1}(x,s) ds
\end{aligned}$$

we get in the end that

$$\sum_{n=0}^{\infty} u_n(x,t) = xt$$

George Adomian

George Adomian (March 21, 1922 – June 17, 1996) was an American mathematician of Armenian descent who developed the Adomian decomposition method (ADM) for solving nonlinear differential equations, both ordinary and partial. The method is explained among other places in his book ***"Solving Frontier Problems in Physics: The Decomposition Method"***. He was a faculty member at the University of Georgia (UGA) from 1966 through 1989. While at UGA, he started the Center for Applied Mathematics. Adomian was also an aerospace engineer.



Consider the following hyperbolic nonlinear problem

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = u(x,t)\dfrac{\partial u(x,t)}{\partial x} \\ u(x,0) = \dfrac{x}{10} \end{cases} \tag{3}$$

The solution of (3) is given by $u(x,t) = \dfrac{x}{10-t}$
Now let's use A.D.M to solve it !
First we integrate with respect to t

$$\begin{aligned} u(x,t) &= u(x,0) + \int_0^t u(x,s)\frac{\partial u(x,s)}{\partial x}\, ds \\ u(x,t) &= \frac{x}{10} + \int_0^t u(x,s)\frac{\partial u(x,s)}{\partial x}\, ds \end{aligned}$$

Let $u(x,t) = \sum_{n=0}^{\infty} u_n \quad \& \quad N(u) = \sum_{n=0}^{\infty} A_n$, where

$$A_n = \left[\frac{1}{n!}\frac{\mathrm{d}^n}{\mathrm{d}\lambda^n}\left(\sum_{n=0}^{\infty}\lambda^j u_j\right)\left(\sum_{n=0}^{\infty}\lambda^j\frac{\partial u_j}{\partial x}\right)\right]_{\lambda=0}$$

And as before we get $A_0, A_1, A_2 \dots$

$$\begin{aligned} A_0 &= u_0\frac{\partial u_0}{\partial x} \\ A_1 &= \left[\frac{d}{d\lambda}\left(\sum_{i=0}^{1}\lambda^i u_i(x,t)\right)\left(\sum_{i=0}^{1}\lambda^i\frac{\partial u_i(x,t)}{\partial x}\right)\right]_{\lambda=0} \\ &= \left[\frac{d}{d\lambda}(u_0+\lambda u_1)\left(\frac{\partial u_0}{\partial x}+\lambda\frac{\partial u_1}{\partial x}\right)\right]_{\lambda=0} \\ &= u_1\frac{\partial u_0}{\partial x} + u_0\frac{\partial u_1}{\partial x} \\ A_2 &= u_0\frac{\partial u_2}{\partial x} + u_1\frac{\partial u_1}{\partial x} + u_2\frac{\partial u_0}{\partial x} \\ &\vdots \end{aligned}$$

now

$$u_0 + u_1 + u_2 + \cdots = \frac{x}{10} + \int_0^t \sum_{i=0}^{\infty} A_n\, ds$$

put

$$\begin{aligned} u_0 &= \frac{x}{10} \\ u_1 &= \int_0^t u_0\frac{\partial u_0}{\partial x} ds = \frac{x}{10}(\frac{t}{10}) \\ u_2 &= \int_0^t u_0\frac{\partial u_1}{\partial x} + u_1\frac{\partial u_0}{\partial x} ds = \frac{x}{10}(\frac{t}{10})^2 \\ &\vdots \\ u_n &= \frac{x}{10}(\frac{t}{10})^n \end{aligned}$$

$$\sum_{n=0}^{\infty} u_n(x,t) = \frac{x}{10}\left[1 + \left(\frac{t}{10}\right) + \left(\frac{t}{10}\right)^2 + \dots\right]$$

$$\therefore u(x,t) = \frac{x}{10}\sum_{n=0}^{\infty}\left(\frac{t}{10}\right)^n$$

Remember that the value of the geometric series is $\displaystyle\sum_{n=s}^{\infty}(r)^n = \frac{r^s}{1-r}$

then in our case $\displaystyle\sum_{n=0}^{\infty}\left(\frac{t}{10}\right)^n = \frac{1}{1-\frac{t}{10}} = \frac{10}{10-t}$

$$\therefore u(x,t) = \frac{x}{10}\frac{10}{10-t} = \frac{x}{10-t}$$

Consider the nonlinear system of equations

$$\begin{cases} \dfrac{\partial u}{\partial t} = u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y} \\ \dfrac{\partial u}{\partial t} = u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y} \end{cases} \tag{4}$$

With initial condition: $u(x,y,0) = v(x,y,0) = x+y$
Integrate (4) with respect to t

$$u(x,y,t) = x + y + \int_0^t \left(u(x,y,\theta)\frac{\partial u}{\partial x} + v(x,y,\theta)\frac{\partial u}{\partial y}\right) d\theta$$
$$v(x,y,t) = x + y + \int_0^t \left(u(x,y,\theta)\frac{\partial v}{\partial x} + v(x,y,\theta)\frac{\partial v}{\partial y}\right) d\theta$$

Let

$$u(x,y,t) = \sum_{n=0}^{\infty} u_n \quad \& \quad v(x,y,t) = \sum_{n=0}^{\infty} v_n$$
$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \sum_{n=0}^{\infty} A_n \quad \& \quad u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = \sum_{n=0}^{\infty} B_n$$

where

$$A_n = \frac{1}{n!}\frac{\mathrm{d}^n}{\mathrm{d}\lambda^n}\left[\left(\sum_{j=0}^{n}\lambda^j u_j\right)\left(\sum_{j=0}^{n}\lambda^j\frac{\partial u_j}{\partial x}\right) + \left(\sum_{j=0}^{n}\lambda^j v_j\right)\left(\sum_{j=0}^{n}\lambda^j\frac{\partial u_j}{\partial y}\right)\right]_{\lambda=0}$$
$$B_n = \frac{1}{n!}\frac{\mathrm{d}^n}{\mathrm{d}\lambda^n}\left[\left(\sum_{j=0}^{n}\lambda^j u_j\right)\left(\sum_{j=0}^{n}\lambda^j\frac{\partial v_j}{\partial x}\right) + \left(\sum_{j=0}^{n}\lambda^j v_j\right)\left(\sum_{j=0}^{n}\lambda^j\frac{\partial v_j}{\partial y}\right)\right]_{\lambda=0}$$

And as before we get $A_0, A_1, A_2\ldots$ and $B_0, B_1, B_2\ldots$

$$A_0 = u_0\frac{\partial u_0}{\partial x} + v_0\frac{\partial u_0}{\partial y}$$
$$A_1 = u_0\frac{\partial u_1}{\partial x} + v_0\frac{\partial u_1}{\partial y} + u_1\frac{\partial u_0}{\partial x} + v_1\frac{\partial u_0}{\partial y}$$
$$A_2 = u_0\frac{\partial u_2}{\partial x} + v_0\frac{\partial u_2}{\partial y} + u_1\frac{\partial u_1}{\partial x} + v_1\frac{\partial u_1}{\partial y} + u_2\frac{\partial u_0}{\partial x} + v_2\frac{\partial u_0}{\partial y}$$
$$\vdots$$

Similarly

$$
\begin{aligned}
B_0 &= u_0 \frac{\partial v_0}{\partial x} + v_0 \frac{\partial v_0}{\partial y} \\
B_1 &= u_0 \frac{\partial v_1}{\partial x} + v_0 \frac{\partial v_1}{\partial y} + u_1 \frac{\partial v_0}{\partial x} + v_1 \frac{\partial v_0}{\partial y} \\
B_2 &= u_0 \frac{\partial v_2}{\partial x} + v_0 \frac{\partial v_2}{\partial y} + u_1 \frac{\partial v_1}{\partial x} + v_1 \frac{\partial v_1}{\partial y} + u_2 \frac{\partial v_0}{\partial x} + v_2 \frac{\partial v_0}{\partial y} \\
&\vdots
\end{aligned}
$$

Now, Let

$$
\begin{aligned}
u_0 &= x + y & v_0 &= x + y \\
u_1 &= (x+y)(2t) & v_1 &= (x+y)(2t) \\
u_2 &= (x+y)(2t)^2 & v_2 &= (x+y)(2t)^2 \\
&\vdots & &\vdots \\
u_n &= (x+y)(2t)^n & v_n &= (x+y)(2t)^n
\end{aligned}
$$

Thus

$$
\begin{aligned}
\sum_{n=0}^{\infty} u_n &= (x+y)\left[1 + (2t) + (2t)^2 + \dots\right] \\
u(x,y,t) &= (x+y)\frac{1}{1-2t} = \frac{x+y}{1-2t} \\
\sum_{n=0}^{\infty} v_n &= (x+y)\left[1 + (2t) + (2t)^2 + \dots\right] \\
v(x,y,t) &= (x+y)\frac{1}{1-2t} = \frac{x+y}{1-2t}
\end{aligned}
$$

Selected Topics In Pure Mathematics

In this book embark on a captivating journey through the intricate web of pure mathematics, climate change models, and dynamical systems. Delve into the depths of stability theory, where the delicate balance between order and chaos shapes the world around us.
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