Lecture 8
Instability & Models
For Mathematical Biology

1 Well-Posed Problems

In mathematics a well-posed problem is one for which the following properties hold:

1. The problem has a solution
2. The solution is unique
3. The solution's behavior changes continuously with the initial conditions

Examples of well-posed problems include the Dirichlet problem for Laplace's equation
This definition of a well-posed problem comes from the work of Jacques Hadamard on mathematical modeling of physical phenomena.

Problems that are not well-posed in the sense of Hadamard are termed ill-posed

For example, the inverse heat equation, deducing a previous distribution of temperature from final data, is not well-posed in that the solution is highly sensitive to changes in the final data.

1.1 Hadamard Laplace Example

In the theory of partial differential equations, an example constructed by J. Hadamard, which shows the instability of the solution of the Cauchy problem for the Laplace equation with respect to small changes in the initial data, is of great importance. Hadamard's example served as the beginning of a systematic study of ill-posed problems in mathematical physics. On the other hand, the study of the Cauchy problem for the Laplace equation arises from problems of geophysics. At the same time, the question arises whether the Cauchy problem is correct for other elliptic equations.

We will see how Hadamard's example showed the the instability of the solution of the Cauchy problem for the Laplace equation.

Consider the following Cauchy problems for the Laplace equation.

$$\begin{cases} \dfrac{\partial^2 u(x,t)}{\partial t^2} + \dfrac{\partial^2 u(x,t)}{\partial x^2} = 0 \\ u(x,0) = 0 \\ \dfrac{\partial u(x,0)}{\partial t} = \phi(x) \end{cases} \tag{1}$$

$$\begin{cases} \dfrac{\partial^2 u^*(x,t)}{\partial t^2} + \dfrac{\partial^2 u^*(x,t)}{\partial x^2} = 0 \\ u^*(x,0) = 0 \\ \dfrac{\partial u^*(x,0)}{\partial t} = \phi(x) + \dfrac{\sin(nx)}{n^k} \end{cases} \tag{2}$$

As we can see the two problems are identical except for the second initial condition in (2) but if we take the limit for $n \to \infty$ in the (2) it suppose to reduce to problem (1) which means that $u(x,t) = u^*(x,t)$ as $n \to \infty$

BUT

let's construct new problem using (1),(2)

Jacques Hadamard

Jacques Hadamard (1865–1963) was French mathematician who made significant contributions to various branches of mathematics. One of his notable contributions lies in the field of partial differential equations and the study of well-posed and ill-posed problems.
Hadamard's work helped establish fundamental principles in the study of mathematical problems. Well-posed problems are essential for ensuring the stability and reliability of mathematical models used in various scientific disciplines.

let $V(x,t) = u^*(x,t) - u(x,t)$ we get that

$$\begin{cases} \dfrac{\partial^2 V(x,t)}{\partial t^2} + \dfrac{\partial^2 V(x,t)}{\partial x^2} = 0 \\ V(x,0) = 0 \\ \dfrac{\partial V(x,0)}{\partial t} = \dfrac{\sin(nx)}{n^k} \end{cases} \tag{3}$$

the solution of this problem is given by

$$V(x,t) = \frac{\sinh(nt)\sin(nx)}{n^{k+1}}$$

now as we Assumed $V(x,t)$ should vanish as $n \to \infty$ but we find that

$$\lim_{n\to\infty} V(x,t) = \infty$$

that's because $\mathbf{O}(\sinh(n)) > \mathbf{O}(n^{k+1})$ which causes the numerator to diverges faster than the denominator

2 Some Basic Differential Models in Mathematical Biology

2.1 Population dynamic models (Malthus model)

Any specie in the natural world does not exist solely, but is closely related to other species in biological communities and then constitute a population ecosystem. However, a single population is the basic unit which composed of the entire ecosystem. In order to predict the change law of the population, the famous demographer Malthus proposed the following basic model for single specie

$$\frac{\mathrm{d}N(t)}{\mathrm{d}t} = rN(t) \tag{1}$$

Where $N(t)$ denotes the population density at the time t, r denotes the intrinsic growth rate, which is the difference between the birth rate and the death rate.

Based on model (1), the famous ecologist Logistic proposed the famous insect population model when studying the growth rate of the insect in the laboratory as follows,

$$\frac{\mathrm{d}N(t)}{\mathrm{d}t} = rN(t)[1 - \frac{N(t)}{K}]$$

Where r denotes the intrinsic growth rate, K is the carrying capacity of the environment

2.2 Epidemic models(SIR model)

As the epidemic models are concerned, it should date back to the famous SIR model, which was proposed by Kermack and Mckendrick in 1927 when studying the propagation law of the Black Death in London from 1655-1666 and the plague in Mumbai in 1906. And the famous SIR model is as follow,

$$\begin{cases} \dfrac{\mathrm{d}S(t)}{\mathrm{d}t} = -\beta S(t)I(t) \\[2ex] \dfrac{\mathrm{d}I(t)}{\mathrm{d}t} = \beta S(t)I(t) - \gamma I(t) \\[2ex] \dfrac{\mathrm{d}R(t)}{\mathrm{d}t} = \gamma I(t) \end{cases} \tag{2}$$

Where $S(t)$ denotes the number of the susceptible individuals which has not yet infected but may be infected by the bacteria at time t, $I(t)$ denotes the number of the infected individuals at time t, and $R(t)$ denotes the number of the removed individuals at time t, β, denotes the infected rate, γ denotes the cure rate and γ^{-1} denotes the average cure rate.

notice that if we add equations (2) we get

$$
\begin{aligned}
\frac{\mathrm{d}S(t)}{\mathrm{d}t} + \frac{\mathrm{d}I(t)}{\mathrm{d}t} + \frac{\mathrm{d}R(t)}{\mathrm{d}t} &= \beta S(t)I(t) - \gamma I(t) - \beta S(t)I(t)\gamma I(t) \\
\frac{\mathrm{d}S(t) + I(t) + R(t)}{\mathrm{d}t} &= 0 \\
S(t) + I(t) + R(t) &= constant
\end{aligned}
$$

that says that the total number of the individuals keeps constant.
and it's Assumed that the infected individuals have permanent immune capacity after the cure.