Lecture 7
Moore & Penrose
Generalized inverse

1 Generalized Inverse

In the classical sence of linear algebra, an n-by-n square matrix A is called invertible also nonsingular if there exists an n-by-n square matrix B such that

$$AB = BA = I_n$$

If this is the case, then the matrix B is uniquely determined by A, and is called the inverse of A, denoted by A^{-1}
In linear algebra for matrix A to own an inverse it had to be square matrix and nonsingular (i.e $\det(A) \neq 0$)
Let us now study the case for any matrix of order $(m \times n)$
we define the generalized inverse A^+

$$\begin{aligned} &\text{I} \qquad AA^+A = A \\ &\text{II} \qquad \exists U, V \quad \text{s.t} \quad A^+ = UA^* \quad , \quad A^+ = A^*V \end{aligned}$$

where A^* is the conjugate transpose if all elements of A are real then $A^* = A^{\mathrm{T}}$

$$A = \begin{pmatrix} 1+2i & 3-i \\ 5 & 6i \end{pmatrix} \implies A^* = \begin{pmatrix} 1-2i & 5 \\ 3+i & -6i \end{pmatrix}$$

Theorem 1.1 — The existence and uniqueness of GI. for matrix A of order $(m \times n)$ the generalized inverse A^+ exists ,and it's unique

Proof the uniqueness. suppose that there is two generalized inverse A_1^+ , A_2^+ thus

$$\begin{aligned} &\text{I} \qquad AA_i^+A = A \\ &\text{II} \qquad \exists U_i, V_i \quad \text{s.t} \quad A_i^+ = U_iA^* \quad , \quad A^+ = A_i^*V_i \qquad \text{for } i = 1, 2 \end{aligned}$$

from I

$$\begin{aligned} A[A_2^+ - A_1^+]A =& 0 \\ AD^+A =& 0 \end{aligned}$$

where $D^+ = A_2^+ - A_1^+$ and we can say that

$$\begin{aligned} D^+ &= A^*V & D^+ &= UA^* \\ V &= V_2 - V_1 & U &= U_2 - U_1 \end{aligned}$$

now

$$\begin{aligned} (D^+A)^*D^+A &= A^*{D^+}^*D^+A \\ &= A^*V^*AD^+A = 0 \end{aligned}$$

because $AD^+A = 0$, then we get

$$\begin{aligned} (D^+A)^*D^+A &= 0 \\ D^+A &= 0 \\ D^+ &= 0 \\ A_2^+ - A_1^+ &= 0 \\ A_2^+ &= A_1^+ \end{aligned}$$

■

Proof the existence. Let A be a non square matrix we can write $A = BC$ where

$$\begin{aligned} A &\text{ is of order } (m \times n) \\ B &\text{ is of order } (m \times r) \\ C &\text{ is of order } (r \times n) \end{aligned}$$

set

$$\begin{cases} \Lambda = C^*(CC^*)^{-1}(B^*B)^{-1}B^* \\ U = C^*(CC^*)^{-1}(B^*B)^{-1}(CC^*)^{-1}C \\ V = B(B^*B)^{-1}(CC^*)^{-1}(B^*B)^{-1}B^* \end{cases}$$

we can see that

$$\begin{aligned} A\Lambda A &= BCC^*(CC^*)^{-1}(B^*B)^{-1}B^*BC \\ &= B \underbrace{\left[CC^*(CC^*)^{-1}\right]}_{I} \underbrace{\left[(B^*B)^{-1}B^*B\right]}_{I} C \\ &= BC = A \end{aligned}$$

now to proof the property II

$$\begin{aligned} UA^* &= C^*(CC^*)^{-1}(B^*B)^{-1}(CC^*)^{-1}CC^*B^* \\ &= C^*(CC^*)^{-1}(B^*B)^{-1} \underbrace{\left[(CC^*)^{-1}CC^*\right]}_{I} B^* \\ &= C^*(CC^*)^{-1}(B^*B)^{-1}B^* = \Lambda \end{aligned}$$

$$\begin{aligned} A^*V &= C^*B^*B(B^*B)^{-1}(CC^*)^{-1}(B^*B)^{-1}B^* \\ &= C^* \underbrace{\left[B^*B(B^*B)^{-1}\right]}_{I} (CC^*)^{-1}(B^*B)^{-1}B^* \\ &= C^*(CC^*)^{-1}(B^*B)^{-1}B^* = \Lambda \end{aligned}$$

thus $\Lambda = A^+$ is the generalized inverse ■

Lemma 1.2 B^*B,CC^* are square matrices and $\det(B^*B), \det(CC^*) \neq 0$

Proof. it's clear that B^*B,CC^* are square matrices of order r now to proof that $\det(B^*B), \det(CC^*) \neq$ 0
consider the homogeneous system

$$B^*BX = 0 \qquad , \qquad X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_r \end{pmatrix}$$

we have that

$$\begin{aligned} B^*BX &= 0 \\ X^*B^*BX &= 0 \\ (BX)^*BX &= 0 \end{aligned}$$

set $BX = Y$

$$Y^*Y = 0$$

$$
\begin{aligned}
y_1^2 + y_2^2 + \cdots + y_r^2 &= 0 \\
y_1^2 = y_2^2 = \cdots = y_r^2 &= 0 \\
\implies BX &= 0
\end{aligned}
$$

this means that only the trevial solution solves this equation
but $\text{rank}(B) = r$ =number of unknowns therefore $x_1 = x_2 = \cdots = x_r = 0$ also $\text{rank}(B^*B) = r$ then $\det(B^*B) \neq 0$ ∎

■ **Example 1.0.1** find A^+ if

$$
A = \begin{pmatrix} 1 & 1 & 4 \\ 2 & 1 & 2 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{B} \underbrace{\begin{pmatrix} 1 & 1 & 4 \\ 2 & 1 & 2 \end{pmatrix}}_{C}
$$

we have taken $B = I$ to make the calculation easier but most of the time it doesn't work
only works if $\det(AA^*) \neq 0$ in case of $\det(AA^*) = 0$ you need to find two matrices B, C s.t $BC = A$ and $\det(CC^*), \det(B^*B) \neq 0$

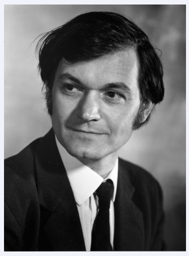
$$
\begin{aligned}
A^+ &= C^*(CC^*)^{-1}(B^*B)^{-1}B^* = A^*(AA^*)^{-1} \\
&= \begin{pmatrix} 1 & 2 \\ 1 & 1 \\ 4 & 2 \end{pmatrix} \left[\begin{pmatrix} 1 & 1 & 4 \\ 2 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & 1 \\ 4 & 2 \end{pmatrix} \right]^{-1} \\
&= \begin{pmatrix} 1 & 2 \\ 1 & 1 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} 18 & 11 \\ 11 & 9 \end{pmatrix}^{-1} \\
&= \begin{pmatrix} 1 & 2 \\ 1 & 1 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} \frac{9}{41} & \frac{-11}{41} \\ \frac{-11}{41} & \frac{18}{41} \end{pmatrix} \\
&= \begin{pmatrix} \frac{-13}{41} & \frac{25}{41} \\ \frac{-2}{41} & \frac{7}{41} \\ \frac{14}{41} & \frac{-8}{41} \end{pmatrix}
\end{aligned}
$$

$$
AA^+ = \begin{pmatrix} 1 & 1 & 4 \\ 2 & 1 & 2 \end{pmatrix} \begin{pmatrix} \frac{-13}{41} & \frac{25}{41} \\ \frac{-2}{41} & \frac{7}{41} \\ \frac{14}{41} & \frac{-8}{41} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
$$

$\begin{pmatrix} \frac{-13}{41} & \frac{25}{41} \\ \frac{-2}{41} & \frac{7}{41} \\ \frac{14}{41} & \frac{-8}{41} \end{pmatrix}$ is the right inverse of A

notice that if you take $B = A$ and $C = I$ will not give an answer because $\det(A^*A) = 0$ ■

Roger Penrose



Sir Roger Penrose is a British mathematical physicist, mathematician, and philosopher who has made significant contributions to various fields, including general relativity, cosmology, and the foundations of quantum mechanics.
Penrose has made significant contributions to the fields of mathematics One of Penrose's notable contributions is his work on the generalized inverse of a matrix, Penrose introduced the concept of the Moore-Penrose pseudoinverse, which is a widely used method for finding a generalized inverse of a matrix. This pseudoinverse has applications in various areas, including linear algebra, statistics, signal processing, and machine learning.

E.H. Moore

Eliakim Hastings Moore (1862–1932) was an American mathematician known for his contributions to algebra and mathematical logic. One of his significant contributions was in the field of linear algebra, particularly his work on the generalized inverse of a non-square matrix. The Moore–Penrose inverse is a concept that extends the idea of the matrix inverse to non-square matrices. The Moore–Penrose generalized inverse is perhaps the most well-known and widely used among various generalizations of the matrix inverse for non-square matrices. It has applications in solving linear systems of equations, least squares problems, and in situations where the original matrix might not have a unique inverse.