Selected Topics *in* Pure Mathematics

Selected Topics In Pure Mathematics
(040101401)
(Fall 2023)

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1 Fractional Calculus

Fractional calculus is a branch of mathematical analysis that extends the concepts of differentiation and integration to non-integer orders.
Unlike classical calculus, which deals with integer-order derivatives and integrals, fractional calculus considers derivatives and integrals of arbitrary order, including non-integer or fractional orders.

1.1 Fractional Integral

Let f be a continuous function on $[a, b]$ and let I donate the integral operator

$$If(t) = \int_0^t f(s)ds = g(t) \quad , \quad t \in [a, b]$$

And if we apply it again

$$\begin{aligned} I^2 f(t) &= \int_0^t g(s)ds \\ &= \int_0^t \int_0^s f(\theta)d\theta ds \\ &= \int_0^t \left(\int_s^t d\theta \right) f(s)ds \\ &= \int_0^t (t-s)f(s)ds \end{aligned}$$

We can get the general formula for integrating n time by

$$\begin{aligned} I^n f(t) &= \frac{1}{(n-1)!} \int_0^t (t-s)^{n-1} f(s)ds \\ &= \frac{1}{\Gamma(n)} \int_0^t (t-s)^{n-1} f(s)ds \end{aligned}$$

Then we can say that the fractional integral of order α is defined as

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds \tag{1}$$

$$0 < \alpha \leq 1$$

■ **Example 1.1.1** Evaluate $I^{\frac{1}{2}}(t^n)$

$$I^{\frac{1}{2}}(t^n) = \frac{1}{\Gamma(\frac{1}{2})} \int_0^t (t-s)^{-\frac{1}{2}} s^n \, ds$$

Let $s = t\theta \rightarrow ds = td\theta$

$$\begin{aligned} I^{\frac{1}{2}}(t^n) &= \frac{1}{\Gamma(\frac{1}{2})} \int_0^1 (t-t\theta)^{-\frac{1}{2}} (t\theta)^n \, td\theta \\ &= \frac{1}{\Gamma(\frac{1}{2})} t^{n+\frac{1}{2}} \int_0^1 (1-\theta)^{-\frac{1}{2}} (\theta)^n \, d\theta \\ &= \frac{1}{\Gamma(\frac{1}{2})} t^{n+\frac{1}{2}} \beta(\frac{1}{2}, n+1) \\ &= \frac{1}{\Gamma(\frac{1}{2})} t^{n+\frac{1}{2}} \frac{\Gamma(\frac{1}{2})\Gamma(n+1)}{\Gamma(n+\frac{3}{2})} \\ &= t^{n+\frac{1}{2}} \frac{\Gamma(n+1)}{\Gamma(n+\frac{3}{2})} \end{aligned}$$

■

1.2 Fractional Derivative

The fractional derivative is defined by

$$D^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \frac{df(s)}{ds} ds \tag{2}$$

Or

$$D_*^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-s)^{-\alpha} f(s) ds \tag{3}$$
$$0 \le \alpha < 1$$

Where α is the order of differentiation
The need of having 2 formulas that each has a problem that the other solves like that formula (2) need the 1^{st} derivative to exist to get the fractional derivative and formula (3) the derivative of the constant not equal zero

$$\begin{aligned} D_*^{\alpha} 1 &= \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-s)^{-\alpha} ds \\ &= \frac{-1}{\Gamma(1-\alpha)} \frac{d}{dt} \left[\frac{(t-s)^{-\alpha+1}}{-\alpha+1} \right]_0^t \\ &= \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \frac{t^{1-\alpha}}{1-\alpha} \\ &= \frac{1}{\Gamma(1-\alpha)} t^{-\alpha} \neq 0 \end{aligned}$$

■ **Example 1.2.1** Evaluate $D^{\frac{1}{2}}(t^n)$ using the defintion D_*^{α}

$$D^{\frac{1}{2}}(t^n) = \frac{1}{\Gamma(\frac{1}{2})} \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t (t-s)^{-\frac{1}{2}} s^n \, ds$$

Let $s = t\xi \rightarrow ds = td\xi$

$$\begin{aligned} D^{\frac{1}{2}}(t^n) &= \frac{1}{\Gamma(\frac{1}{2})} \frac{\mathrm{d}}{\mathrm{d}t} \int_0^1 (t-t\xi)^{-\frac{1}{2}} (t\xi)^n \, td\xi \\ &= \frac{1}{\Gamma(\frac{1}{2})} \frac{\mathrm{d}}{\mathrm{d}t} t^{n+\frac{1}{2}} \int_0^1 (1-\xi)^{-\frac{1}{2}} (\xi)^n \, d\xi \\ &= \frac{1}{\Gamma(\frac{1}{2})} \frac{\mathrm{d}}{\mathrm{d}t} t^{n+\frac{1}{2}} \beta(\frac{1}{2}, n+1) \\ &= (n+\frac{1}{2}) \frac{\Gamma(n+1)}{\Gamma(n+\frac{3}{2})} t^{n-\frac{1}{2}} \\ &= \frac{\Gamma(n+1)}{\Gamma(n+\frac{1}{2})} t^{n-\frac{1}{2}} \end{aligned}$$

■

1.3 Laplace Transform For Fractional Integral

We know that

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)dt = F(s)$$

Then we can do the following

$$\begin{aligned}
\mathcal{L}\{I^\alpha f(t)\} &= \int_0^\infty e^{-st} \frac{1}{\Gamma(\alpha)} \int_0^t (t-\theta)^{\alpha-1} f(\theta) d\theta dt \\
&= \frac{1}{\Gamma(\alpha)} \mathcal{L}\{t^{\alpha-1} \times f(t)\}
\end{aligned}$$

From the convolution property

$$\mathcal{L}\{I^\alpha f\} = \frac{1}{\Gamma(\alpha)} \mathcal{L}\{t^{\alpha-1}\} \times \mathcal{L}\{f(t)\} \tag{4}$$

Let's handle the first transformation

$$\begin{aligned}
\mathcal{L}\{t^{\alpha-1}\} &= \int_0^\infty e^{-st} t^{\alpha-1} dt \\
&\text{put } st = \eta \implies dt = \frac{1}{s} d\eta \\
\mathcal{L}\{t^{\alpha-1}\} &= \int_0^\infty e^{-\eta} \eta^{\alpha-1} s^{1-\alpha} \frac{1}{s} d\eta \\
&= \int_0^\infty e^{-\eta} \eta^{\alpha-1} s^{-\alpha} \frac{1}{s} d\eta = s^{-\alpha}\Gamma(\alpha)
\end{aligned}$$

Now substitute in equation (4)

$$\mathcal{L}\{I^\alpha f\} = s^{-\alpha} \mathcal{L}\{f(t)\} = s^{-\alpha} F(s)$$

1.4 The Integral of Derivative

Now that we defined the integral and the differential operator logically they suppose to cancel each other we need to proof that

$$I^\alpha D^\alpha f(t) = f(t)$$

Using the formula (2)

$$\begin{aligned}
I^\alpha D^\alpha f(t) &= I^\alpha \left[\frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \frac{df(s)}{ds} ds \right] \qquad 0 < \alpha < 1 \\
&= \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \int_0^t (t-s)^{\alpha-1} \int_0^s (s-\theta)^{-\alpha} \frac{df(\theta)}{d\theta} d\theta ds \\
&= \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \int_0^t \underbrace{\int_\theta^t (t-s)^{\alpha-1} (s-\theta)^{-\alpha} ds}_{J} \frac{df(\theta)}{d\theta} d\theta
\end{aligned} \tag{5}$$

Let's handle the inner integral first

$$\begin{aligned}
J &= \int_\theta^t (t-s)^{\alpha-1} (s-\theta)^{-\alpha} ds \\
&\text{put } s - \theta = \eta \implies ds = d\eta \\
&= \int_0^{t-\theta} (t-\theta-\eta)^{\alpha-1} (\eta)^{-\alpha} d\eta \\
&= (t-\theta)^{\alpha-1} \int_0^{t-\theta} (1 - \frac{\eta}{t-\theta})^{\alpha-1} (\eta)^{-\alpha} d\eta
\end{aligned}$$

$$
\begin{aligned}
&\text{put } \eta = (t-\theta)\xi \Longrightarrow d\eta = (t-\theta)d\xi \\
&= (t-\theta)^{\alpha-1}\int_0^1 (1-\xi)^{\alpha-1}(t-\theta)^{1-\alpha}\xi^{-\alpha}d\xi \\
&= \int_0^1 (1-\xi)^{\alpha-1}\xi^{-\alpha}d\xi = \beta(\alpha, 1-\alpha)
\end{aligned}
$$

Substitute in (5) we get that

$$
\begin{aligned}
I^\alpha D^\alpha f &= \frac{\beta(\alpha,1-\alpha)}{\Gamma(\alpha)\Gamma(1-\alpha)}\int_0^t \frac{df(\theta)}{d\theta}d\theta \\
&= \frac{\Gamma(\alpha)\Gamma(1-\alpha)}{\Gamma(\alpha+1-\alpha)\Gamma(\alpha)\Gamma(1-\alpha)}[f(t)-f(0)] \\
&= f(t)-f(0)
\end{aligned}
$$

2 Stability

Consider

$$\begin{cases} \dfrac{dx(t)}{dt} = & f(x(t), t) \qquad t > 0 \\ x(0) = a \end{cases} \tag{1}$$

We say that the solutions of equation (1) are **stable** if and only if

$$\forall \epsilon > 0 \ , \ \exists \delta > 0 \text{ s.t } x(0) = a, x^*(0) = b$$

$$|a - b| < \delta \Longrightarrow |x(t) - x^*(t)| \le \epsilon$$

Where $x(t)$ and $x^*(t)$ are solutions of equation (1) moreover we say that the solutions of equation (1) are **asymptotically stable** if and only if they satisfies the previous Conditions and $\lim_{t \to \infty} (x(t) - x^*(t)) = 0$

2.1 Lipschitz Condition

We say that $f(x)$ satisfies lipschitz condition with lipschitz constant N if and only if

$$|f(x) - f(x^*)| \le N |x - x^*|$$

$$f(x) \ \text{ defined on } \ [a, b] \ , \ x, x^* \in [a, b]$$

If $f(x)$ is differentiable and $f'(x)$ is bounded i.e. $|f'(x)| \le M$

$$f(x) - f(x^*) = (x - x^*) f'(x^{**})$$

$$x \le x^{**} \le x^*$$

$$|f(x) - f(x^*)| \le M |x - x^*|$$

Theorem 2.1 Let $f(x, t)$ be a continues function on $G := \{(x, t) \mid a \le x \le b \ , \ 0 \le t \le T\}$ and Satisfies lipschitz condition with respect to x and with lipschitz constant N , i.e.

$$|f(x, t) - f(x^*, t)| \le N |x - x^*|$$

Suppose that $\begin{cases} \dfrac{ds_1(t)}{dt} = f(s_1(t), t) \ , \ s_1(0) = \beta_1 \\ \dfrac{ds_2(t)}{dt} = f(s_2(t), t) \ , \ s_2(0) = \beta_2 \end{cases}$

if $|\beta_1 - \beta_2| \le \delta$ then $|s_1(t) - s_2(t)| \le \delta e^{Nt}$

Proof.

$$s_1(t) = \beta_1 + \int_0^t f(s_1(\theta), \theta) d\theta \tag{1}$$

$$s_2(t) = \beta_2 + \int_0^t f(s_2(\theta), \theta) d\theta \tag{2}$$

Subtract equation (2) from (1)

$$s_1(t) - s_2(t) = \beta_1 - \beta_2 + \int_0^t [f(s_1(\theta), \theta) - f(s_2(\theta), \theta)] \, d\theta$$

Taking the absolute value to both sides
And because $|\beta_1 - \beta_2| \le \delta$ and f satisfies lipschitz condition $|f(x, t) - f(x^*, t)| \le N |x - x^*|$

Then

$$|s_1(t) - s_2(t)| \le \delta + N \int_0^t |s_1(\theta) - s_2(\theta)| \, d\theta$$

Put $|s_1(t) - s_2(t)| = r(t)$

$$r(t) \leq \delta + N \int_0^t r(\theta) d\theta \tag{3}$$

Set $R(t) = \displaystyle\int_0^t r(\theta) d\theta$ i.e. $r(t) = \dfrac{dR(t)}{dt}$

And Substitute in (3)

$$\frac{dR(t)}{dt} - NR(t) \leq \delta \tag{4}$$

Multiply both sides by e^{-Nt}

$$e^{-Nt}\left[\frac{dR(t)}{dt} - NR(t)\right] \leq \delta e^{-Nt}$$

$$\frac{d}{dt}\left[e^{-Nt}R(t)\right] \leq \delta e^{-Nt}$$

Integrating both sides from $0 \to t$ we get that

$$e^{-Nt}R(t) - R(0) \leq \frac{\delta}{N}\left[1 - e^{-Nt}\right]$$

And we know that

$$R(t) = \int_0^t r(\theta) d\theta$$

Then

$$R(0) = \int_0^0 r(\theta) d\theta = 0$$

Therefore we get

$$R(t) \leq \frac{\delta}{N}\left[e^{Nt} - 1\right]$$

Substitute in (4) to get the following

$$r(t) \leq \delta + \delta\left[e^{Nt} - 1\right]$$

$$|s_1(t) - s_2(t)| \leq \delta e^{Nt}$$

■

Theorem 2.2 Let $f(x,t)$ be a continues function on $G := \{(x,t) \mid a \le x \le b \ , \ 0 \le t \le T\}$ and Satisfies lipschitz condition with respect to x and with lipschitz constant N , i.e.

$$|f(x,t) - f(x^*,t)| \le N\,|x - x^*|$$

Suppose that

$$\frac{dx(t)}{dt} = -\gamma x(t) + f(x(t),t) \tag{1}$$

And let $s_1(t)$ and $s_2(t)$ be solutions for equation (1) corresponding to $\begin{cases} s_1(0) = \beta_1 \\ s_2(0) = \beta_2 \end{cases}$

If $\gamma > N$ and $|\beta_1 - \beta_2| \le \delta$ then $\lim_{t\to\infty} |s_1(t) - s_2(t)| = 0$ and $|s_1(t) - s_2(t)| \le \delta e^{-(\gamma - N)t}$

Proof. Let $y(t) = e^{\gamma t} x(t)$

$$\frac{dy(t)}{dt} = e^{\gamma t}\frac{dx(t)}{dt} + \gamma e^{\gamma t} x(t)$$

Substitute $\frac{dx(t)}{dt}$ from equation (1)

$$\begin{aligned} \frac{dy(t)}{dt} &= e^{\gamma t}\left[-\gamma x(t) + f(x(t),t)\right] + \gamma e^{\gamma t} x(t) \\ &= e^{\gamma t} f(x(t),t) \\ &\because y(t) = e^{\gamma t} x(t) \\ &\therefore x(t) = e^{-\gamma t} y(t) \end{aligned}$$

Therefore

$$\frac{dy(t)}{dt} = e^{\gamma t} f(e^{-\gamma t} y(t), t) \tag{2}$$

Let $\mathbf{S}_1(t)$ and $\mathbf{S}_2(t)$ be solution of equation (2)

$$\begin{cases} \mathbf{S}_1(t) = e^{\gamma t} s_1(t) \ , \ s_1(0) = \beta_1 \\ \mathbf{S}_2(t) = e^{\gamma t} s_2(t) \ , \ s_2(0) = \beta_2 \end{cases} \tag{3}$$

$y(0) = x(0)$ i.e. $\mathbf{S}_1(t) = \beta_1$ and $\mathbf{S}_2(t) = \beta_2$

$$\mathbf{S}_1(t) = \beta_1 + \int_0^t e^{\gamma\theta} f(e^{-\gamma\theta}\mathbf{S}_1(\theta), \theta)\, d\theta \tag{4}$$

$$\mathbf{S}_2(t) = \beta_2 + \int_0^t e^{\gamma\theta} f(e^{-\gamma\theta}\mathbf{S}_2(\theta), \theta)\, d\theta \tag{5}$$

Subtract equation (5) from (4)

$$\mathbf{S}_1(t) - \mathbf{S}_2(t) = \beta_1 - \beta_2 + \int_0^t e^{\gamma\theta}\left[f(e^{-\gamma\theta}\mathbf{S}_1(\theta), \theta) - f(e^{-\gamma\theta}\mathbf{S}_2(\theta), \theta)\right] d\theta$$

Taking the absolute value to both sides
And because $|\beta_1 - \beta_2| \le \delta$ and f satisfies lipschitz condition $|f(x,t) - f(x^*,t)| \le N\,|x - x^*|$

Then

$$\begin{aligned} |\mathbf{S}_1(t) - \mathbf{S}_2(t)| &\le \delta + N\int_0^t e^{\gamma\theta}\left|e^{-\gamma\theta}\mathbf{S}_1(\theta) - e^{-\gamma\theta}\mathbf{S}_2(\theta)\right| d\theta \\ &\le \delta + N\int_0^t |\mathbf{S}_1(\theta) - \mathbf{S}_2(\theta)|\, d\theta \end{aligned}$$

Put $|\mathbf{S}_1(t) - \mathbf{S}_2(t)|\, d\theta = r(t)$

$$r(t) \leq \delta + N \int_0^t r(\theta) d\theta \tag{6}$$

Set $R(t) = \int_0^t r(\theta) d\theta$ i.e. $r(t) = \dfrac{dR(t)}{dt}$

And Substitute in (6)

$$\frac{dR(t)}{dt} - NR(t) \leq \delta \tag{7}$$

Multiply both sides by e^{-Nt}

$$e^{-Nt}\left[\frac{dR(t)}{dt} - NR(t)\right] \leq \delta e^{-Nt}$$

$$\frac{d}{dt}\left[e^{-Nt}R(t)\right] \leq \delta e^{-Nt}$$

Integrating both sides from $0 \to t$ we get that

$$R(t) \leq \frac{\delta}{N}\left[e^{Nt} - 1\right]$$

Substitute in (7) to get the following

$$r(t) \leq \delta + \delta\left[e^{Nt} - 1\right]$$

$$|\mathbf{S}_1(t) - \mathbf{S}_2(t)| \leq \delta e^{Nt}$$

Multiply both sides by $e^{-\gamma t}$

$$e^{-\gamma t}|\mathbf{S}_1(t) - \mathbf{S}_2(t)| \leq \delta e^{-(\gamma - N)t}$$

From equations (3) we get that

$$e^{-\gamma t}e^{\gamma t}|s_1(t) - s_2(t)| \leq \delta e^{-(\gamma - N)t}$$

$$|s_1(t) - s_2(t)| \leq \delta e^{-(\gamma - N)t}$$

Because $\gamma > N$ is given in the theorem then the power of R.H.S is negative therefore when $t \to \infty$ then $e^{-(\gamma - N)} \to 0$ then

$$\lim_{t\to\infty} |s_1(t) - s_2(t)| \leq 0$$

$$\therefore \lim_{t\to\infty} |s_1(t) - s_2(t)| = 0$$

■

Rudolf Lipschitz

Rudolf Otto Sigismund Lipschitz (14 May 1832 – 7 October 1903) was a German mathematician who made contributions to mathematical analysis (where he gave his name to the Lipschitz continuity condition) and differential geometry, as well as number theory, algebras with involution and classical mechanics.

Theorem 2.3 Let A be a constant matrix suppose that all the characteristic roots of A with negative real part
Now consider the equation

$$\frac{dx(t)}{dt} = Ax(t) + f(t, x(t))$$

If $||f(t, x(t))|| = o(||x(t)||)$ and $f(t, 0) = 0$, then the rest point is asymptotically stable

We can define A as $\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$ And x as $\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

And the Norm as $||x(t)|| = \sum_{i=1}^{n} |x_i(t)|$ Or $||x(t)|| = \left(\sum_{i=1}^{n} |x_i(t)|^2 \right)^{\frac{1}{2}}$

And $||A|| = \sum_{i,j=1}^{n} |a_{ij}|$ Or $||A|| = \left(\sum_{i,j=1}^{n} |a_{ij}|^2 \right)^{\frac{1}{2}}$
And rest point is the zero solution of the equation (1)
The rest point is stable if

$$\forall \epsilon > 0 \ , \ \exists \delta > 0 \text{ s.t } ||x(0)|| \leq \delta \implies ||x(t)|| \leq \epsilon$$

And it is asymptotically stable if it satisfies the last condition and

$$\lim_{t \to \infty} ||x(t)|| = 0$$

Proof. We have

$$\frac{dx(t)}{dt} = Ax(t) + f(t, x(t)) \tag{1}$$

We can write it as

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\theta)} f(x(\theta), \theta) d\theta \tag{2}$$

This is a representation for equation (1) to see that they are the same take the derivative of it with respect to t

$$\frac{dx(t)}{dt} = Ae^{At}x(0) + \frac{d}{dt} \int_0^t e^{A(t-\theta)} f(x(\theta), \theta) d\theta$$

Using Leibniz rule

$$\begin{aligned}
\frac{dx(t)}{dt} &= Ae^{At}x(0) + A \int_0^t e^{A(t-\theta)} f(x(\theta), \theta) d\theta + f(x(t), t) \\
&= A \left(e^{At}x(0) + \int_0^t e^{A(t-\theta)} f(x(\theta), \theta) d\theta \right) + f(x(t), t) \\
&= Ax(t) + f(x(t), t)
\end{aligned}$$

Leibniz rule

$$\frac{d}{dt} \int_{\alpha(t)}^{\beta(t)} f(t, \theta) d\theta = \frac{d\beta(t)}{dt} f(t, \beta(t)) - \frac{d\alpha(t)}{dt} f(t, \alpha(t)) + \int_{\alpha(t)}^{\beta(t)} \frac{\partial f(t, \theta)}{\partial t} d\theta$$

We can find $K > 0, \sigma > 0$ such that $||e^{At}|| < Ke^{-\sigma t}$
Take the Norm for equation (2)

$$||x(t)|| \leq Ke^{-\sigma t}||x(0)|| + K\int_0^t e^{-\sigma(t-\theta)}||f(x(\theta),\theta)||d\theta$$

And we know that $||f(t,x(t))|| = o(||x(t)||)$ or in other word $\lim_{||x(t)||\to 0} \dfrac{||f(x(t),t)||}{||x(t)||} = 0$ i.e

$$\forall\epsilon > 0\ ,\ \exists\delta > 0 \text{ such that } ||x(t)|| \leq \delta \implies ||f(x(t),t)|| \leq \epsilon||x(t)||$$

Thus

$$||x(t)|| \leq Ke^{-\sigma t}||x(0)|| + K\epsilon\int_0^t e^{-\sigma(t-\theta)}||x(\theta)||d\theta$$

Put $\epsilon = \dfrac{\epsilon}{K}$ and multiply by $e^{\sigma t}$

$$e^{\sigma t}||x(t)|| \leq K||x(0)|| + \epsilon\int_0^t e^{\sigma\theta}||x(\theta)||d\theta$$

As long as $||x(t)|| \leq \delta$

Set $R(t) = \displaystyle\int_0^t e^{\sigma\theta}||x(\theta)||d\theta$ i.e. $\dfrac{dR(t)}{dt} = e^{\sigma t}||x(t)||$

$$\frac{dR(t)}{dt} \leq K||x(0)|| + \epsilon R(t) \tag{3}$$

$$\frac{dR(t)}{dt} - \epsilon R(t) \leq K||x(0)||$$

Multiply by $e^{-\epsilon t}$

$$e^{-\epsilon t}\left[\frac{dR(t)}{dt} - \epsilon R(t)\right] \leq K||x(0)||e^{-\epsilon t}$$

$$\frac{d}{dt}\left[e^{-\epsilon t}R(t)\right] \leq K||x(0)||e^{-\epsilon t}$$

Integrate with respect to t

$$e^{-\epsilon t}R(t) \leq \frac{K\sigma}{\epsilon}(1 - e^{-\epsilon t})$$

Multiply by $e^{\epsilon t}$

$$R(t) \leq \frac{K\sigma}{\epsilon}(e^{\epsilon t} - 1)$$

Substitute $\dfrac{dR(t)}{dt} = e^{\sigma t}||x(t)||$ and $R(t) \leq \dfrac{K\sigma}{\epsilon}(e^{\epsilon t} - 1)$ in equation (3)

$$\begin{aligned} e^{\sigma t}||x(t)|| &\leq K||x(0)|| + \epsilon R(t) \\ &\leq K\delta + K\delta e^{\epsilon t} - K\delta \\ &\leq K\delta e^{\epsilon t} \end{aligned}$$

$$||x(t)|| \leq K\delta e^{(\epsilon-\sigma)t}$$

Put $\epsilon < \sigma$ and take the limit as $t \to \infty$

$$\lim_{t\to\infty}||x(t)|| \leq 0$$

$$\lim_{t\to\infty}||x(t)|| = 0$$

■

2.2 Lyapunov Function

Consider the Dynamical System or the Autonomous ODE

$$\frac{dx(t)}{dt} = f(x(t))$$

$$x(t) = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad f(x(t)) = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix}$$

Theorem 2.4 — Lyapunov's theorem. Suppose that there exist a function $V(x)$, such that $||V(x)|| \geq 0, \forall x$ and $||V(x)|| = 0$ only at $x = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$

$$\frac{dV}{dt} = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i}\frac{dx_i}{dt} = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} f_i(x,t) \leq 0$$

In some neighborhood of $\begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$ It's supposed that f(0) = $\begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$ then the rest point is stable

If also $\frac{dV}{dt} \leq -\beta, \beta > 0$ outside $||x(t)|| \leq \delta$

Then the rest point is asymptotically stable

■ **Example 2.2.1** Check the stability of the system

$$\begin{cases} \dfrac{dx}{dt} = -y - x^3 \\[2ex] \dfrac{dy}{dt} = x - y^3 \end{cases}$$

Set Lyapunov function as following

$$V = x^2 + y^2$$

For sure $V \geq 0, \forall x, y$ and $V = 0$ only at $x = y = 0$

$$\begin{aligned} \frac{dV}{dt} &= 2x\frac{dx}{dt} + 2y\frac{dy}{dt} \\ &= 2x(-y - x^3) + 2y(x - y^3) \\ &= -2xy - 2x^4 + 2xy - 2y^4 \\ &= -2(x^4 + y^4) \end{aligned}$$

$\frac{dV}{dt}$ is negative and $< \beta$ then the system is asymptotically stable ■

■ **Example 2.2.2** Check the stability of the system

$$\begin{cases} \dfrac{dx}{dt} = -xy^4 \\ \\ \dfrac{dy}{dt} = yx^4 \end{cases}$$

Set Lyapunov function as following

$$V = x^4 + y^4$$

For sure $V \geq 0, \forall x, y$ and $V = 0$ only at $x = y = 0$

$$\begin{aligned} \frac{dV}{dt} &= 4x^3\frac{dx}{dt} + 4y^3\frac{dy}{dt} \\ &= -4x^4y^4 + 4x^4y^4 \\ &= 0 \end{aligned}$$

Then the system is stable but not asymptotically stable ■

■ **Example 2.2.3** Check the stability of the system

$$\begin{cases} \dfrac{dx(t)}{dt} = -y(t) - x^3(t) + z(t) \\ \\ \dfrac{dy(t)}{dt} = x(t) - y^3(t) - z(t) \\ \\ \dfrac{dz(t)}{dt} = y(t) - x(t) - z^3(t) \end{cases}$$

Set Lyapunov function as following

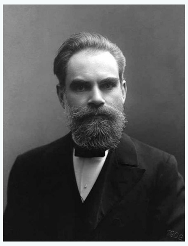
$$V = x^2 + y^2 + z^2$$

For sure $V \geq 0, \forall x, y, z$ and $V = 0$ only at $x = y = z = 0$

$$\begin{aligned} \frac{\partial V}{\partial t} &= 2x\dot{x} + 2y\dot{y} + 2z\dot{z} \\ &= 2x[-y - x^3 + z] + 2y[x - y^3 - z] + 2z[y - x - z^3] \\ &= -2xy - 2x^4 + 2xz + 2xy - 2y^4 - 2yz + 2zy - 2zx - 2z^4 \\ &= -2x^4 - 2y^4 - 2z^4 \\ \therefore \dot{V} &= -2(x^4 + y^4 + z^4) \leq 0 \end{aligned}$$

i.e. $\dot{V}$ outside $(0, 0, 0)$ is < 0 then the system is asymptotically stable ■

Aleksandr Lyapunov



Aleksandr Mikhailovich Lyapunov was a Russian mathematician, mechanician and physicist. Lyapunov contributed to several fields, including differential equations, potential theory, dynamical systems and probability theory. His main preoccupations were the stability of equilibria and the motion of mechanical systems, especially rotating fluid masses, and the study of particles under the influence of gravity. Lyapunov's impact was significant, and the following mathematical concepts are named after him: Lyapunov equation , Lyapunov exponent , Lyapunov function , Lyapunov fractal , Lyapunov stability , Lyapunov's central limit theorem , Lyapunov vector

3 Dynamical System And Climate Change Models

Consider the following Autonomous

$$R\frac{dT(t)}{dt} = a - bT(t)$$
$$a = (1-\alpha)Q - A \qquad , \qquad b = B$$

This equation represents a simple energy balance model used in climate science. This type of model is often used to study the Earth's energy budget, taking into account various factors that influence the planet's temperature over time. In this equation:

- $\frac{dT(t)}{dt}$ represents the rate of change of temperature with respect to time.
- Q represents the incoming solar radiation.
- α is the albedo, which represents the fraction of incoming solar radiation that is reflected back to space.
- R is the averaged heat capacity of the Earth/atmosphere system (heat capacity is the amount of heat required to raise the temperature of an object or substance 1 kelvin(= 1 C))
- A and B are empirically determined parameters.

Now let's try to solve it

$$\frac{dT(t)}{a - bT(t)} = \frac{1}{R}dt$$

Multiply both sides with $-b$ and integrating with respect to t

$$\int_0^t \frac{-bdT(t)}{a - bT(t)} = \frac{-bt}{R}$$
$$ln(a - bT(t)) - ln(a - bT(0)) = \frac{-bt}{R}$$
$$ln\left(\frac{a - bT(t)}{a - bT(0)}\right) = \frac{-bt}{R}$$
$$a - bT(t) = (a - bT(0))e^{\frac{-bt}{R}}$$
$$T(t) = \frac{a}{b} + \frac{1}{b}(bT(0) - a)e^{\frac{-bt}{R}}$$

When taking the limit of $T(t)$ as t goes to ∞

$$\lim_{t\to\infty} T(t) = \frac{a}{b}$$

This is called the equilibrium point (or the zero solution that makes $T(t)$ constant)

3.1 Kaper and Engler Climate Model

Consider the next model

$$R\frac{dT(t)}{dt} = (1-\alpha)Q - \sigma T^4(t) \qquad 0 < \alpha < 1$$

The Kaper and Engler climate model is a simplified mathematical representation of the Earth's climate system. The model describes the rate of change of the Earth's temperature $T(t)$ over time t where :

- σ is the Stefan-Boltzmann constant, which relates the temperature of a black body (in this case, the Earth) to the amount of radiation it emits.

This equation captures two main factors influencing the Earth's temperature change:

1. Solar Radiation (First Term): The term $(1-\alpha)Q$ represents the solar radiation absorbed by the Earth. $(1-\alpha)$ is the fraction of incoming solar radiation that is absorbed (since α is the albedo, the fraction that is reflected), and Q represents the total incoming solar radiation.
2. Radioactive Cooling (Second Term): The term $-\sigma T^4(t)$ represents the Earth's radioactive cooling. This term describes how the Earth emits thermal radiation into space as a function of its temperature $T(t)$. According to the Stefan-Boltzmann law, the rate at which a black body radiates energy is proportional to the fourth power of its temperature.

The equilibrium point of this model is

$$(1-\alpha)Q - \sigma T^4(t) = 0$$
$$T^4(t) = \frac{(1-\alpha)Q}{\sigma}$$

3.2 Adomian Decomposition Method(A.D.M)

Consider the problem

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} + L(u(x,t)) + N(u(x,y)) = g(x,t) \\ u(x,0) = \phi(x) \end{cases} \tag{1}$$

This is general form of any equation where $L(u(x,t))$ is the linear part and $N(u(x,t))$ is the non-linear part

This equation can be solved by successive approximation or the method that we will discuss which is A.D.M

Now, Integrate (1) from $0 \to t$

$$u(x,t) = \phi(x) - \int_0^t L(u(x,s))ds - \int_0^t N(u(x,s))ds + \int_0^t g(x,s)ds$$

Set

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,y) \tag{2}$$

$$N(u(x,t)) = \sum_{n=0}^{\infty} A_n(x,y) \tag{3}$$

Where $A_0, A_1, A_2, \ldots$ are **Adomian polynomials** defined as:

$$A_n(x,y) = \frac{1}{n!}\frac{\mathrm{d}^n}{\mathrm{d}\lambda^n}\left(N\left(\sum_{j=0}^{n}\lambda^j u_j\right)\right) \tag{4}$$

Substitute equations (2),(3) into (4) we get

$$u(x,y) = \sum_{n=0}^{\infty} u_i(x,t) = \phi(x) + \int_0^t g(x,s)ds - \int_0^t L\sum_{n=0}^{\infty} u_n(z,t) - \int_0^t \sum_{n=0}^{\infty} A_n(x,s)ds$$

Now,

$$u_0 = \phi(x) + \int_0^t g(x,s)ds$$
$$u_1 = -\int_0^t Lu_0 ds - \int_0^t A_0 ds$$
$$u_2 = -\int_0^t Lu_1 ds - \int_0^t A_1 ds$$

$$\vdots$$

$$u_n = -\int_0^t Lu_{n-1}ds - \int_0^t A_{n-1}ds$$

$$\therefore u(x,t) = \sum_{n=0}^{\infty} u_n(x,y) = u_0 + u_1 + u_2 + \dots$$

This will get the solution for the equation (1)

Note that The Adomian Decomposition Method (ADM) is a numerical technique used to approximate solutions of differential equations. Whether ADM converges or diverges depends on the specific problem and how it is applied. The convergence and divergence of ADM can be influenced by several factors, including the complexity of the problem, the choice of the decomposition functions, and the behavior of the nonlinear terms in the differential equation.

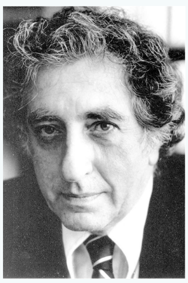
1. Convergence :
    - ADM is more likely to converge for problems with simple nonlinear terms. If the nonlinear terms are well-behaved, ADM tends to produce more accurate results.
    - Convergence is more likely when the decomposition functions are carefully chosen and match the problem's characteristics.
    - For linear problems, ADM will always converge.
2. Divergence :
    - ADM may diverge if the nonlinear terms in the differential equation are very complex or ill-behaved. Chaotic behavior in the nonlinear terms can lead to divergence.
    - Poor choices of decomposition functions can also result in divergence.
    - In some cases, ADM may exhibit slow convergence, meaning that it requires a large number of iterations to reach an acceptable solution. If the nonlinear terms in the differential equation are highly oscillatory, slow convergence may occur.

To enhance the convergence of ADM, you can try different strategies, such as refining the choice of decomposition functions, using regularization techniques, or employing other numerical methods in combination with ADM.

It's important to note that ADM is not guaranteed to converge for all problems, and its convergence behavior can be problem-dependent. Therefore, it is often recommended to perform a convergence analysis for a specific differential equation and problem setup to determine whether ADM is a suitable method for solving it. Additionally, other numerical methods, such as finite difference, finite element, or spectral methods, may be more appropriate for certain types of problems.

George Adomian

George Adomian (March 21, 1922 – June 17, 1996) was an American mathematician of Armenian descent who developed the Adomian decomposition method (ADM) for solving nonlinear differential equations, both ordinary and partial. The method is explained among other places in his book ***"Solving Frontier Problems in Physics: The Decomposition Method"***. He was a faculty member at the University of Georgia (UGA) from 1966 through 1989. While at UGA, he started the Center for Applied Mathematics. Adomian was also an aerospace engineer.



■ **Example 3.2.1** Consider the nonlinear differential equation

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = x^2 - \dfrac{1}{4}(\dfrac{\partial u(x,t)}{\partial x})^2 \\ u(x,0) = 0 \end{cases} \tag{1}$$

Set

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t)$$
$$N(u) = \sum_{n=0}^{\infty} A_n(x,t)$$

Where $N(u)$ represents the nonlinear form of u in our case in equation (1) $N(u) = \left(\frac{\partial u}{\partial x}\right)^2$

$$A_n(x,t) = \left[\frac{1}{n!}\frac{d^n}{d\lambda^n} N\left(\sum_{i=0}^{n} \lambda^i u_i(x,t)\right)\right]_{\lambda=0}$$
$$A_n(x,t) = \left[\frac{1}{n!}\frac{d^n}{d\lambda^n}\left(\sum_{i=0}^{n} \lambda^i \frac{\partial u_i(x,t)}{\partial x}\right)^2\right]_{\lambda=0}$$

Integrating equation (1) from $0 \to t$

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) = x^2 t - \frac{1}{4}\int_0^t \sum_{n=0}^{\infty} A_n(x,\theta) d\theta$$

Now we get A_0,A_1,A_2...

$$\begin{aligned}
A_0(x,\theta) &= \left[\sum_{i=0}^{0} \lambda^i \frac{\partial u_i(x,\theta)}{\partial x}^2\right]_{\lambda=0} = \left(\frac{\partial u_0(x,\theta)}{\partial x}\right)^2 \\
A_1(x,\theta) &= \left[\frac{d}{d\lambda}\left(\sum_{i=0}^{1} \lambda^i \frac{\partial u_i(x,\theta)}{\partial x}\right)^2\right]_{\lambda=0} \\
&= \left[\frac{d}{d\lambda}\left(\frac{\partial u_0(x,\theta)}{\partial x} + \lambda \frac{\partial u_1(x,\theta)}{\partial x}\right)^2\right]_{\lambda=0} \\
&= 2\left[\left(\frac{\partial u_0(x,\theta)}{\partial x} + \lambda \frac{\partial u_1(x,\theta)}{\partial x}\right)\frac{\partial u_1(x,\theta)}{\partial x}\right]_{\lambda=0} = 2\frac{\partial u_0(x,\theta)}{\partial x}\frac{\partial u_1(x,\theta)}{\partial x} \\
A_2(x,\theta) &= \left[\frac{1}{2!}\frac{d^2}{d\lambda^2}\left(\sum_{i=0}^{2} \lambda^i \frac{\partial u_i(x,\theta)}{\partial x}\right)^2\right]_{\lambda=0} \\
&= \left[\frac{1}{2}\frac{d^2}{d\lambda^2}\left(\frac{\partial u_0(x,\theta)}{\partial x} + \lambda\frac{\partial u_1(x,\theta)}{\partial x} + \lambda^2\frac{\partial u_2(x,\theta)}{\partial x}\right)^2\right]_{\lambda=0} \\
&= \left(\frac{\partial u_1(x,\theta)}{\partial x}\right)^2 + 2\left(\frac{\partial u_0(x,\theta)}{\partial x}\frac{\partial u_2(x,\theta)}{\partial x}\right) \\
A_3(x,\theta) &= 2\frac{\partial u_1(x,\theta)}{\partial x}\frac{\partial u_2(x,\theta)}{\partial x} + 2\frac{\partial u_0(x,\theta)}{\partial x}\frac{\partial u_3(x,\theta)}{\partial x}
\end{aligned}$$

Now because

$$u_0 + u_1 + u_2 + \cdots = u(x,t) = x^2 t - \frac{1}{4}[A_0 + A_1 + A_2 + \ldots]$$

Then

$$u_0 = x^2 t$$

$$u_1 = -\frac{1}{4}\int_0^t A_0 d\theta = -\frac{1}{4}\int_0^t \left(\frac{\partial u_0(x,\theta)}{\partial x}\right)^2 = -\int_0^t x^2\theta^2 d\theta = \frac{-1}{3}x^2t^3$$
$$u_2 = \frac{2}{15}x^2t^5 \quad , \quad u_3 = \frac{-17}{315}x^2t^7 \quad , \quad \ldots$$
$$u(x,t) = x^2\left[t - \frac{1}{3}t^3 + \frac{2}{15}t^5 - \frac{17}{315}t^7 \ldots\right] = x^2\tanh(t)$$

■

■ **Example 3.2.2** Consider the following hyperbolic nonlinear problem

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = u(x,t)\dfrac{\partial u(x,t)}{\partial x} \\ u(x,0) = \dfrac{x}{10} \end{cases} \tag{1}$$

The solution of (1) is given by $u(x,t) = \dfrac{x}{10-t}$
Now let's use A.D.M to solve it !
First we integrate with respect to t

$$u(x,t) = u(x,0) + \int_0^t u(x,s)\frac{\partial u(x,s)}{\partial x}\, ds$$
$$u(x,t) = \frac{x}{10} + \int_0^t u(x,s)\frac{\partial u(x,s)}{\partial x}\, ds$$

Let $u(x,t) = \sum_{n=0}^{\infty} u_n \quad \& \quad N(u) = \sum_{n=0}^{\infty} A_n$, where

$$A_n = \left[\frac{1}{n!}\frac{\mathrm{d}^n}{\mathrm{d}\lambda^n}\left(\sum_{n=0}^{\infty}\lambda^j u_j\right)\left(\sum_{n=0}^{\infty}\lambda^j\frac{\partial u_j}{\partial x}\right)\right]_{\lambda=0}$$

And as before we get $A_0, A_1, A_2 \ldots$

$$\begin{aligned}
A_0 &= u_0\frac{\partial u_0}{\partial x} \\
A_1 &= \left[\frac{d}{d\lambda}\left(\sum_{i=0}^{1}\lambda^i u_i(x,t)\right)\left(\sum_{i=0}^{1}\lambda^i\frac{\partial u_i(x,t)}{\partial x}\right)\right]_{\lambda=0} \\
&= \left[\frac{d}{d\lambda}\left(u_0 + \lambda u_1\right)\left(\frac{\partial u_0}{\partial x} + \lambda\frac{\partial u_1}{\partial x}\right)\right]_{\lambda=0} \\
&= u_1\frac{\partial u_0}{\partial x} + u_0\frac{\partial u_1}{\partial x} \\
A_2 &= u_0\frac{\partial u_2}{\partial x} + u_1\frac{\partial u_1}{\partial x} + u_2\frac{\partial u_0}{\partial x} \\
&\vdots
\end{aligned}$$

Now

$$u_0 + u_1 + u_2 + \cdots = \frac{x}{10} + \int_0^t \sum_{i=0}^{\infty} A_n\, ds$$

Put

$$u_0 = \frac{x}{10}$$
$$u_1 = \int_0^t u_0\frac{\partial u_0}{\partial x} ds = \frac{x}{10}\left(\frac{t}{10}\right)$$

$$
\begin{aligned}
u_2 &= \int_0^t u_0 \frac{\partial u_1}{\partial x} + u_1 \frac{\partial u_0}{\partial x} ds = \frac{x}{10}(\frac{t}{10})^2 \\
&\vdots \\
u_n &= \frac{x}{10}(\frac{t}{10})^n
\end{aligned}
$$

$$
\begin{aligned}
\sum_{n=0}^{\infty} u_n(x,t) &= \frac{x}{10}\left[1 + \left(\frac{t}{10}\right) + \left(\frac{t}{10}\right)^2 + \dots\right] \\
\therefore u(x,t) &= \frac{x}{10}\sum_{n=0}^{\infty}\left(\frac{t}{10}\right)^n
\end{aligned}
$$

Remember that the value of the geometric series is $\displaystyle\sum_{n=s}^{\infty}(r)^n = \frac{r^s}{1-r}$

Then in our case $\displaystyle\sum_{n=0}^{\infty}\left(\frac{t}{10}\right)^n = \frac{1}{1-\frac{t}{10}} = \frac{10}{10-t}$

$$
\therefore u(x,t) = \frac{x}{10}\frac{10}{10-t} = \frac{x}{10-t}
$$

■

■ **Example 3.2.3** Consider the nonlinear system of equations

$$
\begin{cases}
\dfrac{\partial u}{\partial t} = u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y} \\
\dfrac{\partial v}{\partial t} = u\dfrac{\partial v}{\partial x} + v\dfrac{\partial v}{\partial y} \\
u(x,y,0) = v(x,y,0) = x + y \Longrightarrow I.C
\end{cases} \tag{1}
$$

Integrate each equation in (1) with respect to t

$$
\begin{aligned}
u(x,y,t) &= x + y + \int_0^t \left(u(x,y,\theta)\frac{\partial u}{\partial x} + v(x,y,\theta)\frac{\partial u}{\partial y}\right) d\theta \\
v(x,y,t) &= x + y + \int_0^t \left(u(x,y,\theta)\frac{\partial v}{\partial x} + v(x,y,\theta)\frac{\partial v}{\partial y}\right) d\theta
\end{aligned}
$$

Let

$$
\begin{aligned}
u(x,y,t) &= \sum_{n=0}^{\infty} u_n \quad \& \quad v(x,y,t) = \sum_{n=0}^{\infty} v_n \\
u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} &= \sum_{n=0}^{\infty} A_n \quad \& \quad u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = \sum_{n=0}^{\infty} B_n
\end{aligned}
$$

Where

$$
\begin{aligned}
A_n &= \frac{1}{n!}\frac{\mathrm{d}^n}{\mathrm{d}\lambda^n}\left[\left(\sum_{j=0}^{n}\lambda^j u_j\right)\left(\sum_{j=0}^{n}\lambda^j \frac{\partial u_j}{\partial x}\right) + \left(\sum_{j=0}^{n}\lambda^j v_j\right)\left(\sum_{j=0}^{n}\lambda^j \frac{\partial u_j}{\partial y}\right)\right]_{\lambda=0} \\
B_n &= \frac{1}{n!}\frac{\mathrm{d}^n}{\mathrm{d}\lambda^n}\left[\left(\sum_{j=0}^{n}\lambda^j u_j\right)\left(\sum_{j=0}^{n}\lambda^j \frac{\partial v_j}{\partial x}\right) + \left(\sum_{j=0}^{n}\lambda^j v_j\right)\left(\sum_{j=0}^{n}\lambda^j \frac{\partial v_j}{\partial y}\right)\right]_{\lambda=0}
\end{aligned}
$$

And as before we get $A_0, A_1, A_2 \ldots$ and $B_0, B_1, B_2 \ldots$

$$\begin{aligned}
A_0 &= u_0 \frac{\partial u_0}{\partial x} + v_0 \frac{\partial u_0}{\partial y} \\
A_1 &= u_0 \frac{\partial u_1}{\partial x} + v_0 \frac{\partial u_1}{\partial y} + u_1 \frac{\partial u_0}{\partial x} + v_1 \frac{\partial u_0}{\partial y} \\
A_2 &= u_0 \frac{\partial u_2}{\partial x} + v_0 \frac{\partial u_2}{\partial y} + u_1 \frac{\partial u_1}{\partial x} + v_1 \frac{\partial u_1}{\partial y} + u_2 \frac{\partial u_0}{\partial x} + v_2 \frac{\partial u_0}{\partial y} \\
&\vdots
\end{aligned}$$

Similarly

$$\begin{aligned}
B_0 &= u_0 \frac{\partial v_0}{\partial x} + v_0 \frac{\partial v_0}{\partial y} \\
B_1 &= u_0 \frac{\partial v_1}{\partial x} + v_0 \frac{\partial v_1}{\partial y} + u_1 \frac{\partial v_0}{\partial x} + v_1 \frac{\partial v_0}{\partial y} \\
B_2 &= u_0 \frac{\partial v_2}{\partial x} + v_0 \frac{\partial v_2}{\partial y} + u_1 \frac{\partial v_1}{\partial x} + v_1 \frac{\partial v_1}{\partial y} + u_2 \frac{\partial v_0}{\partial x} + v_2 \frac{\partial v_0}{\partial y} \\
&\vdots
\end{aligned}$$

Now, We get that

$$\begin{aligned}
u_0 &= x + y & v_0 &= x + y \\
u_1 &= (x + y)(2t) & v_1 &= (x + y)(2t) \\
u_2 &= (x + y)(2t)^2 & v_2 &= (x + y)(2t)^2 \\
&\vdots & &\vdots \\
u_n &= (x + y)(2t)^n & v_n &= (x + y)(2t)^n
\end{aligned}$$

Thus

$$\begin{aligned}
\sum_{n=0}^{\infty} u_n &= (x + y)\left[1 + (2t) + (2t)^2 + \ldots\right] \\
u(x, y, t) &= (x + y)\frac{1}{1 - 2t} = \frac{x + y}{1 - 2t} \\
\sum_{n=0}^{\infty} v_n &= (x + y)\left[1 + (2t) + (2t)^2 + \ldots\right] \\
v(x, y, t) &= (x + y)\frac{1}{1 - 2t} = \frac{x + y}{1 - 2t}
\end{aligned}$$

$\blacksquare$

■ **Example 3.2.4** Consider the following system (Archana Varsoliwala & Twinkle R. Singh 2022)

$$\begin{cases} \dfrac{\partial m(x,t)}{\partial t} + m\dfrac{\partial m(x,t)}{\partial x} - fp(x,t) + g\dfrac{\partial \eta(x,t)}{\partial x} = 0 \\[2ex] \dfrac{\partial p(x,t)}{\partial t} + m\dfrac{\partial p(x,t)}{\partial x} + fm(x,t) = 0 \\[2ex] \dfrac{\partial \eta(x,t)}{\partial t} + m\dfrac{\partial \eta(x,t)}{\partial x} + \eta\dfrac{\partial \eta(x,t)}{\partial x} = 0 \end{cases} \tag{$\bigstar$}$$

The problems that we has in this system are the nonlinear terms which are the terms colored in red we will work on it as we did before but on bigger scale (system of 3 equations)

Set

$$m(x,t) = \sum_{n=0}^{\infty} m_n(x,t) \quad , \quad p(x,t) = \sum_{n=0}^{\infty} p_n(x,t) \quad , \quad \eta(x,t) = \sum_{n=0}^{\infty} \eta_n(x,t)$$

And now we set the Summation that is equivalent to each nonlinear term

$$m\frac{\partial m(x,t)}{\partial x} = \sum_{n=0}^{\infty} A_n(x,t) \quad , \quad m\frac{\partial p(x,t)}{\partial x} = \sum_{n=0}^{\infty} B_n(x,t) \quad , \quad m\frac{\partial \eta(x,t)}{\partial x} + \eta\frac{\partial \eta(x,t)}{\partial x} = \sum_{n=0}^{\infty} C_n(x,t)$$

Then we get the formula of A_n, B_n, C_n

$$A_n(x,t) = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[\left(\sum_{i=0}^{n} \lambda^i m_i(x,t)\right)\left(\sum_{i=0}^{n} \lambda^i \frac{\partial m_i(x,t)}{\partial x}\right)\right]_{\lambda=0}$$

$$B_n(x,t) = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[\left(\sum_{i=0}^{n} \lambda^i m_i(x,t)\right)\left(\sum_{i=0}^{n} \lambda^i \frac{\partial p_i(x,t)}{\partial x}\right)\right]_{\lambda=0}$$

$$C_n(x,t) = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[\left(\sum_{i=0}^{n} \lambda^i m_i(x,t)\right)\left(\sum_{i=0}^{n} \lambda^i \frac{\partial \eta_i(x,t)}{\partial x}\right) + \left(\sum_{i=0}^{n} \lambda^i \eta_i(x,t)\right)\left(\sum_{i=0}^{n} \lambda^i \frac{\partial \eta_i(x,t)}{\partial x}\right)\right]_{\lambda=0}$$

Now, Integrate each equation in ($\bigstar$) from $0 \to t$

$$m(x,t) = m(x,0) - \int_0^t m(x,s)\frac{\partial m(x,s)}{\partial x}ds + f\int_0^t p(x,s)ds - g\int_0^t \frac{\partial \eta(x,s)}{\partial x}ds$$

$$p(x,t) = p(x,0) - \int_0^t m(x,s)\frac{\partial p(x,s)}{\partial x}ds - f\int_0^t m(x,s)ds$$

$$\eta(x,t) = \eta(x,0) - \int_0^t m(x,s)\frac{\partial \eta(x,s)}{\partial x}ds - \int_0^t \eta(x,s)\frac{\partial \eta(x,s)}{\partial x}ds$$

Now replace each nonlinear term with it's equivalent Summation

$$\sum_{n=0}^{\infty} m_n(x,t) = m(x,0) - \int_0^t \sum_{n=0}^{\infty} A_n(x,s)ds + f\int_0^t \sum_{n=0}^{\infty} p_n(x,s)ds - g\int_0^t \sum_{n=0}^{\infty} \frac{\partial \eta_n(x,s)}{\partial x}ds$$

$$\sum_{n=0}^{\infty} p_n(x,t) = p(x,0) - \int_0^t \sum_{n=0}^{\infty} B_n(x,s)ds - f\int_0^t \sum_{n=0}^{\infty} m_n(x,s)ds$$

$$\sum_{n=0}^{\infty} \eta_n(x,t) = \eta(x,0) - \int_0^t \sum_{n=0}^{\infty} C_n(x,s)ds$$

Set initial conditions $m(x,0) = p(x,0) = \eta(x,0) = x$ and $f = g = 1$
Now,

$$m_0 = x \qquad p_0 = x \qquad \eta_0 = x$$

$$\begin{aligned}
A_0 &= m_0 \frac{\partial m_0}{\partial x} \\
A_1 &= m_1 \frac{\partial m_0}{\partial x} + m_0 \frac{\partial m_1}{\partial x} \\
&\vdots \\
B_0 &= m_0 \frac{\partial p_0}{\partial x} \\
B_1 &= m_1 \frac{\partial p_0}{\partial x} + m_0 \frac{\partial p_1}{\partial x} \\
&\vdots \\
C_0 &= m_0 \frac{\partial \eta_0}{\partial x} + \eta_0 \frac{\partial \eta_0}{\partial x} \\
C_1 &= m_1 \frac{\partial \eta_0}{\partial x} + \eta_1 \frac{\partial \eta_0}{\partial x} + m_0 \frac{\partial \eta_1}{\partial x} + \eta_0 \frac{\partial \eta_1}{\partial x} \\
&\vdots
\end{aligned}$$

Substitute in system, we get

$$\begin{aligned}
m_1 &= -\int_0^t A_0 ds + \int_0^t P_0 ds - \int_0^t \frac{\partial \eta_0}{\partial x} ds \\
&= -\int_0^t x.1 ds + \int_0^t x ds - \int_0^t 1 ds = -T \\
m_2 &= \dots \\
&\vdots \\
p_1 &= -\int_0^t B_0 ds - \int_0^t m_0 ds \\
&= -\int_0^t x.1 ds - \int_0^t x ds = -2xt \\
p_2 &= \dots \\
&\vdots \\
\eta_1 &= -\int_0^t c_0 ds \\
&= -\int_0^t x.1 + x.1 ds = -2xt \\
\eta_2 &= \dots \\
&\vdots
\end{aligned}$$

Getting $m_1, m_2, m_3 \dots\,, p_1, p_2, p_3 \dots\,, \eta_1, \eta_2, \eta_3 \dots$ will get us the solution of the system $(\bigstar)$ $\blacksquare$

4 Linear ODE System

Consider the system

$$\begin{cases} \dfrac{dX(t)}{dt} = A(t)X(t) \\ X(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} \quad , \quad A(t) = \begin{bmatrix} a_{11}(t) & a_{12}(t) & \dots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \dots & a_{2n}(t) \\ \vdots & \vdots & & \vdots \\ a_{n1}(t) & a_{n2}(t) & \dots & a_{nn}(t) \end{bmatrix} \end{cases} \tag{1}$$

Where $a_{ij}(t)$ are continuous functions on $[\alpha, \beta]$

Suppose that $\mathbf{S}(t)$ is a square matrix of order n such that

$$\frac{d\mathbf{S}(t)}{dt} = A(t)\mathbf{S}(t) \quad , \quad \mathbf{S}(t) = \begin{bmatrix} s_{11}(t) & s_{12}(t) & \dots & s_{1n}(t) \\ s_{21}(t) & s_{22}(t) & \dots & s_{2n}(t) \\ \vdots & \vdots & & \vdots \\ s_{n1}(t) & s_{n2}(t) & \dots & s_{nn}(t) \end{bmatrix}$$

If the columns of $\begin{bmatrix} s_{11}(t) \\ s_{21}(t) \\ \vdots \\ s_{n1}(t) \end{bmatrix}, \begin{bmatrix} s_{12}(t) \\ s_{22}(t) \\ \vdots \\ s_{n2}(t) \end{bmatrix}, \dots, \begin{bmatrix} s_{1n}(t) \\ s_{2n}(t) \\ \vdots \\ s_{nn}(t) \end{bmatrix}$ are Linearly independent then $\mathbf{S}(t)$ is called a fundamental solution of system (1)

Theorem 4.1 The solution $\mathbf{S}(t)$ is fundamental iff $\det(\mathbf{S}(t)) \neq 0$ on the interval $[\alpha, \beta]$
i.e Linearly independent $\iff \det(\mathbf{S}(t)) \neq 0$

Theorem 4.2 If $\mathbf{S}(t)$ is a fundamental solution of system (1) and if $\mathbf{C}$ is constant matrix such that $\det(\mathbf{C}) \neq 0$ then $\mathbf{S}(t)\mathbf{C}$ is a fundamental solution of system (1)

Proof. Set

$$f(t) = \mathbf{S}(t)\mathbf{C}$$

Where $\det(\mathbf{C}) \neq 0$

$$\begin{aligned} \det f(t) &= \det[\mathbf{S}(t)\mathbf{C}] \\ &= \det \mathbf{S}(t) \det \mathbf{C} \\ &\neq 0 \end{aligned}$$

Now we proof that it's a solution

$$\frac{df(t)}{dt} = \frac{d\mathbf{S}(t)}{dt}\mathbf{C} = A(t)\mathbf{S}(t)\mathbf{C} = A(t)f(t)$$

Then $\mathbf{S}(t)\mathbf{C}$ is a fundamental solution ∎

Theorem 4.3 If $\mathbf{S}_1(t)$ and $\mathbf{S}_2(t)$ are fundamental solution of system (1) then $\mathbf{S}_2(t) = \mathbf{S}_1(t)\mathbf{C}$ where $\mathbf{C}$ is constant matrix such that $\det(\mathbf{C}) \neq 0$

Proof. Let $\mathbf{S}_1(t)$ and $\mathbf{S}_2(t)$ be fundamental solutions
Set

$$\mathbf{S}_2(t) = \mathbf{S}_1(t)\mathbf{G}(t)$$

Then

$$\begin{aligned}
\frac{d\mathbf{S}_2(t)}{dt} &= \mathbf{S}_1(t)\frac{d\mathbf{G}(t)}{dt} + \frac{d\mathbf{S}_1(t)}{dt}\mathbf{G}(t) \\
A(t)\mathbf{S}_2(t) &= \mathbf{S}_1(t)\frac{d\mathbf{G}(t)}{dt} + A(t)\mathbf{S}_1(t)\mathbf{G}(t) \\
A(t)\mathbf{S}_1(t)\mathbf{G}(t) &= \mathbf{S}_1(t)\frac{d\mathbf{G}(t)}{dt} + A(t)\mathbf{S}_1(t)\mathbf{G}(t) \\
\therefore \mathbf{S}_1(t)\frac{d\mathbf{G}(t)}{dt} &= 0
\end{aligned}$$

Because $\det(\mathbf{S}_1(t)) \neq 0$ then it has inverse
Then multiply by this inverse from the left

$$\begin{aligned}
\mathbf{S}_1^{-1}(t)\mathbf{S}_1(t)\frac{d\mathbf{G}(t)}{dt} &= 0 \\
\frac{d\mathbf{G}(t)}{dt} &= 0 \\
\therefore \mathbf{G}(t) &= \mathbf{C}
\end{aligned}$$

Then $\mathbf{G}(t)$ is a constant matrix ■

4.1 Matrix and Determinant Differentiation

If a differentiation applied on matrix then the result will be as following

$$\frac{d}{dt}\begin{bmatrix} a_{11}(t) & a_{12}(t) & \dots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \dots & a_{2n}(t) \\ \vdots & \vdots & & \vdots \\ a_{n1}(t) & a_{n2}(t) & \dots & a_{nn}(t) \end{bmatrix} = \begin{bmatrix} \dot{a}_{11}(t) & \dot{a}_{12}(t) & \dots & \dot{a}_{1n}(t) \\ \dot{a}_{21}(t) & \dot{a}_{22}(t) & \dots & \dot{a}_{2n}(t) \\ \vdots & \vdots & & \vdots \\ \dot{a}_{n1}(t) & \dot{a}_{n2}(t) & \dots & \dot{a}_{nn}(t) \end{bmatrix}$$

On the other hand if it's applied on Determinant

$$\frac{d}{dt}\begin{vmatrix} a_{11}(t) & a_{12}(t) & \dots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \dots & a_{2n}(t) \\ \vdots & \vdots & & \vdots \\ a_{n1}(t) & a_{n2}(t) & \dots & a_{nn}(t) \end{vmatrix} = \begin{vmatrix} \dot{a}_{11}(t) & \dot{a}_{12}(t) & \dots & \dot{a}_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \dots & a_{2n}(t) \\ \vdots & \vdots & & \vdots \\ a_{n1}(t) & a_{n2}(t) & \dots & a_{nn}(t) \end{vmatrix} + \begin{vmatrix} a_{11}(t) & a_{12}(t) & \dots & a_{1n}(t) \\ \dot{a}_{21}(t) & \dot{a}_{22}(t) & \dots & \dot{a}_{2n}(t) \\ \vdots & \vdots & & \vdots \\ a_{n1}(t) & a_{n2}(t) & \dots & a_{nn}(t) \end{vmatrix} + \dots$$
$$+ \begin{vmatrix} a_{11}(t) & a_{12}(t) & \dots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \dots & a_{2n}(t) \\ \vdots & \vdots & & \vdots \\ \dot{a}_{n1}(t) & \dot{a}_{n2}(t) & \dots & \dot{a}_{nn}(t) \end{vmatrix}$$

Lemma 4.4 Let $\mathbf{S}(t) = e^{\mathbf{A}t}$ where $\mathbf{A}$ is a constant matrix then

$$\frac{d\mathbf{S}(t)}{dt} = \mathbf{A}e^{\mathbf{A}t} = \mathbf{A}\mathbf{S}(t)$$

Where $\det\left(e^{\mathbf{A}t}\right) \neq 0$ then $e^{\mathbf{A}t}$ is a fundamental solution of $\dfrac{dX(t)}{dt} = AX(t)$

Theorem 4.5 If $\mathbf{S}(t)$ is fundamental solution of system (1) i.e

$$\frac{d\mathbf{S}(t)}{dt} = \mathbf{A}(t)\mathbf{S}(t)$$

Then $\frac{d}{dt}\det(\mathbf{S}(t)) = \mathrm{tr}[\mathbf{A}(t)]\det(\mathbf{S}(t))$

And $\mathrm{Det}(\mathbf{S}(t)) = \det(\mathbf{S}(t_0))e^{\int_{t_0}^{t} tr[\mathbf{A}(\theta)]d\theta}$

Where $\mathrm{tr}[\mathbf{A}(t)] = a_{11}(t) + a_{22}(t) + \cdots + a_{nn}(t)$

Proof. Because

$$\frac{d\mathbf{S}(t)}{dt} = \mathbf{A}(t)\mathbf{S}(t)$$

Let

$$\mathbf{A}(t)\mathbf{S}(t) = b_{ij}(t) = \sum_{k=1}^{n} a_{ik}(t)s_{kj}(t)$$

Then

$$\frac{d}{dt}\det(\mathbf{S}(t)) = \begin{vmatrix} \sum_{k=1}^{n} a_{1k}(t)s_{k1}(t) & \sum_{k=1}^{n} a_{1k}(t)s_{k2}(t) & \dots & \sum_{k=1}^{n} a_{1k}(t)s_{kn}(t) \\ s_{21}(t) & s_{22}(t) & \dots & s_{2n}(t) \\ \vdots & \vdots & & \vdots \\ s_{n1}(t) & s_{n2}(t) & \dots & s_{nn}(t) \end{vmatrix} + \dots$$

$$+ \begin{vmatrix} s_{11}(t) & s_{12}(t) & \dots & s_{1n}(t) \\ s_{21}(t) & s_{22}(t) & \dots & s_{2n}(t) \\ \vdots & \vdots & & \vdots \\ \sum_{k=1}^{n} a_{nk}(t)s_{k1}(t) & \sum_{k=1}^{n} a_{nk}(t)s_{k2}(t) & \dots & \sum_{k=1}^{n} a_{nk}(t)s_{kn}(t) \end{vmatrix}$$

Using the properties of determinants we get

$$\frac{d}{dt}\det(\mathbf{S}(t)) = a_{11}\det(\mathbf{S}(t)) + a_{22}\det(\mathbf{S}(t)) + \cdots + a_{nn}\det(\mathbf{S}(t)) = \mathrm{tr}(\mathbf{A}(t))\det(\mathbf{S}(t))$$

Then we can write that

$$\begin{aligned} \frac{d}{dt}\det(\mathbf{S}(t)) &= \mathrm{tr}(\mathbf{A}(t))\det(\mathbf{S}(t)) \\ \frac{d\det(\mathbf{S}(t))}{\det(\mathbf{S}(t))} &= \mathrm{tr}(\mathbf{A}(t))dt \\ \ln\left(\frac{\det(\mathbf{S}(t))}{\det(\mathbf{S}(t_0))}\right) &= \int_{t_0}^{t} \mathrm{tr}[\mathbf{A}(\theta)]d\theta \\ \det(\mathbf{S}(t)) &= \det(\mathbf{S}(t_0))e^{\int_{t_0}^{t} \mathrm{tr}[\mathbf{A}(\theta)]d\theta} \end{aligned}$$

■

special case

If $\mathbf{A}$ is a constant matrix and $t_0 = 0$ then

$$\det(\mathbf{S}(t)) = \det(\mathbf{S}(0))e^{t\times\mathrm{tr}[\mathbf{A}]}$$

Consider the linear homogeneous system

$$\begin{cases} \dfrac{dX(t)}{dt} = \mathbf{A}X(t) \end{cases} \tag{2}$$

We try to find a solution $X(t)$ of the form

$$X(t) = e^{\lambda t}\mathbf{v} \quad , \quad \text{Where } \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

We notice that

$$\frac{\mathrm{d}X(t)}{\mathrm{d}t} = \lambda e^{\lambda t}\mathbf{v} = e^{\lambda t}\mathbf{A}\mathbf{v}$$

Hence $X(t) = e^{\lambda t}\mathbf{v}$ is a solution of system $\iff$ $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$
A non zero vector $\mathbf{v}$ satisfying the last equation is called **eigenvector** of matrix $\mathbf{A}$ corresponding to **eigenvalue** λ
Now, to find $e^{\mathbf{A}}$, $\mathbf{A}$ is of order n

$$\frac{dX(t)}{dt} = \mathbf{A}X(t)$$

we find the n^{th} solution $\begin{bmatrix} x_{11}(t) \\ x_{21}(t) \\ \vdots \\ x_{n1}(t) \end{bmatrix}$, $\begin{bmatrix} x_{12}(t) \\ x_{22}(t) \\ \vdots \\ x_{n2}(t) \end{bmatrix}$, $\ldots$, $\begin{bmatrix} x_{1n}(t) \\ x_{2n}(t) \\ \vdots \\ x_{nn}(t) \end{bmatrix}$ and by compining them in one matrix we get the solution

$$X(t) = e^{\mathbf{A}} = \begin{bmatrix} x_{11}(t) & x_{12}(t) & \ldots & x_{1n}(t) \\ x_{21}(t) & x_{22}(t) & \ldots & x_{2n}(t) \\ \vdots & \vdots & & \vdots \\ x_{n1}(t) & x_{n2}(t) & \ldots & x_{nn}(t) \end{bmatrix}$$

■ **Example 4.1.1** Find $\det e^{\mathbf{A}}$ where $\mathbf{A} = \begin{bmatrix} 8 & 5 & 2 \\ 9 & 1 & 3 \\ 4 & -5 & 6 \end{bmatrix}$

$$\det e^{\mathbf{A}} = \operatorname{tr}(\mathbf{A}) = 8 + 1 + 6 = 15$$

■

■ **Example 4.1.2** Solve the IVP

$$\begin{cases} \dfrac{dX(t)}{dt} = \mathbf{A}X(t) \\ A = \begin{bmatrix} 1 & 12 \\ 3 & 1 \end{bmatrix} \\ X(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{cases} \tag{3}$$

Characteristic polynomial of $\mathbf{A}$ is given by $(\mathbf{A} - \lambda I) = 0$ which is

$$\begin{vmatrix} 1-\lambda & 12 \\ 3 & 1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)^2 = 36$$

$$1 - \lambda = \pm 6$$

$$\lambda = -5 \quad , \quad \lambda = 7$$

At $\lambda = 7$, we seek a non zero vector $\mathbf{v}$ such that

$$(\mathbf{A} - 7I)\mathbf{v} = \begin{bmatrix} -6 & 12 \\ 3 & -6 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

By solving this system we get

$$v_1 = 2v_2$$

Consequently, every vector $\mathbf{v} = \mathbf{c} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is an eigen vector of $\mathbf{A}$ corresponding to eigenvalue $\lambda = 7$
Thus,

$$X(t) = \mathbf{c_1} e^{7t} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Is a solution of our system
Similarly, At $\lambda = -5$

$$(\mathbf{A} + 5I)\mathbf{v} = \begin{bmatrix} 6 & 12 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

By solving this system we get

$$v_1 = -2v_2$$

Consequently, every vector $\mathbf{v} = \mathbf{c} \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ is an eigen vector of $\mathbf{A}$ corresponding to eigenvector $\lambda = -5$ Thus,

$$X(t) = \mathbf{c_2} e^{-5t} \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Then the general solution is given by

$$X(t) = \mathbf{c_1} e^{7t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \mathbf{c_2} e^{-5t} \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

The constants $\mathbf{c_1}$ and $\mathbf{c_2}$ are determined from initial condition

$$X(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \mathbf{c_1} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \mathbf{c_2} \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

We get

$$2c_1 - 2c_2 = 0$$

$$c_1 + c_2 = 1$$

We get that $c_1 = c_2 = \frac{1}{2}$

Thus, The general solution is given by

$$X(t) = \frac{1}{2}e^{7t}\begin{bmatrix}2\\1\end{bmatrix} + \frac{1}{2}e^{-5t}\begin{bmatrix}-2\\1\end{bmatrix}$$

■

Moreover, Evaluate $e^{\mathbf{A}}$
Remember we considered that the solution of the form $X(t) = e^{\lambda t}\mathbf{v}$ and we know that $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$
$X(t) = e^{\mathbf{A}t}\mathbf{v}$
We can get $\mathbf{v}$ easily from initial condition $\mathbf{v} = \begin{bmatrix}0\\1\end{bmatrix}$

$$e^{\mathbf{A}t}\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}e^{7t} - e^{-5t}\\ \\ \frac{1}{2}e^{7t} + \frac{1}{2}e^{-5t}\end{bmatrix}$$

Assume $e^{\mathbf{A}t} = \begin{bmatrix}b_{11} & b_{12}\\ b_{21} & b_{22}\end{bmatrix}$ We can deduce that

$$b_{12} = e^{7t} - e^{-5t}$$
$$b_{22} = \frac{1}{2}e^{7t} + \frac{1}{2}e^{-5t}$$

If we consider a new initial condition. It is easy to obtain that

$$e^{\mathbf{A}t}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}\frac{1}{2}e^{7t} + \frac{1}{2}e^{7t}\\ \frac{1}{4}e^{7t} - \frac{1}{4}e^{-5t}\end{bmatrix}$$

Thus, we get

$$b_{11} = \frac{1}{2}e^{7t} + \frac{1}{2}e^{7t}$$
$$b_{21} = \frac{1}{4}e^{7t} - \frac{1}{4}e^{-5t}$$

Therefore we get that

$$e^{\mathbf{A}t} = \begin{bmatrix}\frac{1}{2}e^{7t} + \frac{1}{2}e^{7t} & e^{7t} - e^{-5t}\\ \\ \frac{1}{4}e^{7t} - \frac{1}{4}e^{-5t} & \frac{1}{2}e^{7t} + \frac{1}{2}e^{-5t}\end{bmatrix}$$

At t=1

$$e^{\mathbf{A}} = \begin{bmatrix}\frac{1}{2}e^{7} + \frac{1}{2}e^{7} & e^{7} - e^{-5}\\ \\ \frac{1}{4}e^{7} - \frac{1}{4}e^{-5} & \frac{1}{2}e^{7} + \frac{1}{2}e^{-5}\end{bmatrix}$$

5 Generalized Inverse

In the classical sense of linear algebra, an $n \times n$ square matrix A is called invertible also non-singular if there exists an $n \times n$ square matrix B such that

$$AB = BA = I_n$$

If this is the case, then the matrix B is uniquely determined by A, and is called the inverse of A, denoted by A^{-1}
In linear algebra for matrix A to own an inverse it had to be square matrix and non-singular (i.e $\det(A) \neq 0$)

5.1 Moore–Penrose generalized inverse

Let us now study the concept of the generalized inverse that allow us to get inverse for any matrix of order $(m \times n)$
We define the generalized inverse A^+

$$\text{I} \qquad AA^+A = A$$
$$\text{II} \qquad \exists U, V \quad \text{s.t} \quad A^+ = UA^* \quad , \quad A^+ = A^*V$$

Where A^* is the conjugate transpose if all elements of A are real then $A^* = A^{\mathrm{T}}$

$$A = \begin{pmatrix} 1+2i & 3-i \\ 5 & 6i \end{pmatrix} \implies A^* = \begin{pmatrix} 1-2i & 5 \\ 3+i & -6i \end{pmatrix}$$

Theorem 5.1 — The existence and uniqueness of GI. For matrix A of order $(m \times n)$ the generalized inverse A^+ exists ,and it's unique

Proof the uniqueness. Suppose that there is two generalized inverse A_1^+ , A_2^+ thus

$$\text{I} \qquad AA_i^+A = A$$
$$\text{II} \qquad \exists U_i, V_i \quad \text{s.t} \quad A_i^+ = U_iA_i^* \quad , \quad A_i^+ = A_i^*V_i \qquad \text{for } i = 1, 2$$

From I

$$\begin{aligned} A[A_2^+ - A_1^+]A &= 0 \\ ADA &= 0 \end{aligned}$$

Where $D = A_2^+ - A_1^+$ and we can say that

$$\begin{aligned} D &= A^*V & \qquad D &= UA^* \\ V &= V_2 - V_1 & \qquad U &= U_2 - U_1 \end{aligned}$$

E.H. Moore

Eliakim Hastings Moore (1862–1932) was an American mathematician known for his contributions to algebra and mathematical logic. One of his significant contributions was in the field of linear algebra, particularly his work on the generalized inverse of a non-square matrix. The Moore–Penrose inverse is a concept that extends the idea of the matrix inverse to non-square matrices. it's perhaps the most well-known and widely used among various generalizations of the matrix inverse. It has applications in solving linear systems of equations, least squares problems.

Now

$$\begin{aligned}(DA)^*DA &= A^*D^*DA \\ &= A^*V^*ADA = 0\end{aligned}$$

Because $ADA = 0$, then we get

$$\begin{aligned}(DA)^*DA &= 0 \\ DA &= 0 && \text{(multiply } U^* \text{ from the right)} \\ DAU^* &= 0 \\ DD^* &= 0 \\ \therefore D &= 0 \\ A_2^+ - A_1^+ &= 0 \\ A_2^+ &= A_1^+\end{aligned}$$

■

Proof the existence synthetically. Let A be a non square matrix we can write $A = BC$ where

$$\begin{aligned}A &\text{ is of order } (m \times n) \\ B &\text{ is of order } (m \times r) \\ C &\text{ is of order } (r \times n)\end{aligned}$$

Set

$$\begin{cases}\Lambda = C^*(CC^*)^{-1}(B^*B)^{-1}B^* \\ U = C^*(CC^*)^{-1}(B^*B)^{-1}(CC^*)^{-1}C \\ V = B(B^*B)^{-1}(CC^*)^{-1}(B^*B)^{-1}B^*\end{cases}$$

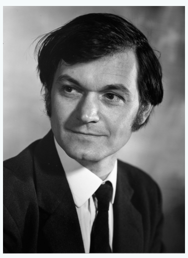
We can see that

$$\begin{aligned}A\Lambda A &= BCC^*(CC^*)^{-1}(B^*B)^{-1}B^*BC \\ &= B\underbrace{\left[CC^*(CC^*)^{-1}\right]}_{I}\underbrace{\left[(B^*B)^{-1}B^*B\right]}_{I}C \\ &= BC = A\end{aligned}$$

Now to proof the property II

$$UA^* = C^*(CC^*)^{-1}(B^*B)^{-1}(CC^*)^{-1}CC^*B^*$$

Roger Penrose



Sir Roger Penrose is a British mathematical physicist, mathematician, and philosopher who has made significant contributions to various fields, including general relativity, cosmology, and the foundations of quantum mechanics.
Penrose has made significant contributions to the fields of mathematics One of Penrose's notable contributions is his work on the generalized inverse of a matrix, Penrose introduced the concept of the Moore-Penrose pseudo-inverse, which is a widely used method for finding a generalized inverse of a matrix. This pseudo-inverse has applications in various areas, including linear algebra, statistics, signal processing, and machine learning.

$$
\begin{aligned}
&= C^*(CC^*)^{-1}(B^*B)^{-1}\underbrace{\left[(CC^*)^{-1}CC^*\right]}_{I}B^* \\
&= C^*(CC^*)^{-1}(B^*B)^{-1}B^* = \Lambda
\end{aligned}
$$

$$
\begin{aligned}
A^*V &= C^*B^*B(B^*B)^{-1}(CC^*)^{-1}(B^*B)^{-1}B^* \\
&= C^*\underbrace{\left[B^*B(B^*B)^{-1}\right]}_{I}(CC^*)^{-1}(B^*B)^{-1}B^* \\
&= C^*(CC^*)^{-1}(B^*B)^{-1}B^* = \Lambda
\end{aligned}
$$

Thus $\Lambda = A^+$ is the generalized inverse ■

Synthetic Proofs

Synthetic proofs refer to a style of proof that relies on creative and constructive methods rather than analytical or deductive reasoning. In a synthetic proof, mathematicians often use geometric or intuitive arguments to establish the truth of a statement. This approach involves constructing figures, diagrams, or models that illustrate the relationships and properties relevant to the mathematical concept being proven.

Proof the existence analytically. Let A be a non square matrix we can write $A = BC$ where

$$
\begin{gathered}
A \text{ is of order } (m \times n) \\
B \text{ is of order } (m \times r) \\
C \text{ is of order } (r \times n)
\end{gathered}
$$

Let us try to find two matrices such B^+, Q that

$$BB^+B = B \quad , \quad B^+ = QB^*$$

We notice that

$$
\begin{aligned}
BQB^*B &= B \\
B^*BQB^*B &= B^*B
\end{aligned}
$$

Since $\det(B^*B) \neq 0$ (will be proofed in next lemma) and B^*B is square thus Q is the ordinary inverse of B^*B

$$Q = (B^*B)^{-1}$$

Thus

$$B^+ = (B^*B)^{-1}B^*$$

Similarly

$$C^+ = C^*(C^*C)^{-1}$$

Which makes that

$$A^+ = C^+B^+ = C^*(C^*C)^{-1}(B^*B)^{-1}B^*$$

The rest of the proof as in the synthetically part ■

Analytic Proofs

Analytic proofs refer to a method of proof that relies on logical reasoning and the application of established mathematical principles and definitions. The term ”analytic” is derived from ”analysis,” emphasizing the breakdown of a mathematical statement or proposition into simpler, well-understood components.

Lemma 5.2 B^*B,CC^* are square matrices and $\det(B^*B), \det(CC^*) \neq 0$

Proof. It's clear that B^*B,CC^* are square matrices of order r because $B^*_{r\times m}B_{m\times r} = D_{r\times r}$
Now to proof that $\det(B^*B), \det(CC^*) \neq 0$
Consider the homogeneous system

$$B^*BX = 0 \qquad , \qquad X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_r \end{pmatrix}$$

We have that

$$\begin{aligned} B^*BX &= 0 && \text{(multiply } X^* \text{ from the left)} \\ X^*B^*BX &= 0 \\ (BX)^*BX &= 0 \end{aligned}$$

Set $BX = Y$

$$\begin{aligned} Y^*Y &= 0 \\ y_1^2 + y_2^2 + \cdots + y_r^2 &= 0 \\ y_1^2 = y_2^2 = \cdots = y_r^2 &= 0 \\ \implies BX &= 0 \end{aligned}$$

This means that only the trivial solution solves this equation
But $\operatorname{rank}(B) = r =$ number of unknowns therefore $x_1 = x_2 = \cdots = x_r = 0$ also $\operatorname{rank}(B^*B) = r$ then $\det(B^*B) \neq 0$

■

■ **Example 5.1.1** Find A^+ if

$$A = \begin{pmatrix} 1 & 1 & 4 \\ 2 & 1 & 2 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{B} \underbrace{\begin{pmatrix} 1 & 1 & 4 \\ 2 & 1 & 2 \end{pmatrix}}_{C}$$

We have taken $B = I$ to make the calculation easier but most of the time it doesn't work
Only works if $\det(AA^*) \neq 0$ in case of $\det(AA^*) = 0$ you need to find two matrices B, C such that $BC = A$ and $\det(CC^*), \det(B^*B) \neq 0$

$$\begin{aligned} A^+ &= C^*(CC^*)^{-1}(B^*B)^{-1}B^* = A^*(AA^*)^{-1} \\ &= \begin{pmatrix} 1 & 2 \\ 1 & 1 \\ 4 & 2 \end{pmatrix} \left[\begin{pmatrix} 1 & 1 & 4 \\ 2 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & 1 \\ 4 & 2 \end{pmatrix} \right]^{-1} \\ &= \begin{pmatrix} 1 & 2 \\ 1 & 1 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} 18 & 11 \\ 11 & 9 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} 1 & 2 \\ 1 & 1 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} \frac{9}{41} & \frac{-11}{41} \\ \frac{-11}{41} & \frac{18}{41} \end{pmatrix} \end{aligned}$$

$$= \begin{pmatrix} \frac{-13}{41} & \frac{25}{41} \\ \frac{-2}{41} & \frac{7}{41} \\ \frac{14}{41} & \frac{-8}{41} \end{pmatrix}$$

$$AA^+ = \begin{pmatrix} 1 & 1 & 4 \\ 2 & 1 & 2 \end{pmatrix} \begin{pmatrix} \frac{-13}{41} & \frac{25}{41} \\ \frac{-2}{41} & \frac{7}{41} \\ \frac{14}{41} & \frac{-8}{41} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$\begin{pmatrix} \frac{-13}{41} & \frac{25}{41} \\ \frac{-2}{41} & \frac{7}{41} \\ \frac{14}{41} & \frac{-8}{41} \end{pmatrix}$ is the right inverse of A

Notice that if you take $B = A$ and $C = I$ will not give an answer because $\det(A^*A) = 0$ that says that it's hard to find suitable B, C to get the inverse of A. ■

6 Well-Posed Problems

In mathematics a well-posed problem is one for which the following properties hold:

1. The problem has a solution
2. The solution is unique
3. The solution's behavior changes continuously with the initial conditions (stable)

Examples of well-posed problems the Dirichlet problem for Laplace's equation
This definition of a well-posed problem comes from the work of Jacques Hadamard on mathematical modeling of physical phenomena.

Problems that are not well-posed in the sense of Hadamard are termed ill-posed

For example, the inverse heat equation, deducing a previous distribution of temperature from final data, is not well-posed in that the solution is highly sensitive to changes in the final data.

6.1 Hadamard Laplace Example

In the theory of partial differential equations, an example constructed by J. Hadamard, which shows the instability of the solution of the Cauchy problem for the Laplace equation with respect to small changes in the initial data, is of great importance.

Hadamard's example served as the beginning of a systematic study of ill-posed problems in mathematical physics. On the other hand, the study of the Cauchy problem for the Laplace equation arises from problems of geophysics. At the same time, the question arises whether the Cauchy problem is correct for other elliptic equations.

We will see how Hadamard's example showed the the instability of the solution of the Cauchy problem for the Laplace equation.

Consider the following Cauchy problems for the Laplace equation.

$$\begin{cases} \dfrac{\partial^2 u(x,t)}{\partial t^2} + \dfrac{\partial^2 u(x,t)}{\partial x^2} = 0 \\ u(x,0) = 0 \\ \dfrac{\partial u(x,0)}{\partial t} = \phi(x) \end{cases} \tag{4}$$

$$\begin{cases} \dfrac{\partial^2 u^*(x,t)}{\partial t^2} + \dfrac{\partial^2 u^*(x,t)}{\partial x^2} = 0 \\ u^*(x,0) = 0 \\ \dfrac{\partial u^*(x,0)}{\partial t} = \phi(x) + \dfrac{\sin(nx)}{n^k} \end{cases} \tag{5}$$

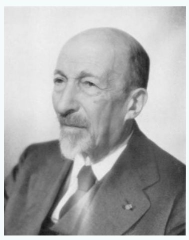
As we can see the two problems are nearly identical except for the second initial condition in (2) but if we take the limit for $n \to \infty$ in (2) it suppose to reduce to problem (1) which means that $u(x,t) = u^*(x,t)$ as $n \to \infty$

BUT

Let's construct new problem using (1),(2)

Jacques Hadamard

Jacques Hadamard (1865–1963) was French mathematician who made significant contributions to various branches of mathematics. One of his notable contributions lies in the field of partial differential equations and the study of well-posed and ill-posed problems.
Hadamard's work helped establish fundamental principles in the study of mathematical problems. Well-posed problems are essential for ensuring the stability and reliability of mathematical models used in various scientific disciplines.



Let $V(x,t) = u^*(x,t) - u(x,t)$ we get that

$$\begin{cases} \dfrac{\partial^2 V(x,t)}{\partial t^2} + \dfrac{\partial^2 V(x,t)}{\partial x^2} = 0 \\ V(x,0) = 0 \\ \dfrac{\partial V(x,0)}{\partial t} = \dfrac{\sin(nx)}{n^k} \end{cases} \tag{6}$$

The solution of this problem is given by

$$V(x,t) = \frac{\sinh(nt)\sin(nx)}{n^{k+1}}$$

Now as we Assumed $V(x,t)$ should vanish as $n \to \infty$ but we find that

$$\lim_{n\to\infty} V(x,t) = \infty$$

That's because $\mathbf{O}(\sinh(n)) > \mathbf{O}(n^{k+1})$ which causes the numerator to diverges faster than the denominator

7 Some Basic Differential Models in Mathematical Biology

7.1 Population dynamic models (Malthus model)

Any specie in the natural world does not exist solely, but is closely related to other species in biological communities and then constitute a population ecosystem. However, a single population is the basic unit which composed of the entire ecosystem. In order to predict the change law of the population, the famous demographer Malthus proposed the following basic model for single specie

$$\frac{\mathrm{d}N(t)}{\mathrm{d}t} = rN(t) \tag{1}$$

Where $N(t)$ denotes the population density at the time t, r denotes the intrinsic growth rate, which is the difference between the birth rate and the death rate.

Based on model (1), the famous ecologist Logistic proposed the famous insect population model when studying the growth rate of the insect in the laboratory as follows,

$$\frac{\mathrm{d}N(t)}{\mathrm{d}t} = rN(t)[1 - \frac{N(t)}{K}]$$

Where r denotes the intrinsic growth rate, K is the carrying capacity of the environment

7.2 Epidemic models(SIR model)

As the epidemic models are concerned, it should date back to the famous SIR model, which was proposed by Kermack and Mckendrick in 1927 when studying the propagation law of the Black Death in London from 1655-1666 and the plague in Mumbai in 1906. And the famous SIR model is as follow,

$$\begin{cases} \dfrac{\mathrm{d}S(t)}{\mathrm{d}t} = -\beta S(t)I(t) \\ \\ \dfrac{\mathrm{d}I(t)}{\mathrm{d}t} = \beta S(t)I(t) - \gamma I(t) \\ \\ \dfrac{\mathrm{d}R(t)}{\mathrm{d}t} = \gamma I(t) \end{cases} \tag{2}$$

Where $S(t)$ denotes the number of the susceptible individuals which has not yet infected but may be infected by the bacteria at time t, $I(t)$ denotes the number of the infected individuals at time t, and $R(t)$ denotes the number of the removed individuals at time t, β, denotes the infected rate, γ denotes the cure rate and γ^{-1} denotes the average cure rate.

notice that if we add equations (2) we get

$$\begin{aligned}\frac{\mathrm{d}S(t)}{\mathrm{d}t}+\frac{\mathrm{d}I(t)}{\mathrm{d}t}+\frac{\mathrm{d}R(t)}{\mathrm{d}t}&=\beta S(t)I(t)-\gamma I(t)-\beta S(t)I(t)+\gamma I(t)\\ \frac{\mathrm{d}}{\mathrm{d}t}\left(S(t)+I(t)+R(t)\right)&=0\\ S(t)+I(t)+R(t)&=constant\end{aligned}$$

that says that the total number of the individuals keeps constant.
and it's Assumed that the infected individuals have permanent immune capacity after the cure.

8 Integro-Partial Differential Equations

Consider the equation

$$\frac{\partial u(x,t)}{\partial t}=\sum_{|q|\le m}a_q(x,y,t)D_x^q u(x,t)+\int_G\sum_{|q|\le m}b_q(x,y,t)D_y^q u(x,t)\,dy$$

Where

- $G\subset\mathbb{R}^n$ is bounded region with smooth surface
- $D_x^q=\dfrac{\partial^{|q|}}{\partial x_1^{q_1}\partial x_2^{q_2}\partial x_3^{q_3}\ldots\partial x_n^{q_n}}$
- $|q|=q_1+q_2+\cdots+q_n$
- $x=(x_1,x_2,\ldots,x_n)$
- $y=(y_1,y_2,\ldots,y_n)$
- $dy=dy_1dy_2\ldots dy_n$
- $\int_G=\underbrace{\int\int\cdots\int}_{n}$

■ **Example 8.0.1** Consider Integro-Partial differential equation

$$\begin{cases}\dfrac{\partial u(x,t)}{\partial t}=\dfrac{\partial^2u(x,t)}{\partial x^2}+\displaystyle\int_{a_1}^{b_1}K_1(x,y,t)\frac{\partial u(y,t)}{\partial y}dy+\int_{a_2}^{b_2}K_2(x,y,t)u(y,t)\,dy\\ u(x,0)=\phi(x)\end{cases}\tag{3}$$

And $\phi(x)$, K_1 , K_2 are bounded and continuous known functions on $(-\infty,\infty)$
Consider the Cauchy problem

$$\begin{cases}\dfrac{\partial u(x,t)}{\partial t}=\dfrac{\partial^2u(x,t)}{\partial x^2}+V(x,t)\\ u(x,0)=\phi(x)\end{cases}\tag{4}$$

The solution of this problem is given by

$$u(x,t)=\underbrace{\frac{1}{2\sqrt{\pi t}}\int_{-\infty}^{\infty}e^{-\frac{(x-\xi)^2}{4t}}\phi(\xi)d\xi}_{=\psi(x,t)}+\int_0^t\int_{-\infty}^{\infty}\frac{1}{2\sqrt{\pi(t-\theta)}}e^{-\frac{(x-\xi)^2}{4(t-\theta)}}V(\xi,\theta)d\xi d\theta\tag{5}$$

(More information about this problem in the end of the lecture)
Comparing problem (1) with (2) we get that

$$V(x,t)=\int_{a_1}^{b_1}K_1(x,y,t)\frac{\partial u(y,t)}{\partial y}dy+\int_{a_2}^{b_2}K_2(x,y,t)u(y,t)\,dy$$

Now substitute for u from (3)

$$
\begin{aligned}
V(x,t) = &\int_{a_1}^{b_1} K_1(x,y,t)\frac{\partial\psi(y,t)}{\partial y}dy \\
&+ \int_{a_1}^{b_1} K_1(x,y,t)\frac{\partial}{\partial y}\int_0^t\int_{-\infty}^{\infty}\frac{1}{2\sqrt{\pi(t-\theta)}}e^{-\frac{(y-\xi)^2}{4(t-\theta)}}V(\xi,\theta)d\xi d\theta dy \\
&+ \int_{a_2}^{b_2} K_2(x,y,t)\psi(y,t)\,dy \\
&+ \int_{a_2}^{b_2} K_2(x,y,t)\int_0^t\int_{-\infty}^{\infty}\frac{1}{2\sqrt{\pi(t-\theta)}}e^{-\frac{(y-\xi)^2}{4(t-\theta)}}V(\xi,\theta)d\xi d\theta\,dy
\end{aligned}
$$

Put $K_1 = 0$ to make it a little simpler

$$
V(x,t) = \psi^*(y,t) + \int_{a_2}^{b_2} K_2(x,y,t)\int_0^t\int_{-\infty}^{\infty}\frac{1}{2\sqrt{\pi(t-\theta)}}e^{-\frac{(y-\xi)^2}{4(t-\theta)}}V(\xi,\theta)d\xi d\theta\,dy
$$

Now we have a volttera integral equation that can be solved by the integral equation methods like successive approximation ■

Remember From PDE Course(Extra Information)

8.1 Cauchy In-Homogeneous Problem

Also known as Heat with a source Cauchy problem
Consider the in-homogeneous heat equation on the whole line

$$
\begin{cases}
\dfrac{\partial u\,(x,t)}{\partial t} = c^2\dfrac{\partial^2 u(x,t)}{\partial x^2} + f(x,t), \quad c \neq 0, \quad -\infty < x < \infty, \quad t > 0 \\
u\,(x,0) = \phi\,(x)
\end{cases} \tag{1}
$$

Where $f(x,t)$ and $\phi(x)$ are arbitrary given functions.
$f(x,t)$ is called the source term, and it measures the physical effect of an external heat source.

From the superposition principle, we know that the solution of the in-homogeneous equation can be written as the sum of the solution of the homogeneous equation, and a particular solution of the in-homogeneous equation.
Thus we can break problem (1) into the following two problems

$$
\begin{cases}
\dfrac{\partial u_h\,(x,t)}{\partial t} = c^2\dfrac{\partial^2 u_h(x,t)}{\partial x^2} \\
u_h\,(x,0) = \phi\,(x)
\end{cases} \tag{2}
$$

$$
\begin{cases}
\dfrac{\partial u_p\,(x,t)}{\partial t} = c^2\dfrac{\partial^2 u_p(x,t)}{\partial x^2} + f(x,t) \\
u_p\,(x,0) = 0
\end{cases} \tag{3}
$$

Obviously, $u = u_h + u_p$ will solve the original problem (1).
We have solved problem (2) using Poisson formula which is

$$
u_h(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\phi(y)dy \tag{4}
$$

Where $S(x,t)$ is the heat kernel and it's equal to $\dfrac{e^{-\frac{x^2}{4tc^2}}}{2\sqrt{\pi tc^2}}$.
Notice that the physical meaning of expression (4) is that the heat kernel averages out the initial temperature distribution along the entire rod.

Since $f(x,t)$ plays the role of an external heat source, it is clear that this heat contribution must be averaged out too. But in this case one needs to average not only over the entire rod, but over time as well, since the heat contribution at an earlier time will effect the temperatures at all later times. We claim that the solution to (3) is given by

$$u_p(x,t) = \int_0^t \int_{-\infty}^{\infty} S(x-y, t-s) f(y,s) dyds \tag{5}$$

Combining (4) and (5) we obtain the solution to the IVP (1)

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\phi(y)dy + \int_0^t \int_{-\infty}^{\infty} S(x-y, t-s) f(y,s) dyds \tag{6}$$

Now substitute the heat kernel

$$u(x,t) = \frac{1}{2\sqrt{\pi t c^2}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4tc^2}} \phi(y)dy + \frac{1}{2\sqrt{\pi(t-s)c^2}} \int_0^t \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4(t-s)c^2}} f(y)dyds$$

We now need to show that (6) indeed solves problem (1) by direct substitution.
Since we have solved the homogeneous equation before, it suffices to show that u_p solves problem (3).

By differentiating (5) with respect to t gives

$$\frac{\partial u_p}{\partial t} = \int_{-\infty}^{\infty} S(x-y,0) f(y,t)dy + \int_0^t \int_{-\infty}^{\infty} \frac{\partial}{\partial t} S(x-y,t-s) f(y,s)dyds$$

Note that the heat kernel solves the heat equation and has the Dirac delta function as its initial means that $S_t = c^2 S_{xx}$ and $S(x-y,0) = \delta(x-y)$
When integrating the Dirac Delta function we would get

$$\int_{-\infty}^{\infty} \delta(x-y)dy = 1$$

If we have another function $f(y,t)$ multiplied to the Dirac Delta function and integrating them we would get

$$\int_{-\infty}^{\infty} \delta(x-y) f(y,t)dy = f(x,t) \int_{-\infty}^{\infty} \delta(x-y)dy = f(x,t)$$

$$\begin{aligned}
\frac{\partial u_p}{\partial t} &= \int_{-\infty}^{\infty} \delta(x-y) f(y,t)dy + \int_0^t \int_{-\infty}^{\infty} c^2 \frac{\partial^2}{\partial x^2} S(x-y,t-s) f(y,s)dyds \\
&= f(x,t) + c^2 \frac{\partial^2}{\partial x^2} \int_0^t \int_{-\infty}^{\infty} S(x-y,t-s) f(y,s)dyds \\
&= f(x,t) + c^2 \frac{\partial^2 u_p}{\partial x^2}
\end{aligned}$$

Which shows that $u_p(x,t)$ solves the in-homogeneous heat equation. It is also clear that

$$\lim_{t\to 0} u_p(x,t) = \lim_{t\to 0} \int_0^t \int_{-\infty}^{\infty} S(x-y,t-s) f(y,s)dyds = 0$$

Therefore $u_p(x,t)$ indeed solves problem (3) which finishes the proof that (6) solves the original IVP (1).

Duhamel's principle

If one can solve an initial value problem for a homogeneous linear differential equation then an in-homogeneous linear differential equation can be solved as well.

For the things of this world cannot be made
Knowm wihtour a knowledge of mathematics.

M.M.ElBorai