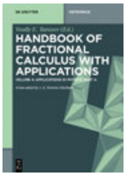
Anatoly Kochubei, Yuri Luchko (Eds.)
Handbook of Fractional Calculus with Applications

Handbook of Fractional Calculus with Applications

Edited by *J. A. Tenreiro Machado*

Volume 1: Theory
Anatoly Kochubei, Yuri Luchko (Eds.), 2019
ISBN 978-3-11-057081-6, e-ISBN (PDF) 978-3-11-057162-2,
e-ISBN (EPUB) 978-3-11-057063-2

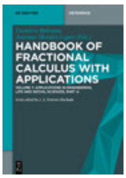
Volume 3: Numerical Methods
George Em Karniadakis (Ed.), 2019
ISBN 978-3-11-057083-0, e-ISBN (PDF) 978-3-11-057168-4,
e-ISBN (EPUB) 978-3-11-057106-6



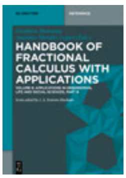
Volume 4: Applications in Physics, Part A
Vasily E. Tarasov (Ed.), 2019
ISBN 978-3-11-057088-5, e-ISBN (PDF) 978-3-11-057170-7,
e-ISBN (EPUB) 978-3-11-057100-4

Volume 5: Applications in Physics, Part B
Vasily E. Tarasov (Ed.), 2019
ISBN 978-3-11-057089-2, e-ISBN (PDF) 978-3-11-057172-1,
e-ISBN (EPUB) 978-3-11-057099-1

Volume 6: Applications in Control
Ivo Petráš (Ed.), 2019
ISBN 978-3-11-057090-8, e-ISBN (PDF) 978-3-11-057174-5,
e-ISBN (EPUB) 978-3-11-057093-9



Volume 7: Applications in Engineering, Life and Social Sciences, Part A
Dumitru Băleanu, António Mendes Lopes (Eds.), 2019
ISBN 978-3-11-057091-5, e-ISBN (PDF) 978-3-11-057190-5,
e-ISBN (EPUB) 978-3-11-057096-0



Volume 8: Applications in Engineering, Life and Social Sciences, Part B
Dumitru Băleanu, António Mendes Lopes (Eds.), 2019
ISBN 978-3-11-057092-2, e-ISBN (PDF) 978-3-11-057192-9,
e-ISBN (EPUB) 978-3-11-057107-3

Anatoly Kochubei, Yuri Luchko (Eds.)

Handbook of Fractional Calculus with Applications

Volume 2: Fractional Differential Equations

Series edited by Jose Antonio Tenreiro Machado

DE GRUYTER

Editors
Prof. Dr. Anatoly Kochubei
National Academy of Sciences of Ukraine
Institute of Mathematics
Tereschenkivska str., 3
Kiev 01004
Ukraine
kochubei@i.com.ua

Prof. Dr. Yuri Luchko
Beuth Hochschule für Technik Berlin
FB II Mathematik-Physik-Chemie
Luxemburger Str. 10
13353 Berlin
Germany
luchko@beuth-hochschule.de

Series Editor
Prof. Dr. Jose Antonio Tenreiro Machado
Department of Electrical Engineering
Instituto Superior de Engenharia do Porto
Instituto Politécnico do Porto
4200-072 Porto
Portugal
jtm@isep.ipp.pt

ISBN 978-3-11-057082-3
e-ISBN (PDF) 978-3-11-057166-0
e-ISBN (EPUB) 978-3-11-057105-9

Library of Congress Control Number: 2018967840

Bibliographic information published by the Deutsche Nationalbibliothek
The Deutsche Nationalbibliothek lists this publication in the Deutsche Nationalbibliografie; detailed bibliographic data are available on the Internet at http://dnb.dnb.de.

Cover image: djmilic / iStock / Getty Images Plus
Typesetting: VTeX UAB, Lithuania
Printing and binding: CPI books GmbH, Leck

www.degruyter.com

Preface

Fractional Calculus (FC) originated in 1695, nearly at the same time as the conventional calculus. However, FC attracted limited attention and remained a pure mathematical exercise in spite of the contributions of important mathematicians, physicists, and engineers. FC had a rapid development during the last few decades, both in mathematics and applied sciences, being nowadays recognized as an excellent tool for describing complex systems, phenomena involving long range memory effects and nonlocality. A huge number of research papers and books devoted to this subject have been published, and presently several specialized conferences and workshops are organized each year. The FC popularity in all fields of science is due to its successful application in mathematical models, namely in the form of FC operators and fractional integral and differential equations. Presently, we are witnessing considerable progress both on theoretical aspects and applications of FC in areas such as physics, engineering, biology, medicine, economy, or finance.

The popularity of FC has attracted many researchers from all over the world and there is a demand for works covering all areas of science in a systematic and rigorous form. In fact, the literature devoted to FC and its applications is huge, but readers are confronted with a high heterogeneity and, in some cases, with misleading and inaccurate information. The Handbook of Fractional Calculus with Applications (HFCA) intends to fill that gap and provides the readers with a solid and systematic treatment of the main aspects and applications of FC. Motivated by these ideas, the editors of the volumes involved a team of internationally recognized experts for a joint publishing project offering a survey of their own and other important results in their fields of research. As a result of these joint efforts, a modern encyclopedia of FC and its applications, reflecting present day scientific knowledge, is now available with the HFCA. This work is distributed by several distinct volumes each one developed under the supervision of its editors.

The second volume of HFCA presents both some basic and advances results in fractional ordinary and partial differential equations. Fractional differential equations (FDE) play an important role in many FC applications. Furthermore, they are of great interest in pure mathematics, as a source of meaningful problems connected with many other branches of mathematics. The first part of the volume is devoted to the fractional ordinary differential equations, related operational methods, and dynamical systems. In particular, symmetries and group invariant solutions of fractional ordinary and partial differential equations are presented. The next part of the volume treats the partial time-FDE and systems of such equations. Among other topics, the Cauchy problems for the FDE and systems of parabolic and hyperbolic type, as well as for equations with more general fractional derivatives are discussed in detail. An important particular case of these equations corresponds to equations of distributed order that are often employed, say, for modeling of the ultraslow diffusion processes.

https://doi.org/10.1515/9783110571660-201

Qualitative properties of the solutions for the Cauchy problems (regularity, long time behavior) are considered. The third part of the volume concerns boundary value problems for FDE and the techniques for studying them (versions of the maximum principle, potential theory, and stability problems). Several chapters of the volume are devoted to inverse problems for the time-fractional partial differential equations, such as the problems of determining coefficients or parameters of the equations. In addition, surveys on linear and nonlinear abstract FDE in Banach spaces are included into the volume. Finally, the space-FDE are also considered including the more general settings like the pseudo-differential operators and extension problems. This kind of equations is important for a variety of subjects ranging from mathematical physics to probability theory.

Our special thanks go to the authors of individual chapters that are excellent surveys of selected classical and new results in several important fields of FC. The editors believe that the HFCA will represent a valuable and reliable reference work for all scholars and professionals willing to develop research in the challenging and timely scientific area.

Anatoly Kochubei and Yuri Luchko

Contents

Kai Diethelm

General theory of Caputo-type fractional differential equations

Abstract: This article describes the fundamentals of the theory of ordinary fractional differential equations of Caputo's type. Starting from the existence and uniqueness of solutions and the well-posedness in general, the flow of topics continues via a derivation of explicit solution formulas for certain important classes of problems and the discussion of their smoothness properties to the stability properties of these solutions. The main focus is on initial value problems, but terminal value problems are briefly considered as well. In addition to dealing with standard single-order problems, the presentation also contains a short discussion of multiterm equations and multiorder systems.

Keywords: Fractional differential equation, Caputo derivative, initial value problem, terminal value problem, existence of solutions, uniqueness of solutions, smoothness, Mittag-Leffler function, stability

MSC 2010: 34A08, 26A33, 34A05, 34A12

1 Introduction and statement of the problem

Ordinary differential equations involving fractional differential operators of Caputo's type are known to have many potential applications in mathematical modeling, for example, in areas like mechanics [1, 10, 18] and in the life sciences [7, 8, 12, 19]. Apart from these (and other) applications outside of the field of mathematics, they are also of interest as mathematical objects in their own right. It is therefore of significant interest to collect some basic information about the solutions of such equations. This article will provide such information. Most of the results will be given without proof; unless otherwise stated, the interested reader may find these proofs in [5, Chapters 6 and 7].

The primary object of this article are differential equations of the form

$$ {}^{C}D^{\alpha} y(t) = f(t, y(t)) \tag{1} $$

with a given $\alpha > 0$ and $\alpha \notin \mathbb{N}$ although some subsections will deal with a slightly more general class of problems. In the formulation (1) and throughout this entire article, the

Acknowledgement: The preparation of this article took place while the author was affiliated to AG Numerik of Institut Computational Mathematics at Technische Universität Braunschweig.

Kai Diethelm, GNS Gesellschaft für numerische Simulation mbH, Am Gaußberg 2, 38114 Braunschweig, Germany; and Fakultät ANG, Hochschule Würzburg-Schweinfurt, Ignaz-Schön-Str. 11, 97421 Schweinfurt, Germany, e-mail: diethelm@gns-mbh.com

https://doi.org/10.1515/9783110571660-001

Caputo differential operator ${}^{\mathrm{C}}D^{\alpha}$ is given by

$${}^{\mathrm{C}}D^{\alpha}z(t) := {}^{\mathrm{RL}}D^{\alpha}[z - T_{\lceil\alpha\rceil-1}z](t)$$

where

$${}^{\mathrm{RL}}D^{\alpha}z(t) := \frac{\mathrm{d}^{\lceil\alpha\rceil}}{\mathrm{d}t^{\lceil\alpha\rceil}} \frac{1}{\Gamma(\lceil\alpha\rceil - \alpha)} \int_0^t (t-s)^{\lceil\alpha\rceil-\alpha-1} z(s)\,\mathrm{d}s$$

is the Riemann–Liouville differential operator and

$$T_n z(t) := \sum_{k=0}^{n} \frac{z^{(k)}(0)}{k!} t^k$$

is the nth degree Taylor polynomial for z, centered at the origin. Hence, ${}^{\mathrm{C}}D^{\alpha}$ is tacitly assumed to be a left-sided differential operator and to have its starting point at $t = 0$, so that one naturally seeks solutions to the differential equation on an interval of the form $[0, T]$ with some $T > 0$.

Our main focus, in particular with respect to the more advanced elements of the theory of such equations, resides on the case $0 < \alpha < 1$, which appears to be the case that is relevant for the vast majority of the applications. Nevertheless, we try to formulate the results in a way that is applicable to general α whenever this is possible without introducing too many technical difficulties.

In the classical theory of integer-order ordinary differential equations, it is well known that unique solutions can only be expected if the differential equation is considered in combination with certain additional conditions. The same observation is true in the fractional case. The question is then where on the t-axis such condition(s) should be imposed.

Traditionally, one has answered this question by choosing the differential operator's starting point, that is, the point 0. Interpreting the free variable t as a time variable, this amounts to providing information at the beginning of the process that the differential equation describes and to seeking the process behavior for times that are in the future of this instant. Thus, one talks about an *initial value problem*; the properties of such problems will be discussed in Sections 2 and 4.

A different type of problems has also received some attention recently: Observing the state of the system at a point $t^* > 0$, that is, augmenting the differential equations with some condition related to a point to the right of the starting point on the time axis, one may be interested in reconstructing the process from the starting point 0 up to this observation point t^*, that is, one seeks a solution on the interval $[0, t^*]$. Since the point t^* at which the information is given is the end point of this interval, a problem of this type is commonly referred to as a *terminal value problem*. Section 3 will be devoted to this class of problems.

2 Initial value problems for single-term equations

Our first object of study will thus be initial value problems for the differential equation (1). Since exactly one differential operator occurs in this equation, this type of equations is known as a *single-term fractional differential equation*.

It turns out that, due to the differential operators being defined in Caputo's sense, the most natural way of formulating the initial conditions is exactly the same way that is well known from the classical theory of integer-order differential equations. The initial value problem to be discussed hence reads

$$^{\mathrm{C}}D^{\alpha} y(t) = f(t, y(t)), \tag{2a}$$

$$y^{(k)}(0) = y_0^{(k)}, \quad k = 0, 1, \dots, \lceil \alpha \rceil - 1. \tag{2b}$$

This formulation works both for integer and for noninteger values of $\alpha > 0$, and as such it encompasses the classical theory as a special case. Indeed, many (but not all) classical results (and their proofs) can be generalized to this fractional setting in a relatively straightforward manner.

We explicitly remark that the validity of this approach for handling initial value problems strongly depends on the fact that *Caputo-type* differential operators are used; if we had chosen, for example, Riemann–Liouville derivatives instead, then we would have to use initial conditions of a different structure; cf., for example, the brief discussion in [5, Chapter 5] or the very detailed exposition in [16, Section 42.4].

Almost all results in this section are formulated for d-dimensional systems of equations with arbitrary $d \in \mathbb{N}$, that is, we assume the function y in (2) to map an interval $[0, T]$ to $\mathbb{R}^d$. Consequently, the initial values $y_0^{(k)}$ are vectors in $\mathbb{R}^d$, and the function f is assumed to map a subset of $[0, T] \times \mathbb{R}^d$ to $\mathbb{R}^d$.

2.1 Existence and uniqueness of solutions

The most important results in the classical theory, Peano's existence theorem and the Picard–Lindelöf uniqueness theorem, remain valid in the fractional setting too. Their proofs are based on an equivalence between the initial value problem and a Volterra integral equation. Even though we shall not provide these proofs explicitly, we shall formally state this equivalence because it has useful applications in many other areas of the theory and the numerical treatment of fractional differential equations. Evidently, the special case $\alpha \in \mathbb{N}$ of this property also reduces to the well-known result from the classical theory.

Lemma 1. *Consider the initial value problem* (2) *and assume that the function* $f : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ *with some* $T > 0$ *is continuous. Then the function* $y \in C[0, T]$ *is a solution of this initial value problem if and only if it is a solution of the nonlinear Volterra integral* *equation of the second kind*

$$y(t) = \sum_{k=0}^{\lceil\alpha\rceil-1} \frac{t^k}{k!} y_0^{(k)} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y(s))\, \mathrm{d}s. \tag{3}$$

Based on this equivalence result, it is possible to prove—using Schauder's fixed-point theorem—the fractional analogue of Peano's existence theorem: Continuity and boundedness of the given function f on the right-hand side of the differential equation (2a) suffice to assert the existence of a continuous solution to the initial value problem (2).

Here and in the following, $\|\cdot\|$ denotes an arbitrary norm on $\mathbb{R}^d$.

Theorem 1 (Peano's existence theorem). *Let* $0 < \alpha$ *and* $m = \lceil\alpha\rceil$. *Moreover, let* $y_0^{(0)}, \ldots, y_0^{(m-1)} \in \mathbb{R}^d$, $K > 0$ *and* $T^* > 0$. *Assume that either*
(a) $G = [0, T^*] \times \mathbb{R}^d$ *or*
(b) $G = \{(t, y) : t \in [0, T^*], \|y - \sum_{k=0}^{m-1} t^k y_0^{(k)}/k!\| \le K\}$

and that $f : G \to \mathbb{R}^d$ *is continuous and bounded, with* $M := \sup_{(t,z)\in G} |f(t, z)|$. *Furthermore, let*

$$T := \begin{cases} T^* & \text{if } M = 0 \text{ or } G = [0, h^*] \times \mathbb{R}^d, \\ \min\{T^*, (K\Gamma(\alpha+1)/M)^{1/\alpha}\} & \text{else.} \end{cases} \tag{4}$$

Then there exists a function $y \in C[0, T]$ *solving the initial value problem* (2).

Note that the set G is compact in case (b). Hence, in this case the boundedness of f on G is an immediate consequence of its continuity and does not need to be shown explicitly.

The classical Picard–Lindelöf theorem can be generalized to the fractional setting in the same way: If the given function f on the right-hand side of the differential equation (2a) is continuous and bounded and satisfies a Lipschitz condition with respect to the (d-dimensional) second variable, then uniqueness of the continuous solution to the initial value problem (2) can be guaranteed.

Theorem 2 (Picard–Lindelöf uniqueness theorem). *Assume the hypotheses of Theorem* 1. *Moreover, let* f *fulfill a Lipschitz condition with respect to the second variable, that is,*

$$\|f(t, y_1) - f(t, y_2)\| \le L\|y_1 - y_2\|$$

with some constant $L > 0$ *independent of* t, y_1, *and* y_2. *Then there exists a uniquely defined function* $y \in C[0, T]$ *solving the initial value problem* (2), *where* T *is once again given according to* (4).

The example initial value problem

$$ {}^{\mathrm{C}}D^{\alpha}y(t) = \big(y(t)\big)^{\mu}, \quad y(0) = 0, \tag{5} $$

with arbitrary $\alpha \in (0,1]$ and $\mu \in (0,1)$ does not satisfy the Lipschitz condition mentioned in Theorem 2. Since it has the solutions

$$ y(t) = 0 \quad \text{and} \quad y(t) = \left(\frac{\Gamma(j+1-\alpha)}{\Gamma(j+1)} \right)^{1/(1-\mu)} t^{j} $$

with $j = \alpha/(1-\mu)$, it shows that we must expect nonuniqueness if the Lipschitz condition is omitted without any replacement.

However, it is possible to recover the uniqueness if the Lipschitz condition is replaced by a different suitable condition. For a one-dimensional problem, a possible choice in the case $0 < \alpha < 1$ that is also known from the classical theory is Nagumo's condition which leads to the following result taken from [6].

Theorem 3 (Nagumo's uniqueness theorem). *Let $\alpha \in (0,1)$, $T > 0$ and $y_0^{(0)} \in \mathbb{R}$. If the function $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ is continuous at $(0, y_0^{(0)})$ and satisfies the inequality*

$$ t^{\alpha}\big|f(t,y_1) - f(t,y_2)\big| \le \Gamma(\alpha+1)|y_1 - y_2| \tag{6} $$

for all $t \in [0,T]$ and all $y_1, y_2 \in \mathbb{R}$, then the initial value problem (2) *has at most one continuous solution y on $[0,T]$ satisfying ${}^{\mathrm{C}}D^{\alpha}y \in C[0,T]$.*

Theorem 3 is a pure uniqueness statement; it does not claim the existence of a solution. This limitation is caused by the fact that the hypotheses require the function f to be continuous only at the single point $(0, y_0^{(0)})$ but not—as in Theorem 1—throughout its entire domain of definition.

2.2 Explicit solutions for linear equations

In the particular case where the given differential equation is linear, it is often possible to write up the solutions in closed form. Specifically, an explicit computation using the power series expansion of the one-parameter Mittag-Leffler function E_{α}, defined by

$$ E_{\alpha}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, $$

(cf. [14]) reveals the following result.

Theorem 4. *For arbitrary $\lambda \in \mathbb{C}$ and $y_0 \in \mathbb{C}$, the function $y : [0,\infty) \to \mathbb{C}$ with*

$$ y(t) := y_0 E_{\alpha}(\lambda t^{\alpha}) $$

is the unique solution to the initial value problem

$$ {}^{\mathrm{C}}D^{\alpha}y(t) = \lambda y(t), \quad y(0) = y_0, \; y^{(k)}(0) = 0 \quad (k = 1, 2, \ldots, \lceil \alpha \rceil - 1). $$

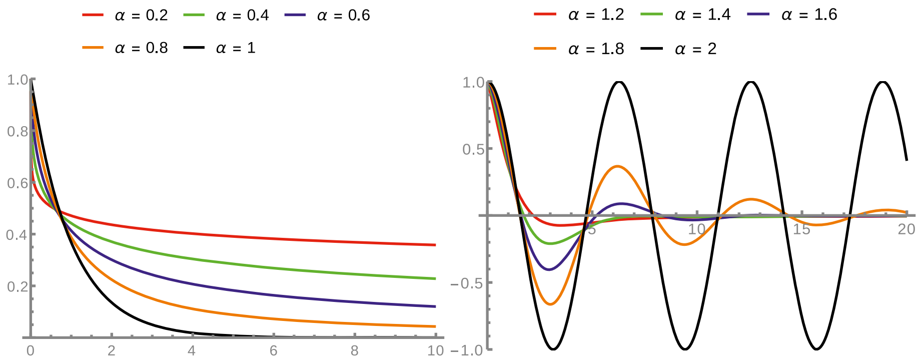


Figure 1: Plots of $y(t) = E_\alpha(-t^\alpha)$ for various $\alpha \in (0,1]$ (left) and $\alpha \in (1,2]$ (right).

In view of the identities $E_1(-z) = \exp(-z)$ and $E_2(-z^2) = \cos z$, the cases $\alpha = 1$ and $\alpha = 2$ of Theorem 4 reduce to the well-known statements that the exponential function and the cosine solve the given first- and second-order initial value problems, respectively, for $\lambda = -1$. This observation indicates that the solutions to the general problem of order α decay in a monotonic way for $\alpha = 1$ and exhibit persistent oscillations for $\alpha = 2$ if λ is a negative real number. One may therefore wonder which behavior to expect for these λ in the cases $1 < \alpha < 2$ and $0 < \alpha < 1$. The associated results are illustrated in Figure 1 and formally summarized in Theorem 5; cf. [14, Proposition 3.23] and [15, Section 4].

Theorem 5. *Let $\mu > 0$ and $y(t) = E_\alpha(-\mu t)$.*
(a) *For $0 < \alpha \le 1$, the function y is completely monotonic on $(0,\infty)$, that is, $(-1)^k D^k y(t) \ge 0$ for every $t > 0$ and every $k \in \mathbb{N}_0$. In particular, y does not have any zeros on $[0,\infty)$.*
(b) *For any fixed $\alpha \in (1,2)$, denote by Z_α the number of zeros of y on $[0,\infty)$ (counting multiplicities). Then, for all α, Z_α is finite and odd. Moreover, for $\alpha \to 2-$ we have $Z_\alpha = 12\pi^{-2}(2-\alpha)^{-1}(1+o(1))\ln(2-\alpha)^{-1}$.*

The last statement of part (b) in particular implies that $\lim_{\alpha\to 2-} Z_\alpha = \infty$ as expected from the observation that the function y becomes a cosine in this limit case.

Theorem 4 only addresses homogeneous linear differential equations with constant coefficients. Much as in the case of first-order equations, it can however also be used as the basis for a corresponding result for inhomogeneous equations, namely, the *variation of constants* formula.

Theorem 6. *Let $\alpha > 0$ and $\lambda \in \mathbb{C}$. The solution of the initial value problem*

$$ {}^{\mathrm{C}}D^\alpha y(t) = \lambda y(t) + q(t), \quad y^{(k)}(0) = y_0^{(k)} \quad (k = 0, 1, \ldots, \lceil\alpha\rceil - 1), $$

where $q \in C[0,h]$ is a given function, can be expressed in the form

$$ y(t) = \sum_{k=0}^{m-1} y_0^{(k)} u_k(t) + \tilde{y}(t) $$

with

$$\tilde{y}(t) = \begin{cases} I^\alpha q(t) & \text{if } \lambda = 0, \\ \frac{1}{\lambda}\int_0^t q(t-s)u_0'(s)\,\mathrm{d}s & \text{if } \lambda \neq 0, \end{cases}$$

where $u_k(t) := I^k e_\alpha(t)$, $k = 0, 1, \ldots, \lceil \alpha \rceil - 1$, *and* $e_\alpha(t) := E_\alpha(\lambda t^\alpha)$.

An alternative representation is available that has certain advantages over the form given in Theorem 6 in the sense that (a) there is no need to distinguish between the cases $\lambda = 0$ and $\lambda \neq 0$, and (b) it cannot only handle inhomogeneous linear equations but also a certain class of nonlinear equations [2].

Theorem 7. *Let* $0 < \alpha < 1$ *and* $\lambda \in \mathbb{C}$. *The solution of the initial value problem*

$$^{\mathrm{C}}D^\alpha y(t) = \lambda y(t) + f(y(t)), \quad y(0) = y_0,$$

where $f : \mathbb{C} \to \mathbb{C}$ *is assumed to satisfy a global Lipschitz condition satisfies the identity*

$$y(t) = y_0 E_\alpha(\lambda t^\alpha) + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(\lambda(t-s)^\alpha) f(y(s))\,\mathrm{d}s.$$

The function $E_{\alpha,\alpha}$ that appears in this theorem is a special case of the two-parameter Mittag-Leffler function [14]

$$E_{\alpha,\beta}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}$$

where both parameters coincide with each other.

2.3 Well-posedness

Traditionally, a problem is called *well-posed* if it has the following three properties:
- A solution exists;
- the solution is unique;
- the solution depends on the given data in a continuous way.

The first two aspects have already been discussed in the previous sections; the third one requires further attention. In this connection, one important difference between the fractional and the classical setting must be pointed out, it being the precise meaning of the expression “the given data”: In the classical theory, one usually assumes the initial values and the function f on the right-hand side of the differential equation

$$D^k y(x) = f(x, y(x), Dy(x), \ldots, D^{k-1} y(x))$$

to be given, and then the behavior of the solution under perturbations of these expressions is discussed. In the fractional setting, however, it is additionally possible to perturb the order α of the differential equation, and so this novel feature must be taken into account as well. This is particularly relevant in connection with tasks such as, for example, the mathematical modeling of viscoelastic materials where α can be interpreted as a material constant that is known only up to a limited accuracy.

We thus have to compare the solution to the initial value problem (2) with the solution to its perturbed form

$$ {}^{\mathrm{C}}D^{\tilde{\alpha}}\tilde{y}(t) = \tilde{f}(t,\tilde{y}(t)), \quad \tilde{y}^{(k)}(0) = \tilde{y}_0^{(k)}, \quad k = 0,1,\ldots,\lceil\tilde{\alpha}\rceil - 1. \tag{7}$$

In this comparison, we must take care because the number of initial conditions imposed in (2) may differ from the number of conditions in (7); in fact, this happens when the values $\lceil\alpha\rceil$ and $\lceil\tilde{\alpha}\rceil$ are not identical. If such a situation occurs, we shall—without loss of generality—assume that $\tilde{\alpha} > \alpha$.

Fortunately, the usual conditions, that is, the conditions that we have already encountered in the fractional version of the Picard–Lindelöf uniqueness theorem (Theorem 2), suffice to assert the desired well-posedness properties:

Theorem 8. *Consider the initial value problems* (2) *and* (7)*, and assume that both problems satisfy the assumptions of Theorem* 2*. Then there exists an interval* $[0, T^*]$ *where both problems have continuous solutions* y *and* $\tilde{y}$*, respectively, and there holds*

$$\sup_{t\in[0,T^*]} \|y(t) - \tilde{y}(t)\| \leq O\Big(\sup_{(t,z)\in G} \|f(t,z) - \tilde{f}(t,z)\|\Big) + O(|\alpha - \tilde{\alpha}|) + O\left(\sum_{k=0}^{\lceil\alpha\rceil-1} \|y_0^{(k)} - \tilde{y}_0^{(k)}\|\right) + O\Big(\max_{\lceil\alpha\rceil\leq k<\lceil\tilde{\alpha}\rceil} \|\tilde{y}_0^{(k)}\|\Big).$$

In the inequality in Theorem 8, the last summand on the right-hand side contains the additional initial conditions that are present in the initial value problem (7) but not in (2); it must be interpreted as 0 if $\lceil\alpha\rceil = \lceil\tilde{\alpha}\rceil$, that is, if both initial value problems have the same number of initial conditions.

The key ingredient in the proof of Theorem 8 is a fractional version of Gronwall's lemma that is frequently useful in other contexts as well and that thus is of interest in its own right. It reads as follows.

Lemma 2 (Gronwall's inequality). *Let* $\alpha, T, \epsilon_1, \epsilon_2 > 0$*. Moreover, assume that* $\delta : [0,T] \to \mathbb{R}$ *is a continuous function satisfying the inequality*

$$|\delta(t)| \leq \epsilon_1 + \frac{\epsilon_2}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |\delta(s)| \, \mathrm{d}s$$

for all $t \in [0, T]$. *Then*

$$|\delta(t)| \le \epsilon_1 E_\alpha(\epsilon_2 t^\alpha)$$

for $t \in [0, T]$.

As above, E_α denotes the one-parameter Mittag-Leffler function of order α.

2.4 Smoothness of solutions

One aspect with respect to which there are significant differences between classical differential equations of integer order and their fractional counterparts is the smoothness of their solutions.

While the solution y to a first-order initial value problem

$$D^1 y(t) = f(t, y(t)), \quad y(0) = y_0,$$

is in $C^k[0, T]$ for some T if the given function f on the right-hand side of the differential equation is in C^{k-1}, similar smoothness results for the solutions to differential equations of fractional order generally hold only on the *half-open* interval $(0, T]$, that is, some of the smoothness is lost at the initial point.

Theorem 9. *Consider the initial value problem* (2) *under the assumptions of Theorem* 2. *Moreover, let* $k \in \mathbb{N}_0$ *and assume that* $\alpha > k$, $\alpha \notin \mathbb{N}$ *and* $f \in C^k(G)$. *Then* $y \in C^{\lceil\alpha\rceil+k-1}(0, T] \cap C^{\lceil\alpha\rceil-1}[0, T]$.

Furthermore, for $\ell = 1, 2, \dots, k$, *the solution satisfies* $y \in C^{\lceil\alpha\rceil+\ell-1}[0, T]$ *if and only if the function* z *with* $z(t) := f(t, y(t))$ *has an* ℓ*-fold zero at the origin.*

This result tells us that the solution y must be expected to be nonsmooth at the initial point unless a very restrictive condition is satisfied. This condition is formulated in terms of the given function f on the right-hand side of the differential equation and the initial values that are known as well. Thus, in a concrete application scenario, it is possible to explicitly verify whether or not the condition holds. If the outcome of this process is the nonsmoothness of the solution, then it is also possible to precisely describe the form that the nonsmooth part of $y(t)$ takes in an asymptotic sense as $t \to 0+$.

Theorem 10. *Assume the hypotheses of Theorem* 9. *Then, for* $\ell = \lceil\alpha\rceil, \lceil\alpha\rceil+1, \dots, \lceil\alpha\rceil+k-1$ *we have* $y^{(\ell)}(t) = O(t^{\alpha-\ell})$ *as* $t \to 0+$.

These facts are often summarized in the statement that it is not possible for a truly fractional differential equation, that is, a differential equation of order $\alpha \notin \mathbb{N}$, to have a smooth given function f and a smooth solution y simultaneously. This statement is not completely accurate though, in fact there is one significant exception (which can, in some sense, be seen as a pathological case).

Theorem 11. *Consider the initial value problem* (2) *and assume that f is analytic on a suitable set G. Define $T(t) := \sum_{j=0}^{\lceil\alpha\rceil-1} y_0^{(j)} t^j/j!$. Then y is analytic if and only if $f(t, T(t)) = 0$ for all t.*

The nonsmooth behavior near the initial point has a significant influence on the construction and analysis of numerical methods for such equations. In particular, if one assumes more smoothness in a convergence analysis than the theory provides in general, then—as seen from Theorem 9—this amounts to severely restricting the set of initial values for which the error analysis is valid [17].

2.5 Stability

In many important situations, the solutions to the initial value problem (2) exist on the unbounded interval $[0, \infty)$. In such a case, it is often required to investigate the properties of the solution in the very long run, that is, to determine the behavior of $y(t)$ as $t \to \infty$. Usually, the following questions are then particularly important:
- Does $y(t)$ remain bounded as $t \to \infty$?
- Does $y(t)$ converge to 0 as $t \to \infty$?
- If $y(t)$ converges to 0 as $t \to \infty$, what is the rate of convergence?

The common practice in the investigation of such questions for first-order differential equations is to start with d-dimensional systems of linear equations and then to gradually extend the results to larger classes of problems. In this context, a fundamental role is played by the matrix exponential function; the corresponding matrix-valued Mittag-Leffler functions take over this role in the fractional setting. The theory can be most naturally developed if all functions are assumed to be complex valued.

Theorem 12. *Let $0 < \alpha \leq 1$, and let $f : [0, \infty) \to \mathbb{C}^d$ be a continuous function with the property $\lim_{t\to\infty} \|f(t)\| = 0$. Given an arbitrary constant $(d \times d)$-matrix Λ, consider the d-dimensional fractional differential equation system*

$$ {}^{C}D^{\alpha} y(t) = \Lambda y(t) + f(t). $$

(a) *For all $y_0 \in \mathbb{C}^d$, there exists a continuous solution $y : [0, \infty) \to \mathbb{C}^d$ to this differential equation that satisfies the initial condition $y(0) = y_0$.*

(b) *The following two statements are equivalent:*
  (i) *All eigenvalues λ_j ($j = 1, 2, \dots, d$) of Λ satisfy $|\arg \lambda_j| > \alpha\pi/2$.*
  (ii) *All solutions y of the system satisfy $\lim_{t\to\infty} y(t) = 0$.*

(c) *The following two statements are equivalent:*
  (i) *All eigenvalues of Λ satisfy $|\arg \lambda_j| \geq \alpha\pi/2$ and all eigenvalues with $|\arg \lambda_j| = \alpha\pi/2$ have a geometric multiplicity that coincides with their algebraic multiplicity (i. e., an eigenvalue that is an ℓ-fold zero of the characteristic polynomial has ℓ linearly independent eigenvectors).*

(ii) *All solutions y of the system remain bounded on* $[0, \infty)$.

(d) *If the matrix Λ has an eigenvalue λ_j with $|\arg \lambda_j| < \alpha\pi/2$, then there exists a unique initial value $y_0^* \in \mathbb{C}^d$ such that the solution y of the differential equation subject to the initial condition $y(0) = y_0^*$ is bounded on* $[0, \infty)$. *The solutions to the differential equation subject to all other initial values are unbounded.*

If one of the conditions of part (b) is satisfied, we say that the system is *asymptotically stable*; if one of the conditions of (c) holds then we say that the system is *stable*. The situation of case (d) is known as an *unstable* system.

The key message of Theorem 12 is that the stability properties of a given system of fractional differential equations can be computed using an approach that is essentially the same as the one used in the classical case, but with the additional feature that the location of the border between the stable and the unstable regime depends on the order α of the differential equation; specifically, the size of the stable region increases as α decreases.

Regarding the rate of convergence in the case of asymptotic stability, we restrict the attention to homogeneous linear equations with constant coefficients. It is well known that an exponential rate can be found in the classical case $\alpha = 1$. However, as already indicated in the plots in Figure 1, the convergence takes place at a much slower speed if α is not an integer.

Theorem 13. *Let $\alpha > 0$, let Λ be a complex $(d \times d)$-matrix, and let y be the solution to the initial value problem*

$$^{\mathrm{C}}D^{\alpha} y(t) = \Lambda y(t), \quad y(0) = y_0,\ y^{(k)}(0) = 0 \quad \text{for } k = 1, 2, \dots, \lceil \alpha \rceil - 1$$

with arbitrary $y_0 \in \mathbb{C}^d$.

(a) *For all $t \ge 0$, we have $y(t) = E_\alpha(\Lambda t^\alpha) y_0$.*

(b) *If $0 < \alpha < 1$ or $1 < \alpha < 2$ and all eigenvalues λ_j of Λ satisfy $|\arg \lambda_j| > \alpha\pi/2$, then we have*

$$\|y(t)\| = c t^{-\alpha}(1 + o(1)) \quad \text{for } t \to \infty$$

with a certain constant c; in particular, $c \neq 0$ if $y_0 \neq 0$.

2.6 Separation of solutions

Another classical result from the theory of first-order differential equations states that the graphs of two solutions of the same differential equation that satisfy different initial conditions can never meet or cross each other if the given function f satisfies a Lipschitz condition. This statement is indeed valid only for first-order problems, that is, for problems with exactly one initial condition, and not for higher-order initial value

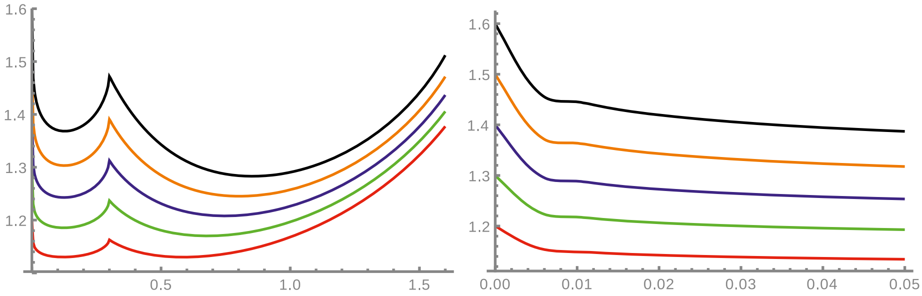


Figure 2: Graphs of solutions to the differential equation ${}^{C}D^{0.28}y(t) = \sqrt{|0.3 - t|}\sin 3y(t) + 0.3t^3$ with initial conditions $y(0) = 1.2$ (red), $y(0) = 1.3$ (green), $y(0) = 1.4$ (blue), $y(0) = 1.5$ (orange) and $y(0) = 1.6$ (black), plotted over the interval $[0, 1.6]$ (left) and zoom of this picture to the interval $[0, 0.05]$ (right). It can be observed that the graphs never meet or cross each other.

problems with two or more initial conditions. Thus it is not surprising that a similar result for fractional differential equations can also only be shown for initial value problems with exactly one initial condition, that is, for problems with an order $\alpha \in (0, 1)$, cf. [3, 9]. The result reads as follows.

Theorem 14. *Consider the equation* (2a) *for some* $\alpha \in (0, 1)$ *and assume that the function* f *satisfies the conditions of the Picard–Lindelöf theorem* (*Theorem* 2). *Then, for any two different initial values* $y_{10}, y_{20} \in \mathbb{R}$, *there exists a common nonempty interval* $[0, T^*]$ *on which the two associated initial value problems have a solution. The trajectories of these solutions to* (2a) *do not meet on* $[0, T^*]$. *More precisely, denoting the solutions to the two initial value problems by* y_1 *and* y_2, *respectively,*

$$|y_1(t) - y_2(t)| \geq |y_{10} - y_{20}| \cdot E_\alpha(-Lt^\alpha)$$

holds for each $t \in [0, T^*]$, *where* L *is the Lipschitz constant for the given function* f *with respect to its second variable.*

This property will turn out to be very important in the investigation of terminal value problems; cf. Section 3. It can be nicely explained by the visualization indicated in Figure 2.

One needs to stress that the result of Theorem 14 differs significantly from most of the other statements in this text in the sense that it is known to hold only for scalar differential equations; for multidimensional systems of equations, counterexamples are known [3].

3 Terminal value problems for single-term equations

In certain applications, it is undesirable or impossible to measure the state of a system modeled with a fractional differential equation at the starting point of the fractional operator. One is then forced to determine the system state at a certain point in time $t^* > 0$, say. The search for the model equation's solution y then requires to combine the given differential equation with the condition

$$y(t^*) = y^*$$

instead of the initial condition considered in the previous sections.

For the sake of simplicity, we restrict our attention to the case where exactly one such condition is imposed; this naturally leads to the consideration of differential equations of order $\alpha \in (0, 1)$, and one needs to discuss under which conditions one may expect such a *terminal value problem* to have a unique solution. In fact, the notion "terminal" stems from the fact that one is typically interested in the solution on the interval $[0, t^*]$, that is, the data are given at the terminal point of the interval of interest. This does not preclude the search for the solution to the right of the point t^*; however, if one can manage to find the solution on $[0, t^*]$ then the system's state at $t = 0$ is known and can be used to define a standard initial condition that then replaces the terminal condition, so that the new initial value problem can be solved on an interval $[0, T]$ with some $T > t^*$ by the methods described in the previous section. Thus, in the context of terminal value problems it suffices to work on the interval $[0, t^*]$. In this context, it is of interest to derive an analog of the integral equation formulation of the initial value problem that Lemma 1 provided. This result looks as follows.

Lemma 3. *Let $t^* > 0$ and $0 < \alpha < 1$, and assume $f : [0, t^*] \times \mathbb{R} \to \mathbb{R}$ to be continuous and satisfy a Lipschitz condition with respect to the second variable. Then the differential equation*

$$^{C}D^{\alpha} y(t) = f(t, y(t)) \tag{8}$$

subject to the condition

$$y(t^*) = y^* \tag{9}$$

is equivalent to the weakly singular integral equation

$$y(t) = y^* + \frac{1}{\Gamma(\alpha)} \int_0^{t^*} G(t, s) f(s, y(s))\, ds$$

where

$$G(t, s) = \begin{cases} -(t^* - s)^{\alpha - 1} & \textit{for } s > t, \\ (t - s)^{\alpha - 1} - (t^* - s)^{\alpha - 1} & \textit{for } s \le t. \end{cases}$$

The substantial difference between the integral equation of Lemma 3 and the integral equation derived in Lemma 1 in the case $t^* = 0$ is that we now have a *Fredholm* integral equation of Hammerstein type whereas the initial value problem gave rise to a *Volterra* equation. Thus, in analogy with the corresponding results for integer-order equations, it would be natural to interpret the terminal condition as a boundary condition and not an initial condition. Hence, in contrast to the situation observed for first-order differential equations, the terminal value problem consisting of equations (8) and (9) is much more closely related to a boundary value problem than it is to an initial value problem.

In spite of this difference, it is possible to provide an existence and uniqueness theorem for terminal value problems that formally coincides with the corresponding result for initial value problems [9].

Theorem 15. *Under the assumptions of Lemma* 3*, the differential equation* (8) *subject to the terminal condition* (9) *has exactly one continuous solution on* $[0, t^*]$.

This result is strongly related to our Theorem 14 above. Indeed, like Theorem 14, Theorem 15 also holds in the case of a scalar problem only, but not necessarily when the functions are assumed to be vector-valued.

4 Equations with more than one differential operator

The final considerations of this article will be devoted to differential equations that contain two or more differential operators of different orders. In this context, it is helpful to distinguish two instances of the problem: In a *multiterm fractional differential equation*, the operators of different orders appear in just one differential equation, or in at least one equation of an equation system, whereas a *multiorder system of fractional differential equations* is a (coupled) system of fractional differential equations where each individual equations contains only one differential operator, but where the orders of these operators may differ from one equation to the next one. The following two subsections will address these two types of problems.

4.1 Multiterm equations

Some applications require the use of mathematical models based on differential equations that contain more than one fractional differential operator. Such equations, having the form

$$^{\mathrm{C}}D^{\alpha_N} y(t) = f(t, y(t), {}^{\mathrm{C}}D^{\alpha_1} y(t), {}^{\mathrm{C}}D^{\alpha_2} y(t), \dots, {}^{\mathrm{C}}D^{\alpha_{N-1}} y(t)) \tag{10a}$$

with $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_N$, are usually called *multiterm equations*. This form is not the most general possible form since it can only be used if the differential equation can be explicitly solved for the highest derivative of y, but it is sufficient for the vast majority of use cases and it avoids severe technical difficulties. Differential equations of the form (10a) are usually combined with initial conditions

$$y^{(k)}(0) = y_0^{(k)}, \quad k = 0, 1, \ldots, \lceil \alpha_N \rceil - 1. \tag{10b}$$

In this formulation, it is convenient to assume—without loss of generality— that all integers in the interval $(0, \alpha_N)$ are contained in the finite sequence $(\alpha_1, \alpha_2, \ldots, \alpha_N)$.

A direct approach for solving such an equation or for investigating its properties is usually possible only if the equation is linear, so that Laplace transform techniques can be applied. In the general case, it is advisable to follow an indirect path that consists of first transforming the multiterm equation (10a) together with its initial conditions (10b) into an equivalent multiorder system and then to apply the techniques of Subsection 4.2 below to investigate this system. The required transformation can be implemented in various different ways; the following theorems provide two viable options [4, 13]. For the first approach, we use the notation

$$\beta_j := \begin{cases} \alpha_1 & \text{for } j = 1, \\ \alpha_j - \alpha_{j-1} & \text{else.} \end{cases}$$

Our assumption on the α_j above then implies that $\beta_j \leq 1$ for all j.

Theorem 16. *Subject to the conditions above, the multiterm equation* (10a) *with initial conditions* (10b) *is equivalent to the system*

$$\begin{aligned} {}^{\mathrm{C}}D^{\beta_1} y_1(t) &= y_2(t), \\ {}^{\mathrm{C}}D^{\beta_2} y_2(t) &= y_3(t), \\ &\vdots \\ {}^{\mathrm{C}}D^{\beta_{N-1}} y_{N-1}(t) &= y_N(t), \\ {}^{\mathrm{C}}D^{\beta_N} y_N(t) &= f(t, y_1(t), y_2(t), \ldots, y_N(t)) \end{aligned} \tag{11a}$$

with the initial conditions

$$y_j(0) = \begin{cases} y_0^{(0)} & \textit{if } j = 1, \\ y_0^{(\ell)} & \textit{if } \alpha_{j-1} = \ell \in \mathbb{N}, \\ 0 & \textit{else} \end{cases} \tag{11b}$$

in the following sense:

1. *Whenever the function $y \in C^{\lceil \alpha_N \rceil}[0, T]$ is a solution of the multiterm equation* (10a) *with initial conditions* (10b), *the vector-valued function $Y := (y_1, \ldots, y_N)^{\mathrm{T}}$ with*

$$y_j(t) := \begin{cases} y(t) & \textit{if } j = 1, \\ {}^{\mathrm{C}}D^{\alpha_{j-1}} y(t) & \textit{if } j \geq 2, \end{cases} \tag{12}$$

is a solution of the multiorder fractional differential system (11a) *with initial conditions* (11b).

2. *Whenever the vector-valued function* $Y := (y_1, \dots, y_N)^{\mathrm{T}}$ *is a solution of the multiorder fractional differential system* (11a) *with initial conditions* (11b), *the function* $y := y_1$ *is a solution of the multiterm equation* (10a) *with initial conditions* (10b).

For a general formal description of the second method, it is helpful to express the multiterm equation in the form

$$\begin{aligned}{}^{\mathrm{C}}D^{N+\delta_{N,\ell_N}} y(t) = f\big(t, & D^0 y(t), {}^{\mathrm{C}}D^{\delta_{0,1}} y(t), \dots, {}^{\mathrm{C}}D^{\delta_{0,\ell_0}} y(t), \\ & D^1 y(t), {}^{\mathrm{C}}D^{1+\delta_{1,1}} y(t), \dots, {}^{\mathrm{C}}D^{1+\delta_{1,\ell_1}} y(t), \dots, \\ & D^N y(t), \dots, {}^{\mathrm{C}}D^{N+\delta_{N,\ell_N-1}} y(t)\big),\end{aligned} \tag{13a}$$

where $0 < \delta_{j,1} < \delta_{j,2} < \dots < \delta_{j,\ell_j} < 1$ for all j. The corresponding initial conditions are then

$$y_j(0) = y_0^{(j)}, \quad j = 0, 1, \dots, N. \tag{13b}$$

In order to achieve our goal, we define

$$s(\mu, \sigma) := \sigma + \mu + 1 + \sum_{j=0}^{\mu-1} \ell_j \quad \text{and} \quad M := s(N, \ell_N) - 1 = N + \sum_{j=0}^{N} \ell_j.$$

Using this notation, we come to the following statement.

Theorem 17. *The multiterm initial value problem* (13) *is equivalent to the* M*-dimensional system*

$$\begin{aligned}{}^{\mathrm{C}}D^{\delta_{\mu,\sigma}} y_{s(\mu,0)}(t) &= y_{s(\mu,\sigma)}(t), \quad \mu = 0, 1, \dots, N, \quad \sigma = 1, 2, \dots, \widehat{\sigma}_\mu, \\ D^1 y_{s(\mu,0)}(t) &= y_{s(\mu+1,0)}(t), \quad \mu = 0, 1, \dots, N-1, \\ {}^{\mathrm{C}}D^{\delta_{N,\ell_N}} y_{s(N,0)}(t) &= f(t, y_1(t), y_2(t), \dots, y_M(t))\end{aligned} \tag{14a}$$

where $\widehat{\sigma}_\mu := \ell_\mu$ *if* $0 \le \mu < N$ *and* $\widehat{\sigma}_N := \ell_N - 1$, *with the initial conditions*

$$y_j(0) = \begin{cases} y_0^{(k)} & \textit{if there exists } k \textit{ such that } j = s(k, 0), \\ 0 & \textit{else} \end{cases} \tag{14b}$$

in the following sense:

1. *Whenever the function* $y \in C^{N+1}[0, T]$ *is a solution of the multiterm equation* (13a) *with initial conditions* (13b), *the vector-valued function* $Y := (y_1, \dots, y_M)^{\mathrm{T}}$ *with*

$$y_{s(\mu,\sigma)}(t) := \begin{cases} D^\mu y(t) & \textit{for } \sigma = 0, \\ {}^{\mathrm{C}}D^{\mu+\delta_{\mu,\sigma}} y(t) & \textit{for } \sigma = 1, 2, \dots, \widehat{\sigma}_\mu, \end{cases} \quad \mu = 0, 1, \dots, N, \tag{15}$$

is a solution of the multiorder system (14a) *with initial conditions* (14b).

2. *Whenever the vector-valued function* $Y := (y_1, \dots, y_M)^{\mathrm{T}}$ *is a solution of the multiorder system* (14a) *with initial conditions* (14b), *the function* $y := y_1$ *is a solution of the multiterm equation* (13a) *with initial conditions* (13b).

Notice that M is the total number of differential operators of strictly positive order in (13a). Thus, the approaches of Theorems 16 and 17 lead to systems of identical (and, in general, relatively low) dimensions. However, both approaches generally lead to multiorder systems, and those systems tend to be more cumbersome to analyze and investigate than single-order systems. Therefore, a third approach shall be presented that, under certain additional conditions, can provide a single-order system that is equivalent to a given multiterm differential equation. Unfortunately, the advantage that the simpler structure introduces comes with the price of a (possibly very severe) increase of the system's dimension. Thus, in each concrete situation, one must carefully deliberate which choice is more practical.

Theorem 18. *Consider the equation* (10a) *subject to the initial conditions* (10b) *under the assumption that all* α_j *are rational numbers. Then define* μ *to be the least common multiple of the denominators of* $\alpha_1, \alpha_2, \dots, \alpha_N$ *and set* $\gamma := 1/\mu$ *and* $M := \mu\alpha_N$. *Then this initial value problem is equivalent to the system of equations*

$$\begin{aligned}
{}^{\mathrm{C}}D^{\gamma} y_0(t) &= y_1(t),\\
{}^{\mathrm{C}}D^{\gamma} y_1(t) &= y_2(t),\\
&\ \vdots\\
{}^{\mathrm{C}}D^{\gamma} y_{M-2}(t) &= y_{M-1}(t),\\
{}^{\mathrm{C}}D^{\gamma} y_{M-1}(t) &= f(t, y_0(t), y_{\alpha_1/\gamma}(t), \dots, y_{\alpha_{N-1}/\gamma}(t)),
\end{aligned} \tag{16a}$$

together with the initial conditions

$$y_j(0) = \begin{cases} y_0^{(j/\mu)} & \text{if } j/\mu \in \mathbb{N}_0,\\ 0 & \text{else,} \end{cases} \tag{16b}$$

in the following sense:

1. *Whenever* $Y := (y_0, \dots, y_{M-1})^{\mathrm{T}}$ *with* $y_0 \in C^{\lceil \alpha_N \rceil}[0, T]$ *for some* $T > 0$ *is the solution of the system* (16), *the function* $y := y_0$ *solves the multiterm differential equation* (10a) *subject to the initial conditions* (10b).
2. *Whenever* $y \in C^{\lceil \alpha_N \rceil}[0, T]$ *is a solution of the multiterm differential equation* (10a) *subject to the initial conditions* (10b), *the vector function* $Y := (y_0, \dots, y_{M-1})^{\mathrm{T}} := (y, {}^{\mathrm{C}}D^{\gamma} y, {}^{\mathrm{C}}D^{2\gamma} y, \dots, {}^{\mathrm{C}}D^{(M-1)\gamma} y)^{\mathrm{T}}$ *solves the multidimensional initial value problem* (16).

In fact, the condition of Theorem 18 that all α_j be rational may be relaxed slightly if $\alpha_N < 1$; in this case, an analog equivalent system can be set up under the weaker assumption that all quotients α_j/α_k are rational.

4.2 Multiorder equation systems

The theory of multiorder fractional differential equation systems, that is, of systems of the form

$$ {}^{\mathrm{C}}D^{\alpha_j} y_j(t) = f_j(t, y_1(t), y_2(t), \dots, y_d(t)), \quad j = 1, 2, \dots, d, \tag{17} $$

with a general vector $(\alpha_1, \alpha_2, \dots, \alpha_d) \in (0, \infty)^d$ (or $(0, 1]^d$) is much less thoroughly established than the corresponding special case where $\alpha_1 = \alpha_2 = \dots = \alpha_d$, that is, the case of single order systems. However, a few fundamental results are available [11].

Theorem 19. *Consider the equation system* (17) *under the hypothesis that* $0 < \alpha_j \le 1$ *for all j. Moreover, assume that all functions* f_j *are continuous on* $[0, T] \times \mathbb{R}^d$ *and satisfy a Lipschitz condition with respect to the last d variables. Then, for any* $(y_{01}, \dots, y_{0d})^{\mathrm{T}} \in \mathbb{R}^d$, *this differential equation system has a unique continuous solution* $y : [0, T] \to \mathbb{R}^d$ *that satisfies the initial condition* $y(0) = (y_{01}, \dots, y_{0d})^{\mathrm{T}}$.

The behavior of the solutions to the system (17) near the initial point can be described.

Theorem 20. *If the functions* $f_1, \dots, f_d$ *are sufficiently smooth, then the jth component of the solution y to the system* (17) *subject to the initial condition* $y(0) = (y_{01}, \dots, y_{0d})^{\mathrm{T}}$ *behaves as*

$$ y_j(t) = y_{0j} + \sum_{k=0}^{\infty} \sum_{\ell_1, \ell_2, \dots, \ell_{j-1}, \ell_{j+1}, \dots, \ell_d = 1}^{\infty} b_{k, \ell_1, \dots, \ell_{j-1}, \ell_{j+1}, \dots, \ell_d} t^{k\alpha_j + \sum_{\mu=1, \mu \ne j}^{d} \ell_\mu \alpha_\mu} $$

for $t \to 0$.

The investigation of the properties of the system's solutions as $t \to \infty$ is much more difficult; only some partial results seem to be known.

Theorem 21. *Let*

$$ A = \begin{pmatrix} A_{11} & \cdots & & A_{1j} \\ 0 & A_{22} & \cdots & A_{2j} \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & A_{jj} \end{pmatrix} $$

be a block triangular $(d \times d)$*-matrix whose submatrices* A_{kk} *have dimension* $d_k \times d_k$ *so that* $d = \sum_{k=1}^{j} d_k$. *Moreover, let* $(\alpha_1, \dots, \alpha_d)^{\mathrm{T}} = (\tilde\alpha_1, \dots, \tilde\alpha_1, \tilde\alpha_2, \dots, \tilde\alpha_2, \dots, \tilde\alpha_j, \dots, \tilde\alpha_j)^{\mathrm{T}}$

where each entry $\tilde{\alpha}_k$ is repeated d_k times and each $\tilde{\alpha}_k \in (0,1]$, and let the functions $f : [0,\infty) \to \mathbb{R}^d$ be continuous and satisfy $\lim_{t\to\infty} f(t) = 0$. Then the following three statements are equivalent:

(i) *Every solution of the inhomogeneous linear system*

$$\begin{pmatrix} {}^{C}D^{\alpha_1} y_1(t) \\ \vdots \\ {}^{C}D^{\alpha_d} y_d(t) \end{pmatrix} = A \begin{pmatrix} y_1(t) \\ \vdots \\ y_d(t) \end{pmatrix} + f(t) \tag{18}$$

converges to zero at infinity.

(ii) *All solutions of the associated homogeneous system tend to zero as $t \to \infty$.*

(iii) *For all $k = 1, 2, \dots, j$, all eigenvalues $\lambda_{k\ell}$, $\ell = 1, 2, \dots, d_k$, of the matrix A_{kk} satisfy $|\arg \lambda_{k\ell}| > \tilde{\alpha}_k \pi/2$.*

On the other hand, whenever there exist some $k \in \{1, 2, \dots, j\}$ and $\ell \in \{1, 2, \dots, d_k\}$ with $|\arg \lambda_{k\ell}| < \tilde{\alpha}_k \pi/2$, there exists an initial value whose corresponding solution to the system (18) *is unbounded.*

In the case of a general, not necessarily block triangular, linear system, or even more generally in the case of a nonlinear system, similar results do not seem to be known. The available examples indicate that the connection between (asymptotic) stability and properties of the matrix (eigenvalues etc.) are likely to be very involved.

Bibliography

[1] M. Caputo, Linear models of dissipation whose Q is almost frequency independent—II, *Geophys. J. R. Astron. Soc.*, **13** (1967), 529–539; reprinted in *Fract. Calc. Appl. Anal.*, **11** (2008), 4–14.

[2] N. D. Cong, T. S. Doan, S. Siegmund, and H. T. Tuan, On stable manifolds for planar fractional differential equations, *Appl. Math. Comput.*, **226** (2014), 157–168.

[3] N. D. Cong and H. T. Tuan, Generation of non-local fractional dynamical systems by fractional differential equations, *J. Integral Equ. Appl.*, **29** (2017), 585–608.

[4] K. Diethelm, Multi-term fractional differential equations, multi-order fractional differential systems and their numerical solution, *J. Eur. Syst. Autom.*, **42** (2008), 665–676.

[5] K. Diethelm, *The Analysis of Fractional Differential Equations*, Springer, Berlin, 2010.

[6] K. Diethelm, The mean value theorems and a Nagumo-type uniqueness theorem for Caputo's fractional calculus, *Fract. Calc. Appl. Anal.*, **15** (2012), 304–313. Erratum: *Fract. Calc. Appl. Anal.* **20** (2017), 1567–1570.

[7] K. Diethelm, A fractional calculus based model for the simulation of an outbreak of dengue fever, *Nonlinear Dyn.*, **71** (2013), 613–619.

[8] K. Diethelm, Properties of the solutions to "Fractionalized" ODE systems, with applications to processes arising in the life sciences, in D. T. Spasic, N. Grahovac, M. Zigic, M. Rapaic, and T. M. Atanackovic (eds.), *Proceedings of the International Conference on Fractional Differentiation and its Applications 2016*, vol. 1, pp. 32–44, Faculty of Technical Sciences, Novi Sad, 2016.

[9] K. Diethelm and N. J. Ford, A note on the well-posedness of terminal value problems for fractional differential equations, *J. Integral Equ. Appl.*, **30** (2018), 371–376.

[10] K. Diethelm and A. D. Freed, On the solution of nonlinear fractional differential equations used in the modeling of viscoplasticity, in F. Keil, W. Mackens, H. Voß, and J. Werther (eds.), *Scientific Computing in Chemical Engineering II: Computational Fluid Dynamics, Reaction Engineering and Molecular Properties*, pp. 217–224, Springer, Heidelberg, 1999.

[11] K. Diethelm, S. Siegmund, and H. T. Tuan, Asymptotic behavior of solutions of linear multi-order fractional differential equation systems, *Fract. Calc. Appl. Anal.*, **20** (2017), 1165–1195.

[12] A. Dokoumetzidis, R. Magin, and P. Macheras, A commentary on fractionalization of multi-compartmental models, *J. Pharmacokinet. Pharmacodyn.*, **37** (2010), 203–207.

[13] J. T. Edwards, N. J. Ford, and A. C. Simpson, The numerical solution of linear multi-term fractional differential equations: systems of equations, *J. Comput. Appl. Math.*, **148** (2002), 401–418.

[14] R. Gorenflo, A. A. Kilbas, F. Mainardi, and S. V. Rogosin, *Mittag-Leffler Functions, Related Topics and Applications*, Springer, Berlin, 2014.

[15] R. Gorenflo and F. Mainardi, Fractional oscillations and Mittag-Leffler functions, in *Proceedings of the International Workshop on the Recent Advances in Applied Mathematics (RAAM ’96)*, pp. 193–208, Kuwait University, Department of Mathematics and Computer Science, Kuwait, 1996.

[16] S. G. Samko, A. A. Kilbas, and O. I. Marichev, *Fractional Integrals and Derivatives: Theory and Applications*, Gordon and Breach, Yverdon, 1993.

[17] M. Stynes, Too much regularity may force too much uniqueness, *Fract. Calc. Appl. Anal.*, **19** (2016), 1554–1562.

[18] P. J. Torvik and R. L. Bagley, On the appearance of the fractional derivative in the behavior of real materials, *J. Appl. Mech.*, **51** (1984), 294–298.

[19] D. Verotta, Fractional compartmental models and multi-term Mittag-Leffler response functions, *J. Pharmacokinet. Pharmacodyn.*, **37** (2010), 209–215.

Temirkhan Aleroev and Hedi Aleroeva

Problems of Sturm–Liouville type for differential equations with fractional derivatives

Abstract: This chapter is devoted to consideration of boundary value problems for differential equations with fractional derivatives. In Section 1, we introduce main definitions, whose used in the present chapter. In Section 2, we consider the operators induced by the linear differential expressions and boundary conditions of Sturm–Liouville type. Section 3 is devoted to the spectral analysis of the operator generated by the differential expression of second order with fractional derivatives in lower terms and boundary conditions of Sturm–Liouville type.

Keywords: Fractional derivative, fractional integral, eigenvalue, eigenfunction

MSC 2010: 34A08, 26A33

1 Introduction

Let $f(x) \in L_1(0,1)$. Then the function

$$\frac{d^{-\alpha}}{dx^{-\alpha}} f(x) \equiv \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)\, dt \in L_1(0,1)$$

is called a fractional integral of order $\alpha > 0$ beginning at $x = 0$ [20]. Here, $\Gamma(\alpha)$ is the Euler gamma-function. As is known (see [20]), the function $g(x) \in L_1(0,1)$ is called the fractional derivative of the function $f(x) \in L_1(0,1)$ of order $\alpha > 0$ beginning at $x = 0$, if

$$f(x) = \frac{d^{-\alpha}}{dx^{-\alpha}} g(x),$$

which is written

$$g(x) = \frac{d^{\alpha}}{dx^{\alpha}} f(x).$$

Acknowledgement: The authors are grateful to the referees for the numerous and very helpful remarks which allowed us to remove shortcomings.

Temirkhan Aleroev, Department of Applied Mathematics, NRU MGSU, Yaroslavskoe Shosse (highway) 26, Moscow, 129337, Russia, e-mail: aleroev@mail.ru
Hedi Aleroeva, Department of Information Technologies, MTUCI, Aviamotornaya street 8a, Moscow, 111024, Russia, e-mail: BinSabanur@gmail.com

https://doi.org/10.1515/9783110571660-002

Then

$$\frac{d^\alpha}{dx^\alpha}$$

denote the fractional integral for $\alpha < 0$ and the fractional derivative for $\alpha > 0$.

Let $\{\gamma_k\}_0^n$ be a set of real numbers satisfying the condition $0 < \gamma_j \le 1$, $(0 \le j \le n)$. We denote

$$\sigma_k = \sum_{j=0}^{k} \gamma_j - 1;$$

$$\mu_k = \sigma_k + 1 = \sum_{j=0}^{k} \gamma_j \quad (0 \le k \le n),$$

and assume that

$$\frac{1}{\rho} = \sum_{j=0}^{n} \gamma_j - 1 = \sigma_n = \mu_n - 1 > 0.$$

Following M. M. Dzhrbashyan [20], we consider the integrodifferential operators

$$\begin{aligned}
D^{(\sigma_0)} f(x) &\equiv \frac{d^{-(1-\gamma_0)}}{dx^{-(1-\gamma_0)}} f(x),\\
D^{(\sigma_1)} f(x) &\equiv \frac{d^{-(1-\gamma_1)}}{dx^{-(1-\gamma_1)}} \frac{d^{\gamma_0}}{dx^{\gamma_0}} f(x),\\
D^{(\sigma_2)} f(x) &\equiv \frac{d^{-(1-\gamma_2)}}{dx^{-(1-\gamma_2)}} \frac{d^{\gamma_1}}{dx^{\gamma_1}} \frac{d^{\gamma_0}}{dx^{\gamma_0}} f(x),\\
&\dots\\
D^{(\sigma_n)} f(x) &\equiv \frac{d^{-(1-\gamma_n)}}{dx^{-(1-\gamma_n)}} \frac{d^{\gamma_{n-1}}}{dx^{\gamma_{n-1}}} \cdots \frac{d^{\gamma_0}}{dx^{\gamma_0}} f(x).
\end{aligned}$$

We denote D_{ax}^{α} the operator of fractional integrodifferentiation of order α beginning at $a \in R$ and with end at $x \in R$ of order $[\alpha]$. By definition, we have

$$D_{ax}^{(\alpha)} \phi(t) = \begin{cases} \frac{\operatorname{sign}(x-a)}{\Gamma(-\alpha)} \int_a^x \frac{\phi(t)dt}{(x-t)^{\alpha-1}}, & \alpha < 0,\ \phi(t) \in L_1[a,b],\\ \phi(t), & \alpha = 0,\ \phi(t) \in L_1[a,b],\\ \operatorname{sign}^{[\alpha]+1}(x-a) \frac{\partial^{[\alpha]+1}}{\partial x^{[\alpha]+1}} D_{ax}^{\alpha-[\alpha]-1} \phi(t), & \alpha > 0,\ \phi(t) \in L_1[a,b], \end{cases}$$

where $[\alpha]$ is the integer part of α, which satisfies $[\alpha] \le \alpha < [\alpha] + 1$, and $x \in [a,b]$.

2 Boundary value problems for differential equations of fractional order

In this section, we study boundary value problems in $L_2(0,1)$ for the equation

$$D^{(\sigma_n)} u - [\lambda + q(x)] u = 0, \quad 0 < \sigma_n < \infty,$$

and integral operators, corresponding to some boundary-value problems for fractional differential equations. These operators have the form

$$A_\gamma^{[\alpha,\beta]}u(x) = c_\alpha \int_0^x (x-t)^{\frac{1}{\alpha}-1}u(t)\,dt + c_{\beta,\gamma}\int_0^1 x^{\frac{1}{\beta}-1}(1-t)^{\frac{1}{\gamma}-1}u(t)\,dt,$$

where c_α and $c_{\beta,\gamma}$ are arbitrary constants. The operators $A_\gamma^{[\alpha,\beta]}$ belong to classes of nonself-adjoint operators whose spectral structure has been little studied. We note that the problem of the eigenvalues of introduced above operators is investigated in [10] (there a powers were parametrized and the case of their coincidence was considered for the first time). In this section, we introduce principally new methods for investigation such operators.

2.1 Spectral analysis of operators, generated by fractional differential equations of order more than 1 but less than 2 and boundary conditions of Sturm–Liouville type

We devote this subsection to the spectral analysis of two boundary value problems [9, 16, 26, 29]

$$L(u;1,1-\alpha,1,0) = \frac{1}{\Gamma(\alpha)}\frac{d}{dx}\int_0^x \frac{u'(\tau)}{(x-\tau)^{1-\alpha}}d\tau + \lambda u(x) = 0,$$

$$u(0) = 0, \quad u(1) = 0,$$

and

$$L(u;1,1,1-\alpha,0) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}u''(t)dt + \lambda u(x) = 0,$$

$$u(0) = 0, \quad u(1) = 0.$$

These problems are the focus of many researchers.

First, we note that in [29] (and references therein) was considered the problem

$$u'' + \lambda\frac{d^\alpha}{dx^\alpha}u = 0, \quad 0 < \alpha < 1, \tag{2.1}$$

$$u(0) = 0, \quad u(1) = 0, \tag{2.2}$$

with studying of the spectrum of the operator

$$D^{(\beta)}u = \frac{d^{-\alpha}}{dx^{-\alpha}}\frac{d^2}{dx^2}u = \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}u''(t)dt, \quad (\beta = 2-\alpha)$$

(the operator $D^{(\sigma_2)}$ transforms to the operator $D^{(\beta)}$ if $\gamma_0 = \gamma_1 = 1$ and $\gamma_2 = 1-\alpha$).

For the first time, the problem (2.1)–(2.2) was considered in [29](see also references therein) in the theory of mixed-type equations. This problem was considered in conjunction with Tricomi gas flow at the sonic line. Also, it was established that many direct and inverse problems associated with a degenerate hyperbolic equation and the equation of mixed hyperbolic-parabolic type equivalently may be reduced to the problem (2.1)–(2.2). In particular, [17] to the problem (2.1)–(2.2) may be reduced as an analog of the Tricomi problem for the equation

$$|y|^{mH(-y)}\frac{\partial^2 u}{\partial x^2} = \frac{\partial^{1+H(-y)}u}{\partial y^{1+H(-y)}}$$

where $m = \text{const} > 0$, $H(y)$ is Heaviside function, $u = u(x,y)$.

The operator $D^{(\beta)}$ generated great interest after F. Mainardi's paper [26]. In this paper, the following equation was considered:

$$\frac{1}{\Gamma(2-\gamma)}\int_0^x \frac{u''(\tau)}{(t-\tau)^{\gamma-1}}d\tau + \omega^\gamma u(t) = 0 \tag{2.3}$$

where ω is a positive constant and $1 < \gamma < 2$, which Mainardi called a fractional oscillatory equation. This paper is, without exaggeration, very interesting for a lot of researchers this period of time. First of all, note that:

1. If $\lambda \neq 0$, then any solution $u(x) \in S^2[0,1]$ (where $S^2[0,1]$ is the class of functions $u(x)$, that are summable (integrable) on $[0,1]$ including their derivatives of first and second order) for the equation

$$D^{(\beta)}u = \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}u''(t)dt = -\lambda u \tag{2.4}$$

   coincides with the solution for the equation (2.1);
2. the equations (2.1) and (2.4) are equal if

$$\lim_{x\to 0}\frac{d^{-(1-\alpha)}}{dx^{-(1-\alpha)}}u(x) = 0.$$

Of course, the fractional oscillatory equation, or equation for fractional oscillator (as an equation, which describes an oscillatory physical system), will have at least main oscillatory properties.

Hereafter, the following integral equations will play the main role:

$$\begin{aligned}u(x) - \frac{\lambda}{\Gamma(2-\alpha)}&\left[\int_0^1 x(1-t)^{1-\alpha}u(t)dt - \int_0^x (x-t)^{1-\alpha}u(t)dt\right]\\ &= u(x) - \frac{\lambda}{\Gamma(2-\alpha)}\int_0^1 G_0(x,t)u(t)dt = 0,\end{aligned}$$

$$u(x) - \frac{\lambda}{\Gamma(2-\alpha)}\left[\int_0^1 x^{1-\alpha}(1-t)^{1-\alpha}u(t)dt - \int_0^x (x-t)^{1-\alpha}u(t)dt\right]$$
$$= u(x) - \frac{\lambda}{\Gamma(2-\alpha)}\int_0^1 G_1(x,t)u(t)dt = 0,$$

where $G_0(x,t)$ is the Green's function of the problem (2.1)–(2.2), which was constructed in [4] and $G_1(x,t)$ is the Green function of the problem

$$L(u;1,1-\alpha,1,0) = \frac{1}{\Gamma(\alpha)}\frac{d}{dx}\int_0^x \frac{u'(\tau)}{(x-\tau)^{1-\alpha}}d\tau + \lambda u(x) = 0, \tag{2.5}$$
$$u(0) = 0, \quad u(1) = 0, \tag{2.6}$$

which was for the first time considered in [7] (see also references therein).

Important note: the operators $L(u;1,1-\alpha,1,0)$ and $L(u;1,1,1-\alpha,0)$ have the same orders, but the $\gamma_0, \gamma_1, \gamma_2$, of those orders are different.

It is easy to show [15] that $G_0(x,t)$ is not with fixed sign, and this fact says that equation (2.4) was incorrectly chosen as an oscillatory equation. Physically, it is clear, that if the order of the operator $L(u;\gamma_0,\gamma_1,\gamma_2,q(x))$ is close to 2 (or when $\gamma_0+\gamma_1+\gamma_2-1$ is close to 2), then the operator $L(u;\gamma_0,\gamma_1,\gamma_2,q(x))$ has main oscillatory properties. We have the following result.

Theorem 2.1. *If*

$$0 < \alpha < \left(\frac{32\pi^2}{9} + \frac{2}{3}\right)^{-1},$$

then the first eigenvalue of the problem (2.1)–(2.2) *is positive and simple* (*the multiplicity of this eigenvalue is equal to* 1), *and basic* (*main*) *tone has no nodes* (*i. e., the first eigenfunction corresponding to the first eigenvalue does not vanish in* (0, 1)).

Proof. That the first eigenvalue of the problem (2.1)–(2.2) is positive and simple for

$$0 < \alpha < \left(\frac{32\pi^2}{9} + \frac{2}{3}\right)^{-1}$$

was proved in [12, 30]. We show now that basic (main) tone of the problem (2.1)–(2.2) has no nodes.

It is known that a number λ will be an eigenvalue of the problem (2.1)–(2.2) [29] if and only if this value λ is the root (zero) of the function $E_{1/\beta}(-\lambda;2)$ and the corresponding eigenfunctions of the problem (2.1)–(2.2) are

$$u_n(x) = xE_{\frac{1}{\beta}}(-\lambda_n x^\beta;2), \quad n = 1,2,3,\dots,$$

where

$$\lambda_1, \lambda_2, \dots, \lambda_n, \dots$$

are zeros of the function $E_\beta(-\lambda; 2)$, numbered by according to the nondecreasing of their modules,

$$E_\rho(z; \mu) = \sum_{k=0}^{\infty} \frac{z^k}{(\mu + k\rho^{-1})}$$

is a Mittag-Leffler function. We shall show that the function

$$u_1(x) = xE_{\frac{1}{\beta}}(-\lambda_1 x^\beta; 2)$$

does not vanish in $(0, 1)$. Let $x_0 \in (0, 1)$ be such that

$$x_0 E_{\frac{1}{\beta}}(-\lambda_1 x_0^\beta; 2) = 0.$$

Then the number $\lambda_1 x_0^\beta$ is a zero of $E_\beta(-\lambda; 2)$; moreover, $\lambda_1 x_0^\beta < \lambda_1$ (since $x_0 \in (0, 1)$). This contradicts the assumption that λ_1 is the first zero of the function $E_{1/\beta}(-\lambda; 2)$. Theorem 2.1 is proved. □

Since for $\alpha > \frac{2}{3}$, the function $E_{1/\beta}(-\lambda; 2)$ has no real zeros [31], then the problem (2.1)–(2.2) has this main (at least first eigenvalues are real) oscillatory property only for small α.

Now we consider the function $G_1(x, t)$. As it was shown in [12], this function has many useful properties, in particular $G_1(x, t) = G_1(1 - t, 1 - x)$ and $G_1(x, t) > 0$, for any $x, t \in (0, 1)$. Some results of the theorem below follow from the well-known Perron's theorem.

Theorem 2.2. *The first eigenvalue λ_1 of the problem* (2.5)–(2.6) *is positive, simple, and satisfies the condition*

$$0 < \lambda_1^{-1} < \frac{\Gamma(2 + 2\alpha)}{\Gamma(1 + \alpha)},$$

and basic (*main*) *tone has no nodes for all* $0 < \alpha < 1$.

Proof. As it was written above, from Perron's theorem follows that the first eigenvalues are positive and simple, and basic (main) tone has no nodes. Let us show that

$$0 < \lambda_1^{-1} < \frac{\Gamma(2 + 2\alpha)}{\Gamma(1 + \alpha)}$$

holds. As it was mentioned above, the problem (2.5)–(2.6) is equivalent to the integral equation of Fredholm (II kind)

$$u(x) + \frac{\lambda}{\Gamma(1 + \alpha)} \left[\int_0^x (x - \xi)^\alpha u(\xi) d\xi - x^\alpha \int_0^1 (1 - \xi)^\alpha u(\xi)\, d\xi \right] = 0,$$

and the value λ is an eigenvalue of the problem (2.5)–(2.6) if and only if it is a zero of the Mittag-Leffler function $E_{\frac{1}{1+\alpha}}(-\lambda, 1+\alpha)$ [7, 12].

Let us rewrite the operator

$$Au = \frac{1}{\Gamma(1+\alpha)}\left[\int_0^x (x-\xi)^\alpha u(\xi)d\xi - x^\alpha \int_0^1 (1-\xi)^\alpha u(\xi)d\xi\right]$$

as

$$Au = A_0 u - A_1 u,$$

where

$$A_0 u = \frac{1}{\Gamma(1+\alpha)}\int_0^x (x-\xi)^\alpha u(\xi)d\xi,$$

and

$$A_1 u = \frac{1}{\Gamma(1+\alpha)}\int_0^1 x^\alpha (1-\xi)^\alpha u(\xi)d\xi.$$

It is clear that for $0 < \alpha < 1$, operators A_0 and A_1 are trace class operators [21]. So,

$$\operatorname{sp} A = \operatorname{sp}(A_0 - A_1) = \operatorname{sp}(A_0) - \operatorname{sp}(A_1).$$

Since A_0 is Volterra's operator, then $\operatorname{sp}(A_0) = 0$, and so

$$\operatorname{sp}(A) = -\operatorname{sp}(A_1).$$

As A_1 is a one-dimensional operator, it is easy to find its trace. For, we consider the equation

$$u(x) - \frac{\lambda}{\Gamma(1+\alpha)}\int_0^1 x^\alpha (1-\xi)^\alpha u(\xi)d\xi = 0.$$

The Fredholm determinant of this equation is

$$d(\lambda) = |1 - \lambda K_{11}|,$$

where

$$K_{11} = \frac{1}{\Gamma(1+\alpha)}\int_0^1 \xi^\alpha (1-\xi)^\alpha d\xi = \frac{\Gamma(1+\alpha)}{\Gamma(2+2\alpha)}.$$

From this, $\mathrm{sp}(A) = \frac{\Gamma(1+\alpha)}{\Gamma(2+2\alpha)}$. Thus $\lambda_1^{-1} + \sum_{i=2}^{\infty} \lambda_i^{-1} = \frac{\Gamma(1+\alpha)}{\Gamma(2+2\alpha)}$. Since the kernel of the operator $-A$ is nonnegative, then λ_1 is positive number. Since $\sum_{i=2}^{\infty} \lambda_i^{-1}$ is positive, then

$$\lambda_1^{-1} < \frac{\Gamma(1+\alpha)}{\Gamma(2+2\alpha)}.$$

Theorem 2.2 is proved. □

Corollary 2.1. *Since λ is an eigenvalue of the problem* (2.5)–(2.6) [7, 12] *if and only if λ is a zero of the function $E_{1/(1+\alpha)}(-\lambda; 1+\alpha)$, and the corresponding eigenfunctions of the problem* (2.5)–(2.6) *are*

$$u_n(x) = x^{\alpha} E_{1/(1+\alpha)}(-\lambda_n x^{1+\alpha}; 1+\alpha)$$

$n = 1, 2, 3, \ldots$, where $\lambda_1, \lambda_2, \ldots, \lambda_n, \ldots$ are zeros of the function $E_{1/(1+\alpha)}(-\lambda; 1+\alpha)$, numbered by their nondecreased modules, then the function $E_{1/(1+\alpha)}(-\lambda; 1+\alpha)$ has positive and simple first zero, and the function

$$u_n(x) = x^{\alpha} E_{1/(1+\alpha)}(-\lambda_n x^{1+\alpha}; 1+\alpha)$$

does not vanish in $(0, 1)$.

Conclusion. The above proved theorems show:
(a) which of equations (2.1) or (2.5) is to choose correctly as an oscillatory;
(b) how depends the spectral structure of $L(u; \gamma_0, \gamma_1, \gamma_2, q(x))$ on γ_0, γ_1, γ_2, that form the order of the operator;
(c) how the nature of the modeling process helps to indentify the γ_0, γ_1, γ_2, that form the order of the operator.

2.2 On completeness of system of eigenfunctions and associated functions of operator, generated by the model fractional differential equation and boundary conditions of Sturm–Liouville type

Let us start from the equation

$$D^{(\sigma_2)} u - [\lambda + q(x)] u(x) = 0, \tag{2.7}$$

where

$$D^{(\sigma_2)} u = \frac{1}{\Gamma(1-\gamma)} \frac{d}{dx} \int_0^x \frac{u'(t)}{(x-t)^{\gamma}} dt, \quad 0 < \gamma < 1,\ \sigma_2 = 1 + \gamma.$$

At first, equation (2.7) was studied in [7] as a model equation of the fractional order $1 < \sigma_2 < 2$. In particular, it was established in [7] that the two-point Dirichlet problem

$$u(0) = 0, \quad u(1) = 0, \tag{2.8}$$

for equation (2.7) with $q(x) = 0$ is equivalent to the integral equation

$$\frac{1}{\Gamma(2-\gamma)}\left[\int_0^x (x-t)^{1-\gamma}u(t)dt - \int_0^1 x^{1-\gamma}(1-t)^{1-\gamma}u(t)dt\right] = \lambda u.$$

We have the following.

Theorem 2.3. *Let* $\gamma_0 = \gamma_1 = 1$, $q(x) \equiv 0$. *Then the system of eigenfunctions and associated functions of the problem* (2.7)–(2.8) *is complete in* $L_2(0,1)$.

A close result (for a semibounded potential $q(x)$) was obtained in [1, 2, 11, 27, 28]. It should be noted that the proofs of these statements are based on the fact that the operator, generated by the problem (2.7)–(2.8), is the sectorial [19].

Theorem 2.4. *All eigenvalues of the problem* (2.7)–(2.8) *for* $q(x) \equiv 0$ *are in the angle* $|\arg z| < \frac{\pi(1-\gamma)}{2}$, $0 < \gamma < 1$.

Proof. Consider the expression $(-D^{(\sigma_2)}f, f)$. It is obvious that

$$\begin{aligned}(-D^{(\sigma_2)}f, f) &= -\left(\frac{1}{\Gamma(1-\gamma)}\frac{d}{dx}\int_0^x \frac{f'(t)}{(x-t)^\gamma}dt, f(x)\right)\\ &= \left(\frac{1}{\Gamma(1-\gamma)}\int_0^x \frac{f'(t)}{(x-t)^\gamma}dt, f'(x)\right) = (J_{0,x}^\alpha f', f')\end{aligned}$$

where $\alpha = 1-\gamma$ and ${}_0J_x^\alpha$ is the operator of fractional integration of order α:

$$(J_{0,x}^\alpha f)(x) = \frac{1}{\Gamma(\alpha)}\int_0^x (t-s)^{1-\alpha}f(s)ds.$$

By a well-known Matsaev–Palant theorem ([22], p. 481), the values of the form $(J_{0,x}^\alpha f', f')$ is in the angle $|\arg z| < \frac{\pi\alpha}{2}$. This proves Theorem 2.4. □

Since the number λ is an eigenvalue of the problem (2.7)–(2.8) iff λ is a zero of the function $E_{1/\mu}(-\lambda;\mu)$ $(\mu = 1+\gamma)$ [6], the following proposition is valid.

Corollary 2.2. *All zeros of the function* $E_{1/\mu}(-\lambda;\mu)$ *are in the angle* $|\arg z| < (\pi(1-\gamma))/(2)$, $0 < \gamma < 1$. *Here,* $\mu = 1+\gamma$.

Theorem 2.5. *The problem* (2.7)–(2.8) *for* $q(x) \equiv 0$ *has no eigenvalues inside the circle with radius* $\Gamma(4-2\gamma)/\Gamma(2-\gamma)$ *centered at the coordinate origin.*

2.3 Spectral analysis of operator, generated by fractional differential expression of order more than 2

In $L_2(0,1)$, we consider the operator

$$\begin{aligned} A_\rho(u) &= \int_0^1 G(x,t)u(t)\,dt \\ &= \frac{1}{\Gamma(\rho^{-1})}\left[\int_0^x (x-t)^{\frac{1}{\rho}-1}u(t)dt - \int_0^1 x^{\frac{1}{\rho}-1}(1-t)^{\frac{1}{\rho}-1}u(t)\,dt\right], \end{aligned}$$

which was for the first time studied in [12]. Here, $0 < \rho < 2$, and

$$G(x,t) = \begin{cases} \frac{(1-t)^{\frac{1}{\rho}-1}x^{\frac{1}{\rho}-1}-(x-t)^{\frac{1}{\rho}-1}}{\Gamma(\rho^{-1})}, & 0 \le t \le x \le 1, \\ \frac{(1-t)^{\frac{1}{\rho}-1}x^{\frac{1}{\rho}-1}}{\Gamma(\rho^{-1})}, & 0 \le x \le t \le 1, \end{cases}$$

is the Green function of the following problem S (for $\lambda = 0$):

$$\frac{1}{\Gamma(n-\rho^{-1})}\frac{d^n}{dx^n}\int_0^x (x-s)^{n-\rho^{-1}-1}u(s)ds + \lambda u = 0,$$

($n-1 \le \rho^{-1} < n$, $n = [\rho^{-1}]+1$, where $[\rho^{-1}]$ is the integer part of the number ρ^{-1})

$$u(0) = 0, \quad u'(0) = 0, \quad \dots, \quad u^{(n-2)}(0) = 0, \quad u(1) = 0.$$

In this case [12], if $\gamma_0 = \gamma_1 = \cdots = \gamma_n = 1$, then the problem S takes the form

$$\begin{aligned} &u^{(n)} + \lambda u = 0, \\ &\quad u(0) = 0, \quad u'(0) = 0, \quad \dots, \quad u^{(n-2)}(0) = 0, \quad u(1) = 0, \end{aligned}$$

whose Green function $G(x,t)$ (for $\lambda = 0$) reads

$$G(x,t) = \begin{cases} \frac{(1-t)^{n-1}x^{n-1}-(x-t)^{n-1}}{(n-1)!}, & 0 \le t \le x \le 1, \\ \frac{(1-t)^{n-1}x^{n-1}}{(n-1)!}, & 0 \le x \le t \le 1. \end{cases}$$

The last function was studied sufficiently well, and we will use it in the sequel. The operator A_ρ was studied in [3, 7, 10, 12]. We will study this operator carefully, because it turns out that the Mainardi equation [26] (fractional oscillatory equation) does not possess many basic oscillatory properties. The search for an equation of fractional order that possesses these properties led us to the study of the operator A_ρ. Now, we would introduce the most significant properties of this operator established by us earlier:

1. for $\rho > 1$, the operator A_ρ is completely nonself-adjoint [7, 10, 12];
2. for $\rho \leq 1$, the operator A_ρ is sectorial [15] (and see the references therein);
3. for $0 < \rho < 2$, the system of eigenfunctions of the operator A_ρ is complete in $L_2(0,1)$ [7, 8].

For $1/\rho < 2$, this operator was considered in Section 2.1. But in this section we study the operator for $1/\rho > 2$.

Let us consider the operator studied in [10, 12]:

$$A_\rho u = \frac{1}{\Gamma(\rho^{-1})}\left[\int_0^x (x-t)^{\frac{1}{\rho}-1}u(t)dt - \int_0^1 x^{\frac{1}{\rho}-1}(1-t)^{\frac{1}{\rho}-1}u(t)\,dt\right].$$

Here and below, we assume $1/\rho = \alpha > 2$. It was announced in [10] that many results in [24] can be used for operators of the kind $A_\gamma^{[\alpha,\beta]}$. Indeed, let K be a cone of nonnegative functions in $L_2(0,1)$. The operator A is called u_0-positive, if there exists the nonzero element $u_0 \in K$ such that, for any nonzero $u \in K$, it is possible to find the numbers $\alpha(u)$ and $\beta(u) > 0$ for which

$$\alpha(u)u_0(x) \leq (Au)(x) \leq \beta(u)u_0(x).$$

The operator A is called u_0-bounded from above if any nonzero $u \in K$ corresponds $\beta(u) \geq 0$ such that

$$Au \leq \beta(u)u_0(x).$$

It was shown in [19] that the operator A_ρ is u_0-bounded, and, in this case,

$$\begin{aligned} u_0(t) &= \frac{1}{\Gamma(\alpha)}\int_0^t [t^{\alpha-1}(1-s) - (t-s)^{\alpha-1}]ds \\ &\quad + \frac{1}{\Gamma(\alpha)}\int_t^1 t^{\alpha-1}(1-s)^{\alpha-1}ds = \frac{t^{\alpha-1}(1-t)}{\alpha\Gamma(\alpha)}. \end{aligned}$$

It was shown in [24] that if the kernel of an operator satisfies the inequalities

$$u_0(t)b_1(s) \leq K(t,s) \leq u_0(t)b_2(s)$$

then

$$\alpha(x) = \int_0^1 b_1(t)x(t)dt,$$

and

$$\beta(x) = \int_0^1 b_2(t)x(t)dt$$

can be taken as $\alpha(u_0)$ and $\beta(u_0)$, respectively. The kernel $K(t,s)$ of the operator A_ρ possesses many useful properties. In particular, it was shown in [12] that $K(t,s) = K(1-s,1-t)$. With the help of this rather obvious property, it is possible to show (see [19]) that we can take

$$b_1(s) = \frac{\alpha}{\Gamma(\alpha)} s(1-s)^{\alpha-1}$$

and

$$b_2(s) = \alpha(\alpha-1).$$

Theorem 2.6. *The first eigenvalue μ_1 of the operator*

$$A_\rho u = \frac{1}{\Gamma(\rho^{-1})}\left[\int_0^x (x-t)^{\frac{1}{\rho}-1}u(t)dt - \int_0^1 x^{\frac{1}{\rho}-1}(1-t)^{\frac{1}{\rho}-1}u(t)\,dt\right],$$

$(1/\rho = \alpha > 2)$, *satisfies the inequalities*

$$\frac{\alpha^2}{\Gamma(2+2\alpha)} \le \mu_1 \le \frac{\alpha-1}{\Gamma(\alpha)} B(2;\alpha),$$

where B is the Euler beta-function.

Proof. Let us calculate $\alpha(u_0)$ and $\beta(u_0)$:

$$\alpha(u_0) = \int_0^1 \frac{\alpha}{\Gamma(\alpha)} s(1-s)^{\alpha-1} \frac{1}{\alpha\Gamma(\alpha)} s^{\alpha-1}(1-s)ds$$
$$= \frac{1}{[\Gamma(\alpha)]^2}\int_0^1 s^\alpha(1-s)^\alpha ds = \frac{\alpha^2}{\Gamma(2+2\alpha)},$$
$$\beta(u_0) = \int_0^1 \alpha(\alpha-1)\frac{s^{\alpha-1}(1-s)}{\alpha\Gamma(\alpha)}ds = \frac{\alpha-1}{\Gamma(\alpha)}B(2;\alpha).$$

Theorem 2.6 is proved. □

Corollary 2.3. *Let $1/\rho = \alpha > 2$. Then, for the first zero λ_1 of the function $E_\rho(\lambda;\frac{1}{\rho})$, the following two-sided estimates hold:*

$$\frac{\alpha-1}{\Gamma(\alpha)} B(2;\alpha) \le \lambda_1 \le \frac{\alpha^2}{\Gamma(2+2\alpha)}.$$

The assertion of Corollary 2.3 follows from Theorem 2.6 and the fact that the number λ is an eigenvalue of the operator A iff λ is a zero of the function $E_\rho(\lambda;\frac{1}{\rho})$.

2.4 Methods of the theory of perturbations in fractional calculus

Now, let us study integral operators corresponding to boundary value problems for fractional differential equations using methods of the theory of perturbations.

The holomorphic dependence of these operators on the order of fractional differentiation is proved. There are several useful criteria for holomorphy. In accordance with this, various types of holomorphic families are considered. We will use type (A). The type (A) is defined in terms of the boundedness of the perturbation with respect to the unperturbed operator.

Let us formulate very important criterion, which we will use later [23].

Theorem (Criterion of holomorphy (A)). *Let T be a closable operator from X in Y, and let $T^{(n)}$, $n = 1, 2, \ldots$, be operators from X in Y, whose domains of definition contain $D(T) = D$. Assume that there are exist constants $a, b, c \geq 0$, such that*

$$T^{(n)}u \leq c^{n-1}(a\|u\| + b\|Tu\|), \quad u \in D,\ n = 1, 2, \ldots. \tag{2.9}$$

Then for $|\kappa| < 1/c$ the series

$$T(\kappa)u = Tu + \kappa T^{(1)}u + \kappa^2 T^{(2)}u + \cdots, \quad u \in D$$

defines the operator $T(\kappa)$ with the domain of definition D. If $|\kappa| < (b + c)^{-1}$, then the operator $T(\kappa)$ is closable, and the closures $\widetilde{T}(\kappa)$ form a holomorphic family of type (A).

We shall note that the holomorphic families of this type and, in particular bounded-holomorphic families, were studied since Rellich's papers [25] (and references therein). A wide list of references is presented in papers of M. K. Gavurin and V. B. Loginov [25] (and references therein).

Theorem 2.7. *If $|\varepsilon| < 1$, then the operator*

$$A(\varepsilon)u = -\int_0^x (x - t)^{1+\varepsilon}u(t)dt + \int_0^1 x^{1+\varepsilon}(1 - t)^{1+\varepsilon}u(t)dt$$

forms a holomorphic family of type (A)*, that is,*

$$A(\varepsilon)u = A(0)u + \varepsilon A_1 u + \varepsilon^2 A_2 u + \cdots + \varepsilon^n A_n u + \cdots$$

where

$$A(0)u = -\int_0^x (x - t)u(t)dt + \int_0^1 x(1 - t)u(t)dt$$

is the unperturbed operator, and

$$A_n u(x) = \int_0^x (\widetilde{K}(x, t)_n - K(x, t)_n)u(t)dt,$$

$$\widetilde{K}(x,t)_n = \frac{x(1-t)\ln^n(1-t)x}{n!},$$

$$K(x,t)_n = \begin{cases} \frac{x(1-t)\ln^n(1-t)x}{n!}, & t < x, \\ 0, & t \geq x. \end{cases}$$

Theorem 2.8. *If* $|\varepsilon| < 3/2$, *then the operator*

$$\widetilde{B}(\varepsilon)u = -\int_0^x (x-t)^{1+\varepsilon} u(t)dt + \int_0^1 x(1-t)^{1+\varepsilon} u(t)dt$$

forms a holomorphic family of type (A) *where*

$$B(0)u = -\int_0^x (x-t)u(t)dt + \int_0^1 x(1-t)u(t)dt$$

is the unperturbed operator, and

$$B_n u(x) = \int_0^x (\overline{K}(x,t)_n - K(x,t)_n)u(t)dt,$$

where

$$\overline{K}(x,t)_n = \frac{x(1-t)\ln^n(1-t)x}{n!},$$

and

$$K(x,t)_n = \begin{cases} \frac{x(1-t)\ln^n(1-t)x}{n!}, & t < x, \\ 0, & t \geq x. \end{cases}$$

Since [10], the Fredholm spectrum of the operators under study coincides with the zeros of appropriate functions of Mittag-Leffler type; the presented method allows one to efficiently study a distribution of zeros for such functions. To confirm this assertion, we give two examples. Following [30], we introduce the following notation: $\lambda_n(\alpha)$ are the eigenvalues of the problem (2.1)–(2.2). In [30], it was written that "... in the limiting case $\alpha = 0$, the problem (2.1)–(2.2) becomes the Sturm–Liouville boundary value problem with the sequence of eigenvalues $\lambda_n(\alpha) = (\pi n)^2$. Is it true that $\lim_{\alpha\to 0+} \lambda_n(\alpha) = (\pi n)^2$ for any fixed n? The answer will be positive."

Let us prove a stronger proposition.

Theorem 2.9. $\lim_{\alpha\to\alpha_0+} \lambda_n(\alpha) = \lim_{\alpha\to\alpha_0-} \lambda_n(\alpha) = \lambda_n(\alpha_0)$ *for any* $\alpha_0 \in [0,1]$.

Proof. Theorem 2.9 is a simple corollary of Theorem 4.2 (see [21], p. 35) and the fact that the operator function $\widetilde{B}(\varepsilon)$ is strongly continuous for $|\varepsilon| < 1$. □

Finally, we consider a one more significant question of the multiplicity of eigenvalues of the operator $\widetilde{B}(\varepsilon)$ (as was mentioned above, this question is related to the question of the multiplicity of zeros of a corresponding function of the Mittag-Leffler type [12]).

It is known that all zeros of a function of the Mittag-Leffler type $E_\rho(z,\mu)$ (where $\rho > 1/2$, $\rho \equiv 1$, $\mathrm{Im}(\mu) = 0$ that are sufficiently large in modulus are simple. Therefore, we pay the main attention to the multiplicity of the first eigenvalues of the operator $\widetilde{B}(\varepsilon)$.

Theorem 2.10. *Let $|\varepsilon| < (\frac{32\pi^2}{9} + \frac{2}{3})^{-1}$. Then the first eigenvalue $\lambda_1(\varepsilon)$ of the operator $\widetilde{B}(\varepsilon)$ is simple.*

Proof. It is known [12] that if the spectrum of the operator $\widetilde{B}(0)$ is divided into two parts by a closed curve Γ, then the spectrum of the operator $\widetilde{B}(\varepsilon)$ is also divided by the curve Γ for sufficiently small ε. In this case, the estimate of the smallness of ε is as follows [12]:

$$|\varepsilon| < \min_{\zeta\in\Gamma}\left(a\|R(\zeta,\widetilde{B}(0))\| + b\|\widetilde{B}(0)R(\zeta,\widetilde{B}(0))\| + c\right)^{-1} \tag{2.10}$$

(where a, b, and c are parameters that enter inequality (2.9)). As the contour Γ in the formula (2.10), we take the circumference $|\zeta - \frac{1}{\pi^2}| = \frac{\rho}{2}$, where ρ is the distance from $\frac{1}{\pi^2}$ to the set of the rest eigenvalues of the operator $\widetilde{B}(0)$. The parameters a, b, and c are already calculated [12]. Theorem 2.10 is proved. □

We note that in the same way it can be shown that the second eigenvalue of the operator $\widetilde{B}(\varepsilon)$ is simple, also. It is the principal point that this method gives possibility to include the study of nonself-adjoint operators of the form $A_\gamma^{[\alpha,\beta]}$ (and not only operators of the form $A_\gamma^{[\alpha,\beta]}$) in the general scheme of perturbation theory.

3 Boundary value problems for differential equations of second order with fractional derivatives in lower terms

Many problems of mathematical physics [5, 13, 14, 18] associated with perturbations of normal operators with discrete spectrum lead to the consideration in Hilbert space $\mathfrak{H}$ of the compact operator

$$A = (I + S)H,$$

called a *a weak perturbation* H (for a compact S) or as the operator of *Keldysh type*. In present paper, we consider the operator of Keldysh type B, generated by the differential expression

$$u'' + \sum_{j=1}^{n} a_j(x) D_{0x}^{\alpha_j} u + q(x)u = \lambda u \tag{3.1}$$

and the boundary conditions of Sturm–Liouville type:

$$u(0) = 0, \quad u(1) = 0. \tag{3.2}$$

Here, $0 < \alpha_n < \cdots < \alpha_1 = \alpha < 2$.

For the first time (see [29] and references therein), for $0 < \alpha_n < \cdots < \alpha_1 = \alpha < 1$, this operator was considered in [29] in conjunction with the study of Tricomi's gas (fluid) at the sonic line. There was established that many direct and inverse problems associated with a degenerating hyperbolic equation and equation of the mixed hyperbolic-parabolic type are reduced to equations of the type (3.1). In particular in [17], an analogue of a problem of Tricomi type for the hyperbolic-parabolic equation

$$|y|^{mH(-y)} \frac{\partial^2 u}{\partial^2 x} = \frac{\partial^{1+H(-y)} u}{\partial^{1+H(-y)} y} u$$

where $m = \text{const} > 0$, and $H(y)$ is the Heavyside function, $u = u(x, y)$, may be reduced to problem (3.1)–(3.2).

Later, in 1984 the book of A. I. Tseytlin [32] was published. There, using problem (3.1)–(3.2), were studied problems of structural mechanics. But only recently, problem (3.1)–(3.2) is the center of attention of many authors (see [18] and references therein) because it simulates many physical processes. For example, the fractional Langevin equation is the focus of many authors, also (see, e. g., [18] and references therein):

$$m \frac{d^2 x(t)}{dt^2} = F(x) - \gamma D_{0t}^{\alpha} x + \xi(t).$$

In [18] (and references therein) using problem (3.1)–(3.2), the oscillator motion under elastic forces, typical for viscoelastic media was investigated.

Note, that for $0 < \alpha < 1$, the spectral structure of operator $\tilde{B}$ generated by the problem

$$u'' + \varepsilon D_{0x}^{\alpha} u = \lambda u, \tag{3.3}$$

$$u(0) = 0, \quad u(1) = 0, \tag{3.4}$$

was considered in detail in our paper [19]. In particular, there was shown the following theorem.

Theorem 3.1. *If $|\varepsilon| < \frac{10}{20}$, then all eigenvalues of operator $\tilde{B}$ are simple and real.*

From this theorem follows that the operator $\tilde{B}$ generated no associated functions.

Theorem 3.2. *The number λ is an eigenvalue of problem* (3.3)–(3.4) *iff λ is a zero of the function*

$$\omega(\lambda) = 1 + \sum_{n=1}^{\infty} \sum_{m=0}^{n} (-\varepsilon)^n \frac{C_n^m \lambda^{n-m}}{\Gamma(2n - m\alpha + 2)}. \tag{3.5}$$

The eigenfunctions of problem (A) take the form

$$\chi_i(x) = x + \sum_{n=1}^{\infty} \sum_{m=0}^{n} (-\varepsilon)^n \frac{C_n^m \lambda_i^{n-m}}{\Gamma(2n - m\alpha + 2)} x^{2n-m\alpha+1}, \tag{3.6}$$

where λ_i are zeros of the function $\omega(\lambda)$. In [19], it was proved that the system of eigenfunctions (3.6) is complete in $L_2(0,1)$. But this system is not orthogonal. Therefore, in paper [19] were considered, together with problem (3.3)–(3.4), the problem conjugate to it.

Let us consider operator B generated by the differential equation (3.3) and boundary conditions (3.4).

Theorem 3.3. *Let $0 < \alpha_n < \cdots < \alpha_1 = \alpha < 1$; all coefficients $a_j(x)$ and $q(x)$ are continuously differentiable on $[0,1]$. Then the system of eigenfunctions and associated functions of operator B is complete in $L_2(0,1)$.*

Proof. The study of the spectrum of the operator B reduces to study the spectrum of the linear beam operator $L(\lambda) = I + M - \lambda N$, where

$$Mu = \int_0^1 G(x,t)\left(\sum_{j=1}^{n} a_j(t) D_{0t}^{\alpha_j} u + q(t)u\right)dt,$$

$$Nu = \int_0^1 G(x,t)u(t)dt,$$

and

$$G(x,t) = \begin{cases} t(x-1), & t \le x, \\ x(t-1), & t > x. \end{cases}$$

Let us show that the operator M is compact. We can show that

$$\begin{aligned} Mu = \sum_{j=1}^{n} \frac{1}{\Gamma(1-\alpha_j)} \Bigg[x \int_x^1 u(\xi)d\xi \int_\xi^1 \frac{((1-t)a_j(t))}{(t-\xi)^{\alpha_j}} \\ + \int_0^x u(\xi) \left\{ (1-x) \int_\xi^x \frac{[ta_j(t)]'}{(t-\xi)^{\alpha_j}} dt - \int_x^1 \frac{[(t-1)a_j(t)]'}{(t-\xi)^{\alpha_j}} dt \right\} d\xi \Bigg] \\ + \int G(x,t)q(t)u(t)dt. \end{aligned}$$

Since $0 < \alpha_n < \cdots < \alpha_1 = \alpha < 1$ and all coefficients $a_j(x)$ and $q(x)$ are continuously differentiable in $[0,1]$, then the operator M is compact; it is clear that the operator N is self-adjoint. Thus, from Keldysh's theorem ([21], p. 318) it follows the completeness of system of eigenfunctions and associated functions. □

Remark 3.1. Let $1 < \alpha < 2$, then it is obvious that

$$\begin{aligned} ND_{0x}^{\alpha}u &= \frac{1}{\Gamma(2-\alpha)} \int_0^1 G(x,t)\left\{\frac{d^2}{dt^2}\int_0^1 \frac{u(\zeta)}{(t-\zeta)^{\alpha-1}}d\zeta\right\} \\ &= \frac{1}{\Gamma(2-\alpha)} \int_0^x t(x-1)\left\{\int_0^t \frac{u(\zeta)}{(t-\zeta)^{\alpha-1}}d\zeta\right\}'' dt \\ &\quad + \frac{1}{\Gamma(2-\alpha)} \int_x^1 x(t-1)\left\{\int_0^t \frac{u(\zeta)}{(t-\zeta)^{\alpha-1}}d\zeta\right\}'' dt \\ &= \frac{1}{\Gamma(2-\alpha)}\left[\int_0^x (x-t)^{1-\alpha}u(t)dt - \int_0^1 x(1-t)^{1-\alpha}u(t)dt\right], \end{aligned}$$

and from this it follows that the operator ND_{0x}^{α} is compact for $1 < \alpha < 2$. Thus, the system of eigenfunctions and associated functions of the operator B is complete in $L_2(0,1)$ even for $0 < \alpha_n < \cdots < \alpha_1 = \alpha < 2$.

Remark 3.2. Next, let us denote $n(r,B)$ the exact number of characteristic values of the operator B lying in circle $|\lambda| \leq r$. The problem of allocation of characteristic values of the operator B formulates as inestigationo of asymptotic properties of $n(r,B)$ for $r \to \infty$. In [18], this problem was solved when the order of fractional derivative D_{0x}^{α} is less than 1. In [18], the study of the function $n(r,B)$ is reduced to the one of the spectrum for the linear beam operator $L(\lambda) = I + M - \lambda N$.

Since M is a compact operator and N is a positive operator, then by Keldysh's theorem ([21], p. 318) we have

$$\lim_{x\to 0} \frac{n(r,B)}{n(r,N)} = 1$$

if for the distribution function $n(r,B)$ of characteristic values of the operator N we may choose nondecreasing function $\varphi(r)$ $(0 \leq r \leq \infty)$ with such that [21]:

1. $\lim_{r\to\infty} \varphi(r) = \infty$;
2. $\lim_{r\to\infty} (\ln \varphi(r))' < \infty$;
3. $\lim_{r\to\infty} \frac{n(r,B)}{\varphi(r)} = 1$.

Obviously, in our case as in [18], we may take the function $\sqrt{r}$ as $\varphi(r)$. Any linearized mechanical system in which there is energy dissipation is described by a linear operator A, densely defined in H, with values of the form (Af, f) in the left half-plane:

$$\mathrm{Re}(Af, f) \le 0, \quad (f \in D_A).$$

In quantum mechanics, energy dissipation is characterized by the fact that the form of the linear operator describing the physical system lies in the upper half-plane, that is,

$$\mathrm{Im}(Af, f) \ge 0, \quad (f \in D_A).$$

For definiteness, when speaking of dissipative operators, we shall have in mind the operators of the latter type; dissipative operators of quantum mechanics.

Remark 3.3. It is known that any linearized mechanical system that has an energy dissipation is described by linear operator $\tilde{A}$, densely defined in a Hilbert space $\mathfrak{H}$ with values of the form $(\tilde{A}f, f)$ in the left half-plane

$$\mathrm{Re}(\tilde{A}f, f) \le 0, \quad (f \in \mathfrak{D}_{\tilde{A}}).$$

As the operator $\tilde{B}$ describes oscillations of mechanical system, then it should be dissipative.

In this paragraph, we show that the operator $\tilde{B}$ is dissipative.

First, we shall note papers of F. Tricomi, Matsaev, and Palant [18] (and references therein) (where it was shown that values of the form $(I^\alpha f, f)$ are lying in the angle $|\arg\lambda| \le \frac{\alpha\pi}{2}$, here I^α is a fractional integral in Riemann–Liouville sense of order α) and papers of authors [18] (and references therein), where it was established

$$\mathrm{Re}\big(D^\alpha_{0x}u, u\big) \ge 0, \quad 0 < \alpha < 1, \tag{3.7}$$

and

$$\mathrm{Re}\big(D^\alpha_{0x}u, u\big) \le 0, \quad 1 < \alpha < 2. \tag{3.8}$$

Theorem 3.4. *If* $\varepsilon > 0$ *and* $0 < \alpha < 1$, *then the operator, generated by the problem*

$$\begin{aligned} u'' - \varepsilon D^\alpha_{0x}u &= \lambda u, \\ u(0) = 0, \quad & u(1) = 0, \end{aligned}$$

is dissipative.

Proof. This theorem follows from the relation (3.7) and the fact that the operator

$$Tu = \begin{cases} -u'', \\ u(0) = 0, \quad u(1) = 0, \end{cases}$$

is dissipative. □

Theorem 3.5. *If $\varepsilon < 0$ and $1 < \alpha < 2$, then the operator generated by the problem*

$$u'' - \varepsilon D_{0x}^{\alpha} u = \lambda u,$$
$$u(0) = 0, \quad u(1) = 0,$$

is dissipative.

Proof. The scheme of proof of Theorem 3.5 is the same as the one of Theorem 3.4. □

Remark 3.4. Let us consider operator A, generated by the problem

$$u'' - \varepsilon D_{0x}^{\alpha} u = \lambda u, \tag{3.9}$$
$$u(0) = 0, \quad u(1) = 0, \tag{3.10}$$

where $0 < \alpha < 2$.

Finally, let us show that the operator A is oscillatory (if the operator describes the oscillatory motions, then it has a whole complex of the oscillatory properties).

It is known that [18] (and references therein) if $0 \le \varepsilon \le \frac{1}{3}$, and $1 < \alpha < 2$, then the Green function of the problem (3.9)–(3.10) is of fixed sign (we shall note that Green's function of problem (3.9)–(3.10) was first constructed by one of the authors in his paper [18] (and references therein)). Unfortunately, this very important property of Green's function is possible to get only for a small enough ε. This is primarily due to the fact that Green's function $G_2(x,\tau)$ [29] of the problem (3.9)–(3.10), for $1 < \alpha < 2$, has the following complex structure:

$$G_2(x,\tau) = G_1(x,\tau) - \frac{\varepsilon}{E_{1/2}(\varepsilon,2)} \int_{\tau}^{1} E_\beta[\varepsilon(\eta-\tau)]^{\beta} d\eta \int_{0}^{1} G(x,t) D_{0t}^{\alpha-1} E_\beta[\varepsilon t^{\beta}] dt,$$

$$G_1(x,\tau) = \begin{cases} (1-x)\int_\tau^x E_\beta[\varepsilon(t-\tau)]dt - x - \int_x^1 E_\beta[\varepsilon(t-\tau)^\beta]dt, & x \ge \tau, \\ -x\int_\tau^1 E_\beta[\varepsilon(t-\tau)^\beta]dt, & x \le \tau. \end{cases}$$

For $0 < \alpha < 1$, $|\varepsilon| < 1/4$, Green's function of the problem (3.9)–(3.10), was constructed in [18] (and references therein). Let us show how this function was constructed. Since the problem (3.9)–(3.10), for $0 < \alpha < 1$, is equivalent to the equation

$$u(x) + \frac{\varepsilon}{\Gamma(2-\alpha)} \left\{ \int_0^x (x-t)^{1-\alpha} u(t)dt - \int_0^1 x(1-t)^{1-\alpha} u(t)dt \right\}$$
$$= \lambda \int_0^1 G(x,t)u(t)dt$$

then

$$u(x) = \lambda(I - \varepsilon K)^{-1} \int_0^1 G(x,t)u(t)dt,$$

where

$$G(x,t) = \begin{cases} t(x-1), & t \le x, \\ x(t-1), & t > x, \end{cases}$$

$$\begin{aligned} Ku &= -\frac{1}{\Gamma(2-\alpha)} \int_0^x (x-t)^{1-\alpha} u(t)dt + \frac{1}{\Gamma(2-\alpha)} \int_0^1 x(1-t)^{1-\alpha} u(t)dt \\ &= (xJ_{0,1}^{2-\alpha} - J_{0,x}^{2-\alpha})u. \end{aligned}$$

We can show that

$$\begin{aligned} K^2u &= K \cdot Ku = (Kx)J_{0,1}^{2-\alpha}u - xJ_{0,1}^{4-2\alpha}u + J_{0,x}^{4-2\alpha}u, \\ K^3u &= K \cdot K^2u = (K^2x)J_{0,1}^{2-\alpha}u - (Kx)J_{0,1}^{4-2\alpha}u + xJ_{0,1}^{6-3\alpha}u - J_{0,x}^{6-3\alpha}u. \end{aligned}$$

By induction, we have

$$K^n = \sum_{i=1}^{n} (-1)^{i+1}(K^{n-i}x)J_{0,1}^{(2-\alpha)i}u + (-1)^{n+2}J_x^{(2-\alpha)n}u.$$

Thus,

$$\begin{aligned} (I - \varepsilon K)^{-1} &= I + \sum_{n=1}^{\infty} (\varepsilon K)^n u \\ &= I + \sum_{n=1}^{\infty} \varepsilon^n \left[\sum_{i=1}^{n} (-1)^{i+1}(K^{n-i}x)J_{0,1}^{(2-\alpha)i}u + (-1)^{n+2}J_{0,x}^{(2-\alpha)n}u \right] \\ &= I + \sum_{n=1}^{\infty} \sum_{i=1}^{n} \varepsilon^n (-1)^{i+1}(K^{n-i}x)J_{0,1}^{(2-\alpha)i}u + \sum_{n=1}^{\infty} (-1)^{n+2}J_{0,x}^{(2-\alpha)n}u. \end{aligned}$$

Since, for $|\varepsilon| < 1/4$ the kernel $k(x,t)$ of the operator K satisfies the condition

$$\left|k(x,t)\right| < 2,$$

we have that the Green's function of problem (3.9)–(3.10) is of fixed-sign for $0 < \alpha < 1$, also.

Remark 3.5. In [18] (and references therein) for the problem,

$$u'' - \varepsilon \cdot {}^{C}D_{0x}^{\alpha}u = \lambda u, \tag{3.11}$$

$$u(0) = 0, \quad u(1) = 0, \tag{3.12}$$

where

$${}^{C}D_{0t}^{\alpha}u = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{u^{(n)}(\tau)\,d\tau}{(t-\tau)^{\alpha+1-n}}$$

is the fractional derivative in the Caputo sequence of order α, and was proved in the following very important theorem.

Theorem 3.6. *Let $\lambda_m(\varepsilon)$ be the mth eigenvalue of problem* (3.11)–(3.12)*, and* $1 < \alpha < 2$*. Then for*

$$|\varepsilon| < \frac{\Gamma(2-\alpha)}{1+\frac{2m^2+2m+1}{2m+1}},$$

$\lambda_m(\varepsilon)$ is simple and real.

Proof. The proof will be provided as in [18] (and references therein). Let us consider $T(\varepsilon) = T + \varepsilon T_1$, where T is the differential operator

$$Tu = \begin{cases} -u'', \\ u(0) = 0, \quad u(1) = 0, \end{cases}$$

and $T_1 = {}^C D_{0x}^{\alpha}$, $1 < \alpha < 2$. Since all eigenvalues of the operator T are isolated and have multiplicity equal to 1, then all corresponding eigennumbers $\lambda_m(\varepsilon)$ and all eigenfunction $\varphi_m(\varepsilon, x)$ of the operator $T(\varepsilon)$ are holomorphic at least for ε small [25]

$$\lambda_m(\varepsilon) = \lambda_m^{(0)} + \varepsilon\lambda_m^{(1)} + \varepsilon^2\lambda_m^{(2)} + \cdots + \varepsilon^n\lambda_m^{(n)} + \cdots, \tag{3.13}$$

$$\varphi_m(\varepsilon, x) = \varphi_m^{(0)} + \varepsilon\varphi_m^{(1)} + \varepsilon^2\varphi_m^{(2)} + \cdots + \varepsilon^n\varphi_m^{(n)} + \cdots. \tag{3.14}$$

There are various formulas to calculate the lower limit for the radius of convergence r_m for the Taylor series (3.13)–(3.14).

We will use formula [23]

$$r_m = \min_{\zeta\in\Gamma}\left(a\|R(\zeta, T)\| + b\|TR(\zeta, T)\| + c\right)^{-1}, \tag{3.15}$$

where $R(\zeta, T) = (T - \zeta I)^{-1}$. In (3.15) as a contour Γ, we may consider the circle $|\zeta - \pi n^2| = \frac{\rho}{2}$, where ρ—the distance from πn^2 to all other eigennumbers of the operator T, and parameters a, b, c will be calculated below. In our case $c = 0$, [18] (and see references therein). To calculate a, b, we will note that

$$\|{}^C D_{0x}^{\alpha} u\| \le \frac{1}{\Gamma(2-\alpha)}\|u''\|.$$

Thus, we have

$$r_m \ge \frac{\Gamma(2-\alpha)}{1+\frac{2m^2+2m+1}{2m+1}}.$$

Here, we use the fact that

$$\|R(\zeta)\| = \frac{1}{\operatorname{dist}(\zeta; \Sigma(T))},$$

where $\Sigma(T)$ is the spectrum of the operator A ([23], p. 51),

$$TR(\zeta) = I + \zeta R(\zeta).$$

It remains to prove that all eigenvalues of this problem are real. Let us refer to formulas (3.13)–(3.14). For the coefficients $\lambda_m^{(n)}$ and $\varphi_m^{(n)}$, we will calculate by formulas, showed in [18] (and see references therein)

$$\lambda_m^{(n)} = \sum_{k=1}^{n} (\widetilde{A}_k \varphi_m^{(n-k)}, \varphi_m^{(0)}), \quad \varphi_m^{(n)} = \sum_{k=1}^{n} R(\lambda_m^{(k)} - \widetilde{A}_k)\varphi_m^{(n-k)}. \tag{3.16}$$

Here, R is the reduced resolvent of operator T, corresponding to eigennumber $\lambda_m^{(0)}$, and it is the integral operator with the kernel $S(x, y)$

$$S(x, y) = \Big[-y \cos \sqrt{\lambda_m^{(0)}} y \sin \sqrt{\lambda_m(0)} x$$
$$+ (1 - x) \sin \sqrt{\lambda_m^{(0)}} y \cos \sqrt{\lambda_m^{(0)}} x + \frac{1}{2} \sin \sqrt{\lambda_m^{(0)}} y \sin \sqrt{\lambda_m^{(0)}} x \Big], \quad y \le x$$

(for $x \le y$ on the right-hand side of this formula, we have to interchange y and x). It is clear that R transforms bijectively the H_0 (H_0—the orthogonal complement of the function $\sin \sqrt{\lambda_m^{(0)}} x$) into itself and "annul" $\sin \sqrt{\lambda_m^{(0)}} x$, and $\widetilde{A}_1 = T_1$, and $\widetilde{A}_k = 0$ for $k = 2, 3, \ldots$). By (3.16), we obtain that

$$\lambda_m^{(1)} = (T_1 \varphi_m^{(0)}, \varphi_m^{(0)}).$$

As the kernel of operator T_1 is a real-value, then $\operatorname{Im} \lambda_m^{(1)} = 0$. From (3.16) follows that

$$\varphi_m^{(1)} = R(\lambda_m^{(1)} - B_1)\varphi_m^{(0)}$$

as kernels of the operators R and T_1 are real-value, then $\operatorname{Im} \varphi_m^{(1)} = 0$. Thus, sequentially, we may set that $\operatorname{Im} \lambda_m^{(n)} = \operatorname{Im} \varphi_m^{(n)} = 0$, for all n ($n = 1, 2, 3, \ldots$). Thus, if ε is real, then $\lambda_m(\varepsilon)$ is real, also. □

Remark 3.6. Since

$$\lim_{n\to\infty} \frac{\Gamma(2-\alpha)}{1 + \frac{2n^2+2n+1}{2n+1}} = 0,$$

we cannot provide a statement, analogous to Theorem 3.1, for the case $1 < \alpha < 2$.

Bibliography

[1] A. V. Agibalova, Analog of the Birkhoff theorem and completeness of root subspaces of a differential operator of fractional order with matrix coefficients (in Russian), *Ukr. Mat. Visn.*, **6**(3) (2009), 283–310; translation in *Ukr. Math. Bull.*, **6**(2) (2009), 279–305.

[2] A. V. Agibalova, On the completeness of a system of eigenfunctions and associated functions of differential operators of the orders $(2-\varepsilon)$ and $(1-\varepsilon)$, *J. Math. Sci.*, **174**(4) (2011), 425–436.
[3] M. Aleroev, T. Aleroev, M. Kirane, and Y.-F. Tang, On one class of persymmetric matrices generated by boundary value problems for differential equations of fractional order, *Appl. Math. Comput.*, **268** (2015), 151–163.
[4] T. S. Aleroev, The Sturm–Liouville problem for differential fractional-order equations with ss in lower terms, *Differ. Uravn.*, **18**(2) (1982), 341–342.
[5] T. S. Aleroev, On the eigenfunctions of a nonself-adjoint operator, *Differ. Uravn.*, **25**(11) (1989), 1996–1997.
[6] T. S. Aleroev, On the eigenvalues of a boundary-value problem for a differential equation of fractional order, *Differ. Uravn.*, **36**(10) (2000), 1422–1423.
[7] T. S. Aleroev, Boundary-value problems for differential equations with fractional derivatives, Dissert. on Doctoral Degree (Phys.-Math. Sci.), MGU, Moscow, 2000.
[8] T. S. Aleroev, On the completeness of eigenfunctions of a differential fractional-order operator, *Differ. Uravn.*, **36**(6) (2000), 829–830.
[9] T. S. Aleroev, On the problem of zeros of the Mittag-Leffler function and the spectrum of a differential fractional-order operator, *Differ. Uravn.*, **36**(9) (2000), 1414–1415.
[10] T. S. Aleroev, On a class of operators connected with differential equations of fractional-order, *Sib. Mat. Zh.*, **46**(6) (2005), 1201–1207.
[11] T. S. Aleroev and H. T. Aleroeva, A problem on the zeros of the Mittag-Leffler function and the spectrum of a fractional-order differential operator, *Electron. J. Qual. Theory Differ. Equ.*, **2009** (2009), 25, 1–18.
[12] T. S. Aleroev and H. T. Aleroeva, On a class of nonself-adjoint operators that accompany differential equations of fractional order, *Ukr. Math. Bull.*, **12**(3) (2015), 293–310.
[13] T. S. Aleroev and H. T. Aleroeva, On the eigenfunctions and eigenvalues of a class of non-selfadjoint operators, *Lobachevskii J. Math.*, 37(3) (2016), 227–230.
[14] T. S. Aleroev and H. T. Aleroeva, Erratum to: "On the eigenfunctions and eigenvalues of a class of nonself-adjoint operators", *Lobachevskii J. Math.*, **37**(6) (2016), 815–815.
[15] T. S. Aleroev and H. T. Aleroeva, The Sturm–Liouville problem for a second-order differential equation with fractional derivatives in the lower terms, *Russ. Math. (Izv. VUZ)*, **10** (2014), 3–12.
[16] T. S. Aleroev, H. T. Aleroeva, J. Huang, M. V. Tamm, Y. Tang, and Y. Zhao, Boundary value problems of fractional Fokker–Planck equations, *Comput. Math. Appl.*, **73**(6) (2017), 959–969.
[17] T. S. Aleroev, H. T. Aleroeva, N.-M. Nie, and Y.-F. Tang, Boundary value problems for differential equations of fractional order, *Mem. Differ. Equ. Math. Phys.*, **49** (2010), 19–82.
[18] T. S. Aleroev and E. R. Kekharsaeva, Boundary value problems for differential equations with fractional derivatives, *Integral Transforms Spec. Funct.*, **28**(12) (2017), 900–908.
[19] T. S. Aleroev, M. Kirane, and Y.-F. Tang, Boundary-value problems for differential equations of fractional order, *J. Math. Sci.*, **10**(2) (2013), 158–175.
[20] M. M. Dzhrbashyan, The boundary-value problem for a differential fractional-order operator of the Sturm–Liouville type, *Izv. Akad. Nauk Arm. SSR, Ser. Mat.*, **5**(2) (1970), 71–96.
[21] I. Ts. Gokhberg and M. G. Krein, *Introduction to the Theory of Linear Nonself-Adjoint Operators*, Amer. Math. Soc., Providence, RI, 1969.
[22] I. Ts. Gokhberg and M. G. Krein, *Theory and Applications of Volterra Operators in Hilbert Space*, Amer. Math. Soc., Providence, RI, 1970.
[23] T. Kato, *Perturbation theory for linear operators*, Mir, Moscow, 1972.
[24] M. A. Krasnosel'sky and G. M. Vaiiniko, *Approximate Solutions of Operator Equations* [in Russian], Moscow, Nauka, 1969.
[25] B. V. Loginov, To the estimate of the exactness of the method of perturbations, *Izv. Akad. Nauk Uzb. SSR, Ser. Fiz.-Mat. Nauk*, **6** (1963), 14–19.

[26] F. Mainardi, Fractional relaxation-oscillation and fractional diffusion-wave phenomena, *Chaos Solitons Fractals*, **7**(9) (1996), 1461–1477.
[27] M. M. Malamud, Similarity of Volterra operators and related questions of the theory of differential equations of fractional order, *Trans. Mosc. Math. Soc.*, **55** (1994), 57–132.
[28] M. M. Malamud and L. L. Oridoroga, Analog of the Birkhoff theorem and the completeness results for fractional order differential equations, *Russ. J. Math. Phys.*, **8**(3) (2001), 287–308.
[29] A. M. Nakhushev, *Fractional Calculus and Its Application* [in Russian], Fizmatlit, Moscow, 2003.
[30] A. Yu. Popov, On the number of real eigenvalues of the boundary-value problem for a second-order equation with fractional derivative, *Fundam. Prikl. Mat.*, **12**(6) (2006), 137–155.
[31] A. V. Pskhu, *Partial Differential Equations of Fractional Order*, Science, Moscow, 2005.
[32] A. I. Tseytlin, *Applied Methods for Boundary Value Problems Solving in Structural Mechanics*, Stroyizdat, Moscow, 1984.

Mark Edelman

Maps with power-law memory: direct introduction and Eulerian numbers, fractional maps, and fractional difference maps

Abstract: In fractional dynamics, as in regular dynamics, discrete maps can be used to investigate general properties of dynamical systems. Maps with power-law memory related to fractional dynamics can be introduced directly as convolutions. The same maps are solutions of fractional differential equations with periodic delta-function kicks. Solutions of fractional difference equations also can be represented in the form of maps with asymptotically power-law memory. Fractional generalizations of the logistic map (quadratic nonlinearity) and the standard map (harmonic nonlinearity) are introduced in this chapter to investigate the general properties of nonlinear fractional dynamics.

Keywords: Power-law memory, fractional maps, fractional difference

MSC 2010: 47H99, 60G99, 34A99, 39A70

1 Introduction

Memory is an important characteristic of many natural and social systems (some references can be found in the review papers [16, 22]). In many cases, memory obeys a power law (PL) and the corresponding evolution can be described by fractional differential/difference equations. To find how a continuous system with PL memory evolves, one should solve a fractional differential equation (FDE), which is an integrodifferential equation. This is not an easy task—there are no high-order algorithms to solve FDE. The existing algorithms are discrete approximations of continuous equations and they can be viewed as maps with PL memory. The first review of the maps with memory in the context of the evolution of fractional systems has been written by A. Stanislavsky [56], who applied one of the algorithms of the numerical fractional integration to derive a fractional version of the logistic map. For various forms of fractional maps as approximations to solutions of FDE, see, for example, [26, 53, 55].

Acknowledgement: The author expresses his gratitude to the administration of the Courant Institute for the opportunity to complete this work at Courant and to Virginia Donnelly for technical help.

Mark Edelman, Department of Physics, Stern College at Yeshiva University, 245 Lexington Ave, New York, NY 10016, USA; and Courant Institute of Mathematical Sciences, New York University, 251 Mercer St., New York, NY 10012, USA; and Department of Mathematics, BCC, CUNY, 2155 University Avenue, Bronx, New York, 10453, USA, e-mail: edelman@cims.nyu.edu

https://doi.org/10.1515/9783110571660-003

In regular dynamics discrete maps are a powerful instrument to investigate general properties of nonlinear systems exhibiting period doubling bifurcations and chaos. Maps are also a good starting point to investigate the behavior of nonlinear systems with memory. To avoid the errors related to numerical approximations in the investigation of general properties of fractional dynamical systems, we choose the exact equations modeling systems with PL memory. In this chapter, we introduce three different ways to derive the exact forms of fractional maps which then are used in the author's chapter in Volume 4 of this book [23] to investigate general properties of fractional dynamics.

In Section 2, we discuss the direct introduction of maps with PL memory as convolutions and consider their relationship to fractional calculus. One of the ways in which the universal map, generating the whole class of the area preserving maps, can be derived in regular dynamics is to consider equations describing the evolution of continuous systems under periodic delta-function kicks. In Section 3, we describe the derivation of fractional maps by the application of the same idea to fractional systems. It is interesting that the maps obtained in Section 3 are identical to the maps directly introduced in Section 2. Evolution of systems which are essentially discrete can be considered in the framework of the well-developed regular discrete calculus dealing with the finite differences of integer orders [2, 38, 45]. Fractional discrete calculus is a generalization of regular discrete calculus [35]. In Section 4, we describe how to derive fractional difference maps which are equivalent to fractional difference equations. The memory in these maps is falling/rising factorial-law memory which is asymptotically PL memory. In conclusion, Section 5, we use the results of previous sections to introduce the fractional/fractional difference logistic and standard α-families of maps, which are used in the author's chapter in Volume 4 of this book [23] to investigate general properties of fractional dynamics.

2 Direct introduction of maps with power-law memory

The most general form of an explicit map with memory would be

$$\mathbf{x}_{n+1} = \mathbf{f}_{n+1}(\mathbf{x}_n, \mathbf{x}_{n-1}, \dots, \mathbf{x}_0, P), \tag{1}$$

where $\mathbf{f}_k$ and $\mathbf{x}_k$ are N-dimensional vectors, $n, k, N \in \mathbb{Z}$, $k \geq 0$, $n > 0$, and P is a set of parameters. The maps of this type are called long-term memory maps, but we will refer to them simply as maps with memory because this is the only type of memory maps considered in this chapter. There is practically no way to derive the general properties of systems with memory from equation (1). A simpler, one-dimensional form of a map with memory

$$x_{n+1} = \sum_{k=0}^{n} V_\alpha(n,k) G_K(x_k), \tag{2}$$

where $V_\alpha(n,k)$ and α characterize memory effects and $G_K(x)$ is a real valued function depending on the parameter K, was investigated in relation to various problems in physics and mathematics in papers [28, 30, 32–34, 37, 56].

The maps related to fractional dynamics, convolutions with PL weights, which we will investigate can be written as

$$x_n = \sum_{k=0}^{n-1} (n-k)^{\alpha-1} G_K(x_k, h), \tag{3}$$

where K is a parameter and h is a constant time step between time instants $t_k = hk$ and $t_{k+1} = h(k+1)$. What follows in this section is based on the results obtained in the article [19].

For $\alpha = 1$, equation (3) is equivalent to

$$x_1 = G_K(x_0, h), \quad x_n - x_{n-1} = G_K(x_{n-1}, h), \quad (n > 1) \tag{4}$$

and requires one initial condition x_0. In the case $\alpha = 2$, equation (3) yields

$$\begin{gathered} x_1 = G_K(x_0, h), \quad x_2 = 2G_K(x_0, h) + G_K(x_1, h), \\ x_n - 2x_{n-1} + x_{n-2} = G_K(x_{n-1}, h), \quad (n > 2) \end{gathered} \tag{5}$$

with the initial condition x_0. Calculations of the third and the fourth backward differences of x_n yield

$$\begin{gathered} x_1 = G_K(x_0, h), \quad x_2 = 4G_K(x_0, h) + G_K(x_1, h), \\ x_3 = 9G_K(x_0, h) + 4G_K(x_1, h) + G_K(x_2, h), \\ x_n - 3x_{n-1} + 3x_{n-2} - x_{n-3} = G_K(x_{n-1}, h) + G_K(x_{n-2}, h), \quad (n > 3) \end{gathered} \tag{6}$$

for $\alpha = 3$ and

$$\begin{aligned} x_1 &= G_K(x_0, h), \quad x_2 = 8G_K(x_0, h) + G_K(x_1, h), \\ x_3 &= 27G_K(x_0, h) + 8G_K(x_1, h) + G_K(x_2, h), \\ x_4 &= 64G_K(x_0, h) + 27G_K(x_1, h) + 8G_K(x_2, h) + G_K(x_3, h), \\ x_n &- 4x_{n-1} + 6x_{n-2} - 4x_{n-3} + x_{n-4} \\ &= G_K(x_{n-1}, h) + 4G_K(x_{n-2}, h) + G_K(x_{n-3}, h), \quad (n > 4) \end{aligned} \tag{7}$$

for $\alpha = 4$.

The generalization of equations (4)–(7) proven in [19] is formulated as the following theorem.

Theorem 1. *Any long-term memory map*

$$x_n = \sum_{k=0}^{n-1} (n-k)^{m-1} G_K(x_k, h), \quad (n > 0), \tag{8}$$

where $m \in \mathbb{N}$, is equivalent to the m-step memory map

$$\begin{aligned} x_n &= \sum_{k=0}^{n-1} (n-k)^{m-1} G_K(x_k, h), \quad (0 < n \le m), \\ &\sum_{k=0}^{m} (-1)^k \binom{m}{k} x_{n-k} \\ &= \delta_{m-1} G_K(x_{n-1}, h) + \sum_{k=0}^{m-2} A(m-1, k) G_K(x_{n-k-1}, h), \quad (n > m). \end{aligned} \tag{9}$$

In the last equation, the alternating sum on the left-hand side (LHS) is the mth backward difference for the x_n, δ_i is the Kronecker delta ($\delta_0 = 1$; $\delta_{i \ne 0} = 0$), and $A(n, k)$ are the Eulerian numbers

$$A(n, k) = \sum_{j=0}^{k} (-1)^j \binom{n+1}{j} (k + 1 - j)^n \tag{10}$$

defined for $k, n \in \mathbb{N}_0$ ($\mathbb{N}_0 := \mathbb{N} \cup \{0\}$), which satisfy the recurrence formula

$$A(n, k) = (k+1) A(n-1, k) + (n-k) A(n-1, k-1). \tag{11}$$

Fractional generalization of Theorem 1 is formulated as the following.

Theorem 2. *Any long-term memory map*

$$x_n = \sum_{k=0}^{n-1} (n-k)^{\alpha-1} G_K(x_k, h), \quad (n > 0) \tag{12}$$

where $\alpha \in \mathbb{R}$ and $n \in \mathbb{N}$, is equivalent to the map

$$\begin{aligned} &\sum_{k=0}^{n} (-1)^k \binom{\alpha}{k} x_{n-k} \\ &= (-1)^n \binom{\alpha}{n} x_0 + \sum_{k=0}^{n-1} G_K(x_{n-k-1}, h) A(\alpha - 1, k). \end{aligned} \tag{13}$$

Here, the fractional binomial coefficients are defined as (see [31, 54])

$$\binom{\alpha}{n} = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!} = \frac{\Gamma(\alpha+1)}{\Gamma(n+1)\Gamma(\alpha-n+1)} \tag{14}$$

and the Eulerian numbers with fractional order parameters are defined as [8]

$$A(\alpha,k)=\sum_{j=0}^{k}(-1)^j\binom{\alpha+1}{j}(k+1-j)^{\alpha}. \tag{15}$$

Maps with $m-1<\alpha\le m$, where $m\in\mathbb{N}$, are equivalent to m-dimensional maps (see [14, 59] and Section 3 of this article). For integer $\alpha=m>1$, these maps are m-dimensional volume preserving maps with no (one-step) memory. After the introduction,

$$\begin{aligned}
x_k^{(0)}&=x_k,\\
x_k^{(1)}&=x_k^{(0)}-x_{k-1}^{(0)},\\
&\ldots,\\
x_k^{(r)}&=x_k^{(r-1)}-x_{k-1}^{(r-1)},\\
&\ldots,\\
x_k^{(m-1)}&=x_k^{(m-2)}-x_{k-1}^{(m-2)},
\end{aligned} \tag{16}$$

where $k\ge m-1$, the map equation (9) can be written as the m-dimensional map

$$\begin{cases}
x_n^{(m-1)}=x_{n-1}^{(m-1)}+\sum_{k=0}^{m-2}A(m-1,k)G_K(\sum_{i=0}^{k}(-1)^i\binom{k}{i}x_{n-1}^{(i)},h)\\
\qquad\;\;=x_{n-1}^{(m-1)}+F(x_{n-1}^{(0)},\ldots,x_{n-1}^{(m-2)}),\\
x_n^{(m-2)}=x_{n-1}^{(m-2)}+x_n^{(m-1)},\\
\ldots,\\
x_n^{(m-k)}=x_{n-1}^{(m-k)}+x_n^{(m-k+1)},\\
\ldots,\\
x_n^{(0)}=x_{n-1}^{(0)}+x_n^{(1)},
\end{cases} \tag{17}$$

whose Jacobian is equal to one.

To establish correspondence between maps with memory and fractional calculus, let us assume, as in the definition of the Grünwald–Letnikov fractional derivative, that

$$x=x(t),\quad x_k=x(t_k),\quad t_k=a+kh,\quad nh=t-a \tag{18}$$

for $0\le k\le n$. Let us also assume

$$G_K(x,h)=\frac{1}{\Gamma(\alpha)}h^{\alpha}G_K(x), \tag{19}$$

where $G_K(x)$ is a continuous function and $x(t)\in C^m$. The map equation (3) can be written as

$$x(t)=\frac{1}{\Gamma(\alpha)}h\sum_{k=0,nh=t-a}^{n-1}(t-t_k)^{\alpha-1}G_K(x(t_k)) \tag{20}$$

and in the limit $h\to 0$, as shown in [19], Theorem 2 yields the following theorem.

Theorem 3. *For $\alpha \in \mathbb{R}$, $\alpha > 0$ the Volterra integral equation of the second kind*

$$x(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{G_K(x(\tau))d\tau}{(t-\tau)^{1-\alpha}}, \quad (t > a), \tag{21}$$

where $G_K(x(\tau))$ is a continuous on $x \in \{[x_{\min}(\tau), x_{\max}(\tau)] : \tau \in [a,t]\}$ function satisfying the Lipschitz condition is equivalent to the fractional differential equation

$${}_aD_t^{\alpha}x(t) = G_K(x(t)), \tag{22}$$

where the derivative on the LHS is the Grünwald–Letnikov fractional derivative, with the zero initial conditions

$$c_k = \frac{d^k x(t)}{dt^k}(t=a) = 0, \quad k = 0, 1, \ldots, \lceil\alpha\rceil - 1. \tag{23}$$

This theorem is a particular form of the well-known results obtained by Kilbas with co-authors. In [41, 42], Kilbas and Marzan proved that the FDE with the Caputo fractional derivative (FD)

$${}^C D_a^{\alpha} x(t) = G_K(t, x(t)), \quad 0 < \alpha, \quad t \in [a, T] \tag{24}$$

with the initial conditions

$$\frac{d^k x(t)}{dt^k}(t=a) = c_k, \quad k = 0, 1, \ldots, \lceil\alpha\rceil - 1 \tag{25}$$

is equivalent to the Volterra integral equation of the second kind

$$x(t) = \sum_{k=0}^{\lceil\alpha\rceil-1} \frac{c_k}{\Gamma(k+1)}(t-a)^k + \frac{1}{\Gamma(\alpha)} \int_a^t \frac{G_K(\tau, x(\tau))d\tau}{(t-\tau)^{1-\alpha}}, \quad (t > a) \tag{26}$$

where $x \in C^{\lceil\alpha\rceil-1}[a,T]$. The equivalence of the FDE with the Riemann–Liouville FD

$${}^{\mathrm{RL}}D_a^{\alpha} x(t) = G_k(t, x(t)), \quad 0 < \alpha \tag{27}$$

with the initial conditions

$$({}^{\mathrm{RL}}D_a^{\alpha-k} x)(a+) = c_k, \quad k = 1, 2, \ldots, \lceil\alpha\rceil \tag{28}$$

to the Volterra integral equation of the second kind

$$x(t) = \sum_{k=1}^{\lceil\alpha\rceil} \frac{c_k}{\Gamma(\alpha-k+1)}(t-a)^{\alpha-k} + \frac{1}{\Gamma(\alpha)} \int_a^t \frac{G_K(\tau, x(\tau))d\tau}{(t-\tau)^{1-\alpha}}, \quad (t > a) \tag{29}$$

for $x \in L(a, T)$, $G : (a, T] \times Q \to \mathbb{R}$ (here Q is an open set in $\mathbb{R}$), and $G(t, x) \in L(a, T)$ for any $x \in Q$, was proven by Kilbas, Bonilla, and Trujillo in [39, 40]. These results and the notation used in equations (24)–(29) are now textbook materials which can be found, for example, in Chapter 3 of [43].

The generalization of Theorem 3 for nonzero initial conditions—the equivalence of the problem equations (27) and (28) (with the Grünwald–Letnikov FD) to the problem equation (29) in the case $c_{\lceil \alpha \rceil} = 0$, which corresponds to a finite value of $x(a)$, is the limiting case ($h \to 0$) of the following theorem.

Theorem 4. *Any long-term memory map*

$$x_n = \sum_{k=1}^{\lceil \alpha \rceil - 1} \frac{c_k}{\Gamma(\alpha - k + 1)} (nh)^{\alpha - k} + \sum_{k=0}^{n-1} (n-k)^{\alpha - 1} G_K(x_k, h), \tag{30}$$

where $\alpha \in \mathbb{R}$, is equivalent to the map

$$\begin{aligned} &\sum_{k=0}^{n} (-1)^k \binom{\alpha}{k} x_{n-k} \\ &\quad - \sum_{i=1}^{\lceil \alpha \rceil - 1} \frac{c_i h^{\alpha - i}}{\Gamma(\alpha - i + 1)} \sum_{k=0}^{i-1} (-1)^k \binom{i-1}{k} A(\alpha - i, n - k - 1) \\ &= (-1)^n \binom{\alpha}{n} x_0 + \sum_{k=0}^{n-1} G_K(x_{n-k-1}, h) A(\alpha - 1, k). \end{aligned} \tag{31}$$

The map equation (30) is a general form of the map with power-law memory, identical to the fractional map of the following Section 3, whose properties (with the assumption equation (19)) have been investigated (see author's chapter in Volume 4 of this book [23]).

3 Fractional maps

In regular dynamics, many area-preserving two-dimensional maps can be considered as particular forms of the universal map (see, e. g., [10, 46, 63])

$$p_{n+1} = p_n - TG(x_n), \tag{32}$$

$$x_{n+1} = x_n + p_{n+1} T, \tag{33}$$

where $G(x)$ is a real valued function and T is a parameter defined below. The universal map is the solution of the differential equation

$$\ddot{x} + G(x) \sum_{n=-\infty}^{\infty} \delta\Big(\frac{t}{T} - (n + \varepsilon)\Big) = 0, \tag{34}$$

describing evolution of a particle under periodic (with the period T) delta-function kicks with arbitrary initial conditions. Tarasov and Zaslavsky [60] proposed derivation of the fractional universal map based on equation (34) in which the second derivative on the LHS is substituted by a derivative of the fractional order α. This idea was further developed by Tarasov in papers [57, 58] and reviewed in his book [59] and by the author of this paper in [14, 16]. Tarasov's results were summarized as two theorems. The following theorem defines the Caputo universal map.

Theorem 5. *For $\alpha \in \mathbb{R}$, $\alpha > 1$, $m = \lceil \alpha \rceil$ the solution $x_n = x(t = nh)$ of the FDE*

$$ {}^{C}D_0^{\alpha} x(t) + G_K\big(x(t)\big) \sum_{n=-\infty}^{\infty} \delta\left(\frac{t}{h} - n\right) = 0 \tag{35}$$

with the initial conditions

$$D^k x(0+) = b_k, \quad k = 0, \dots, m-1 \tag{36}$$

is equivalent to the Caputo universal map

$$x_{n+1} = \sum_{k=0}^{m-1} \frac{b_k}{k!} h^k (n+1)^k - \frac{h^{\alpha}}{\Gamma(\alpha)} \sum_{k=0}^{n} G_K(x_k)(n-k+1)^{\alpha-1}, \tag{37}$$

where $n \in \mathbb{Z}$, $n > 0$.

The map equation (37) can be written as the m-dimentional map

$$x_{n+1}^{(s)} = \sum_{k=0}^{m-s-1} \frac{x_0^{(k+s)}}{k!} h^k (n+1)^k - \frac{h^{\alpha-s}}{\Gamma(\alpha-s)} \sum_{k=0}^{n} G_K(x_k)(n-k+1)^{\alpha-s-1}, \tag{38}$$

where $x^{(s)}(t) = D^s x(t)$, $s = 0, 1, \dots, m-1$.

The theorem which defines the Riemann–Liouville universal map can be formulated as the following.

Theorem 6. *For $\alpha \in \mathbb{R}$, $\alpha > 1$, $m = \lceil \alpha \rceil$, the solution $x_n = x(t = nh)$ of the FDE*

$$ {}^{\mathrm{RL}}D_0^{\alpha} x(t) + G_K\big(x(t)\big) \sum_{n=-\infty}^{\infty} \delta\left(\frac{t}{h} - n\right) = 0 \tag{39}$$

with the initial conditions

$$ {}^{\mathrm{RL}}D_0^{\alpha-k} x(0+) = c_k, \quad k = 1, \dots, m \quad (c_m = 0) \tag{40}$$

is equivalent to the Riemann–Liouville universal map

$$\begin{aligned} x_{n+1} = {} & \sum_{k=1}^{m-1} \frac{c_k}{\Gamma(\alpha-k+1)} h^{\alpha-k}(n+1)^{\alpha-k} \\ & + \frac{h^{\alpha}}{\Gamma(\alpha)} \sum_{k=0}^{n} (n+1-k)^{\alpha-1} G_K(x_k), \end{aligned} \tag{41}$$

where $n \in \mathbb{Z}$, $n > 0$.

The map equation (41) can be written as the m-dimensional map

$$
\begin{aligned}
x_{n+1} &= \sum_{k=1}^{m-1} \frac{c_k}{\Gamma(\alpha-k+1)} h^{\alpha-k}(n+1)^{\alpha-k} \\
&\quad - \frac{h^{\alpha}}{\Gamma(\alpha)} \sum_{k=0}^{n} G_K(x_k)(n-k+1)^{\alpha-1},
\end{aligned} \tag{42}
$$

$$
\begin{aligned}
p_{n+1}^{s} &= \sum_{k=1}^{m-s-1} \frac{c_k}{(m-s-1-k)!} h^{m-s-1-k}(n+1)^{m-s-1-k} \\
&\quad - \frac{h^{m-s-1}}{(m-s-2)!} \sum_{k=0}^{n} G_K(x_k)(n-k+1)^{m-s-2},
\end{aligned} \tag{43}
$$

where $p_k^s = p^{(s)}(t = kh)$, $p^{(0)}(t) = {}^{\mathrm{RL}}D_0^{\alpha-m+1}x(t)$, and $p^{(s)}(t) = D^s p(t)$ for $s = 1, 2, \ldots, m-2$. The Riemann–Liouville universal map equation (41) is equivalent, with the substitution equation (19) and up to the change in the sign of $G_K(x)$, to the directly introduced map with PL memory equation (30).

Solution of equations (35) and (39) with $\alpha \le 1$ would require calculations of the function $G(x)$ at the time instances of the kicks at which $x(t)$ is discontinuous. In [14], it was shown that Theorems 5 and 6 are valid for any $\alpha \ge 0$ if the limit $\varepsilon \to 0$ of the equations with an infinitely small time delay $\Delta < \varepsilon$

$$
{}^{C/\mathrm{RL}}D_0^{\alpha}x(t) + G_K\big(x(t-\Delta)\big) \sum_{n=-\infty}^{\infty} \delta\left(\frac{t}{h} - (n+\varepsilon)\right) = 0, \tag{44}
$$

were used instead of equations (35) and (39).

In this section, we generalized the derivation of the universal map in the regular dynamics equations (32) and (33) from the differential equation describing evolution of a particle under periodic delta-function kicks equation (34) by substituting the second derivative in this equation by fractional derivatives. The summary of this generalization is the statement that for $\alpha \ge 0$ the map equations (37) and (38) can be treated as the (fractional) Caputo universal map and the map equation (41) and equations (42) and (43) can be treated as the (fractional) Riemann–Liouville universal map.

4 Fractional difference maps

Evolution of discrete systems is described by means of discrete fractional calculus. The basic operators of discrete calculus are difference and sum operators. The two major definitions of the fractional difference/sum operator are fractional generalizations of the forward difference operator [48]

$$
\Delta x(t) = x(t+1) - x(t) \tag{45}
$$

and the backward difference operator $\nabla x(t) = x(t) - x(t-1)$ [36]. There are multiple equivalent ways to define these operators (see, e. g., [1]) and there is an obvious relationship between the fractional forward (defined below) and backward difference operators [5]: for $\alpha > 0$ and $m - 1 < \alpha \le m$,

$$\Delta_a^{\alpha} y(t-\alpha) = \nabla_{a-1}^{\alpha} y(t); \quad t \in \mathbb{N}_{m+a}, \tag{46}$$

$$\Delta_a^{-\alpha} y(t+\alpha) = \nabla_{a-1}^{-\alpha} y(t); \quad t \in \mathbb{N}_{a}, \tag{47}$$

where $\mathbb{N}_a = \{a, a+1, a+2, \ldots\}$. Because of the close relationship between the fractional forward and backward difference operators and taking into account that extensive numerical simulations revealing some general properties of fractional systems were performed using fractional forward difference maps [23], here we will consider only fractional generalizations of the forward difference/sum operators and the corresponding maps.

The fractional sum ($\alpha > 0$)/difference ($\alpha < 0$) operator defined by Miller and Ross [48]

$$\Delta_a^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \sum_{s=a}^{t-\alpha} (t-s-1)^{(\alpha-1)} f(s) \tag{48}$$

is a fractional generalization of the n-fold summation formula [20, 35, 36]

$$\begin{aligned} \Delta_a^{-n} f(t) &= \frac{1}{(n-1)!} \sum_{s=a}^{t-n} (t-s-1)^{(n-1)} f(s) \\ &= \sum_{s^0=a}^{t-n} \sum_{s^1=a}^{s^0} \cdots \sum_{s^{n-1}=a}^{s^{n-2}} f(s^{n-1}), \end{aligned} \tag{49}$$

where $n \in \mathbb{N}$. In equation (48), f is defined on $\mathbb{N}_a$ and $\Delta_a^{-\alpha}$ on $\mathbb{N}_{a+\alpha}$. The definition of the falling factorial $t^{(\alpha)}$ is

$$t^{(\alpha)} = \frac{\Gamma(t+1)}{\Gamma(t+1-\alpha)}, \quad t \neq -1, -2, -3 \ldots. \tag{50}$$

Falling factorial is asymptotically a power function:

$$\lim_{t\to\infty} \frac{\Gamma(t+1)}{\Gamma(t+1-\alpha) t^{\alpha}} = 1, \quad \alpha \in \mathbb{R}. \tag{51}$$

For $0 \le m - 1 < \alpha \le m$, the fractional (left) Riemann–Liouville difference operator is defined (see [4, 5]) as

$$\begin{aligned} \Delta_a^{\alpha} x(t) &= \Delta^m \Delta_a^{-(m-\alpha)} x(t) \\ &= \frac{1}{\Gamma(m-\alpha)} \Delta^m \sum_{s=a}^{t-(m-\alpha)} (t-s-1)^{(m-\alpha-1)} x(s). \end{aligned} \tag{52}$$

The fractional (left) Caputo-like difference operator (see [3]) is defined as

$$\begin{aligned} {}^C\Delta_a^{\alpha} x(t) &= \Delta^{-(m-\alpha)}{}_a\Delta^m x(t) \\ &= \frac{1}{\Gamma(m-\alpha)} \sum_{s=a}^{t-(m-\alpha)} (t-s-1)^{(m-\alpha-1)} \Delta^m x(s). \end{aligned} \tag{53}$$

The fractional difference operators Δ_a^{α}, equation (52), and ${}^C\Delta_a^{\alpha}$, equation (53), as follows from the definition of the fractional sum operator, equation (48), are defined on $\mathbb{N}_{a+\alpha}$, while x is defined on $\mathbb{N}_a$. Due to the fact that ${}_a\Delta_t^{\lambda}$ in the limit $\lambda \to 0$ approaches the identity operator (see [20, 48]), the definition equation (53) can be extended to all real $\alpha \geq 0$ with ${}^C\Delta_a^m x(t) = \Delta^m x(t)$ for $m \in \mathbb{N}_0$.

As was noticed in [20, 61, 62], Lemma 2.4 from [9] on the equivalency of the fractional Caputo-like difference and sum equations can be formulated for all real $\alpha \geq 0$:

Theorem 7. *The Caputo-like difference equation*

$$ {}^C\Delta_a^{\alpha} x(t) = f\big(t+\alpha-1, x(t+\alpha-1)\big) \tag{54}$$

with the initial conditions

$$\Delta^k x(a) = c_k, \quad k = 0, 1, \dots, m-1, \quad m = \lceil \alpha \rceil \tag{55}$$

is equivalent to the fractional sum equation

$$\begin{aligned} x(t) = &\sum_{k=0}^{m-1} \frac{(t-a)^{(k)}}{k!} \Delta^k x(a) \\ &+ \frac{1}{\Gamma(\alpha)} \sum_{s=a+m-\alpha}^{t-\alpha} (t-s-1)^{(\alpha-1)} f\big(s+\alpha-1, x(s+\alpha-1)\big), \end{aligned} \tag{56}$$

where $t \in \mathbb{N}_{a+m}$.

Considering autonomous nonlinear system with $f(t, x(t)) = -G_K(x(t))$ (K is a nonlinearity parameter), adopting the Miller and Ross proposition to let $a = 0$, and assuming $x_n = x(n)$, Theorem 7 can be formulated as

Theorem 8. *For* $\alpha \in \mathbb{R}$, $\alpha \geq 0$ *the Caputo-like difference equation*

$$ {}^C\Delta_0^{\alpha} x(t) = -G_K\big(x(t+\alpha-1)\big), \tag{57}$$

where $t \in \mathbb{N}_m$, *with the initial conditions*

$$\Delta^k x(0) = c_k, \quad k = 0, 1, \dots, m-1, \quad m = \lceil \alpha \rceil \tag{58}$$

is equivalent to the map with falling factorial-law memory

$$\begin{aligned} x_{n+1} = &\sum_{k=0}^{m-1} \frac{\Delta^k x(0)}{k!} (n+1)^{(k)} \\ &- \frac{1}{\Gamma(\alpha)} \sum_{s=0}^{n+1-m} (n-s-m+\alpha)^{(\alpha-1)} G_K(x_{s+m-1}). \end{aligned} \tag{59}$$

The map equation (59) is called the fractional difference Caputo universal α-family of maps.

As shown in [17], the integer ($\alpha = m$) members of the fractional difference Caputo universal α-family of maps are:

$$x_{n+1} = -G_K(x_n); \quad m = 0, \tag{60}$$

$$x_{n+1} = x_n - G_K(x_n); \quad m = 1, \tag{61}$$

$$x^s_{n+1} = \sum_{k=s}^{m-1} x^k_n - G_K(x^0_n); \quad s = 0, 1, \ldots, m-1; \quad m > 1, \tag{62}$$

where

$$x^0_n = x_n, \quad x^s_n = \Delta x^{s-1}_{n-1}; \quad s = 1, 2, \ldots, m-1. \tag{63}$$

Note that the maps equation (62) are m-dimensional volume preserving maps.

Introduced and investigated in [6, 7, 27, 49–52], fractional h-difference operators are generalizations of the fractional difference operators. The h-sum operator is defined as

$$({}_a\Delta_h^{-\alpha} f)(t) = \frac{h}{\Gamma(\alpha)} \sum_{s=\frac{a}{h}}^{\frac{t}{h}-\alpha} (t-(s+1)h)_h^{(\alpha-1)} f(sh), \tag{64}$$

where $\alpha \geq 0$, $({}_a\Delta_h^0 f)(t) = f(t)$, f is defined on $(h\mathbb{N})_a$, and ${}_a\Delta_h^{-\alpha}$ on $(h\mathbb{N})_{a+\alpha h}$. $(h\mathbb{N})_t = \{t, t+h, t+2h, \ldots\}$. The h-factorial $t_h^{(\alpha)}$ is defined as

$$t_h^{(\alpha)} = h^\alpha \frac{\Gamma(\frac{t}{h}+1)}{\Gamma(\frac{t}{h}+1-\alpha)} = h^\alpha \left(\frac{t}{h}\right)^{(\alpha)}, \quad \frac{t}{h} \neq -1, -2, -3, \ldots. \tag{65}$$

With $m = \lceil \alpha \rceil$ the Riemann–Liouville (left) h-difference is defined as

$$\begin{aligned}({}_a\Delta_h^\alpha x)(t) &= (\Delta_h^m ({}_a\Delta_h^{-(m-\alpha)} x))(t) \\ &= \frac{h}{\Gamma(m-\alpha)} \Delta_h^m \sum_{s=\frac{a}{h}}^{\frac{t}{h}-(m-\alpha)} (t-(s+1)h)_h^{(m-\alpha-1)} x(sh)\end{aligned} \tag{66}$$

and the Caputo (left) h-difference is defined as

$$\begin{aligned}({}_a\Delta_{h,*}^\alpha x)(t) &= ({}_a\Delta_h^{-(m-\alpha)} (\Delta_h^m x))(t) \\ &= \frac{h}{\Gamma(m-\alpha)} \sum_{s=\frac{a}{h}}^{\frac{t}{h}-(m-\alpha)} (t-(s+1)h)_h^{(m-\alpha-1)} (\Delta_h^m x)(sh),\end{aligned} \tag{67}$$

where $(\Delta_h^m x)(t)$ is the mth power of the forward h-difference operator

$$(\Delta_h x)(t) = \frac{x(t+h) - x(t)}{h}. \tag{68}$$

As has been noticed in [6, 27, 29], due to the convergence of solutions of fractional Riemann–Liouville h-difference equations when $h \to 0$ to solutions of the corresponding differential equations, they can be used to solve fractional Riemann–Liouville differential equations numerically. A proof of the convergence (as $h \to 0$) of the Caputo fractional h-difference operators to the corresponding fractional Caputo differential operators for $0 < \alpha \le 1$ can be found in [50] (Proposition 17).

The generalization of Theorem 8 based on the property (see [27])

$$({}_0\Delta^{\alpha}_{h,*}x)(t) = h^{-\alpha}\,{}^{C}_{0}\Delta^{\alpha}_{t}\bar{x}\left(\frac{t}{h}\right), \tag{69}$$

where x is defined on $(h\mathbb{N})_a$, ${}_a\Delta^{\alpha}_{h,*}$ on $(h\mathbb{N})_{a+\alpha h}$, and $\bar{x}(s) = x(sh)$ was obtained in [22].

Theorem 9. *For $\alpha \in \mathbb{R}$, $\alpha \ge 0$ the Caputo-like h-difference equation*

$$({}_0\Delta^{\alpha}_{h,*}x)(t) = -G_K\big(x(t + (\alpha - 1)h)\big), \tag{70}$$

where $t \in (h\mathbb{N})_m$, with the initial conditions

$$({}_0\Delta^{k}_{h}x)(0) = c_k, \quad k = 0, 1, \dots, m-1, \quad m = \lceil \alpha \rceil \tag{71}$$

is equivalent to the map with h-factorial-law memory

$$\begin{aligned} x_{n+1} = {} & \sum_{k=0}^{m-1} \frac{c_k}{k!}\big((n+1)h\big)^{(k)}_h \\ & - \frac{h^{\alpha}}{\Gamma(\alpha)} \sum_{s=0}^{n+1-m} (n - s - m + \alpha)^{(\alpha-1)} G_K(x_{s+m-1}), \end{aligned} \tag{72}$$

where $x_k = x(kh)$, which is called the h-difference Caputo universal α-family of maps.

5 Conclusion

The main goal of this paper is a review of the various forms of the exact fractional maps as instruments to investigate general properties of fractional (with PL memory) dynamics. The simplest maps used to investigate general properties of regular dynamics are the one-dimensional logistic map (quadratic nonlinearity) and the two-dimensional standard map (harmonic nonlinearity). Fractional/fractional difference standard and logistic α-families of maps were proposed and investigated in papers [13–22, 24, 25, 57–62] and are reviewed in the author's chapter in Volume 4 [23] of this book. In what follows, we assume $h = 1$.

All of the fractional/fractional difference α-families of maps considered above for $\alpha = 1$ turn into the logistic map

$$x_{n+1} = Kx_n(1 - x_n) \tag{73}$$

iff (see [14–16] for fractional maps and [20] for fractional difference maps)

$$G_K(x) = x - Kx(1-x). \tag{74}$$

The corresponding α-families of maps are called the logistic families of maps. This definition allows us to uniquely define the two-dimensional logistic map ($\alpha = 2$) as

$$p_{n+1} = p_n + Kx_n(1-x_n) - x_n, \tag{75}$$
$$x_{n+1} = x_n + p_{n+1}. \tag{76}$$

In zero dimensions, the fractional logistic maps are identical zeros and the Caputo fractional difference logistic map is

$$x_{n+1} = -x_n + Kx_n(1-x_n). \tag{77}$$

All of the fractional/fractional difference α-families of maps considered above for $\alpha = 2$ turn into the standard map (see [11])

$$p_{n+1} = p_n - K\sin(x_n) \pmod{2\pi}, \tag{78}$$
$$x_{n+1} = x_n + p_{n+1} \pmod{2\pi} \tag{79}$$

iff (see [14–16, 57–60] for fractional maps and [17, 20] for fractional difference maps)

$$G_K(x) = K\sin(x). \tag{80}$$

The corresponding α-families of maps are called the standard families of maps. This definition allows one to uniquely define the one-dimensional standard map ($\alpha = 1$) as

$$x_{n+1} = x_n - K\sin(x_n) \pmod{2\pi}, \tag{81}$$

which is a particular form of the Circle Map with zero driving phase. In zero dimensions, the fractional standard maps are identical zeros and the Caputo fractional difference standard map is the sine map (see, e. g., [44])

$$x_{n+1} = -K\sin(x_n) \pmod{2\pi}. \tag{82}$$

It is remarkable that the three well-known maps investigated in regular dynamics: the standard map, the circle map, and the sine map, are members of the same α-family of maps and it is possible to observe changes in properties of maps from this family with changes in the value of α. Properties of the fractional maps with $\alpha \leq 3$ were investigated in [15, 16] (the three-dimensional logistic and standard maps could be used to investigate turbulence [12, 47]).

The results of the investigation of general properties of fractional/fractional difference maps, including the existence of the cascade of bifurcations type trajectories and two-dimensional bifurcation diagrams, are presented in the author's chapter in Volume 4 of this book [23].

Bibliography

[1] T. Abdeljawad, D. Baleanu, F. Jarad, and R. P. Agarwal, Fractional sums and differences with binomial coefficients, *Discrete Dyn. Nat. Soc.*, **2013** (2013), 104173.

[2] R. P. Agarwal, *Difference Equations and Inequalities: Theory, Methods, and Applications*, Marcel Dekker, New York, 2000.

[3] G. A. Anastassiou, Discrete fractional calculus and inequalities, http://arxiv.org/abs/0911.3370, 2009.

[4] F. Atici and P. Eloe, Initial value problems in discrete fractional calculus, *Proc. Am. Math. Soc.*, **137** (2009), 981–989.

[5] F. M. Atici and P. W. Eloe, Discrete fractional calculus with the nabla operator, *Electron. J. Qual. Theory Differ. Equ.*, **Spec. Ed. I. 2009** (2009), No. 3, 1–12.

[6] N. R. O. Bastos, R. A. C. Ferreira, and D. F. M. Torres, Discrete-time fractional variational problems, *Signal Process.*, **91** (2011), 513–524.

[7] N. R. O. Bastos, R. A. C. Ferreira, and D. F. M. Torres, Necessary optimality conditions for fractional difference problems of the calculus of variations, *Discrete Contin. Dyn. Syst.*, **29** (2011), 417–437.

[8] P. L. Butzer and M. Hauss, Eulerian numbers with fractional order parameters, *Aequ. Math.*, **46** (1993), 119–142.

[9] F. Chen, X. Luo, and Y. Zhou, Existence results for nonlinear fractional difference equation, *Adv. Differ. Equ.*, **2011** (2011), 713201.

[10] B. V. Chirikov, A universal instability of many dimensional oscillator systems, *Phys. Rep.*, **52** (1979), 263–379.

[11] B. V. Chirikov, A universal instability of many dimensional oscillator systems, *Phys. Rep.*, **52** (1979), 263–379.

[12] H. R. Dullin and J. D. Meiss, Resonances and twist in volume-preserving maps, *SIAM J. Appl. Dyn. Syst.*, **11** (2012), 319–359.

[13] M. Edelman, Fractional standard map: Riemann–Liouville vs. Caputo, *Commun. Nonlinear Sci. Numer. Simul.*, **16** (2011), 4573–4580.

[14] M. Edelman, Universal fractional map and cascade of bifurcations type attractors, *Chaos*, **23** (2013), 033127.

[15] M. Edelman, Fractional maps and fractional attractors. Part I: α-families of maps, *Discontinuity Nonlinearity Complexity*, **1** (2013), 305–324.

[16] M. Edelman, Fractional maps as maps with power-law memory, in V. Afraimovich, A. C. J. Luo, and X. Fu (eds.) *Nonlinear Dynamics and Complexity*, Nonlinear Systems and Complexity, pp. 79–120, Springer, New York, 2014.

[17] M. Edelman, Caputo standard α-family of maps: fractional difference vs. fractional, *Chaos*, **24** (2014), 023137.

[18] M. Edelman, Universality in fractional dynamics, in *International Conference on Fractional Differentiation and Its Applications (ICFDA), 2014*, pp. 1–6, 2014, DOI:10.1109/ICFDA.2014.6967376.

[19] M. Edelman, On the fractional Eulerian numbers and equivalence of maps with long term power-law memory (integral Volterra equations of the second kind) to Grünwald–Letnikov fractional difference (differential) equations, *Chaos*, **25** (2015), 073103.

[20] M. Edelman, Fractional maps and fractional attractors. Part II: fractional difference α-families of maps, *Discontinuity Nonlinearity Complexity*, **4** (2015), 391–402.

[21] M. Edelman, On nonlinear fractional maps: nonlinear maps with power-law memory, in X. Leoncini, C. Eloy, and G. Boedec (eds.) *Chaos, Complexity and Transport, Proceedings of*

the CCT ’15, Conference on Chaos, Complexity, and Transport 2015, Marseilles, France, 1–5 June 2015, pp. 119–130, World Scientific, Singapore, 2017.

[22] M. Edelman, Universality in systems with power-law memory and fractional dynamics, in M. Edelman, E. Macau, and M. A. F. Sanjuan (eds.) *Chaotic, Fractional, and Complex Dynamics: New Insights and Perspectives*, Understanding Complex Systems, pp. 147–171, Springer, eBook, 2018.

[23] M. Edelman, Dynamics of nonlinear systems with power-law memory, in V. E. Tarasov (ed.), *Handbook of Fractional Calculus with Applications, Vol. 4, Applications in Physics* De Gruyter, Berlin, 2018.

[24] M. Edelman and L. A. Taieb, New types of solutions of non-linear fractional differential equations, in A. Almeida, L. Castro, and T.-O. Speck (eds.) *Advances in Harmonic Analysis and Operator Theory*, Operator Theory: Advances and Applications, vol. 229, pp. 139–155, Springer, Basel, 2013.

[25] M. Edelman and V. E. Tarasov, Fractional standard map, *Phys. Lett. A*, **374** (2009), 279–285.

[26] A. M. A. El-Sayed and S. M. Salman, On a discretization process of fractional-order Riccati differential equation, *J. Fract. Calc. Appl.*, **4** (2013), 251–259.

[27] R. A. C. Ferreira and D. F. M. Torres, Fractional *h*-difference equations arising from the calculus of variations, *Appl. Anal. Discrete Math.*, **5** (2011), 110–121.

[28] E. Fick, M. Fick, and G. Hausmann, Logistic equation with memory, *Phys. Rev. A*, **44** (1991), 2469–2473.

[29] G. S. F. Frederico and D. F. M. Torres, A formulation of Noether’s theorem for fractional problems of the calculus of variations, *J. Math. Anal. Appl.*, **334** (2007), 834–846.

[30] A. Fulinski and A. S. Kleczkowski, Nonlinear maps with memory, *Phys. Scr.*, **335** (1987), 119–122.

[31] S. G Samko, A. A. Kilbas, and O. I. Marichev, *Fractional Integrals and Derivatives Theory and Applications*, Gordon and Breach, New York, 1993.

[32] J. A. C. Gallas, Simulating memory effects with discrete dynamical systems, *Physica A*, **195** (1993), 417–430.

[33] J. A. C. Gallas, Simulating memory effects with discrete dynamical systems, *Physica A*, **198** (1993), 339–339 (erratum).

[34] M. Giona, Dynamics and relaxation properties of complex systems with memory, *Nonlinearity*, **4** (1991), 911–925.

[35] C. Goodrich and A. C. Peterson, *Discrete Fractional Calculus*, Springer, New York, 2015.

[36] H. L. Gray and N. F. Zhang, On a new definition of the fractional difference, *Math. Comput.*, **50** (1988), 513–529.

[37] K. Hartwich and E. Fick, Hopf bifurcations in the logistic map with oscillating memory, *Phys. Lett. A*, **177** (1993), 305–310.

[38] C. Jordan, *Calculus of Finite Differences*, Chelsea, New York, 1979.

[39] A. A. Kilbas, B. Bonilla, and J. J. Trujillo, *Dokl. Akad. Nauk, Ross. Akad. Nauk*, **374** (2000), 445–449 (in Russian); A. A. Kilbas, B. Bonilla, and J. J. Trujillo, Nonlinear differential equations of fractional order is space of integrable functions, *Dokl. Math.* **62** (2000), 222–226 (Engl. Transl.).

[40] A. A. Kilbas, B. Bonilla, and J. J. Trujillo, Existence and uniqueness theorems for nonlinear fractional differential equations, *Demonstr. Math.*, **33** (2000), 583–602.

[41] A. A. Kilbas and S. A. Marzan, *Dokl. Akad. Nauk, Ross. Akad. Nauk*, **399** (2004), 7–11 (in Russian); A. A. Kilbas and S. A. Marzan, The Cauchy problem for differential equations with fractional Caputo derivative, *Dokl. Math.* **70** (2004), 841–845 (Engl. Transl.).

[42] A. A. Kilbas and S. A. Marzan, *Differ. Uravn.*, **41** (2005), 82–86 (in Russian); A. A. Kilbas and S. A. Marzan, Nonlinear differential equations with the Caputo fractional derivative in the space of continuously differentiable functions, *Diff. Eqns.*, **41** (2005), 84–89 (Engl. Transl.).
[43] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Application of Fractional Differential Equations*, Elsevier, Amsterdam, 2006.
[44] C. C. Lalescu, Patterns in the sine map bifurcation diagram, arXiv:1011.6552, 2010.
[45] H. Levy and F. Lessman, *Finite Difference Equations*, Dover, New York, 1992.
[46] A. J. Lichtenberg and M. A. Lieberman, *Regular and Chaotic Dynamics*, Springer, Berlin, 1992.
[47] H. E. Lomeli and J. D. Meiss, Quadratic volume-preserving maps, *Nonlinearity*, **11** (1998), 557–574.
[48] K. S. Miller and B. Ross, Fractional difference calculus, in H. M. Srivastava and S. Owa (eds.) *Univalent Functions, Fractional Calculus, and Their Applications*, pp. 139–151, Ellis Howard, Chichester, 1989.
[49] D. Mozyrska and E. Girejko, Overview of the fractional *h*-difference operators, in A. Almeida, L. Castro, and F.-O. Speck (eds.) *Advances in Harmonic Analysis and Operator Theory*, Operator Theory: Advances and Applications, vol. 229, pp. 253–267, Springer, Basel, 2013.
[50] D. Mozyrska, E. Girejko, and M. Wirwas, Fractional nonlinear systems with sequential operators, *Cent. Eur. J. Phys.*, **11** (2013), 1295–1303.
[51] D. Mozyrska and E. Pawluszewicz, Local controllability of nonlinear discrete-time fractional order systems, *Bull. Pol. Acad. Sci., Tech. Sci.*, **61** (2013), 251–256.
[52] D. Mozyrska, E. Pawluszewicz, and E. Girejko, Stability of nonlinear h-difference systems with N fractional orders, *Kibernetica*, **51** (2015), 112–136.
[53] J. Munkhammar, Chaos in a fractional order logistic map, *Fract. Calc. Appl. Anal.*, **16** (2013), 511–519.
[54] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, 1999.
[55] W. S. Sayed, A. G. Radwan, and H. A. H. Fahmy, Design of positive, negative, and alternating sign generalized logistic maps, *Discrete Dyn. Nat. Soc.*, **2015** (2015), 586783.
[56] A. A. Stanislavsky, Long-term memory contribution as applied to the motion of discrete dynamical system, *Chaos*, **16** (2006), 043105.
[57] V. E. Tarasov, Discrete map with memory from fractional differential equation of arbitrary positive order, *J. Math. Phys.*, **50** (2009), 122703.
[58] V. E. Tarasov, Differential equations with fractional derivative and universal map with memory, *J. Phys. A*, **42** (2009), 465102.
[59] V. E. Tarasov, *Fractional Dynamics: Application of Fractional Calculus to Dynamics of Particles, Fields and Media*, HEP, Springer, Beijing, Berlin, Heidelberg, 2011.
[60] V. E. Tarasov and G. M. Zaslavsky, Fractional equations of kicked systems and discrete maps, *J. Phys. A*, **41** (2008), 435101.
[61] G.-C. Wu and D. Baleanu, Discrete fractional logistic map and its chaos, *Nonlinear Dyn.*, **75** (2014), 283–287.
[62] G.-C. Wu, D. Baleanu, and S.-D. Zeng, Discrete chaos in fractional sine and standard maps, *Phys. Lett. A*, **378** (2014), 484–487.
[63] G. M. Zaslavsky, *Hamiltonian Chaos and Fractional Dynamics*, Oxford University Press, Oxford, 2008.

Rafail K. Gazizov, Alexey A. Kasatkin, and Stanislav Yu. Lukashchuk

Symmetries and group invariant solutions of fractional ordinary differential equations

Abstract: The current results and techniques of applying Lie group analysis methods to ordinary FDEs are discussed. The prolongation of the local Lie group to the fractional integrals and derivatives is done; the constructive algorithm of finding linearly autonomous symmetries is proposed and illustrated. Some classification results are presented together with examples of constructed solutions.

Keywords: Ordinary fractional differential equation, symmetry, invariant solution, group classification problem

MSC 2010: 34A08, 70G65

1 Introduction

Lie group analysis of differential equations provides effective tools for integrating classical ordinary differential equations and constructing their particular solutions. The theory and main methods were developed by Sophus Lie and later extended; see books [3, 14, 18, 19, 21, 24]. These methods can also be successfully used for some classes of integrodifferential equations (see [11] and references therein).

The main idea of the group analysis is to search and utilize transformations (change of variables) that conserves the given equation. Such transformations and their infinitesimal generators are usually called *symmetries* of equation under consideration. Using the infinitesimal approach allows to reduce essentially the nonlinear problem of constructing symmetries to solving the so-called determining equation, which leads to some overdetermined system of *linear* partial differential equations.

The main problem of generalizing these procedures for FDEs is that determining equations should be considered in infinite-dimensional spaces of variables. However, a constructive algorithm for finding symmetries of the special form (so-called linearly autonomous symmetries) can be proposed. General theory of constructing and using symmetries for FDEs is still far from being complete. Here, the basic results in this area obtained during last decade are presented.

In this chapter, only the equations with one independent variable x (ordinary FDEs) are considered. It is necessary to note that nowadays there are only several papers where the problems of symmetry properties of ordinary FDEs are studied [2, 7–10, 12, 15, 22].

Rafail K. Gazizov, Alexey A. Kasatkin, Stanislav Yu. Lukashchuk, Ufa State Aviation Technical University, Karl Marx str. 12, Ufa 450000, Russia, e-mail: gazizovrk@gmail.com

https://doi.org/10.1515/9783110571660-004

The chapter is organized as follows. Section 2 contains basic definitions and facts about Lie groups of transformations and symmetries of ordinary differential equations with integer-order derivatives. In Subsection 3.1, the change of variables in the fractional integral is illustrated and different forms of prolongation formulas are presented for fractional derivatives and integrals. Subsection 3.2 contains some discussion regarding the definition of FDE's symmetries and illustrates problems of finding them. The constructive algorithm of calculating linearly autonomous symmetries is proposed here and illustrated for ordinary differential equations with the Riemann–Liouville fractional derivatives. Subsection 3.3 is devoted to methods of constructing solutions by using symmetries of ordinary FDEs. Section 4 contains the results of symmetry analysis for some classes of fractional differential equations and systems with examples of constructing solutions.

2 Preliminaries: Lie group analysis of differential equations

Here, for simplicity, we illustrate the basic principles of Lie group analysis on ordinary differential equations with single dependent variable $y(x)$. For more details regarding Lie group analysis for ordinary differential equations, see [14, 24].

2.1 One-parameter groups of transformations, generator of the group

We consider a set of invertible transformations T_a of variables (x, y),

$$T_a : (x, y) \to (\bar{x}, \bar{y}) : \bar{x} = \varphi(x, y, a), \quad \bar{y} = \psi(x, y, a), \tag{1}$$

depending on real parameter $a \in \Delta \subset \mathbb{R}$ from some neighborhood Δ of the point $a = 0$.

Definition 1. It is said that the transformations (1) form a one-parameter local Lie group G with respect to the parameter a if

$$T_0 = I \in G, \quad T_a T_b = T_{a+b} \in G, \quad T_a^{-1} = T_{-a} \in G$$

for every $a, b, a + b \in \Delta$, and I is the identity transformation.

Hereafter, we suppose that all the considered functions are sufficiently smooth.

Definition 2. Differential operator

$$X = \xi(x, y)\frac{\partial}{\partial x} + \eta(x, y)\frac{\partial}{\partial y} \tag{2}$$

with coefficients

$$\xi(x,y) = \left.\frac{d\varphi(x,y,a)}{da}\right|_{a=0}, \quad \eta(x,y) = \left.\frac{d\psi(x,y,a)}{da}\right|_{a=0}$$

is called the infinitesimal operator (generator) of the group G of transformations (1).

According to the well-known theory developed by Sophus Lie, every local group G can be defined by its generator.

Theorem 1. *Transformations* (1) *satisfy the so-called Lie equations*

$$\frac{d\bar{x}}{da} = \xi(\bar{x},\bar{y}), \quad \frac{d\bar{y}}{da} = \eta(\bar{x},\bar{y}) \tag{3}$$

with initial conditions

$$\bar{x}|_{a=0} = x, \quad \bar{y}|_{a=0} = y. \tag{4}$$

Conversely, for given operator (2), *the solution of the Lie equations* (3), *satisfying the condition* (4), *forms a local Lie group of transformations* (1).

The solution of (3), (4) can be written in the infinitesimal form

$$\bar{x} = x + a\xi(x,y) + o(a), \quad \bar{y} = y + a\eta(x,y) + o(a).$$

Example 1. Infinitesimal generator

$$X = x^2\frac{\partial}{\partial x} + \gamma xy\frac{\partial}{\partial y} \tag{5}$$

corresponds to the projective transformations

$$\bar{x} = \frac{x}{1-ax}, \quad \bar{y} = \frac{y}{(1-ax)^{\gamma}}, \quad a \in (-\infty, x^{-1}). \tag{6}$$

The infinitesimal form of these transformations can be written as

$$\bar{x} = x + ax^2 + o(a), \quad \bar{y} = y + \gamma xy + o(a). \tag{7}$$

Definition 3. Function $F(x,y)$ is called an invariant of the group G if at each point (x,y) the equality $F(\bar{x},\bar{y}) = F(x,y)$ holds for every $a \in \Delta$.

Theorem 2. *$F(x,y)$ is an invariant of G if and only if F is a solution of the first order homogeneous partial differential equation*

$$XF = 0.$$

2.2 Prolongation of the group. Symmetry groups of differential equations

The transformations (1) from the group G can be prolonged to derivatives $y', y'', \ldots, y^{(k)}$, and the corresponding prolonged transformations have the form

$$\bar{y}' = \theta_1(x, y, y', a), \quad \bar{y}'' = \theta_2(x, y, y', y'', a), \quad \ldots, \quad \bar{y}^{(k)} = \theta_k(x, y, y', y'', \ldots, y^{(k)}, a).$$

Remark 1. Transformations of variable $y^{(k)}$ depend only on x, y, and derivatives of orders not greater than k, and the prolonged transformations form a local Lie group $\tilde{G}$ of transformations in finite-dimensional space of variables $x, y, y', y'', \ldots, y^{(k)}$ for arbitrary k. This group $\tilde{G}$ is called the prolonged group.

Theorem 3. *The prolonged group of transformations is generated by infinitesimal operator*

$$\tilde{X} = \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} + \zeta_1\frac{\partial}{\partial y'} + \zeta_2\frac{\partial}{\partial y''} + \cdots + \zeta_k\frac{\partial}{\partial y^{(k)}} \tag{8}$$

which is called the kth order prolongation of operator X. Coefficients ζ_s are defined by prolongation formulas

$$\zeta_1 = D_x(\eta) - y'D_x(\xi), \quad \zeta_s = D_x(\zeta_{s-1}) - y^{(s)}D_x(\xi), \quad s = 2, \ldots, k, \tag{9}$$

where D_x is the operator of total differentiation given by the following formal infinite sum:

$$D_x = \frac{\partial}{\partial x} + y'\frac{\partial}{\partial y} + y''\frac{\partial}{\partial y'} + \cdots.$$

Remark 2. The prolongation formula for arbitrary integer order of derivatives can be written in the following form:

$$\zeta_s = D_x^s(\eta - \xi y') + \xi y^{(s+1)}. \tag{10}$$

The highest order of derivative in ζ_s is $y^{(s)}$ like in (9). The term $y^{(s+1)}$ is always eliminated from (10) by using the classical Leibnitz product rule.

We consider the ordinary differential equation

$$F(x, y, y', \ldots, y^{(k)}) = 0 \tag{11}$$

and assume that it has the classical (smooth) solution.

There is the following formal classical definition of a symmetry group of differential equation.

Definition 4. A group G of transformations (1) is called a symmetry group for differential equation (11) if G converts every solution of equation into a solution of the same equation.

Remark 3. As usual in Lie group analysis, in this definition we assume that set of its solutions is not empty without verifying that conditions of corresponding existence and uniqueness theorems are fulfilled.

In practice, another definition is more convenient for calculation of the symmetry group.

Definition 5. Differential equation (11) has a symmetry group (admits the group) G if the prolonged group transforms every point of the surface $F(x, y, y', \dots, y^{(k)}) = 0$ (considered in space of variables $x, y, y', \dots, y^{(k)}$) to some point $(\bar{x}, \bar{y}, \bar{y}', \dots, \bar{y}^{(k)})$ on the same surface:

$$F(\bar{x}, \bar{y}, \bar{y}', \dots, \bar{y}^{(k)}) = 0.$$

Problems of these two definitions equivalence are discussed, for example, in [14, 19].

Remark 4. Infinitesimal transformation of the function F under the action of transformations forming the prolonged group $\tilde{G}$

$$\bar{x} = x + a\xi + o(a), \quad \bar{y} = y + a\eta + o(a), \quad \dots, \quad \bar{y}^{(k)} = y^{(k)} + a\zeta_k + o(a)$$

can be presented as

$$F(\bar{x}, \bar{y}, \bar{y}', \dots, \bar{y}^{(k)}) = F(x, y, y', \dots, y^{(k)}) + a\tilde{X}F + o(a), \tag{12}$$

where $\tilde{X}$ is the prolonged infinitesimal operator (8).

Due to finite set of variables, Definition 5 allows to prove the following infinitesimal criterion.

Theorem 4. *The group of transformations G with an infinitesimal generator* (2) *is a symmetry group of equation* (11) *if and only if*

$$\tilde{X}F|_{F=0} = 0. \tag{13}$$

The equality (13) *means that* $\tilde{X}F = 0$ *should be satisfied for all points* $(x, y, y', \dots, y^{(k)})$ *on the surface* $F(x, y, y', \dots, y^{(k)}) = 0$.

Equation (13) for $\xi(x, y)$, $\eta(x, y)$ is called the *determining equation*.

Remark 5. The determining equations (13) are always linear homogeneous partial differential equations for unknown functions $\xi(x, y)$, $\eta(x, y)$. Expression $\tilde{X}F$ in the left-hand side of (13) involves also the derivatives $y', y'', \dots, y^{(k)}$. One of such derivatives (usually $y^{(k)}$) can be eliminated by using the equation $F = 0$ and the remaining derivatives are considered as independent variables. It gives the opportunity to split the obtained determining equation into several ones, and the resulting system is usually an overdetermined system which can be easily solved. For detailed theory and examples, see books [14, 19].

2.3 Problem of group classification

A lot of differential equations used as mathematical models contain some arbitrary parameters or functions. In practical problems, they are usually obtained from experimental data or physical models. The methods of Lie group analysis allow to find the forms of arbitrary functions or specific values of coefficients for which the symmetry group of equation is the most wide (or at least is wider than for general case with arbitrary parameters). This problem is called the *problem of group classification* [21].

The problem of group classification is always solved up to the *equivalence transformations* that conserve the form of equation, transforming only the arbitrary elements [1, 21].

3 Lie group analysis of fractional differential equations

3.1 Prolongation formulas for fractional integrals and derivatives

For using methods of Lie group analysis to construct symmetry groups for fractional differential equations, the local Lie group of transformations (1) should be prolonged to all fractional derivatives involved in the equations.

We restrict our attention to the most used Riemann–Liouville and Caputo types of derivatives [17, 23]. Since these derivatives are defined using corresponding fractional integrals, we illustrate constructing prolongation formulas by example of the left-sided fractional integral. We consider the fractional integral with independent variable $\bar{x}$ and dependent variable $\bar{y}$:

$$I^{\alpha}_{c+}\bar{y}(\bar{x}) = \frac{1}{\Gamma(\alpha)} \int\limits_{c}^{\bar{x}} \bar{y}(\bar{s})(\bar{x}-\bar{s})^{\alpha-1} d\bar{s}. \tag{14}$$

Let variables (x, y) and $(\bar{x}, \bar{y})$ be connected by the group transformation (1). Then

$$\bar{y}(\bar{s}) = \psi(s, y(s), a),$$

where the new integration variable s is connected with $\bar{s}$ by equation

$$\bar{s} = \varphi(s, y(s), a).$$

It gives the limits of integration $\tilde{c}(a)$ and x, where $\tilde{c}(a)$ is found from equation

$$\varphi(\tilde{c}, y(\tilde{c}), a) = c.$$

Using the infinitesimal representations,

$$\bar{x} = x + a\xi[x] + o(a), \quad \bar{s} = s + a\xi[s] + o(a), \quad \bar{y}(\bar{s}) = y(s) + a\eta[s] + o(a)$$

(hereafter for brevity the notation $f[x] = f(x, y(x))$ is used), one can rewrite the integral (14) as

$$I^{\alpha}_{c+}\bar{y}(\bar{x}) = \frac{1}{\Gamma(\alpha)} \int\limits_{\tilde{c}(a)}^{x} (y(s) + a\eta[s])\big(x - s + a(\xi[x] - \xi[s])\big)^{\alpha-1}$$
$$\times (1 + aD_s\xi[s])ds + o(a). \tag{15}$$

If $\tilde{c}(a) < c$, then the last integral is not defined because as it follows from (14), $y(s)$ is not defined for $s \in (\tilde{c}(a), c)$. In this way, we should conserve the form of integral after transformations (1) (including the lower limit). It means that $\tilde{c}(a) = c$, that is,

$$\varphi(c, y(c), a) = c, \quad a \in \Delta. \tag{16}$$

In terms of infinitesimal operator coefficients, the condition (16) takes the form

$$\xi(x, y(x))\big|_{x \to c+} = 0. \tag{17}$$

Remark 6. Any group transformation of the form

$$\bar{x} = x + a, \quad \bar{y}(\bar{x}) = \psi(x, y(x), a)$$

does not save the domain of integration in general case and can be used only when considered function $y(x)$ is known for $x < c$.

After some further calculations in (15), one obtains the infinitesimal form for transformations of fractional integral [7].

Theorem 5. *Let $\alpha > 0$, $I^{\alpha}_{c+}y(x)$ and $I^{\alpha}_{c+}(\eta - \xi y')$ exist, $I^{\alpha}_{c+}y(x) \in C^1(c, d)$, and the condition* (17) *holds. Then the infinitesimal transformation of fractional integral* (14) *has the form*

$$I^{\alpha}_{c+}\bar{y}(\bar{x}) = I^{\alpha}_{c+}y(x) + a\zeta_{-\alpha} + o(a), \tag{18}$$

and the coefficient $\zeta_{-\alpha}$ is defined by prolongation formula

$$\zeta_{-\alpha} = I^{\alpha}_{c+}(\eta - \xi y') + \xi D_x I^{\alpha}_{c+} y. \tag{19}$$

Remark 7. The prolongations for the right-sided fractional integral is

$$I_{d-}^{\alpha}\bar{y}(\bar{x}) = I_{d-}^{\alpha}y(x) + a\zeta_{-\alpha}^{-} + o(a), \quad \zeta_{-\alpha}^{-} = I_{d-}^{\alpha}(\eta - \xi y') + \xi D_x I_{d-}^{\alpha} y. \tag{20}$$

Since the Riemann–Liouville fractional derivatives can be written as

$$D_{c+}^{\alpha}y(x) = D_x^m I_{c+}^{m-\alpha}y(x), \quad D_{d-}^{\alpha}y(x) = (-1)^m D_x^m I_{d-}^{m-\alpha}y(x), \quad m = [\alpha] + 1,$$

the prolongations for these derivatives are obtained by combining prolongation formulas (19), (20), and (10) (like in (9)).

Theorem 6. *Let $\alpha > 0$, $D_{c+}^{\alpha}y(x)$ and $D_{c+}^{\alpha}(\eta - \xi y')$ exist, $D_{c+}^{\alpha}y(x) \in C^1(c,d)$, and the condition* (17) *holds. Then the infinitesimal transformation of the left-sided Riemann–Liouville fractional derivative has the form*

$$D_{c+}^{\alpha}\bar{y}(\bar{x}) = D_{c+}^{\alpha}y(x) + a\zeta_{\alpha} + o(a)$$

with ζ_{α} defined by

$$\zeta_{\alpha} = D_{c+}^{\alpha}(\eta - \xi y') + \xi D_{c+}^{\alpha+1}y. \tag{21}$$

If the condition $\xi|_{x\to d-} = 0$ holds, $D_{d-}^{\alpha}y(x)$ and $D_{d-}^{\alpha}(\eta - \xi y')$ exist and $D_{d-}^{\alpha}y(x) \in C^1(c,d)$, the prolongation for the right-sided fractional derivative is

$$\zeta_{\alpha}^{-} = D_{d-}^{\alpha}(\eta - \xi y') - \xi D_{d-}^{\alpha+1}y. \tag{22}$$

Remarks to Theorem 6 (other forms of prolongation formulas).

1. For limit case $\alpha = n$, $n \in \mathbb{N}$, the prolongation formula (21) becomes classical integer-order prolongation formula (10).
2. In some cases, the fractional derivatives can exist for $\eta[x] - \xi[x]y'(x)$, but not for its terms separately [9].
3. If fractional derivatives $D_{c+}^{\alpha}\eta$, $D_{c+}^{\alpha}(\xi y'(x))$ exist, the condition (17) is satisfied, and additionally

   $$\xi[x]\, y(x)|_{x\to c+0} = 0, \tag{23}$$

   then the term with y' in ζ_{α} can be expanded

   $$D_{c+}^{\alpha}(\xi y') = D_{c+}^{\alpha+1}(\xi y) - D_{c+}^{\alpha}(yD\xi),$$

   and the prolongation formula (21) has the form

   $$\zeta_{\alpha} = D_{c+}^{\alpha}\eta + D_{c+}^{\alpha}(yD_x\xi) - D_{c+}^{\alpha+1}(\xi y) + \xi D_{c+}^{\alpha+1}y. \tag{24}$$

4. For sufficiently smooth ξ and y, the prolongation formula (24) can be represented as a series

$$\zeta_\alpha = D^\alpha_{c+}\eta + \sum_{n=0}^{\infty} \binom{\alpha}{n} \frac{n-\alpha}{n+1} D^{\alpha-n}_{c+} y \, D^{n+1}_x \xi \tag{25}$$

by using the generalized Leibnitz rule [23]

$$D^\alpha_{c+}(f(x)g(x)) = \sum_{n=0}^{\infty} \binom{\alpha}{n} D^{\alpha-n}_{c+} f(x) \, D^n_x g(x). \tag{26}$$

Remark 8. We do not analyze the convergence conditions for series like (25), (26) but suppose that for all considered functions such conditions are fulfilled and operations with these series performed below are valid.

Theorem 7. *Let $\alpha > 0$, ${}^C D^\alpha_{c+} y$ and ${}^C D^\alpha_{c+}(\eta - \xi y')$ exist, ${}^C D^\alpha_{c+} y \in C^1(c,d)$, and the condition* (17) *holds. Then the infinitesimal transformation of the left-sided Caputo fractional derivative has the form*

$${}^C D^\alpha_{c+} \bar{y}(\bar{x}) = {}^C D^\alpha_{c+} y(x) + a \, {}^C\zeta_\alpha + o(a)$$

with ${}^C\zeta_\alpha$ defined by

$${}^C\zeta_\alpha = {}^C D^\alpha_{c+}(\eta - \xi y') + \xi D_x {}^C D^\alpha_{c+} y. \tag{27}$$

If the condition

$$\xi[x]|_{x \to d-} = 0 \tag{28}$$

holds, ${}^C D^\alpha_{d-} y(x)$ and ${}^C D^\alpha_{d-}(\eta - \xi y')$ exist, ${}^C D^\alpha_{d-} y(x) \in C^1(c,d)$, then the prolongation formula for the right-sided Caputo fractional derivative has the form

$${}^C\zeta^-_\alpha = {}^C D^\alpha_{d-}(\eta - \xi y') + \xi D_x {}^C D^\alpha_{d-} y. \tag{29}$$

Remark 9. The term $D^\alpha_{c+}\eta(x, y(x))$ in (25) can be expanded by using different forms of generalized chain rule. Unlike the integer order derivative, the result contains multiple series. For example, one of the forms is presented in [7]

$$D^\alpha_x(\eta) = \partial^\alpha_x(\eta) + \partial^\alpha_x(y(x)\eta_y) - y\partial^\alpha_x(\eta_y) + \mu, \tag{30}$$

where $\partial^\alpha_x(f(x,y))$ is the partial fractional derivative (taken for fixed y, not a function $y(x)$). The last term in (30) can be written as

$$\begin{aligned} \mu = {} & \sum_{n=2}^{\infty} \sum_{m=2}^{n} \sum_{k=2}^{m} \sum_{r=0}^{k-1} \binom{\alpha}{n} \binom{n}{m} \binom{k}{r} \frac{1}{k!} \frac{x^{n-\alpha}}{\Gamma(n+1-\alpha)} [-y(x)]^r \\ & \times D^m_x [(y(x))^{k-r}] \frac{\partial^{n-m+k} \eta(x,y)}{\partial x^{n-m} \partial y^k}. \end{aligned} \tag{31}$$

Similar expansion can be obtained for the second term in (25).

The formulas like (31) are almost unusable in practical calculations of prolonged infinitesimal transformations. This problem is partially solved in a simple case when

$$\xi = \xi(x), \quad \eta(x,y) = p(x)y + q(x). \tag{32}$$

Then according to (31), one has $\mu = 0$ and all prolongations ζ_α are linear with respect to y.

Definition 6. The transformations defined by infinitesimal generators (2) with coefficients of the form (32) are called linearly autonomous transformations.

Theorem 8. *For linearly autonomous transformations with the infinitesimals* (32) *satisfying the conditions* (17) *and* (23)*, the prolongation formula* (25) *can be written as a linear combination of fractional derivatives and integrals:*

$$\zeta_\alpha = D^\alpha_{c+}q(x) + \sum_{n=0}^{\infty}\binom{\alpha}{n}D^{\alpha-n}_{c+}y\Big[p^{(n)}(x) + \frac{n-\alpha}{n+1}\xi^{(n+1)}(x)\Big]. \tag{33}$$

For the right-sided derivatives (*with conditions* $\xi = 0$, $(\xi y) = 0$ *at point* d),

$$\zeta^-_\alpha = D^\alpha_{d-}q(x) + \sum_{n=0}^{\infty}\binom{\alpha}{n}(-1)^n D^{\alpha-n}_{d-}y\Big[p^{(n)}(x) + \frac{n-\alpha}{n+1}\xi^{(n+1)}(x)\Big]. \tag{34}$$

Note that even for linearly autonomous transformations of x, y in the general case the transformations of fractional derivatives cannot be written as transformations in finite-dimensional space of "natural" differential and integral variables $D^{\alpha_1}_{c+}y, \dots, D^{\alpha_m}_{c+}y$ with some $\alpha_i \in \mathbb{R}$.

Example 2. Simple projective transformation (6) is generated by the infinitesimal operator

$$X = x^2\frac{\partial}{\partial x} + \gamma xy\frac{\partial}{\partial y}.$$

The prolongation formula (33) for fractional derivative $D^\alpha_{0+}y$ gives

$$\zeta_\alpha = (\gamma - 2\alpha)xD^\alpha_{0+}y + (\gamma - \alpha + 1)\alpha D^{\alpha-1}_{0+}y.$$

For arbitrary γ, the corresponding Lie equations form an infinite system and the transformation of the fractional derivative cannot be found in closed form in "natural" variables x, y, $D^\alpha_{c+}y$, $D^{\alpha-k}_{c+}y$, $k \in \mathbb{N}$. Note that the corresponding transformation can be written in closed form by using the new fractional variable:

$$D^\alpha_{0+}\bar{y}(\bar{x}) = (1-ax)^{\alpha+1}D^\alpha_{0+}\left(\frac{y(x)}{(1-ax)^{\gamma+1-\alpha}}\right). \tag{35}$$

For $\gamma = \alpha - n$, the system of Lie equations contains a finite number of variables and has the closed form solution in "natural" variables. For example, if $\gamma = \alpha - 1$, the transformation is

$$D^\alpha_{0+}\bar{y}(\bar{x}) = (1-ax)^{\alpha+1}D^\alpha_{0+}y(x).$$

3.2 Symmetries of ordinary FDEs

Let us consider an equation with fractional derivatives of different orders:

$$F(x, y, \mathcal{D}_x^{\alpha_1} y, \mathcal{D}_x^{\alpha_2} y, \dots, \mathcal{D}_x^{\alpha_m} y) = 0, \quad \alpha_i \in \mathbb{R}. \tag{36}$$

Here, $\mathcal{D}_x^{\alpha_i} y$ can be left-sided or right-sided fractional derivatives of the Riemann–Liouville or Caputo types ($\alpha_i > 0$, $\alpha_i \notin \mathbb{N}$), fractional integrals ($\alpha_i < 0$), or integer-order derivatives ($\alpha_i \in \mathbb{N}$).

Applying the infinitesimal transformation of the group G generated by (2) to equation (36) written in variables $\bar{x}$, $\bar{y}$, gives the following equality:

$$F(\bar{x}, \bar{y}, \mathcal{D}_{\bar{x}}^{\alpha_1} \bar{y}, \dots, \mathcal{D}_{\bar{x}}^{\alpha_m} \bar{y}) = F(x, y, \mathcal{D}_x^{\alpha_1} y, \dots, \mathcal{D}_x^{\alpha_m} y) + a\tilde{X}F + o(a), \tag{37}$$

where

$$\tilde{X} = X + \sum_{i=1}^{m} \zeta_{\alpha_i} \frac{\partial}{\partial(\mathcal{D}_x^{\alpha_i} y)}$$

is the generator of appropriate prolonged group.

The equality (37) looks like equation (12) for ordinary differential equations. However, the principal difference between (37) and (12) is that terms in $\tilde{X}F$ depend on an infinite number of derivatives and integrals (see prolongation formula (25)).

This problem is analogous to one that arises in theory of nonlocal symmetries of differential equations (see discussion in [1], introduction section). Therefore, for FDEs it is not possible to use Definition 5 based on a geometrical approach.

In [11] and references therein, the Lie group analysis methods are developed for integrodifferential equations and the definition of symmetry group for these equations is similar to Definition 4. It means that such group transforms any solution of the considered equation into solution of the same equation. Using such approach, the equality (37) provides the infinitesimal condition

$$\tilde{X}F|_{S_F} = 0, \tag{38}$$

where S_F is the set of all solutions $y(x)$ of (36).

Unfortunately, the condition (38) cannot be used as a criterion of invariance for equation (36). First of all, in the general case we do not know all solutions of the equation. An approach that is based on using particular solutions is discussed and illustrated in [11], but in the general case this method is not applicable to ordinary FDEs.

Secondly, the main problem is that in the general case, transformations satisfying the condition (38) transform equation (36) to another equation of different form (that has the same family of solutions).

Example 3. The ordinary FDE

$$D_{0+}^{\alpha+1} y(x) = 0 \tag{39}$$

has the general solution

$$y(x) = C_1 x^{\alpha} + C_2 x^{\alpha-1}. \tag{40}$$

This set of functions admits 8 one-parameter transformation groups [7], and one of them is generated by

$$X = y x^{2-\alpha} \frac{\partial}{\partial x} + \alpha y^2 x^{1-\alpha} \frac{\partial}{\partial y},$$

and the corresponding transformation is

$$x = \frac{\bar{x}}{1 + a\bar{x}^{1-\alpha}\bar{y}}, \quad y = \frac{\bar{y}}{(1 + a\bar{x}^{1-\alpha}\bar{y})^{\alpha}}. \tag{41}$$

It is easy to show by direct calculations that the condition (38) is satisfied.

However, as shown in [9], equation (39) in new variables takes the form with derivative of a function with respect to another function [23]:

$$\frac{1}{\Gamma(1-\alpha)} \left[\frac{1}{D_{\bar{x}}\varphi[\bar{x}]} \frac{d}{d\bar{x}} \right]^2 \int_0^{\bar{x}} \frac{\psi[s] D_s \varphi[s]}{(\varphi[\bar{x}] - \varphi[s])^{\alpha}} ds = 0.$$

Here, the transformation (41) is written as $x = \varphi[\bar{x}]$, $y = \psi[\bar{y}]$ for brevity. The obtained equation has exactly the same general solution (40). Nevertheless, one can see that this equation has essentially different structure and cannot be reduced to original equation by obvious calculations. Therefore, we do not include transformation (41) into the set of admitted transformation for equation (39).

Example 3 shows that the condition (38) is not sufficient for defining the FDE symmetry group and additional restrictions should be stated. However, such restrictions depend on a considered class of FDEs and cannot be formulated in general and clear form. Therefore, we will use the following formal extension of Definition 4.

Definition 7. A group G of transformations (1) is called a symmetry group for fractional differential equation (36) if G converts every solution of the equation into a solution of the same equation and preserves the form of the equation.

In view of this definition, the condition (38) is only necessary condition for the symmetry group, and for operators satisfying the condition (38), we need to check that the corresponding transformation conserves the form of the equation. Note that conditions like (17) and (28) are also implicitly included in Definition 7.

Remark 10. Definition 7 can be considered as a formal extension of the classical Lie symmetry group definition to FDEs. Nevertheless, such definition is not the only one possible. Similar to modern generalizations of group analysis (see, e. g., [4–6, 13, 20]), other definitions of FDE's symmetries can be proposed.

In case when the solution is unknown (general case), we should consider all conditions which are differential and "integral" consequences of the original equation (36). However, following this way, we again arrive to the equation with an infinite set of "natural" integral and differential variables (each ζ_α can be expressed by using the prolongation formula (25) with the chain rule (30), e. g.). There are no constructive algorithms to obtain these consequences and to apply them for solving the determining equation (38).

The problems discussed above can be avoided when we consider equations solved with respect to the highest derivatives and seek only linearly autonomous symmetries. We will illustrate our approach with an example of the FDE with the left-sided Riemann–Liouville derivatives.

Algorithm for constructing linearly autonomous symmetries.

1. Let an equation be in the form

$$D^{\alpha_m}_{c+}y = f(x, y, D^{\alpha_1}_{c+}y, \dots, D^{\alpha_{m-1}}_{c+}y), \quad 0 < \alpha_1 < \cdots < \alpha_{m-1} < \alpha_m.$$

2. Admitted operators X satisfy the conditions:

$$\xi = \xi(x), \quad \eta(x,y) = p(x)y + q(x), \quad \xi(c+) = 0, \quad (\xi y)(c+) = 0. \tag{42}$$

3. Prolongation formula for ζ_α is used in the form (33).
4. Coefficients $\xi(x)$, $p(x)$, $q(x)$ of the admitted operator are found from the determining equation

$$(\tilde{X}F)|_{D^{\alpha_m}_{c+}y=f} = 0, \tag{43}$$

   where only the highest derivative is substituted from the equation.
5. The symbols $x, y, D^{\alpha_i}_{c+}y, D^{\alpha_i-n}_{c+}y,\ n \in \mathbb{N}$ in determining equation (43) are considered as independent variables. The determining equation (43) is split with respect to these variables and the resulting infinite over-determined system is solved to find ξ and η.

Definition 8. We call the operators constructed by this algorithm *linearly autonomous symmetries* of equation (36).

Note that if ξ is a combination of power functions $(x-c)^n$, $n \in \mathbb{N}$, the condition $(\xi y)(c+) = 0$ always holds for all integrable functions $y(x)$.

Example 4. For equation

$$D^{\alpha}_{0+}y = y^2, \tag{44}$$

the determining equation (43) can be written as $(\zeta_\alpha - 2y\eta)|_{D^{\alpha}_{c+}y=y^2} = 0$ or

$$D^{\alpha}_{0+}q + \sum_{n=1}^{\infty}\binom{\alpha}{n}D^{\alpha-n}_{0+}y\left[p^{(n)} + \frac{n-\alpha}{n+1}\xi^{(n+1)}\right] + y^2(p - \alpha\xi') - 2y(py + q) = 0. \tag{45}$$

Splitting this equation with respect to $D^{\alpha-1}_{0+}y, D^{\alpha-2}_{0+}y, \dots$, one obtains the infinite system of linear ordinary differential equations

$$p^{(n)}(x) + \frac{n-\alpha}{n+1}\xi^{(n+1)}(x) = 0, \quad n = 1, 2, \dots, \tag{46}$$

and splitting the remaining equation with respect to y gives

$$D^{\alpha}_{0+}q = 0, \quad p + \alpha\xi' = 0, \quad q = 0. \tag{47}$$

From (47), it is obvious that $q(x) = 0$. From the first two equations of the system (46), one gets $\xi''' = 0$ and using the condition $\xi(0) = 0$, the general solution of the infinite system (46) is obtained:

$$\xi = C_1x + C_2x^2, \quad p = (\alpha - 1)C_2x + C_3.$$

The relation between C_2 and C_3 (47) is obtained from (47) and has the form

$$(3\alpha - 1)C_2x + \alpha C_1 + C_3 = 0. \tag{48}$$

For arbitrary α, one has $C_2 = 0$, $C_3 = -\alpha C_1$, and the symmetry group is generated by operator

$$X_1 = x\frac{\partial}{\partial x} - \alpha y\frac{\partial}{\partial y}.$$

It is the one-parameter group of scaling transformations

$$\bar{x} = e^{a}x, \quad \bar{y} = e^{-\alpha a}y. \tag{49}$$

If $\alpha = 1/3$, then C_2 can be arbitrary in (48) and $C_3 = -C_1/3$. The symmetry group extends and the equation has two linearly-autonomous symmetries:

$$X_1 = x\frac{\partial}{\partial x} - \frac{1}{3}y\frac{\partial}{\partial y}, \quad X_2 = x^2\frac{\partial}{\partial x} - \frac{2}{3}xy\frac{\partial}{\partial y}. \tag{50}$$

3.3 Solution construction

The symmetries of fractional differential equations can be used to construct its solutions.

The first way is based on Definition 7 of FDE's symmetry. Namely, when some solution $y = \phi(x)$ of an equation is known, the transformations (1) of a symmetry group allow us to obtain a one-parameter set of solutions.

Example 5. In case of equation $D^{\alpha}_{0+}y = y^2$ considered in Example 4, the symmetry transformation (49) gives a family of solutions

$$y = e^{\alpha a}\phi(e^{a}x).$$

The second method is to construct the so-called invariant solution (the solution that is not changed when the corresponding group transformation is applied). Let the symmetry group of the equation have invariant $J(x, y)$ with $J_y \neq 0$. Then the invariant solution

$$y(x) = \Phi(x, C) \tag{51}$$

can be found from the implicit form

$$J(x, y) = C,$$

and the constant C is determined after substitution of (51) into the original FDE.

It is known that the existence of invariant representation does not guarantee uniqueness of the solution or even its existence. Corresponding examples can be found in [14, 19, 21]. For the case of FDE, there are additional conditions of solution existence [17] related to fractional derivatives defining and they should be checked after constructing the form of invariant solution.

Example 6. For equation $D^{\alpha}_{0+}y = y^2$, the symmetry

$$X_1 = x\frac{\partial}{\partial x} - \alpha y\frac{\partial}{\partial y}$$

has an invariant $J = yx^{\alpha}$ and the corresponding invariant solution has the form

$$y = Cx^{-\alpha}.$$

Substituting it into equation (44), one obtains $C = \Gamma(1-\alpha)/\Gamma(1-2\alpha)$ or $C = 0$.

Example 7. For equation $D^{1/3}_{0+}y = y^2$ with symmetries (50), the solution from the previous example can be used to construct a family of solutions by using the transformation generated by X_2:

$$\bar{x} = \frac{x}{1-ax}, \quad \bar{y} = y(1-ax)^{2/3}.$$

Starting from $\bar{y} = C\bar{x}^{-\alpha}$, we obtain the solution family

$$y(x) = \frac{C}{x^{1/3}(1-ax)^{1/3}}, \quad C = \frac{\Gamma(1-\alpha)}{\Gamma(1-2\alpha)}.$$

Note that this solution can also be obtained as invariant one with respect to the admitted operator $X_1 - aX_2$.

4 Symmetry analysis results

4.1 Equation $D^{\alpha}_{0+}y(x) = f(x,y)$

For the class of equations,

$$D^{\alpha}_{0+}y(x) = f(x,y), \quad 0 < \alpha < 1, \ x > 0, \tag{52}$$

the problem of group classification with respect to linearly autonomous symmetries is solved in [8, 16].

Equivalence transformations.

$$\begin{aligned} \bar{x} &= a_1 x(1-a_2x)^{-1}, \quad \bar{y} = a_3(1-a_2x)^{1-\alpha}(y + v(x)), \\ \bar{f} &= a_1^{-\alpha}a_3(1-a_2x)^{1+\alpha}(f + D^{\alpha}_x v(x)), \end{aligned} \tag{53}$$

where $a_1 > 0$, $a_3 \neq 0$ and $v(x)$ is an arbitrary function.

Symmetries. For arbitrary function $f(x,y)$, equation (52) has no linearly autonomous symmetries.

The symmetry group appears only for the following nonequivalent cases:

1. $f(x,y) = y\Psi(x)$:

$$X_1 = y\frac{\partial}{\partial y}, \quad X_2 = q(x)\frac{\partial}{\partial y},$$

where $q(x)$ is an arbitrary solution of the same equation $D^{\alpha}_x q = q\Psi(x)$.

1.1. $\Psi(x) = kx^{-\alpha}$:

$$X_3 = x\frac{\partial}{\partial x}.$$

1.2. $\Psi(x) = \pm x^{-2\alpha}$:

$$X_3 = x^2\frac{\partial}{\partial x} + (\alpha - 1)xy\frac{\partial}{\partial y}.$$

1.3. $\Psi(x) = 0$:

$$X_3 = x^2\frac{\partial}{\partial x} + (\alpha - 1)xy\frac{\partial}{\partial y}, \quad X_4 = x\frac{\partial}{\partial x}.$$

2. $f(x,y) = x^{-1-\alpha}\Psi(yx^{1-\alpha})$:

$$X_1 = x^2\frac{\partial}{\partial x} + (\alpha - 1)xy\frac{\partial}{\partial y}.$$

2.1. $\Psi(z) = e^z$:

$$X_2 = x\frac{\partial}{\partial x} + \big((\alpha - 1)y + \alpha x^{\alpha-1}\big)\frac{\partial}{\partial y}.$$

2.2. $\Psi(z) = z^\lambda$, $\lambda \in \mathbb{R}$, $\lambda \neq 0, 1$:

$$X_2 = x\frac{\partial}{\partial x} - \frac{1 - \lambda(1 - \alpha)}{1 - \lambda}y\frac{\partial}{\partial y}.$$

3. $f(x,y) = x^{\beta-\alpha}\Psi(y/x^\beta)$, $\beta \in \mathbb{R}$:

$$X_1 = x\frac{\partial}{\partial x} + \beta y\frac{\partial}{\partial y}.$$

4. $f(x,y) = x^{-1-\alpha}e^{\mp 1/x}\Psi(yx^{1-\alpha}e^{\pm 1/x})$:

$$X_1 = x^2\frac{\partial}{\partial x} + \big((\alpha - 1)xy \pm y\big)\frac{\partial}{\partial y}.$$

Examples of solutions. For the equation (see case 4),

$$D^\alpha_{0+}y = x^{-1-\alpha}e^{-1/x}\Psi(yx^{1-\alpha}e^{1/x}),$$

the solution invariant with respect to generator,

$$X_1 = x^2\frac{\partial}{\partial x} + (\alpha - 1)xy\frac{\partial}{\partial y} + y\frac{\partial}{\partial y}$$

has the form

$$y = Cx^{\alpha-1}e^{-1/x}, \quad C = \Psi(C).$$

For the equation (see case 2.1),

$$D^\alpha_{0+}y = x^{-1-\alpha}e^{yx^{1-\alpha}},$$

the invariant solution with respect to operator

$$X_2 = x\frac{\partial}{\partial x} + \big((\alpha - 1)y + \alpha x^{\alpha-1}\big)\frac{\partial}{\partial y}$$

has the form

$$y = x^{\alpha-1}(\alpha \ln x + \ln \Gamma(\alpha+1))$$

and, using the projective transformation generated by

$$X_1 = x^2 \frac{\partial}{\partial x} + (\alpha - 1)xy\frac{\partial}{\partial y},$$

it can be transformed to a family of solutions with parameter a:

$$y = x^{\alpha-1}(\alpha \ln x - \alpha \ln(1+ax) + \ln \Gamma(\alpha+1)), \quad a = \text{const}.$$

Some other examples of invariant solutions for equations of the class (52) are presented in [2, 12, 22].

4.2 Equations $D_{0+}^{\alpha+1} y(x) = \varphi(x, y, D_{0+}^{\alpha} y)$

For the class of equations,

$$D_{0+}^{\alpha+1} y = \varphi(x, y, D_{0+}^{\alpha} y), \quad 0 < \alpha < 1, x > 0. \tag{54}$$

the problem of group classification with respect to linearly autonomous symmetries is considered in [10, 16].

Equivalence transformations.

$$\begin{aligned} \bar{x} &= a_1 x(1 - a_2 x)^{-1}, \quad \bar{y} = a_3(1 - a_2 x)^{1-\alpha}(y + v(x)), \\ \bar{\varphi} &= a_1^{-\alpha-1} a_3 [(1 - a_2 x)^{3+\alpha}(\varphi + D^{\alpha+1} v(x)) \\ &\quad - a_2(\alpha+1)(1 - a_2 x)^{2+\alpha}(D_{0+}^{\alpha} y + D_{0+}^{\alpha} v(x))]. \end{aligned} \tag{55}$$

Here, $a_1 > 0$, $a_2 < x^{-1}$, $a_3 \neq 0$, the function $v(x)$ is arbitrary.

Symmetries. The group classification procedure leads to three classes of equations with linearly autonomous symmetries.

I. $\varphi(x, y, z) = f(x, y)$.
The classification result for $D_{0+}^{\alpha+1} y = f(x, y)$ has the same form as in the previous section with α replaced by $\alpha + 1$.

II. $\varphi(x, y, z)$: $\varphi_{zz} \neq 0$ or $\varphi_{zy} \neq 0$ ($z = D_{0+}^{\alpha} y$).
The symmetry group appears only for the following cases:

1. $\varphi(x, y, z) = yF(x, z/y)$:

$$X_1 = y\frac{\partial}{\partial y}.$$

1.1. $F(x,w) = x^{-\alpha-1}\Psi(x^{\alpha}w)$:

$$X_2 = x\frac{\partial}{\partial x}.$$

1.1.1. $\Psi(s) = -(\alpha+1)s + As^{1+\frac{1}{\alpha}}$:

$$X_3 = x^2\frac{\partial}{\partial x} + (\alpha-1)xy\frac{\partial}{\partial y}.$$

1.1.2. $\Psi(s) = As + B$:

$$X_q = q(x)\frac{\partial}{\partial y}, \text{ where } q(x) \text{ is a solution of}$$
$$D_{0+}^{\alpha+1}q = Ax^{-1}D_{0+}^{\alpha}q + Bx^{-\alpha-1}q.$$

1.1.3. $\Psi(s) = -(\alpha+1)s$:

$$X_3 = x^2\frac{\partial}{\partial x} + (\alpha-1)xy\frac{\partial}{\partial y}, \quad X_q = q(x)\frac{\partial}{\partial y},$$
where $q(x)$ is a solution of $D_{0+}^{\alpha+1}q = -(\alpha+1)x^{-1}D_{0+}^{\alpha}q$.

1.2. $F(x,w) = -(\alpha+1)x^{-1}w + x^{-2\alpha-2}\Psi(x^{2\alpha}w)$:

$$X_2 = x^2\frac{\partial}{\partial x} + (\alpha-1)xy\frac{\partial}{\partial y}.$$

1.2.1. $\Psi(s) = \pm s + A$,

$$X_q = q(x)\frac{\partial}{\partial y}, \text{ where } q(x) \text{ is a solution of}$$
$$D_{0+}^{\alpha+1}q = -(\alpha+1)x^{-1}D_{0+}^{\alpha}q \pm x^{-2}D_{0+}^{\alpha}q + Aqx^{-2\alpha-2}.$$

1.2.2. $\Psi(s) = \pm 1$:

$$X_q = q(x)\frac{\partial}{\partial y}, \text{ where } q(x) \text{ is a solution of}$$
$$D_{0+}^{\alpha+1}q = -(\alpha+1)x^{-1}D_{0+}^{\alpha}q \pm qx^{-2\alpha-2}.$$

1.3. $F(x,w) = \Phi(x)w + \Psi(x)$:

$$X_q = q(x)\frac{\partial}{\partial y}, \text{ where } q(x) \text{ is a solution of}$$
$$D_{0+}^{\alpha+1}q = \Phi(x)D_{0+}^{\alpha}q + q\Psi(x).$$

2. $\varphi(x,y,z) = -(\alpha+1)zx^{-1} + x^{-\alpha-3}F(x^{1-\alpha}y, x^{1+\alpha}z)$:

$$X_1 = x^2\frac{\partial}{\partial x} + (\alpha-1)xy\frac{\partial}{\partial y}.$$

2.1. $F(v,w) = v^{\frac{\beta+2}{\beta+1-\alpha}}\Psi(wv^{\frac{\beta+1}{\alpha-\beta-1}})$, $\beta \neq \alpha - 1$:

$$X_2 = x\frac{\partial}{\partial x} + \beta y\frac{\partial}{\partial y}.$$

2.2. $F(v,w) = e^{(\alpha+1)v}\Psi(we^{-\alpha v})$:

$$X_2 = x\frac{\partial}{\partial x} + \big((\alpha-1)y + x^{\alpha-1}\big)\frac{\partial}{\partial y}.$$

2.3. $F(v,w) = v + \Psi(w - v)$:

$$X_2 = x^{\alpha-1}e^{-1/x}\frac{\partial}{\partial y}.$$

2.4. $F(v,w) = \Psi(w)$:

$$X_2 = x^{\alpha-1}\frac{\partial}{\partial y}.$$

3. $\varphi(x,y,z) = x^{\beta-\alpha-1}F(x^{-\beta}y, x^{\alpha-\beta}z)$:

$$X_1 = x\frac{\partial}{\partial x} + \beta y\frac{\partial}{\partial y}.$$

3.1. $F(v,w) = (\lambda - \alpha)\mu v + \Psi(w - \mu v)$, $\mu = \Gamma(\lambda+1)/\Gamma(\lambda-\alpha+1)$:

$$X_2 = x^{\lambda}\frac{\partial}{\partial y}, \quad \lambda > -1.$$

4. $\varphi(x,y,z) = -(\alpha+1)x^{-1}z + x^{-\alpha-3}e^{\mp\frac{1}{x}}F(yx^{1-\alpha}e^{\pm\frac{1}{x}}, zx^{1+\alpha}e^{\pm\frac{1}{x}})$:

$$X_1 = x^2\frac{\partial}{\partial x} + \big((\alpha-1)xy \pm y\big)\frac{\partial}{\partial y}.$$

4.1. $F(v,w) = \gamma^{\alpha+1}v + \Psi(w - \gamma^{\alpha}v)$, $\gamma \geq 0$:

$$X_2 = x^{\alpha-1}e^{-\gamma/x}\frac{\partial}{\partial y}.$$

5. $\varphi(x,y,z) = \frac{D_{0+}^{\alpha+1}w}{w}y + F(x, z - \frac{D_{0+}^{\alpha}w}{w}y)$ with some $w(x)$:

$$X_1 = w(x)\frac{\partial}{\partial y}.$$

Here, β, A, B are arbitrary constants.

III. The equation is linear with respect to $z = D_{0+}^{\alpha}y$:

$$\varphi(x,y,z) = \phi_1(x)D_{0+}^{\alpha}y + \phi_2(x,y).$$

There are the same symmetries that are listed in case II (subcases with linear φ), but there may be additional linear autonomous symmetries of the different form. For example, the group generated by

$$X = x\frac{\partial}{\partial x} + \frac{xy}{k-x}\frac{\partial}{\partial y},$$

is admitted by equation

$$D_{0+}^{\alpha+1}y = \frac{\alpha+1}{k-x}D_{0+}^{\alpha}y.$$

Note that this equation can be transformed to $D_{0+}^{\alpha+1}y = 0$ from case I by the projective equivalence transformation, and the operator X arrives to $x^2\frac{\partial}{\partial x} + \alpha xy\frac{\partial}{\partial y}$.

For this case, the problem of group classification is not solved completely.

Some examples of invariant solutions are presented in [2, 22].

4.3 Equations $D_{0+}^{\alpha}y(x) + \gamma \cdot D_{1-}^{\alpha}y(x) = f(x,y)$

For the class of equations with combination of the left-sided and right-sided Riemann–Liouville fractional derivatives,

$$D_{0+}^{\alpha}y(x) + \gamma \cdot D_{1-}^{\alpha}y(x) = f(x,y), \quad 0 < \alpha < 1, \ x > 0, \ y \neq 0, \tag{56}$$

the problem of group classification with respect to linearly autonomous symmetries is solved in [16].

Equivalence transformations.

$$\begin{aligned} \bar{x} &= \frac{x}{x+(1-x)a_1}, \quad \bar{y} = \frac{a_2 a_1^{\alpha-1}(y+v(x))}{(x+(1-x)a_1)^{\alpha-1}}, \\ \bar{f} &= \frac{a_2}{a_1}\left(f + D_{0+}^{\alpha}v(x)\right)\left(x+(1-x)a_1\right)^{\alpha+1}. \end{aligned} \tag{57}$$

Here, $a_1 > 0$, $a_2 \in \mathbb{R}$, and the function $v(x)$ is arbitrary.

Symmetries. For arbitrary function $f(x,y)$, the equation does not have linearly autonomous admitted operators.

Group extension cases (up to equivalence transformations (57)):

1. $f(x,y) = y\Psi(x)$:

$$X_1 = y\frac{\partial}{\partial y}, \quad Z_q = q(x)\frac{\partial}{\partial y},$$

where $q(x)$ satisfies the equation

$$D_{0+}^{\alpha}q(x) + \gamma \cdot D_{1-}^{\alpha}q(x) = q\Psi(x).$$

1.1. $\Psi(x) = kx^{-\alpha}(1-x)^{-\alpha}$, $k \in \mathbb{R}$:

$$X_2 = (x^2 - x)\frac{\partial}{\partial x} + (\alpha - 1)xy\frac{\partial}{\partial y}.$$

2. $f(x,y) = x^{-\beta-\alpha}(1-x)^{\beta-1}\Psi(y(1-x)^{1-\alpha-\beta}x^{\beta})$, $k \in \mathbb{R}$:

$$X_1 = (x^2 - x)\frac{\partial}{\partial x} + ((\alpha - 1)xy + \beta y)\frac{\partial}{\partial y}.$$

Note that symmetry group is very restricted compared to the equation with one fractional derivative because of conditions $\xi(0) = 0$, $\xi(1) = 0$.

Solution. For the homogeneous equation,

$$D^{\alpha}_{0+}y(x) + D^{\alpha}_{1-}y(x) = 0, \quad 0 < \alpha < 1,$$

invariant solutions with respect to the symmetries $X_1 + \lambda X_2$, $\lambda \in (-\alpha, 1)$, have the form

$$y = Cx^{-\lambda}(1-x)^{\lambda+\alpha-1}.$$

For λ other than $-\alpha/2$ and $1-\alpha/2$, the reduced equation for C has no solutions, and for these two cases, C is arbitrary. Thus we have constructed the general solution

$$y(x) = \frac{C_1 x^{\alpha/2}}{(1-x)^{1-\alpha/2}} + \frac{C_2(1-x)^{\alpha/2}}{x^{1-\alpha/2}}, \quad C_1, C_2 \in \mathbb{R}.$$

4.4 Symmetries of FDE systems

The problem of group classification for the system of two ordinary FDEs

$$D^{\alpha}_{0+}u(x) = f(x,u,v), \quad D^{\alpha}_{0+}v(x) = g(x,u,v) \tag{58}$$

is considered in [15, 16]. The outline of results is presented here.

All linearly autonomous symmetries of the system (58) have the form

$$\begin{aligned}
&X = C_1X_1 + C_2X_2 + \cdots + C_6X_6 + \langle q^u(x)\rangle_u + \langle q^v(x)\rangle_v,\\
&X_1 = x\frac{\partial}{\partial x}, \quad X_2 = x^2\frac{\partial}{\partial x} + (\alpha-1)tu\frac{\partial}{\partial u} + (\alpha-1)tv\frac{\partial}{\partial v},\\
&X_3 = u\frac{\partial}{\partial u}, \quad X_4 = v\frac{\partial}{\partial u}, \quad X_5 = u\frac{\partial}{\partial v}, \quad X_6 = v\frac{\partial}{\partial v},\\
&X^u_{q^u(x)} = q^u(x)\frac{\partial}{\partial u}, \quad X^v_{q^v(x)} = q^v(x)\frac{\partial}{\partial v},
\end{aligned} \tag{59}$$

with coefficients $C_1, \dots C_6, q^u(x), q^v(x)$ and functions f, g satisfying the relations

$$\begin{cases} (C_1 + C_2x)xf_t + [(\alpha - 1)C_2xu + C_3u + C_4v + q^u(x)]f_u \\ \qquad + [(\alpha - 1)C_2xv + C_5u + C_6v + q^v(x)]f_v \\ \quad = D^\alpha_{0+}q^u(x) + (C_3 - \alpha C_1 - (\alpha + 1)C_2x)f + C_4g, \\ (C_1 + C_2x)xg_t + [(\alpha - 1)C_2xu + C_3u + C_4v + q^u(x)]g_u \\ \qquad + [(\alpha - 1)C_2xv + C_5u + C_6v + q^v(x)]g_v \\ \quad = D^\alpha_{0+}q^v(x) + (C_6 - \alpha C_1 - (\alpha + 1)C_2x)g + C_5f, \end{cases} \tag{60}$$

for arbitrary x, u, v.

The equivalence transformations are presented in [15].

The group classification problem can in this case be solved by preliminary group classification method [1] based on constructing the optimal system of subalgebras. The result contains more than 88 cases and is presented in [15].

Examples.

1. The system from classification [15]

$$D^\alpha_{0+}u(x) = x^{-\alpha}v^{\alpha/(1-\alpha)}(Au + Bv), \quad D^\alpha_{0+}v(x) = x^{-\alpha}v^{\alpha/(1-\alpha)}Av$$

with arbitrary constants A, B admits Lie algebra of symmetries spanned by

$$Z_1 = x\frac{\partial}{\partial x}, \quad Z_2 = v\frac{\partial}{\partial u}, \quad Z_3 = x^2\frac{\partial}{\partial x} + (\alpha - 1)xu\frac{\partial}{\partial u} + (\alpha - 1)xv\frac{\partial}{\partial v}.$$

2. For system

$$D^\alpha_{0+}u(x) = uv, \quad D^\alpha_{0+}v(x) = v^2,$$

equations (60) can be written as

$$D^\alpha_{0+}q_1(x) - q_2(x)u - q_1(x)v - C_5u^2 + (-C_6 - \alpha C_1 + (1 - 3\alpha)C_2x)uv = 0,$$
$$D^\alpha_{0+}q_2(x) - 2q_2(x)v - C_5uv + (-C_6 - \alpha C_1 + (1 - 3\alpha)C_2x)v^2 = 0.$$

After splitting with respect to u, v, one obtains

$$C_5 = 0, \quad -C_6 - \alpha C_1 = 0, \quad (1 - 3\alpha)C_2 = 0, \quad q_1(x) = 0, \quad q_2(x) = 0.$$

If $\alpha \neq 1/3$, the general form of admitted operator is

$$X = C_1X_1 + C_3X_3 + C_4X_4 - \alpha C_1X_6$$

and the symmetries are

$$Z_1 = x\frac{\partial}{\partial x} - \alpha v\frac{\partial}{\partial v}, \quad Z_2 = u\frac{\partial}{\partial u}, \quad Z_3 = v\frac{\partial}{\partial u}.$$

The invariant solution with respect to $Z_1 + \beta Z_2$ has the form

$$u = Cx^{\beta}, \quad v = \lambda x^{-\alpha}, \quad C \in \mathbb{R}, \tag{61}$$

but the suitable constant λ exists only if β satisfies the equation

$$\frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)} = \lambda, \quad \lambda = \frac{\Gamma(1-\alpha)}{\Gamma(1-2\alpha)}.$$

Applying transformation generated by Z_3, the family of solutions is easily obtained:

$$u = Cx^{\beta} + a\lambda x^{-\alpha}, \quad v = \lambda x^{-\alpha}, \quad a, C \in \mathbb{R}.$$

For $\alpha = 1/3$, one more symmetry is

$$Z_4 = x^2\frac{\partial}{\partial x} - \frac{2}{3}xu\frac{\partial}{\partial u} + (\alpha - 1)xv\frac{\partial}{\partial v}.$$

Using the corresponding transformation, different set of solutions can be obtained from (61):

$$u = Cx^{\beta}(1+ax)^{-\beta-2/3}, \quad v = \lambda x^{-1/3}(1+ax)^{-1/3},$$

and an extra parameter can be introduced by using Z_3:

$$u = Cx^{\beta}(1+ax)^{-\beta-2/3} + b\lambda x^{-1/3}(1+ax)^{-1/3}, \quad v = \lambda x^{-1/3}(1+ax)^{-1/3}.$$

Bibliography

[1] I. Sh. Akhatov, R. K. Gazizov, and N. Kh. Ibragimov, Nonlocal symmetries. Heuristic approach, *Journal of Soviet Mathematics*, **55**(1) (1991), 1401–1450.

[2] T. Bakkyaraj and R. Sahadevan, Invariant analysis of nonlinear fractional ordinary differential equations with Riemann–Liouville fractional derivative, *Nonlinear Dynamics*, **80**(1–2) (2015), 447–455.

[3] G. Bluman and S. Kumei, *Symmetries and Differential Equations*, Springer-Verlag, New York, Heidelberg, Berlin, 1989.

[4] P. A. Clarkson and P. Winternitz, Symmetry reduction and exact solutions of nonlinear partial differential equations, in *The Painlevé Property*, pp. 591–660, 1999.

[5] W. I. Fushchich and A. G. Nikitin, *Symmetries of Equations of Quantum Mechanics*, Allerton Press, 1994.

[6] W. I. Fushchich, W. M. Shtelen, and N. I. Serov, *Symmetry Analysis and Exact Solutions of Equations of Nonlinear Mathematical Physics*, Springer Science & Business Media, 2013.

[7] R. K. Gazizov, A. A. Kasatkin, and S. Yu. Lukashchuk, Continuous transformation groups of fractional-order differential equations, *Vestnik USATU*, **3**(21) (2007), 125–135 (in Russian).

[8] R. K. Gazizov, A. A. Kasatkin, and S. Yu. Lukashchuk, Group-invariant solutions of fractional differential equations, in J. Machado, A. Luo, R. Barbosa, M. Silva, L. Figueiredo (eds) *Nonlinear Science and Complexity*, pp. 51–59, Springer, Dordrecht, 2011.

[9] R. K. Gazizov, A. A. Kasatkin, and S. Yu. Lukashchuk, Fractional differential equations: change of variables and nonlocal symmetries, *Ufa Mathematical Journal*, **4**(4) (2012), 54–67.

[10] R. K. Gazizov, A. A. Kasatkin, and S. Yu. Lukashchuk, Linearly autonomous symmetries of the ordinary fractional differential equations, in *Proc. of 2014 International Conference on Fractional Differentiation and Its Applications (ICFDA 2014)*, pp. 1–6, IEEE, 2014.

[11] Y. N. Grigoriev, N. H. Ibragimov, V. F. Kovalev, and S. V. Meleshko, *Symmetries of Integro-Differential Equations With Applications in Mechanics and Plasma Physics*, Springer Science & Business Media, 2010.

[12] M. S. Hashemi, F. Bahrami, and R. Najafib, Lie symmetry analysis of steady-state fractional reaction-convection-diffusion equation, *Optik-International Journal for Light and Electron Optics*, **138** (2017), 240–249.

[13] N. H. Ibragimov (ed.), *CRC Handbook of Lie Group Analysis of Differential Equations, Volume III*, CRC Press, 1995.

[14] N. H. Ibragimov, *Elementary Lie Group Analysis and Ordinary Differential Equations*, Wiley, 1999.

[15] A. A. Kasatkin, Symmetry properties for systems of two ordinary fractional differential equations, *Ufa Mathematical Journal*, **4**(1) (2012), 71–81.

[16] A. A. Kasatkin, Symmetries and exact solutions of equations with Riemann–Liouville fractional derivatives (in Russian), Thesis (Cand. Phys.-Mathem. Science), 2013.

[17] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematical Studies, 2006.

[18] S. Lie and G. Scheffers, *Lectures on Differential Equations with Known Infinitesimal Transformations*, Teubner, Leipzig, 1891.

[19] P. J. Olver, *Applications of Lie Groups to Differential Equations*, Springer Science & Business Media, 2000.

[20] P. J. Olver and P. Rosenau, Group-invariant solutions of differential equations, *SIAM Journal on Applied Mathematics*, **47**(2) (1987), 263–278.

[21] L. V. Ovsyannikov, *Group Analysis of Differential Equations*, Academic press, New York, 1982.

[22] P. Prakash and R. Sahadevan, Lie symmetry analysis and exact solution of certain fractional ordinary differential equations, *Nonlinear Dynamics*, **89**(1) (2017), 305–319.

[23] S. G. Samko, A. A. Kilbas, and O. I. Marichev, *Fractional Integrals and Derivatives. Theory and Applications*, Gordon and Breach Science Publishers, London, 1993.

[24] H. Stephani, *Differential Equations. Their Solution Using Symmetries*, Cambridge University Press, 1989.

Yuri Luchko

Operational method for fractional ordinary differential equations

Abstract: In this chapter, a survey of some applications of the operational calculi of Mikusiński type to fractional ordinary differential equations is presented. We start with constructing of an operational calculus for the general multiple Erdélyi–Kober fractional derivative that includes both differential operators of the hyper-Bessel type and the Riemann–Liouville fractional derivative, and then proceed with its modification for the Caputo fractional derivative. The operational calculi allow to easily treat the Cauchy initial value problems for the linear ordinary fractional differential equations with these fractional derivatives. In particular, the multi-term linear fractional differential equations with the Caputo fractional derivatives of arbitrary orders are considered. In all cases, the obtained solutions are expressed in terms the Mittag-Leffler-type functions.

Keywords: Operational calculus, convolutions, fractional derivatives and integrals, Erdélyi–Kober fractional operators, Mittag-Leffler type functions, fractional differential equations

MSC 2010: 26A33, 44A40, 44A35, 33E30, 45J05, 45D05

1 Introduction

In the 1950s, Jan Mikusiński proposed a new algebraic construction of an operational calculus for the operator of differentiation [26]. His approach was based on interpretation of the Laplace convolution as a multiplication in the ring of the functions continuous on the real half-axis. Later on, Mikusiński's scheme was extended to operational calculi for several differential operators with variable coefficients [11, 25, 29] that are particular cases of the hyper-Bessel differential operator

$$(By)(x) = x^{-\beta} \prod_{i=1}^{n} \left(\gamma_i + \frac{1}{\beta} x \frac{d}{dx} \right) y(x), \quad \beta > 0,\ \gamma_i \in \mathbb{R},\ i = 1, \ldots, n. \tag{1}$$

An operational calculus of Mikusiński type for the hyper-Bessel differential operator (1) was constructed in [6].

The next phase in advancement of the operational method was initiated in the publications [1, 12, 14, 19–23, 30], where operational calculi for the Riemann–Liouville

Yuri Luchko, Beuth University of Applied Sciences Berlin, Department of Mathematics, Physics, and Chemistry, Luxemburger Str. 10, 13353 Berlin, Germany, e-mail: luchko@beuth-hochschule.de

https://doi.org/10.1515/9783110571660-005

fractional derivative, the Caputo fractional derivative, and the general multiple Erdélyi–Kober fractional derivative were developed and applied for solving fractional differential equations and integral equations of the Abel type.

Recently, some new operational calculi of the Mikusiński type for other fractional derivatives were constructed. In particular, we refer to the results presented in the papers [15] (operational calculus for the Caputo-type fractional Erdélyi–Kober derivative) and [16] (operational calculus for the Hilfer fractional derivative) that do not fit the general schema of the operational calculus for the multiple Erdélyi–Kober fractional derivative.

It is also worth mentioning that instead of the Mikusiński-type operational calculi, the corresponding integral transforms can be employed to solve the Cauchy initial value problems for the linear ordinary fractional differential equations with fractional derivatives of different types. Say, for the equations with the Riemann–Liouville fractional derivatives and the Caputo fractional derivatives, the Laplace integral transform can be used [2, 5, 28] whereas the Obrechkoff integral transform [8, 10, 17, 30] can be employed for treating the equations with the multiple Erdélyi–Kober fractional derivatives. In this chapter, we restrict ourselves to the Mikusiński approach for development of the operational calculi for fractional derivatives of different types.

2 Operational calculus for the multiple Erdélyi–Kober fractional derivative

In this section, some elements of the operational calculus for the general multiple Erdélyi–Kober fractional derivative are presented. For details and proofs, we refer the readers to [1, 19], and [23].

We start with a definition of the multiple Erdélyi–Kober fractional integral.

Definition 1. Let $\mu > 0$, $a_i > 0$, $\alpha_i \in \mathbb{R}$, $i = 1, \ldots, n$. The multiple Erdélyi–Kober fractional integral of a function $y = y(t)$, $t > 0$ at the point x is defined by the formula

$$(L_\mu y)(x) = \frac{x^\mu}{\prod_{i=1}^n \Gamma(a_i\mu)} \int_0^1 \cdots \int_0^1 \prod_{i=1}^n (1-t_i)^{a_i\mu-1} t_i^{-\alpha_i}\, y\left(x \prod_{i=1}^n t_i^{a_i}\right) dt_1 \ldots dt_n. \tag{2}$$

Remark 1. The multiple Erdélyi–Kober fractional integral (2) can be represented as a composition of the Erdélyi–Kober fractional integrals:

$$(L_\mu y)(x) = x^\mu \left(\prod_{i=1}^n I_{1/a_i}^{-\alpha_i, a_i\mu} y \right)(x) \tag{3}$$

with

$$(I_\beta^{\gamma,\delta} y)(x) = \begin{cases} \frac{1}{\Gamma(\delta)} \int_0^1 (1-t)^{\delta-1} t^\gamma\, y(xt^{1/\beta})\, dt, & \text{if } \delta > 0, \\ y(x), & \text{if } \delta = 0 \end{cases} \tag{4}$$

being the Erdélyi–Kober fractional integral of order $\delta \geq 0$ ($\beta > 0, \gamma \in \mathbb{R}$) of the function $y = y(t)$, $t > 0$ at the point x.

Both the Riemann–Liouville fractional integral and the hyper-Bessel integral operator are particular cases of the multiple Erdélyi–Kober fractional integral:

Setting $n = 1$, $a_1 = 1$, $\alpha_1 = 0$ in the formula (2) leads to the Riemann–Liouville fractional integral

$$(L_\mu y)(x) = (I^\mu y)(x) = \frac{1}{\Gamma(\mu)} \int\limits_0^x (x-t)^{\mu-1}\, y(t)\, dt. \tag{5}$$

Setting $a_i = 1/\beta$, $\alpha_i = -\gamma_i$, $1 \leq i \leq n$, $\mu = \beta$ in the formula (2) leads to the hyper-Bessel integral operator ([6, 7], [17, Chapter 3]):

$$(L_\mu y)(x) = (\hat{B} y)(x) = x^\beta \int\limits_0^1 \cdots \int\limits_0^1 \prod_{i=1}^n t_i^{\gamma_i}\, y\left(x \prod_{i=1}^n t_i^{1/\beta} \right) dt_1 \ldots dt_n. \tag{6}$$

Remark 2. A similar but more general definition of the multiple Erdélyi–Kober fractional integral was given in [17]:

$$(\mathfrak{I} y)(x) = x^\mu (I_{(\beta_i),n}^{(\gamma_i),(\delta_i)} y)(x) = x^\mu \left(\prod_{i=1}^n I_{\beta_i}^{\gamma_i,\delta_i} y \right)(x)$$

with $\mu > 0$, $\gamma_i \in \mathbb{R}$, $\delta_i \geq 0$, $\beta_i > 0$, $i = 0, 1, \ldots, n$. It turned out that an operational calculus of the Mikusiński type can be developed only for its particular case given by (3) and in the rest of the chapter we restrict our attention solely to this case.

For an operational calculus for the multiple Erdélyi–Kober fractional derivative (left inverse operator to the multiple Erdélyi–Kober fractional integral (2)), a space of functions that is invariant regarding the action of the multiple Erdélyi–Kober fractional integral is needed.

Definition 2. A real or complex-valued function $y = y(t)$, $t > 0$ is said to belong to the space $C_\alpha(0, \infty)$, $\alpha \in \mathbb{R}$ if it can be represented in the form $y(t) = t^p y_1(t)$ with a real number p, $p > \alpha$ and a function $y_1 \in C[0, \infty)$.

Evidently, $C_\alpha(0, \infty)$ is a vector space and the inclusion

$$C_\alpha(0, \infty) \subseteq C_\beta(0, \infty)$$

holds true for $\alpha \geq \beta$.

Theorem 1 ([18]). *For $\alpha \geq \max_{1\leq i\leq n}\{\frac{\alpha_i-1}{a_i}\}$, the multiple Erdélyi–Kober fractional integral (2) is a linear map of the space $C_\alpha(0,\infty)$ into itself:*

$$L_\mu : C_\alpha(0,\infty) \to C_{\alpha+\mu}(0,\infty) \subset C_\alpha(0,\infty). \tag{7}$$

Now the multiple Erdélyi–Kober fractional derivative is introduced.

Definition 3. Let $\mu > 0$, $a_i > 0$, $\alpha_i \in \mathbb{R}$, $i = 1,\dots,n$. The multiple Erdélyi–Kober fractional derivative is defined by the formula

$$(D_\mu y)(x) = x^{-\mu}\prod_{i=1}^{n}\prod_{k=1}^{\eta_i}\left(k - \alpha_i - a_i\mu + a_i x\frac{d}{dx}\right)\left(\prod_{i=1}^{n} I_{1/a_i}^{-\alpha_i,\eta_i-a_i\mu} y\right)(x), \tag{8}$$

where

$$\eta_i = \begin{cases} [a_i\mu]+1, & a_i\mu \notin \mathbb{N}, \\ a_i\mu, & a_i\mu \in \mathbb{N}. \end{cases}$$

Important particular cases of the multiple Erdélyi–Kober fractional derivative are:

(a) The hyper-Bessel differential operator ([6, 7], [17, Chapter 3]):

$$(D_\mu y)(x) = (By)(x) = x^{-\beta}\prod_{i=1}^{n}\left(\gamma_i + \frac{1}{\beta}x\frac{d}{dx}\right)y(x) \tag{9}$$

for $a_i = 1/\beta$, $\alpha_i = -\gamma_i$, $\eta_i = 1$, $1 \leq i \leq n$, $\mu = \beta$.

(b) The Riemann–Liouville fractional derivative

$$(D_\mu y)(x) = (D^\mu y)(x) = \frac{d^\eta}{dx^\eta}(I^{\eta-\mu}y)(x) \tag{10}$$

for $n = 1$, $a_1 = 1$, $\alpha_1 = 0$, and

$$\eta_1 = \eta = \begin{cases} [\mu]+1, & \mu \notin \mathbb{N}, \\ \mu, & \mu \in \mathbb{N}. \end{cases}$$

The multiple Erdélyi–Kober fractional integral (6) is a right inverse operator to the multiple Erdélyi–Kober fractional derivative (8) as stated in the following theorem.

Theorem 2 ([18]). *Let $y \in C_\alpha(0,\infty)$ with $\alpha \geq \max_{1\leq i\leq n}\{\frac{\alpha_i-1}{a_i}\}$ and $g(x) = (L_\mu y)(x)$. Then*

$$(D_\mu L_\mu y)(x) = (D_\mu g)(x) = y(x), \tag{11}$$

that is, the operator L_μ is a right inverse of the operator D_μ on $C_\alpha(0,\infty)$.

For operations with the multiple Erdélyi–Kober fractional derivative, a suitable space of functions is needed.

Definition 4. Let $m \in \mathbb{N}$, $\mu > 0$, $\alpha \in \mathbb{R}$. By $\Omega_\mu^m(C_\alpha(0,\infty))$, we denote the space of functions y, such that $D_\mu^k y \in C_\alpha(0,\infty)$, $k = 0,\dots,m$, where D_μ^k means the composition of k multiple Erdélyi–Kober fractional derivatives (8) for $k = 1,2\dots,n$ and D_μ^0 is the identity operator on the space $C_\alpha(0,\infty)$.

Because of Theorem 2, the space of functions $\Omega_\mu^1(C_\alpha(0,\infty))$ contains in particular all functions that can be represented in the form $y(x) = (L_\mu g)(x)$ with $g \in C_\alpha(0,\infty)$. On this subspace of $\Omega_\mu^1(C_\alpha(0,\infty))$, the multiple Erdélyi–Kober fractional integral L_μ is also a left inverse of the multiple Erdélyi–Kober fractional derivative D_μ, that is, $(L_\mu D_\mu y)(x) = y(x)$ if $y(x) = (L_\mu g)(x)$ with $g \in C_\alpha(0,\infty)$. This property is not valid for the whole space $\Omega_\mu^1(C_\alpha(0,\infty))$. Instead, we have the following result.

Theorem 3 ([18]). *Let $y \in \Omega_\mu^1(C_\alpha(0,\infty))$ and the conditions*

$$\frac{\mu a_i - \eta_i + \alpha_i}{a_i} > \alpha, \quad \eta_i = \begin{cases} [a_i\mu] + 1, & a_i\mu \notin \mathbb{N}, \\ a_i\mu, & a_i\mu \in \mathbb{N}. \end{cases} \quad i = 1,\dots,n \tag{12}$$

are fulfilled. Then the relation

$$(Fy)(x) = \big((\mathrm{Id} - L_\mu D_\mu)y\big)(x) = \sum_{i=1}^{n}\sum_{k=1}^{\eta_i} C_{ik}\Big[\lim_{x\to 0+}(A_{ik}y)(x)\Big]x^{\mu - \frac{k-\alpha_i}{a_i}} \tag{13}$$

holds true, where the operator $F = \mathrm{Id} - L_\mu D_\mu$ is the projector of the operator L_μ and Id *is the identity operator on the space $\Omega_\mu^1(C_\alpha(0,\infty))$,*

$$C_{ik} = \frac{\prod_{j=1}^{i-1}\Gamma(1-\alpha_j - \frac{a_j}{a_i}(k-\alpha_i) + \eta_j)\prod_{j=i+1}^{n}\Gamma(1-\alpha_j - \frac{a_j}{a_i}(k-\alpha_i))}{\prod_{j=1}^{n}\Gamma(1-\alpha_j - \frac{a_j}{a_i}(k-\alpha_i) + a_j\mu)},$$

$$(A_{ik}y)(x) = x^{-\mu + \frac{k-\alpha_i}{a_i}} \prod_{j=1}^{\eta_i - k}\left(k + j - \alpha_i - a_i\mu + a_i x\frac{d}{dx}\right)$$
$$\times \prod_{l=i+1}^{n}\prod_{j=1}^{\eta_i}\left(j - \alpha_l - a_l\mu + a_l x\frac{d}{dx}\right)\left(\prod_{j=1}^{n} I_{1/a_j}^{-\alpha_j,\eta_j - a_j\mu} y\right)(x).$$

For the Riemann–Liouville fractional integral and derivative, the representation (13) takes the known form

$$(Fy)(x) = \big((\mathrm{Id} - I^\mu D^\mu)y\big)(x) = \sum_{k=1}^{m}\frac{x^{\mu-k}}{\Gamma(\mu - k + 1)}\lim_{x\to 0+}(D^{\mu-k}y)(x). \tag{14}$$

The main component of an operational calculus of the Mikusiński type is a suitable convolution operation. In the theorem below, we introduce a one-parametric family of convolutions for the multiple Erdélyi–Kober fractional integral [18, 22, 30].

Theorem 4 ([18]). *Let the conditions*

$$\alpha \geq \max_{1\leq i\leq n}\left\{\frac{\alpha_i - 1}{a_i}\right\}, \quad \lambda \geq \max_{1\leq i\leq n}\left\{\frac{1-\alpha_i}{a_i}\right\} \tag{15}$$

be fulfilled. Then the operation

$$(f \overset{\lambda}{*} g)(x) = x^{\lambda}\left(\prod_{i=1}^{n} I_{1/a_i}^{1-2\alpha_i,\alpha_i+a_i\lambda-1}(f \circ g)\right)(x) \tag{16}$$

with

$$(f \circ g)(x) = \int_0^1 \cdots \int_0^1 f\left(x\prod_{i=1}^{n} t_i^{a_i}\right) g\left(x\prod_{i=1}^{n}(1-t_i)^{a_i}\right)\prod_{i=1}^{n}\big(t_i(1-t_i)\big)^{-\alpha_i}\, dt_1 \ldots dt_n$$

is a convolution without divisors of zero of the multiple Erdélyi–Kober fractional integral L_μ defined by (2) *on the space $C_\alpha(0,\infty)$ in the following sense* (*cf.* [9])*:*

$$(L_\mu(f \overset{\lambda}{*} g))(x) = ((L_\mu f) \overset{\lambda}{*} g)(x) = (f \overset{\lambda}{*} (L_\mu g))(x), \quad \forall f, g \in C_\alpha(0,\infty). \tag{17}$$

Remark 3. For the Riemann–Liouville fractional integral ($n = 1$, $a_1 = 1$, $\alpha_1 = 0$ in (2)), we also get a one-parametric family of convolutions defined by (16). The simplest convolution of this family is the conventional Laplace convolution

$$(f * g)(x) = \int_0^x f(t)\, g(x-t)\, dt \tag{18}$$

that is obtained from the formula (16) for $\lambda = 1$.

The convolution defined by Theorem 4 allows to introduce an algebraic structure on the space of functions $C_\alpha(0,\infty)$ that is needed for construction of the operational calculus we deal with in this section.

Theorem 5 ([18]). *The space of functions $C_\alpha(0,\infty)$ with the operations $\overset{\lambda}{*}$ defined by* (16) *and the ordinary addition of functions is a commutative ring without divisors of zero.*

In what follows, we denote this ring by $(C_\alpha, \overset{\lambda}{*}, +)$. On the ring $(C_\alpha, \overset{\lambda}{*}, +)$, the multiple Erdélyi–Kober fractional integral (2) can be interpreted as a simple convolution operation.

Theorem 6 ([18]). *Let the conditions* (15) *hold true and*

$$\lambda < \mu - \alpha. \tag{19}$$

Then the multiple Erdélyi–Kober fractional integral (2) *can be represented as convolution with a power function:*

$$(L_\mu y)(x) = (y \overset{\lambda}{*} h_\mu)(x), \quad h_\mu(x) = \frac{x^{\mu-\lambda}}{\prod_{i=1}^{n}\Gamma(1-\alpha_i + a_i(\mu-\lambda))}. \tag{20}$$

In the rest of this section, we always assume that the conditions (15) and (19) hold true.

Remark 4. The relation (20) and direct calculations lead to the representation

$$(L_\mu^p y)(x) = (L_{p\mu} y)(x) = (y \overset{\lambda}{*} h_\mu^p)(x) \tag{21}$$

of the composition L_μ^p of p multiple Erdélyi–Kober fractional integrals *(2)* with

$$h_\mu^p(x) = \frac{x^{p\mu-\lambda}}{\prod_{i=1}^n \Gamma(1-\alpha_i + a_i(p\mu-\lambda))}. \tag{22}$$

The key element in applications of an operational calculus for solving fractional differential equations with the fractional derivative D_μ is its representation as a multiplication operator. It turns out that such representations are not possible on the conventional spaces of functions, say, on $\Omega_\mu^1(C_\alpha(0,\infty))$, where the fractional derivative D_μ is well-defined. Instead, the needed representation can be constructed on a space of generalized functions that is defined as an extension of the ring $(C_\alpha, \overset{\lambda}{*}, +)$ to the field $\mathcal{M}_{\alpha,\lambda}$ of convolution quotients by factorizing the set $C_\alpha(0,\infty) \times (C_\alpha(0,\infty) \setminus \{0\})$ with respect to the equivalence relation

$$(f,g) \sim (f_1,g_1) \;\Leftrightarrow\; (f \overset{\lambda}{*} g_1)(t) = (g \overset{\lambda}{*} f_1)(t), \tag{23}$$

the operation $\overset{\lambda}{*}$ being the convolution defined by (16). An element of $\mathcal{M}_{\alpha,\lambda}$ is thus a set of all pairs (f,g), $f \in C_\alpha(0,\infty)$, $g \in C_\alpha(0,\infty) \setminus \{0\}$ that are equivalent to each other in the sense of (23). It is convenient to denote a pair (f,g) as a convolution quotient $\frac{f}{g}$. On $\mathcal{M}_{\alpha,\lambda}$, the operations $+$ and $\cdot$ for the representatives $\frac{f}{g}, \frac{f_1}{g_1}$ of any two elements of $\mathcal{M}_{\alpha,\lambda}$ are naturally defined as follows:

$$\frac{f}{g} + \frac{f_1}{g_1} = \frac{f \overset{\lambda}{*} g_1 + g \overset{\lambda}{*} f_1}{g \overset{\lambda}{*} g_1}, \qquad \frac{f}{g} \cdot \frac{f_1}{g_1} = \frac{f \overset{\lambda}{*} f_1}{g \overset{\lambda}{*} g_1}. \tag{24}$$

The operations $+$ and $\cdot$ do not depend on the representatives of the elements from $\mathcal{M}_{\alpha,\lambda}$. Moreover, the following result holds true.

Theorem 7 ([18]). *The set $\mathcal{M}_{\alpha,\lambda}$ of convolution quotients is a commutative field with respect to the operations "+" and "·" defined by* (24).

Evidently, the ring $(C_\alpha, \overset{\lambda}{*}, +)$ can be embedded into the field $\mathcal{M}_{\alpha,\lambda}$ by the map

$$f \mapsto \frac{h_\mu \overset{\lambda}{*} f}{h_\mu}$$

with h_μ defined as in the formula (20).

The field $\mathcal{M}_{\alpha,\lambda}$ can be interpreted as a vector space of functions. The addition on $\mathcal{M}_{\alpha,\lambda}$ is defined by (24) and multiplication with a scalar λ from the field $\mathbb{R}$ (or $\mathbb{C}$) by the relation

$$\lambda\frac{f}{g} = \frac{\lambda f}{g}, \quad \frac{f}{g} \in \mathcal{M}_{\alpha,\lambda}.$$

Since the constant function $f(x) \equiv \lambda$, $x > 0$ belongs to the space $C_\alpha(0,\infty)$ when $\alpha < 0$, we have to distinguish the operations of multiplication with a scalar λ in the vector space $\mathcal{M}_{\alpha,\lambda}$ and multiplication with a constant function λ in the field $\mathcal{M}_{\alpha,\lambda}$ that will be written in the form

$$\{\lambda\} \cdot \frac{f}{g} = \frac{\lambda h_{\mu+1}}{h_\mu} \cdot \frac{f}{g} = \{1\} \cdot \frac{\lambda f}{g}. \tag{25}$$

It can be easily checked that the representative $I = \frac{h_\mu}{h_\mu}$ of the field $\mathcal{M}_{\alpha,\lambda}$ defines the unit element of this field with respect to multiplication. Because I cannot be reduced to a function from the ring $C_\alpha(0,\infty)$ (see [30] for a proof), we interpret it as a generalized function. Another generalized function (element of $\mathcal{M}_{\alpha,\lambda}$ that cannot be reduced to an element of $C_\alpha(0,\infty)$) that plays an important role in applications of the operational calculus is introduced in the following definition.

Definition 5. The element $S_\mu \in \mathcal{M}_{\alpha,\lambda}$ that is reciprocal to the element h_μ defined by (20) is called the algebraic inverse of the multiple Erdélyi–Kober fractional integral L_μ:

$$S_\mu = \frac{I}{h_\mu} = \frac{h_\mu}{(h_\mu \overset{\lambda}{*} h_\mu)} = \frac{h_\mu}{h_\mu^2}, \tag{26}$$

where $I = \frac{h_\mu}{h_\mu}$ denotes the identity element of the field $\mathcal{M}_{\alpha,\lambda}$ with respect to multiplication.

Remark 5. The convolution representation (20) of L_μ introduced in Theorem 6 can be rewritten in the field $\mathcal{M}_{\alpha,\lambda}$ as a simple multiplication operation:

$$(L_\mu y)(x) = (y \overset{\lambda}{*} h_\mu)(x) = y \cdot h_\mu = y \cdot \frac{I}{S_\mu}. \tag{27}$$

This representation was employed in [12] for solving generalized Abel integral equations of the second kind.

Because the elements h_μ and S_μ are reciprocal, it is natural to expect a representation of the multiple Erdélyi–Kober fractional derivative D_μ as multiplication with S_μ in the field of convolution quotients.

Theorem 8 ([18]). *Let $y \in \Omega_\mu^m(C_\alpha(0,\infty))$. Then the relation*

$$(D_\mu^m y)(x) = S_\mu^m \cdot y - \sum_{k=0}^{m-1} S_\mu^{m-k} \cdot y_k, \quad y_k(x) = (FD_\mu^k y)(x) \tag{28}$$

holds true in the field $\mathcal{M}_{\alpha,\lambda}$ of convolution quotients with $F = \mathrm{Id} - L_\mu D_\mu$ being the projector (13) of the operator L_μ.

For the Riemann–Liouville fractional derivative, the formula (28) takes the following form ($m-1<\mu\le m$, $m\in\mathbb{N}$):

$$(D^{\mu}y)(x) = S_\mu\cdot y - S_\mu\cdot\tilde{y}_\mu,\quad \tilde{y}_\mu(x) = \sum_{k=1}^{m}\frac{x^{\mu-k}}{\Gamma(\mu-k+1)}\lim_{x\to0+}(D^{\mu-k}y)(x). \tag{29}$$

For applications of the developed operational calculus, representations of some elements of the field $\mathcal{M}_{\alpha,\lambda}$ of convolution quotients as elements of the space $C_\alpha(0,\infty)$ are important. A class of such representations is described in the following theorem [18, 22, 30].

Theorem 9 ([18]). *Let a power series in $z\in\mathbb{C}$ with the coefficients $a_k\in\mathbb{C}$, $k=0,1,\dots$ be convergent at a point $z_0\neq0$, that is,*

$$\sum_{k=0}^{\infty}a_k z_0^k = A,\quad A\in\mathbb{C}$$

Then the function

$$\sum_{k=1}^{\infty}a_k\left(\frac{I}{S_\mu}\right)^k = \sum_{k=1}^{\infty}a_k h_\mu^k(x),\quad h_\mu^k(x) = \frac{x^{k\mu-\lambda}}{\prod_{i=1}^{n}\Gamma(1-\alpha_i+a_i(k\mu-\lambda))} \tag{30}$$

is an element of the space $C_\alpha(0,\infty)$.

Let us write down some operational relations that follow from Theorem 9.

For $\rho\in\mathbb{C}$, the representation (21) and the known formula for the geometric series lead to the basic operational relation

$$\begin{aligned}\frac{I}{S_\mu-\rho} &= \frac{h_\mu}{I-\rho h_\mu} = h_\mu(I+\rho h_\mu+\rho^2h_\mu^2+\cdots)\\ &= x^{\mu-\lambda}\sum_{k=0}^{\infty}\frac{(\rho x^\mu)^k}{\prod_{i=1}^{n}\Gamma(1-\alpha_i+a_i(\mu-\lambda)+a_i\mu k)}\\ &= x^{\mu-\lambda}E((1-\alpha_i+a_i(\mu-\lambda),a_i\mu)_n;\rho x^\mu),\end{aligned} \tag{31}$$

where

$$E((\alpha,\beta)_n;z) = E((\alpha_1,\beta_1),\dots,(\alpha_n,\beta_n);z) = \sum_{k=0}^{\infty}\frac{z^k}{\prod_{i=1}^{n}\Gamma(\alpha_i+\beta_i k)}$$

is the Mittag-Leffler function of vector index [18, 23, 30].

Calculating the powers of the operational relation (31), we get its generalization:

$$\begin{aligned}\frac{I}{(S_\mu-\rho)^m} &= x^{\mu m-\lambda}\sum_{k=0}^{\infty}\frac{(m)_k(\rho x^\mu)^k}{k!\prod_{i=1}^{n}\Gamma(1-\alpha_i+a_i(\mu m-\lambda)+a_i\mu k)}\\ &= x^{\mu m-\lambda}E_m((1-\alpha_i+a_i(\mu m-\lambda),a_i\mu)_n;\rho x^\mu)\end{aligned} \tag{32}$$

with

$$E_m((\alpha,\beta)_n;z) = E_m((\alpha_1,\beta_1),\dots,(\alpha_n,\beta_n);z) = \sum_{k=0}^{\infty} \frac{(m)_k z^k}{k!\prod_{i=1}^{n}\Gamma(\alpha_i+\beta_i k)}.$$

For the Riemann–Liouville fractional derivative and the Caputo fractional derivative, the operational relation (31) is reduced to the formula (if $1 \le \lambda < 1+\mu$):

$$\frac{I}{S_\mu - \rho} = x^{\mu-\lambda} \sum_{k=0}^{\infty} \frac{(\rho x^\mu)^k}{\Gamma(1+\mu-\lambda+\mu k)} = x^{\mu-\lambda} E_{1+\mu-\lambda,\mu}(\rho x^\mu), \tag{33}$$

where $E_{\alpha,\beta}(z)$ is the generalized Mittag-Leffler function defined by the following convergent series:

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)}, \quad \alpha > 0,\ \beta \in \mathbb{C}. \tag{34}$$

The operational relation (32) takes the form

$$\frac{I}{(S_\mu - \rho)^m} = x^{\mu m-\lambda} \sum_{k=0}^{\infty} \frac{(m)_k (\rho x^\mu)^k}{k!\Gamma(1+\mu m-\lambda+\mu k)} = x^{\mu m-\lambda} E^m_{1+m\mu-\lambda,\mu}(\rho x^\mu). \tag{35}$$

Let us note that for $\mu \in \mathbb{Q}$ the generalized Mittag-Leffler function from the right-hand side of (33) can be represented as a linear combination of the generalized hypergeometric functions ${}_pF_q(z)$ with the power weights that can be sometimes written in terms of other simpler special functions. In particular, the following formula is well known [30]:

$$E_{1/2,1/2}(z) = \frac{1}{\sqrt{\pi}}\,{}_1F_1(1;1/2;z^2) + ze^{z^2} = \frac{1}{\sqrt{\pi}} + z(1+\operatorname{erf}(z))e^{z^2}, \tag{36}$$

where

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2}\,dt \tag{37}$$

is the error function.

Because any rational function with the power of numerator less then the power of denominator can be represented as a finite sum of the partial fractions, any rational function of S_μ in the field $\mathcal{M}_{\alpha,\lambda}$ can be represented as an element of the space $C_\alpha(0,\infty)$ because of the operational relations (31) and (32). This fact will play an important role in applications of the operational calculi to the fractional differential equations. Let us demonstrate this procedure by some examples. Based on the operational relation (33), we get

$$\frac{I}{S_\mu^2 + \rho^2} = \frac{1}{2\rho i}\left(\frac{I}{S_\mu - i\rho} - \frac{I}{S_\mu + i\rho}\right) = \frac{1}{\rho} x^{\mu-\lambda} \sum_{k=0}^{\infty} \frac{(-1)^k (\rho x^\mu)^{2k+1}}{\Gamma(1+2\mu-\lambda+2\mu k)}. \tag{38}$$

The series from the right-hand side of the last formula can be written in terms of the generalized Mittag-Leffler function and then interpreted as the fractional sin-function. Thus we arrive at the operational relation

$$\frac{I}{S_\mu^2 + \rho^2} = x^{2\mu-\lambda} E_{1+2\mu-\lambda,\,2\mu}(-\rho^2 x^{2\mu}) = \frac{1}{\rho} x^{\mu-\lambda} \sin_{\lambda,\mu}(\rho x^\mu) \tag{39}$$

with

$$\sin_{\lambda,\mu}(z) = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+1}}{\Gamma(1 + 2\mu - \lambda + 2\mu k)}.$$

For $\lambda = \mu = 1$, the function $\sin_{\lambda,\mu}$ is reduced to the conventional sin-function:

$$\sin_{1,1}(z) = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+1}}{(2k+1)!} = \sin(z). \tag{40}$$

For some other values of the parameters λ and μ, the function $\sin_{\lambda,\mu}$ can be written in terms of other elementary and special functions, for example,

$$\sin_{1,1/2}(z) = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+1}}{k!} = z e^{-z^2}. \tag{41}$$

Again starting with the operational relation (33), we get another important operational relation:

$$\begin{aligned}\frac{S_\mu}{S_\mu^2 + \rho^2} &= \frac{1}{2}\left(\frac{I}{S_\mu - i\rho} + \frac{I}{S_\mu + i\rho}\right) \\ &= x^{\mu-\lambda} \sum_{k=0}^{\infty} \frac{(-1)^k (\rho x^\mu)^{2k}}{\Gamma(1 + \mu - \lambda + 2\mu k)} = x^{\mu-\lambda} \cos_{\lambda,\mu}(\rho x^\mu),\end{aligned} \tag{42}$$

where

$$\cos_{\lambda,\mu}(z) = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{\Gamma(1 + \mu - \lambda + 2\mu k)}.$$

In particular,

$$\cos_{1,1}(z) = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{(2k)!} = \cos(z), \tag{43}$$

$$\cos_{1,1/2}(z) = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{\Gamma(1/2 + k)} = \frac{1}{\sqrt{\pi}} {}_1F_1(1; 1/2; -z^2) = \frac{1}{\sqrt{\pi}}(1 - 2zF(z)) \tag{44}$$

with

$$F(z) = e^{-z^2} \int_0^z e^{t^2}\, dt \tag{45}$$

being the Doson function (see [27]).

Remark 6. The right-hand side of the operational relation (32) can be represented in terms of the Fox H-function:

$$\frac{I}{(S_\mu - \rho)^m} = \frac{x^{\mu m-\lambda}}{(m-1)!} H^{1,1}_{n+1,1}\left(-\rho x^\mu \middle| \begin{matrix} (1,1), (1-\alpha_i + a_i(\mu m - \lambda), a_i\mu)_{1,n} \\ (m,1) \end{matrix} \right). \tag{46}$$

Remark 7. As already mentioned in the Introduction, an operational calculus for the multiple Erdélyi–Kober fractional derivative can be also developed on the basis of the generalized Obrechkoff integral transform with a power weight

$$(\mathcal{O}_\beta y)(x) = x^\beta \int_0^\infty H^{0,n}_{n,0}\left(\frac{x}{t} \middle| \begin{matrix} (\alpha, a)_{1,n} \\ - \end{matrix} \right) y(t) \frac{dt}{t}, \tag{47}$$

which plays in this calculus the same role as the Laplace transform in the operational calculus for the first derivative.

3 Operational calculus for the Caputo fractional derivative

The Riemann–Liouville fractional derivative is a particular case of the multiple Erdélyi–Kober fractional derivative (see the formula (10)), and thus we can employ the operational calculus developed in the previous section for the Riemann–Liouville fractional derivative. As to the Caputo fractional derivative, it is not a particular case of the multiple Erdélyi–Kober fractional derivative. Even if an operational calculus of the Mikusiński type for the Caputo fractional derivative has a lot in common with the one presented in the previous section, there are also some peculiarities in construction of this operational calculus [19, 20] that will be discussed in this section.

The so-called Caputo fractional derivative was introduced in [3] and adopted in [4] in the framework of the theory of linear viscoelasticity:

$$({}^{\mathrm{C}}D^\mu y)(x) = (I^{m-\mu} y^{(m)})(x), \quad m-1 < \mu \le m \in \mathbb{N},\ x > 0, \tag{48}$$

the operator I^α being the Riemann–Liouville fractional integral defined by (5).

A natural space of functions, where the Caputo fractional derivative is well-defined, is introduced as follows.

Definition 6. By $C^m_\alpha(0,\infty)$, $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, we define the space of all functions $y = y(t)$, $t > 0$ that satisfy the inclusion $y^{(m)} \in C_\alpha(0,\infty)$ with $C_\alpha(0,\infty)$ being defined as in Definition 2.

Remark 8. The space $C^m_\alpha(0,\infty)$ does not coincide either with the space $C^{(m)}_\alpha = \{y : y(t) = t^p \tilde{y}(t),\ t > 0,\ p > \alpha,\ \tilde{y} \in C^m[0,\infty)\}$ considered in [17] or with the space $\Omega^m_\mu(C_\alpha(0,\infty))$ from Definition 4.

Evidently, $C_\alpha^m(0,\infty)$ is a vector space and $C_\alpha^0(0,\infty) \equiv C_\alpha(0,\infty)$. For further properties of this space of functions, we refer to [19] and [20]. In particular, the following results were proved in [20].

Theorem 10. *Let $y \in C_{-1}^m(0,\infty)$, $m \in \mathbb{N}_0$. Then the Caputo fractional derivative $({}^{\mathrm{C}}D^\mu y)(x)$, $0 \le \mu \le m$ is well-defined and the inclusion*

$$({}^{\mathrm{C}}D^\mu y)(x) \in \begin{cases} C_{-1}(0,\infty), & m-1 < \mu \le m, \\ C^{k-1}[0,\infty), & m-k-1 < \mu \le m-k,\ k = 1,\dots,m-1 \end{cases}$$

holds true.

Theorem 11. *Let $y \in C_{-1}^m(0,\infty)$, $m \in \mathbb{N}$ and $m-1 < \mu \le m$. Then the Riemann–Liouville fractional derivative D^μ and the Caputo fractional derivative ${}^{\mathrm{C}}D^\mu$ are connected by the relation*

$$(D^\mu y)(x) = ({}^{\mathrm{C}}D^\mu y)(x) + \sum_{k=0}^{m-1} \frac{y^{(k)}(0+)}{\Gamma(1+k-\mu)} x^{k-\mu}, \quad x > 0. \tag{49}$$

Theorem 12. *Let $m-1 < \mu \le m$, $m \in \mathbb{N}$, $\alpha \ge -1$ and $y \in C_\alpha^m(0,\infty)$. Then*

$$(I^\mu\, {}^{\mathrm{C}}D^\mu y)(x) = y(x) - \sum_{k=0}^{m-1} y^{(k)}(0+) \frac{x^k}{k!}, \quad x \ge 0, \tag{50}$$

I^μ being the Riemann–Liouville fractional integral.

For the sake of simplicity, in the further discussions we deal with the space of functions $C_\alpha^m(0,\infty)$ only in the case $\alpha = -1$ that is the most relevant one for applications of the operational method to the fractional differential equations with the Caputo derivatives.

Because the convolutions for an operational calculus are defined for the integral operators, not for the differential ones, the one-parametric family of convolutions introduced in Theorem 4 for the case $n = 1$, $a_1 = 1$, $\alpha_1 = 0$ (the Riemann–Liouville fractional integral) will be used for construction of the operational calculus for the Caputo fractional derivative, also. Moreover, we restrict ourselves to the case $\lambda = 1$ in the formula (16) that corresponds to the conventional Laplace convolution (18) (see Remark 3). Then we repeat the argumentation presented in the previous section and get the following result.

Theorem 13 ([20]). *The space $C_{-1}(0,\infty)$ with the operations of the Laplace convolution $*$ and ordinary addition becomes a commutative ring $(C_{-1}, *, +)$ without divisors of zero.*

This ring can be extended to the field $\mathcal{M}_{-1}$ of convolution quotients by following the procedure that was described in the previous section for the multiple Erdélyi–Kober fractional derivative in general and for the Riemann–Liouville fractional derivative in particular.

The only difference compared to the case of the operational calculus for the Riemann–Liouville fractional derivative is in representation of the Caputo fractional derivative in the field $\mathcal{M}_{-1}$ of convolution quotients that is given in the following theorem (cf. to (29)).

Theorem 14 ([20]). *Let* $f \in C_{-1}^{m}(0,\infty)$, $m-1 < \mu \le m$, $m \in \mathbb{N}$. *Then the operational relation*

$$({}^{\mathrm{C}}D^{\mu}y)(x) = S_{\mu}\cdot y - S_{\mu}\cdot y_{\mu},\ y_{\mu}(x) = \sum_{k=0}^{m-1} y^{(k)}(0+)\frac{x^k}{k!} \tag{51}$$

holds true in the field $\mathcal{M}_{-1}$ *of convolution quotients with*

$$h_{\mu}(x) = \frac{x^{\mu-1}}{\Gamma(\mu)} \quad \textit{and} \quad S_{\mu} = \frac{I}{h_{\mu}} = \frac{h_{\mu}}{h_{\mu} * h_{\mu}}. \tag{52}$$

We already know (see (22)) that for $\mu > 0$, $p \in \mathbb{N}$,

$$h_{\mu}^{p}(x) = \underbrace{h_{\mu} * \cdots * h_{\mu}}_{p} = h_{p\mu}(x).$$

Let us extend this relation to an arbitrary positive real power exponent:

$$h_{\mu}^{\lambda}(x) = h_{\lambda\mu}(x), \quad \lambda > 0. \tag{53}$$

For any $\lambda > 0$, we then have the inclusion $h_{\mu}^{\lambda} \in C_{-1}(0,\infty)$ and the formulas ($\alpha > 0$, $\beta > 0$):

$$h_{\mu}^{\alpha} * h_{\mu}^{\beta} = h_{\alpha\mu} * h_{\beta\mu} = h_{(\alpha+\beta)\mu} = h_{\mu}^{\alpha+\beta}, \tag{54}$$

$$h_{\mu_1}^{\alpha} = h_{\mu_2}^{\beta} \Leftrightarrow \mu_1\alpha = \mu_2\beta. \tag{55}$$

Then a power function of the element S_{μ} given by (52) with an arbitrary real power exponent λ can be defined:

$$S_{\mu}^{\lambda} = \begin{cases} h_{\mu}^{-\lambda}, & \lambda < 0, \\ I, & \lambda = 0, \\ \frac{I}{h_{\mu}^{\lambda}}, & \lambda > 0. \end{cases} \tag{56}$$

Using this definition and the relations (54) and (55), we get the operational relations ($\alpha, \beta \in \mathbb{R}$):

$$S_{\mu}^{\alpha}\cdot S_{\mu}^{\beta} = S_{\mu}^{\alpha+\beta}, \tag{57}$$

$$S_{\mu_1}^{\alpha} = S_{\mu_2}^{\beta} \Leftrightarrow \mu_1\alpha = \mu_2\beta. \tag{58}$$

Remark 9. The operational formulas of type (57) and (58) with the suitably defined element S_μ are also valid for the operational calculus for the multiple Erdélyi–Kober fractional derivative that we constructed in the previous section. The operational relations we present in the rest of this section can be written down (with the evident adjustments) for the operational calculi for the multiple Erdélyi–Kober fractional derivative and the Riemann–Liouville fractional derivative, also.

The next theorem is a generalization of Theorem 9 that was formulated for the operational calculus for the multiple Erdélyi–Kober fractional derivative.

Theorem 15 ([20]). *Let the multiple power series*

$$\sum_{i_1,\dots,i_n=0}^{\infty} a_{i_1,\dots,i_n} z_1^{i_1} \times \cdots \times z_n^{i_n}, \quad z_1,\dots,z_n \in \mathbb{C},\ a_{i_1,\dots,i_n} \in \mathbb{C}$$

be convergent at a point $z_0 = (z_{10},\dots,z_{n0})$ with all $z_{k0} \neq 0$, $k = 1,\dots,n$ and let the inequalities $\beta > 0$, $\alpha_i > 0$, $i = 1,\dots,n$ be satisfied. Then the function of S_μ given by

$$S_\mu^{-\beta} \sum_{i_1,\dots,i_n=0}^{\infty} a_{i_1,\dots,i_n} (S_\mu^{-\alpha_1})^{i_1} \times \cdots \times (S_\mu^{-\alpha_n})^{i_n} = \sum_{i_1,\dots,i_n=0}^{\infty} a_{i_1,\dots,i_n} h_{\hat{\mu}}(x), \tag{59}$$

with $\hat{\mu} = (\beta + \alpha_1 i_1 + \cdots + \alpha_n i_n)\mu$ and $h_\mu(x)$ being defined by (52) *can be interpreted as an element of the ring $C_{-1}(0,\infty)$.*

This theorem allows us to deduce both the operational relations of the type we mentioned in the previous section and the more general ones.

For $\rho \in \mathbb{C}$, we get the operational relation

$$\frac{I}{S_\mu - \rho} = x^{\mu-1} E_{\mu,\mu}(\rho x^\mu), \tag{60}$$

where $E_{\alpha,\beta}(z)$ is the generalized Mittag-Leffler function defined by (34) (cf. to (33) with $\lambda = 1$).

The m-fold convolution of the right-hand side of the relation (60) leads to the next operational relation

$$\frac{I}{(S_\mu - \rho)^m} = x^{\mu m-1} E_{\mu,m\mu}^{m}(\rho x^\mu), \quad m \in \mathbb{N}, \tag{61}$$

where

$$E_{\alpha,\beta}^{m}(z) = \sum_{k=0}^{\infty} \frac{(m)_k z^k}{k!\Gamma(\alpha k + \beta)}, \quad \alpha > 0,\ z \in \mathbb{C},\ (m)_k = \prod_{i=0}^{k-1}(m+i). \tag{62}$$

The next operational relation is very important for solving the fractional differential equations with the Caputo fractional derivatives of arbitrary orders. Let the inequalities $\beta > 0$, $\alpha_i > 0$, $i = 1,\dots,n$ hold true. We then have the operational relation

$$\frac{S_\mu^{-\beta}}{I - \sum_{i=1}^{n} \lambda_i S_\mu^{-\alpha_i}} = x^{\beta\mu-1} E_{(\alpha_1\mu,\dots,\alpha_n\mu),\beta\mu}(\lambda_1 x^{\alpha_1\mu},\dots,\lambda_n x^{\alpha_n\mu}) \tag{63}$$

with the multivariate Mittag-Leffler function

$$E_{(a_1,\dots,a_n),b}(z_1,\dots,z_n) = \sum_{k=0}^{\infty} \sum_{\substack{l_1+\dots+l_n=k\\ l_1\geq 0,\dots,l_n\geq 0}} (k;l_1,\dots,l_n)\frac{\prod_{i=1}^{n} z_i^{l_i}}{\Gamma(b+\sum_{i=1}^{n} a_i l_i)}$$

and the multinomial coefficients

$$(k;l_1,\dots,l_n) = \frac{k!}{l_1!\times\dots\times l_n!}.$$

4 Fractional differential equations with the multiple Erdélyi–Kober fractional derivatives

The Mikusiński operational calculus was successfully employed among other things for solving initial value problems for linear ordinary differential equations with constant and polynomial coefficients. In this section, we apply the Mikusiński schema to derive solutions of some classes of fractional differential equations with the multiple Erdélyi–Kober fractional derivatives defined by (8).

Let $P(z) = \sum_{i=0}^{m} c_i z^i$ be a polynomial of mth degree with real or complex coefficients. In this section, we deal with the following initial value problem:

$$\begin{cases} (P(D_\mu)y)(x) = f(x), \\ (FD_\mu^k y)(x) = y_k(x), \quad k = 0,1,\dots,m-1,\ y_k(x) \in \ker D_\mu, \end{cases} \tag{64}$$

where D_μ is the multiple Erdélyi–Kober fractional derivative (8), $F = E - L_\mu D_\mu$ is the projector (13) of the multiple Erdélyi–Kober fractional integral L_μ, the function $f = f(x)$ belongs to the space $C_\alpha(0,\infty)$, and the unknown function $y = y(x)$ is looked for in the space $\Omega_\mu^m(C_\alpha(0,\infty))$.

Remark 10. An explicit representation of $\ker D_\mu$ easily follows from Definition 3 of the multiple Erdélyi–Kober fractional derivative:

$$\ker D_\mu = \left\{ \sum_{i=1}^{n} \sum_{k=1}^{\eta_i} d_{ik} x^{\mu - \frac{k-\alpha_i}{a_i}} \right\}, \quad d_{ik} \in \mathbb{C},\ \eta_i = \begin{cases} [a_i\mu]+1, & a_i\mu \notin \mathbb{N}, \\ a_i\mu, & a_i\mu \in \mathbb{N}. \end{cases} \tag{65}$$

Employing the operational relation (28), we rewrite the problem (64) as an algebraic equation in the field $\mathcal{M}_{\alpha,\lambda}$ of convolution quotients as follows:

$$P(S_\mu)\cdot y = f + \sum_{k=0}^{m-1}\left(\sum_{j=1}^{m-k} c_{k+j} S_\mu^j\right)\cdot y_k. \tag{66}$$

This algebraic (in fact, linear in y) equation evidently has a unique solution in the field $\mathcal{M}_{\alpha,\lambda}$:

$$y = \frac{I}{P(S_\mu)} \cdot f + \sum_{k=0}^{m-1} \frac{P_k(S_\mu)}{P(S_\mu)} \cdot y_k, \quad P_k(S_\mu) = \sum_{j=1}^{m-k} c_{k+j} S_\mu^j. \tag{67}$$

However, we are looking for a solution of the problem (64) in the space $\Omega_\mu^m(C_\alpha(0,\infty))$. This solution is obtained by employing the operational relations (31), (32) and is given in the following theorem.

Theorem 16 ([18]). *Under the conditions* (15) *and* (19), *the unique solution of the initial value problem* (64) *in the space* $\Omega_\mu^m(C_\alpha(0,\infty))$ *is represented in the form*

$$\begin{aligned} y(x) = &\sum_{j=1}^{l}\sum_{r=1}^{m_j} c_{jr}(f \overset{\lambda}{*} x^{\mu r-\lambda} E_r((1-\alpha_i + a_i(\mu r - \lambda), a_i\mu)_n; z_j x^\mu))(x) \\ &+ y_0(x) + \sum_{k=0}^{m-1}\sum_{j=1}^{l}\sum_{r=1}^{m_j} A_{jrk}(y_k \overset{\lambda}{*} x^{\mu r-\lambda} E_r((1-\alpha_i + a_i(\mu r - \lambda), a_i\mu)_n; z_j x^\mu))(x), \end{aligned} \tag{68}$$

where the constants z_j, c_{jr}, A_{jrk} *are determined by the representations of the rational functions* $\frac{1}{P(z)}$ *and* $\frac{P_k(z)}{P(z)}$ *as the sums of partial fractions*

$$\frac{1}{P(z)} = \frac{1}{c_m(z-z_1)^{m_1} \times \cdots \times (z-z_l)^{m_l}} = \sum_{j=1}^{l}\sum_{r=1}^{m_j} c_{jr}\frac{1}{(z-z_j)^r}; \tag{69}$$

$$\frac{P_k(z)}{P(z)} = \sum_{j=1}^{l}\sum_{r=1}^{m_j} A_{jrk}\frac{1}{(z-z_j)^r}, \quad k = 1,..,m-1; \quad \frac{P_0(z)}{P(z)} = 1 + \sum_{j=1}^{l}\sum_{r=1}^{m_j} A_{jr0}\frac{1}{(z-z_j)^r}$$

and the functions $E_r((1-\alpha_i + a_i(\mu r - \lambda), a_i\mu)_n; z_j x^\mu)$ *are defined by* (32).

In general, the solution (68) has a complicated form. However, in some cases it can be simplified that leads to nice solution representations. In the next sections, we consider the fractional differential equations with the Riemann–Liouville and the Caputo fractional derivatives. For applications of the operational method to the ordinary hyper-Bessel differential equations, we refer to [19, 30].

5 Fractional differential equations with the Riemann–Liouville fractional derivatives

In this section, we specialize the general results obtained in the previous section for the case of the Riemann–Liouville fractional derivative defined by (10). To make the presentation simpler, the results will be formulated for the case of the largest of the spaces $C_\alpha(0,\infty)$, where the Riemann–Liouville fractional integral is defined, namely, for the case $\alpha = -1$. The initial value problem (64) takes then the following form:

$$\begin{cases} (P(D^\mu)y)(x) = f(x), \\ (F\prod_{j=1}^{k} D^\mu y)(x) = y_k(x), \quad k = 0, 1, \ldots, m-1, \ y_k \in \ker D^\mu, \end{cases} \tag{70}$$

where $F = E - I^\mu D^\mu$ is the projector (14) of the Riemann–Liouville fractional integral I^μ, the function $f = f(x)$ belongs to the space $C_{-1}(0, \infty)$, and the unknown function $y = y(x)$ is looked for in the space $\Omega_\mu^m(C_{-1}(0, \infty))$.

Employing the representation (65) of $\ker D^\mu$ and the formula (14) for the projector F of the Riemann–Liouville fractional integral, the initial value problem (70) can be reformulated in the usual form:

$$\begin{cases} (P(D^\mu)y)(x) = f(x), \\ \lim\limits_{x\to 0+} (D^{\mu-i}\prod_{j=1}^{k} D^\mu y)(x) = d_{ki}, \quad k = 0, \ldots, m-1, \ i = 1, \ldots, \eta, \ d_{ki} \in \mathbb{C} \end{cases} \tag{71}$$

with

$$\eta = \begin{cases} [\mu] + 1, & \mu \notin \mathbb{N}, \\ \mu, & \mu \in \mathbb{N}. \end{cases} \tag{72}$$

The unique solution of the problem (70) (or the problem (71)) is given by Theorem 16 that we rewrite for the case of the Riemann–Liouville fractional derivative.

Theorem 17 ([18]). *Let the conditions* $1 \le \lambda < \mu + 1$ *be satisfied. The unique solution of the initial value problem* (71) *for the fractional differential equation with the Riemann–Liouville fractional derivatives belongs to the function space* $\Omega_\mu^m(C_{-1}(0, \infty))$ *and is given by the formula*

$$\begin{aligned} y(x) = {} & \sum_{j=1}^{l}\sum_{r=1}^{m_j} c_{jr}(f(t) \overset{\lambda}{*} t^{\mu r-\lambda} E^r_{1+\mu r-\lambda,\mu}(z_j t^\mu))(x) + \sum_{i=1}^{\eta} \frac{d_{0i}}{\Gamma(\mu - i + 1)} x^{\mu-i} \\ & + \sum_{i=1}^{\eta}\sum_{k=0}^{m-1}\sum_{j=1}^{l}\sum_{r=1}^{m_j} A_{jrk} \frac{d_{ki}}{\Gamma(\mu - i + 1)} x^{\mu r+\mu-i} E^r_{1+\mu r+\mu-i,\mu}(z_j x^\mu), \end{aligned} \tag{73}$$

where the constants z_j, c_{jr}, A_{jrk} *are determined by the representations of the rational functions* $\frac{1}{P(z)}$ *and* $\frac{P_k(z)}{P(z)}$ *as the sums of partial fractions* (69) *and the functions* $E^r_{1+\mu r-\lambda,\mu}(z_j t^\mu)$ *are defined by* (62).

Let us now consider some examples and start with the basic initial value problem for the Riemann–Liouville fractional derivative:

$$\begin{cases} (D^\mu y)(x) - \alpha y(x) = f(x), \\ \lim\limits_{x\to 0+} (D^{\mu-i} y)(x) = d_i, \quad d_i \in \mathbb{C}, \ i = 1, 2, \ldots, \eta \end{cases} \tag{74}$$

with η being defined by (72).

Following the general procedure we described in the previous section, the initial value problem (74) is reduced to an algebraic equation in the field $\mathcal{M}_{-1,\lambda}$ of convolution quotients:

$$S_\mu \cdot y - \alpha y = f + S_\mu \cdot y_0,\ y_0(x) = \sum_{i=1}^{\eta} \frac{x^{\mu-i}}{\Gamma(\mu - i + 1)} d_i. \tag{75}$$

The unique solution of the equation (75) in the field $\mathcal{M}_{-1,\lambda}$ has the form

$$y = \frac{I}{S_\mu - \alpha} \cdot f + \frac{S_\mu}{S_\mu - \alpha} \cdot y_0. \tag{76}$$

For the sake of simplicity, we set now $\lambda = 1$, and thus the solution (76) is a generalized function from the field $\mathcal{M}_{-1,1}$. To interpret the solution as a conventional function, we first set $\lambda = 1$ in the operational relation (33):

$$\frac{I}{S_\mu - \alpha} = x^{\mu-1} E_{\mu,\mu}(\alpha x^\mu) \tag{77}$$

and then employ it to calculate the second term of the right-hand side of (76):

$$\begin{aligned}
\frac{S_\mu}{S_\mu - \alpha} \cdot y_0 &= y_0 + \frac{\alpha}{S_\mu - \alpha} \cdot y_0 = y_0(x) + \alpha(x^{\mu-1} E_{\mu,\mu}(\alpha x^\mu) \overset{1}{*} y_0)(x) \\
&= \sum_{i=1}^{\eta} \frac{x^{\mu-i}}{\Gamma(\mu - i + 1)} d_i + \alpha \sum_{i=1}^{\eta} d_i x^{2\mu-i} E^1_{1+2\mu-i,\mu}(\alpha x^\mu) \\
&= \sum_{i=1}^{\eta} \left(\frac{x^{\mu-i}}{\Gamma(\mu - i + 1)} d_i + \alpha d_i x^{2\mu-i} \sum_{k=0}^{\infty} \frac{(1)_k (\alpha x^\mu)^k}{k!\Gamma(\mu k - i + 1 + 2\mu)} \right) \\
&= \sum_{i=1}^{\eta} d_i x^{\mu-i} \sum_{k=0}^{\infty} \frac{(\alpha x^\mu)^k}{\Gamma(\mu k - i + 1 + \mu)} = \sum_{i=1}^{\eta} d_i x^{\mu-i} E_{\mu+1-i,\mu}(\alpha x^\mu).
\end{aligned} \tag{78}$$

Using the representation of the convolution (16) for the case of the Riemann–Liouville fractional derivative and $\lambda = 1$, the solution (76) of the initial value problem (74) can be now represented in the known form:

$$y(x) = \int_0^x (x - t)^{\mu-1} E_{\mu,\mu}(\alpha (x - t)^\mu) f(t)\, dt + \sum_{i=1}^{\eta} d_i x^{\mu-i} E_{\mu+1-i,\mu}(\alpha x^\mu). \tag{79}$$

The next example is the initial value problem for the following fractional differential equation of type (71):

$$\begin{cases} (D^{\frac{3}{2}} y)(x) - \alpha y'(x) + \beta^2 (D^{\frac{1}{2}} y)(x) - \alpha\beta^2 y(x) = f(x), \\ \lim\limits_{x\to 0+} (I^{\frac{1}{2}} y)(x) = 0, \quad \lim\limits_{x\to 0+} y(x) = 0, \quad \lim\limits_{x\to 0+} (D^{\frac{1}{2}} y)(x) = 0. \end{cases} \tag{80}$$

The initial value problem (80) is reduced to an algebraic equation in the field $\mathcal{M}_{-1,1}$ of convolution quotients for the Riemann–Liouville fractional derivative D^{μ} with $\mu = \frac{1}{2}$ of the form

$$S_{\mu}^{3} \cdot y - \alpha S_{\mu}^{2} \cdot y + \beta^{2} S_{\mu} \cdot y - \alpha\beta^{2} y = f.$$

Evidently, this linear equation has a unique solution in the field $\mathcal{M}_{-1,1}$ given by the formula

$$y = \frac{f}{S_{\mu}^{3} - \alpha S_{\mu}^{2} + \beta^{2} S_{\mu} - \alpha\beta^{2}} = \frac{1}{\alpha^{2}+\beta^{2}}\left(-\frac{S_{\mu}}{S_{\mu}^{2}+\beta^{2}} - \frac{\alpha}{S_{\mu}^{2}+\beta^{2}} + \frac{I}{S_{\mu}-\alpha}\right)\cdot f. \tag{81}$$

The operational relations (33), (38), and (42) with $\lambda = 1$, $\mu = 1/2$ as well as the formulas (36), (41), and (44) lead to the operational relation

$$\begin{aligned} &\frac{1}{\alpha^{2}+\beta^{2}}\left(-\frac{S_{\mu}}{S_{\mu}^{2}+\beta^{2}} - \frac{\alpha}{S_{\mu}^{2}+\beta^{2}} + \frac{I}{S_{\mu}-\alpha}\right) \\ &\quad = \frac{\sqrt{x}}{\alpha^{2}+\beta^{2}}\left(\alpha e^{\alpha^{2}x}(1+\operatorname{erf}(\alpha\sqrt{x})) - \alpha e^{-x} + \frac{2\beta}{\sqrt{\pi}}F(\beta\sqrt{x})\right) = K(x), \end{aligned} \tag{82}$$

where $\operatorname{erf}(z)$ is the error function (37) and $F(z)$ is the Doson function (45). The embedding of the space $C_{-1}(0,\infty)$ into the field $\mathcal{M}_{-1,1}$ along with the formulas (81), (82) leads finally to the solution of the initial value problem (80) in the form

$$y(x) = \int_{0}^{x} K(x-t) f(t)\, dt, \tag{83}$$

where the function $K(x)$ is given by the formula (82).

6 Fractional differential equations with the Caputo fractional derivatives

In this section, the operational calculus for the Caputo fractional derivative will be applied for solving the initial value problems for the linear fractional differential equations with the Caputo derivatives. The schema of the method is the same as the one we employed in the previous section. Still, in this section an essentially larger class of fractional differential equations is treated compared to ones from the previous sections.

Again, we start with the initial value problem for the basic equation ($\mu > 0$):

$$\begin{cases} ({}^{\mathrm{C}}D^{\mu}y)(x) - \lambda y(x) = g(x), & \lambda \in \mathbb{R}, \\ y^{(k)}(0) = c_k \in \mathbb{R}, & k = 0,\ldots,m-1,\ m-1 < \mu \le m. \end{cases} \tag{84}$$

The function g from the initial condition is assumed to belong to $C_{-1}(0,\infty)$ if $\mu \in \mathbb{N}$ and to $C^1_{-1}(0,\infty)$ if $\mu \notin \mathbb{N}$, and the unknown function y is to be determined in the space $C^m_{-1}(0,\infty)$.

Making use of the operational relation (51), the initial value problem (84) can be reduced to the following algebraic equation in the field $\mathcal{M}_{-1}$ of the convolution quotients:

$$S_\mu \cdot y - \lambda y = S_\mu \cdot y_\mu + g, \quad y_\mu(x) = \sum_{k=0}^{m-1} c_k \frac{x^k}{k!}, \quad m-1 < \mu \le m. \tag{85}$$

The unique solution of this linear equation in the field $\mathcal{M}_{-1}$ has then the form:

$$y = y_g + y_h = \frac{I}{S_\mu - \lambda} \cdot g + \frac{S_\mu}{S_\mu - \lambda} \cdot y_\mu. \tag{86}$$

As before, the right-hand side of the last relation can be interpreted as a conventional function from the space $C^m_{-1}(0,\infty)$.

The operational relation (60) along with the embedding of the space $C_{-1}(0,\infty)$ into the field $\mathcal{M}_{-1}$ provides us with a suitable representation of y_g (solution of the inhomogeneous fractional differential equation (84) with zero initial conditions):

$$y_g(x) = \int_0^x t^{\mu-1} E_{\mu,\mu}(\lambda t^\mu) g(x-t)\, dt. \tag{87}$$

The second term, y_h, of the solution (86) (solution of the homogeneous fractional differential equation (84) with the given initial conditions) has the form

$$y_h(x) = \sum_{k=0}^{m-1} c_k u_k(x), \quad u_k(x) = \frac{S_\mu}{S_\mu - \lambda} \cdot \left\{ \frac{x^k}{k!} \right\}. \tag{88}$$

The relation

$$\frac{x^k}{k!} = h_{k+1}(x) = h^\mu_{(k+1)/\mu}(x) = \frac{I}{S_\mu^{(k+1)/\mu}}, \tag{89}$$

the formula (57), and the operational relation (63) lead to the following representation of the functions $u_k(x)$, $k = 0, \dots, m-1$ in terms of the generalized Mittag-Leffler function:

$$u_k(x) = \frac{S_\mu}{S_\mu - \lambda} \cdot \left\{ \frac{x^k}{k!} \right\} = \frac{S_\mu^{-(k+1)/\mu}}{I - \lambda S_\mu^{-1}} = x^k E_{\mu,k+1}(\lambda t^\mu). \tag{90}$$

Furthermore, due to representation (27) of the Riemann–Liouville fractional integral, we have a formula for u_k in terms of u_0:

$$u_k(x) = (I^k u_0)(x), \quad u_0(x) = E_{\mu,1}(\lambda t^\mu) = E_\mu(\lambda t^\mu),$$

where $E_\alpha(z)$ is the Mittag-Leffler function. Because of the last formula, we can calculate u_k and its derivatives at the point $x = 0$:

$$u_k^{(l)}(0) = \delta_{kl}, \quad k, l = 0, \dots, m-1.$$

Therefore, the m functions $u_k(x)$, $k = 0, \dots, m-1$ build the fundamental set of solutions of the homogeneous fractional differential equation (84). Summarizing the obtained results, the unique solution of the initial value problem (84) is represented in the form:

$$y(x) = \int_0^x t^{\mu-1} E_{\mu,\mu}(\lambda t^\mu) g(x-t)\, dt + \sum_{k=0}^{m-1} c_k x^k E_{\mu,k+1}(\lambda x^\mu).$$

In the case $\lambda \neq 0$, the solution can be rewritten in terms of the Mittag-Leffler function E_μ:

$$y(x) = \frac{1}{\lambda} \int_0^x \frac{d}{dt}(E_\mu(\lambda t^\mu)) g(x-t)\, dt + \sum_{k=0}^{m-1} c_k (I^k E_\mu(\lambda t^\mu))(x).$$

The next example is an initial value problem for the generalized Basset equation

$$\begin{cases} y'(x) - \lambda_1 ({}^{\mathrm{C}}D^\mu y)(x) - \lambda_2 y(x) = g(x), \\ y(0) = c_0 \in \mathbb{R}, \quad 0 < \mu < 1,\ \lambda_1, \lambda_2 \in \mathbb{R}. \end{cases} \tag{91}$$

In the case $\mu = 1/2$ and $\lambda_1 < 0, \lambda_2 < 0$, the equation from the first line of (91) is called the Basset equation, a classical equation in fluid dynamics [13, 24]. We treat the general problem (91) with a function $g \in C_{-1}(0, \infty)$ and look for the unknown function y in the space $C^1_{-1}(0, \infty)$.

Once again, we employ the relation (51) and reduce the initial value problem (91) to the linear equation in the field $\mathcal{M}_{-1}$:

$$S_1 \cdot y - \lambda_1 S_\mu \cdot y - \lambda_2 y = g + S_1 \cdot y_1 - \lambda_1 S_\mu \cdot y_\mu,\ y_1(x) = y_\mu(x) = c_0. \tag{92}$$

Applying the operational relation (58), we represent a unique solution of this equation in the field $\mathcal{M}_{-1}$ in the form:

$$y = y_g + y_h = \frac{I}{S_1 - \lambda_1 S_1^\mu - \lambda_2} \cdot g + \frac{S_1 - \lambda_1 S_1^\mu}{S_1 - \lambda_1 S_1^\mu - \lambda_2} \cdot y_1, \quad y_1(x) = c_0. \tag{93}$$

Using now the operational relations (57) and (63), we arrive at the representation

$$\frac{I}{S_1 - \lambda_1 S_1^\mu - \lambda_2} = \frac{S_1^{-1}}{I - \lambda_1 S_1^{-(1-\mu)} - \lambda_2 S_1^{-1}} = E_{(1-\mu,1),1}(\lambda_1 x^{1-\mu}, \lambda_2 x) \tag{94}$$

of the first factor of y_g in terms of the multivariate Mittag-Leffler function. The term y_h of the solution (93) is obtained using the same technique as before and the operational relations (25) and (89):

$$y_h(t) = \frac{S_1 - \lambda_1 S_1^{\mu}}{S_1 - \lambda_1 S_1^{\mu} - \lambda_2} \cdot \{c_0\} = \left(I + \frac{\lambda_2}{S_1 - \lambda_1 S_1^{\mu} - \lambda_2}\right) \cdot \frac{c_0 I}{S_1}$$
$$= c_0\left(\frac{I}{S_1} + \lambda_2 \frac{S_1^{-2}}{I - \lambda_1 S_1^{\mu-1} - \lambda_2 S_1^{-1}}\right) = c_0(1 + \lambda_2 x E_{(1-\mu,1),2}(\lambda_1 x^{1-\mu}, \lambda_2 x)). \tag{95}$$

The unique solution of the initial value problem (91) has then the form

$$y(x) = \int_0^x E_{(1-\mu,1),1}(\lambda_1 t^{1-\mu}, \lambda_2 t) g(x-t)\,dt \;+\; c_0(1 + \lambda_2 x E_{(1-\mu,1),2}(\lambda_1 x^{1-\mu}, \lambda_2 x)).$$

Now we consider the general n-term linear fractional differential equation with constant coefficients and the Caputo fractional derivatives. Its solution is obtained by the same method that was employed for the previous examples and is given in the following theorem (see [19] or [20] for a proof).

Theorem 18. *Let* $0 \le \mu_n < \cdots < \mu_1 < \mu$, $m-1 < \mu \le m$, $m \in \mathbb{N}$, $m_i - 1 < \mu_i \le m_i$, $m_i \in \mathbb{N}_0$, $\lambda_i \in \mathbb{R}$, $i = 1, \dots, n$ *and the function g belong to the function space* $C_{-1}(0,\infty)$ *if* $\mu \in \mathbb{N}$ *and to* $C_{-1}^1(0,\infty)$ *if* $\mu \notin \mathbb{N}$. *The initial value problem for the n-term linear fractional differential equation*

$$\begin{cases} ({}^{C}D^{\mu} y)(x) - \sum_{i=1}^{n} \lambda_i ({}^{C}D^{\mu_i} y)(x) = g(x), \\ y^{(k)}(0) = c_k \in \mathbb{R}, \quad k = 0, \dots, m-1,\ m-1 < \mu \le m,\ m \in \mathbb{N}, \end{cases} \tag{96}$$

has a solution, unique in the space $C_{-1}^m(0,\infty)$, *in the form:*

$$y(x) = y_g(x) + \sum_{k=0}^{m-1} c_k u_k(x), \quad x \ge 0, \tag{97}$$

where

$$y_g(x) = \int_0^x t^{\mu-1} E_{(\cdot),\mu}(t) g(x-t)\,dt \tag{98}$$

is the solution of the initial value problem (96) *with zero initial conditions, and the fundamental set of solutions*

$$u_k(x) = \frac{x^k}{k!} + \sum_{i=l_k+1}^{n} \lambda_i x^{k+\mu-\mu_i} E_{(\cdot),k+1+\mu-\mu_i}(x), \quad k = 0, \dots, m-1 \tag{99}$$

fulfills the initial conditions $u_k^{(l)}(0) = \delta_{kl}$, $k, l = 0, \ldots, m-1$. *The function*

$$E_{(\cdot),\beta}(x) = E_{(\mu-\mu_1,\ldots,\mu-\mu_n),\beta}(\lambda_1 x^{\mu-\mu_1}, \ldots, \lambda_n x^{\mu-\mu_n}) \tag{100}$$

is a particular case of the multivariate Mittag-Leffler function (63) *and the natural numbers* l_k, $k = 0, \ldots, m-1$ *are determined from the conditions*

$$\begin{cases} m_{l_k} \geq k+1, \\ m_{l_k+1} \leq k. \end{cases} \tag{101}$$

In the case $m_i \leq k$, $i = 0, \ldots, m-1$, *we set* $l_k = 0$ *and if* $m_i \geq k+1$, $i = 0, \ldots, m-1$ *then we set* $l_k = n$.

The solution (97)–(99) of the initial value problem (96) is represented in terms of the Mittag-Leffler-type function $E_{(\cdot),\beta}(x)$ that is defined as a convergent series in the form

$$E_{(\cdot),\beta}(x) = \sum_{k=0}^{\infty} \sum_{\substack{l_1+\cdots+l_n=k \\ l_1\geq 0,\ldots,l_n\geq 0}} (k; l_1, \ldots, l_n) \frac{\prod_{i=1}^n (\lambda_i x^{\mu-\mu_i})^{l_i}}{\Gamma(\beta + \sum_{i=1}^n (\mu-\mu_i) l_i)}. \tag{102}$$

It is worth mentioning that the function $E_{(\cdot),\beta}(x)$ has an integral representation [19, 20]:

$$E_{(\cdot),\beta}(x) = \frac{x^{1-\beta}}{2\pi i} \int\limits_{Ha(\lambda+)} \frac{e^{sx} s^{\mu-\beta}\, ds}{s^{\mu} - \sum_{i=1}^n \lambda_i s^{\mu_i}}, \quad \lambda = \max\left\{1, \left(\sum_{i=1}^n |\lambda_i|\right)^{\frac{1}{\mu-\mu_1}}\right\}, \tag{103}$$

where $Ha(\epsilon+)$ is the Hankel path, a loop that starts in $-\infty$ and goes along the lower side of the negative real axis, then encircles the circular disc $|\zeta| = \zeta_0 > \epsilon$ in the positive sense and ends at $-\infty$ along the upper side of the negative real axis.

Remark 11. Applying the integral representation (103) to the formulas (98) and (99), we can rewrite the solution (97) of the initial value problem (96) as follows:

$$y(x) = \int\limits_0^x u_\delta(t) g(x-t) + \sum_{k=0}^{m-1} c_k u_k(x), \quad x \geq 0, \tag{104}$$

where

$$u_\delta(x) = \frac{1}{2\pi i} \int\limits_{Ha(\lambda+)} \frac{e^{sx}\, ds}{s^{\mu} - \sum_{i=1}^n \lambda_i s^{\mu_i}}, \tag{105}$$

$$\begin{aligned} u_k(x) &= \frac{x^k}{k!} + \frac{1}{2\pi i} \int\limits_{Ha(\lambda+)} \frac{e^{sx} \sum_{i=l_k+1}^n \lambda_i s^{\mu_i}}{s^{\mu} - \sum_{i=1}^n \lambda_i s^{\mu_i}} \frac{ds}{s^{k+1}} \\ &= \frac{1}{2\pi i} \int\limits_{Ha(\lambda+)} \frac{e^{sx}(s^{\mu} - \sum_{i=1}^{l_k} \lambda_i s^{\mu_i})}{s^{\mu} - \sum_{i=1}^n \lambda_i s^{\mu_i}} \frac{ds}{s^{k+1}}, \quad k = 0, \ldots, m-1. \end{aligned} \tag{106}$$

In particular, in the case $\mu_n = 0$, $\lambda_n \neq 0$, we get $l_0 = n-1$, and thus arrive at the representation

$$u_0(x) = \frac{1}{2\pi i} \int\limits_{Ha(\lambda+)} \frac{e^{sx}(s^\mu - \sum_{i=1}^{n-1} \lambda_i s^{\mu_i})}{s^\mu - \sum_{i=1}^{n-1} \lambda_i s^{\mu_i} - \lambda_n} \frac{ds}{s}.$$

Moreover, the relation

$$u_\delta(x) = \frac{1}{\lambda_n} u_0'(x)$$

holds true in the case $\mu_n = 0$, $\lambda_n \neq 0$. If $\mu_n > 0$, then $u_0(x) \equiv 1$.

Remark 12. The initial value problem (96) with the parameters (1) $n = 1$, $\mu_1 = 0$, $\lambda_1 = -1$, (2) $n = 2$, $\mu = 1$, $\lambda_2 = -1$, $\mu_2 = 0$, and (3) $n = 2$, $\mu = 2$, $\lambda_2 = -1$, $\mu_2 = 0$ was treated in [13] by using the Laplace transform method. For these cases, the form (104)–(106) of the solution was obtained and used for representing the solution as a sum of oscillatory and monotone parts. In addition, asymptotic expansions, plots and interesting particular cases of the solution were presented in [13], also.

In the rest of this section, some important particular cases of the initial value problem (96) are discussed.

Let us start with the case when the function g is a power function

$$g(x) = \frac{x^\alpha}{\Gamma(\alpha+1)}, \quad \alpha > -1 \text{ if } \mu \in \mathbb{N},\ \alpha \geq 0 \text{ if } \mu \notin \mathbb{N}.$$

Employing the operational relation

$$\frac{x^\alpha}{\Gamma(\alpha+1)} = h_{\alpha+1}(x) = h^\mu_{(\alpha+1)/\mu}(x) = S_\mu^{-(\alpha+1)/\mu}$$

and the formulas (57), (63), and (103), we get the following representation of the part y_g of the solution (97):

$$\begin{aligned} y_g(x) &= \frac{I}{S_\mu - \sum_{i=1}^n \lambda_i S_\mu^{\mu_i/\mu}} \cdot g = \frac{I}{S_\mu - \sum_{i=1}^n \lambda_i S_\mu^{\mu_i/\mu}} \cdot \frac{I}{S_\mu^{(\alpha+1)/\mu}} \\ &= \frac{S_\mu^{-(\mu+\alpha+1)/\mu}}{I - \sum_{i=1}^n \lambda_i S_\mu^{-(\mu-\mu_i)/\mu}} = x^{\mu+\alpha} E_{(\cdot),\mu+\alpha+1}(x) = \frac{1}{2\pi i} \int\limits_{Ha(\lambda+)} \frac{e^{sx} s^{-\alpha-1}\, ds}{s^\mu - \sum_{i=1}^n \lambda_i s^{\mu_i}}. \end{aligned}$$

The next particular case is the initial value problem (96) for the n-term fractional differential equation, where the quotients of all derivatives orders are some rational numbers. Without loss of generality, we set $\mu_i = (n-i)\alpha$, $i = 1, \ldots, n$, $\mu = n\alpha$, $q - 1 < \mu \leq q$, $q \in \mathbb{N}$. Then the solution (97) can be represented in terms of the generalized Mittag-Leffler function $E^m_{\alpha,\beta}(x)$ defined by (61). Indeed, using the operational relation

(58) and representing the rational functions in S_μ as sums of partial fractions, we get the following representation of y_g in the field of convolution quotients:

$$y_g = \frac{I}{S_\mu - \sum_{i=1}^n \lambda_i S_\mu^{\mu_i/\mu}} \cdot g = \frac{I}{S_\alpha^n - \sum_{i=1}^n \lambda_i S_\alpha^{n-i}} \cdot g$$
$$= \left(\sum_{j=1}^{p} \sum_{m=1}^{n_j} \frac{c_{jm}}{(S_\alpha - \beta_j)^m} \right) \cdot g, \quad n_1 + \cdots + n_p = n.$$

The embedding of the space $C_{-1}(0,\infty)$ into the field of convolution quotients and the operational relation (61) lead then to the representation

$$y_g(x) = \int_0^x u_\delta(t) g(x-t)\, dt,\ u_\delta(x) = \sum_{j=1}^{p} \sum_{m=1}^{n_j} c_{jm} x^{\alpha m-1} E^m_{\alpha,m\alpha}(\beta_j x^\alpha).$$

For the solution part y_h, we get $(k = 0, \ldots, q-1)$

$$y_h(x) = \sum_{k=0}^{q-1} c_k u_k(x),\ u_k(x) = \frac{x^k}{k!} + \left\{ \frac{x^k}{k!} \right\} \cdot \frac{\sum_{i=l_k+1}^n \lambda_i S_\alpha^{n-i}}{S_\alpha^n - \sum_{i=1}^n \lambda_i S_\alpha^{n-i}} = \frac{x^k}{k!} + (I^{k+1} v_k)(x)$$

with

$$v_k(x) = \frac{\sum_{i=l_k+1}^n \lambda_i S_\alpha^{n-i}}{S_\alpha^n - \sum_{i=1}^n \lambda_i S_\alpha^{n-i}} = \sum_{j=1}^{p_k} \sum_{m=1}^{n_{jk}} \frac{c_{jmk}}{(S_\alpha - \beta_{jk})^m}$$
$$= \sum_{j=1}^{p_k} \sum_{m=1}^{n_{jk}} c_{jmk} x^{\alpha m-1} E^m_{\alpha,m\alpha}(\beta_{jk} x^\alpha), \quad \sum_{j=1}^{p_k} n_{jk} = n.$$

Finally, we mention that the generalized Mittag-Leffler function $E^m_{\alpha,m\alpha}(x)$ with $\alpha \in \mathbb{Q}$ is reduced to a linear combination of the generalized hypergeometric functions [13, 19, 30], and thus the solution of the problem (96) for the n-term fractional differential equation with rational derivatives orders can be represented in terms of the conventional special functions.

Bibliography

[1] M. A. Al-Bassam and Yu. F. Luchko, On generalized fractional calculus and its application to the solution of integro-differential equations, *Journal of Fractional Calculus*, **7** (1995), 69–88.

[2] T. Atanackovic, D. Dolicanin, S. Pilipovic, and B. Stankovic, Cauchy problems for some classes of linear fractional differential equations, *Fract. Calc. Appl. Anal.*, **17** (2014), 1039–1059.

[3] M. Caputo, Linear models of dissipation whose Q is almost frequency independent, part II, *Geophys. J. R. Astr. Soc.*, **13** (1967), 529–539.

[4] M. Caputo and F. Mainardi, Linear models of dissipation in inelastic solids, *Riv. Nuovo Cimento (Ser II)*, **1** (1971), 161–198.

[5] K. Diethelm, *The Analysis of Fractional Differential Equations*, Springer, Berlin, 2010.

[6] I. H. Dimovski, On an operational calculus for a class of differential operators, *C. R. Acad. Bulg. Sci.*, **19** (1966), 1111–1114.

[7] I. H. Dimovski, On a Bessel-type integral transformation due to N. Obrechkoff, *C. R. Acad. Bulg. Sci.*, **27** (1974), 23–26.

[8] I. H. Dimovski, A transform approach to operational calculus for the general Bessel-type differential operator, *C. R. Acad. Bulg. Sci.*, **27** (1974), 155–158.

[9] I. H. Dimovski, *Convolutional Calculus*, Bulgarian Academy of Sciences, Sofia, 1982, 2nd ed. by Kluwer Acad. Publ., Dordrecht, 1990.

[10] I. H. Dimovski and V. Kiryakova, The Obrechkoff integral transform: Properties and relation to a generalized fractional calculus, *Numer. Funct. Anal. and Optimization*, **21** (2000), 121–144.

[11] V. A. Ditkin and A. P. Prudnikov, The theory of operational calculus, which is generated by the Bessel equation, *J. Vichisl. Mat. i Mat. Fiz.*, **3** (1963), 223–238 (in Russian).

[12] R. Gorenflo and Yu. Luchko, Operational method for solving generalized Abel integral equations of second kind, *Integr. Transf. Spec. Funct.*, **5** (1997), 47–58.

[13] R. Gorenflo and F. Mainardi, Fractional calculus: integral and differential equations of fractional order, in A. Carpinteri and F. Mainardi (eds.), *Fractals and Fractional Calculus in Continuum Mechanics*, pp. 223–276, Springer Verlag, New York, 1997.

[14] S. B. Hadid and Yu. F. Luchko, An operational method for solving fractional differential equations of an arbitrary real order, *Panamer. Math. J.*, **6** (1996), 57–73.

[15] L. A.-M. Hanna and Yu. Luchko, Operational calculus for the Caputo-type fractional Erdélyi–Kober derivative and its applications, *Integr. Transf. Spec. Funct.*, **25** (2014), 359–373.

[16] R. Hilfer, Yu. Luchko, and Z. Tomovski, Operational method for the solution of fractional differential equations with generalized Riemann–Liouville fractional derivatives, *Fract. Calc. Appl. Anal.*, **12** (2009), 299–318.

[17] V. Kiryakova, *Generalized Fractional Calculus and Applications*, Longman, Harlow & J. Wiley, New York, 1994.

[18] Yu. Luchko, *Theory of the integral transformations with the Fox H-function as a kernel and some of its applications including operational calculus*, PhD. thesis, Byelorussian State University, Minsk, 1993.

[19] Yu. Luchko, Operational method in fractional calculus, *Fract. Calc. Appl. Anal.*, **2** (1999), 463–488.

[20] Yu. F. Luchko and R. Gorenflo, An operational method for solving fractional differential equations, *Acta Mathematica Vietnamica*, **24** (1999), 207–234.

[21] Yu. F. Luchko and H. M. Srivastava, The exact solution of certain differential equations of fractional order by using operational calculus, *Comput. Math. Appl.*, **29** (1995), 73–85.

[22] Yu. F. Luchko and S. B. Yakubovich, Convolution of the generalized fractional differentiation operators and applications, *Mathematica Balkanica*, **7** (1993), 119–130.

[23] Yu. F. Luchko and S. B. Yakubovich, An operational method for solving some classes of integro-differential equations, *Differential Equations*, **30** (1994), 247–256.

[24] F. Mainardi, Fractional calculus: some basic problems in continuum and statistical mechanics, in A. Carpinteri and F. Mainardi (eds.), *Fractals and Fractional Calculus in Continuum Mechanics*, pp. 291–348, Springer Verlag, New York, 1997.

[25] N. A. Meller, On an operational calculus for the operator $B_\alpha = t^{-\alpha} D t^{\alpha+1} D$, *J. Vychisl. Mat. i Mat. Fiz.*, **6** (1960), 161–180 (in Russian).

[26] J. Mikusiński, *Operational Calculus*, Pergamon Press, Oxford, 1959.

[27] F. W. J. Olver, *Introduction to asymptotic and special functions*, Academic Press, London, 1974.

[28] I. Podlubny, *Fractional Differential Equations*, Academic Press, New York, 1999.

[29] J. Rodrigues, Operational calculus for the generalized Bessel operator, *Serdica Bulg. Math. Publ.*, **15** (1989), 179–186.
[30] S. B. Yakubovich and Yu. F. Luchko, *The Hypergeometric Approach to Integral Transforms and Convolutions*, Kluwer Acad. Publ., Dordrecht, 1994.

Mohamed Jleli, Mokhtar Kirane, and Bessem Samet

Lyapunov-type inequalities for fractional boundary value problems

Abstract: Lyapunov's inequality is one of the outstanding results in mathematics. It has proved to be useful for studying spectral properties of ordinary differential equations, including bounds for eigenvalues, stability criteria for periodic differential equations, estimates for intervals of disconjugacy, etc. In this chapter, we present a review of results concerning Lyapunov-type inequalities for fractional boundary value problems. Moreover, we show the applicability of such inequalities in fractional eigenvalue problems. We discuss the one-dimensional case as well as the multidimensional case.

Keywords: Lyapunov-type inequalities, fractional differential equations, fractional partial differential equations, eigenvalues

MSC 2010: 26A33, 34K38, 35A23

1 Introduction

Let us consider the Hill equation

$$y''(t) + q(t)y(t) = 0, \quad t \in \mathbb{R}, \tag{1}$$

where $q : \mathbb{R} \to \mathbb{R}$ is a given function. In his classical work [35], Lyapunov proved the following result.

Theorem 1.1. *Let q be continuous, of period ω, and satisfies*

$$q \not\equiv 0, \quad q(t) \geq 0, \quad t \in \mathbb{R}$$

and

$$\omega \int_0^\omega q(s)\,ds \leq 4.$$

Then the roots of the characteristic equation corresponding to (1) *are purely imaginary with modulus unity.*

Mohamed Jleli, Bessem Samet, Department of Mathematics, College of Science, King Saud University, P. O. Box 2455, Riyadh 11451, Saudi Arabia, e-mails: jleli@ksu.edu.sa, bsamet@ksu.edu.sa
Mokhtar Kirane, LaSIE, Faculté des Sciences, Pole Sciences et Technologies, Université de La Rochelle, Avenue M. Crepeau, 17042, La Rochelle Cedex, France, e-mail: mkirane@univ-lr.fr

https://doi.org/10.1515/9783110571660-006

By Floquet theory, Theorem 1.1 provides a sufficient condition for the stability of (1). Recall that (1) is said to be stable, if all solutions to (1) are bounded on $\mathbb{R}$.

In [2], Borg established the following stability result for (1).

Theorem 1.2. *Let q be continuous, of period π, and satisfies*

$$q \not\equiv 0, \qquad \int_0^\pi q(s)\,ds \geq 0$$

and

$$\pi \int_0^\pi q(s)\,ds \leq 4.$$

Then (1) *is stable.*

The above result completes the Lyapunov criterion for stability of (1) by replacing the Lyapunov condition

$$q \not\equiv 0, \quad q(t) \geq 0, \quad t \in \mathbb{R}$$

by

$$q \not\equiv 0, \qquad \int_0^\pi q(s)\,ds \geq 0.$$

In his proof of Theorem 1.2, Borg [2] introduced the Lyapunov inequality that he attributes to Beurling. This inequality is embodied in the following result.

Theorem 1.3 (Lyapunov inequality). *Let $q : [a,b] \to \mathbb{R}$, $(a,b) \in \mathbb{R}^2$, $a < b$, be a continuous function. If y is a nontrivial solution to*

$$\begin{aligned} y''(t) + q(t)y(t) &= 0, \quad a < t < b, \\ y(a) = y(b) &= 0, \end{aligned} \tag{2}$$

then

$$\int_a^b |q(s)|\,ds > \frac{4}{b-a}. \tag{3}$$

Observe that in Theorem 1.3, the function q is allowed to change sign, and the inequality (3) involves $|q(s)|$.

In [45], Wintner improved the inequality (3) by considering only the positive part of q. More precisely, he proved that under the assumptions of Theorem 1.3, we have

$$\int_a^b q^+(s)\,ds > \frac{4}{b-a}, \tag{4}$$

where

$$q^+(t) = \max\{0, q(t)\}, \quad a \le t \le b.$$

Hartman and Wintner [20] obtained that under the assumptions of Theorem 1.3, we have

$$\int_a^b (b-s)(s-a)q^+(s)\,ds > b-a. \tag{5}$$

Using that

$$\max_{a\le s\le b}(b-s)(s-a) = \frac{(b-a)^2}{4},$$

it can be easily seen that (4) follows from (5).

Several other generalizations and extensions of Theorem 1.3 have been proposed in the literature. In [10], Das and Vatsala studied the higher order differential equation

$$\begin{aligned} (-1)^n y^{(2n)}(t) &= q(t)y, \quad a < t < b, \\ y^{(k)}(a) = y^{(k)}(b) &= 0, \quad 0 \le k \le n-1. \end{aligned} \tag{6}$$

By computing the Green's function of the considered problem and using Mercer's theorem (see, e. g., [9]), they proved that if (6) admits a nontrivial solution, then

$$\int_a^b [(s-a)(b-s)]^{2n-1} q^+(s)\,ds \ge [(n-1)!]^2(2n-1)(b-a)^{2n-1}.$$

Note that the technique of the Green's function was first used by Nehari [37] in his proof of the inequality (3). Pinasco [40] studied the p-Laplacian problem

$$\begin{aligned} (|y'|^{p-2}y')' + q(t)|y|^{p-2}y &= 0, \quad a < t < b, \\ y(a) = y(b) &= 0, \end{aligned} \tag{7}$$

where $q \in L^1(a,b)$ is a positive function and $1 < p < \infty$. Observe that for $p = 2$, (7) reduces to (2). He proved that if (7) admits a nontrivial solution, then

$$\int_a^b q(s)\,ds \ge \frac{2^p}{(b-a)^{p-1}}.$$

For other results concerning Lyapunov-type inequalities for differential equations, we refer the reader to [3, 11, 12, 16, 33, 39, 41, 46] and the references therein.

Beside the one-dimensional case, similar inequalities for partial differential equations were obtained by some authors. Cañada et al. [4, 5] studied the boundary value problem

$$\Delta u + q(x)u = 0, \quad x \in \Omega,$$
$$\frac{\partial u}{\partial n} = 0, \quad x \in \partial\Omega,$$

where $\Omega \subset \mathbb{R}^N$ ($N \geq 2$) is a bounded and regular domain, and the function $q : \Omega \to \mathbb{R}$ satisfies

$$q \in L^{\frac{N}{2}}(\Omega)\backslash\{0\}, \qquad \int\limits_{\Omega} q(x)\,dx \geq 0.$$

In [13], Nápoli and Pinasco studied the partial differential equation

$$\begin{aligned} \Delta_p u + w(x)|u|^{p-2}u = 0, \quad x \in \Omega,\\ u = 0, \quad x \in \partial\Omega, \end{aligned} \tag{8}$$

where Ω is a bounded open domain in $\mathbb{R}^N$, $N \geq 2$, $1 < p < \infty$, Δ_p is the p-Laplacian operator, and $w \in L^{\theta}(\Omega)$, for some θ depending on p and N. They obtained the following results.

Theorem 1.4. *Let $\Omega \subset \mathbb{R}^N$ be an open set, let $w \in L^1(\Omega)$ be a nonnegative weight, and let $u \in W_0^{1,p}(\Omega)$, $p > N$ be a nontrivial solution to* (8). *Then*

$$\|w\|_{L^1(\Omega)} \geq \frac{C}{r_\Omega^{p-N}}, \tag{9}$$

where C is an universal constant depending only on p and N, and r_Ω is the inner radius of Ω, given by $r_\Omega = \max\{d(x, \partial\Omega) : x \in \Omega\}$.

Theorem 1.5. *Let $\Omega \subset \mathbb{R}^N$ be a smooth domain, $N < p\theta$, and $w \in L^{\theta}(\Omega)$. Let $u \in W_0^{1,p}(\Omega)$ be a nontrivial solution to* (8). *Then*

$$\|w\|_{L^{\theta}(\Omega)} \geq \frac{C}{r_\Omega^{\frac{\theta p - N}{\theta}}}, \tag{10}$$

where C is a constant depending on p, N, and the capacity of $\mathbb{R}^N\backslash\Omega$ (see [21]).

On the other hand, due to the importance of fractional calculus in modeling various problems from many fields of science and engineering (see, e. g., the monograph [32]), recently, a great interest has been devoted to the study of Lyapunov-type inequalities for fractional differential equations. The first contribution in this direction is due to Ferreira [17], where he considered the fractional boundary value problem

$$\begin{aligned} (D_a^{\alpha} y)(t) + q(t)y(t) = 0, \quad a < t < b,\\ y(a) = y(b) = 0, \end{aligned} \tag{11}$$

where $1 < \alpha < 2$, D_a^α is the Riemann–Liouville fractional derivative of order α, and $q : [a,b] \to \mathbb{R}$ is a continuous function. Note that in the limit case $\alpha = 2$, (11) reduces to (2). Using the method of Green's function, Ferreira obtained the following result.

Theorem 1.6. *If* (11) *admits a nontrivial solution, then*

$$\int_a^b |q(s)|\,ds > \Gamma(\alpha)\left(\frac{4}{b-a}\right)^{\alpha-1}, \tag{12}$$

where Γ *is the Gamma function.*

Observe that in the limit case $\alpha = 2$, (12) reduces to the standard Lyapunov inequality (3).

In [18], Ferreira studied the fractional boundary value problem

$$\begin{aligned} ({}^C D_a^\alpha y)(t) + q(t)y(t) = 0, &\quad a < t < b, \\ y(a) = y(b) = 0, & \end{aligned} \tag{13}$$

where $1 < \alpha < 2$, ${}^C D_a^\alpha$ is the Caputo fractional derivative of order α, and $q : [a,b] \to \mathbb{R}$ is a continuous function. The main result in [18] is the following.

Theorem 1.7. *If* (13) *admits a nontrivial solution, then*

$$\int_a^b |q(s)|\,ds > \frac{\Gamma(\alpha)\alpha^\alpha}{[(\alpha-1)(b-a)]^{\alpha-1}}.$$

Following the contributions [17, 18], several Lyapunov-type inequalities were derived for different classes of fractional boundary value problems. In this direction, we refer the reader to [8, 14, 26, 27, 30, 31, 36, 38, 43], and the references therein.

Unfortunately, there are only few works devoted to the study of similar inequalities for fractional partial differential equations; see [28, 29].

In this chapter, we present a review of results concerning Lyapunov-type inequalities for fractional boundary value problems in the one-dimensional case as well as the multidimensional case. Moreover, we show the applicability of such inequalities to fractional eigenvalue problems. The chapter is organized as follows. In Section 2, we recall some concepts on fractional calculus and some useful results and properties related to the fractional derivative operator with respect to another function. In Section 3, we study a fractional boundary value problem involving a ψ-fractional derivative. A Lyapunov-type inequality is established for this problem using the Green's function method. We discuss also some particular cases related to different choices of the function ψ. Next, we present some applications to fractional eigenvalue problems. Our considerations in Section 3 are based on our results obtained in [27]. In Section 4, we consider a fractional partial differential equation involving the fractional

p-Laplacian operator $(-\Delta_p)^s$, $1 < p < \infty$, $0 < s < 1$, and posed in an open bounded subset Ω in $\mathbb{R}^N$, $N \geq 2$. We discuss the case $sp > N$ and the case $sp < N$. For each case, a Lyapunov-type inequality is derived. Next, the obtained inequalities are used to obtain lower bounds for the smallest eigenvalue of the corresponding problem. Let us mention that the results presented in Section 4 are closely patterned on our paper [28]. In Section 5, we extend the obtained results in Section 4 to a system of fractional partial differential equations. Next, we obtain some geometric properties of the generalized spectrum associated to the considered system. Section 5 can be treated as an overview of our results obtained in [29].

2 Some preliminaries on fractional calculus

Let $I = [a,b]$, where $(a,b) \in \mathbb{R}^2$ and $a < b$. We denote by $\mathrm{AC}(I)$ the set of real valued and absolutely continuous functions on I. For $n = 1,2,\dots$, we denote by $\mathrm{AC}^n(I)$ the set of real valued functions $f(t)$ having continuous derivatives up to order $n-1$ on I such that $D^{n-1}f \in \mathrm{AC}(I)$, where $D = \frac{d}{dt}$.

The following concepts can be found in [32].

Definition 2.1. Let $f \in L^1(I)$ and $\alpha > 0$. The Riemann–Liouville fractional integral of order α of f is defined by

$$(I_a^\alpha f)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s)\, ds, \quad \text{a.e. } t \in I.$$

For $\alpha > 0$, we denote by $[\alpha]+1$ the smallest integer greater than or equal to α.

Definition 2.2. Let $\alpha > 0$ and $f : I \to \mathbb{R}$ be a function such that $I_a^{n-\alpha} f \in \mathrm{AC}^n(I)$, where $n = [\alpha]+1$. Then the Riemann–Liouville fractional derivative of order α of f is defined by

$$(D_a^\alpha f)(t) = D^n I_a^{n-\alpha} f(t), \quad \text{a.e. } t \in I.$$

Let $\alpha > 0$ and $n = [\alpha]+1$. We denote by $\mathrm{AC}^\alpha(I)$ (see [24]) the set of all functions $f : I \to \mathbb{R}$ having the representation:

$$f(t) = \sum_{i=0}^{n-1} \frac{c_i}{\Gamma(\alpha-n+1+i)} (t-a)^{\alpha-n+i} + I_a^\alpha \varphi(t), \quad \text{a.e. } t \in I, \tag{14}$$

where $c_i \in \mathbb{R}$, $i = 0,1,\dots,n-1$, are some constants, and $\varphi \in L^1(I)$.

The following theorem (see [24]) provides a necessary and sufficient condition for the existence of $D_a^\alpha f$, where $f \in L^1(I)$.

Theorem 2.3. *Let $\alpha > 0$, $n = [\alpha] + 1$, and $f \in L^1(I)$. Then $D_a^\alpha f(t)$ exists almost everywhere on I if and only if $f \in \mathrm{AC}^\alpha(I)$; that is, f has representation* (14). *In such a case, one has*

$$(D_a^\alpha f)(t) = \varphi(t), \quad \text{a.e. } t \in I.$$

Further, let $\psi : I \to \mathbb{R}$ be a continuously differentiable function such that

$$\psi'(t) > 0, \quad t \in I.$$

Definition 2.4. Let $f \in L^1(I)$ and $\alpha > 0$. The ψ-fractional integral of order α of f is defined by

$$(I_a^{\alpha,\psi} f)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (\psi(t) - \psi(s))^{\alpha-1} \psi'(s) f(s)\, ds, \quad \text{a.e. } t \in I.$$

Definition 2.5. Let $\alpha > 0$ and $n = [\alpha] + 1$. Let $f : I \to \mathbb{R}$ be a function such that $(\frac{1}{\psi'(t)} \frac{d}{dt})^n I_a^{n-\alpha,\psi} f$ exists almost everywhere on I. In this case, the ψ-fractional derivative of order α of f is given by

$$(D_a^{\alpha,\psi} f)(t) = \left(\frac{1}{\psi'(t)} \frac{d}{dt} \right)^n I_a^{n-\alpha,\psi} f(t), \quad \text{a.e. } t \in I.$$

It can be easily seen that in the case $\psi(t) = t$, the above definition reduces to the Riemann–Liouville fractional derivative of order α. In the case $\psi : [a,b] \to \mathbb{R}$, $0 < a < b$, is given by

$$\psi(t) = \ln t, \quad a \le t \le b,$$

we have $D_a^{\alpha,\psi} = {}^H D_a^\alpha$, where ${}^H D_a^\alpha$ is the Hadamard fractional derivative of order α.

The following result gives the relation between ψ-fractional derivative and Riemann–Liouville fractional derivative.

First, for $\alpha > 0$, let us introduce the functional space

$$E_\psi(I) := \{f : I \to \mathbb{R} : f \circ \psi^{-1} \in \mathrm{AC}^\alpha(J)\},$$

where $J = \psi(I)$.

Theorem 2.6 (see [27]). *Let $\alpha > 0$, $n = [\alpha] + 1$ and $f \in E_\psi(I)$. Then*

$$D_a^{\alpha,\psi} f(\psi^{-1}(y)) = D_{\psi(a)}^\alpha (f \circ \psi^{-1})(y), \quad \text{a.e. } y \in J.$$

3 A Lyapunov-type inequality for a fractional differential equation involving ψ-fractional derivative

In this section, we are concerned with the fractional boundary value problem

$$\begin{aligned} (D_a^{\alpha,\psi} u)(t) + q(t)u(t) = 0, \quad a < t < b, \\ u(a) = u(b) = 0, \end{aligned} \tag{15}$$

where $1 < \alpha < 2$, $(a, b) \in \mathbb{R}^2$, $a < b$, $q : [a, b] \to \mathbb{R}$ is a continuous function, $D_a^{\alpha,\psi}$ is the ψ-fractional derivative of order α, and $\psi : [a, b] \to \mathbb{R}$ is a continuously differentiable function such that

$$\psi'(t) > 0, \quad t \in [a, b].$$

Using the Green's function method, we derive a Lyapunov-type inequality for (15). Next, the obtained inequality is used to obtain a lower bound of the eigenvalues associated to (15).

The following lemmas will be used later. For the proofs, we refer the reader to [17].

Lemma 3.1. *Let $h : [A, B] \to \mathbb{R}$, $(A, B) \in \mathbb{R}^2$, $A < B$, be a continuous function. Then $w \in \mathrm{AC}^\alpha([A, B]) \cap C([A, B])$, where $1 < \alpha < 2$, is a solution to the fractional boundary value problem*

$$\begin{aligned} (D_A^\alpha w)(t) + h(t) = 0, \quad A < t < B, \\ w(A) = w(B) = 0, \end{aligned}$$

if and only if w satisfies the integral equation

$$w(t) = \int_A^B G(t, s)h(s)\, ds, \quad A \le t \le B,$$

where $G(t, s)$ is the Green's function given by

$$\Gamma(\alpha)G(t, s) = \begin{cases} \frac{(t-A)^{\alpha-1}}{(B-A)^{\alpha-1}}(B - s)^{\alpha-1} - (t - s)^{\alpha-1}, & A \le s \le t \le B, \\ \frac{(t-A)^{\alpha-1}}{(B-A)^{\alpha-1}}(B - s)^{\alpha-1}, & A \le t \le s \le B. \end{cases} \tag{16}$$

Lemma 3.2. *The Green's function G defined by* (16) *satisfies the following properties:*

(i) $G(t, s) \ge 0$, *for all* $(t, s) \in [A, B] \times [A, B]$.

(ii) *For all* $s \in [A, B]$, *we have*

$$\max_{A \le t \le B} G(t, s) = G(s, s) = \frac{1}{\Gamma(\alpha)(B - A)^{\alpha-1}}(s - A)^{\alpha-1}(B - s)^{\alpha-1}.$$

Now, we are able to state and prove the main result of this section.

Theorem 3.3. *Let $u \in C([a,b]) \cap E_\psi([a,b])$ be a nontrivial solution to* (15). *Then*

$$\int_a^b \left[(\psi(s)-\psi(a))(\psi(b)-\psi(s))\right]^{\alpha-1}\psi'(s)\left|q(s)\right| ds \geq \Gamma(\alpha)\left(\psi(b)-\psi(a)\right)^{\alpha-1}. \tag{17}$$

Proof. Let us introduce the function $v : [\psi(a),\psi(b)] \to \mathbb{R}$ defined by

$$v(y) = u(\psi^{-1}(y)), \quad \psi(a) \leq y \leq \psi(b).$$

Using Theorem 2.6, we obtain that v is a nontrivial solution to the Riemann–Liouville fractional boundary value problem

$$\begin{aligned} (D_A^\alpha v)(y) + Q(y)v(y) &= 0, \quad A < y < B, \\ v(A) = v(B) &= 0, \end{aligned}$$

where $A = \psi(a)$, $B = \psi(b)$, and $Q : [A,B] \to \mathbb{R}$ is the function defined by

$$Q(y) = q(\psi^{-1}(y)), \quad A \leq y \leq B.$$

By Lemma 3.1, we have

$$v(y) = \int_A^B G(y,s)Q(s)v(s)\, ds, \quad A \leq y \leq B,$$

where G is the Green's function defined by (16). Next, let us consider the Banach space $C([A,B])$ equipped with the standard norm

$$\|f\|_\infty := \max\{|f(y)| : A \leq y \leq B\}, \quad f \in C([A,B]).$$

By Lemma 3.2, we have

$$|v(y)| \leq \|v\|_\infty \int_A^B G(s,s)|Q(s)|\, ds, \quad A \leq y \leq B,$$

which yields

$$\int_A^B G(s,s)|Q(s)|\, ds \geq 1.$$

Finally, using the change of variable $s = \psi(t)$, the desired inequality (17) follows. □

Next, using Theorem 3.3, some Lyapunov-type inequalities are deduced for different choices of the function ψ.

3.1 The case $\psi(x) = x^\beta$, $\beta > 0$

Let us consider the case when the function $\psi : [a, b] \to \mathbb{R}$ is given by

$$\psi(x) = x^\beta, \quad a \le x \le b, \tag{18}$$

where $\beta > 0$ and $(a, b) \in \mathbb{R}^2$, $a < b$. In the case $\beta \ne 1$, it is supposed that $a > 0$. By Theorem 3.3, we deduce the following result.

Corollary 3.4. *Let $u \in C([a, b]) \cap E_\psi([a, b])$ be a nontrivial solution to* (15)*, where ψ is given by* (18)*. Then*

$$\int_a^b [(s^\beta - a^\beta)(b^\beta - s^\beta)]^{\alpha-1} s^{\beta-1} |q(s)|\, ds \ge \frac{\Gamma(\alpha)(b^\beta - a^\beta)^{\alpha-1}}{\beta}. \tag{19}$$

Next, let us define the function $\mu_{\alpha,\beta} : [a, b] \to \mathbb{R}$ by

$$\mu_{\alpha,\beta}(s) = [(s^\beta - a^\beta)(b^\beta - s^\beta)]^{\alpha-1} s^{\beta-1}, \quad a \le s \le b.$$

It can be easily seen that the function $\mu_{\alpha,\beta}$ is continuous on $[a, b]$. Moreover, we have $\mu_{\alpha,\beta}(a) = \mu_{\alpha,\beta}(b) = 0$. Therefore, there exists $s^*(\alpha, \beta) \in]a, b[$ such that

$$\mu_{\alpha,\beta}(s^*(\alpha, \beta)) = \max\{\mu_{\alpha,\beta}(s) : a \le s \le b\} > 0.$$

Then, by (19), the following result follows.

Corollary 3.5. *Let $u \in C([a, b]) \cap E_\psi([a, b])$ be a nontrivial solution to* (15)*, where ψ is given by* (18)*. Then*

$$\int_a^b |q(s)|\, ds \ge \frac{\Gamma(\alpha)(b^\beta - a^\beta)^{\alpha-1}}{\beta \mu_{\alpha,\beta}(s^*(\alpha, \beta))}. \tag{20}$$

To compute the value of $s^*(\alpha, \beta)$ appearing in the inequality (20), for $1 < \alpha < 2$ and $\beta > 0$, we need to study the variations of the function $\mu_{\alpha,\beta}$. Two possible cases hold.

Case 1. If $(2\alpha - 1), \beta = 1$.

In this case, we obtain

$$s^*(\alpha, \beta) = \left(\frac{2MN}{M + N}\right)^{\frac{1}{\beta}} = \left(\frac{2}{a^\beta + b^\beta}\right)^{\frac{1}{\beta}} ab.$$

Case 2. If $0 < \beta < \frac{1}{2\alpha-1}$ or $(2\alpha - 1)\beta > 1$.

In this case, we obtain

$$s^*(\alpha, \beta) = \left(\frac{(1 - \alpha\beta)(a^\beta + b^\beta) - \sqrt{(a^\beta - b^\beta)^2(1 - \alpha\beta)^2 + 4a^\beta b^\beta \beta^2 (1 - \alpha)^2}}{2[\beta(1 - 2\alpha) + 1]}\right)^{\frac{1}{\beta}}.$$

Remark 3.6. Observe that if $\beta = 1$, we are in Case 2. Therefore,

$$s^*(\alpha, 1) = \frac{a+b}{2}$$

and

$$\mu_{\alpha,1}(s^*(\alpha, 1)) = \left[\frac{(b-a)^2}{4}\right]^{\alpha-1}.$$

Moreover, in this situation, we have $D_a^{\alpha,\psi} = D_a^\alpha$. Therefore, taking $\beta = 1$ in Corollary 3.5, we obtain the Lyapunov-type inequality (in large sense) given by Theorem 1.6.

3.2 The case $\psi(x) = \ln x$

Let us consider the case when the function $\psi : [a, b] \to \mathbb{R}$ is given by

$$\psi(x) = \ln x, \quad a \le x \le b, \tag{21}$$

where $(a, b) \in \mathbb{R}^2$, $0 < a < b$. In this case, (15) reduces to the Hadamard fractional boundary value problem

$$\begin{aligned} ({}^H D_a^\alpha u)(t) + q(t)u(t) &= 0, \quad a < t < b, \\ u(a) = u(b) &= 0, \end{aligned} \tag{22}$$

where ${}^H D_a^\alpha$ is the Hadamard fractional derivative of order α.

By Theorem 3.3, we deduce the following result.

Corollary 3.7. *Let $u \in C([a, b]) \cap E_\psi([a, b])$ be a nontrivial solution to* (22)*, where ψ is given by* (21)*. Then*

$$\int_a^b \left[\left(\ln\frac{s}{a}\right)\left(\ln\frac{b}{s}\right)\right]^{\alpha-1} \frac{|q(s)|}{s}\, ds \ge \Gamma(\alpha)\left(\ln\frac{b}{a}\right)^{\alpha-1}.$$

Next, by studying the variations of the function $f : [a, b] \to \mathbb{R}$ given by

$$f(s) = \left[\left(\ln\frac{s}{a}\right)\left(\ln\frac{b}{s}\right)\right]^{\alpha-1} s^{-1}, \quad a \le s \le b,$$

we deduce the following result.

Corollary 3.8. *Let $u \in C([a, b]) \cap E_\psi([a, b])$ be a nontrivial solution to* (22)*, where ψ is given by* (21)*. Then*

$$\int_a^b |q(s)|\, ds \ge \Gamma(\alpha)\left[\frac{\ln b - \ln a}{(\lambda(a, b) - \ln a)(\ln b - \lambda(a, b))}\right]^{\alpha-1} e^{\lambda(a,b)},$$

where

$$\lambda(a,b) = \frac{2(\alpha-1)+\ln a+\ln b-\sqrt{4(\alpha-1)^2+(\ln a-\ln b)^2}}{2}.$$

Further, we give some applications of the above obtained results to fractional eigenvalue problems.

3.3 Applications to eigenvalue problems

Let $1<\alpha<2$, $(a,b)\in\mathbb{R}^2$, $a<b$, and $\psi:[a,b]\to\mathbb{R}$ be a continuously differentiable function such that

$$\psi'(t)>0,\quad t\in[a,b].$$

We say that $\lambda\in\mathbb{R}$ is an eigenvalue of the fractional operator $D_a^{\alpha,\psi}$ if the fractional boundary value problem

$$(D_a^{\alpha,\psi}u)(t)+\lambda u(t)=0,\quad a<t<b,$$
$$u(a)=u(b)=0,$$

admits at least one nontrivial solution $u_\lambda\in C([a,b])\cap E_\psi([a,b])$. In this case, u_λ is said to be an eigenfunction associated to λ.

We have the following result, which provides s a lower bound of the eigenvalues of $D_a^{\alpha,\psi}$.

Theorem 3.9. *Let λ be an eigenvalue of the fractional operator $D_a^{\alpha,\psi}$. Then*

$$|\lambda|\geq\Gamma(\alpha)\big(\psi(b)-\psi(a)\big)^{\alpha-1}\left(\int_{\psi(a)}^{\psi(b)}(s-\psi(a))^{\alpha-1}(\psi(b)-s)^{\alpha-1}\,ds\right)^{-1}.$$

Proof. It follows immediately from Theorem 3.3 by taking $q\equiv\lambda$. □

Let us consider now the case $\psi(t)=t$ and $(a,b)=(0,1)$. In this case, $D_0^{\alpha,\psi}=D_0^\alpha$. Let λ be an eigenvalue of the Riemann–Liouville fractional operator D_0^α. Then the fractional boundary value problem

$$(D_0^\alpha u)(t)+\lambda u(t)=0,\quad 0<t<1,$$
$$u(0)=u(1)=0,$$

admits at least one nontrivial solution $u_\lambda\in C([0,1])\cap AC^\alpha([0,1])$.

Let us consider now the Mittag-Leffler function defined by

$$E_\alpha(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k\alpha+\alpha)},\quad z\in\mathbb{C}.$$

It is well known (see [32]) that $\lambda \in \mathbb{R}$ is an eigenvalue of D_0^α if and only if

$$E_\alpha(-\lambda) = 0.$$

Therefore, by Theorem 3.9, if λ is a real zero of E_α, then

$$|\lambda| \geq \Gamma(\alpha)\left(\int_0^1 s^{\alpha-1}(1-s)^{\alpha-1}\,ds\right)^{-1}.$$

Note that

$$\int_0^1 s^{\alpha-1}(1-s)^{\alpha-1}\,ds = B(\alpha,\alpha) = \frac{[\Gamma(\alpha)]^2}{\Gamma(2\alpha)},$$

where B is the Beta function.

Therefore, we have the following result.

Theorem 3.10. *Let* $1 < \alpha < 2$. *Then the Mittag-Leffler function* $E_\alpha(z)$ *has no real zeros for*

$$|z| < \frac{\Gamma(2\alpha)}{\Gamma(\alpha)}.$$

4 Lyapunov-type inequalities for fractional partial differential equations

In this section, we are concerned with the fractional partial differential equation

$$\begin{aligned} (-\Delta_p)^s u &= w|u|^{p-2}u \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \mathbb{R}^N \backslash \Omega, \end{aligned} \tag{23}$$

where Ω is an open bounded set in $\mathbb{R}^N$, $N \geq 2$, $1 < p < \infty$, $0 < s < 1$, $w \in L^\infty(\Omega)$, and $(-\Delta_p)^s$ is the fractional p-Laplacian operator. Two cases are discussed: the case $sp > N$ and the case $sp < N$. For each case, we obtain a Lyapunov-type inequality involving the inner radius of the domain and L^θ norms of the weight w. Next, the obtained inequalities are used to provide lower bounds for the first eigenvalue of the fractional p-Laplacian under homogeneous Dirichlet boundary conditions.

For the reader's convenience, we collect some basic results that will be used in the sequel.

Let $1 < p < \infty$ and $0 < s < 1$. The fractional p-Laplacian operator $(-\Delta_p)^s$ (up to normalization factors) can be defined as

$$(-\Delta_p)^s u(x) = 2\lim_{\varepsilon \downarrow 0} \int\limits_{\mathbb{R}^N \backslash B(x,\varepsilon)} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+sp}}\,dy, \quad x \in \mathbb{R}^N,$$

for u smooth enough. For $p = 2$, $(-\Delta_p)^s = (-\Delta)^s$ is the fractional Laplacian operator of order s.

The Gagliardo seminorm is defined for all measurable function $u : \mathbb{R}^N \to \mathbb{R}$ by

$$[u]_{s,p} = \left(\int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} \, dx \, dy \right)^{\frac{1}{p}}.$$

We define the fractional Sobolev space

$$W^{s,p}(\mathbb{R}^N) = \{u \in L^p(\mathbb{R}^N) : \ u \text{ is measurable, } [u]_{s,p} < \infty\}$$

equipped with the norm

$$\|u\|_{s,p} = \left(\|u\|_{L^p(\mathbb{R}^N)} + [u]_{s,p}\right)^{\frac{1}{p}}, \quad u \in W^{s,p}(\mathbb{R}^N).$$

Let Ω be a bounded open domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. We shall work in the closed linear subspace

$$W_0^{s,p}(\Omega) = \{u \in W^{s,p}(\mathbb{R}^N) : \ u = 0 \text{ a.e in } \mathbb{R}^N \backslash \Omega\},$$

which can be equivalently renormed by $[\cdot]_{s,p}$. For more details on fractional Sobolev spaces, we refer the reader to [1, 15, 44].

The following fractional Sobolev-type inequalities will be useful later. The proof of the following result can be found in [15].

Lemma 4.1. *Let D be an open bounded subset in $\mathbb{R}^N$, $sp < N$, $0 < s < 1$, and $1 < p < \infty$. Then there exists a constant $C_H > 0$ such that*

$$\|u\|^p_{L^{p_s^*}(\mathbb{R}^N)} \le C_H [u]^p_{s,p}, \quad u \in W_0^{s,p}(D),$$

where $p_s^* = \frac{Np}{N-sp}$.

The proof of following result can be found in [25].

Lemma 4.2. *Let $0 < s < 1$ and $1 < p < \infty$ be such that $sp < N$. Assume that $D \subset \mathbb{R}^N$ is a (bounded) uniform domain with a (locally) (s,p)-uniformly fat boundary. Then D admits an (s,p)-Hardy inequality, that is, there is a constant $C_S > 0$ such that*

$$\int_D \frac{|u(x)|^p}{d(x, \partial D)^{sp}} \, dx \le C_S [u]^p_{s,p}, \quad u \in W_0^{s,p}(D),$$

where $d(x, \partial D)$ is the distance from $x \in D$ to the boundary ∂D.

The proof of the following result can be found in [34].

Lemma 4.3. *Let D be an open bounded subset in $\mathbb{R}^N$, $sp > N$, and $0 < s < 1$. Then there is a constant $C_M > 0$ such that for all $u \in W_0^{s,p}(D)$,*

$$|u(x) - u(y)| \le C_M |x - y|^\beta [u]_{s,p}, \quad x, y \in \mathbb{R}^N,$$

where $\beta = \frac{sp-N}{p}$.

In the sequel, we take $1 < p < \infty$ and $0 < s < 1$. We suppose that Ω is an open bounded domain in $\mathbb{R}^N$ satisfying the regularities required by the fractional Sobolev inequalities given by Lemmas 4.1, 4.2, and 4.3. We denote by r_Ω the inner radius of the domain Ω, that is,

$$r_\Omega = \max\{d(x, \partial\Omega) : x \in \Omega\}.$$

Definition 4.4. A weak solution to (23) is a function $u \in W_0^{s,p}(\Omega)$ satisfying

$$\int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+sp}}\, dx\, dy = \int_\Omega w(x)|u(x)|^{p-2} u(x) v(x)\, dx,$$

for all $v \in W_0^{s,p}(\Omega)$.

4.1 The case $sp > N$

In the case $sp > N$, we have the following result.

Theorem 4.5. *Let $w \in L^1(\Omega)$ be a nonnegative weight. Suppose that* (23) *with $sp > N$ has a nontrivial weak solution $u \in W_0^{s,p}(\Omega)$. Then*

$$\int_\Omega w(x)\, dx \ge \frac{C}{r_\Omega^{sp-N}}, \tag{24}$$

where $C = C_M^{-p}$ and C_M is the universal constant given by Lemma 4.3.

Proof. The proof makes use of the fractional Morrey inequality given by Lemma 4.3. □

Remark 4.6. Note that in the limit case $s = 1$, the inequality (24) reduces to the inequality (9).

4.2 The case $sp < N$

Now, we discuss the case $sp < N$. We have the following result.

Theorem 4.7. *Let $w \in L^{\theta}(\Omega)$, $\frac{N}{sp} < \theta < \infty$, be a nonnegative weight. Suppose that* (23) *with $sp < N$ has a nontrivial weak solution $u \in W_0^{s,p}(\Omega)$. Then*

$$\left(\int_{\Omega} w^{\theta}(x)\,dx\right)^{\frac{1}{\theta}} \geq \frac{C}{r_{\Omega}^{sp-\frac{N}{\theta}}}, \tag{25}$$

where

$$C = \left(\frac{1}{C_S^s C_H^{\frac{N}{(\theta-1)s}}}\right)^{\frac{\theta-1}{\theta}} > 0$$

and C_H, C_S are the universal constants given by Lemmas 4.1 *and* 4.2.

Proof. Let

$$q = \alpha p + (1-\alpha)p_s^*,$$

where

$$\alpha = \frac{1}{\theta-1}\left(\theta - \frac{N}{sp}\right)$$

and

$$p_s^* = \frac{Np}{N-sp}.$$

It can be easily seen that $0 < \alpha < 1$ and $q = p\theta'$, where $\frac{1}{\theta} + \frac{1}{\theta'} = 1$. On the other hand, we have

$$\frac{1}{r_{\Omega}^{\alpha sp}} \int_{\Omega} |u(x)|^q\,dx \leq \int_{\Omega} \frac{|u(x)|^q}{d(x,\partial\Omega)^{\alpha sp}}\,dx. \tag{26}$$

By Hölder's inequality with exponents $\mu = \alpha^{-1}$ and $\mu' = \frac{\mu}{\mu-1}$, we get

$$\int_{\Omega} \frac{|u(x)|^q}{d(x,\partial\Omega)^{\alpha sp}}\,dx \leq \left(\int_{\Omega} \frac{|u(x)|^p}{d(x,\partial\Omega)^{sp}}\,dx\right)^{\alpha} \left(\int_{\Omega} |u(x)|^{p_s^*}\,dx\right)^{(1-\alpha)}.$$

Further, using Lemmas 4.1 and 4.2, we obtain

$$\int_{\Omega} \frac{|u(x)|^q}{d(x,\partial\Omega)^{\alpha sp}}\,dx \leq C_S^{\alpha} C_H^{\frac{(1-\alpha)p_s^*}{p}} \|w\|_{L^{\theta}(\Omega)}^{\theta'} \int_{\Omega} |u(x)|^{p\theta'}\,dx. \tag{27}$$

Next, combining (26) with (27), we obtain the desired result. □

Remark 4.8. In the limit case $s = 1$, the inequality (25) reduces to the inequality (10).

For $p = 2$, the fractional p-Laplacian $(-\Delta_p)^s$, $0 < s < 1$, coincides with the fractional Laplacian operator $(-\Delta)^s$. In this case, we have $sp = 2s < 2 \leq N$. Therefore, by Theorem 4.7, we obtain the following Lyapunov-type inequality for the fractional Laplacian operator.

Corollary 4.9. *Let $w \in L^\theta(\Omega)$, $\frac{N}{2s} < \theta < \infty$, be a nonnegative weight. Let $u \in W_0^{s,2}(\Omega)$ be a nontrivial weak solution to*

$$\begin{aligned} (-\Delta)^s u &= wu \quad \textit{in } \Omega, \\ u &= 0 \quad \textit{on } \mathbb{R}^N \backslash \Omega. \end{aligned}$$

Then there exists $C > 0$ (an universal constant) such that

$$\left(\int_\Omega w^\theta(x)\,dx \right)^{\frac{1}{\theta}} \geq \frac{C}{r_\Omega^{2s - \frac{N}{\theta}}}.$$

4.3 Applications to eigenvalue problems

We are concerned with the nonlinear eigenvalue problem

$$\begin{aligned} (-\Delta_p)^s u &= \lambda |u|^{p-2} u \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \mathbb{R}^N \backslash \Omega, \end{aligned} \tag{28}$$

depending on the parameter $\lambda \in \mathbb{R}$, where $1 < p < \infty$ and $0 < s < 1$. If (28) admits a nontrivial weak solution $u_\lambda \in W_0^{s,p}(\Omega)$, then λ is said to be an eigenvalue of (28) and u_λ is said to be a λ-eigenfunction associated to λ. Let $\sigma(s,p)$ be the set of all eigenvalues of (28). Some properties of the structure of $\sigma(s,p)$ are obtained by some authors; see, for example, [19, 23, 34], and the references therein.

From the fractional Poincaré inequality [22], we know that

$$\lambda_1 = \inf_{u \in W_0^{s,p}(\Omega), u \neq 0} \frac{[u]_{s,p}^p}{\|u\|_{L^p(\mathbb{R}^N)}^p}$$

belongs to $]0, +\infty[$.

We have the following result which provides lower bounds for λ_1.

Theorem 4.10. *Let λ_1 be the first eigenvalue of* (28).

(i) *If $sp > N$, then*

$$\lambda_1 \geq \frac{1}{C_M^p |\Omega| r_\Omega^{sp-N}}.$$

(ii) *If $sp < N$, then*

$$\lambda_1 \geq \sup_{\frac{N}{sp}<\theta<\infty} \frac{\left(\frac{1}{C_S^s C_H^{\frac{N}{(\theta-1)s}}}\right)^{\frac{\theta-1}{\theta}}}{|\Omega|^{\frac{1}{\theta}} r_\Omega^{sp-\frac{N}{\theta}}}.$$

Proof. If $sp > N$, (i) follows from Theorem 4.5 by taking $w \equiv \lambda_1$. If $sp < N$, since $w \equiv \lambda_1 \in L^\theta(\Omega)$, for every $\frac{N}{sp} < \theta < \infty$ (since Ω is bounded), using Theorem 4.7, we obtain

$$\lambda_1 \geq \frac{\left(\frac{1}{C_S^s C_H^{\frac{N}{(\theta-1)s}}}\right)^{\frac{\theta-1}{\theta}}}{|\Omega|^{\frac{1}{\theta}} r_\Omega^{sp-\frac{N}{\theta}}}, \quad \frac{N}{sp} < \theta < \infty,$$

which yields (ii). □

5 Lyapunov-type inequalities for a fractional p-Laplacian system

In this section, we are concerned with the system of fractional partial differential equations

$$\begin{cases} (-\Delta_{p_1})^{s_1} u_1(x) = q_1(x)|u_1(x)|^{\alpha_1-2}|u_2(x)|^{\alpha_2}\cdots|u_n(x)|^{\alpha_n}u_1(x), \\ (-\Delta_{p_2})^{s_2} u_2(x) = q_2(x)|u_1(x)|^{\alpha_1}|u_2(x)|^{\alpha_2-2}|u_3(x)|^{\alpha_3}\cdots|u_n(x)|^{\alpha_n}u_2(x), \\ \vdots \\ (-\Delta_{p_n})^{s_n} u_n(x) = q_n(x)|u_1(x)|^{\alpha_1}\cdots|u_{n-1}(x)|^{\alpha_{n-1}}|u_n(x)|^{\alpha_n-2}u_n(x), \end{cases} \tag{29}$$

posed in an open bounded subset Ω in $\mathbb{R}^N$ ($N \geq 2$), under Dirichlet boundary conditions

$$u_i = 0 \quad \text{on } \mathbb{R}^N \backslash \Omega, \quad i = 1, 2, \ldots, n, \tag{30}$$

where $0 < s_i < 1$, $1 < p_i < \infty$, $(-\Delta_{p_i})^{s_i}$ is the fractional p_i-Laplacian operator of order s_i, $q_i \in L^1(\Omega)$, $q_i \geq 0$, and the positive parameters α_i satisfy

$$\sum_{i=1}^{n} \frac{\alpha_i}{p_i} = 1. \tag{31}$$

A Lyapunov-type inequality is derived for (29)–(30). Next, we obtain some geometric properties of the generalized spectrum associated to (29)–(30) with $q_i(x) = \lambda_i \alpha_i \varphi(x)$, $\varphi \in L^1(\Omega)$.

We suppose that $\Omega \subset \mathbb{R}^N$ satisfies the regularities required by the fractional Sobolev inequalities given by Lemmas 4.1, 4.2, and 4.3.

Definition 5.1. We say that $(u_1, u_2, \dots, u_n) \in \prod_{i=1}^n W_0^{s_i,p_i}(\Omega)$ is a weak solution to (29)–(30) if and only if for every $(v_1, v_2, \dots, v_n) \in \prod_{i=1}^n W_0^{s_i,p_i}(\Omega)$, we have

$$\int_{\mathbb{R}^{2N}} \frac{|u_i(x) - u_i(y)|^{p_i-2}(u_i(x) - u_i(y))(v_i(x) - v_i(y))}{|x-y|^{N+s_ip_i}}\,dx\,dy$$
$$= \int_{\Omega} q_i(x)\big|u_1(x)\big|^{\alpha_1} \cdots \big|u_{i-1}(x)\big|^{\alpha_{i-1}}\big|u_i(x)\big|^{\alpha_i-2}\big|u_{i+1}(x)\big|^{\alpha_{i+1}} \cdots \big|u_n(x)\big|^{\alpha_n} u_i(x)v_i(x)\,dx,$$

for all $i = 1, 2, \dots, n$.

5.1 The case $s_ip_i > N$, $i = 1, 2, \dots, n$

In this case, the following Lyapunov-type inequality holds.

Theorem 5.2. *Suppose that $s_ip_i > N$, for all $i = 1, 2, \dots, n$. If (29)–(30) admits a nontrivial weak solution, then*

$$\prod_{i=1}^n \left(\int_{\Omega} q_i(x)\,dx\right)^{\frac{\alpha_i}{p_i}} \geq \frac{C}{r_{\Omega}^{\sum_{i=1}^n \alpha_i s_i - N}}, \tag{32}$$

where $C > 0$ is an universal constant.

Proof. The proof makes use of the fractional Morrey inequality given by Lemma 4.3. □

Remark 5.3. Observe that in the case $n = 1$, $\alpha_1 = p_1 = p$, $s_1 = s$, and $q_1 = w$, (29)–(30) reduces to (23). Moreover, in this case, the inequality (32) reduces to the inequality (24).

5.2 The case $s_ip_i < N$, $i = 1, 2, \dots, n$

In this case, we have the following result.

Theorem 5.4. *Suppose that $s_ip_i < N$, for all $i = 1, 2, \dots, n$. Moreover, suppose that $q_i \in L^{\theta}(\Omega)$, for all $i = 1, 2, \dots, n$, where*

$$\max\left\{\frac{N}{s_ip_i} : i = 1, 2, \dots, n\right\} < \theta < \infty.$$

If (29)–(30) admits a nontrivial weak solution, then

$$\prod_{i=1}^n \left(\int_{\Omega} q_i(x)^{\theta}\,dx\right)^{\frac{\alpha_i}{p_i}} \geq \frac{C}{r_{\Omega}^{\theta\sum_{j=1}^n s_j\alpha_j - N}}, \tag{33}$$

where $C > 0$ is an universal constant.

Proof. The proof is based on Lemmas 4.1 and 4.2. □

Remark 5.5. In the case $n = 1$, $\alpha_1 = p_1 = p$, $s_1 = s$, and $q_1 = w$, the inequality (33) reduces to the inequality (25).

5.3 Generalized eigenvalues

The concept of generalized eigenvalues was introduced by Protter [42] for a system of linear elliptic operators. Then this concept was extended in [6, 7] to more general elliptic systems. The first work dealing with generalized eigenvalues for p-Laplacian systems is due to Nápoli and Pinasco [12]. Inspired by the work [12], we present some applications to generalized eigenvalues related to problem (29)–(30).

Let us consider the system of fractional partial differential equations

$$\begin{cases} (-\Delta_{p_1})^{s_1} u_1(x) = \lambda_1 \alpha_1 \varphi(x) |u_1(x)|^{\alpha_1 - 2} |u_2(x)|^{\alpha_2} \cdots |u_n(x)|^{\alpha_n} u_1(x), \\ (-\Delta_{p_2})^{s_2} u_2(x) = \lambda_2 \alpha_2 \varphi(x) |u_1(x)|^{\alpha_1} |u_2(x)|^{\alpha_2 - 2} |u_3(x)|^{\alpha_3} \cdots |u_n(x)|^{\alpha_n} u_2(x), \\ \vdots \\ (-\Delta_{p_n})^{s_n} u_n(x) = \lambda_n \alpha_n \varphi(x) |u_1(x)|^{\alpha_1} \cdots |u_{n-1}(x)|^{\alpha_{n-1}} |u_n(x)|^{\alpha_n - 2} u_n(x), \end{cases} \tag{34}$$

posed in an open bounded subset Ω in $\mathbb{R}^N$ ($N \geq 2$), under the boundary conditions (30), where $\varphi \in L^1(\Omega)$, $\varphi \geq 0$, $\lambda_i > 0$, $\alpha_i > 0$, $0 < s_i < 1$, and $1 < p_i < \infty$, for all $i = 1, 2, \ldots, n$. We suppose that the condition (31) is satisfied.

We say that $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ is a generalized eigenvalue if and only if problem (34)–(30) admits at least one nontrivial weak solution $(u_1, u_2, \ldots, u_n) \in \prod_{i=1}^{n} W_0^{s_i, p_i}(\Omega)$. The set of generalized eigenvalues is called generalized spectrum, and it is denoted by σ.

Using the Lyapunov-type inequalities established previously, we obtain lower bounds of the generalized spectrum. Moreover, some geometric properties of this set are proved.

5.3.1 Lower bounds of the generalized spectrum

In the case $s_i p_i > N$, $i = 1, 2, \ldots, n$, we have the following result.

Theorem 5.6. *Suppose that $s_i p_i > N$, $i = 1, 2, \ldots, n$. Then there exists a function $F :]0, \infty[\to \mathbb{R}$ such that*

$$\lambda_n \geq F\left(\prod_{i=1}^{n-1} \lambda_i^{\frac{\alpha_i}{p_i}} \right), \quad (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \sigma, \tag{35}$$

where

$$F(t)=\frac{1}{\alpha_n}\left(\frac{1}{t}\right)^{\frac{p_n}{\alpha_n}}\left(\frac{C}{r_\Omega^{\sum_{i=1}^n \alpha_i s_i-N}\prod_{i=1}^{n-1}\alpha_i^{\frac{\alpha_i}{p_i}}\int_\Omega \varphi(x)\,dx}\right)^{\frac{p_n}{\alpha_n}},\quad t>0,$$

and $C>0$ *is an universal constant.*

Proof. Let $(\lambda_1,\lambda_2,\dots,\lambda_n)\in\sigma$. By replacing in (32) the functions

$$q_i(x)=\lambda_i\alpha_i\varphi(x),\quad x\in\Omega,$$

we obtain

$$\lambda_n^{\frac{\alpha_n}{p_n}}\alpha_n^{\frac{\alpha_n}{p_n}}\prod_{i=1}^{n-1}(\lambda_i\alpha_i)^{\frac{\alpha_i}{p_i}}\prod_{i=1}^{n}\left(\int_\Omega \varphi(x)\,dx\right)^{\frac{\alpha_i}{p_i}}\geq\frac{C}{r_\Omega^{\sum_{i=1}^n \alpha_i s_i-N}}.$$

Taking in consideration the condition (31), we obtain

$$\lambda_n^{\frac{\alpha_n}{p_n}}\alpha_n^{\frac{\alpha_n}{p_n}}\prod_{i=1}^{n-1}(\lambda_i\alpha_i)^{\frac{\alpha_i}{p_i}}\int_\Omega \varphi(x)\,dx\geq\frac{C}{r_\Omega^{\sum_{i=1}^n \alpha_i s_i-N}},$$

which yields

$$\lambda_n^{\frac{\alpha_n}{p_n}}\geq\frac{C}{\alpha_n^{\frac{\alpha_n}{p_n}}r_\Omega^{\sum_{i=1}^n \alpha_i s_i-N}\prod_{i=1}^{n-1}(\lambda_i\alpha_i)^{\frac{\alpha_i}{p_i}}\int_\Omega \varphi(x)\,dx}.$$

Therefore,

$$\begin{aligned}\lambda_n&\geq\frac{1}{\alpha_n}\left(\frac{1}{\prod_{i=1}^{n-1}\lambda_i^{\frac{\alpha_i}{p_i}}}\right)^{\frac{p_n}{\alpha_n}}\left(\frac{C}{r_\Omega^{\sum_{i=1}^n \alpha_i s_i-N}\prod_{i=1}^{n-1}\alpha_i^{\frac{\alpha_i}{p_i}}\int_\Omega \varphi(x)\,dx}\right)^{\frac{p_n}{\alpha_n}}\\&=F\left(\prod_{i=1}^{n-1}\lambda_i^{\frac{\alpha_i}{p_i}}\right),\end{aligned}$$

which proves the desired result. □

Similarly, applying Theorem 5.4 with $q_i(x)=\lambda_i\alpha_i\varphi(x)$, $i=1,2,\dots,n$, we obtain the following result.

Theorem 5.7. *Suppose that* $s_ip_i<N$, $i=1,2,\dots,n$. *Moreover, suppose that* $\varphi\in L^\theta(\Omega)$, *where*

$$\max\left\{\frac{N}{s_ip_i}\,:\,i=1,2,\dots,n\right\}<\theta<\infty.$$

Then there exists a function $G :]0, \infty[\to \mathbb{R}$ *such that*

$$\lambda_n \geq G\left(\prod_{i=1}^{n-1} \lambda_i^{\frac{\alpha_i}{p_i}}\right), \quad (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \sigma,$$

where

$$G(t) = \frac{1}{\alpha_n}\left(\frac{1}{t}\right)^{\frac{p_n}{\alpha_n}}\left(\frac{C}{r_\Omega^{\theta \sum_{i=1}^n \alpha_i s_i - N} \prod_{i=1}^{n-1} \alpha_i^{\frac{\theta \alpha_i}{p_i}} \int_\Omega \varphi^\theta(x)\, dx}\right)^{\frac{p_n}{\theta \alpha_n}}, \quad t > 0,$$

and $C > 0$ *is an universal constant.*

5.3.2 Some geometric properties of the generalized spectrum

In this part, as consequences of the previous obtained results, we deduce Protter's type results for the generalized spectrum. We consider only the case $s_i p_i > N$, $i = 1, 2, \ldots, n$. The case $s_i p_i < N$, $i = 1, 2, \ldots, n$, can be treated similarly.

Corollary 5.8. *Suppose that* $s_i p_i > N$, $i = 1, 2, \ldots, n$. *Then there exists a constant* $C(\Omega) > 0$ *that depends on the domain* Ω *such that no point of the generalized spectrum* σ *is contained in the ball* $B(0, C(\Omega))$, *where*

$$B(0, C(\Omega)) = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : \|x\|_\infty < C(\Omega)\}$$

and $\|\cdot\|_\infty$ *is the Chebyshev norm in* $\mathbb{R}^n$.

Proof. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \sigma$. Suppose that $s_i p_i > N$, $i = 1, 2, \ldots, n$. From (35), we obtain easily that

$$\prod_{i=1}^{n} \lambda_i^{\frac{\alpha_i}{p_i}} \geq \left(\frac{1}{\alpha_n}\right)^{\frac{\alpha_n}{p_n}}\left(\frac{C}{r_\Omega^{\sum_{i=1}^n \alpha_i s_i - N} \prod_{i=1}^{n-1} \alpha_i^{\frac{\alpha_i}{p_i}} \int_\Omega \varphi(x)\, dx}\right). \tag{36}$$

Using the condition (31), we have

$$\begin{aligned}\prod_{i=1}^{n} \lambda_i^{\frac{\alpha_i}{p_i}} &\leq \prod_{i=1}^{n} \|\lambda\|_\infty^{\frac{\alpha_i}{p_i}} \\ &= \|\lambda\|_\infty.\end{aligned}$$

Therefore, we have

$$\|\lambda\|_\infty \geq C(\Omega),$$

where

$$C(\Omega)=\left(\frac{1}{\alpha_n}\right)^{\frac{\alpha_n}{p_n}}\left(\frac{C}{r_\Omega^{\sum_{i=1}^n\alpha_i s_i-N}\prod_{i=1}^{n-1}\alpha_i^{\frac{\alpha_i}{p_i}}\int_\Omega\varphi(x)\,dx}\right).$$

Hence,

$$\sigma\subset\mathbb{R}^n\backslash B(0,C(\Omega)),$$

which yields the desired result. □

Corollary 5.9. *Suppose that $s_ip_i>N$, $i=1,2,\dots,n$, and let $\lambda=(\lambda_1,\lambda_2,\dots,\lambda_n)$ be fixed. If Ω is contained in a ball B of sufficiently small radius, then there are no nontrivial weak solutions to* (34)–(30).

Proof. Since

$$\left(\frac{1}{\alpha_n}\right)^{\frac{\alpha_n}{p_n}}\left(\frac{C}{r_\Omega^{\sum_{i=1}^n\alpha_i s_i-N}\prod_{i=1}^{n-1}\alpha_i^{\frac{\alpha_i}{p_i}}\int_\Omega\varphi(x)\,dx}\right)\to+\infty\quad\text{as } r_\Omega\to0^+,$$

there exists a certain $\delta>0$ such that

$$r_\Omega<\frac{\delta}{2}\implies\left(\frac{1}{\alpha_n}\right)^{\frac{\alpha_n}{p_n}}\left(\frac{C}{r_\Omega^{\sum_{i=1}^n\alpha_i s_i-N}\prod_{i=1}^{n-1}\alpha_i^{\frac{\alpha_i}{p_i}}\int_\Omega\varphi(x)\,dx}\right)>\prod_{i=1}^n\lambda_i^{\frac{\alpha_i}{p_i}}.\tag{37}$$

Therefore, if $\Omega\subset B(0,\delta)$, then there are no nontrivial weak solutions to (34)–(30), otherwise, from (37) we obtain a contradiction with (36). □

Bibliography

[1] A. Adams and J. F. Fournier, *Sobolev Spaces*, vol. 140, Academic Press, 2003.
[2] G. Borg, On a Liapounoff criterion of stability, *Am. J. Math.*, **71** (1949), 67–70.
[3] R. C. Brown and D. B. Hinton, Lyapunov inequalities and their applications, in T. M. Rassias (ed.) *Survey on Classical Inequalities*, pp. 1–25, Springer, 2000.
[4] A. Cañada, J. A. Montero, and S. Villegas, Lyapunov inequalities for partial differential equations, *J. Funct. Anal.*, **237** (2006), 17–193.
[5] A. Cañada and S. Villegas, Lyapunov inequalities for Neumann boundary conditions at higher eigenvalues, *J. Eur. Math. Soc.*, **12** (2010), 163–178.
[6] R. Cantrell, On coupled multiparameter nonlinear elliptic system, *Trans. Am. Math. Soc.*, **294**(1) (1986), 263–285.
[7] R. Cantrell and C. Cosner, On the generalized spectrum for second-order elliptic systems, *Trans. Am. Math. Soc.*, **303**(1) (1987), 345–363.
[8] A. Chdouh and D. F. M. Torres, A generalized Lyapunov's inequality for a fractional boundary value problem, *J. Comput. Appl. Math.*, **312** (2017), 192–197.

[9] R. Courant and D. Hilbert, *Methods of Mathematical Physics*, vol. I, Interscience Publishers, Inc., New York, 1953.

[10] K. M. Das and A. S. Vatsala, Green's function for *n-n* boundary value problem and an analogue of Hartman's result, *J. Math. Anal. Appl.*, **51** (1975), 670–677.

[11] P. L. De Nápoli and J. P. Pinasco, A Lyapunov inequality for monotone quasilinear operators, *Differ. Integral Equ.*, **18**(10) (2005), 1193–1200.

[12] P. L. De Nápoli and J. P. Pinasco, Estimates for eigenvalues of quasilinear elliptic systems, *J. Differ. Equ.*, **227**(1) (2006), 102–115.

[13] P. L. De Nápoli and J. P. Pinasco, Lyapunov-type inequalities for partial differential equations, *J. Funct. Anal.*, **270**(6) (2016), 1995–2018.

[14] S. Dhar, Q. Kong, and M. McCabe, Fractional boundary value problems and Lyapunov-type inequalities with fractional integral boundary conditions, *Electron. J. Qual. Theory Differ. Equ.*, **2016** (2016), 43.

[15] E. Di Nezza, G. Palatucci, and E. Valdinoci, Hitchhiker's guide to the fractional Sobolev spaces, *Bull. Sci. Math.*, **136** (2012), 521–573.

[16] A. Elbert, A half-linear second-order differential equation, *Colloq. Math. Soc. János Bolyai*, **30** (1979), 158–180.

[17] R. A. C. Ferreira, A Lyapunov-type inequality for a fractional boundary value problem, *Fract. Calc. Appl. Anal.*, **16**(4) (2013), 978–984.

[18] R. A. C. Ferreira, On a Lyapunov-type inequality and the zeros of a certain Mittag-Leffler function, *J. Math. Anal. Appl.*, **412** (2014), 1058–1063.

[19] G. Franzina and G. Palatucci, Fractional *p*-eigenvalues, *Riv. Mat. Univ. Parma*, **5**(8) (2014), 315–328.

[20] P. Hartman and A. Wintner, On an oscillation criterion of Liapunoff, *Am. J. Math.*, **73** (1951), 885–890.

[21] J. Heinonen, T. Kilpeläinen, and O. Martio, *Nonlinear Potential Theory of Degenerate Elliptic Equations*, Oxford Univ. Press, 1993.

[22] R. Hurri-Syrjanen and A. V. Vähäkangas, On fractional Poincaré inequalities, *J. Anal. Math.*, **120** (2013), 85–104.

[23] A. Iannizzotto and M. Squassina, Weyl-type laws for fractional *p*-eigenvalue problems, *Asymptot. Anal.*, **88** (2014), 233–245.

[24] D. Idczak and S. Walczak, Fractional Sobolev spaces via Riemann–Liouville derivatives, *J. Funct. Spaces Appl.*, **2013** (2013), Article ID 128043, 15 pages.

[25] L. Ihnatsyeva, J. Lehrback, H. Tuominen, and A. V. Vaahakangas, Fractional Hardy inequalities and visibility of the boundary, available at http://arxiv.org/abs/1305.4616.

[26] M. Jeli and B. Samet, Lyapunov-type inequalities for fractional boundary value problems, *Electron. J. Differ. Equ.*, **2015** (2015), 88.

[27] M. Jleli, M. Kirane, and B. Samet, Hartman–Wintner-type inequality for a fractional boundary value problem via a fractional derivative with respect to another function, *Discrete Dyn. Nat. Soc.*, **2017** (2017), Article ID 5123240.

[28] M. Jleli, M. Kirane, and B. Samet, Lyapunov-type inequalities for fractional partial differential equations, *Appl. Math. Lett.*, **66**(88) (2017), 30–39.

[29] M. Jleli, M. Kirane, and B. Samet, Lyapunov-type for a fractional *p*-Laplacian system, *Fract. Calc. Appl. Anal.*, to appear.

[30] M. Jleli, J. J. Nieto, and B. Samet, Lyapunov-type inequalities for a higher order fractional differential equation with fractional integral boundary conditions, *Electron. J. Qual. Theory Differ. Equ.*, **2017** (2017), 16.

[31] M. Jleli and B. Samet, Lyapunov-type inequalities for a fractional differential equation with mixed boundary conditions, *Math. Inequal. Appl.*, **18**(2) (2015), 443–451.

[32] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies, vol. 204, Elsevier Science B.V., Amsterdam, 2006.
[33] C.-F. Lee, C.-C. Yeh, C.-H. Hong, and R. P. Agarwal, Lyapunov and Wirtinger inequalities, *Appl. Math. Lett.*, **17** (2004), 847–853.
[34] E. Lindgren and P. Lindqvist, Fractional eigenvalues, *Calc. Var. Partial Differ. Equ.*, **49**(1–2) (2014), 795–826.
[35] A. M. Lyapunov, Problème général de la stabilité du mouvement, *Ann. Fac. Sci. Univ. Toulouse Sci. Math. Sci. Phys.*, **2** (1907), 203–474.
[36] D. Ma, A generalized Lyapunov inequality for a higher-order fractional boundary value problem, *J. Inequal. Appl.*, **2016** (2016), 261.
[37] Z. Nehari, On the zeros of solutions of second-order linear differential equations, *Am. J. Math.*, **76** (1954), 689–697.
[38] D. O'Regan and B. Samet, Lyapunov-type inequalities for a class of fractional differential equations, *J. Inequal. Appl.*, **2015** (2015), 247.
[39] B. G. Pachpatte, Lyapunov type integral inequalities for certain differential equations, *Georgian Math. J.*, **4** (1997), 139–148.
[40] J. P. Pinasco, Lower bounds for eigenvalues of the one-dimensional p-Laplacian, *Abstr. Appl. Anal.*, **2004** (2004), 147–153.
[41] J. P. Pinasco, *Lyapunov-Type Inequalities, With Applications to Eigenvalue Problems*, Springer Briefs in Mathematics, Springer, New York, 2013.
[42] M. Protter, The generalized spectrum of second-order elliptic systems, *Rocky Mt. J. Math.*, **9**(3) (1979), 503–518.
[43] J. Rong and C. Bai, Lyapunov-type inequality for a fractional differential equation with fractional boundary conditions, *Adv. Differ. Equ.*, **2015** (2015), 82.
[44] T. Runst and W. Sickel, *Sobolev Spaces of Fractional Order, Nemytskij Operators, and Nonlinear Partial Differential Equations*, vol. 3, Walter de Gruyter, 1996.
[45] A. Wintner, On the Non-existence of conjugate points, *Am. J. Math.*, **73** (1951), 368–380.
[46] X. Yang, On inequalities of Lyapunov type, *Appl. Math. Comput.*, **134** (2003), 293–300.

Anatoly N. Kochubei

Fractional-parabolic equations and systems. Cauchy problem

Abstract: The paper is a survey of results on fundamental solutions of the Cauchy problem for fractional-parabolic equations and systems with the Caputo–Djrbashian fractional time derivative. We describe the parametrix method, applications to the Cauchy problem, and uniqueness theorems.

Keywords: Fractional-parabolic equation, Cauchy problem, fundamental solution, parametrix method, Caputo–Djrbashian fractional derivative

MSC 2010: 35R11, 35K99

1 Introduction

Fractional diffusion equations of the form

$$(\mathbb{D}_t^{(\alpha)}u)(t,x) - Au(t,x) = f(t,x), \quad 0 \le t \le T, x \in \mathbb{R}^n, \tag{1}$$

where $0 < \alpha < 1$, $\mathbb{D}_t^{(\alpha)}$ is the Caputo–Djrbashian fractional derivative, that is,

$$(\mathbb{D}_t^{(\alpha)}u)(t,x) = \frac{1}{\Gamma(1-\alpha)}\left[\frac{\partial}{\partial t}\int_0^t (t-\tau)^{-\alpha}u(\tau,x)\,d\tau - t^{-\alpha}u(0,x)\right],$$

A is a second-order elliptic operator, are among the basic subjects in the theory of fractional differential equations. The initial motivation came from physics—the equations of the above type were first used for modeling anomalous diffusion on fractals by Nigmatullin [28] and for a description of Hamiltonian chaos by Zaslavsky [37]. At present, such equations are among classical equations of mathematical physics.

The first mathematical works in this direction dealt either with the case of an abstract operator A, that is with a kind of an abstract Cauchy problem [15] (see [13] for further references), or with the case where $A = \Delta$ is a Laplacian. For the latter case, a fundamental solution of the Cauchy problem (FSCP) is expressed via Fox's H-function [16, 35] or the Wright functions [33]; uniqueness theorems were proved in [16] for an equation with a general second-order elliptic operator A; see also [33]. The first example of an initial-boundary value problem for this equation was considered in [36]. Later the initial-boundary value problems for fractional diffusion equations were studied in

Anatoly N. Kochubei, Institute of Mathematics, National Academy of Sciences of Ukraine, Tereshchenkivska 3, Kyiv, 01004, Ukraine, e-mail: kochubei@imath.kiev.ua

https://doi.org/10.1515/9783110571660-007

[25], with an emphasis on the probabilistic aspects, and in [22]; for the probabilistic interpretations, see also [26, 34] and references therein. FSCPs for time- and space-fractional equations are found in [5, 11, 14, 24]. Of course, here we cannot touch huge literature on nonlinear equations, results on qualitative theory of fractional-parabolic equations and many other subjects. Some of them are discussed in other sections of this handbook.

In [8] (see also [7]), Eidelman and the author constructed and investigated a FSCP for fractional diffusion equations with variable coefficients. We followed the classical parametrix method using an H-function representation for the parametrix kernel and the detailed information about asymptotic properties of the H-function available from [4, 12]. The fractional diffusion equation shares many essential properties with second-order parabolic equations (though some properties are different, like, e. g., the singularity of the FSCP at $x = 0$ appearing for $n \geq 2$).

In this survey, we choose, as our basic framework, the class of fractional-parabolic systems introduced and investigated in [17] as a natural fractional extension of the class of parabolic systems. For classical (differential) parabolic systems of equations, there must be a well-posed Cauchy problem whose fundamental solution is an ordinary function smooth outside the singular point $t = 0$. Such a class of systems was first found by Petrowsky [31] in 1938. In fact, after Petrowsky's work it was understood that a complete theory of partial differential equations must include a thorough study of systems of equations. Petrowsky introduced and investigated not only parabolic systems but also hyperbolic ones, systems with correct Cauchy problems, etc. For several decades, these subjects were central in the theory of partial differential equations; see, in particular, [10, 30]. The class of parabolic systems was studied in the greatest detail [6, 7, 9, 21].

We consider systems of the form (1) where $u = (u_1, \dots, u_N)$ is a vector-valued function,

$$A = A(x, D_x) = A_0(x, D_x) + A_1(x, D_x), \tag{2}$$

is a differential operator of even order $2b$ with matrix-valued coefficients,

$$(A_0(x, D_x)u)_i = \sum_{j=1}^{N} \sum_{|\beta|=2b} a_\beta^{ij}(x) D_x^\beta u_j, \quad i = 1, \dots, N, \tag{3}$$

$$(A_1(x, D_x)u)_i = \sum_{j=1}^{N} \sum_{|\beta|<2b} a_\beta^{ij}(x) D_x^\beta u_j, \quad i = 1, \dots, N, \tag{4}$$

$$D_x^\beta = D_{x_1}^{\beta_1} \cdots D_{x_n}^{\beta_n}, \quad D_{x_j} = \frac{1}{\sqrt{-1}} \frac{\partial}{\partial x_j}, \quad |\beta| = \beta_1 + \cdots + \beta_n.$$

We assume that all the coefficients $a_\beta^{ij}(x)$ are bounded and satisfy the global Hölder condition

$$\left| a_\beta^{ij}(x) - a_\beta^{ij}(y) \right| \leq C|x - y|^\gamma$$

(below the letters C, c, σ will denote various positive constants while $\gamma > 0$ will denote all the Hölder exponents; for simplicity, we denote by $|\cdot|$ norms of all finite vectors and matrices). We also assume the uniform strong parabolicity condition: for all $\eta \in \mathbb{R}^n$, $z \in \mathbb{C}^N$,

$$\mathbb{R}\langle A_0(x,\eta)z,z\rangle \leq -\delta|\eta|^{2b}|z|^2, \quad \delta > 0. \tag{5}$$

In fact, the key ingredients in the construction of a FSCP for a problem with variable coefficients are precise estimates for the model problem

$$(\mathbb{D}_t^{(\alpha)}u(t,x)) = A_0(y,D_x)u(t,x) \tag{6}$$

containing only the homogeneous highest order differential operator in x, with "frozen" coefficients depending on a parameter point y. As the first step, one has to consider the case where the coefficients $a_\beta^{ij}(x)$, $|\beta| = 2b$, are constant. Already in this case, the study of a FSCP is far from trivial. The approach used in [7, 8] based on the H-function representation, does not work for systems.

Instead, we use the subordination representation [1, 3, 32] expressing the FSCP for the model system via the FSCP for the first-order (in t) parabolic system. At the first sight, it looks an easy approach to all fractional problems. However, the subordination identity involves the integration in t over the half-axis $(0,\infty)$, while usually a FSCP for a parabolic equation or system is constructed only on a finite time interval. Nevertheless, for our model case of constant coefficients the subordination method, together with integral representations resulting from the use of the Fourier transform, works efficiently giving, by the way, new proofs of the estimates known for fractional diffusion equations. Note also that the probabilistic side of subordination, not touched here, is also an important subject of fractional analysis; see [2, 19, 27, 29].

We give some attention also to the case of a single fractional diffusion equation. In particular, we include the H-function and the Wright function formulas for the FSCP in the constant coefficients case. The uniqueness theorems for general systems [17], for the fractional diffusion equation [16] and for the model fractional heat equation [33] will be formulated. The results for more narrow classes of equations are not contained in the one for systems because the maximum principle method or the one using an explicit FSCP do not work in the general situation.

The contents of the paper are as follows. In Section 2, we explain the subordination method and consider systems with constant coefficients. Section 3 contains the main result on the existence and estimates of the Green matrix. In Section 4, we give its explicit expression, for a scalar fractional heat equation, in terms of the Fox H-function and the Wright function. Section 5 is devoted to uniqueness theorems.

2 The model system – subordination

Let us consider systems of the form

$$\big(\mathbb{D}_t^{(\alpha)}u(t,x)\big) = A_0(D_x)u(t,x) \tag{7}$$

where

$$(A_0(D_x)u)_i = \sum_{j=1}^{N}\sum_{|\mu|=2b} a_\mu^{ij}D_x^\mu u_j, \quad i = 1,\dots,N,$$

$a_\mu^{ij} \in \mathbb{C}$, and for any $\eta \in \mathbb{R}^n$, $z \in \mathbb{C}^N$,

$$\Re\langle A_0(\eta)z,z\rangle \le -\delta|\eta|^{2b}|z|^2, \quad \delta > 0. \tag{8}$$

Under the assumption (8) (in fact, even under a weaker assumption of parabolicity in the sense of Petrowsky), the differential expression $A_0(D)$ defines on the space $L^2(\mathbb{R}^n,\mathbb{C}^N)$ of square integrable vector-functions with values in $\mathbb{C}^N$, an infinitesimal generator $\mathcal{A}_0$ of a C_0-semigroup $S_1(t) = e^{t\mathcal{A}_0}$ (see [20]).

By the subordination theorem (see Theorem 3.1 in [3]), the system (7) interpreted as an equation in $L^2(\mathbb{R}^n,\mathbb{C}^N)$, possesses a solution operator $S_\alpha(t)$, such that for any element $u_0 = u_0(x)$ from the domain $D(\mathcal{A}_0)$, the function $u(t,x) = (S_\alpha(t)u_0)(x)$, $t \ge 0$, $x \in \mathbb{R}^n$, is a solution of the equation (7) satisfying the initial condition $u(0,x) = u_0(x)$. In addition,

$$S_\alpha(t) = \int_0^\infty \varphi_{t,\alpha}(s)S_1(s)\,ds, \quad t \ge 0, \tag{9}$$

where $\varphi_{t,\alpha}(s) = t^{-\alpha}\Phi_\alpha(st^{-\alpha})$,

$$\Phi_\alpha(\zeta) = \sum_{k=0}^{\infty}\frac{(-\zeta)^k}{k!\Gamma(-\alpha k + 1 - \alpha)},$$

so that Φ_α can be written as the Wright function

$$\Phi_\alpha(\zeta) = {}_0\Psi_1\left[\begin{matrix} - \\ (1-\alpha,-\alpha)\end{matrix}\,\middle|\,-\zeta\right].$$

See [13] for general information regarding the definition and properties of the Wright functions.

The function Φ_α is a probability density:

$$\Phi_\alpha(t) \ge 0,\ t > 0; \quad \int_0^\infty \Phi_\alpha(t)\,dt = 1.$$

It is connected also with the Mittag-Leffler function

$$E_\alpha(\zeta) = \sum_{k=0}^{\infty} \frac{\zeta^k}{\Gamma(1+\alpha k)}, \quad \zeta \in \mathbb{C},$$

via the Laplace transform identity

$$E_\alpha(-\zeta) = \int_0^\infty \Phi_\alpha(t) e^{-\zeta t}\, dt, \quad \zeta \in \mathbb{C}.$$

By the classical theory of parabolic equations, the semigroup $S_1(t)$ possesses the integral representation

$$(S_1(t)\eta)(x) = \int_{\mathbb{R}^n} Z(t, x-\xi)\eta(\xi)\, d\xi, \quad \eta \in L^2(\mathbb{R}^n, \mathbb{C}^N),$$

in terms of the FSCP $Z(t,x)$ of the parabolic system $\frac{\partial u}{\partial t} = A_0(D_x)u$. It follows from the estimates of Φ_α and Z (see below) that, for example, if $\eta \in \mathcal{S}(\mathbb{R}^n)$, then

$$(S_\alpha(t)\eta)(x) = \int_{\mathbb{R}^n} Z_\alpha(t, x-\xi)\eta(\xi)\, d\xi$$

where

$$Z_\alpha(t,x) = \int_0^\infty \varphi_{t,\alpha}(s) Z(s,x)\, ds, \quad x \neq 0$$

(as it was seen already for the fractional diffusion equations [7, 8], Z_α may have a singularity at $x = 0$).

The kernel Z_α is a FSCP for the system (7).

In order to obtain an integral representation of a solution $u(t,x)$ of the inhomogeneous equation

$$(\mathbb{D}_t^{(\alpha)} u(t,x)) - A_0(D_x)u(t,x) = f(t,x), \quad u(0,x) = u_0(x),$$

in the form

$$u(t,x) = \int_{\mathbb{R}^n} Z_\alpha(t, x-\xi)u_0(\xi)\, d\xi + \int_0^t d\tau \int_{\mathbb{R}^n} Y_\alpha(t-\tau, x-y) f(\tau, y)\, dy$$

(the definition of a classical solution will be given below for a more general situation), we need another kernel

$$Y_\alpha(t,x) = (\mathbb{D}_t^{(1-\alpha)} Z_\alpha)(t,x), \quad x \neq 0.$$

We have

$$Y_\alpha(t,x) = \int_0^\infty \psi_{t,\alpha}(s) Z(s,x)\, ds, \quad x \neq 0$$

where

$$\psi_{t,\alpha}(s) = t^{-1}{}_0\Psi_1 \left[\begin{matrix} - \\ (0,-\alpha) \end{matrix} \middle| -st^{-\alpha} \right].$$

Note that $\varphi_{t,\alpha}$ is the law of the inverse to a α-stable subordinator; see [26]. In the context of fractional calculus models of physical phenomena, the functions $\varphi_{t,\alpha}$ and $\psi_{t,\alpha}$ appeared for the first time in [23].

Using properties of the functions Z and Φ_α we get the integral identities

$$\int_{\mathbb{R}^n} Z_\alpha(t,x)\, dx = I, \qquad \int_{\mathbb{R}^n} Y_\alpha(t,x)\, dx = \frac{t^{\alpha-1}}{\Gamma(\alpha)} I.$$

The second identity contains a correction of the wrong formula from [7, 8, 17]. This correction was first made in [18].

Theorem 1. *The matrix-functions $Z_\alpha(t,x)$, $Y_\alpha(t,x)$ are infinitely differentiable for $t > 0$, $x \neq 0$, and satisfy the following estimates. Denote*

$$R = t^{-\alpha}|x|^{2b}, \quad \rho(t,x) = \left(t^{-\alpha}|x|^{2b}\right)^{\frac{1}{2b-\alpha}}.$$

(i) *If $R \geq 1$, then*

$$\left|D_x^\beta Z_\alpha(t,x)\right| \leq Ct^{-\alpha\frac{(n+|\beta|)}{2b}} \exp(-\sigma\rho(t,x)), \quad \sigma > 0;$$
$$\left|D_x^\beta Y_\alpha(t,x)\right| \leq Ct^{-\alpha\frac{(n+|\beta|)}{2b}+\alpha-1} \exp(-\sigma\rho(t,x)).$$

(ii) *If $R \leq 1$, $n + |\beta| < 2b$, then*

$$\left|D_x^\beta Z_\alpha(t,x)\right| \leq Ct^{-\alpha\frac{(n+|\beta|)}{2b}},$$
$$\left|D_x^\beta Y_\alpha(t,x)\right| \leq Ct^{-\alpha\frac{(n+|\beta|)}{2b}+\alpha-1}.$$

(iii) *If $R \leq 1$, $n + |\beta| > 2b$, then*

$$\left|D_x^\beta Z_\alpha(t,x)\right| \leq Ct^{-\alpha}|x|^{-n+2b-|\beta|},$$
$$\left|D_x^\beta Y_\alpha(t,x)\right| \leq Ct^{-1}|x|^{-n+2b-|\beta|}.$$

(iv) *If $R \leq 1$, $n + |\beta| = 2b$, then*

$$\left|D_x^\beta Z_\alpha(t,x)\right| \leq Ct^{-\alpha}, \quad \text{if } n = 1;$$
$$\left|D_x^\beta Z_\alpha(t,x)\right| \leq Ct^{-\alpha}\left[\left|\log(t^{-\alpha}|x|^{2b})\right| + 1\right], \quad \text{if } n \geq 2;$$
$$\left|D_x^\beta Y_\alpha(t,x)\right| \leq Ct^{-1}.$$

(v) *If $R \geq 1$, then*

$$\left|\frac{\partial Z_\alpha(t,x)}{\partial t}\right| \leq Ct^{-\frac{\alpha n}{2b}-1}\exp(-\sigma\rho(t,x)).$$

(vi) *If $R \leq 1$, $n < 2b$, then*

$$\left|\frac{\partial Z_\alpha(t,x)}{\partial t}\right| \leq Ct^{-\frac{\alpha n}{2b}-1}.$$

If $R \leq 1$, $n > 2b$, then

$$\left|\frac{\partial Z_\alpha(t,x)}{\partial t}\right| \leq Ct^{-\alpha-1}|x|^{-n+2b}.$$

If $R \leq 1$, $n = 2b$, then

$$\left|\frac{\partial Z_\alpha(t,x)}{\partial t}\right| \leq Ct^{-\alpha-1}\left[\left|\log(t^{-\alpha}|x|^{2b})\right| + 1\right].$$

In all the above estimates, the constants depend only on N, n, $\max|a_\mu^{ij}|$, and the strong parabolicity constant δ.

Note that the fractional derivative $\mathbb{D}_t^{(\alpha)}Z_\alpha$ satisfies the same estimate as the derivatives $D_x^\beta Z_\alpha$, $|\beta| = 2b$.

In Theorem 1, the estimates are given separately for large and small values of R. In order to justify the iteration procedures of the Levi method (see the end of Section 3 below), we need unified estimates valid for all values of the variables.

Proposition. *If $n + |\beta| < 2b$, then*

$$\left|D_x^\beta Z_\alpha(t,x)\right| \leq Ct^{-\alpha\frac{n+|\beta|}{2b}}e^{-c\rho(t,x)}, \quad c > 0; \tag{10}$$

$$\left|D_x^\beta Y_\alpha(t,x)\right| \leq Ct^{-1+\alpha-\alpha\frac{n+|\beta|}{2b}}e^{-c\rho(t,x)}. \tag{11}$$

If $n + |\beta| > 2b$, then

$$\left|D_x^\beta Z_\alpha(t,x)\right| \leq Ct^{-\alpha}|x|^{-n+2b-|\beta|}e^{-c\rho(t,x)}. \tag{12}$$

If $n + |\beta| = 2b$, then

$$\left|D_x^\beta Z_\alpha(t,x)\right| \leq Ct^{-\alpha}\left[\left|\log(t^{-\alpha}|x|^{2b})\right| + 1\right]e^{-c\rho(t,x)}. \tag{13}$$

If $n + |\beta| \geq 2b$, then

$$\left|D_x^\beta Y_\alpha(t,x)\right| \leq Ct^{-1}|x|^{-n+2b-|\beta|}e^{-c\rho(t,x)}. \tag{14}$$

The constants can depend only on the parameters listed in the formulation of Theorem 1.

3 The general case

As stated in Introduction, we consider the system (1)–(4) with bounded Hölder continuous coefficients, under the uniform strong parabolicity condition (5).

We call a vector-function $u(t,x)$, $0 \le t \le T$, $x \in \mathbb{R}^n$, *a classical solution* of the system (1), with the initial condition

$$u(0,x) = u_0(x), \quad x \in \mathbb{R}^n, \tag{15}$$

if:

(i) $u(t,x)$ is continuously differentiable in x up to the order $2b$, for each $t > 0$;

(ii) for each $x \in \mathbb{R}^n$, $u(t,x)$ is continuous in t on $[0,T]$, and its fractional integral

$$(I_{0+}^{1-\alpha}u)(t,x) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} u(\tau,x)\,d\tau \tag{16}$$

is continuously differentiable in t for $0 \le t \le T$.

(iii) $u(t,x)$ satisfies the equation (1) and the initial condition (15).

A classical solution $u(t,x)$ is called *a uniform classical solution*, if it is continuous in t uniformly with respect to $x \in \mathbb{R}^n$, and the first derivative of the fractional integral (16) exists uniformly with respect to $x \in \mathbb{R}^n$.

Our main task is to construct *a Green matrix* for the problem (1), (15), that is such a pair

$$\{Z_\alpha^{(1)}(t,x;\xi), Y_\alpha^{(1)}(t,x;\xi)\}, \quad t \in [0,T],\ x,\xi \in \mathbb{R}^n,$$

that for any bounded function f, jointly continuous in (t,x) and locally Hölder continuous in x uniformly with respect to t, and any bounded locally Hölder continuous function u_0, the function

$$u(t,x) = \int_{\mathbb{R}^n} Z_\alpha^{(1)}(t,x;\xi)u_0(\xi)\,d\xi + \int_0^t d\lambda \int_{\mathbb{R}^n} Y_\alpha^{(1)}(t-\lambda,x;y)f(\lambda,y)\,dy \tag{17}$$

is a classical solution of the problem (1), (15).

Denote by $Z_\alpha^{(0)}(t,x-\xi;y)$ and $Y_\alpha^{(0)}(t,x-\xi;y)$ the kernels defined just as $Z_\alpha(t,x-\xi)$ and $Y_\alpha(t,x-\xi)$, but for the system (6) with the coefficients a_β^{ij}, $|\beta| = 2b$, “frozen” at a point $y \in \mathbb{R}^n$, and other coefficients set equal to zero; in (6), y appears as a parameter.

Theorem 2. (a) *There exists a Green matrix* $\{Z_\alpha^{(1)}(t,x;\xi), Y_\alpha^{(1)}(t,x;\xi)\}$ *of the form*

$$Z_\alpha^{(1)}(t,x;\xi) = Z_\alpha^{(0)}(t,x-\xi;\xi) + V_Z(t,x;\xi),$$
$$Y_\alpha^{(1)}(t,x;\xi) = Y_\alpha^{(0)}(t,x-\xi;\xi) + V_Y(t,x;\xi),$$

where the kernels $Z_\alpha^{(0)}(t,x;\xi)$, $Y_\alpha^{(0)}(t,x;\xi)$ *satisfy the estimates listed in Theorem* 1 *with coefficients independent on the parameter point* ξ. *The functions* V_Z, V_Y *satisfy the following estimates:*

(i) *If* $n+|\beta|<2b$, *then*

$$\begin{aligned} &\left|D_x^\beta V_Z(t,x;\xi)\right| \le Ct^{-\frac{\alpha}{2b}(|\beta|+\gamma_0)}|x-\xi|^{-n+\gamma-\gamma_0}e^{-\sigma\rho(t,x-\xi)}, \\ &\quad 0<\gamma_0<\gamma,\ \sigma>0; \\ &\left|D_x^\beta V_Y(t,x;\xi)\right| \le Ct^{-1+\alpha-\frac{\alpha|\beta|}{2b}}|x-\xi|^{-n+\gamma}e^{-\sigma\rho(t,x-\xi)}. \end{aligned} \tag{18}$$

(ii) *If* $n+|\beta|\ge 2b$, $|\beta|<2b$, *then*

$$\left|D_x^\beta V_Z(t,x;\xi)\right| \le Ct^{-\alpha+\frac{\alpha\gamma_0}{2b}}|x-\xi|^{-n+2b-|\beta|+\gamma-\gamma_0}e^{-\sigma\rho(t,x-\xi)};$$
$$\left|D_x^\beta V_Y(t,x;\xi)\right| \le Ct^{-1+\frac{\alpha\gamma_0}{b}}|x-\xi|^{-n++2b-|\beta|+\gamma-2\gamma_0}e^{-\sigma\rho(t,x-\xi)}.$$

(iii) *If* $|\beta|=2b$, *then*

$$\left|D_x^\beta V_Z(t,x;\xi)\right| \le Ct^{-\alpha+\mu_1}|x-\xi|^{-n+\mu_2}e^{-\sigma\rho(t,x-\xi)};$$
$$\left|D_x^\beta V_Y(t,x;\xi)\right| \le Ct^{-1+\mu_1}|x-\xi|^{-n+\mu_2}e^{-\sigma\rho(t,x-\xi)}$$

where $\mu_1,\mu_2>0$.

(b) *If the functions* $u_0(x), f(t,x)$ *are bounded and globally Hölder continuous* (*for* f, *uniformly with respect to* t), *and* f *is continuous in* t *uniformly with respect to* $x\in\mathbb{R}^n$, *then the solution* (17) *is a uniform classical solution. All its derivatives in* x, *up to the order* $2b$, *are bounded and globally Hölder continuous, uniformly with respect to* $t\in[0,T]$.

Note that the estimates in Theorem 2 can be written in a variety of ways. For example, in (18) we may write

$$t^{-\frac{\alpha}{2b}|\beta|} = \left(t^{-\frac{\alpha}{2b}}|x-\xi|\right)^{|\beta|}|x-\xi|^{-|\beta|}$$

and obtain, taking $0<\sigma'<\sigma$, that

$$\left|D_x^\beta V_Z(t,x;\xi)\right| \le Ct^{-\frac{\alpha}{2b}\gamma_0}|x-\xi|^{-n-|\beta|+\gamma-\gamma_0}e^{-\sigma'\rho(t,x-\xi)}.$$

This kind of transformation is often used in proofs of various estimates.

Just as for the classical parabolic equations and systems, the scheme of Levi's parametrix method consists of the following construction.

We look for the functions $Z_\alpha^{(1)}(t,x;\xi)$, $Y_\alpha^{(1)}(t,x;\xi)$ appearing in Theorem 2 assuming the following integral representations:

$$Z_\alpha^{(1)}(t,x;\xi) = Z_\alpha^{(0)}(t,x-\xi;\xi) + \int_0^t d\lambda \int_{\mathbb{R}^n} Y_\alpha^{(0)}(t-\lambda,x-y;y)Q(\lambda,y;\xi)\,dy;$$
$$Y_\alpha^{(1)}(t,x;\xi) = Y_\alpha^{(0)}(t,x-\xi;\xi) + \int_0^t d\lambda \int_{\mathbb{R}^n} Y_\alpha^{(0)}(t-\lambda,x-y;y)\Phi(\lambda,y;\xi)\,dy.$$

For the functions Q, Φ, we assume the integral equations

$$Q(t,x;\xi) = M(t,x;\xi) + \int_0^t d\lambda \int_{\mathbb{R}^n} K(t-\lambda,x;y)Q(\lambda,y;\xi)\,dy,$$

$$\Phi(t,x;\xi) = K(t,x;\xi) + \int_0^t d\lambda \int_{\mathbb{R}^n} K(t-\lambda,x;y)\Phi(\lambda,y;\xi)\,dy,$$

where

$$M(t,x;\xi) = [A(x,D_x) - A_0(\xi,D_x)]Z_\alpha^{(0)}(t,x-\xi;\xi),$$
$$K(t,x;\xi) = [A(x,D_x) - A_0(\xi,D_x)]Y_\alpha^{(0)}(t,x-\xi;\xi)$$

The iterative solution of the integral equations is based on the above estimates and a number of derived ones, such as estimates for differences of the kernels etc. For the details, see [17].

4 Second-order equations

For a scalar fractional-parabolic equation of second order with respect to spatial variables with real-valued constant coefficients, a FSCP can be written more explicitly, in terms of certain special functions. Since asymptotic properties of the latter are known in great detail, such representations of the FSCP can be useful for applications. For the definition and properties of Fox's H-functions, see [12, 13]. The Wright functions are considered in [13, 33]. The main source on asymptotic behavior of the above classes of special functions is the seminal paper by Braaksma [4]. See also other sections of this handbook.

It is sufficient to consider the fractional heat equation

$$(\mathbb{D}_t^{(\alpha)}u)(t,x) = \Delta u(t,x), \quad t > 0, x \in \mathbb{R}^n. \tag{19}$$

Other equations with constant coefficients are reduced to (19) by a change of variables.

An expression of a FSCP $Z(t,x)$ for the equation (19) in terms of an H-function was found in [16, 35]:

$$Z(t,x) = \pi^{-\frac{n}{2}}|x|^{-n}H_{12}^{20}\left[\frac{1}{4}t^{-\alpha}|x|^2 \,\middle|\, \begin{matrix}(1,\alpha) \\ (\frac{n}{2},1), \quad (1,1)\end{matrix}\right].$$

For a proof, see also [7] where the corresponding function Y is also given:

$$Y(t,x) = \pi^{-\frac{n}{2}}|x|^{-n/2}t^{\alpha-1}H_{12}^{20}\left[\frac{1}{4}t^{-\alpha}|x|^2 \,\middle|\, \begin{matrix}(\alpha,\alpha) \\ (\frac{n}{2},1), \quad (1,1)\end{matrix}\right].$$

In this case, the functions Z and Y are nonnegative.

Representations of these functions in terms of simpler Wright functions were found by Mainardi [23] for the case where $n = 1$ and by Pskhu [33] in the general case. The Wright function appearing in this framework is denoted by $\Phi(-\alpha/2, \delta, z)$ or ${}_0\Psi_1\left[\begin{smallmatrix} - \\ (-\frac{\alpha}{2},\delta) \end{smallmatrix} \,\middle|\, z\right]$:

$$\Phi(-\alpha/2, \delta, z) = \sum_{m=0}^{\infty} \frac{z^m}{m!\Gamma(\delta - \frac{\alpha m}{2})}.$$

Next, denote

$$f_{\alpha/2}(z; \mu, \delta) = \begin{cases} \frac{2}{\Gamma(\mu/2)} \int_1^\infty \Phi(-\alpha/2, \delta, -zt)(t^2 - 1)^{\frac{\mu}{2}-1} dt, & \text{if } \mu > 0; \\ \Phi(-\alpha/2, \delta, -z), & \text{if } \mu = 0. \end{cases}$$

Then

$$Z(t,x) = c_n t^{-\frac{\alpha n}{2}} f_{\alpha/2}\left(t^{-\alpha/2}|x|; n-1, 1 - \frac{\alpha n}{2}\right),$$
$$Y(t,x) = c_n t^{\alpha - \frac{\alpha n}{2} - 1} f_{\alpha/2}\left(t^{-\alpha/2}|x|; n-1, \alpha - \frac{\alpha n}{2}\right),$$

where $c_n = 2^{-n}\pi^{\frac{1-n}{2}}$.

In [33], equations with some other fractional derivatives are considered (in particular, the Riemann–Liouville derivative) where the formulation of the Cauchy problem is different.

5 Uniqueness theorems

There are several uniqueness results stating the uniqueness of a solution of the Cauchy problem in a certain class of functions. The wider is the class of equations, the more narrow is the functional class. The results are proved by different methods working for the corresponding class of equations. We begin with the class of general fractional-parabolic systems.

Here, we maintain the same assumptions as in Theorem 2.

Theorem 3. *Let $u(t,x)$, $0 \le t \le T$, $x \in \mathbb{R}^n$, be a uniform classical solution of the problem* (1), (15) *with $f(t,x) \equiv 0$, $u_0(x) \equiv 0$. Suppose that the function $u(t,x)$ and all its derivatives of orders $\le 2b$ are bounded and globally Hölder continuous. Then $u(t,x)$ equals zero identically.*

The proof is based on the reduction to the case of an abstract fractional-parabolic equation in a Banach space and the use of methods from [15].

The next two results [7, 16] are based on the maximum principle arguments which are possible for scalar second-order equations. We consider the Cauchy problem

$$(\mathbb{D}_t^{(\alpha)}u)(t,x) - Bu(t,x) = f(t,x), \quad t > 0, x \in \mathbb{R}^n, \tag{20}$$

$$u(0,x) = u_0(x), \quad x \in \mathbb{R}^n, \tag{21}$$

where $0 < \alpha < 1$,

$$B = \sum_{j,l=1}^{n} a_{jl}(t,x)\partial_{x_j}\partial_{x_l} + \sum_{j=1}^{n} b_j(t,x)\partial_{x_j} + c(t,x).$$

Here, the uniform parabolicity condition may be weakened to parabolicity:

$$\sum_{j,l=1}^{n} a_{jl}\xi_j\xi_l \geq 0;$$

it is sufficient to assume continuity and boundedness of coefficients of the operator B (the Hölder continuity is not required).

Denote by $H_{\varkappa}^{\alpha+\lambda}[0,T]$ the class of such functions $y(t)$ that $t^{\varkappa}y(t)$ is Hölder continuous on $[0,T]$ with the exponent $\alpha+\lambda$, $\lambda > 0$.

Theorem 4. *Let $u(t,x)$ be a classical solution of the problem* (20)–(21) *with $u_0(x) \equiv 0$. Suppose that $|u(t,x)| \leq M$ for all $t \in [0,T]$, $x \in \mathbb{R}^n$, and that for each $x \in \mathbb{R}^n$, $u(\cdot,x) \in H_{\varkappa}^{\alpha+\lambda}[0,T]$ where $0 < \lambda < 1-\alpha$, $0 \leq \varkappa < \lambda+1$. Then u equals zero identically.*

Let $n = 1$. For this case, we have the uniqueness in the class of functions with exponential growth.

Theorem 5. *Let the conditions of Theorem* 4*, with $n = 1$, be satisfied, except that the solution $u(t,x)$ is not necessarily bounded, but*

$$|u(t,x)| \leq M_1 \exp\{M_2|x|^{2/(2-\alpha)}\}, \quad M_1, M_2 > 0. \tag{22}$$

Then $u(t,x) \equiv 0$.

There are counterexamples [16, 33] showing that the exponent $\frac{2}{2-\alpha}$ cannot be increased.

Using the explicit FSCP, Pskhu [33] proved the following uniqueness theorem for the equation (19) (with an arbitrary n).

Theorem 6. *The equation* (19) *with the zero initial condition can have only one solution $u(t,x)$ satisfying, for some $M_1, M_2 > 0$, the condition* (22)*, namely $u(t,x) \equiv 0$.*

Bibliography

[1] B. Baeumer and M. M. Meerschaert, Stochastic solutions for fractional Cauchy problems, *Fract. Calc. Appl. Anal.*, **4** (2001), 481–500.

[2] B. Baeumer, M. M. Meerschaert, and E. Nane, Brownian subordinators and fractional Cauchy problems, *Trans. Am. Math. Soc.*, **361** (2009), 3915–3930.
[3] E. G. Bazhlekova, Subordination principle for fractional evolution equations, *Fract. Calc. Appl. Anal.*, **3** (2000), 213–230.
[4] B. L. J. Braaksma, Asymptotic expansions and analytic continuation for a class of Barnes integrals, *Compos. Math.*, **15** (1964), 239–341.
[5] X. Cheng, Z. Li, and M. Yamamoto, Asymptotic behavior of solutions to space-time fractional diffusion-reaction equations, *Math. Methods Appl. Sci.*, **40** (2017), 1019–1031.
[6] S. D. Eidelman, *Parabolic Systems*, North-Holland, Amsterdam, 1969.
[7] S. D. Eidelman, S. D. Ivasyshen, and A. N. Kochubei, *Analytic Methods in the Theory of Differential and Pseudo-Differential Equations of Parabolic Type*, Birkhäuser, Basel, 2004.
[8] S. D. Eidelman and A. N. Kochubei, Cauchy problem for fractional diffusion equations, *J. Differ. Equ.*, **199** (2004), 211–255.
[9] A. Friedman, *Partial Differential Equations of Parabolic Type*, Prentice-Hall, Englewood Cliffs, NJ, 1964.
[10] I. M. Gel'fand and G. E. Shilov, *Generalized Functions. Vol. 3. Theory of Differential Equations*, Academic Press, New York, 1967.
[11] J. Kemppainen, J. Siljander, and R. Zacher, Representation of solutions and large-time behavior for fully nonlocal diffusion equations, *J. Differ. Equ.*, **263** (2017), 149–201.
[12] A. A. Kilbas and M. Saigo, *H-Transforms. Theory and Applications*, Chapman and Hall, Boca Raton, Fl, 2004.
[13] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier, Amsterdam, 2006.
[14] K.-H. Kim and S. Lim, Asymptotic behavior of fundamental solution and its derivatives related to space-time fractional differential equations, *J. Korean Math. Soc.*, **53** (2016), 929–967.
[15] A. N. Kochubei, A Cauchy problem for evolution equations of fractional order, *Differ. Equ.*, **25** (1989), 967–974.
[16] A. N. Kochubei, Fractional-order diffusion, *Differ. Equ.*, **26** (1990), 485–492.
[17] A. N. Kochubei, Fractional-parabolic systems, *Potential Anal.*, **37** (2012), 1–30.
[18] A. N. Kochubei, Cauchy problem for fractional diffusion-wave equation with variable coefficients, *Appl. Anal.*, **93** (2014), 2211–2242.
[19] V. N. Kolokoltsov, Generalized continuous-time random walks, subordination by hitting times, and fractional dynamics, *Theory Probab. Appl.*, **53** (2009), 594–609.
[20] S. G. Krein, *Linear Differential Equations in Banach Space*, American Mathematical Society, Providence, 1972.
[21] O. A. Ladyzhenskaya, V. A. Solonnikov, and N. N. Uraltseva, *Linear and Quasilinear Equations of Parabolic Type*, American Mathematical Society, Providence, 1968.
[22] Yu. Luchko, Some uniqueness and existence results for the initial-boundary value problems for the generalized time-fractional diffusion equation, *Comput. Math. Appl.*, **59** (2010), 1766–1772.
[23] F. Mainardi, Fractional relaxation-oscillation and fractional diffusion-wave phenomena, *Chaos Solitons Fractals*, **7** (1996), 1461–1477.
[24] F. Mainardi, Y. Luchko, and G. Pagnini, The fundamental solution of the space-time fractional diffusion equation, *Fract. Calc. Appl. Anal.*, **4**(2) (2001), 153–192.
[25] M. M. Meerschaert, E. Nane, and P. Vellaisamy, Fractional Cauchy problems on bounded domains, *Ann. Probab.*, **37** (2009), 979–1007.
[26] M. M. Meerschaert and H.-P. Scheffler, Limit theorems for continuous-time random walks with infinite mean waiting times, *J. Appl. Probab.*, **41** (2004), 623–638.
[27] A. Mura, M. S. Taqqu, and F. Mainardi, Non-Markovian diffusion equations and processes: analysis and simulations, *Physica A*, **387** (2008), 5033–5064.

[28] R. R. Nigmatullin, The realization of the generalized transfer in a medium with fractal geometry, *Phys. Status Solidi B*, **133** (1986), 425–430.
[29] E. Orsingher and L. Beghin, Fractional diffusion equations and processes with randomly varying time, *Ann. Probab.*, **37** (2009), 206–249.
[30] V. P. Palamodov, *Linear Differential Operators with Constant Coefficients*, Springer, Berlin, 1970.
[31] I. Petrowsky, Über das Cauchysche Problem fur ein System linearer partieller Differentialgleichungen im Gebiete der nichtanalytischen Funktionen, *Bull. Univ. Etat. Moscou, Ser. Int., Sect. A, Math. et Mécan. 1, Fasc.*, **7** (1938), 1–74.
[32] J. Prüss, *Evolutionary Integral Equations and Applications*, Birkhäuser, Basel, 1993.
[33] A. V. Pskhu, The fundamental solution of a diffusion-wave equation of fractional order, *Izv. Math.*, **73** (2009), 351–392.
[34] W. R. Schneider, Grey noise, in *Ideas and Methods in Mathematics and Physics. Memorial Volume Dedicated to Raphael Høegh-Krohn*, vol. 1, pp. 261–282, Cambridge University Press, 1992.
[35] W. R. Schneider and W. Wyss, Fractional diffusion and wave equations, *J. Math. Phys.*, **30** (1989), 134–144.
[36] W. Wyss, The fractional diffusion equation, *J. Math. Phys.*, **27** (1986), 2782–2785.
[37] G. M. Zaslavsky, Fractional kinetic equation for Hamiltonian chaos, *Physica D*, **76** (1994), 110–122.

Rico Zacher

Time fractional diffusion equations: solution concepts, regularity, and long-time behavior

Abstract: In this chapter, we give a survey of results on various analytical aspects of time fractional diffusion equations. We describe the approach via abstract Volterra equations and collect results on strong solutions in the L_p sense. We further discuss the concept of weak solutions for equations with rough coefficients and give an account of recent developments towards a De Giorgi–Nash–Moser theory for such equations. The last part summarizes recent results on the long-time behavior of solutions, which turns out to be significantly different from that in the heat equation case.

Keywords: Time fractional diffusion, weak solution, strong solution, maximal L_p-regularity, Hölder regularity, Harnack inequalities, decay estimates

MSC 2010: 35R11, 35K10, 47G20

1 Introduction

The purpose of this chapter is to give a survey of results on various analytical aspects of time fractional diffusion equations. We discuss different solution concepts such as weak solutions and strong L_p-solutions and give an account of recent developments toward a De Giorgi–Nash–Moser theory for such equations. We also describe some very recent results on the long-time behavior of solutions, which turns out to be markedly different from that in the classical parabolic case.

The prototype of the equations we will look at is given by

$$\partial_t^\alpha (u - u_0) - \Delta u = f, \quad t \in (0,T),\ x \in \Omega. \tag{1}$$

Here, $T > 0$, Ω is a domain in $\mathbb{R}^d$ and $u : [0,T] \times \Omega \to \mathbb{R}$ is the unknown. Further, $\partial_t^\alpha v$ denotes the Riemann–Liouville fractional derivative of order $\alpha \in (0,1)$ w.r.t. time. For (sufficiently smooth) $v : [0,T] \to \mathbb{R}$, it is defined by

$$\partial_t^\alpha v(t) = \partial_t (g_{1-\alpha} * v)(t),$$

where ∂_t stands for the usual derivative, g_β denotes the standard kernel

$$g_\beta(t) = \frac{t^{\beta-1}}{\Gamma(\beta)}, \quad t > 0,\ \beta > 0,$$

Rico Zacher, Institute of Applied Analysis, University of Ulm, 89069 Ulm, Germany, e-mail: rico.zacher@uni-ulm.de

https://doi.org/10.1515/9783110571660-008

and $k * v$ denotes the convolution on the positive half-line $\mathbb{R}_+ := [0,\infty)$, that is, $(k * v)(t) = \int_0^t k(t-\tau)v(\tau)\,d\tau$, $t \geq 0$. The functions u_0 and f are given data; u_0 plays the role of the initial value for u, that is,

$$u|_{t=0} = u_0 \quad \text{in } \Omega. \tag{2}$$

We point out that for sufficiently smooth $v : [0,T] \to \mathbb{R}$,

$$\partial_t^\alpha (v - v(0)) = g_{1-\alpha} * \partial_t v, \tag{3}$$

that is, $\partial_t^\alpha (v - v(0))$ coincides with the Caputo fractional derivative of v of order α. The formulation on the left-hand side of (3) has the advantage that it requires less regularity of v.

Replacing the Laplacian in (1) by a more general elliptic operator of second order (w.r.t. the spatial variables) leads to the class of problems we will refer to as *time fractional diffusion equations*. A considerable part of this chapter will be concerned with the following problem in divergence form:

$$\partial_t^\alpha (u - u_0) - \operatorname{div}\big(A(t,x)\nabla u\big) = f, \quad t \in (0,T),\ x \in \Omega, \tag{4}$$

where the coefficient matrix $A \in L_\infty((0,T) \times \Omega; \mathbb{R}^{d\times d})$ satisfies a uniform parabolicity condition. Here, the main problem consists in proving suitable *a priori* estimates. We will explain how these can be obtained and discuss the corresponding (natural) notion of weak solution.

Let us fix some notation. For a Banach space X, we denote by $\mathcal{B}(X)$ the space of all bounded linear operators from X into X. For an interval $J \subset \mathbb{R}$, $s > 0$, $p \in (1,\infty)$, and a UMD space X, by $H_p^s(J;X)$ and $B_{pp}^s(J;X)$, we mean the vector-valued Bessel potential space respectively Besov space of X-valued functions on J; see, for example, [35, 47]. For $T > 0$ and $s \in (0,1]$, we set ${}_0H_p^s((0,T);X) = \{g_s * h : h \in L_2(J;X)\}$. Note that for $s \in (1/p, 1]$, ${}_0H_p^s((0,T);X) = \{v \in H_p^s((0,T);X) : v(0) = 0\}$; cf. [47].

2 Strong solutions and maximal L_p-regularity

Suppose for the moment that $\Omega = \mathbb{R}^d$. Convolving (1) with the kernel g_α and using the identity $g_\alpha * g_{1-\alpha} = 1$, we obtain

$$u - g_\alpha * \Delta u = u_0 + g_\alpha * f, \quad t \in (0,T),\ x \in \mathbb{R}^d. \tag{5}$$

In fact, for sufficiently smooth u we have

$$g_\alpha * \partial_t\big(g_{1-\alpha} * [u - u_0]\big) = \partial_t\big(g_\alpha * g_{1-\alpha} * [u - u_0]\big) = u - u_0.$$

Equation (5) can be viewed as an abstract Volterra equation. Take as base space, for example, $X = L_q(\mathbb{R}^d)$ with $q \in (1, \infty)$ and define the operator A with domain $D(A) = H_q^2(\mathbb{R}^d)$ by $Av = -\Delta v$, $v \in D(A)$. Setting $h = u_0 + g_\alpha * f$, equation (5) can be reformulated as

$$u(t) + (g_\alpha * Au)(t) = h(t), \quad t \in [0, T], \tag{6}$$

where now u is regarded as an X-valued function of time.

There is a rich theory of abstract Volterra equations that generalizes semigroup theory and applies to our situation, the standard reference being the monograph by Prüss [34]; see also [5–7, 15]. The operator A is a sectorial operator with spectral angle 0 and the kernel g_α is completely monotone and sectorial with angle $\alpha\pi/2$. Since the sum of the two angles is less than π, the equation is parabolic and thus admits a resolvent family $(S(t))_{t\geq 0} \subset \mathcal{B}(X)$, which is the solution operator in case $h = u_0$ (i. e., $u(t) = S(t)u_0$ solves the problem) and which, in case $\alpha = 1$, coincides with the C_0-semigroup generated by $-A$. Depending on the regularity of h, the results from [34] immediately give existence and uniqueness in the classical and mild sense.

Here, we want to consider strong L_p-solutions, that is, we ask for maximal L_p-regularity. Given a Banach space X and a closed linear operator A with domain $D(A) \subset X$, the time fractional evolution equation (with $\alpha \in (0, 1]$)

$$\partial_t^\alpha u(t) + Au(t) = f(t), \quad t \in J := (0, T), \tag{7}$$

is said to have the property of *maximal L_p-regularity*, if for each $f \in L_p(J; X)$ equation (7) possesses a unique solution u in the space ${}_0H_p^\alpha(J; X) \cap L_p(J; D_A)$, that is, both terms on the left-hand side of (7) belong to $L_p(J; X)$; here, D_A denotes the domain of A equipped with the graph norm.

In the case $\alpha = 1$, important contributions on maximal L_p-regularity have been made by Weis [45], who established an operator-valued version of the Mikhlin Fourier multiplier theorem, and by Denk, Hieber, and Prüss [9]. We also refer to the monograph by Prüss and Simonett [35]. The case $\alpha \in (0, 1)$ has been intensively studied by the author [47, 48]; we also refer to [34]. Applying the abstract theory from [47, Theorem 3.4, Theorem 3.6] to the time fractional diffusion equation (1) in the full space $\mathbb{R}^d$ and using results on the interpolation of Sobolev spaces (see, e. g., [40] and [10]), we obtain the following result.

Theorem 2.1. *Let $p, q \in (1, \infty)$ and $\alpha \in (\frac{1}{p}, 1)$. Then the problem* (1), (2) *with $\Omega = \mathbb{R}^d$ admits a unique solution*

$$u \in Z := H_p^\alpha(J; L_q(\mathbb{R}^d)) \cap L_p(J; H_q^2(\mathbb{R}^d)),$$

if and only if $f \in L_p(J; L_q(\mathbb{R}^d))$ and $u_0 \in B_{qp}^{2-\frac{2}{p\alpha}}(\mathbb{R}^d)$. Furthermore, we have the continuous embedding

$$Z \hookrightarrow C([0, T]; B_{qp}^{2-\frac{2}{p\alpha}}(\mathbb{R}^d)).$$

This result extends to second-order elliptic operators in nondivergence form under suitable regularity assumptions on the coefficients like continuity of the top order coeffcients; the case $q = p$ can be found in [48]. Note that the condition $\alpha > 1/p$ ensures that functions $u \in Z$ have a time trace with the Besov space $B_{qp}^{2-\frac{2}{p\alpha}}(\mathbb{R}^d)$ being the natural trace space. We remark that $L_p(L_q)$-estimates for time fractional diffusion equations in $\mathbb{R}^d$ have also been proved recently in [22] by PDE methods.

In the case $q = p$, there also exist corresponding results for problems on domains with nonhomogenous boundary conditions, see [48]. As an example, we formulate such a result for a Dirichlet boundary condition, that is, we consider the problem

$$\begin{cases} \partial_t^\alpha(u - u_0) - \Delta u = f, & t \in (0,T),\ x \in \Omega \\ u|_{\partial\Omega} = g, & t \in (0,T),\ x \in \partial\Omega \\ u|_{t=0} = u_0, & x \in \Omega. \end{cases} \tag{8}$$

Theorem 2.2. *Let $\Omega \subset \mathbb{R}^d$ $(d \geq 2)$ be a domain with compact C^2-boundary $\partial\Omega$. Let $J = (0,T)$, $p \in (1,\infty)$ and assume that $\alpha \in (\frac{1}{p},1)\setminus\{\frac{2}{2p-1}\}$. Then (8) possesses a unique solution u in the space $H_p^\alpha(J;L_p(\Omega)) \cap L_p(J;H_p^2(\Omega))$ if and only if the functions f, g, u_0 are subject to the following conditions:*

$$\text{(i)}\quad f \in L_p(J;L_p(\Omega)),\quad \text{(ii)}\quad g \in B_{pp}^{\alpha(1-\frac{1}{2p})}(J;L_p(\partial\Omega)) \cap L_p(J;B_{pp}^{2-\frac{1}{p}}(\partial\Omega)),$$

$$\text{(iii)}\quad u_0 \in B_{pp}^{2-\frac{2}{p\alpha}}(\Omega),\quad \text{(iv)}\quad g|_{t=0} = u_0|_{\partial\Omega}\quad \text{if } \alpha > \frac{2}{2p-1}.$$

To prove this theorem, one can use the localization method and perturbation arguments to reduce the problem to related problems on the full space $\mathbb{R}^d$ and the half-space $\mathbb{R}_+^d = \{(x',y) \in \mathbb{R}^d : x' \in \mathbb{R}^{d-1}, y > 0\}$. These problems in turn can then be treated by means of operator theoretic methods; see [48].

3 Weak solutions in the Hilbert space setting

We turn now to weak solutions. Let $T > 0$ and Ω be a bounded domain in $\mathbb{R}^d$. We consider the problem

$$\begin{cases} \partial_t^\alpha(u - u_0) - \operatorname{div}(A\,\nabla u) + cu = f, & t \in (0,T),\ x \in \Omega \\ u|_{\partial\Omega} = 0, & t \in (0,T),\ x \in \partial\Omega \\ u|_{t=0} = u_0, & x \in \Omega. \end{cases} \tag{9}$$

The coefficients and data are supposed to satisfy the following assumptions:

(Hd) $u_0 \in L_2(\Omega), f \in L_2((0,T);L_2(\Omega)), c \in L_\infty((0,T)\times\Omega)$.
(HA) $A \in L_\infty((0,T)\times\Omega;\mathbb{R}^{d\times d})$, and there exists a $\nu > 0$ such that

$$(A(t,x)\xi|\xi) \geq \nu|\xi|^2, \quad \text{for a.a. } t \in (0,T),\ x \in \Omega, \text{ and all } \xi \in \mathbb{R}^d.$$

Here, $(\cdot|\cdot)$ denotes the standard scalar product in $\mathbb{R}^d$.

In what follows, we denote by y_+ and $y_- := [-y]_+$ the positive and negative part, respectively, of $y \in \mathbb{R}$. We say that u is a *weak solution (subsolution, supersolution)* of (9) if

(a) $u \in W := \{w \in L_2((0,T);H_2^1(\Omega)) :\ g_{1-\alpha} * w \in C([0,T];L_2(\Omega))$ and $(g_{1-\alpha} * w)|_{t=0} = 0\}$,
(b) $u\,(u_+,\ u_-) \in L_2((0,T);\ \overset{\circ}{H}{}_2^1(\Omega))$, where $\overset{\circ}{H}{}_2^1(\Omega) := \overline{C_0^\infty(\Omega)}^{H_2^1(\Omega)}$,
(c) for any nonnegative test function

$$\eta \in H_2^1((0,T);L_2(\Omega)) \cap L_2((0,T);\ \overset{\circ}{H}{}_2^1(\Omega))$$

with $\eta|_{t=T} = 0$ there holds

$$\int_0^T \int_\Omega \big(-\eta_t(g_{1-\alpha} * [u-u_0]) + (A\nabla u|\nabla\eta) + cu\eta\big)\,dx\,dt = (\leq, \geq) \int_0^T \int_\Omega f\eta\,dx\,dt.$$

The following theorem is due to the author; see [50, Section 4]. Here, the symbol $L_{p,\infty}$ refers to the weak L_p space and $H_2^{-1}(\Omega)$ denotes the dual space of $\overset{\circ}{H}{}_2^1(\Omega)$.

Theorem 3.1. *Let $T > 0$ and Ω be a bounded domain in $\mathbb{R}^d$. Let $\alpha \in (0,1)$ and assume that* (Hd) *and* (HA) *hold. Then the problem* (9) *has a unique weak solution $u \in W$ and*

$$|g_{1-\alpha} * u|_{C([0,T];L_2(\Omega))} + |u|_{L_2((0,T);H_2^1(\Omega))} \leq C\big(|u_0|_{L_2(\Omega)} + |f|_{L_2((0,T);L_2(\Omega))}\big),$$

where the constant C is independent of u, u_0, and f. Moreover, we have

$$u \in L_{\frac{2}{1-\alpha},\infty}((0,T);L_2(\Omega)) \quad \text{and} \quad u - u_0 \in {}_0H_2^\alpha((0,T);H_2^{-1}(\Omega)). \tag{10}$$

Note that $u \in W$ does not entail $u \in C([0,T];L_2(\Omega))$ in general, so it is not so clear how to interpret the initial condition. However, once one knows that the solution u is sufficiently smooth (e. g., if $\alpha > \frac{1}{2}$), then $u|_{t=0} = u_0$ is satisfied in an appropriate sense (see [50]). We also point out that the statement of Theorem 3.1 remains true, if we only assume that $f \in L_2((0,T);H_2^{-1}(\Omega))$; the integral $\int_\Omega f\eta\,dx$ in the weak formulation above then has to be replaced by the duality pairing $\langle f,\eta\rangle$ between $H_2^{-1}(\Omega)$ and $\overset{\circ}{H}{}_2^1(\Omega)$.

The first statement in (10) follows from considerations for more general problems (cf. [50]) and can be slightly improved in the time fractional case. In fact, the solution u even enjoys the property

$$u \in L_{\frac{2}{1-\alpha}}((0,T);L_2(\Omega)), \tag{11}$$

which is also in accordance with the estimates in [2] for weak solutions of bifractional porous medium equations; see also [1]. To see (11), we use Theorem 3.1, cross interpolation (see, e. g., the mixed derivative theorem in [39]) and Sobolev embedding, thereby obtaining that

$$\begin{aligned} u &\in H_2^\alpha((0,T);H_2^{-1}(\Omega)) \cap L_2((0,T);H_2^1(\Omega)) \\ &\hookrightarrow H_2^{\frac{\alpha}{2}}((0,T);L_2(\Omega)) \hookrightarrow L_{\frac{2}{1-\alpha}}((0,T);L_2(\Omega)). \end{aligned}$$

Theorem 3.1 follows from a rather general result on weak solutions for abstract evolutionary integrodifferential equations in Hilbert spaces (see [50, Theorem 3.1]), which is the nonlocal in time analogue of the classical result on weak solutions for abstract parabolic equations given via a bounded and coercive bilinear form; cf. for example, Theorem 4.1 and Remark 4.3 in Chapter 4 in Lions and Magenes [27] or Zeidler [55, Section 23]. The theory from [50] covers a wide range of nonlocal in time subdiffusion problems, including also problems with sums of fractional derivatives and ultra-slow diffusion equations (cf. [24]) and with other boundary conditions like a Neumann boundary condition.

The proof of Theorem 3.1 in [50] is based on the Galerkin method and suitable *a priori* estimates, which can be derived by means of a basic identity for integrodifferential operators B of the form $Bv = \partial_t(k * v)$. Before explaining several versions of this so-called *fundamental identity* we collect some further basic results on (9).

The first is the weak maximum principle for (9) with $f = 0$. It is contained in [49, Theorem 3.2], which also covers the case of nonhomogenous boundary data and more general subdiffusion equations. Its proof relies on the fundamental identity described in the next section.

Theorem 3.2. *Let $T > 0$ and Ω be a bounded domain in $\mathbb{R}^d$. Let $\alpha \in (0,1)$ and assume that* (Hd) *and* (HA) *are satisfied. Assume further that $f = 0$ and $c \geq 0$. Then for any weak subsolution* (*supersolution*) *u of* (9) *there holds for a.a. $(t,x) \in (0,T) \times \Omega$,*

$$u(t,x) \leq \max\Big\{0, \operatorname*{ess\,sup}_{\Omega} u_0\Big\} \quad \Big(u(t,x) \geq \min\Big\{0, \operatorname*{ess\,inf}_{\Omega} u_0\Big\}\Big),$$

provided this maximum (*minimum*) *is finite.*

Results on the maximum principle in a stronger setting have also been found in [28, 29] by different methods.

The next result provides the comparison principle for (9). It is a special case of [44, Theorem 3.3].

Theorem 3.3. *Let $T > 0$ and Ω be a bounded domain in $\mathbb{R}^d$. Let $\alpha \in (0,1)$ and assume that* (Hd) *and* (HA) *are satisfied. Suppose that $u \in W$ is a weak subsolution of* (9) *and that $v \in W$ is a weak supersolution of* (9)*. Then $u \leq v$ a.e. in $(0,T) \times \Omega$.*

The comparison principle for (9) has also been proved in [30] under much stronger assumptions; for example, in [30], the coefficient matrix A may only depend on x and has to be symmetric as well as $C^1(\bar{\Omega})$-smooth.

We remark that weak solutions for time fractional diffusion equations with nonhomogenous Dirichlet boundary condition have been studied recently in [46].

4 The fundamental identity

An important tool for deriving *a priori* estimates for time fractional diffusion equations (in particular in the weak setting) is the so-called *fundamental identity* for integro-differential operators of the form $\partial_t(k * \cdot)$; see, for example, [53]. It can be viewed as the analogue to the chain rule $(H(u))' = H'(u)u'$. The time derivative of k (also in the generalized sense) will be denoted by $\dot{k}$.

Lemma 4.1. *Let $T > 0$, $J = (0,T)$ and U be an open subset of $\mathbb{R}$. Let further $k \in H_1^1(J)$, $H \in C^1(U)$, and $u \in L_1(J)$ with $u(t) \in U$ for a.a. $t \in J$. Suppose that the functions $H(u)$, $H'(u)u$, and $H'(u)(\dot{k} * u)$ belong to $L_1(J)$ (which is the case if, e.g., $u \in L_\infty(J)$). Then we have for a.a. $t \in J$,*

$$\begin{aligned} &H'(u(t))\partial_t(k * u)(t) \\ &\quad = \partial_t(k * H(u))(t) + \big(-H(u(t)) + H'(u(t))u(t)\big)k(t) \\ &\qquad + \int_0^t \big(H(u(t-s)) - H(u(t)) - H'(u(t))[u(t-s) - u(t)]\big)[-\dot{k}(s)]\,ds. \end{aligned} \tag{12}$$

The proof is a straightforward computation. We remark that (12) remains valid for singular kernels k, like, for example, $k = g_{1-\alpha}$ with $\alpha \in (0,1)$, provided that u is sufficiently smooth. Recalling that $\partial_t^\alpha u = \partial_t(g_{1-\alpha} * u)$, in the latter case (12) thus applies to the Riemann–Liouville fractional derivative.

In the weak setting, a key idea is to reformulate the problem in such a way that the fractional derivative is replaced by its *Yosida approximations*, which take the form $B_n u = \partial_t(g_{1-\alpha,n} * u)$, $n \in \mathbb{N}$, where $g_{1-\alpha,n} = ns_n$ (see Section 6 below for the definition of s_n) is nonnegative, nonincreasing, and belongs to $H_1^1(J)$ for each $T > 0$. We refer to [42] for the computation of the Yosida approximation and to [49] for the derivation of (in time) regularized weak formulations.

A direct consequence of the fundamental identity is the following *convexity inequality* (cf. [19]), which is in particular very useful when dealing with fractional derivatives in the Caputo sense.

Corollary 4.1. *Let T, J, U, k, H, and u be as in Lemma 4.1. Let $u_0 \in \mathbb{R}$, and assume in addition that k is nonnegative and nonincreasing and that H is convex. Then*

$$H'(u(t))\partial_t(k * [u - u_0])(t) \geq \partial_t(k * [H(u) - H(u_0)])(t), \quad a.a.\ t \in J. \tag{13}$$

Proof. By the fundamental identity, convexity of H, and the properties of k, we have for a.a. $t \in J$,

$$
\begin{aligned}
&H'(u(t))\partial_t(k * [u - u_0])(t) \\
&\quad = H'(u(t))\partial_t(k * u)(t) - H'(u(t))u_0 k(t) \\
&\quad \ge \partial_t(k * H(u))(t) + (-H(u(t)) + H'(u(t))[u(t) - u_0])k(t) \\
&\quad \ge \partial_t(k * H(u))(t) - H(u_0)k(t) \\
&\quad = \partial_t(k * [H(u) - H(u_0)])(t),
\end{aligned}
$$

which shows the asserted inequality. □

Another important consequence of Lemma 4.1 is the so-called *L_p-norm inequality* for operators of the form $\partial_t(k * \cdot)$, which has been established in Vergara and Zacher [43]. In the special case $p = 2$, it states the following.

Theorem 4.1. *Let $T > 0$ and $\Omega \subset \mathbb{R}^d$ be an open set. Let $k \in H^1_{1,\mathrm{loc}}(\mathbb{R}_+)$ be nonnegative and nonincreasing. Then for any $v \in L_2((0,T) \times \Omega)$ and any $v_0 \in L_2(\Omega)$ we have for a.a. $t \in (0,T)$*

$$
\int_\Omega v\partial_t(k * [v - v_0])\,dx \ge |v(t)|_{L_2(\Omega)}\partial_t(k * [|v|_{L_2(\Omega)} - |v_0|_{L_2(\Omega)}])(t). \tag{14}
$$

Proof. The following argument is simpler than that in the more general case considered in [43].

By the fundamental identity, applied twice (!), Fubini's theorem, and the triangle inequality for the $L_2(\Omega)$-norm we have for a.a. $t \in (0,T)$:

$$
\begin{aligned}
\int_\Omega v\partial_t(k * v)\,dx &= \int_\Omega \Big(\frac{1}{2}\partial_t(k * v^2) + \frac{1}{2}k(t)v^2\Big)\,dx \\
&\quad + \int_0^t \int_\Omega |v(t,x) - v(t-s,x)|^2\,dx\,[-\dot{k}(s)]\,ds \\
&\ge \frac{1}{2}\partial_t(k * |v|^2_{L_2(\Omega)}) + \frac{1}{2}k(t)|v(t)|^2_{L_2(\Omega)} \\
&\quad + \frac{1}{2}\int_0^t \big(|v(t)|_{L_2(\Omega)} - |v(t-s)|_{L_2(\Omega)}\big)^2[-\dot{k}(s)]\,ds \\
&= |v(t)|_{L_2(\Omega)}\partial_t(k * |v|_{L_2(\Omega)})(t).
\end{aligned}
$$

From this and Hölder's inequality, we infer that for a.a. $t \in (0,T)$,

$$
\int_\Omega v\partial_t(k * [v - v_0])\,dx = \int_\Omega v\partial_t(k * v)\,dx - k(t)\int_\Omega vv_0\,dx
$$

$$\geq |v(t)|_2 \partial_t(k * |v|_2)(t) - k(t)|v(t)|_2|v_0|_2$$
$$= |v(t)|_2 \partial_t(k * [|v|_2 - |v_0|_2])(t).$$

This proves the theorem. □

As in the case of the fundamental identity, Corollary 4.1 and Theorem 4.1 extend to singular kernels k including $k = g_{1-\alpha}$ for sufficiently smooth functions.

The following identity is basic to energy estimates in the Hilbert space setting. For $\mathcal{H} = \mathbb{R}$, it coincides with (12) with $H(y) = \frac{1}{2}y^2$, $y \in \mathbb{R}$; see [42].

Lemma 4.2. *Let $\mathcal{H}$ be a real Hilbert space with scalar product $(\cdot,\cdot)_{\mathcal{H}}$ and $T > 0$. Then for any $k \in H_1^1((0,T))$ and any $v \in L_2((0,T);\mathcal{H})$ there holds*

$$\begin{aligned}(\partial_t(k * v)(t), v(t))_{\mathcal{H}} &= \frac{1}{2}\,\partial_t(k * |v(\cdot)|^2_{\mathcal{H}})(t) + \frac{1}{2}\,k(t)|v(t)|^2_{\mathcal{H}} \\ &\quad + \frac{1}{2}\int_0^t [-\dot{k}(s)]\,|v(t) - v(t-s)|^2_{\mathcal{H}}\,ds, \quad a.a.\ t \in (0,T). \end{aligned} \tag{15}$$

5 De Giorgi–Nash–Moser estimates

We consider again the time fractional diffusion equation from (9) and set $c = 0$ for simplicity, that is we look at

$$\partial_t^\alpha(u - u_0) - \operatorname{div}\big(A(t,x)\nabla u\big) = f, \quad t \in (0,T),\ x \in \Omega. \tag{16}$$

As before, the coefficient matrix A is merely assumed to satisfy condition (HA). That is, we do not assume any regularity on the coefficients; in this situation, one also speaks of *rough coefficients*.

In the elliptic and in the classical parabolic case (i. e., in the case $\alpha = 1$), there is a powerful theory of *a priori* estimates, often referred to as *De Giorgi–Nash–Moser theory*, which provides local and global estimates for weak solutions of the respective equations such as local and global boundedness, Harnack and weak Harnack inequalities as well as Hölder continuity of weak solutions; see [14, 17] for the elliptic and [25, 26] for the parabolic case. Hölder estimates are of utmost importance for the study of quasilinear problems. In fact, in the elliptic case their discovery opened up the theory of quasilinear equations in higher dimensions; in the parabolic case they allow to prove global in time existence.

Since the time fractional case with $\alpha \in (0,1)$ can be viewed in some sense as an intermediate case between the elliptic ($\alpha = 0$) and the classical parabolic case ($\alpha = 1$), one might think that corresponding results can also be obtained in the time fractional situation. However, there is a significant difference to the cases $\alpha = 0$ and $\alpha = 1$: the time fractional equations are *nonlocal* in time, due to the nonlocal nature of the operator ∂_t^α. This feature complicates the matter considerably, as the theory described above heavily relies on *local* estimates. Another difficulty consists in the lack of a simple calculus for integrodifferential operators like ∂_t^α. Instead of the simple chain rule for the usual derivative, one has to employ the fundamental identity from the previous section in order to use the test-function method, the latter being the basic tool for deriving *a priori* bounds for weak solutions of equations in divergence form.

In the following, we will describe the main results of the De Giorgi–Nash–Moser theory for (16), which has been developed by the author; see [49, 52–54], see also [1, 2] for the fully nonlocal case.

Throughout this section, we will assume that $\alpha \in (0,1)$, $u_0 \in L_2(\Omega)$ and that the function f satisfies the following condition:
(Hf) $f \in L_r((0,T);L_q(\Omega))$, where $r,q \geq 1$ fulfill

$$\frac{1}{\alpha r} + \frac{d}{2q} = 1 - \kappa,$$

and

$$r \in \left[\frac{1}{\alpha(1-\kappa)}, \infty\right], \quad q \in \left[\frac{d}{2(1-\kappa)}, \infty\right], \quad \kappa \in (0,1) \quad \text{for } d \geq 2,$$
$$r \in \left[\frac{1}{\alpha(1-\kappa)}, \frac{2}{\alpha(1-2\kappa)}\right], \quad q \in [1,\infty], \quad \kappa \in \left(0,\frac{1}{2}\right) \quad \text{for } d = 1.$$

We say that a function u is a *weak solution (subsolution, supersolution)* of (16) in $(0,T)\times\Omega$, if u belongs to the space

$$W_\alpha := \{v \in L_{\frac{2}{1-\alpha},\infty}((0,T);L_2(\Omega)) \cap L_2((0,T);H_2^1(\Omega)) \text{ such that}$$
$$g_{1-\alpha} * v \in C([0,T];L_2(\Omega)), \text{ and } (g_{1-\alpha} * v)|_{t=0} = 0\},$$

and for any nonnegative test function

$$\eta \in H_2^1((0,T);L_2(\Omega)) \cap L_2((0,T);\mathring{H}_2^1(\Omega))$$

with $\eta|_{t=T} = 0$ there holds

$$\int_0^T\int_\Omega (-\eta_t(g_{1-\alpha} * [u-u_0]) + (A\nabla u|\nabla\eta))\,dx\,dt = (\leq, \geq) \int_0^T\int_\Omega f\eta\,dx\,dt.$$

We point out that here (16) is considered without any boundary conditions. In this sense, weak solutions of (16) as defined just before are *local* ones (w.r.t. x). Note that this weak formulation is consistent with the one from Section 3, in view of Theorem 3.1.

Before stating the first result on global boundedness, we need some preliminaries.

The boundary $\partial\Omega$ of a bounded domain $\Omega \subset \mathbb{R}^d$ is said to satisfy the property of *positive geometric density*, if there exist $\beta \in (0,1)$ and $\rho_0 > 0$ such that for any $x_0 \in \partial\Omega$, any open ball $B(x_0,\rho)$ with $\rho \in (0,\rho_0]$ we have that $\lambda_d(\Omega \cap B(x_0,\rho)) \leq \beta\lambda_d(B(x_0,\rho))$, where λ_d denotes the Lebesgue measure in $\mathbb{R}^d$; cf., for example, [11, Section I.1].

In what follows, we say that a function $u \in W_\alpha$ satisfies $u \leq K$ a.e. on $(0,T) \times \partial\Omega$ for some number $K \in \mathbb{R}$ if $(u-K)_+ \in L_2((0,T); \mathring{H}_2^1(\Omega))$, likewise for lower bounds on $(0,T) \times \partial\Omega$.

The subsequent result provides sup-bounds for weak subsolutions. The proof given in [49] uses De Giorgi's iteration technique.

Theorem 5.1. *Let $T > 0$ and $\Omega \subset \mathbb{R}^d$ be a bounded domain satisfying the property of positive geometric density. Let further $\alpha \in (0,1)$, $u_0 \in L_2(\Omega)$ and assume that the conditions* (HA) *and* (Hf) *are satisfied. Suppose $K \geq 0$ is such that $u_0 \leq K$ a.e. in Ω. Then there exists a constant $C = C(\alpha,q,r,T,d,\nu,\Omega,f)$ such that for any weak subsolution $u \in W_\alpha$ of* (16) *in $(0,T) \times \Omega$ satisfying $u \leq K$ a.e. on $(0,T) \times \partial\Omega$ there holds $u \leq C(1+K)$ a.e. in $(0,T) \times \Omega$.*

There is a corresponding result for weak supersolutions u of (16) in the situation where $u_0 \geq K$ a.e. in Ω and $u \geq K$ a.e. on $(0,T) \times \partial\Omega$, for some $K \leq 0$. This follows immediately from Theorem 5.1 by replacing u with $-u$, and u_0 with $-u_0$. As shown in [49], Theorem 5.1 extends to a wide class of subdiffusion equations.

As an immediate consequence of Theorem 5.1 and the remark following, it we obtain the global boundedness of weak solutions of (16) that are bounded on the parabolic boundary of $(0,T) \times \Omega$.

Corollary 5.1. *Let $T > 0$ and $\Omega \subset \mathbb{R}^d$ be a bounded domain satisfying the property of positive geometric density. Let further $\alpha \in (0,1)$, $u_0 \in L_\infty(\Omega)$ and assume that the conditions* (HA) *and* (Hf) *are satisfied. Suppose $K \geq 0$ is such that $|u_0| \leq K$ a.e. in Ω. Then there exists a constant $C = C(\alpha,q,r,T,d,\nu,\Omega,f)$ such that for any weak solution $u \in W_\alpha$ of* (16) *in $(0,T)\times\Omega$ which satisfies $|u| \leq K$ a.e. on $(0,T)\times\partial\Omega$ we have $|u| \leq C(1+K)$ a.e. in $(0,T) \times \Omega$.*

We turn now to Hölder regularity of bounded weak solutions. For $\beta_1,\beta_2 \in (0,1)$ and $Q \subset (0,T) \times \Omega$, we set

$$[u]_{C^{\beta_1,\beta_2}(Q)} := \sup_{(t,x),(s,y)\in Q,\,(t,x)\neq(s,y)} \left\{ \frac{|u(t,x)-u(s,y)|}{|t-s|^{\beta_1} + |x-y|^{\beta_2}} \right\}.$$

The main regularity theorem reads as follows; see Zacher [53].

Theorem 5.2. *Let $\alpha \in (0,1)$, $T > 0$ and Ω be a bounded domain in $\mathbb{R}^d$. Let $u_0 \in L_\infty(\Omega)$ and suppose that the assumptions* (HA) *and* (Hf) *are satisfied. Let $u \in W_\alpha$ be a bounded weak solution of* (16) *in $(0,T) \times \Omega$. Then there holds for any $Q \subset (0,T) \times \Omega$ separated from the parabolic boundary $(\{0\} \times \Omega) \cup ((0,T) \times \partial\Omega)$ by a positive distance D,*

$$[u]_{C^{\frac{\alpha\epsilon}{2},\epsilon}(\bar{Q})} \leq C\big(|u|_{L_\infty((0,T)\times\Omega)} + |u_0|_{L_\infty(\Omega)} + |f|_{L_r((0,T);L_q(\Omega))}\big)$$

with positive constants $\epsilon = \epsilon(|A|_\infty, \nu, \alpha, r, q, d, \operatorname{diam}\Omega, \inf_{(\tau,z)\in Q}\tau)$ *and* $C = C(|A|_\infty, \nu, \alpha, r, q, d, \operatorname{diam}\Omega, \lambda_{d+1}(Q), D)$.

Theorem 5.2 gives an interior Hölder estimate for bounded weak solutions of (16) in terms of the data and the L_∞-bound of the solution. It can be viewed as the time fractional analogue of the classical parabolic version ($\alpha = 1$) of the celebrated De Giorgi–Nash theorem on the Hölder continuity of weak solutions to elliptic equations in divergence form (De Giorgi [8], Nash [33]); see also [14] for the elliptic, and [25] as well as the seminal contribution by Moser [31] for the parabolic case.

The proof of Theorem 5.2 is quite involved. It uses De Giorgi's technique and the method of *nonlocal growth lemmas*, which has been developed in [38] for integrodifferential operators like the fractional Laplacian. The fundamental identity is frequently used to derive various a priori estimates for u and certain logarithmic expressions involving u.

The following result gives conditions on the data which are sufficient for Hölder continuity up to the parabolic boundary of $(0, T) \times \Omega$. It has been taken from [52].

Theorem 5.3. *Let* $\alpha \in (0, 1)$, $T > 0$, $d \geq 2$, *and* $\Omega \subset \mathbb{R}^d$ *be a bounded domain with* C^2*-smooth boundary* $\partial\Omega$. *Let the assumptions* (HA) *and* (Hf) *be satisfied. Suppose further that*

$$u_0 \in B_{pp}^{2-\frac{2}{p\alpha}}(\Omega), \quad g \in Y := B_{pp}^{\alpha(1-\frac{1}{2p})}\big((0,T); L_p(\partial\Omega)\big) \cap L_p\big((0,T); B_{pp}^{2-\frac{1}{p}}(\partial\Omega)\big)$$

for some $p > \frac{1}{\alpha} + \frac{d}{2}$, *and that the compatibility condition*

$$u_0|_{\partial\Omega} = g|_{t=0} \quad \text{on } \partial\Omega$$

is satisfied. Then for any bounded weak solution u *of* (16) *in* $(0, T) \times \Omega$ *such that* $u = g$ *a.e. on* $(0, T) \times \partial\Omega$, *there holds*

$$[u]_{C^{\frac{\alpha\epsilon}{2},\epsilon}([0,T]\times\overline{\Omega})} \leq C\big(|u|_{L_\infty((0,T)\times\Omega)} + |u_0|_{B_{pp}^{2-\frac{2}{p\alpha}}(\Omega)} + |f|_{L_r((0,T);L_q(\Omega))} + |g|_Y\big), \tag{17}$$

where $\epsilon = \epsilon(|A|_\infty, \nu, \alpha, p, r, q, d, \Omega)$ *and* $C = C(|A|_\infty, \nu, \alpha, p, r, q, d, \Omega, T)$ *are positive constants.*

The proof uses Theorem 5.2 and extension techniques both in space and time, together with the maximal regularity result Theorem 2.2. This explains the regularity required for the initial and boundary data.

The regularity condition (Hf) imposed on the right-hand side f in Theorem 5.2 and Theorem 5.3 cannot be weakened significantly. In fact, given $f \in L_r((0, T); L_q(\Omega))$ the best possible regularity for the solution u (in general) is that of maximal $L_r((0, T); L_q(\Omega))$-regularity, that is,

$$u \in H_r^\alpha((0, T); L_q(\Omega)) \cap L_r((0, T); H_q^2(\Omega)).$$

By the mixed derivative theorem (cf. [39]), we have

$$u \in H_r^{\alpha(1-\zeta)}((0,T); H_q^{2\zeta}(\Omega)) \quad \text{for all } \zeta \in [0,1].$$

Now observe that the condition

$$\frac{1}{\alpha r} + \frac{d}{2q} < 1$$

from (Hf) just ensures the existence of a $\zeta \in (0,1)$ such that $\alpha(1-\zeta)-\frac{1}{r} > \delta$ and $2\zeta - \frac{d}{q} > \delta$ for some $\delta > 0$, which implies Hölder continuity of u by Sobolev embedding.

Another important result in the De Giorgi–Nash–Moser theory for time fractional diffusion equations is the *weak Harnack inequality* due to Zacher [54]. To formulate the result, recall that $B(x,r)$ denotes the open ball with radius $r > 0$ centered at $x \in \mathbb{R}^d$, and λ_d stands for the Lebesgue measure in $\mathbb{R}^d$. For $\delta \in (0,1)$, $t_0 \geq 0$, $\tau > 0$, and a ball $B(x_0, r)$, we define the boxes:

$$\begin{aligned} Q_-(t_0, x_0, r) &= (t_0, t_0 + \delta\tau r^{2/\alpha}) \times B(x_0, \delta r), \\ Q_+(t_0, x_0, r) &= (t_0 + (2-\delta)\tau r^{2/\alpha}, t_0 + 2\tau r^{2/\alpha}) \times B(x_0, \delta r). \end{aligned}$$

We have now the following result for the equation:

$$\partial_t^\alpha (u - u_0) - \operatorname{div}(A(t,x)\nabla u) = 0, \quad t \in (0,T),\ x \in \Omega. \tag{18}$$

Theorem 5.4. *Let $\alpha \in (0,1)$, $T > 0$, and $\Omega \subset \mathbb{R}^d$ be a bounded domain. Let $u_0 \in L_2(\Omega)$ and suppose that the assumption* (HA) *is satisfied. Let further $\delta \in (0,1)$, $\eta > 1$, and $\tau > 0$ be fixed. Then for any $t_0 \geq 0$ and $r > 0$ with $t_0 + 2\tau r^{2/\alpha} \leq T$, any ball $B(x_0, \eta r) \subset \Omega$, any $0 < p < \frac{2+d\alpha}{2+d\alpha-2\alpha}$, and any nonnegative weak supersolution u of* (18) *in* $(0, t_0 + 2\tau r^{2/\alpha}) \times B(x_0, \eta r)$ *with $u_0 \geq 0$ in $B(x_0, \eta r)$, there holds*

$$\left(\frac{1}{\lambda_{d+1}(Q_-(t_0,x_0,r))} \int_{Q_-(t_0,x_0,r)} u^p \, d\lambda_{d+1} \right)^{1/p} \leq C \operatorname*{ess\,inf}_{Q_+(t_0,x_0,r)} u, \tag{19}$$

where the constant $C = C(\nu, |A|_\infty, \delta, \tau, \eta, \alpha, d, p)$.

Theorem 5.4 says that nonnegative weak supersolutions of (18) with $u_0 \geq 0$ satisfy a weak form of the Harnack inequality in the sense that we do not have an estimate for the supremum of u on $Q_-(t_0, x_0, r)$ but only an L_p estimate. It is also shown in [54] that the critical exponent $\frac{2+d\alpha}{2+d\alpha-2\alpha}$ is optimal, that is, the inequality in general fails to hold for $p \geq \frac{2+d\alpha}{2+d\alpha-2\alpha}$.

Theorem 5.4 can be regarded as the time fractional analogue of the corresponding result in the classical parabolic case $\alpha = 1$; see, for example, [26, Theorem 6.18] and [41]. Sending $\alpha \to 1$, the critical exponent tends to $1 + 2/d$, which coincides with the well-known critical exponent for the heat equation. As pointed out in [54], the statement of Theorem 5.4 remains valid for (appropriately defined) weak supersolutions of (18) with $u_0 \geq 0$ on $(t_0, t_0 + 2\tau r^{2/\alpha}) \times B(x_0, \eta r)$ which are nonnegative on $(0, t_0 + 2\tau r^{2/\alpha}) \times B(x_0, \eta r)$. We also remark that the global positivity assumption cannot be replaced by a local one, as simple examples show; cf. [51]. This significant difference to the case $\alpha = 1$ is due to the non-local nature of ∂_t^α. The same phenomenon is known for integrodifferential operators like $(-\Delta)^\alpha$ with $\alpha \in (0, 1)$; see, for example, [18].

The proof of Theorem 5.4 relies on suitable *a priori* estimates for powers of u and logarithmic estimates, which are derived by means of the fundamental identity for the regularized fractional derivative. It further uses Moser's iteration technique and an elementary but subtle lemma of Bombieri and Giusti [4] (see also [36, Lemma 2.2.6]) which allows to avoid the rather technically involved approach via *BMO*-functions.

From the weak Harnack inequality, one can easily derive the strong maximum principle for weak subsolutions of (18); see [54, Theorem 5.1].

Theorem 5.5. *Let* $\alpha \in (0, 1)$, $T > 0$, *and* $\Omega \subset \mathbb{R}^d$ *be a bounded domain. Let* $u_0 \in L_2(\Omega)$ *and suppose that the assumption* (HA) *is satisfied. Let* $u \in W_\alpha$ *be a weak subsolution of* (18) *in* $(0, T) \times \Omega$ *and assume that* $0 \leq \operatorname{ess\,sup}_{(0,T)\times\Omega} u < \infty$ *and that* $\operatorname{ess\,sup}_\Omega u_0 \leq \operatorname{ess\,sup}_{(0,T)\times\Omega} u$. *Then, if for some cylinder* $Q = (t_0, t_0 + \tau r^{2/\alpha}) \times B(x_0, r) \subset (0, T) \times \Omega$ *with* $t_0, \tau, r > 0$ *and* $\overline{B(x_0, r)} \subset \Omega$ *we have*

$$\operatorname*{ess\,sup}_{Q} u = \operatorname*{ess\,sup}_{(0,T)\times\Omega} u, \tag{20}$$

the function u is constant on $(0, t_0) \times \Omega$.

It is an interesting problem, whether nonnegative weak solutions of (18) with $u_0 \geq 0$, satisfy the (full) Harnack inequality. The latter means that (19) holds with $p = \infty$, that is, the term on the left is replaced by $\operatorname{ess\,sup}_{Q_-} u$. Very recently, the author and coauthors [12] observed that in contrast to the classical case $\alpha = 1$, the full Harnack inequality (in the form described before) fails to hold in general in the time fractional case if the space dimension d is at least 2, even in the case where the elliptic operator is the Laplacian. A corresponding counterexample can be found in [12]. Its construction uses the fact that for $d \geq 2$ the fundamental solution $Z(t, x)$ of the equation has a singularity at $x = 0$ for all $t > 0$. The one-dimensional case is still an open problem. It is conjectured that the Harnack inequality is true in this case. The author could show that the Harnack inequality holds in the purely time dependent case "$d = 0$", that is, in the case without elliptic operator; see [51].

We conclude this section by illustrating the strength of the described regularity results. Theorem 5.3 provides the key estimate to prove the global strong solvability of the following *quasilinear* time fractional diffusion problem

$$\begin{cases} \partial_t^\alpha(u-u_0) - \operatorname{div}(A(u)\,\nabla u) = f, & t \in (0,T),\ x \in \Omega \\ u|_{\partial\Omega} = g, & t \in (0,T),\ x \in \partial\Omega \\ u|_{t=0} = u_0, & x \in \Omega. \end{cases} \tag{21}$$

Letting $p > d + \frac{2}{\alpha}$ we assume that

(Q1) $g \in B_{pp}^{\alpha(1-\frac{1}{2p})}((0,T); L_p(\partial\Omega)) \cap L_p((0,T); B_{pp}^{2-\frac{1}{p}}(\partial\Omega))$, $f \in L_p((0,T); L_p(\Omega))$, $u_0 \in B_{pp}^{2-\frac{2}{p\alpha}}(\Omega)$, and $u_0|_{\partial\Omega} = g|_{t=0}$ on $\partial\Omega$;

(Q2) $A \in C^1(\mathbb{R}; \mathrm{Sym}\{d\})$, and there exists $\nu > 0$ such that $(A(y)\xi|\xi) \geq \nu|\xi|^2$ for all $y \in \mathbb{R}$ and $\xi \in \mathbb{R}^d$.

Here, $\mathrm{Sym}\{d\}$ denotes the space of d-dimensional real symmetric matrices.

The following result has been established in [52].

Theorem 5.6. *Let $\Omega \subset \mathbb{R}^d$ $(d \geq 2)$ be a bounded domain with C^2-smooth boundary. Let $\alpha \in (0,1)$, $T > 0$ be an arbitrary number, $p > d + \frac{2}{\alpha}$, and suppose that the assumptions* (Q1) *and* (Q2) *are satisfied. Then the problem* (21) *possesses a unique strong solution u in the class*

$$u \in H_p^\alpha((0,T); L_p(\Omega)) \cap L_p((0,T); H_p^2(\Omega)).$$

6 Decay estimates for bounded domains

Let $\alpha \in (0,1)$, $\Omega \subset \mathbb{R}^d$ be a bounded domain, $u_0 \in L_2(\Omega)$ and consider the problem

$$\begin{cases} \partial_t^\alpha(u-u_0) - \operatorname{div}(A\,\nabla u) = 0, & t > 0,\ x \in \Omega \\ u|_{\partial\Omega} = 0, & t > 0,\ x \in \partial\Omega \\ u|_{t=0} = u_0, & x \in \Omega, \end{cases} \tag{22}$$

where the coefficient matrix $A = A(t,x)$ is assumed to satisfy the parabolicity condition (HA). From Theorem 3.1, we know that (22) has a unique weak solution u on $(0,T) \times \Omega$ for each $T > 0$. In this sense, u is a global (in time) weak solution of (22). We are now interested in the long-time behavior of u, in particular in decay estimates for the $L_2(\Omega)$-norm of u.

Let us first consider the special case $A = I$, that is, the case of the Laplacian. Let $\{\phi_n\}_{n=1}^\infty \subset \mathring{H}_2^1(\Omega)$ be an orthonormal basis of $L_2(\Omega)$ consisting of eigenfunctions of the negative Dirichlet–Laplacian with eigenvalues $\lambda_n > 0$, $n \in \mathbb{N}$, and denote by λ_1 the smallest such eigenvalue. Further, we define for $\mu \geq 0$ the so-called *relaxation function* $s_\mu : [0,\infty) \to \mathbb{R}$ as the solution of the Volterra equation

$$s_\mu(t) + \mu(g_\alpha * s_\mu)(t) = 1, \quad t \geq 0. \tag{23}$$

Note that $s_0 \equiv 1$ and that (23) is equivalent to the integrodifferential equation

$$\partial_t^\alpha (s_\mu - 1)(t) + \mu s_\mu(t) = 0, \quad t > 0, \quad s_\mu(0) = 1.$$

It is known that for all $\mu \geq 0$ the function s_μ is positive and nonincreasing, $s_\mu \in H^1_{1,\mathrm{loc}}(\mathbb{R}_+)$, and $\partial_\mu s_\mu(t) \leq 0$; this follows, for example, from the theory of completely positive kernels, described in [34]; see also [16]. Alternatively, one can argue with the well-known formula

$$s_\mu(t) = E_\alpha(-\mu t^\alpha), \quad \text{where } E_\alpha(z) := \sum_{j=0}^\infty \frac{z^j}{\Gamma(\alpha j + 1)}, \, z \in \mathbb{C},$$

is the Mittag-Leffler function; see, for example, [21].

Then the solution u of (22) with $A = I$ can be represented via Fourier series as

$$u(t,x) = \sum_{n=1}^\infty s_{\lambda_n}(t)\,(u_0|\phi_n)\phi_n(x), \quad t \geq 0, \, x \in \Omega, \tag{24}$$

where $(\cdot|\cdot)$ stands for the standard inner product in $L_2(\Omega)$; cf. [43, Section 1] and [32, Theorem 4.1]. By Parseval's identity and since $\partial_\mu s_\mu \leq 0$, it follows from (24) that

$$|u(t,\cdot)|^2_{L_2(\Omega)} = \sum_{n=1}^\infty s^2_{\lambda_n}(t)\,|(u_0|\phi_n)|^2 \leq s^2_{\lambda_1}(t) \sum_{n=1}^\infty |(u_0|\phi_n)|^2 = s^2_{\lambda_1}(t)|u_0|^2_{L_2(\Omega)},$$

and thus

$$|u(t,\cdot)|_{L_2(\Omega)} \leq s_{\lambda_1}(t)|u_0|_{L_2(\Omega)}, \quad t \geq 0, \tag{25}$$

cf. [43]. This decay estimate is optimal as the example $u_0 = \phi_1$ with solution $u(t,x) = s_{\lambda_1}(t)\phi_1(x)$ shows. It is further known (see, e. g., [43, Remark 6.1]) that

$$\frac{1}{1 + \mu\Gamma(1-\alpha)t^\alpha} \leq s_\mu(t) \leq \frac{1}{1 + \mu\Gamma(1+\alpha)^{-1}t^\alpha}, \quad t > 0.$$

This shows that, in contrast to the case $\alpha = 1$, where $s_\mu(t) = e^{-\mu t}$, we only have an *algebraic decay* with rate $t^{-\alpha}$ (up to some bounded positive factor) as $t \to \infty$.

In the general case with *rough coefficients,* we have the following result due to Vergara and Zacher [43, Corollary 1.1].

Theorem 6.1. *Let $\alpha \in (0,1)$, $\Omega \subset \mathbb{R}^d$ be a bounded domain, $u_0 \in L_2(\Omega)$ and assume that* (HA) *is fulfilled. Then the global weak solution u of* (22) *satisfies the estimate*

$$|u(t,\cdot)|_{L_2(\Omega)} \leq s_{\nu\lambda_1}(t)\,|u_0|_{L_2(\Omega)}, \quad a.a.\, t > 0. \tag{26}$$

Theorem 6.1 shows that the $L_2(\Omega)$-norm of the solution $u(t,\cdot)$ decays at least as fast as the relaxation function $s_\mu(t)$ with $\mu = \nu\lambda_1$. This decay estimate is again optimal as the special case $A = \nu I$ shows, in fact specializing further to $\nu = 1$ we recover the estimate (25).

The proof of Theorem 6.1 is based on energy estimates and the L_p-norm inequality; see Theorem 4.1. The basic idea is as follows. Testing (formally) the PDE with u, integrating over Ω, and using $A \geq \nu I$ as well as Poincaré's inequality, we obtain

$$\int_\Omega u\,\partial_t^\alpha(u-u_0)\,dx + \nu\lambda_1 \int_\Omega |u|^2\,dx \leq 0, \quad t>0.$$

By (14), this implies (with $|u(t)|_{L_2(\Omega)} := |u(t,\cdot)|_{L_2(\Omega)}$)

$$|u(t)|_{L_2(\Omega)}\partial_t^\alpha\big(|u(\cdot)|_{L_2(\Omega)} - |u_0|_{L_2(\Omega)}\big)(t) + \nu\lambda_1|u(t)|^2_{L_2(\Omega)} \leq 0, \quad t>0.$$

Assuming $|u(t)|_{L_2(\Omega)} > 0$, we thus arrive at the fractional differential inequality

$$\partial_t^\alpha\big(|u|_{L_2(\Omega)} - |u_0|_{L_2(\Omega)}\big)(t) + \nu\lambda_1|u(t)|_{L_2(\Omega)} \leq 0, \quad t>0,$$

which implies (26), by a comparison principle argument. The rigorous proof in the weak setting requires much more effort, in particular, the problem has to be regularized suitably in time.

We point out that Theorem 6.1 can be generalized to a much wider class of subdiffusion equations, which covers, for example, equations of distributed order; see [43].

7 Decay estimates in the full space case

In this section, we consider the classical time fractional diffusion equation in $\mathbb{R}^d$,

$$\begin{cases} \partial_t^\alpha(u-u_0) - \Delta u = 0, & t>0,\ x\in\mathbb{R}^d \\ u|_{t=0} = u_0, & x\in\mathbb{R}^d, \end{cases} \tag{27}$$

where again $\alpha \in (0,1)$. Under appropriate conditions on the initial value u_0, the solution of (27) can be represented as

$$u(t,x) = \int_{\mathbb{R}^d} Z(t,x-y)u_0(y)\,dy, \tag{28}$$

where Z denotes the fundamental solution corresponding to (27); see [13]. It is known (see, e. g., [23, 37]) that

$$Z(t,x) = \pi^{-\frac{d}{2}} t^{\alpha-1}|x|^{-d} H^{20}_{12}\left(\frac{1}{4}|x|^2 t^{-\alpha}\,\Big|^{(\alpha,\alpha)}_{(d/2,1),(1,1)}\right), \quad t>0,\ x\in\mathbb{R}^d\setminus\{0\},$$

where H denotes the Fox H-function ([20, 21]). $Z(t,x)$ is nonnegative and $|Z(t,\cdot)|_{L_1(\mathbb{R}^d)} = 1$ for all $t > 0$; see, for example, [19, Section 2].

In what follows, we write $f \star g$ for the convolution in $\mathbb{R}^d$ of the functions f, g. Given $u_0 \in L_2(\mathbb{R}^d)$, we do not have in general any decay for $|Z(t,\cdot) \star u_0|_{L_2(\mathbb{R}^d)}$, like in the case of the heat equation ($\alpha = 1$). Now suppose that $u_0 \in L_2(\mathbb{R}^d) \cap L_1(\mathbb{R}^d)$. Then it is well known that $u(t,\cdot) := Z_H(t,\cdot) \star u_0$, where Z_H denotes the classical heat kernel, decays in the L_2-norm as

$$\big|u(t,\cdot)\big|_2 \lesssim t^{-\frac{d}{4}}, \quad t > 0,$$

and this estimate is the best one can obtain in general (see, e. g., [3]). Here, $|v|_2 := |v|_{L_2(\mathbb{R}^d)}$, and $v(t) \lesssim w(t)$, $t > 0$ means that there exists a constant $C > 0$ such that $v(t) \le Cw(t)$, $t > 0$. In the case of time fractional diffusion, we have the following surprising result; cf. [19, Corollary 3.2, Theorem 4.1].

Theorem 7.1. *Let $d \in \mathbb{N}$ and $u_0 \in L_1(\mathbb{R}^d) \cap L_2(\mathbb{R}^d)$ and $u(t) = Z(t) \star u_0$. Then*

$$\begin{aligned} \big|u(t)\big|_2 &\lesssim t^{-\min\{\frac{\alpha d}{4},\alpha\}}, \quad t > 0,\ d \in \mathbb{N}\setminus\{4\}, \\ \big|u(t)\big|_{2,\infty} &\lesssim t^{-\alpha}, \quad t > 0,\ d = 4. \end{aligned} \tag{29}$$

Moreover, the estimate (29) *is the best one can get in general.*

Whereas in the case $\alpha = 1$, the decay rate increases with the dimension d, time fractional diffusion leads to the phenomenon of a *critical dimension*, which is $d = 4$ in this case. Below the critical dimension the rate increases with d, the exponent being α times the one from the heat equation, while above the critical dimension the decay rate is the same for all d, namely $t^{-\alpha}$. The reason why the decay rate does not increase any further with d lies in the fact that $t^{-\alpha}$ (up to a constant) coincides with the decay rate in the case of a bounded domain and homogeneous Dirichlet boundary condition; see Section 6. This also shows that for $\alpha \in (0,1)$ the diffusion is so slow that in higher dimensions (d above the critical dimension) restriction to a bounded domain and the requirement of a homogeneous Dirichlet boundary condition do not improve the rate of decay. This is markedly different in the classical diffusion case, where we always have exponential (and thus a better) decay in the case of a bounded domain.

The decay rates in Theorem 7.1 can be proved in different ways; cf. [19]. Using the analytic and asymptotic properties of H (see, e. g., [13, 23]), which is a rather complicated object, one can derive sharp $L_p(\mathbb{R}^d)$-estimates for $Z(t,\cdot)$ for $t > 0$ and all $1 \le p < \kappa(d)$, where $\kappa(d) := d/(d-2)$, $d \ge 3$, and $\kappa(1) = \kappa(2) = \infty$. For $d \ge 3$, one also finds that $|Z(t,\cdot)|_{\kappa(d),\infty} \lesssim t^{-\alpha}$. These estimates and Young's inequality for convolutions then yield the desired decay rates. Alternatively, one can employ tools from Harmonic Analysis such as Plancherel's theorem and argue with properties of the Fourier transform of $Z(t,\cdot)$, which coincides with $s_{|\xi|^2}(t)$ (up to a constant, depending on the used definition of the Fourier transform); cf. Section 6 for the definition of the relaxation function $s_\mu(t)$.

Theorem 7.1 is only a special case of more general results obtained in [19], which also provide decay rates for the L_p-norm and allow for a wider class of subdiffusion equations.

The subsequent result states that for integrable initial data u_0 the asymptotic behavior of $Z(t) \star u_0$ is described by a multiple of $Z(t,x)$; see [19, Theorem 3.6]. We set $\kappa_1(d) := d/(d-1)$, $d \geq 2$, and $\kappa_1(1) := \infty$.

Theorem 7.2. *Let $d \in \mathbb{N}$ and $1 \leq p < \kappa_1(d)$. Let further $u_0 \in L_1(\mathbb{R}^d)$ and set $M = \int_{\mathbb{R}^d} u_0(y)\,dy$.*

(i) *There holds*

$$t^{\frac{\alpha d}{2}(1-\frac{1}{p})}\big|u(t) - MZ(t)\big|_p \to 0, \quad \text{as } t \to \infty.$$

(ii) *Assume in addition that $||x|u_0|_1 < \infty$. Then*

$$t^{\frac{\alpha d}{2}(1-\frac{1}{p})}\big|u(t) - MZ(t)\big|_p \lesssim t^{-\frac{\alpha}{2}}, \quad t > 0.$$

Moreover, in the limit case $p = \kappa_1(d)$ we have

$$t^{\frac{\alpha}{2}}\big|u(t) - MZ(t)\big|_{\kappa_1(d),\infty} \lesssim t^{-\frac{\alpha}{2}}, \quad t > 0.$$

Bibliography

[1] M. Allen, L. Caffarelli, and A. Vasseur, A parabolic problem with a fractional time derivative, *Arch. Ration. Mech. Anal.*, **221** (2016), 603–630.

[2] M. Allen, L. Caffarelli, and A. Vasseur, Porous medium flow with both a fractional potential pressure and fractional time derivative, *Chin. Ann. Math., Ser. B*, **38** (2017), 45–82.

[3] C. Bjorland and M. E. Schonbek, Poincaré's inequality and diffusive evolution equations, *Adv. Differ. Equ.*, **14** (2009), 241–260.

[4] E. Bombieri and E. Giusti, Harnack's inequality for elliptic differential equations on minimal surfaces, *Invent. Math.*, **15** (1972), 24–46.

[5] Ph. Clément, S.-O. Londen, and G. Simonett, Quasilinear evolutionary equations and continuous interpolation spaces, *J. Differ. Equ.*, **196** (2004), 418–447.

[6] Ph. Clément and J. A. Nohel, Abstract linear and nonlinear Volterra equations preserving positivity, *SIAM J. Math. Anal.*, **10** (1979), 365–388.

[7] Ph. Clément and J. A. Nohel, Asymptotic behavior of solutions of nonlinear Volterra equations with completely positive kernels, *SIAM J. Math. Anal.*, **12** (1981), 514–534.

[8] E. De Giorgi, Sulla differenziabilità e l'analiticità delle estremali degli integrali multipli regolari, *Mem. Accad. Sci. Torino, Cl. Sci. Fis. Mat. Nat.*, **3** (1957), 25–43.

[9] R. Denk, M. Hieber, and J. Prüss, *$\mathcal{R}$-Boundedness, Fourier Multipliers and Problems of Elliptic and Parabolic Type*, Mem. Amer. Math. Soc., vol. 166, 2003.

[10] R. Denk, M. Hieber, and J. Prüss, Optimal L_p-L_q-estimates for parabolic boundary value problems with inhomogeneous data, *Math. Z.*, **257** (2007), 193–224.

[11] E. DiBenedetto, *Degenerate Parabolic Equations*, Springer, New York, 1993.
[12] D. Dier, J. Kemppainen, J. Siljander, and R. Zacher, On the parabolic Harnack inequality for non-local diffusion equations, preprint, 2018, available at https://arxiv.org/pdf/1806.04603.pdf.
[13] S. E. Eidelman and A. N. Kochubei, Cauchy problem for fractional diffusion equations, *J. Differ. Equ.*, **199** (2004), 211–255.
[14] D. Gilbarg and N. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer, 1977.
[15] G. Gripenberg, Volterra integro-differential equations with accretive nonlinearity, *J. Differ. Equ.*, **60** (1985), 57–79.
[16] G. Gripenberg, S.-O. Londen, and O. Staffans, *Volterra Integral and Functional Equations*, Encyclopedia of Mathematics and Its Applications, vol. 34, Cambridge University Press, Cambridge, 1990.
[17] Q. Han and F. Lin, *Elliptic Partial Differential Equations*, Courant Lecture Notes in Mathematics, vol. 1, 1998.
[18] M. Kassmann, The classical Harnack inequality fails for non-local operators, preprint, 2007, available at webdoc.sub.gwdg.de/ebook/serien/e/sfb611/360.pdf.
[19] J. Kemppainen, J. Siljander, V. Vergara, and R. Zacher, Decay estimates for time-fractional and other non-local in time subdiffusion equations in $\mathbb{R}^d$, *Math. Ann.*, **366** (2016), 941–979.
[20] A. A. Kilbas and M. Saigo, *H-Transforms: Theory and Application*, CRC Press, LLC, 2004.
[21] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier, 2006.
[22] I. Kim, K.-H. Kim, and S. Lim, An $L_q(L_p)$-theory for the time fractional evolution equations with variable coefficients, *Adv. Math.*, **306** (2017), 123–176.
[23] A. N. Kochubei, Fractional-order diffusion, *Differ. Equ.*, **26** (1990), 485–492.
[24] A. N. Kochubei, Distributed order calculus and equations of ultraslow diffusion, *J. Math. Anal. Appl.*, **340** (2008), 252–281.
[25] O. A. Ladyzenskaja, V. A. Solonnikov, and N. N. Uralceva, *Linear and Quasilinear Equations of Parabolic Type*, Translations of Mathematical Monographs, vol. 23, American Mathematical Society, Providence, R.I., 1968.
[26] G. M. Lieberman, *Second Order Parabolic Differential Equations*, World Scientific, London, 1996.
[27] J. L. Lions and E. Magenes, *Non-Homogeneous Boundary Value Problems and Applications*, vol. I, Springer, Berlin, 1972.
[28] Y. Luchko, Maximum principle for the generalized time-fractional diffusion equation, *J. Math. Anal. Appl.*, **351** (2009), 218–223.
[29] Y. Luchko, Maximum principle and its application for the time-fractional diffusion equations, *Fract. Calc. Appl. Anal.*, **14** (2011), 110–124.
[30] Y. Luchko and M. Yamamoto, On the maximum principle for a time-fractional diffusion equation, *Fract. Calc. Appl. Anal.*, **20** (2017), 1131–1145.
[31] J. Moser, A Harnack inequality for parabolic differential equations, *Commun. Pure Appl. Math.*, **17** (1964), 101–134. Correction in *Comm. Pure Appl. Math.*, **20** (1967), 231–236.
[32] J. Nakagawa, K. Sakamoto, and M. Yamamoto, Overview to mathematical analysis for fractional diffusion equations—new mathematical aspects motivated by industrial collaboration, *J. Math-for-Ind.*, **2** (2010A-10), 99–108.
[33] J. Nash, Continuity of solutions of parabolic and elliptic equations, *Am. J. Math.*, **80** (1958), 931–954.
[34] J. Prüss, *Evolutionary Integral Equations and Applications*, Monographs in Mathematics, vol. 87, Birkhäuser, Basel, 1993.

[35] J. Prüss and G. Simonett, *Moving Interfaces and Quasilinear Parabolic Evolution Equations*, Monographs in Mathematics, vol. 105, Birkhäuser/Springer, 2016.

[36] L. Saloff-Coste, *Aspects of Sobolev-Type Inequalities*, London Mathematical Society Lecture Note Series, vol. 289, Cambridge University Press, 2002.

[37] W. R. Schneider and W. Wyss, Fractional diffusion and wave equations, *J. Math. Phys.*, **30** (1989), 134–144.

[38] L. Silvestre, Hölder estimates for solutions of integro-differential equations like the fractional Laplace, *Indiana Univ. Math. J.*, **55** (2006), 1155–1174.

[39] P. E. Sobolevskii, Coerciveness inequalities for abstract parabolic equations, *Sov. Math. Dokl.*, **5** (1964), 894–897.

[40] H. Triebel, *Interpolation Theory, Function Spaces, Differential Operators*, North-Holland, Amsterdam–New York, 1978.

[41] N. Trudinger, Pointwise estimates and quasilinear parabolic equations, *Commun. Pure Appl. Math.*, **21** (1968), 205–226.

[42] V. Vergara and R. Zacher, Lyapunov functions and convergence to steady state for differential equations of fractional order, *Math. Z.*, **259** (2008), 287–309.

[43] V. Vergara and R. Zacher, Optimal decay estimates for time-fractional and other non-local subdiffusion equations via energy methods, *SIAM J. Math. Anal.*, **47** (2015), 210–239.

[44] V. Vergara and R. Zacher, Stability, instability, and blowup for time fractional and other nonlocal in time semilinear subdiffusion equations, *J. Evol. Equ.*, **17** (2017), 599–626.

[45] L. Weis, Operator-valued Fourier multiplier theorems and maximal L_p-regularity, *Math. Ann.*, **319** (2001), 735–758.

[46] M. Yamamoto, Weak solutions to non-homogeneous boundary value problems for time-fractional diffusion equations, *J. Math. Anal. Appl.*, **460** (2018), 365–381.

[47] R. Zacher, Maximal regularity of type L_p for abstract parabolic Volterra equations, *J. Evol. Equ.*, **5** (2005), 79–103.

[48] R. Zacher, Quasilinear parabolic integro-differential equations with nonlinear boundary conditions, *Differ. Integral Equ.*, **19** (2006), 1129–1156.

[49] R. Zacher, Boundedness of weak solutions to evolutionary partial integro-differential equations with discontinuous coefficients, *J. Math. Anal. Appl.*, **348** (2008), 137–149.

[50] R. Zacher, Weak solutions of abstract evolutionary integro-differential equations in Hilbert spaces, *Funkc. Ekvacioj*, **52** (2009), 1–18.

[51] R. Zacher, The Harnack inequality for the Riemann–Liouville fractional derivation operator, *Math. Inequal. Appl.*, **14** (2011), 35–43.

[52] R. Zacher, Global strong solvability of a quasilinear subdiffusion problem, *J. Evol. Equ.*, **12** (2012), 813–831.

[53] R. Zacher, A De Giorgi–Nash type theorem for time fractional diffusion equations, *Math. Ann.*, **356** (2013), 99–146.

[54] R. Zacher, A weak Harnack inequality for fractional evolution equations with discontinuous coefficients, *Ann. Sc. Norm. Super. Pisa, Cl. Sci. (5)*, **12** (2013), 903–940.

[55] E. Zeidler, *Nonlinear Functional Analysis and Its Applications, II/A: Linear Monotone Operators*, Springer, New York, 1990.

Jukka Kemppainen

Layer potentials for the time-fractional diffusion equation

Abstract: We study the solvability of the time-fractional diffusion equation (TFDE) in the subdiffusive case by the method of layer potentials. We show how the use of the layer potentials leads on the lateral boundary to integral equations which are uniquely solvable. The boundary datum is allowed to be either continuous and bounded or L^p-functions. In the former case, we assume that the lateral boundary has Lyapunov smoothness, which allows to use classical analysis. In the latter case, (TFDE) is analyzed in C^1 cylinders, which is more involved and requires tools of harmonic analysis. The results presented here can be viewed as the first steps in the study of (TFDE) in nonsmooth domains.

Keywords: Boundary integral equation, double layer potential, single layer potential, time fractional diffusion equation

MSC 2010: 35R09, 26A33, 45A05, 42B37, 31B10, 35A08, 33C60

1 Introduction

We give an overview of the layer potential technique applied for the Dirichlet boundary value problem for the time-fractional diffusion equation (TFDE) in the subdiffusive case in a space-time cylinder $Q_T = \Omega \times (0, T)$, where $\Omega \subset \mathbb{R}^n$ is a bounded domain with the lateral boundary $\Sigma_T = \Gamma \times (0, T)$, where $\Gamma = \partial\Omega$ is the boundary of the domain Ω and $0 < T < \infty$. The problem reads

$$\begin{cases} \partial_t^\alpha u(x,t) - \Delta u(x,t) = 0 & \text{in } Q_T, \\ u(x,t) = g(x,t) & \text{on } \Sigma_T, \\ u(x,0) = 0 & x \in \Omega, \end{cases} \tag{1}$$

where ∂_t^α is the Caputo fractional derivative operator of order $0 < \alpha < 1$ defined for a causal function by

$$\partial_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} \frac{\mathrm{d}f}{\mathrm{d}\tau}(\tau)\mathrm{d}\tau.$$

We consider (1) in the pointwise sense. The layer potentials are suitable for this purpose since they are integrals which live only on the boundary. Let us briefly de-

Jukka Kemppainen, Applied and Computational Mathematics, PB 8000, 90014 University of Oulu, Oulu, Finland, e-mail: Jukka.T.Kemppainen@oulu.fi

https://doi.org/10.1515/9783110571660-009

scribe how we are formally led to the layer potentials. For the moment, suppose u is a classical solution of (1) and let $G(x,t)$ be the fundamental solution of (1), that is satisfying

$$(\partial_\tau^\alpha - \Delta_y)G(y,\tau) = \delta(\tau)\delta(y) \tag{2}$$

and $G(y,\tau) \equiv 0$ for all $\tau \leq 0$. Testing (2) with $u(x-y,t-\tau)$ and integrating by parts yields

$$\begin{aligned} u(x,t) = &\int_0^t \int_\Gamma G(x-y,t-\tau)\partial_{n(y)}u(y,\tau)\mathrm{d}\sigma(y)\mathrm{d}\tau \\ &- \int_0^t \int_\Gamma \partial_{n(y)}G(x-y,t-\tau)u(y,\tau)\mathrm{d}\sigma(y)\mathrm{d}\tau, \end{aligned}$$

where $n(y)$ denotes the *outward unit normal* and $\partial_{n(y)}G(x,t) = \langle \nabla G(x,t), n(y)\rangle$. Here, we have applied the Green's theorem for the Laplacian and used the fact that

$$\int_0^t \kappa_t u(\tau)\partial_\tau^\alpha v(\tau)\mathrm{d}\tau = \int_0^t \kappa_t(\partial_\tau^\alpha u)(\tau)v(\tau)\mathrm{d}\tau$$

for causal u and v and the time reversal operator $\kappa_t u(\tau) = u(t-\tau)$ [15]. So the solution of (1) can be represented in terms of the single layer potential

$$(S\varphi)(x,t) = \int_0^t \int_\Gamma G(x-y,t-\tau)\varphi(y,\tau)\mathrm{d}\sigma(y)\mathrm{d}\tau \tag{3}$$

and the double layer potential

$$(D\varphi)(x,t) = \int_0^t \int_\Gamma \partial_{n(y)}G(x-y,t-\tau)\varphi(y,\tau)\mathrm{d}\sigma(y)\mathrm{d}\tau \tag{4}$$

as

$$u(x,t) = \big(S(\partial_n u|_{\Sigma_T})\big)(x,t) - \big(D(u|_{\Sigma_T})\big)(x,t), \tag{5}$$

which we call a *Green's formula for the solution of* (1). A problem is that the Neumann data $\partial_n u|_{\Sigma_T}$ is not known. But if we know the boundary behavior of the layer potentials, then (5) is reduced to a boundary integral equation, which we hope to be solved with respect to the unknown quantity $\varphi = \partial_n u|_{\Sigma_T}$. This is called as a *direct approach*.

The *indirect approach* is to make a guess and seek the solution of (1), let us say, in terms of the double layer potential

$$u(x,t) = (D\varphi)(x,t), \tag{6}$$

where φ is to be determined. Again, if the boundary behavior of the double layer potential is known, then (6) is reduced to a boundary integral equation to be solved with respect to φ.

What makes the layer potential technique useful is that one can reduce not only the problem (1) but also the same problem formulated in the unbounded exterior domain $Q_T^+ := (\mathbb{R}^n \setminus \overline{\Omega}) \times (0, T)$ to the integral equation on the bounded cylinder Σ_T. Another benefit is that one is allowed to relax the smoothness assumption of the lateral boundary Σ_T and to give a meaning for the problem (1) with datum g in the classical Lebesgue space $L^p(\Sigma_T)$. A drawback is that the approach is applicable only if the fundamental solution is available. However, as the latest achievements in the elliptic case show, the exact form of the fundamental solution is not needed. It is crucial to have some estimates for the fundamental solution. See [1] and references therein, where the technique is applied for the elliptic operator $\nabla \cdot (A\nabla)$ with a rough complex-valued matrix $A \in L^\infty(\mathbb{R}^n; \mathbb{C}^{n\times n})$. Note that if in the local coordinates $\Omega = \{(x', x_n) := (x_1, \ldots, x_n) \in \mathbb{R}^n \ : \ x_n > \varphi(x')\}$ with a Lipschitz continuous function, then the change of coordinates $(x', x_n) \mapsto (x', x_n - \varphi(x'))$, which “flattens the boundary,” transforms Δ into a divergence form elliptic operator $\nabla \cdot A\nabla$ with merely bounded and measurable coefficients.

The same philosophy as above applies also for (TFDE), since the fundamental solution contains quite complicated special function called a Fox H-function or just an H-function for short. Fortunately, the sharp asymptotics of the H-function is known, which allows us to apply the method of the layer potential also for (TFDE).

2 Asymptotic estimates for the fundamental solution

The fundamental solution is given by (see [3, Formula (2.4)])

$$G(x,t) = \pi^{-n/2} t^{\alpha-1} |x|^{-n} H_{12}^{20}\left(\frac{1}{4}|x|^2 t^{-\alpha} \,\middle|\, \begin{matrix} (\alpha,\alpha), & \\ (\frac{n}{2},1), & (1,1) \end{matrix}\right), \tag{7}$$

where H_{12}^{20} is called the H-function, which is defined via a Mellin–Barnes-type integral as

$$H_{12}^{20}(z) = \frac{1}{2\pi i} \int_{\mathcal{L}} \mathcal{H}_{12}^{20}(s) z^{-s} \mathrm{d}s, \tag{8}$$

where

$$\mathcal{H}_{12}^{20}(s) = \frac{\Gamma(\frac{n}{2}+s)\Gamma(1+s)}{\Gamma(\alpha+\alpha s)}$$

is the Mellin transform of H_{12}^{20} and $\mathcal{L}$ is an infinite contour in the complex plane. For further details on the H-functions, we refer to [17]. Since the sharp asymptotic behavior of the H-functions is known (see, e. g., [3, 17]) and G is strictly positive for all positive times, we have sharp asymptotics for the fundamental solution.

Lemma 2.1. *Denote $R := |x|^2t^{-\alpha}$. The function G has the following asymptotic behavior:*
(i) *If $R \le 1$, then*

$$G(x,t) \sim \begin{cases} t^{-\frac{\alpha n}{2}+\alpha-1}, & \text{if } n=2 \text{ or } n=3, \\ -t^{-\alpha-1}\log(\frac{1}{4}|x|^2t^{-\alpha}), & \text{if } n=4, \\ t^{-\alpha-1}|x|^{4-n} & \text{if } n>4. \end{cases}$$

(ii) *If $R \ge 1$, then*

$$G(x,t) \sim t^{\alpha-1}|x|^{-n}R^{\frac{n+2+2\alpha}{2(2-\alpha)}}\exp(-\lambda R^{\frac{1}{2-\alpha}})$$

for some positive constant $\lambda > 0$.

For the double layer potential, we need the asymptotics of (see [13])

$$\partial_{n(y)}G(x-y,t) = -2\pi^{-n/2}\frac{\langle x-y, n(y)\rangle}{|x-y|^{n+2}}t^{\alpha-1}F\left(\frac{1}{4}|x-y|^2t^{-\alpha}\right) \tag{9}$$

where F is a function given by its Mellin transform

$$\mathcal{M}(F)(s) = 2\frac{\Gamma(\frac{n+2}{2}+s)\Gamma(1+s)}{\Gamma(\alpha+\alpha s)}, \tag{10}$$

which is exactly the Mellin transform of the H-function appearing in (7) with n replaced by $n+2$. Therefore, if we denote the fundamental solution by G_n to emphasize the dependence on n, we see from (9) and (10) that

$$\partial_{n(y)}G_n(x-y,t) = -2\pi\langle x-y, n(y)\rangle G_{n+2}(x-y,t), \tag{11}$$

and we can use directly the asymptotics of G given in Lemma 2.1 also for the normal derivative of G. Note that the heat kernel obeys the same structure (11) as it can be readily seen by a direct computation.

Lemma 2.1 is crucial in our analysis. It may seem that the singularity in t is too bad in the definitions (3) and (4), if we look at the case (i) of Lemma 2.1. But this is just an illusion, since we can kill the seemingly strong singularity in t. For example, if $n > 4$, then using $R = |x|^2t^{-\alpha} \le 1$ we have $1 \le R^{-2} = |x|^{-4}t^{2\alpha}$. Therefore, the local singularity in t is weak as long as $x \notin \Gamma$. The other cases of n's can be treated similarly, so we can deduce that the single layer potential is well-defined for all $x \notin \Gamma$ and locally integrable φ. Using (11), we see that also the double layer potential is well-defined.

If the boundary Γ of a bounded domain Ω is Lyapunov smooth, then

$$|\langle x-y, n(y)\rangle| \le C|x-y|^{1+\lambda} \tag{12}$$

for some $0 < \lambda \leq 1$ and a positive constant C. Then we see that the right-hand side of (4) makes sense also for $x \in \Gamma$. For example, if $R := |x-y|^2 t^{-\alpha} \leq 1$ and $n > 2$, then (11) and case (i) of Lemma 2.1 gives

$$\left|\partial_{n(y)} G(x-y,t)\right| \leq t^{-\alpha-1}|x-y|^{3+\lambda-n}.$$

Since $R \leq 1$, then $1 \leq R^{-\gamma}$ for any $\gamma > 0$. By choosing $1 < \gamma < 1+\frac{\lambda}{2}$, we obtain an estimate

$$\left|\partial_{n(y)} G(x-y,t)\right| \leq Ct^{-\alpha-1+\alpha\gamma}|x-y|^{3+\lambda-n-2\gamma}, \tag{13}$$

which shows that we have a weakly singular kernel so the integral exists as a L^p-function for any $\varphi \in L^p(\Sigma_T)$.

3 The boundary behavior of the potentials

The boundary behavior of the potentials is the key ingredient of the theory. We start with the case of Lyapunov smooth boundary Γ.

3.1 In Lyapunov smooth domains

We study the limiting values of the potentials, when $x \in \mathrm{R}^n \setminus \Gamma$ tends to a point $x_0 \in \Gamma$ nontangentially. For the double layer potential, we have the following.

Theorem 3.1. *Let Ω be a bounded domain with Lyapunov smooth boundary Γ. The double layer potential with continuous density φ has the following non-tangential limiting values from the interior domain $\Omega^- := \Omega$ and the exterior domain $\Omega^+ := \mathbb{R}^n \setminus \overline{\Omega}$ to the boundary Γ:*

$$\lim_{K^\pm \ni x \to x_0} (D\varphi)(x,t) = \pm\frac{1}{2}\varphi(x_0,t) + \int_0^t \int_\Gamma \partial_{n(y)} G(x_0-y,t-\tau)\varphi(y,\tau)\mathrm{d}\sigma(y)\mathrm{d}\tau, \tag{14}$$

where $x_0 \in \Gamma$, $t \in (0,T)$ and

$$K^\pm := K^\pm(x_0) = \{(x \in \Omega^\pm : |x-x_0| < \delta, \mp\langle x_0 - x, n(x_0)\rangle > \beta|x-x_0|\}$$

is the nontangential cone with the outward unit normal $n(x_0)$, $0 < \beta < 1$ and a positive constant δ depending on Ω.

Idea of the proof. The proof for the single layer potential is given in [14]. We give an outline of the proof here for the double layer potential. We split the double layer potential into two parts

$$(D\varphi)(x,t) = I_\delta(x,t) + J_\delta(x,t),$$

where

$$I_\delta(x,t) = \int_0^t \int_{\Gamma_\delta} \partial_{n(y)} G(x-y, t-\tau)\varphi(y,\tau)\mathrm{d}\sigma(y)\mathrm{d}\tau,$$

J_δ is its complementary part, $\Gamma_\delta = B(x_0,\delta) \cap \Gamma$ and δ is small enough. Here, $B(x_0,\delta)$ denotes the ball of radius δ centered at x_0, as usual.

We shall compare $I_\delta(x,t)$ to

$$I'_\delta(x,t) = \int_0^t \int_{\Gamma'_\delta} \partial_{n(y')} G(x-y', t-\tau)\varphi(y',\tau)\mathrm{d}\sigma'(y')\mathrm{d}\tau,$$

where Γ'_δ and y' are the projections of Γ_δ and y to the tangent plane $T(x_0)$ and $\mathrm{d}\sigma'(y)$ is the surface element on $T(x_0)$, by proving that

$$I_\delta(x,t) = I'_\delta(x,t) + I_\delta(x,t) - I'_\delta(x,t) \approx -\frac{1}{2}\varphi(x_0,t) + I_\delta(x_0,t),$$

when x is close to x_0. Note that $I'_\delta(x_0,t) = 0$, since $n(y') \perp x_0 - y'$ for all $y' \in \Gamma'_\delta$. Here, we will describe how

$$I'_\delta(x,t) \approx -\frac{1}{2}\varphi(x_0,t) \tag{15}$$

by reducing the problem to the properties of the double layer potential of the Laplacian. By the change of variables $\tau \leftrightarrow \rho = \frac{|x-y'|^2}{4(t-\tau)^\alpha}$ and the continuity of φ, we can approximate

$$I'_\delta(x,t) \approx -\frac{\varphi(x_0,t)}{4\alpha\pi^{n/2}} \int_0^\infty \rho^{-2}F(\rho)\mathrm{d}\rho \int_{\Gamma'_\delta} \frac{\langle x-y', n(y')\rangle}{|x-y'|^n}\mathrm{d}\sigma(y'), \tag{16}$$

where F is as in (9). Noting that the first integral in (16) is nothing but the Mellin transform of F evaluated at $s = -1$, we have

$$I'_\delta(x,t) = -\frac{\Gamma(\frac{n}{2})}{2\pi^{n/2}}\varphi(x_0,t) \int_{\Gamma'_\delta} \frac{\langle x-y', n(y')\rangle}{|x-y'|^n}\mathrm{d}\sigma(y').$$

Choose x so close to x_0 that x lies inside the domain bounded by Γ'_δ and the hemisphere $S_\delta(x_0)$ of radius δ and centered at x_0. Since

$$-\frac{1}{\omega_n}\frac{\langle x-y', n(y')\rangle}{|x-y'|^n}$$

is the normal derivative of the fundamental solution of the Laplace equation, there holds

$$
\begin{aligned}
\int_{\Gamma'_\delta} \frac{\langle x-y', n(y')\rangle}{|x-y'|^n} \mathrm{d}\sigma(y') &= \omega_n - \int_{S_\delta(x_0)} \frac{\langle x-y', n(y')\rangle}{|x-y'|^n} \mathrm{d}\sigma(y') \\
&\approx \omega_n - \int_{S_\delta(x_0)} \frac{\langle x_0-y', n(y')\rangle}{|x_0-y'|^n} \mathrm{d}\sigma(y') \\
&= \frac{\omega_n}{2},
\end{aligned}
$$

where ω_n is the surface area of the unit sphere. Using the formula (see [8, Appendix A.3]),

$$
\omega_n = \frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})}
$$

establishes (15). The rest of the proof uses the detailed asymptotics of G. For details, we refer to [7, Chapter 5.2], where the proof is given for the normal derivative of the single layer potential for the heat equation, and to [14], where the same proof is adapted to the time-fractional case. □

The corresponding result for the single layer potential is proved in [14]. Using the same notation as in Theorem 3.1, it reads as the following.

Theorem 3.2. *The single layer potential is continuous up to the boundary. The normal derivative of the single layer potential has the following limiting values:*

$$
\lim_{\substack{x\to x_0 \\ x\in K^\pm}} \langle \nabla S\varphi(x,t), n(x_0)\rangle = \mp\frac{1}{2}\varphi(x_0,t) + \int_0^t \int_\Gamma \partial_{n(x_0)} G(x_0-y, t-\tau)\varphi(y,\tau)\mathrm{d}\sigma(y)\mathrm{d}\tau. \tag{17}
$$

For later use, we denote

$$
(W\varphi)(x,t) = \int_0^t \int_\Gamma \partial_{n(y)} G(x-y, t-\tau)\varphi(y,\tau)\mathrm{d}\sigma(y)\mathrm{d}\tau, \quad (x,t)\in \Sigma_T, \tag{18}
$$

and call W as the double layer potential operator. Then (14) of Theorem 3.1 takes the form

$$
\lim_{K^\pm \ni x\to x_0} (D\varphi)(x,t) = \pm\frac{1}{2}\varphi(x_0,t) + (W\varphi)(x_0,\tau).
$$

If we apply the indirect approach by using the double layer potential (6), then solvability of (1) is reduced to finding φ from the boundary integral equation

$$
-\frac{1}{2}\varphi + W\varphi = g. \tag{19}
$$

Hence we need to prove that the operator $-\frac{1}{2}I + W$ is invertible. We will prove this in Section 4.

3.2 In $\mathcal{C}^1$ domains

We would like to obtain the same boundary integral equation (19). However, if $\Gamma \in \mathcal{C}^1$, then $\lambda = 0$ in (12). The same argument that led to the estimate (13) leads now by choosing $\gamma = 1$ to the estimate

$$\left|\partial_{n(y)} G(x - y, t)\right| \le Ct^{-1}|x - y|^{1-n},$$

which shows that we are exactly *on the edge of integrability* so we need a more delicate analysis.

To give a meaning for the integral $W\varphi$ defined in (18), we follow the general theory of singular integrals of nonconvolution type described, for example, in [8]. The corresponding technique for the heat equation is outlined in [6]. To this end, we introduce the *truncated operator*

$$(W^\epsilon \varphi)(x, t) = \int_0^{t-\epsilon} \int_\Gamma \partial_{n(y)} G(x - y, t - \tau)\varphi(y, \tau)\mathrm{d}\sigma(y)\mathrm{d}\tau, \quad (x, t) \in \Sigma_T,$$

and the *maximal singular operator* associated with W,

$$(W^* \varphi)(x, t) = \sup_{\epsilon>0}\left|(W^\epsilon \varphi)(x, t)\right|.$$

We follow the same technique as in the case of the heat equation presented in [6]. The technique is adapted to (TFDE) in [10, 11]. We will give a brief outline. We interpret $W\varphi$ as the limit

$$(W\varphi)(x, t) = \lim_{\epsilon\to 0}(W^\epsilon \varphi)(x, t), \quad (x, t) \in \Sigma_T. \tag{20}$$

Since we will study the problem (1) with L^p datum g in Section 4, we need to prove that (20) is well-defined for $\varphi \in L^p$. This is guaranteed by the following result.

Theorem 3.3. *Let $1 < p < \infty$. There holds:*
(i) *The operator $W^\epsilon : L^p(\Sigma_\infty) \to L^p(\Sigma_\infty)$ is bounded independently of ϵ.*
(ii) *The operator W^* is bounded in $L^p(\Sigma_\infty)$.*
(iii) *The limit (20) exists pointwise a.e. on Σ_∞ and in $L^p(\Sigma_\infty)$ for all $\varphi \in L^p(\Sigma_\infty)$.*

Idea of the proof. The proof is based on the detailed asymptotics of the H-function and the mean value theorem. Lemma 2.1 is not enough anymore since we need also higher order derivatives. But formal differentiation of (8) and

$$\frac{\mathrm{d}}{\mathrm{d}z}(z^{-s}) = -sz^{-s-1} = -z^{-1}\frac{\Gamma(s+1)}{\Gamma(s)}z^{-s}$$

shows that each differentiation yields an H-function whose sharp asymptotics is known.

Then we may follow the *Calderón–Zygmund theory* (see, e. g., [8]) and prove that
(a) W^ϵ is bounded in $L^2(\Sigma_\infty)$ independently of ϵ.
(b) W^ϵ is of weak type $(1, 1)$.

Case (a)*:* We use the Laplace transform in time to utilize the mapping properties of the double layer potential of the Laplacian similarly as in the proof of Theorem 3.1. Taking the Laplace transform in time and then making the change of variables $t \leftrightarrow |x-y|^{2/\alpha}\tau$, we obtain

$$\mathcal{L}(W^\epsilon\varphi)(x, \mathrm{i}\eta) = \int_\Gamma \frac{\langle x-y, n(y)\rangle}{|x-y|^n} H(\epsilon/|x-y|^{2/\alpha}, |x-y|^{2/\alpha}\eta)(\mathcal{L}\varphi)(x, \mathrm{i}\eta)\mathrm{d}\sigma(y), \tag{21}$$

where

$$H(\epsilon, \eta) = \frac{1}{\pi^{n/2}} \int_\epsilon^\infty \exp(-\mathrm{i}\eta\tau)\tau^{\alpha-1} F\left(\frac{1}{4}\tau^{-\alpha}\right)\mathrm{d}\tau \tag{22}$$

and F is the same as in (9). Careful study of the function H leads to the pointwise estimate

$$\left|\mathcal{L}(W^\epsilon)(x, \mathrm{i}\eta)\right| \le C(M_\Gamma((\mathcal{L}\varphi)(\cdot, \mathrm{i}\eta))(x) + \widetilde{K}(\mathcal{L}(\varphi)(\cdot, \mathrm{i}\eta))(x)), \tag{23}$$

where C is independent of ϵ,

$$M_\Gamma(g)(x) = \sup_{r>0} \int_{|x-y|<r} |g(y)|\mathrm{d}\sigma(y), \quad x \in \Gamma, \tag{24}$$

is the *Hardy–Littlewood maximal operator* on Γ and

$$\widetilde{K}(g)(x) = \sup_{\epsilon>0}\left| \int_{|x-y|>\epsilon} \frac{\langle x-y, n(y)\rangle}{|x-y|^n} g(y)\mathrm{d}\sigma(y)\right|$$

is the maximal singular operator associated with the double layer potential of the Laplacian. The proof of the key estimate (23) is based on splitting the integral (21) into two parts depending on whether $|x-y| < \epsilon^{\alpha/2}$ or not. We give here details for the first part I_1 since it is easier in the sense that we need the asymptotics of F in (22) only near zero. Indeed, since $\epsilon/|x-y|^{2/\alpha} > 1$, the lower limit in the integral (22) defining H in (21) is greater than 1. Then we may use the asymptotics of F near zero. In particular, if $n \ge 3$, we obtain from (10) and [3, Formula (3.14)] the estimate

$$\left|H(\epsilon, \eta)\right| \le C\epsilon^{-\alpha}.$$

Using this in (21) gives

$$\begin{aligned}
|I_1| &\le C\epsilon^{-\alpha} \int\limits_{|x-y|<\epsilon^{\alpha/2}} |x-y|^{3-n} \big|(\mathcal{L}\varphi)(y, \mathrm{i}\eta)\big| \mathrm{d}\sigma(y) \\
&= C\epsilon^{-\alpha} \sum_{k=0}^{\infty} \int\limits_{2^{-(k+1)}\epsilon^{\alpha/2} \le |x-y| < 2^{-k}\epsilon^{\alpha/2}} |x-y|^{3-n} \big|(\mathcal{L}\varphi)(y, \mathrm{i}\eta)\big| \mathrm{d}\sigma(y) \\
&\le C\epsilon^{-\alpha} \sum_{k=0}^{\infty} \big(2^{-(k+1)}\epsilon^{\alpha/2}\big)^{3-n} \int\limits_{|x-y|<2^{-k}\epsilon^{\alpha/2}} \big|(\mathcal{L}\varphi)(y, \mathrm{i}\eta)\big| \mathrm{d}\sigma(y) \\
&= C \sum_{k=0}^{\infty} 4^{-k} \frac{1}{|B(x, 2^{-k}\epsilon^{\alpha/2})|} \int\limits_{|x-y|<2^{-k}\epsilon^{\alpha/2}} \big|(\mathcal{L}\varphi)(y, \mathrm{i}\eta)\big| \mathrm{d}\sigma(y) \\
&\le C M_\Gamma\big((\mathcal{L}\varphi)(\cdot, \mathrm{i}\eta)\big)(x),
\end{aligned}$$

if $n > 3$. If $n = 3$, the same upper bound follows immediately from the first line of the above estimate. For the proof of the second part, we refer to [6], where the technique is used for the heat equation. The same method is applied to (TFDE) in [11]. Case (a) then follows from Parseval's theorem, since M_Γ and $\widetilde{K}$ are bounded on $L^2(\Gamma)$.

Case (b)*:* We use the fact that the Hardy–Littlewood maximal function is of the weak type $(1, 1)$. To do that, we split $W^\epsilon\varphi$ into two parts as follows:

$$(W^\epsilon\varphi)(x, t) = W\big(\varphi(\chi_{t-\tau>\epsilon} - \chi_{B_\alpha((x,t),\epsilon)^c})\big)(x, t) + W(\varphi\chi_{B_\alpha((x,t),\epsilon)^c})(x, t),$$

where χ denotes the characteristic function and

$$B_\alpha((x, t), \epsilon) = \big\{(y, \tau) \ : \ |x-y|^{2/\alpha} + |t-\tau| < \epsilon\big\}$$

denotes the *anisotropic ball of radius ϵ centered at* (x, t).

For the first integral, we use the asymptotics of G to obtain a pointwise bound

$$C\big(M_1(M_\Gamma\varphi)\big)(x, t),$$

where M_Γ is given in (24) and M_1 is one-dimensional Hardy–Littlewood maximal function applied to the time variable. This shows that this integral operator is of the weak type $(1, 1)$.

For the second integral, we may use the technique presented in [21]. Here, the truncation of the integral $(W\varphi)(x, t)$ is made with respect to anisotropic balls. The sets $U_\epsilon := B_\alpha((x, t), \epsilon)$ form a Vitali family. We need to control differences of ∇G. Fortunately, we are able to prove the *Hörmander's condition* (see [11, Proposition 1])

$$\int\limits_{\Sigma_\infty \setminus U_{2r}} \big|\nabla G(x-y, t-\tau) - \nabla G(x_1 - y, t_1 - \tau)\big| \mathrm{d}\sigma(y)\mathrm{d}\tau \le C < \infty$$

for any $(x_1, t_1) \in U_r$. It follows that the second integral operator is of weak type $(1, 1)$, which implies case (b). $\square$

For the analogue of Theorem 3.1 in $\mathcal{C}^1$-domains, we need to control the nontangential maximal function

$$N_\pm(u)(x,t) = \sup\{|u(y,t)| : y \in K^\pm(x),\ t > 0\}. \tag{25}$$

Theorem 3.4. *Let Ω be a bounded domain with $\mathcal{C}^1$-boundary Γ. The nontangential maximal function of the double layer potential $N_\pm(D\varphi)$ with the density $\varphi \in L^p(\Sigma_T)$ belongs to $L^p(\Sigma_T)$. The double layer potential has the following nontangential limiting values:*

$$\lim_{K^\pm \ni x \to x_0} (D\varphi)(x,t) = \pm\frac{1}{2}\varphi(x_0,t) + \int_0^t \int_\Gamma \partial_{n(y)} G(x_0 - y, t-\tau)\varphi(y,\tau)\mathrm{d}\sigma(y)\mathrm{d}\tau$$

for almost every $(x_0, t) \in \Sigma_T$.

Idea of the proof. The technique is demonstrated in [6] for the heat equation. The same technique is applied for (TFDE) in [11]. The proof is again based on the detailed asymptotics of the fundamental solution G and to the properties of the Hardy–Littlewood maximal functions. By partition of unity, we may localize the argument to a small neighborhood of a boundary point y_0. If x_0 is far from y_0, then $|x - y|$ is bounded from below by a positive constant for all x in cone $K^\pm(x_0)$, so we are away from the spatial singularity of G. The estimate in this case is rather simple and we end up to the estimate

$$N(D\varphi)(x_0,t) \le C\|\varphi\|_{L^p(\Sigma_T)}.$$

If x_0 is close to y_0, we have to take into account the possibility that $|x-y|$ is arbitrarily small, when $K^\pm \ni x \to x_0$. This case is the hard part. In this case, we end up to the estimate

$$N(D\varphi)(x_0,t) \le C\{(W^*\varphi)(x_0,t) + (M_1(M_\Gamma\varphi))(x_0,t)\},$$

which implies the boundedness of the nontangential maximal function.

The pointwise limit can be proved first for smooth density φ. The smoothness of φ compensates the lack of integrability of the kernel. Therefore, we can proceed similarly as in Theorem 3.1 to obtain the limit. Then we can use the density argument similarly as in [4, Theorem 1.3]. □

4 Existence and uniqueness of the solution

4.1 Lyapunov-smooth domain and continuous boundary data

The case of Lyapunov smooth boundary Γ is classical since the integral in (18) is converging improper integral; see, for example, the estimate (13). By [18, Theorems 2.22], the operator $W : C(\overline{\Sigma_T}) \to C(\overline{\Sigma_T})$ is compact. Since we have imposed the zero initial condition, we need to impose the compatibility condition

$$g(\cdot, 0) = 0 \quad \text{on } \Gamma.$$

The Fredholm theory (see, e. g., [12] and [18]) implies that the operator $-\frac{1}{2}I + W$ is invertible. The maximum principle (see [20]) gives the uniqueness of the solution for (1). Hence there holds the existence and uniqueness result.

Theorem 4.1. *The problem* (1) *is uniquely solvable and the solution is given by*

$$u = D\left(\left(-\frac{1}{2}I + W\right)^{-1} g\right).$$

Remark 4.1. The same technique can be used to solve the Neumann problem. The solution is searched now in terms of the single layer potential. Noting that the integral operator in (17) is the $L^2(\Sigma_T)$-adjoint of W and denoting the boundary datum again by g, the solution is given by

$$S\left(\left(\frac{1}{2}I + W'\right)^{-1} g\right).$$

4.2 C^1-domain and L^p boundary data

When one relaxes the smoothness assumption of the datum, one usually generalizes the notion of the solution. Commonly used terms are the mild solution or the weak solution. Here, we take another viewpoint and do not change the notion of the solution. We relax not only the smoothness of the data but also the smoothness of the boundary. Both of them bring new difficulties. We saw in Section 3.2 the effect in terms of the boundary values. Moreover, if the boundary datum is in L^p, then in general knowing the boundary values almost everywhere is not enough to guarantee the well-posedness. For example,

$$u(x,t) = (E_\alpha(t^\alpha) - 1)\frac{x_n}{|x|^n}, \quad x = (x_1, x_2, \dots, x_n),$$

where E_α is the Mittag-Leffler function, satisfies $\partial_t^\alpha u(x,t) - \Delta u(x,t) = 0$ in the upper half-space with the zero initial and boundary data almost everywhere. Hence we need an extra condition to rule out the possibility of nonuniqueness. We need an analogue of a sort of dominated convergence theorem such that the boundary values are taken in a stronger sense than just pointwise almost everywhere.

The nontangential maximal function fits to this purpose perfectly. Therefore, we formulate the problem (1) in the following form:

$$\begin{cases} \partial_t^\alpha u(x,t) - \Delta u(x,t) = 0 & \text{in } Q_T, \\ u(x,t) = g(x,t) & \text{on } \Sigma_T, \\ u(x,0) = 0 & x \in \Omega, \\ \|N(u)\|_{L^p(\Sigma_T)} \le C\|g\|_{L^p(\Sigma_T)}, \end{cases} \tag{26}$$

where we have denoted $N(u) := N_-(u)$. Recall the definition of $N_-(u)$ from (25).

Theorem 4.2. *Let Ω be a bounded domain with C^1-boundary Γ and let $g \in L^p(\Sigma_T)$ with $0 < T < \infty$ and $1 < p < \infty$. Then the problem* (26) *admits a unique solution and the solution is given by the double layer potential*

$$u = D\left(\left(-\frac{1}{2}I + W\right)^{-1} g\right). \tag{27}$$

Idea of the proof. The existence of the solution follows if we can show that the operator $-\frac{1}{2}I + W$ is one-to-one and onto. The proof is based on the facts:

(i) the operator norm of $W : L^p(\Sigma_T) \to L^p(\Sigma_T)$ tends to zero as $T \to 0$,

(ii) the operator W has the *Volterra property:* If the support of φ in t is contained in (a,∞), then the support of $W\varphi$ in t is also contained in (a,∞).

The proof of (i) is based on approximating the domain Ω by smooth domains Ω_k. Details are given in [4, 6] and in the references therein. We present here the main lines of the argument. Since Ω is bounded, we may cover Γ with finitely many balls $B_l = B(x^{(l)}, r_l)$ for some numbers $r_l > 0$. Then the problem reduces to showing that the operators

$$(W_l\varphi)(x,t) = \lim_{\epsilon\to 0} \int_0^{t-\epsilon} \int_{\Gamma\cap B_l} \partial_{n(y)} G(x-y, t-\tau)\varphi(y,\tau)\mathrm{d}\sigma(y)\mathrm{d}\tau$$

satisfy the conditions (i) and (ii). Let $n_k(y)$ be the outward unit normal of $\Gamma_k = \partial\Omega_k$ at the point y. We approximate the operator W_l by

$$(W_{l,k}\varphi)(x,t) = \lim_{\epsilon\to 0} \int_0^{t-\epsilon} \int_{\Gamma_k\cap B_l} \partial_{n_k(y)} G(x-y, t-\tau)\varphi(y,\tau)\mathrm{d}\sigma(y)\mathrm{d}\tau \tag{28}$$

Then similar arguments as in Theorem 3.3 may be used to prove that the operator $W_l - W_{l,k}$ has small L^p norm when k is large. Thus we are left with showing that the operator $W_{l,k}$ in (28) has small $L^p(\Sigma_T)$ norm, when T is small. Since Γ_k is smooth, there holds

$$\left|\langle x-y, n_k(y)\rangle\right| \le C_k|x-y|^2, \quad x,y \in \Gamma_k.$$

Using (11), we can estimate

$$\left|\partial_{n_k(y)} G(x-y,t-\tau)\right| \le C_k |x-y|^2 G_{n+2}(x-y,t-\tau).$$

The representation (7) implies that the upper bound is of the form

$$(t-\tau)^{-\frac{\alpha n}{2}+\alpha-1} F(r),$$

where $r = |x-y|(t-\tau)^{-\alpha/2}$ and

$$F(r) = r^{-n} H_{12}^{20}\left(\frac{1}{4}r^2 \,\middle|\, \begin{matrix} (\alpha,\alpha), & \\ (\frac{n+2}{2},1), & (1,1) \end{matrix}\right).$$

Since F is radial and integrable function and the integral is over $(n-1)$-dimensional domain, we may transfer the spatial integration to $\mathbb{R}^{n-1}$ and use [8, Theorem 2.1.10] to obtain an upper bound

$$\left|(W_{l,k}\varphi)(x,t)\right| \le C_k \int_0^t (t-\tau)^{\frac{\alpha}{2}-1} M_{\Gamma\cap B_l}(\varphi)(x,\tau)\mathrm{d}\tau.$$

Minkowsky's integral inequality implies the bound

$$\left\|(W_{l,k}\varphi)\right\|_{L^p(\Gamma\cap B_l\times(0,T))} \le C_k T^{\alpha/2} \|\varphi\|_{L^p(\Gamma\cap B_l\times(0,T))},$$

which in turn implies (i).

The Volterra property follows immediately since W is of convolution type integral in t. Therefore, we have established that the norm of W can be made small with T. The same argument shows that the $L^p(\Gamma\times(a,b))$-norm of W can be made small for all time intervals $(a,b)\subset(0,T)$ for which the length of the interval is small. Then the inverse of $-\frac{1}{2}I + W$ can be expanded as the Neumann series $-2\sum_{k=0}^{\infty}(2W)^k$ on $L^p(\Gamma\times(a,b))$, when $|a-b|$ is small enough. The iteration argument given in [5, pp. 325–326] implies that $-\frac{1}{2}I + W$ is invertible on $L^p(\Sigma_T)$ for any finite $T > 0$. Hence we have proved the *existence of the solution* in the form (27).

For the *uniqueness*, we can proceed as in [6, Theorem 2.3] and construct *Green's function* $\mathcal{G}$ as

$$\mathcal{G}(x,y,t) = G(x-y,t) - D\left(\left(-\frac{1}{2}I + W\right)^{-1} G(x-\cdot,\cdot)\right)(y,t)$$

for fixed $y\in\Omega$. Then

$$u(x,t) = \int_0^t \int_\Omega \mathcal{G}(x,y,t-\tau) f(y,\tau)\mathrm{d}y\mathrm{d}\tau$$

with smooth f is the classical solution of the nonhomogeneous Dirichlet problem (1), where the boundary datum g is zero and the nonhomogeneity f is the source term. We omit the details. □

5 Conclusions

This chapter illustrated how the method of layer potentials can be adapted to the study of (TFDE) in nonsmooth domains. The method of layer potentials and the use of ideas from harmonic analysis have shown to be powerful tools in analysing the initial-boundary value problems and they often lead to sharp results in the field of partial differential equations (PDEs). Therefore, the idea of using these tools in the nonlocal in time context is very natural.

There are some natural directions for further research related to this article. One can ask:

(i) what happens if the lateral datum has more smoothness, say $g \in H^{1,\frac{\alpha}{2}}(\Sigma_T)$. Here, $H^{1,\frac{\alpha}{2}}(\Sigma_T)$ denotes the L^2-based anisotropic Sobolev space. For their definition, see [19, Section 2]. In the literature, the problem (1) with $g \in H^{1,\frac{\alpha}{2}}(\Sigma_T)$ is called the *regularity problem*. This is studied in [10], where it is proved that the operator $-\frac{1}{2}I + W$ is invertible on $H^{1,\frac{\alpha}{2}}(\Sigma_T)$. The solution also for the regularity problem is given by (27) and satisfies

$$\|N(\nabla u)\|_{L^2(\Sigma_T)} \le C\|g\|_{H^{1,\frac{\alpha}{2}}(\Sigma_T)}.$$

Naturally the same question can be posed for the lateral datum in the L^p-based anisotropic Sobolev space.

(ii) whether one can relax the smoothness of the boundary Γ even more and consider the Lipschitz domain Ω. It is shown in the case of the heat equation that the operator norm of W on $L^p(\Sigma_T)$ does not necessarily tend to zero as T tends to zero. For the heat equation, this defect is removed by a Rellich identity [2, Formula (3.4)]. One could try to follow the arguments presented in [2]. However, we emphasize that the *fractional time derivative is not translation invariant*. Translation causes a history term which needs appropriate estimates.

(iii) that can one extend the technique to cover more general nonlocal in time subdiffusion equations such as the ones studied, for example, in [16]. Here, one could try to follow the technique presented in monograph [1] and in the references therein. It should be stressed that the exact form of the fundamental solution is not important. It is enough that the fundamental solution exists with some appropriate estimates.

(iv) that can the technique be used also for the diffusion-wave equation, that is, for the model (1) with $\alpha \in (1,2)$ and vanishing initial velocity $\partial_t u(x,0) = 0$. Since the fundamental solution has the same form (7) also in this case, it should straightforward to go through all the details in this case, too. However, note that the fundamental solution may no longer be nonnegative in higher dimensions, so we may have only upper estimates in the analogue of Lemma 2.1; see [9, Appendix B].

(v) if it is possible to obtain extrapolation-type solvability results presented in [1], that is, solvalibility of (26) in L^p-spaces automatically implies the solvability in a range of Besov spaces.

Bibliography

[1] A. Barton and S. Mayboroda, Layer potentials and boundary-value problems for second order elliptic operators with data in Besov spaces, *Mem. Am. Math. Soc.*, **243**(1149) (2016), v+110 pp.

[2] R. M. Brown, The method of layer potentials for the heat equation in Lipschitz cylinders, *Am. J. Math.*, **111**(2) (1989), 339–379.

[3] S. D. Eidelman and A. N. Kochubei, Cauchy problem for fractional diffusion equations, *J. Differ. Equ.*, **199**(2) (2004), 211–255.

[4] E. B. Fabes, M. Jodeit Jr., and N. M. Rivière, Potential techniques for boundary value problems on C^1-domains, *Acta Math.*, **141**(3–4) (1978), 165–186.

[5] E. B. Fabes and N. M. Rivière, Systems of parabolic equations with uniformly continuous coefficients, *J. Anal. Math.*, **17**(1) (1966), 305–335.

[6] E. B. Fabes and N. M. Rivière, Dirichlet and Neumann problems for the heat equation in C^1-cylinders, in *Proc. Sympos. Pure Math.*, vol. XXXV, Part 2, pp. 179–196, 1979.

[7] A. Friedman, *Partial Differential Equations of Parabolic Type*, reprint ed., Prentice-Hall, Inc., 1983.

[8] L. Grafakos, *Classical and Modern Fourier Analysis*, Pearson Education, Inc., 2004.

[9] A. Hanyga, Multidimensional solutions of time-fractional diffusion-wave equations, *R. Soc. Lond. Proc., Ser. A, Math. Phys. Eng. Sci.*, **458**(2020) (2002), 933–957.

[10] J. Kemppainen, *Behaviour of the boundary potentials and boundary integral solution of the time fractional diffusion equation*, PhD thesis, University of Oulu, 2010.

[11] J. Kemppainen, Boundary behaviour of the layer potentials for the time fractional diffusion equation in Lipschitz domains, *J. Integral Equ. Appl.*, **23**(4) (2011), 541–563.

[12] J. Kemppainen, Existence and uniqueness of the solution for a time-fractional diffusion equation, *Fract. Calc. Appl. Anal.*, **14**(3) (2011), 411–417.

[13] J. Kemppainen, Existence and uniqueness of the solution for a time fractional diffusion equation with Robin boundary condition, *Abstr. Appl. Anal.*, **2011** (2011), Article ID 321903.

[14] J. Kemppainen, Properties of the single layer potential for the time fractional diffusion equation, *J. Integral Equ. Appl.*, **23**(3) (2011), 437–455.

[15] J. Kemppainen and K. Ruotsalainen, Boundary integral solution of the time-fractional diffusion equation, *Integral Equ. Oper. Theory*, **64**(2) (2009), 239–249.

[16] J. Kemppainen, J. Siljander, and R. Zacher, Representation of solutions and large-time behavior for fully nonlocal diffusion equations, *J. Differ. Equ.*, **263**(1) (2017), 149–201.

[17] A. A. Kilbas and M. Saigo, *H-Transforms. Theory and Applications*, Analytical Methods and Special Functions, vol. 9, Chapman and Hall/CRC, Boca Raton, Florida, 2004.

[18] R. Kress, *Linear Integral Equations*, 2nd ed., Springer-Verlag New York, Inc., New York, 1999.

[19] J.-L. Lions and E. Magenes, *Non-Homogeneous Boundary Value Problems and Applications. Vol. II*, Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen, vol. 182, Springer-Verlag, 1972.

[20] Y. Luchko, Maximum principle for the generalized time-fractional diffusion, *J. Math. Anal. Appl.*, **351**(1) (2009), 218–223.

[21] N. M. Rivière, Singular integrals and multiplier operators, *Ark. Mat.*, **9** (1971), 243–278.

Anatoly N. Kochubei

Fractional-hyperbolic equations and systems. Cauchy problem

Abstract: The paper is a survey of results on the Cauchy problem for fractional-hyperbolic equations and systems with the Caputo–Djrbashian fractional time derivative. We describe a class of evolution systems of linear partial differential equations with the Caputo–Djrbashian fractional derivative of order $\alpha \in (0,1)$ in the time variable t and the first-order derivatives in spatial variables $x = (x_1, \dots, x_n)$, which can be considered as a fractional analogue of the class of hyperbolic systems. For such systems, we construct a fundamental solution of the Cauchy problem having exponential decay outside the fractional light cone $\{(t,x) : |t^{-\alpha}x| \le 1\}$. We consider an evolution equation of order $\beta \in (1,2)$ with respect to the time variable, and the second-order uniformly elliptic operator with variable coefficients acting in spatial variables. Properties of such equations are intermediate between those of parabolic and hyperbolic equations. We describe the parametrix method for constructing a fundamental solution of the Cauchy problem, formulate existence and uniqueness theorems for such equations, describe an analog of the principle of limiting amplitude (well known for the wave equation), and a pointwise stabilization property of solutions (similar to a well-known property of the heat equation).

Keywords: Fractional-hyperbolic system, diffusion-wave equation, Cauchy problem, fundamental solution, parametrix method, Caputo–Djrbashian fractional derivative, principle of limiting amplitude, pointwise stabilization property

MSC 2010: 35R11, 35L99

1 Introduction

Evolution equations with fractional time derivatives are among central objects of the modern theory of partial differential equations—due both to various physical applications (dynamical processes in fractal and viscoelastic media [37, 45, 55]) and to the rich mathematical content of this subject; see, for example, the monographs [11, 24] and references therein. The theory has reached sufficient maturity to investigate not only specific classes of equations like fractional diffusion and diffusion-wave equations, but to study general systems of fractional-differential equations. In particular,

Acknowledgement: The author is grateful to the anonymous referees for helpful remarks.

Anatoly N. Kochubei, Institute of Mathematics, National Academy of Sciences of Ukraine, Tereshchenkivska 3, Kyiv, 01004, Ukraine, e-mail: kochubei@imath.kiev.ua

https://doi.org/10.1515/9783110571660-010

Heibig [22] obtained an L^2-existence theorem for general fractional systems while the author [29] investigated a class of fractional-parabolic systems.

We begin this survey with a fractional version of the class of first-order hyperbolic systems with constant coefficients (for the classical material, see [2, 6, 18, 57]). A prototype is the fractional diffusion-wave equation

$$(\mathbb{D}_t^{(\beta)}u)(t,x) = \Delta u(t,x), \quad t \in (0,T], x \in \mathbb{R}^n, \tag{1}$$

where $1 < \beta < 2$, $\mathbb{D}_t^{(\beta)}$ is the Caputo–Djrbashian fractional derivative, that is,

$$\begin{aligned}(\mathbb{D}_t^{(\beta)}u)(t,x) = {}& \frac{1}{\Gamma(2-\beta)}\frac{\partial^2}{\partial t^2}\int_0^t (t-\tau)^{-\beta+1}u(\tau,x)\,d\tau - t^{-\beta+1}\frac{u_t'(0,x)}{\Gamma(2-\beta)} \\ & - t^{-\beta}\frac{u(0,x)}{\Gamma(1-\beta)};\end{aligned} \tag{2}$$

This definition is equivalent to

$$(\mathbb{D}_t^{(\beta)}u)(t,x) = \frac{1}{\Gamma(2-\beta)}\frac{\partial}{\partial t}\int_0^t (t-\tau)^{-\beta+1}u_\tau'(\tau,x)\,d\tau - t^{-\beta+1}\frac{u_t'(0,x)}{\Gamma(2-\beta)}, \tag{3}$$

whenever the integration by parts in (3) is possible. Below we explain how to reduce the equation (1) to a system of equations of order $\beta/2 \in (0,1)$ in t and the first order in spatial variables.

This equation describing the propagation of stress pulses in a viscoelastic medium [37] and its generalizations (linear and nonlinear fractional operator-differential equations with the operator $\mathbb{D}^{(\beta)}$, $1 < \beta < 2$) have been studied by many authors; see [4, 5, 17, 21, 23, 33, 35, 36, 39, 42, 47, 51, 53, 56] and references therein. Such equations can be interpreted also as special cases of abstract Volterra equations [50]. Here, we do not consider another interesting generalization of the classical equations, the fractional space-time wave equation; see [34, 38].

For the Cauchy problem corresponding to the initial conditions,

$$u(0,x) = u_0(x), \quad u_t'(0,x) = 0, \tag{4}$$

a solution is obtained as a convolution with the Green kernel,

$$u(t,x) = \int_{\mathbb{R}^n} G(t,x-\xi)u_0(\xi)\,d\xi, \tag{5}$$

where, as it was proved in [17] for $n = 1$, and in [51] for the general case,

$$|G(t,x)| \le Ct^{-\frac{\beta n}{2}}\gamma_n(|x|t^{-\beta/2})E(|x|t^{-\beta/2}),$$

$$\gamma_n(z) = \begin{cases} 1, & \text{if } n = 1; \\ |\log z| + 1, & \text{if } n = 2; \\ z^{-n+2}, & \text{if } n \geq 3, \end{cases}$$

$$E(z) = \exp(-az^{\frac{2}{2-\beta}}), \quad C, a > 0$$

(here and below the letters C, a will denote various positive constants). Below we will write an explicit expression of the kernel G in terms of the Wright function.

For a comparison, note that for the wave equation corresponding formally to $\beta = 2$, with the initial conditions (4), a counterpart of the kernel G from (5) is a distribution supported on the light cone $\{|x|t^{-1} \leq 1\}$, if n is even, or its boundary, if n is odd. There is a similar behavior, for example, for some symmetric hyperbolic systems of the first order.

Thus in the fractional case, though the fundamental solution of the Cauchy problem (FSCP) is not concentrated on the "fractional light cone," the set $\{|x|t^{-\frac{\beta}{2}} \leq 1\}$, it decays exponentially outside it, which can be interpreted as a kind of weak hyperbolicity property.

Following [30], we describe below (Sections 2, 3) a class of general time-fractional systems of partial differential equations with constant coefficients, of the form

$$(\mathbb{D}_t^{(\alpha)} u_j)(t,x) = \sum_{k=1}^{m} P_{jk}\left(i\frac{\partial}{\partial x}\right) u_k(t,x), \quad j = 1, \ldots, m, \ 0 < t \leq T, x \in \mathbb{R}^n, \tag{6}$$

where $0 < \alpha < 1$,

$$(\mathbb{D}_t^{(\alpha)} \varphi)(t) = \frac{1}{\Gamma(1-\alpha)} \left[\frac{\partial}{\partial t} \int_0^t (t-\tau)^{-\alpha} \varphi(\tau)\, d\tau - t^{-\alpha} \varphi(0) \right],$$

P_{jk} are first-order polynomials with complex coefficients from the derivatives $i\frac{\partial}{\partial x_1}, \ldots,$ $i\frac{\partial}{\partial x_n}$ ($i = \sqrt{-1}$), for which, as in the above example, there exists a FSCP with an exponential decay away from the set $\{|x|t^{-\alpha} \leq 1\}$. This property implies the existence of solutions for the initial functions with exponential growth. We call such systems "*fractional-hyperbolic*." The method used in [30] is an adaptation of the techniques by Gel'fand and Shilov [18] and Friedman [15], with modifications needed to cover the fractional situation. Note that a class of systems with the Riemann–Liouville fractional derivatives possessing similar properties was studied in [40, 41].

The second part of this survey (Sections 4–7) is devoted to another generalization of the equation (1)—a fractional diffusion-wave equation with variable coefficients of the form

$$Lu \equiv (\mathbb{D}_t^{(\beta)} u)(t,x) - \mathcal{B}u(t,x) = f(t,x) \tag{7}$$

where $1 < \beta < 2$,

$$\mathcal{B}u(t,x) = \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2 u(t,x)}{\partial x_i \partial x_j} + \sum_{j=1}^{n} b_j(x)\frac{\partial u(t,x)}{\partial x_j} + c(x)u(t,x),$$

and there exists such a constant $\delta_0 > 0$ that for any $x, \xi \in \mathbb{R}^n$

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \delta_0|\xi|^2. \tag{8}$$

We assume that $a_{ij} = a_{ji}$, the coefficients a_{ij}, b_k, c are bounded, uniformly Hölder continuous real-valued functions with the Hölder exponent γ satisfying the inequality

$$2 - \frac{2}{\beta} < \gamma \leq 1. \tag{9}$$

Here, we consider the general initial conditions

$$u(0,x) = u_0(x), \quad u_t'(0,x) = u_1(x). \tag{10}$$

The assumptions regarding the right-hand side f and the initial functions will be stated below.

The main task is to construct and study the fundamental solution of the Cauchy problem for the equation (7), thus extending the classical Levi method well known for parabolic equations and systems (see, e. g., [10, 11, 16]). For fractional diffusion equations (the case $0 < \beta < 1$), this method was implemented by Eidelman and the author [12]; see also [11, 29]. The Levi construction for fractional diffusion-wave equations was carried out in [32] using the results by Pskhu [51] who made explicit calculations (in terms of the Wright function) and gave precise estimates for the model equation (1). Here and below, the fundamental solution of the Cauchy problem is understood as a collection of three kernels $Z_1(t,x;\xi)$, $Z_2(t,x;\xi)$, $Y(t,x;\xi)$, such that the function

$$u(t,x) = \int_{\mathbb{R}^n} Z_1(t,x;\xi)u_0(\xi)\,d\xi + \int_{\mathbb{R}^n} Z_2(t,x;\xi)u_1(\xi)\,d\xi + \int_0^t d\tau \int_{\mathbb{R}^n} Y(t-\tau,x,;\xi)f(\tau,\xi)\,d\xi \tag{11}$$

is, under some conditions upon u_0, u_1, f, a classical solution of the Cauchy problem. This means that:

(i) $u(t,x)$ is twice continuously differentiable in x for each $t > 0$;

(ii) for each $x \in \mathbb{R}^n$ $u(t,x)$ is continuously differentiable in (t,x) on $[0,T] \times \mathbb{R}^n$, and the fractional integral

$$(I_{0+}^{2-\beta}u)(t,x) = \frac{1}{\Gamma(2-\beta)}\int_0^t (t-\tau)^{-\beta+1}u_\tau'(\tau,x)\,d\tau$$

is continuously differentiable in t for $t > 0$;

(iii) $u(t,x)$ satisfies the equation (where $\mathbb{D}_t^{(\beta)}$ is defined by (3)) and initial conditions.

If the kernels in (11) depend on the difference $x-\xi$, and also on some parameter η, we will write them also as $Z_1(t,x-\xi;\eta)$ etc. The iteration processes of the Levi method are carried out three times, separately for each of the above kernels.

Under appropriate additional assumptions, existence and uniqueness results for this Cauchy problem will be given.

The final part of this paper (Section 8) is about the asymptotic behavior of solutions of the Cauchy problem for the equation (1). Following [31], we study two kinds of asymptotic behavior typical for hyperbolic and parabolic equations, respectively, the principle of limiting amplitude and the stabilization property. We find that the fractional diffusion-wave equation possesses both of them simultaneously.

The principle of limiting amplitude is a classical property of the wave equation on $(0,\infty)\times\mathbb{R}^3$ [54] extended later to more general equations and domains: if $u(t,x)$ is a solution of the equation $\frac{\partial^2 u}{\partial t^2}-\Delta u = e^{-i\mu t}f(x)$ (f is a function with compact support) with zero initial conditions, then $u(t,x) = e^{-i\mu t}v(x)+o(1)$, $t\to\infty$, where v is a solution of the equation $-\Delta v = \mu^2 v+f$. Thus the solution u behaves asymptotically as the steady-state oscillation.

Turning to the equation (1), we first have to identify a counterpart of the oscillation $t\mapsto e^{-i\mu t}$. It is remarkable that an appropriate system of functions is already known. It has the form $\varphi_\omega(t) = E_\beta(i^\beta\omega^\beta t^\beta)$ where, as before, $1<\beta<2$, E_β is the Mittag-Leffler function, $\omega>0$. For these functions, $\mathbb{D}_t^{(\beta)}\varphi_\omega = i^\beta\omega^\beta\varphi_\omega$. As $t\to\infty$, $\varphi_\omega(t) = \frac{1}{\beta}e^{i\omega t}+O(t^{-\beta})$ (see [9, 24]; we follow the notation from [24] different from those in [9] or [49]). Note that the ray $\arg z = \frac{\pi\beta}{2}$ is the only one, on which the function $E_\beta(z)$ has an oscillatory character; it has no zeroes on this ray ([49], Theorem 4.2.1). Moreover, there exists an extensive theory of integral transforms based on the kernel φ_ω. In various aspects, this theory is parallel to the standard Fourier analysis; see [9, 19, 44].

Therefore, the principle of limiting amplitude for the equation (1) is formulated as follows. We consider a solution of the Cauchy problem for the equation (1) with $n\geq 3$, $f(t,x) = E_\beta(i^\beta\omega^\beta t^\beta)F(x)$. Under certain assumptions on F and the initial functions, we prove that

$$\frac{u(t,x)}{E_\beta(i^\beta\omega^\beta t^\beta)} \longrightarrow v(x), \quad t\to\infty, \tag{12}$$

for every $x\in\mathbb{R}^n$, where v is a solution of the equation $\Delta v - i^\beta\omega^\beta v = -F$. A result of this kind is obtained [31] also for an abstract equation $\mathbb{D}_t^{(\beta)}u + Au = f$ where A is a nonnegative self-adjoint operator on a Hilbert space.

The pointwise stabilization theorem for a solution $u(t,x)$ of the Cauchy problem with the initial condition $u(0,x) = u_0(x)$ for the heat equation $\frac{\partial u}{\partial t} = \Delta u$ is formulated as follows (see [13] and the survey papers [7, 8]).

Let u_0 be a continuous bounded function. The solution $u(t,x)$ tends, for every $x \in \mathbb{R}^n$, to a constant c, as $t \to \infty$, if and only if, for every $x_0 \in \mathbb{R}^n$,

$$\lim_{R\to\infty} \frac{1}{|K_R(x_0)|} \int\limits_{K_R(x_0)} u_0(x)\,dx = c \tag{13}$$

where $K_R(x_0)$ is a ball of radius R centered at x_0, $|K_R(x_0)|$ is its volume.

Note that the wave equation does not possess this property; only certain means of a solution stabilize [20]. Stabilization properties resembling the above one hold for hyperbolic equations with dissipative terms [26, 27].

In this paper, we prove a stabilization property of solutions of the diffusion-wave equation (1) similar to that of the heat equation. Thus, in this respect the equation (1) is closer to parabolic equations. For some other results regarding behavior of solutions of fractional diffusion-wave equations, see [25].

2 Fractional hyperbolicity

2.1 Definition

Let us consider a system (6). It is convenient to use its vector form

$$(\mathbb{D}_t^{(\alpha)} U)(t,x) = \mathbf{P}\left(i\frac{\partial}{\partial x}\right)U(t,x), \quad 0 < t \le T, x \in \mathbb{R}^n, \tag{14}$$

where U is a function with values in $\mathbb{C}^m$, $\mathbf{P}$ is a $m \times m$ matrix whose elements are first order differential operators. In the “dual” representation, $\mathbf{P}(s)$ is a matrix whose elements are polynomials in the variables $s_1, \dots, s_n$ of degree 1.

Denote by $\lambda_1(s), \dots, \lambda_m(s)$ the roots of the characteristic equation

$$\det(\mathbf{P}(s) - \lambda I) = 0.$$

Let

$$\Lambda_\alpha(s) = \max_{|\arg \lambda_k(s)| \le \alpha\pi/2} \operatorname{Re} \lambda_k^{1/\alpha}(s). \tag{15}$$

If $|\arg \lambda_k(s)| > \alpha\pi/2$ for all $k = 1, \dots, m$, then we set $\Lambda_\alpha(s) = 0$. In any case, $\Lambda_\alpha(s) \ge 0$.

We call the system (14) *fractional-hyperbolic*, if

$$\Lambda_\alpha(s) \le C(|\tau|^{1/\alpha} + \log(|\sigma| + 2)), \quad s \in \mathbb{C}^n, \tag{16}$$

where $s = \sigma + i\tau$, $\sigma, \tau \in \mathbb{R}^n$.

2.2 Special classes of systems

Let us consider systems satisfying the condition

$$\operatorname{Re}\lambda_k(s) \le a|\tau| + b \quad (k = 1,\dots,m) \tag{17}$$

where $a, b \ge 0$. Then (16) is satisfied. Indeed, if $|\arg \lambda_k(s)| \le \alpha\pi/2$, then

$$\operatorname{Re}\lambda_k^{1/\alpha}(s) = \left|\lambda_k(s)\right|^{1/\alpha} \cos\left(\frac{1}{\alpha}\arg\lambda_k(s)\right).$$

An elementary investigation shows that $\cos(\frac{1}{\alpha}\varphi) \le (\cos\varphi)^{1/\alpha}$, if $|\varphi| \le \alpha\pi/2$. Therefore,

$$\operatorname{Re}\lambda_k^{1/\alpha}(s) \le \left(\operatorname{Re}\lambda_k(s)\right)^{1/\alpha} \le \left(a|\tau| + b\right)^{1/\alpha} \le C\left(|\tau|^{1/\alpha} + 1\right),$$

which implies (16).

Note that the condition (17) is slightly stronger than the Gel'fand–Shilov hyperbolicity condition for differential systems [18].

An important class of fractional-hyperbolic systems is that of "symmetric systems"

$$(\mathbb{D}_t^{(\alpha)}U)(t,x) + \sum_{\nu=1}^{n} A_\nu \frac{\partial U(t,x)}{\partial x_\nu} + BU(t,x) = 0 \tag{18}$$

where A_ν, $\nu = 1,\dots,n$ are Hermitian matrices, B is an arbitrary matrix. In this case,

$$\mathbf{P}(s) = i\left(\sum_{\nu=1}^{n} s_\nu A_\nu + iB\right).$$

Let $\mu_k(s)$, $k = 1,\dots,n$, be the eigenvalues of the matrix $L(s) = \sum_{\nu=1}^{n} s_\nu A_\nu + iB$. Then $\operatorname{Re}\lambda_k(s) = \operatorname{Im}\mu_k(s)$ for each k. By Hirsch's theorem (see Theorem 1.3.1 in Chapter 3 of [43]), $|\mu_k(s)| \le nM$ where M is the maximum of absolute values of elements of the matrix $\frac{1}{2i}(L(s) - L(s)^*) = \sum_{\nu=1}^{n} \tau_\nu A_\nu + i(B - B^*)$. This implies the inequality (17).

2.3 Reduction of the diffusion-wave equation to a system

Let us consider the diffusion-wave equation (1) (where $1 < \beta < 2$) with the initial conditions

$$u(0,x) = u_0(x), \quad u_t'(0,x) = u_1(x), \tag{19}$$

where u_0, u_1 are continuous functions. In this section, we deal with classical solutions, that is we assume that $u \in C^2$ in the spatial variables, $u \in C^1$ jointly in all the variables, there exists, for each $t > 0$, the Riemann–Liouville fractional derivative

$$(D_{0+,t}^{\beta}u)(t,x) = \frac{1}{\Gamma(2-\beta)} \frac{\partial^2}{\partial t^2} \int_0^t (t-\tau)^{-\beta+1} u(\tau,x)\, d\tau,$$

and the equalities (1) and (19) are satisfied pointwise.

Denote $v(t,x) = (\mathbb{D}^{(\beta/2)}u)(t,x)$. It can be shown [30] that the Cauchy problem (1), (19) is reduced to the inhomogeneous problem

$$
\begin{aligned}
(\mathbb{D}_t^{(\beta/2)}v)(t,x) &= \Delta u(t,x) + u_1(x)\frac{t^{-\beta+1}}{\Gamma(2-\beta)},\\
(\mathbb{D}_t^{(\beta/2)}u)(t,x) &= v(t,x),\\
v(0,x) = 0,&\quad u(0,x) = u_0(x).
\end{aligned}
$$

If $u_1(x) \equiv 0$, so that we consider the conditions (4), we can reduce the problem further, to a system of the form (18). Namely, we set

$$
\begin{aligned}
v^0(t,x) &= (\mathbb{D}_t^{(\beta/2)}u)(t,x),\\
v^j(t,x) &= \frac{\partial u(t,x)}{\partial x_j},\quad j = 1,\dots,n.
\end{aligned}
$$

Denoting $V(t,x) = (v^0(t,x), v^1(t,x),\dots,v^n(t,x))$, we obtain the system

$$
(\mathbb{D}_t^{(\beta/2)}V)(t,x) + \sum_{j=1}^n A_j\frac{\partial V(t,x)}{\partial x_j} = 0
$$

where

$$
A_j = \begin{pmatrix}
0 & 0 & \dots & 0 & -1 & 0 & \dots & 0\\
0 & 0 & \dots & 0 & 0 & 0 & \dots & 0\\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots\\
-1 & 0 & \dots & 0 & 0 & 0 & \dots & 0\\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots\\
0 & 0 & \dots & 0 & 0 & 0 & \dots & 0
\end{pmatrix}
$$

with -1 in the $(j+1)$-th place.

3 Fundamental solution of the Cauchy problem

3.1 Resolvent matrix-function

Looking for a FSCP satisfying (14) with the initial condition $U(0,x) = \delta(x)$, we apply formally the Fourier transform in x getting the Cauchy problem

$$
(\mathbb{D}_t^{(\alpha)}\widetilde{U})(t,s) = \mathbf{P}(s)\widetilde{U}(t,s),\quad \widetilde{U}(0,s) = I,
$$

where I is the unit matrix. Then

$$
\widetilde{U}(t,s) = E_\alpha(t^\alpha\mathbf{P}(s)). \tag{20}
$$

The function (20) is called *the resolvent matrix-function* corresponding to the system (14).

Below we will denote the inverse Fourier transform (in the sense specified later) of the function (20) by $G(t,x)$ leaving the notation $U(t,x)$ for a vector-valued solution with a general initial vector function.

Proposition 1 ([30]). *For all $t \in (0,T)$, $x \in \mathbb{R}^n$,*

$$\|E_\alpha(t^\alpha \mathbf{P}(s))\| \le C(1+t^\alpha|s|)^{m-1} e^{t\Lambda_\alpha(s)}. \tag{21}$$

If our system (14) is fractional-hyperbolic, that is, it satisfies (16). Then it follows from (21) that

$$\|E_\alpha(t^\alpha \mathbf{P}(s))\| \le C(1+|\sigma|)^q e^{at|\tau|^{1/\alpha}} \tag{22}$$

with some $q > 0$. We call the minimal possible nonnegative integer q, for which (22) holds, *the exponent* of the system (14). In particular, if our system satisfies (17), then $q \le m-1$.

Note that

$$\mathbb{D}_t^{(\alpha)} E_\alpha(t^\alpha \mathbf{P}(s)) = \mathbf{P}(s) E_\alpha(t^\alpha \mathbf{P}(s)),$$

so that

$$\|\mathbb{D}_t^{(\alpha)} E_\alpha(t^\alpha \mathbf{P}(s))\| \le C(1+|\sigma|)^{q+1} e^{at|\tau|^{1/\alpha}},$$

since the matrix $\mathbf{P}(i\frac{\partial}{\partial x})$ contains only first-order differential operators.

3.2 The Cauchy problem

Let us consider the system (14) satisfying (16), with the initial condition $U(0,x) = U_0(x)$, $x \in \mathbb{R}^n$. A vector-function $U(t,x)$, $0 \le t \le T$, $x \in \mathbb{R}^n$, is called *a classical solution* of the Cauchy problem, if it is continuous in (t,x), as well as its first derivatives in x, the fractional integral

$$(I_{0+}^{1-\alpha} U)(t,x) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} U(\tau,x)\, d\tau$$

has the first derivative in t, continuous in (t,x), and the equation (14) and the initial condition are satisfied pointwise.

Though our aim is to construct a classical solution of the Cauchy problem, at the first stage it will be defined as a generalized solution. Given the estimate (22), the further reasoning is very similar to that of [15] (in particular, to the case of correctly posed systems with positive genus μ); only the exponent $\frac{p_0}{p_0-\mu}$, where p_0 is the reduced order, is replaced with $\frac{1}{1-\alpha}$.

Denote by $W_{\frac{1}{1-\alpha},c}$ ($c > 0$) the Frechet space of such functions $\varphi \in C^\infty(\mathbb{R})$ that for any $c' < c$,

$$\left|\varphi^{(j)}(z)\right| \le C_{j,c'} e^{-(1-\alpha)|c'z|^{\frac{1}{1-\alpha}}}, \quad j = 0, 1, 2, \dots,$$

for all $z \in \mathbb{R}$. Our basic space Φ of test functions is the direct product of n copies of $W_{\frac{1}{1-\alpha},c}$. We will need also a larger space Φ^0, the direct product of n copies of $W_{\frac{1}{1-\alpha},c-\varepsilon}$, $0 < \varepsilon < c$.

The Fourier transform $\mathcal{F}$ maps $W_{\frac{1}{1-\alpha},c}$ onto the space $W^{\frac{1}{\alpha},\frac{1}{c}}$ of such entire functions $\psi(s)$, $s = \sigma + i\tau$, $\sigma, \tau \in \mathbb{R}$, that for any $d > \frac{1}{c}$, $s \in \mathbb{C}$,

$$\left|s^k \psi(s)\right| \le C_{k,d} \exp\left(\alpha |d\tau|^{\frac{1}{\alpha}}\right), \quad k = 0, 1, 2, \dots.$$

Denote $\Psi = \mathcal{F}\Phi$, $\Psi^0 = \mathcal{F}\Phi^0$. These spaces have an obvious direct product structure.

It follows from the estimate (22) and the description of multipliers in $W^{\frac{1}{\alpha},\frac{1}{c}}$ ([18], Section I.2.4) that multiplication by $E_\alpha(t^\alpha \mathbf{P}(s))$ ($0 \le t \le T$) sends Ψ into Ψ^0, if T is small enough. Therefore, the above multiplication defines an action between the conjugate spaces $(\Psi^0)' \to \Psi'$. Thus, if $U_0 \in (\Phi^0)'$, then $\widetilde{U_0} = \mathcal{F}U_0 \in (\Psi^0)' \subset \Psi'$, so that $\widetilde{U}(t,s) = E_\alpha(t^\alpha \mathbf{P}(s))\widetilde{U_0}(s)$ is a generalized solution over Ψ. Therefore,

$$U(t,x) = \big(G(t,\cdot) * U_0(\cdot)\big)(x) \tag{23}$$

is a generalized solution in the sense of the distribution space Φ', defined for $t \le T$, if T is small enough.

Theorem 1. *There exists such a positive number γ that for any initial function U_0 possessing continuous derivatives $D^\nu U_0$, $|\nu| \le q + n + 3$ (q is the exponent of the system), such that*

$$\left|D^\nu U_0(x)\right| \le C e^{\gamma |x|^{\frac{1}{1-\alpha}}}, \quad x \in \mathbb{R}^n,$$

the generalized solution (23) *is a classical solution of the Cauchy problem, with the estimate*

$$\left|U(t,x)\right| \le C e^{\gamma' |x|^{\frac{1}{1-\alpha}}}, \quad 0 \le t \le T, x \in \mathbb{R}^n,$$

where $C, \gamma' > 0$ do not depend on t, x. The fundamental solution $G(t,x)$ has the form

$$G(t,x) = \sum_{k=1}^{M} R_k\left(\frac{\partial}{\partial x}\right) f_k(t,x) \tag{24}$$

(the differential operators R_k of orders $\le q+n+3$ are understood in the sense of $\mathcal{S}'(\mathbb{R}^n)$), f_k are continuous functions satisfying the estimates

$$\left|f_k(t,x)\right| \le Ce^{-\gamma_1 |t^{-\alpha}x|^{\frac{1}{1-\alpha}}}, \quad \gamma_1 > 0, \tag{25}$$

with the constants independent of t, x.

The properties (24)–(25) express the "fractional-hyperbolic" behavior of the class of systems considered in this paper.

The proof [30] follows the scheme of [15] and uses a theorem by Eskin [14] instead of the Paley–Wiener–Schwartz theorem. Note that the latter gives an interpretation of a subclass of the class of entire functions of exponential type as the set of Fourier transforms of distributions with compact supports. Eskin [14] considered the case of entire functions of order greater than 1.

4 Second-order equations with constant coefficients

4.1 Constructions

Considering the diffusion-wave equations (7), we begin with the case where the coefficients of $\mathcal{B}$ are constant, and only the leading terms are present, so that

$$\mathcal{B} = \sum_{i,j=1}^{n} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j}$$

where $A = (a_{ij})$ is a positive definite symmetric matrix. In fact, we will need a little more general situation, in which the coefficients a_{ij} depend on a parameter $\eta \in \mathbb{R}^n$, so that the functions $\eta \mapsto a_{ij}(\eta)$ are bounded and uniformly Hölder continuous, and the ellipticity condition (8) holds uniformly with respect to η.

Let $A(\eta) = (A^{(ij)})$ be the matrix inverse to (a_{ij}). Denote

$$\mathfrak{A}(y,\eta) = \sum_{i,j=1}^{n} A^{(ij)}(\eta) y_i y_j, \quad y, \eta \in \mathbb{R}^n.$$

By our assumptions,

$$C_1|y|^2 \le \mathfrak{A}(y,\eta) \le C_2|y|^2, \tag{26}$$

$$\left|\mathfrak{A}(y,\eta') - \mathfrak{A}(y,\eta'')\right| \le C\left|\eta' - \eta''\right|^{\gamma} |y|^2, \tag{27}$$

$$\left|\left[\det A(\eta')\right]^{1/2} - \left[\det A(\eta'')\right]^{1/2}\right| \le C\left|\eta' - \eta''\right|^{\gamma}. \tag{28}$$

Here and below, we denote by C (with indices or exponents or without them) various positive constants. Positive constants appearing under the sign of exponential will be denoted σ, while the exponents of the Hölder continuity are all denoted by the same letter γ.

Given a fundamental solution $Z_{1,0}(t,x-\xi)$, $Z_{2,0}(t,x-\xi)$, $Y_0(t,x-\xi)$ of the Cauchy problem for the equation (1) (see the Introduction), we can write a similar triple for our present case setting

$$Z_k^{(0)}(t,x-\xi;\eta) = \frac{1}{[\det A(\eta)]^{1/2}} Z_{k,0}\big(t,[\mathfrak{A}(x-\xi,\eta)]^{1/2}\big), \quad k=1,2, \tag{29}$$

$$Y^{(0)}(t,x-\xi;\eta) = \frac{1}{[\det A(\eta)]^{1/2}} Y_0\big(t,[\mathfrak{A}(x-\xi,\eta)]^{1/2}\big); \tag{30}$$

the motivation for this notation will become clear later. Note that the transition from $Z_{k,0}$ and Y_0 to $Z_k^{(0)}$ and $Y^{(0)}$ can be interpreted as a linear change of variables; see the proof of Theorem 1, Chapter 1, in [16].

An expression for the fundamental solution in terms of Fox's H-function was found in [53]. We use a simpler expression found by Pskhu [51]:

$$\begin{aligned} &Z_{1,0}(t,x) = D_{0t}^{\beta-1}\Gamma_{\beta,n}(t,x), \quad Z_{2,0}(t,x) = D_{0t}^{\beta-2}\Gamma_{\beta,n}(t,x),\\ &Y_0(t,x) = \Gamma_{\beta,n}(t,x), \end{aligned} \tag{31}$$

where, following [51], we use the unified notation for the Riemann–Liouville integrals and derivatives: if s is the initial point, and $p \in \mathbb{N}$, $p-1 < k \le p$, then

$$D_{st}^{k} g(t) = \operatorname{sign}^p(t-s)\left(\frac{\partial}{\partial t}\right)^p D_{st}^{k-p} g(t),$$

$$D_{st}^{\mu} g(t) = \frac{\operatorname{sign}(t-s)}{\Gamma(-\mu)} \int_s^t g(\tau)(t-\tau)^{-\mu-1}\,d\tau, \quad \mu < 0,$$

$D_{st}^0 g(t) = g(t)$. The function $\Gamma_{\beta,n}$ is defined as follows:

$$\Gamma_{\beta,n}(t,x) = c_n t^{\beta-\frac{\beta n}{2}-1} f_{\beta/2}\left(t^{-\beta/2}|x|; n-1, \beta-\frac{\beta n}{2}\right). \tag{32}$$

Here,

$$f_{\beta/2}(z;\mu,\delta) = \begin{cases} \frac{2}{\Gamma(\mu/2)}\int_1^\infty \Phi(-\beta/2,\delta,-zt)(t^2-1)^{\frac{\mu}{2}-1}\,dt, & \text{if } \mu > 0;\\ \Phi(-\beta/2,\delta,-z), & \text{if } \mu = 0, \end{cases}$$

$$\Phi(-\beta/2,\delta,z) = \sum_{m=0}^{\infty} \frac{z^m}{m!\,\Gamma(\delta-\frac{\beta m}{2})}$$

is the Wright function, $c_n = 2^{-n}\pi^{(1-n)/2}$. A series representation of the function $f_{\beta/2}$ (see [51]) shows that, if $n > 1$, the fundamental solution has a singularity in spatial variables.

The work with expressions containing $f_{\beta/2}$ is simplified by the identities [51]

$$\frac{d}{dz} f_{\beta/2}(z;\mu,\delta) = -\frac{z}{2} f_{\beta/2}(z;\mu+2,\delta-\beta); \tag{33}$$

$$D_{st}^{\zeta} |t-s|^{\delta-1} f_{\beta/2}(|t-s|^{-\beta/2} z;\mu,\delta) = |t-s|^{\delta-\zeta-1} f_{\beta/2}(|t-s|^{-\beta/2} z;\mu,\delta-\zeta),$$
$$\zeta \in \mathbb{R}. \tag{34}$$

4.2 Estimates

Using the estimates for the integer and fractional order derivatives of the function $\Gamma_{\beta,n}$ found in [51] and the property (26), we find estimates of the kernels $Z_1^{(0)}$, $Z_2^{(0)}$, $Y^{(0)}$ and their derivatives (some higher derivatives absent in [51] are treated easily using (33)). The estimates are different for $n \geq 3$, $n = 2$, and $n = 1$. Therefore, we consider these cases separately. Denote

$$\rho_\sigma(t,x,\xi) = \exp\{-\sigma(t^{-\beta/2}|x-\xi|)^{\frac{2}{2-\beta}}\}, \quad \sigma > 0.$$

Let $n \geq 3$. Then

$$|D_x^m Z_1^{(0)}(t,x-\xi;\eta)| \leq C t^{-\beta} |x-\xi|^{-n+2-|m|} \rho_\sigma(t,x,\xi), \quad |m| \leq 3; \tag{35}$$

$$|D_x^m Z_2^{(0)}(t,x-\xi;\eta)| \leq C t^{-\beta+1} |x-\xi|^{-n+2-|m|} \rho_\sigma(t,x,\xi), \quad |m| \leq 3; \tag{36}$$

$$|Y^{(0)}(t,x-\xi;\eta)| \leq C t^{\beta-\frac{\beta n}{2}-1} \mu_n(t^{-\beta/2}|x-\xi|) \rho_\sigma(t,x,\xi), \tag{37}$$

where

$$\mu_n(z) = \begin{cases} 1, & \text{if } n = 3; \\ 1 + |\log z|, & \text{if } n = 4; \\ z^{-n+4}, & \text{if } n \geq 5. \end{cases}$$

Next,

$$\left| \frac{\partial}{\partial x_i} Y^{(0)}(t,x-\xi;\eta) \right| \leq C t^{-\beta-1} |x-\xi|^{-n+3} \rho_\sigma(t,x,\xi). \tag{38}$$

Making the estimates a little rougher, we can unify (37) and (38), together with the estimates for second- and third-order derivatives, into the following unified estimate:

$$|D_x^m Y^{(0)}(t,x-\xi;\eta)| \leq C t^{-1} |x-\xi|^{-n+2-|m|} \rho_\sigma(t,x,\xi), \quad |m| \leq 3, \tag{39}$$

which is used in the implementation of Levi's method. However, the initial estimate (38) is also useful (for the proof of the uniqueness theorem).

The above transformation of estimates is based on a procedure frequently used throughout the paper—we can drop a positive power of the expression $t^{-\beta/2}|x-\xi|$, simultaneously taking a smaller $\sigma > 0$ in the factor ρ_σ.

The estimates for time derivatives of the functions $Z_1^{(0)}$, $Z_2^{(0)}$, $Y^{(0)}$ are as follows:

$$\left|\frac{\partial}{\partial t} Z_1^{(0)}(t,x-\xi;\eta)\right| \le Ct^{-\beta-1}|x-\xi|^{-n+2}\rho_\sigma(t,x,\xi); \tag{40}$$

$$\left|\frac{\partial}{\partial t} Z_2^{(0)}(t,x-\xi;\eta)\right| \le Ct^{-\beta}|x-\xi|^{-n+2}\rho_\sigma(t,x,\xi); \tag{41}$$

$$\left|\frac{\partial}{\partial t} Y^{(0)}(t,x-\xi;\eta)\right| \le Ct^{\beta-\frac{\beta n}{2}-2}\mu_n(t^{-\beta/2}|x-\xi|)\rho_\sigma(t,x,\xi), \tag{42}$$

$$\left|\mathbb{D}_t^{(\beta)} Z_1^{(0)}(t,x-\xi;\eta)\right| \le Ct^{-2\beta}|x-\xi|^{-n+2}\rho_\sigma(t,x,\xi); \tag{43}$$

$$\left|\mathbb{D}_t^{(\beta)} Z_2^{(0)}(t,x-\xi;\eta)\right| \le Ct^{-2\beta+1}|x-\xi|^{-n+2}\rho_\sigma(t,x,\xi); \tag{44}$$

$$\left|\mathbb{D}_t^{(\beta)} Y^{(0)}(t,x-\xi;\eta)\right| \le Ct^{-\beta-1}|x-\xi|^{-n+2}\rho_\sigma(t,x,\xi). \tag{45}$$

Let $n = 2$. Then

$$\left|Z_1^{(0)}(t,x-\xi;\eta)\right| \le Ct^{-\beta}\left[\left|\log(t^{-\beta/2}|x-\xi|)\right|+1\right]\rho_\sigma(t,x,\xi); \tag{46}$$

$$\left|Z_2^{(0)}(t,x-\xi;\eta)\right| \le Ct^{-\beta+1}\left[\left|\log(t^{-\beta/2}|x-\xi|)\right|+1\right]\rho_\sigma(t,x,\xi); \tag{47}$$

$$\left|Y^{(0)}(t,x-\xi;\eta)\right| \le Ct^{-1}\rho_\sigma(t,x,\xi); \tag{48}$$

$$\left|D_x^m Y^{(0)}(t,x-\xi;\eta)\right| \le Ct^{-\beta-1}\left[\left|\log(t^{-\beta/2}|x-\xi|)\right|+1\right]\rho_\sigma(t,x,\xi), \quad |m|=1; \tag{49}$$

$$\left|\frac{\partial}{\partial t} Z_1^{(0)}(t,x-\xi;\eta)\right| \le Ct^{-\beta-1}\left[\left|\log(t^{-\beta/2}|x-\xi|)\right|+1\right]\rho_\sigma(t,x,\xi); \tag{50}$$

$$\left|\frac{\partial}{\partial t} Z_2^{(0)}(t,x-\xi;\eta)\right| \le Ct^{-\beta}\left[\left|\log(t^{-\beta/2}|x-\xi|)\right|+1\right]\rho_\sigma(t,x,\xi); \tag{51}$$

$$\left|\frac{\partial}{\partial t} Y^{(0)}(t,x-\xi;\eta)\right| \le Ct^{-2}\rho_\sigma(t,x,\xi); \tag{52}$$

$$\left|\mathbb{D}_t^{(\beta)} Z_1^{(0)}(t,x-\xi;\eta)\right| \le Ct^{-2\beta}\left[\left|\log(t^{-\beta/2}|x-\xi|)\right|+1\right]\rho_\sigma(t,x,\xi); \tag{53}$$

$$\left|\mathbb{D}_t^{(\beta)} Z_2^{(0)}(t,x-\xi;\eta)\right| \le Ct^{-2\beta+1}\left[\left|\log(t^{-\beta/2}|x-\xi|)\right|+1\right]\rho_\sigma(t,x,\xi); \tag{54}$$

$$\left|\mathbb{D}_t^{(\beta)} Y^{(0)}(t,x-\xi;\eta)\right| \le Ct^{-\beta-1}\left[\left|\log(t^{-\beta/2}|x-\xi|)\right|+1\right]\rho_\sigma(t,x,\xi). \tag{55}$$

The estimates (35) and (36) with $1 \le m \le 3$, as well as the estimate (39) with $2 \le m \le 3$, remain valid for $n = 2$.

In the one-dimensional case ($n = 1$), our kernels have no singularity with respect to the spatial variables. The estimates are as follows:

$$\left|D_x^m Z_1^{(0)}(t,x-\xi;\eta)\right| \le Ct^{-\frac{\beta}{2}(m+1)}\rho_\sigma(t,x,\xi); \tag{56}$$

$$\left|D_x^m Z_2^{(0)}(t,x-\xi;\eta)\right| \le Ct^{-\frac{\beta}{2}(m+1)+1}\rho_\sigma(t,x,\xi); \tag{57}$$

$$|D_x^m Y^{(0)}(t,x-\xi;\eta)| \le Ct^{-\frac{\beta}{2}(m-1)-1}\rho_\sigma(t,x,\xi). \tag{58}$$

Here, $0 \le m \le 3$. There is a more refined estimate for $m = 1$:

$$\left|\frac{\partial}{\partial x}Y^{(0)}(t,x-\xi;\eta)\right| \le Ct^{-1-\frac{\beta}{2}}|x|\rho_\sigma(t,x,\xi). \tag{59}$$

Next,

$$\left|\frac{\partial}{\partial t}Z_1^{(0)}(t,x-\xi;\eta)\right| \le Ct^{-1-\frac{\beta}{2}}\rho_\sigma(t,x,\xi); \tag{60}$$

$$\left|\frac{\partial}{\partial t}Z_2^{(0)}(t,x-\xi;\eta)\right| \le Ct^{-\frac{\beta}{2}}\rho_\sigma(t,x,\xi); \tag{61}$$

$$\left|\frac{\partial}{\partial t}Y^{(0)}(t,x-\xi;\eta)\right| \le Ct^{\frac{\beta}{2}-2}\rho_\sigma(t,x,\xi). \tag{62}$$

The estimates of $\mathbb{D}_t^{(\beta)}Z_1^{(0)}$, $\mathbb{D}_t^{(\beta)}Z_2^{(0)}$, $\mathbb{D}_t^{(\beta)}Y^{(0)}$ coincide with those for the appropriate second spatial derivatives.

It follows from Lemma 14 in [51] that the following integral formulas hold:

$$\int\limits_{\mathbb{R}^n} Z_1^{(0)}(t,x-\xi;\eta)\,dx = 1; \qquad \int\limits_{\mathbb{R}^n} Z_2^{(0)}(t,x-\xi;\eta)\,dx = t; \tag{63}$$

$$\int\limits_{\mathbb{R}^n} Y^{(0)}(t,x-\xi;\eta)\,dx = \frac{t^{\beta-1}}{\Gamma(\beta)}. \tag{64}$$

5 Levi's method

5.1 Construction

We look for the functions Z_1, Z_2, Y appearing in (11) assuming the integral representations:

$$Z_l(t,x;\xi) = Z_l^{(0)}(t,x-\xi;\xi) + \int\limits_0^t d\lambda \int\limits_{\mathbb{R}^n} Y^{(0)}(t-\lambda,x-y;y)Q_l(\lambda,y;\xi)\,dy; \tag{65}$$

$$Q_l(t,x;\xi) = M_l(t,x;\xi) + \int\limits_0^t d\lambda \int\limits_{\mathbb{R}^n} K(t-\lambda,x;y)Q_l(\lambda,y;\xi)\,dy, \tag{66}$$

where $l = 1,2$,

$$\begin{aligned} M_l(t,x;\xi) = {} & \sum_{i,j=1}^n \left\{[a_{ij}(x)-a_{ij}(\xi)]\frac{\partial^2}{\partial x_i \partial x_j}Z_l^{(0)}(t,x-\xi;\xi)\right\} \\ & + \sum_{j=1}^n b_j(x)\frac{\partial Z_l^{(0)}(t,x-\xi;\xi)}{\partial x_j} + c(x)Z_l^{(0)}(t,x-\xi;\xi); \end{aligned} \tag{67}$$

$$K(t,x;\xi) = \sum_{i,j=1}^{n} \left\{ [a_{ij}(x) - a_{ij}(\xi)] \frac{\partial^2}{\partial x_i \partial x_j} Y^{(0)}(t, x-\xi;\xi) \right\} + \sum_{j=1}^{n} b_j(x) \frac{\partial Y^{(0)}(t, x-\xi;\xi)}{\partial x_j} + c(x) Y^{(0)}(t, x-\xi;\xi). \tag{68}$$

Thus, the desired kernels consist of those for the equation with coefficients "frozen" at the parametric point ξ, plus correction terms constructed (as solutions of appropriate integral equations) in such a way that $LZ_l = 0$ for $x \neq \xi$.

Similarly,

$$Y(t,x;\xi) = Y^{(0)}(t, x-\xi;\xi) + \int_0^t d\lambda \int_{\mathbb{R}^n} Y^{(0)}(t-\lambda, x-y; y) \Psi(\lambda, y;\xi)\, dy; \tag{69}$$

$$\Psi(t,x;\xi) = K(t,x;\xi) + \int_0^t d\lambda \int_{\mathbb{R}^n} K(t-\lambda, x; y) \Psi(\lambda, y;\xi)\, dy, \tag{70}$$

so that $LY = 0$ for $x \neq \xi$.

The implementation of Levi's method in the present situation [32] is based on the above estimates of $Z_1^{(0)}$, $Z_2^{(0)}$, and $Y^{(0)}$, as well as estimates of their differences and increments.

5.2 Estimates

Theorem 2. *The Levi method kernels have the form*

$$Z_j(t,x,\xi) = Z_j^{(0)}(t, x-\xi;\xi) + V_{Z_j}(t,x;\xi), \quad j = 1,2;$$
$$Y(t,x,\xi) = Y^{(0)}(t, x-\xi;\xi) + V_Y(t,x;\xi),$$

where $Z_j^{(0)}$ and $Y^{(0)}$ satisfy the estimates (35)–(62). *If $n \geq 2$, then*

$$\left| D_x^m V_{Z_1}(t,x;\xi) \right| \leq C t^{\nu_0 \beta - 1} |x-\xi|^{-n-|m|+(\gamma-\nu_1)+(2-\nu_0)} \rho_\sigma(t,x,\xi); \tag{71}$$

$$\left| D_x^m V_{Z_2}(t,x;\xi) \right| \leq C t^{\frac{\nu_0 \beta}{2}+1-\beta} |x-\xi|^{-n-|m|+(\gamma-\nu_1)+(2-\nu_0)} \rho_\sigma(t,x,\xi), \tag{72}$$

$|m| = 0,1$;

$$\left| D_x^m V_{Z_1}(t,x;\xi) \right| \leq C t^{\gamma_1 - \beta} |x-\xi|^{-n+\gamma_2} \rho_\sigma(t,x,\xi); \tag{73}$$

$$\left| D_x^m V_{Z_2}(t,x;\xi) \right| \leq C t^{\gamma_1 - \beta + 1} |x-\xi|^{-n+\gamma_2} \rho_\sigma(t,x,\xi), \tag{74}$$

if $|m| = 2$; here, γ_1 and γ_2 are some positive constants,

$$\left| V_Y(t,x;\xi) \right| \leq C t^{\nu_0 \beta - 1} |x-\xi|^{-n+(\gamma-\nu_1)+(2-\nu_0)} \rho_\sigma(t,x,\xi); \tag{75}$$

$$\left|\frac{\partial}{\partial x_i} V_Y(t,x;\xi)\right| \le Ct^{\frac{\beta\varkappa}{2}+\frac{\nu_0\beta}{2}-1}|x-\xi|^{-n+1-\varkappa+\gamma-\nu_1}\rho_\sigma(t,x,\xi), \quad n \ge 3, \tag{76}$$

where $0 < \varkappa < \gamma - \nu_1$. *If* $n = 2$, *then*

$$\left|\frac{\partial}{\partial x_i} V_Y(t,x;\xi)\right| \le Ct^{\frac{\beta\varkappa}{2}+\frac{\nu_0\beta}{2}-1}|x-\xi|^{-1-\varkappa-\mu+\gamma-\nu_1}\rho_\sigma(t,x,\xi), \tag{77}$$

where $\mu > 0$, $0 < \varkappa + \mu < \gamma - \nu_1$;

$$\left|D_x^m V_Y(t,x;\xi)\right| \le Ct^{-1+\gamma_1}|x-\xi|^{-n+\gamma_2}\rho_\sigma(t,x,\xi), \tag{78}$$

if $|m| = 2$; *here,* γ_1 *and* γ_2 *are positive constants.*

If $n = 1$, *then*

$$\left|D_x^m V_{Z_1}(t,x;\xi)\right| \le Ct^{(\gamma-m-1)\beta/2}\rho_\sigma(t,x,\xi); \tag{79}$$

$$\left|D_x^m V_{Z_2}(t,x;\xi)\right| \le Ct^{(\gamma-m-1)\beta/2+1}\rho_\sigma(t,x,\xi); \tag{80}$$

$$\left|D_x^m V_Y(t,x;\xi)\right| \le Ct^{\beta-1+(\gamma-m-1)\beta/2}\rho_\sigma(t,x,\xi), \tag{81}$$

$m = 0, 1, 2$.

5.3 Potentials

For our situation, an analog of the heat potential is the function

$$W(t,x) = \int_0^t d\lambda \int_{\mathbb{R}^n} Y^{(0)}(t-\lambda, x-y; y) f(\lambda, y)\, dy. \tag{82}$$

We assume that $f(\lambda, y)$ is a bounded function, jointly continuous in $(\lambda, y) \in [0,T] \times \mathbb{R}^n$, and locally Hölder continuous in y, uniformly with respect to λ.

It is straightforward to check, using the estimates for $Y^{(0)}$, that the first derivatives in x of $W(t,x)$ can be obtained by differentiating under the sign of integral in (82). Other derivatives are considered in the next proposition.

Proposition 2. *The following differentiation formulas are valid:*

$$\frac{\partial}{\partial t} W(t,x) = \int_0^t d\lambda \int_{\mathbb{R}^n} \frac{\partial}{\partial t} Y^{(0)}(t-\lambda, x-y; y) f(\lambda, y)\, dy, \tag{83}$$

$$\begin{aligned} \frac{\partial^2}{\partial x_i \partial x_j} W(t,x) = {} & \int_0^t d\lambda \int_{\mathbb{R}^n} \frac{\partial^2 Y^{(0)}(t-\lambda, x-y; y)}{\partial x_i \partial x_j} [f(\lambda, y) - f(\lambda, x)]\, dy \\ & + \int_0^t f(\lambda, x) d\lambda \int_{\mathbb{R}^n} \frac{\partial^2 Y^{(0)}(t-\lambda, x-y; y)}{\partial x_i \partial x_j}\, dy; \end{aligned} \tag{84}$$

$$\mathbb{D}_t^{(\alpha)} W(t,x) = f(t,x) + \int_0^t d\lambda \int_{\mathbb{R}^n} \frac{\partial Z_1^{(0)}(t-\lambda, x-y; y)}{\partial t} [f(\lambda, y) - f(\lambda, x)]\, dy$$
$$+ \int_0^t f(\lambda, x) d\lambda \int_{\mathbb{R}^n} \frac{\partial Z_1^{(0)}(t-\lambda, x-y; y)}{\partial t}\, dy. \tag{85}$$

Note that the formulas of Proposition 2 remain valid if the role of f is played by the functions Q_1, Q_2, and Ψ. In these cases, instead of the simple Hölder condition, we use, in a similar way, the increment estimates; see [32] for the details. The above results show that the Levi method constructions indeed produce solutions of the equation (7).

Note also that the differentiation formula (83) remains valid if

$$|f(\lambda, y)| \le C\lambda^{-a}, \quad 0 < a < 1.$$

This case is important for checking the initial conditions.

6 Cauchy problem for the diffusion-wave equations

6.1 The case of zero initial functions

Let us consider the equation (7), under the assumptions on coefficients formulated in the Introduction, with the initial conditions

$$u(0,x) = \frac{\partial u(0,x)}{\partial t} = 0. \tag{86}$$

Theorem 3. *If f is a bounded function, jointly continuous in (t,x) and locally Hölder continuous in x, uniformly with respect to t, then*

$$u(t,x) = \int_0^t d\tau \int_{\mathbb{R}^n} Y(t-\tau, x; \xi) f(\tau, \xi)\, d\xi$$

is a bounded classical solution of the equation (7) *satisfying the initial conditions* (86).

6.2 The homogeneous equation

Here, we consider the more complicated case of the equation (7) with $f = 0$ and the initial conditions (10). We assume (in addition to the assumptions formulated in the Introduction) that:

(A) $u_0(x)$ is bounded, continuously differentiable, and its first derivatives are bounded and Hölder continuous with the exponent $\gamma_0 > \frac{2-\beta}{\beta}$;
(B) $u_1(x)$ is bounded and continuous. If $n > 1$, u_1 is Hölder continuous.
(C) The coefficients a_{ij} are twice continuously differentiable with bounded derivatives of order ≤ 2.

Theorem 4. *Under the above assumptions, the function*

$$u(t,x) = \int_{\mathbb{R}^n} Z_1(t,x;\xi)u_0(\xi)\,d\xi + \int_{\mathbb{R}^n} Z_2(t,x;\xi)u_1(\xi)\,d\xi$$

is a bounded classical solution of the equation (7) (*with* $f = 0$) *satisfying the initial conditions* (10).

For the proof, see [32].

7 Uniqueness theorem

7.1 The adjoint problem

For equations with variable coefficients and $0 < \beta < 1$, uniqueness theorems were proved in [11, 12, 28] using the maximum principle arguments. For $1 < \beta < 2$, the structure of the fractional derivative $\mathbb{D}_t^{(\beta)}$ is different—its Marchaud forms (see [52]) contain either the first derivative or the second difference, thus being not suitable for the maximum principle. Note also that the positivity of the function $Y^{(0)}$ is violated for $1 < \beta < 2$, $n \geq 4$ [51].

Therefore, it is natural to try for $1 < \beta < 2$ another classical method [16] based on the representation of solutions using a fundamental solution of the adjoint problem. Classically, the adjoint problem is a kind of the Cauchy problem with data on the right end of the interval. The time derivative is preserved in the adjoint operator, only with a different sign.

In the fractional case, it is known [48] that the operator, adjoint to the (left-sided) Caputo–Dzhrbashyan fractional derivative $\mathbb{D}^{(\beta)}$ is the right-sided Riemann–Liouville fractional derivative. In contrast to the classical situation, the form of the latter operator depends on the interval on which the adjoint problem is considered. This makes the use of adjoints more complicated necessitating their subtler definition. Such a definition was proposed by Pskhu [51] for the equation (1). Below his approach is adapted to our general case.

In this section, we assume, in addition to the assumptions from Introduction, that the following holds:

(D) All the functions

$$\frac{\partial a_{ij}}{\partial x_k}, \quad \frac{\partial^2 a_{ij}}{\partial x_k \partial x_l}, \quad \frac{\partial b_i}{\partial x_k} \quad (i,j,k,l = 1,\dots,n)$$

exist, are bounded, and uniformly Hölder continuous.

In the adjoint operator $\mathcal{B}^*$ with respect to the spatial variables,

$$\mathcal{B}^* u(t,x) = \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2 u(t,x)}{\partial x_i \partial x_j} + \sum_{k=1}^{n} b_k^*(x) \frac{\partial u(t,x)}{\partial x_k} + c^*(x) u(t,x),$$

the higher coefficients a_{ij} are the same as in $\mathcal{B}$ (we have assumed that $a_{ij} = a_{ji}$),

$$b_i^*(x) = -b_i(x) + 2 \sum_{j=1}^{n} \frac{\partial a_{ij}(x)}{\partial x_j},$$

$$c^*(x) = c(x) - \sum_{i=1}^{n} \frac{\partial b_i(x)}{\partial x_i} + \sum_{i,j=1}^{n} \frac{\partial^2 a_{ij}(x)}{\partial x_i \partial x_j}.$$

Let $S \subset \mathbb{R}^n$ be a bounded domain with a smooth boundary ∂S. Denote $E = \{(t,x) : 0 < t < T, x \in S\}$, $E_t = \{(\eta, x) : \ 0 < \eta < t, x \in S\}$. In agreement with the earlier definition, we call $u(t,x)$ a classical solution of the equation (7) on E, if: 1) it satisfies on $\overline{S}$ the conditions (i)–(ii) from the definition of the classical solution from the Introduction; in particular, for each $x \in \overline{S}$, there exist continuous on $\overline{S}$ limits u_0, u_1 of $u(t,x)$, and $\frac{\partial u(t,x)}{\partial t}$, respectively, as $t \to 0$; 2) $u(t,x)$ satisfies (7) at all the points $(t,x) \in E$.

The adjoint operator L^* acts on functions $v(t,x;\eta,\xi)$, in the variables η, ξ, where $\eta < t$, as follows:

$$L^* v = D_{t\eta}^{\beta} v(t,x;\eta,\xi) - \mathcal{B}_{\xi}^* v(t,x;\eta,\xi).$$

Here, $D_{t\eta}^{\beta}$ (see the definition in Section 4.1 taken from [51]) is the right-sided Riemann–Liouville derivative in the variable η with the base point t.

If we consider the terminal value problem for the equation $L^* v = g, g = g(t,x;\eta,\xi)$, with zero terminal condition at $\eta = t$, this problem is equivalent, via time reflection, to the homogeneous Cauchy problem considered above, with $\mathcal{B}^*$ substituted for $\mathcal{B}$. Therefore, under our condition (D), for such a terminal value problem there exists a fundamental solution

$$Y_*(t,x;\eta,\xi) = Y_*^{(0)}(t-\eta, x-\xi, x) + V_{Y_*}(t,x;\eta,\xi) \tag{87}$$

satisfying the same estimates as the function Y above. Below, dealing with estimates for $Y_*^{(0)}$ and V_{Y_*} we will refer to the appropriate estimates for $Y^{(0)}$ and V_Y. One should only remember that the operators will act on the function (87) in the variable ξ, and x will be the integration variable.

Suppose that a function $v(t,x;\eta,\xi)$ is continuous on $Q \times \overline{E_t}$, $Q \subset E$, together with its first and second derivatives in ξ and its fractional derivatives $D^{\beta}_{t\eta}$ and $D^{\beta-1}_{t\eta}$. We also assume that

$$\lim_{\eta\to t}(D^{\beta-1}_{t\eta}v)(t,x;\eta,\xi) = \lim_{\eta\to t}(D^{\beta-2}_{t\eta}v)(t,x;\eta,\xi) = 0.$$

Denote

$$w(t,x;\eta,\xi) = v(t,x;\eta,\xi) + Y_*(t,x;\eta,\xi).$$

We use the Green formula for the elliptic operator $\mathcal{B}$ (see [46]). Let Ω be a smooth domain in $\mathbb{R}^n$. Denote by X_i the direction cosines of the outer normal to Ω. For $\xi \in \partial\Omega$, ν_ξ will denote the conormal at ξ, that is a vector with the direction cosines

$$Y_i = \frac{1}{a(\xi)}\sum_{k=1}^{n} a_{ik}(\xi)X_k(\xi), \quad a(\xi) = \left[\sum_{i=1}^{n}\left(\sum_{k=1}^{n} a_{ik}(\xi)X_k(\xi)\right)^2\right]^{1/2}.$$

For smooth functions U, V on $\overline{\Omega}$,

$$\int_{\Omega}(V\cdot\mathcal{B}U - U\cdot\mathcal{B}^*V)\,d\Omega = \int_{\partial\Omega}\left[a\left(V\frac{\partial U}{\partial\nu} - U\frac{\partial V}{\partial\nu}\right) + bUV\right]dS_{\partial\Omega}$$

where

$$b(\xi) = \sum_{i=1}^{n}\left(b_i(\xi) - \sum_{k=1}^{n}\frac{\partial a_{ik}(\xi)}{\partial x_k}\right)X_i(\xi), \quad \xi \in \partial\Omega.$$

Proposition 3. *Under the above assumptions, for each $(t,x) \in Q$, a classical solution u of the equation (7) on E has the representation*

$$\begin{aligned} u(t,x) = &\int_S u_0(\xi)[D^{\beta-1}_{t\eta}w(t,x;\eta,\xi)]_{\eta=0}\,d\xi \\ &+ \int_S u_1(\xi)[D^{\beta-2}_{t\eta}w(t,x;\eta,\xi)]_{\eta=0}\,d\xi + G(u;t,x) + F(u,f,g;t,x) \end{aligned}$$

where

$$\begin{aligned} G(u;t,x) = &\int_0^t d\eta \int_{\partial S}\Big\{a(\xi)\Big[w(t,x;\eta,\xi)\frac{\partial}{\partial\nu_\xi}u(t,\xi) - u(t,\xi)\frac{\partial}{\partial\nu_\xi}w(t,x;\eta,\xi)\Big] \\ &+ b(\xi)w(t,x;\eta,\xi)u(t,\xi)\Big\}\,dS_\xi, \end{aligned}$$

$$F(u,f,g;t,x) = \int_0^t d\eta \int_S [w(t,x;\eta,\xi)f(\eta,\xi) - u(\eta,\xi)g(t,x;\eta,\xi)]\,d\xi.$$

Using Proposition 3, one can prove the following uniqueness theorem [32, 51].

Theorem 5. *Suppose that the condition* (D) *is satisfied. Let* $u(t,x)$ *be a classical solution of the Cauchy problem for the equation* (7) *with* $f = 0$ *and the zero initial conditions. If for some* $\sigma > 0$,

$$\lim_{|x|\to\infty} u(t,x)\exp\{-\sigma|x|^{\frac{2}{2-\beta}}\} = 0 \tag{88}$$

uniformly with respect to $t \in [0,T]$, *then* $u(t,x) \equiv 0$.

Note that the formal substitution $\beta = 1$ corresponds to the classical uniqueness theorem for parabolic equations. The formal substitution $\beta = 2$ corresponds to the fact that no condition at infinity is needed for uniqueness for a hyperbolic equation. On the other hand, it is shown in [51] for the equation (1) that the order $\frac{2}{2-\beta}$ in (88) cannot be improved.

Note also that the above approach works also for the case $0 < \beta < 1$ where the condition (88) and the assumption (D) guarantee the uniqueness, too. For the equation (1) with $0 < \beta < 1$ that is a result by Pskhu [51]. For equations with variable coefficients, the uniqueness in a similar class of functions was known for $n = 1$ (see [11, 12, 28]); other uniqueness results for $0 < \beta < 1$ [11, 12, 28, 29] dealt with bounded solutions.

8 Asymptotic properties

8.1 Principle of limiting amplitude

Let us consider the equation (1) with $n \geq 3$, $f(t,x) = E_\beta(i^\beta\omega^\beta t^\beta)F(x)$, $\omega > 0$, and the initial conditions (10). We assume that the functions F, u_0, u_1 are bounded; F is locally Hölder continuous; $u_0 \in C^1$ and its first derivatives are bounded and Hölder continuous with the exponent $\gamma > \frac{2-\beta}{\beta}$; u_1 is Hölder continuous. Under these assumptions, the problem (1), (10) possesses a classical solution $u(t,x)$ [32, 51]. This means that $u(t,x)$ belongs to C^2 in x for each $t > 0$; $u(t,x)$ belongs to C^1 in (t,x), and for any $x \in \mathbb{R}^n$ the fractional integral

$$(I_{0+}^{2-\beta}u)(t,x) = \frac{1}{\Gamma(2-\beta)}\int_0^t (t-\tau)^{-\beta+1}u'_\tau(\tau,x)\,d\tau$$

is continuously differentiable in t for $t > 0$; $u(t,x)$ satisfies the equation and initial conditions.

Moreover, $u(t,x)$ admits the integral representation

$$u(t,x) = \int_{\mathbb{R}^n} Z_1^{(\beta)}(t,x;\xi)u_0(\xi)\,d\xi + \int_{\mathbb{R}^n} Z_2^{(\beta)}(t,x;\xi)u_1(\xi)\,d\xi$$
$$+ \int_0^t d\tau \int_{\mathbb{R}^n} Y(t-\tau,x;\xi)E_\beta(i^\beta\omega^\beta\tau^\beta)F(\xi)\,d\xi \tag{89}$$

where the kernels satisfy the estimates listed in Section 4.2.

In fact, in the study of asymptotic properties of the function (89) it would be possible to remove the above smoothness assumptions considering u as a kind of generalized solution.

Theorem 6. *Assume, in addition to the boundedness, that $u_0, u_1, F \in L_1(\mathbb{R}^n)$, so that, in particular, $F \in L_2(\mathbb{R}^n)$. Then the limit relation* (12) *is valid for every $x \in \mathbb{R}^n$, with the function v belonging to the Sobolev space $H^2(\mathbb{R}^n)$ and satisfying the equation*

$$\Delta v - i^\beta\omega^\beta v = -F. \tag{90}$$

The proof of this principle of limiting amplitude [31] uses both the estimates of the kernels and their explicit representation via the Wright function, as well as the information [9, 24, 51] on the behavior of the Mittag-Leffler function on its oscillation ray.

8.2 Stabilization

Let $u_\beta(t,x)$, $t \geq 0$, $x \in \mathbb{R}^n$, be a bounded solution of the Cauchy problem

$$\mathbb{D}_t^{(\alpha)}u - \Delta u_\beta = 0, \quad u_\beta(0,x) = u_0(x), \frac{\partial u_\beta(0,x)}{\partial t} = 0 \tag{91}$$

where $1 < \beta < 2$, u_0 is a bounded continuous function. Under additional smoothness conditions, namely if u_0 is continuously differentiable, and its first derivatives are bounded and Hölder continuous with the exponent $\gamma > \frac{2-\beta}{\beta}$, the function

$$u_\beta(t,x) = \int_{\mathbb{R}^n} Z_1^{(\beta)}(t,x;\xi)u_0(\xi)\,d\xi \tag{92}$$

is the unique bounded classical solution of the problem (91); see [32, 51]. Without the above smoothness assumptions, we can investigate the function (92) interpreting it as a generalized solution of the problem (91).

Theorem 7. *The function $u_\beta(t,x)$ possesses the property of pointwise stabilization: there exists a constant c, such that*

$$u_\beta(t,x) \longrightarrow c \quad \textit{for any } x \in \mathbb{R}^n, \textit{ as } t \to \infty,$$

if and only if the initial function u_0 satisfies the condition (13).

The proof [31] uses the estimates for the fundamental solution, the subordination identity [1] connecting the fundamental solution of (fractional) evolution equations of different orders, and asymptotic properties of the Mellin convolution [3].

Bibliography

[1] E. G. Bazhlekova, Subordination principle for fractional evolution equations, *Fract. Calc. Appl. Anal.*, **3** (2000), 213–230.

[2] S. Benzoni-Gavage and D. Serre, *Multidimensional Hyperbolic Partial Differential Equations*, Clarendon Press, Oxford, 2007.

[3] N. H. Bingham, C. M. Goldie, and J. L. Teugels, *Regular Variation*, Cambridge University Press, 1987.

[4] Ph. Clément, G. Gripenberg, and S.-O. Londen, Regularity properties of solutions of fractional evolution equations, in *Lect. Notes in Pure and Appl. Math.*, vol. 215, pp. 235–246, 2001.

[5] Ph. Clément, S.-O. Londen, and G. Simonett, Quasilinear evolutionary equations and continuous interpolation spaces, *J. Differ. Equ.*, **196** (2004), 418–447.

[6] R. Courant and D. Hilbert, *Methods of Mathematical Physics, Vol. II: Partial Differential Equations*, Interscience, New York, 1962.

[7] V. N. Denisov, On the behavior of solutions of parabolic equations for large time values, *Russ. Math. Surv.*, **60** (2005), 721–790.

[8] V. N. Denisov and V. D. Repnikov, Stabilization of the solution of the Cauchy problem for parabolic equations, *Differ. Equ.*, **20** (1984), 16–33.

[9] M. M. Djrbashian, *Harmonic Analysis and Boundary Value Problems in Complex Domain*, Birkhäuser, Basel, 1993.

[10] S. D. Eidelman, *Parabolic Systems*, North-Holland, Amsterdam, 1969.

[11] S. D. Eidelman, S. D. Ivasyshen, and A. N. Kochubei, *Analytic Methods in the Theory of Differential and Pseudo-Differential Equations of Parabolic Type*, Birkhäuser, Basel, 2004.

[12] S. D. Eidelman and A. N. Kochubei, Cauchy problem for fractional diffusion equations, *J. Differ. Equ.*, **199** (2004), 211–255.

[13] S. D. Eidelman and V. D. Repnikov, Necessary and sufficient conditions for the establishment of a solution of the Cauchy problem, *Sov. Math. Dokl.*, **7** (1966), 388–391.

[14] G. I. Eskin, A generalization of the Paley–Wiener–Schwartz theorem, *Usp. Mat. Nauk*, **16**(1) (1961), 185–188. *Amer. Math. Soc. Translations*, **28** (1963), 187–190.

[15] A. Friedman, Existence of smooth solutions of the Cauchy problem for differential systems of any type, *J. Math. Mech.*, **12** (1963), 335–374.

[16] A. Friedman, *Partial Differential Equations of Parabolic Type*, Prentice-Hall, Englewood Cliffs, NJ, 1964.

[17] Y. Fujita, Integrodifferential equation which interpolates the heat equation and the wave equation, *Osaka J. Math.*, **27** (1990), 309–321.

[18] I. M. Gel'fand and G. E. Shilov, *Generalized Functions. Vol. 3. Theory of Differential Equations*, Academic Press, New York, 1967.

[19] G. M. Gubreev, Regular Mittag-Leffler kernels and spectral decomposition of a class of nonselfadjoint operators, *Izv. Math.*, **69** (2005), 15–57.

[20] A. K. Gushchin and V. P. Mikhailov, Uniform stabilization of solutions of the Cauchy problem for a second-order hyperbolic equation, *Proc. Steklov Inst. Math.*, **166** (1986), 79–95.

[21] A. Hanyga, Multidimensional solutions of time-fractional diffusion-wave equations, *Proc. R. Soc. Lond. Ser. A*, **458** (2002), 933–957.

[22] A. Heibig, Existence of solutions for a fractional derivative system of equations, *Integral Equ. Oper. Theory*, **72** (2012), 483–508.

[23] J. Kemppainen, Solvability of a Dirichlet problem for a time fractional diffusion-wave equation in Lipschitz domains, *Fract. Calc. Appl. Anal.*, **15** (2012), 195–206.

[24] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier, Amsterdam, 2006.

[25] I. Kim, K.-H. Kim, and S. Lim, An $L_q(L_p)$- theory for the time fractional evolution equations with variable coefficients, *Adv. Math.*, **306** (2017), 123–176.

[26] A. N. Kochubei, Stabilization of solutions of dissipative hyperbolic equations, *Differ. Equ.*, **22** (1986), 1216–1222.

[27] A. N. Kochubei, Stabilization of solutions of dissipative hyperbolic equations with almost periodic coefficients, *Differ. Equ.*, **23** (1987), 1406–1411.

[28] A. N. Kochubei, Fractional-order diffusion, *Differ. Equ.*, **26** (1990), 485–492.

[29] A. N. Kochubei, Fractional-parabolic systems, *Potential Anal.*, **37** (2012), 1–30.

[30] A. N. Kochubei, Fractional-hyperbolic systems, *Fract. Calc. Appl. Anal.*, **16** (2013), 860–873.

[31] A. N. Kochubei, Asymptotic properties of solutions of the fractional diffusion-wave equation, *Fract. Calc. Appl. Anal.*, **17** (2014), 881–896.

[32] A. N. Kochubei, Cauchy problem for fractional diffusion-wave equations with variable coefficients, *Appl. Anal.*, **93** (2014), 2211–2242.

[33] K. Li and J. Peng, Fractional abstract Cauchy problem, *Integral Equ. Oper. Theory*, **70** (2011), 333–361.

[34] Yu. Luchko, Fractional wave equation and damped waves, *J. Math. Phys.*, **54** (2013), 031505.

[35] Yu. Luchko, F. Mainardi, and Yu. Povstenko, Propagation speed of the maximum of the fundamental solution to the fractional diffusion-wave equation, *Comput. Math. Appl.*, **66** (2013), 774–784.

[36] F. Mainardi, The fundamental solution for the fractional diffusion-wave equation, *Appl. Math. Lett.*, **9** (1996), 23–28.

[37] F. Mainardi, *Fractional Calculus and Waves in Linear Viscoelasticity*, Imperial College Press, London, 2010.

[38] F. Mainardi, Yu. Luchko, and G. Pagnini, The fundamental solution of the space-time fractional diffusion equation, *Fract. Calc. Appl. Anal.*, **4** (2001), 153–192.

[39] F. Mainardi and G. Pagnini, The Wright functions as solutions of the time-fractional diffusion equation, *Appl. Math. Comput.*, **141** (2003), 51–62.

[40] M. O. Mamchuev, Fundamental solution of a system of fractional partial differential equations, *Differ. Equ.*, **46** (2010), 1123–1134.

[41] M. O. Mamchuev, Cauchy problem in nonlocal statement for a system of fractional partial differential equations, *Differ. Equ.*, **48** (2012), 354–361.

[42] M. O. Mamchuev, Solutions of the main boundary value problems for the time-fractional telegraph equation by the Green function method, *Fract. Calc. Appl. Anal.*, **20** (2017), 190–211.

[43] M. Markus and H. Minc, *A Survey of Matrix Theory and Matrix Inequalities*, Allyn and Bacon, Boston, 1964.

[44] V. M. Martirosyan, Integral transforms with kernels of Mittag-Leffler-type in the classes $L_p(0, +\infty)$, $1 < p \le 2$, *Math. USSR Sb.*, **57** (1987), 97–109.

[45] R. Metzler and J. Klafter, The restaurant at the end of the random walk: recent developments in the description of anomalous transport by fractional dynamics, *J. Phys. A*, **37** (2004), R161–R208.

[46] C. Miranda, *Partial Differential Equations of Elliptic Type*, Springer, Berlin, 1970.

[47] H. Petzeltová and J. Prüss, Global stability of a fractional partial differential equation, *J. Integral Equ. Appl.*, **12** (2000), 323–347.

[48] I. Podlubny and Y. Chen, Adjoint fractional differential expressions and operators, in *Proc. ASME 2007 Conf. IDET/CIE 2007, Las Vegas, Nevada, USA*, Paper DETC 2007-35005, 6 pages.
[49] A. Yu. Popov and A. M. Sedletskii, Distribution of roots of Mittag-Leffler functions, *J. Math. Sci. (N.Y.)*, **190**(2) (2013), 3–171.
[50] J. Prüss, *Evolutionary Integral Equations and Applications*, Birkhäuser, Basel, 1993.
[51] A. V. Pskhu, The fundamental solution of a diffusion-wave equation of fractional order, *Izv. Math.*, **73** (2009), 351–392.
[52] S. G. Samko, A. A. Kilbas, and O. I. Marichev, *Fractional Integrals and Derivatives: Theory and Applications*, Gordon and Breach, New York, 1993.
[53] W. R. Schneider and W. Wyss, Fractional diffusion and wave equations, *J. Math. Phys.*, **30** (1989), 134–144.
[54] A. N. Tikhonov and A. A. Samarskii, *Equations of Mathematical Physics*, Pergamon Press, Oxford, 1963.
[55] V. V. Uchaikin, *Fractional Derivatives for Physicists and Engineers*, Springer, Berlin and Higher Education Press, Beijing, 2013.
[56] V. Vergara and R. Zacher, Lyapunov functions and convergence to steady state for differential equations of fractional order, *Math. Z.*, **259** (2008), 287–309.
[57] V. S. Vladimirov, *Generalized Functions in Mathematical Physics*, Mir, Moscow, 1979.

Anatoly N. Kochubei

Equations with general fractional time derivatives—Cauchy problem

Abstract: We describe a theory of a kind of heat equations associated with operators in the time variable of the form $(\mathbb{D}_{(k)}u)(t) = \frac{d}{dt}\int_0^t k(t-\tau)u(\tau)\,d\tau - k(t)u(0)$ where k is a nonnegative locally integrable function. The results are based on the theory of complete Bernstein functions. As a special case, equations with fractional derivatives of distributed order are considered.

Keywords: Differential-convolution operator, distributed order fractional derivative, complete Bernstein fuction, Stieltjes function

MSC 2010: 35R11, 35K99

1 Introduction

This paper is a follow-up to two other chapters of this handbook; see [18, 19]. Let $\mathbb{D}_{(k)}$ be a differential-convolution operator

$$(\mathbb{D}_{(k)}u)(t) = \frac{d}{dt}\int_0^t k(t-\tau)u(\tau)\,d\tau - k(t)u(0) \tag{1}$$

where $k \in L_1^{\mathrm{loc}}(\mathbb{R}_+)$ is a nonnegative function whose Laplace transform

$$\mathcal{K}(p) = \int_0^\infty e^{-pt}k(t)\,dt \tag{2}$$

exists for all $p > 0$, belongs to the Stieltjes class $\mathcal{S}$, and

$$\begin{aligned} &\mathcal{K}(p) \to \infty, \quad \text{as } p \to 0; \quad \mathcal{K}(p) \to 0, \quad \text{as } p \to \infty; \\ &p\mathcal{K}(p) \to 0, \quad \text{as } p \to 0; \quad p\mathcal{K}(p) \to \infty, \quad \text{as } p \to \infty. \end{aligned} \tag{$\star$}$$

For the definition and properties of the Stieltjes class $\mathcal{S}$ and the related class of complete Bernstein functions, see [33] or a summary in [19].

We will be interested also in a subclass of distributed order derivatives $\mathbb{D}^{(\mu)}$,

$$(\mathbb{D}^{(\mu)}\varphi)(t) = \int_0^1 (\mathbb{D}^{(\alpha)}\varphi)(t)\mu(\alpha)\,d\alpha, \tag{3}$$

Anatoly N. Kochubei, Institute of Mathematics, National Academy of Sciences of Ukraine, Tereshchenkivska 3, Kyiv, 01004, Ukraine, e-mail: kochubei@imath.kiev.ua

https://doi.org/10.1515/9783110571660-011

where $\mathbb{D}^{(\alpha)}$ is the Caputo–Djrbashian fractional derivative of order α, μ is a continuous nonnegative weight function different from zero on a set of positive measure. Here,

$$k(t) = \int_0^1 \frac{t^{-\alpha}}{\Gamma(1-\alpha)}\mu(\alpha)\,d\alpha.$$

For the differential-convolution operator (1), we consider, following [17], the Cauchy problem,

$$(\mathbb{D}_{(k)}w)(t,x) = \Delta w(t,x), \quad t > 0,\ x \in \mathbb{R}^n; \quad w(0,x) = w_0(x), \tag{4}$$

where w_0 is a bounded globally Hölder continuous function, that is, $|w_0(\xi) - w_0(\eta)| \le C|\xi - \eta|^\gamma$, $0 < \gamma \le 1$, for any $\xi, \eta \in \mathbb{R}^n$. This problem has a unique bounded solution (the notion of a solution should be defined appropriately). Moreover, the equation in (4) possesses a fundamental solution of the Cauchy problem, a kernel with the property of a probability density.

Note that the well-posedness of the Cauchy problem for equations with the operator $\mathbb{D}_{(k)}$ has been established under much weaker assumptions than those needed for the above properties; see [12].

For the case of distributed order derivatives (3), there exists [16] a much more detailed theory of the heat type equation, with estimates for the fundamental solution and qualitative properties of solutions. There are various uniqueness results taking into account the specific kind of an equation.

The study of distributed order equations has a strong physical motivation. Physically, the most important characteristic of diffusion is the mean square displacement

$$\overline{(\Delta x)^2} = \int_{\mathbb{R}^n} |x-\xi|^2 Z(t, x-\xi)\,d\xi$$

of a diffusive particle, where Z is a fundamental solution of the Cauchy problem for the diffusion equation. In normal diffusion (described by the heat equation or more general parabolic equations), the mean square displacement of a diffusive particle behaves like const $\cdot\, t$ for $t \to \infty$. A typical behavior for anomalous diffusion on some amorphous semiconductors, strongly porous materials, etc. is const $\cdot\, t^\alpha$, and this was the reason to invoke the equation (4) with the order α Caputo–Djrbashian fractional derivative, that is, with $k(t) = \frac{t^{-\alpha}}{\Gamma(1-\alpha)}$, where this anomalous behavior is an easy mathematical fact. There are hundreds of physical papers involving such equations; see the surveys [26, 27]. The mathematical theory was initiated independently by Schneider and Wyss [34] and the author [14, 15]; for more recent developments, see [7, 8, 11] and this handbook.

A number of recent publications by physicists (see [2–4, 28, 35] and references there) is devoted to the case where the mean square displacement has a logarithmic growth. This ultraslow diffusion (also called “a strong anomaly”) is encountered in polymer physics (a polyampholyte hooked around an obstacle), as well as in models of a particle’s motion in a quenched random force field, iterated map models, etc. In order to describe ultraslow diffusion, it is proposed to use evolution equations with distributed order derivatives.

The general existence result for the equation (4) is described in Section 2, while Section 3 is devoted to fundamental solutions of distributed order equations, Section 4 deals with the Cauchy problem for such equations, and Section 5 is about uniqueness results.

Here, we will not consider boundary value problems for the above equations in bounded domains. See [20–23] and other chapters of this handbook. For various physical applications, see [31, 32].

2 $\mathbb{D}_{(k)}$-heat equation

Let us consider the Cauchy problem (4). Applying formally the Laplace transform in t to both sides of (4), we obtain the following equation for the Laplace transform $\widetilde{w}(p,x)$ of a solution of (4):

$$p\mathcal{K}(p)\widetilde{w}(p,x) - \mathcal{K}(p)w_0(x) = \Delta\widetilde{w}(p,x), \quad p > 0,\ x \in \mathbb{R}^n. \tag{5}$$

A bounded function $w(t,x)$ will be called a *LT-solution* of the problem (4), if w is continuous in t on $[0,\infty)$ uniformly with respect to $x \in \mathbb{R}^n$, $w(0,x) = w_0(x)$, while its Laplace transform $\widetilde{w}(p,x)$ is twice continuously differentiable in x, for each $p > 0$, and satisfies the equation (5).

Theorem 1. *Suppose that the assumption* (*) *is satisfied. There exists such a nonnegative function $Z(t,x)$, $t > 0$, $x \in \mathbb{R}^n$, $x \neq 0$, locally integrable in t and infinitely differentiable in $x \neq 0$, that*

$$\int_{\mathbb{R}^n} Z(t,x)\,dx = 1, \quad t > 0, \tag{6}$$

and for any bounded globally Hölder continuous function w_0, the function

$$w(t,x) = \int_{\mathbb{R}^n} Z(t,x-\xi)w_0(\xi)\,d\xi \tag{7}$$

is a LT-solution of the Cauchy problem (4).

For the proof based on the subordination techniques, see [17]. Here, we explain only the idea.

Consider the function

$$g(s,p) = \mathcal{K}(p)e^{-sp\mathcal{K}(p)}, \quad s > 0,\ p > 0. \tag{8}$$

Since $p \mapsto p\mathcal{K}(p)$ is a Bernstein function (actually it is a complete Bernstein function), the function $p \mapsto e^{-sp\mathcal{K}(p)}$ is completely monotone (see conditions for the complete monotonicity in Chapter 13 of [9]). By Bernstein's theorem, for each $s \geq 0$, there exists such a probability measure $\mu_s(d\tau)$ that

$$e^{-sp\mathcal{K}(p)} = \int_0^\infty e^{-p\tau}\mu_s(d\tau).$$

The family of measures $\{\mu_s\}$ is weakly continuous in s.

Set

$$G(s,t) = \int_0^t k(t-\tau)\mu_s(d\tau).$$

The Laplace transform in t of the function G coincides with the function $g(s,p)$:

$$g(s,p) = \int_0^\infty e^{-pt}G(s,t)\,dt.$$

On the other hand, it is seen from (8) that

$$\int_0^\infty g(s,p)\,ds = \frac{1}{p},$$

so that

$$\int_0^\infty e^{-pt}\,dt \int_0^\infty G(s,t)\,ds = \frac{1}{p},$$

which implies, for each t, the equality

$$\int_0^\infty G(s,t)\,ds = 1. \tag{9}$$

We define Z by the subordination equality

$$Z(t,x) = \int_0^\infty (4\pi s)^{-n/2} e^{-\frac{|x|^2}{4s}} G(s,t)\,ds, \quad x \neq 0. \tag{10}$$

The equality (6) follows from (9) and properties of the fundamental solution of the classical heat equation. It follows from (10) and (9) that $Z(t,x)$ is infinitely differentiable in $x \neq 0$.

Further proof is based on the representation (10) and the theory of complete Bernstein functions. This contrasts the classical theory of parabolic equations where the main technical tools are the contour integration and explicit estimates. Note that the latter methods are used for distributed order equations. A probabilistic representation of a fundamental solution of the problem (4) (and more general problems, with Lévy generators instead of the Laplacian) was found in [24, 25]. The fundamental solutions were understood there as those of the equations obtained by applying the Laplace transform in time and the Fourier transform in spatial variables. For our situation, we obtained solutions, strong with respect to the variable x. For the case of distributed order equations, more detailed information is given below.

3 Distributed order equations

Let us consider the equation

$$(\mathbb{D}_t^{(\mu)}u)(t,x) = \Delta u(t,x), \quad x \in \mathbb{R}^n, t > 0. \tag{11}$$

A fundamental solution $Z(t,x)$ is constructed [16] using the method of integral transforms. An explicit expression is known for its Laplace transform in t:

$$\widetilde{Z}(p,x) = (2\pi)^{-\frac{n}{2}}|x|^{1-\frac{n}{2}}\mathcal{K}(p)(p\mathcal{K}(p))^{\frac{1}{2}(\frac{n}{2}-1)}K_{\frac{n}{2}-1}(|x|\sqrt{p\mathcal{K}(p)}). \tag{12}$$

We have simpler formulas in the lowest dimensions—if $n = 2$, then

$$\widetilde{Z}(p,x) = \frac{1}{2\pi}\mathcal{K}(p)K_0(|x|\sqrt{p\mathcal{K}(p)}); \tag{13}$$

where K_ν is the Macdonald function [1, 30]. If $n = 1$, then

$$\widetilde{Z}(p,x) = \frac{1}{2}\frac{\mathcal{K}(p)}{\sqrt{p\mathcal{K}(p)}}e^{-|x|\sqrt{p\mathcal{K}(p)}} \tag{14}$$

because $K_{-1/2}(z) = K_{1/2}(z) = \sqrt{\frac{\pi}{2z}}e^{-z}$ (see [1]).

The function K_ν decays exponentially at infinity:

$$K_\nu(z) \sim \sqrt{\frac{\pi}{2z}}e^{-z}, \quad z \to \infty,$$

while $K_\nu(z) \sim Cz^{-\nu}$, as $z \to 0$ (if $\nu > 0$), and $K_0(z) \sim -\log z$, $z \to 0$. We see that the function on the right in (12) belongs to $L_1(\mathbb{R}^n)$ in x for any n.

Let us consider estimates of the function Z and its derivatives. Qualitatively, the behavior of Z is similar to that of the fundamental solution of the Cauchy problem for the fractional diffusion equation with the Caputo–Djrbashian fractional time derivative (see [7, 8, 15, 18]). In addition to the singularity at $t = 0$, $Z(t,x)$ has, if $n > 1$, a singularity at $x = 0$ (a logarithmic singularity, if $n = 2$, and a power one, if $n \geq 3$). As usual $Z(t,x) \to \delta(x)$, as $t \to 0$. This means that the singularity at $t = 0$ becomes "visible" near the origin in x. In fact, we have separate estimates for a small $|x|$, showing the character of singularities in t and x, and for a large $|x|$. In addition, subsequent applications of the fundamental solutions require estimates of $\mathbb{D}^{(\mu)}Z$, applicable simultaneously for all $x \neq 0$, and uniform in t. Of course, estimates for $\mathbb{D}^{(\mu)}Z$ at the origin and infinity can be obtained from the relation $\mathbb{D}^{(\mu)}Z = \Delta Z$.

All these estimates deal with a finite time interval, $t \in (0, T]$, and it is this kind of estimates, that is needed to study the Cauchy problem. Separately, we will give some estimates of Z for large values of t, just to see the qualitative behavior of Z.

Theorem 2. *Suppose that $\mu \in C^2[0,1]$, $\mu(\alpha) = \alpha^\nu \mu_1(\alpha)$, $\mu_1(\alpha) \geq \rho > 0$, $0 \leq \alpha \leq 1$, $\nu \geq 0$. Denote by ε a small positive number. The function Z is infinitely differentiable for $t \neq 0$ and $x \neq 0$. The following estimates hold for $0 < t \leq T$.*

If $n = 1$, then

$$\left|D_x^m Z(t,x)\right| \leq Ct^{-\frac{m+1}{2}}, \quad |x| \leq \varepsilon, 0 \leq m \leq 3.$$

If $n = 2$, then

$$\left|Z(t,x)\right| \leq Ct^{-1} \log |x|^{-1}, \quad 0 < |x| \leq \varepsilon,$$
$$\left|D_x^m Z(t,x)\right| \leq Ct^{-1}|x|^{-m}, \quad 0 < |x| \leq \varepsilon, 1 \leq m \leq 3.$$

If $n \geq 3$, then

$$\left|D_x^m Z(t,x)\right| \leq Ct^{-1}|x|^{-n+2-m}, \quad 0 < |x| \leq \varepsilon, 0 \leq m \leq 3.$$

In all cases,

$$\left|D_x^m Z(t,x)\right| \leq Ce^{-a|x|} \ (a > 0), \quad |x| \geq \varepsilon^{-1}.$$

The estimate of $\mathbb{D}^{(\mu)}Z$, uniform in t, is as follows:

$$\left|(\mathbb{D}^{(\mu)}Z)(t,x)\right| \leq C|x|^{-n-2}e^{-a|x|} \ (a > 0), \quad |x| \neq 0.$$

If $0 < |x| \leq \varepsilon$, then

$$\left|(\mathbb{D}^{(\mu)}Z)(t,x)\right| \leq Ct^{-2}|x|^{-n+2}.$$

Note that Z coincides with the fundamental solution constructed above using the technique of complete Bernstein functions. In particular, it satisfies the properties of positivity and normalization (9), as well as the subordination identity (10).

Some important properties of the fundamental solution are collected in the following theorem, which gives a rigorous treatment of the physical notion of mean square displacement.

Theorem 3. (i) *Let*

$$m(t) = \int_{\mathbb{R}^n} |x|^2 Z(t,x)\,dx.$$

If $\mu(0) \neq 0$, *then*

$$m(t) \sim C \log t, \quad t \to \infty.$$

If

$$\mu(\alpha) \sim a\alpha^{\nu}, \quad \alpha \to 0, \quad a, \nu > 0, \tag{15}$$

then

$$m(t) \sim C(\log t)^{1+\nu}, \quad t \to \infty.$$

(ii) *Suppose that* (15) *holds with* $\nu > 1$, *if* $n = 1$, *and with an arbitrary* $\nu > 0$, *if* $n \geq 2$. *Then for* $|x| \leq \varepsilon$, $\varepsilon > 0$, *and* $t > \varepsilon^{-1}$,

$$Z(t,x) \leq \begin{cases} C(\log t)^{-\frac{\nu-1}{2}}, & \text{if } n = 1; \\ C|\log|x||(\log t)^{-\nu}\log(\log t), & \text{if } n = 2; \\ C|x|^{-n+2}(\log t)^{-\nu-1}, & \text{if } n \geq 3. \end{cases} \tag{16}$$

4 Cauchy problem for distributed order equations

4.1 The homogeneous equation

Let us consider the equation (11) with the initial condition

$$u(0,x) = \varphi(x), \quad x \in \mathbb{R}^n, \tag{17}$$

where φ is a locally Hölder continuous function of the sub-exponential growth: for any $b > 0$,

$$|\varphi(x)| \leq C_b e^{b|x|}.$$

Compared with a more general case considered in Theorem 1, here we weaken our assumptions regarding the initial function φ. We will assume that the weight function μ defining the distributed order derivative $\mathbb{D}^{(\mu)}$ satisfies the conditions of Theorem 2.

The following result is taken from [16].

Theorem 4. (i) *The function*

$$u(t,x) = \int_{\mathbb{R}^n} Z(t, x-\xi)\varphi(\xi)\,d\xi \tag{18}$$

is a classical solution of the Cauchy problem (11), (17), *that is, the function* (18) *is twice continuously differentiable in x for each $t > 0$, for each $x \in \mathbb{R}^n$ it is continuous in $t > 0$, the function*

$$t \mapsto \int_0^t k(t-\tau)u(\tau,x)\,d\tau, \quad t > 0,$$

is continuously differentiable, the equation (11) *is satisfied, and*

$$u(t,x) \longrightarrow \varphi(x), \quad \text{as } t \to 0,$$

for all $x \in \mathbb{R}^n$.

(ii) *On each finite time interval* $(0,T]$, *the solution $u(t,x)$ satisfies the inequality*

$$|u(t,x)| \le Ce^{d|x|}, \quad x \in \mathbb{R}^n,$$

with some constants $C, d > 0$. If φ is bounded, then

$$|u(t,x)| \le C, \quad x \in \mathbb{R}^n, \quad 0 < t \le T.$$

(iii) *For each $x \in \mathbb{R}^n$, there exists such an $\varepsilon > 0$ that*

$$|\mathbb{D}^{(\mu)}u(t,x)| \le C_x t^{-1+\varepsilon}, \quad 0 < t \le T.$$

4.2 The inhomogeneous equation

Let us consider the Cauchy problem

$$(\mathbb{D}_t^{(\mu)}u)(t,x) - \Delta u(t,x) = f(t,x), \quad x \in \mathbb{R}^n, t > 0, \tag{19}$$

$$u(0,x) = 0. \tag{20}$$

We assume that the function f is continuous in t, bounded and locally Hölder continuous in x, uniformly with respect to t. Our task in this section is to obtain a solution of (19)–(20) in the form of a “heat potential”

$$u(t,x) = \int_0^t d\tau \int_{\mathbb{R}^n} E(t-\tau, x-y)f(\tau,y)\,dy. \tag{21}$$

In contrast to the classical theory of parabolic equations [10], the kernel E in (21) does not coincide with the fundamental solution Z —just as this happens for fractional diffusion equations [7, 8]. However, the behavior of the function E is very similar to that of Z. For its Laplace transform in t, we have [16]

$$\tilde{E}(p,x) = (2\pi)^{-\frac{n}{2}}|x|^{1-\frac{n}{2}}(p\mathcal{K}(p))^{\frac{1}{2}(\frac{n}{2}-1)}K_{\frac{n}{2}-1}(|x|\sqrt{p\mathcal{K}(p)}), \tag{22}$$

which differs from (12) only by the absence of the factor $\mathcal{K}(p)$ with a logarithmic behavior at infinity. Therefore, the function $E(t,x)$ obtained from (22) via contour integration, satisfies the same estimates (see Theorem 2) as the function Z, except the estimates for large values of t. The function $E(t,x)$ is nonnegative.

The counterparts of the estimates (16) are as follows. If (15) holds with $\nu \geq 0$, then for $|x| \leq \varepsilon$, $\varepsilon > 0$, and $t > \varepsilon^{-1}$

$$E(t,x) \leq \begin{cases} Ct^{-1}(\log t)^{\frac{1+\nu}{2}}, & \text{if } n = 1; \\ Ct^{-1}\log\log t \log|x|^{-1}, & \text{if } n = 2; \\ Ct^{-1}|x|^{-n+2}, & \text{if } n \geq 3. \end{cases}$$

The function E has (in x) an exponential decay at infinity.

In fact, the analysis of the potential (21) requires estimates of E and its derivatives, uniform in $t \in (0,T]$.

Proposition 1. *Let μ satisfy the conditions of Theorem 2. Then, uniformly in $t \in (0,T]$,*

$$\left|D_x^j E(t,x)\right| \leq C|x|^{-j-n}|1 + |\log|x|||^{\beta}e^{-a|x|}, \quad x \neq 0,\ j \geq 0,$$
$$\left|\mathbb{D}_t^{(\mu)} E(t,x)\right| \leq C|x|^{-n-2}|1 + |\log|x|||^{\beta}e^{-a|x|}, \quad x \neq 0,$$

where C, a, β are positive constants.

The kernel E satisfies the identity

$$\int_{\mathbb{R}^n} E(t,x)\,dx = \varkappa(t)$$

where $\varkappa$ appears in the definition of the distributed order integral $\mathbb{I}^{(\mu)}$ [16, 19].

In the next theorem, a classical solution is understood similar to Theorem 4.

Theorem 5. *Under the above assumptions regarding f, and the assumptions of Theorem 2 regarding μ, the function* (21) *is a classical solution of the problem* (19)–(20), *bounded near the origin in t for each $x \in \mathbb{R}^n$.*

In the proof of Theorem 5 [16], we constructed a solution u of the problem (19)–(20), such that $u = u_1 + u_2$, $u_1(0,x) = u_2(0,x) = 0$, u_1 is absolutely continuous in t, and $u_2 = \mathbb{I}^{(\mu)}f$. On this solution u,

$$\mathbb{I}^{(\mu)}\mathbb{D}^{(\mu)}u = \mathbb{I}^{(\mu)}(k * u_1') + \mathbb{I}^{(\mu)}f = \varkappa * k * u_1' + u_2 = u_1 + u_2 = u$$

(u' means the derivative in t). Applying $\mathbb{I}^{(\mu)}$ to both sides of the equation (19), we find that

$$u(t,x) - \int_0^t \varkappa(t-s)\Delta u(s,x)\,ds = (\varkappa * f)(t,x). \tag{23}$$

Remark. The equation (23) can be interpreted as an abstract Volterra equation

$$u + \varkappa * (Au) = \varphi, \tag{24}$$

if we assume that u belongs to some Banach space X (in the variable x), and A is the operator $-\Delta$ on X. The operator $-A$ generates a contraction semigroup if, for example, $X = L_2(\mathbb{R}^n)$ or $X = C_\infty(\mathbb{R}^n)$ (the space of continuous functions decaying at infinity; see Section X.8 in [29]). Now the existence of a solution in $L_1(0,T;X)$ can be obtained from a general theory of equations (24) developed in [5]; it is essential that $\varkappa$ is completely monotone (conditions of some other papers devoted to equations (24) do not cover our situation). Of course, the "classical" approach gives a much more detailed information about solutions, while the abstract method is applicable to more general equations.

5 Uniqueness theorems

Let us return to the Cauchy problem (4) with a general differential-convolution operator (1).

Our uniqueness result for the problem (4) holds under much more general assumptions and is an immediate consequence of a deep result by E. E. Shnol (see Theorem 2.9 in [6]). Note that the notion of a LT-solution makes sense also for polynomially bounded solutions, that is such solutions $w(t,x)$ that $|w(t,x)| \le P(|x|)$ where P is some polynomial independent of t. Instead of (*), we make the following weaker assumption:

(**) The function k is nonnegative, locally integrable, nonzero on a set of positive measure, and its Laplace transform $\mathcal{K}(p)$ exists for all $p > 0$.

Theorem 6. *Let* (**) *hold, and suppose that* $w(t,x)$ *is a polynomially bounded LT-solution of the problem* (4) *with* $w_0(x) \equiv 0$. *Then* $w(t,x) \equiv 0$.

Proof. [17]. The Laplace transform $\widetilde{w}(p,x)$ satisfies, for each $p > 0$, the equation $\Delta\widetilde{w}(p,x) = p\mathcal{K}(p)\widetilde{w}(p,x)$. Thus $\widetilde{w}(p,x)$ is a generalized eigenfunction of the operator $-\Delta$ on $L_2(\mathbb{R}^n)$ with the eigenvalue $-p\mathcal{K}(p) < 0$. By Shnol's theorem, a nonzero polynomially bounded generalized eigenfunction is possible only if the eigenvalue belongs to the spectrum of $-\Delta$ equal to $[0,\infty)$. Therefore, $\widetilde{w}(p,x) \equiv 0$, so that $w(t,x) \equiv 0$. □

Theorem 4 can be extended to some equations with coefficients depending on x, for which Shnol's theorem can be applied; see [6, 13].

For equations with the Caputo–Djrbashian fractional derivatives, there are uniqueness results for the class of functions of exponential growth (see [18]). For distributed order equations with variable coefficients, the uniqueness of bounded solutions was proved (using the maximum principle arguments) in [16]. A wider class of uniqueness is known for the equation (11) with $n = 1$ [16].

Theorem 7. *Suppose that $u(t,x)$ is a classical solution of the problem* (11) *with $n = 1$, $\Delta = \frac{\partial^2}{\partial x^2}$, such that for any $a > 0$,*

$$|u(t,x)| \le C_a e^{a|x|}, \quad 0 < t \le T,\ x \in \mathbb{R}^1,$$

and $\mathbb{D}_t^{(\mu)} u \in L_p(0,T)$, $p > 1$, in t, for any fixed x. Then $u(t,x) \equiv 0$.

Bibliography

[1] H. Bateman and A. Erdélyi, *Higher Transcendental Functions*, vol. 2, McGraw-Hill, New York, 1953.

[2] A. V. Chechkin, R. Gorenflo, and I. M. Sokolov, Retarding subdiffusion and accelerating superdiffusion governed by distributed order fractional diffusion equations, *Phys. Rev. E*, **66** (2002), 046129, 1–7.

[3] A. V. Chechkin, R. Gorenflo, I. M. Sokolov, and V. Yu. Gonchar, Distributed order fractional diffusion equation, *Fract. Calc. Appl. Anal.*, **6** (2003), 259–279.

[4] A. V. Chechkin, J. Klafter, and I. M. Sokolov, Fractional Fokker–Planck equation for ultraslow kinetics, *Europhys. Lett.*, **63** (2003), 326–332.

[5] Ph. Clément and J. A. Nohel, Abstract linear and nonlinear Volterra equations preserving positivity, *SIAM J. Math. Anal.*, **10** (1979), 365–388.

[6] H. L. Cycon, R. G. Froese, W. Kirsch, and B. Simon, *Schrödinger Operators with Applications to Quantum Mechanics and Global Geometry*, Springer, Berlin, 1987.

[7] S. D. Eidelman, S. D. Ivasyshen, and A. N. Kochubei, *Analytic Methods in the Theory of Differential and Pseudo-Differential Equations of Parabolic Type*, Birkhäuser, Basel, 2004.

[8] S. D. Eidelman and A. N. Kochubei, Cauchy problem for fractional diffusion equations, *J. Differ. Equ.*, **199** (2004), 211–255.

[9] W. Feller, *An Introduction to Probability Theory and Its Applications*, vol. 2, Wiley, New York, 1971.

[10] A. Friedman, *Partial Differential Equations of Parabolic Type*, Prentice-Hall, Englewood Cliffs, NJ, 1964.

[11] R. Gorenflo and F. Mainardi, Simply and multiply scaled diffusion limits for continuous time random walks, *Inst. Phys. Conf. Ser.*, **7** (2005), 1–16.

[12] G. Gripenberg, Volterra integro-differential equations with accretive nonlinearity, *J. Differ. Equ.*, **60** (1985), 57–79.

[13] A. Hinz and G. Stolz, Polynomial boundedness of eigenfunctions and the spectrum of Schrödinger operators, *Math. Ann.*, **294** (1992), 195–211.

[14] A. N. Kochubei, A Cauchy problem for evolution equations of fractional order, *Differ. Equ.*, **25** (1989), 967–974.

[15] A. N. Kochubei, Fractional-order diffusion, *Differ. Equ.*, **26** (1990), 485–492.

[16] A. N. Kochubei, Distributed order calculus and equations of ultraslow diffusion, *J. Math. Anal. Appl.*, **340** (2008), 252–281.

[17] A. N. Kochubei, General fractional calculus, evolution equations, and renewal processes, *Integral Equ. Oper. Theory*, **71** (2011), 583–600.

[18] A. N. Kochubei, Fractional-parabolic equations and systems. Cauchy problem, in *Handbook of Fractional Calculus with Applications. Vol. 2*, De Gruyter, 2019.

[19] A. N. Kochubei, General fractional calculus, in *Handbook of Fractional Calculus with Applications. Vol. 1*, De Gruyter, 2019.

[20] Z. Li, Yu. Luchko, and M. Yamamoto, Asymptotic estimates of solutions to initial-boundary value problems for distributed order time-fractional diffusion equations, *Fract. Calc. Appl. Anal.*, **17** (2014), 1114–1136.

[21] Y. Luchko, Boundary value problems for the generalized time-fractional diffusion equation of distributed order, *Fract. Calc. Appl. Anal.*, **12** (2009), 409–422.

[22] Yu. Luchko and M. Yamamoto, General time-fractional diffusion equation: some uniqueness and existence results for the initial-boundary value problems, *Fract. Calc. Appl. Anal.*, **19** (2016), 676–695.

[23] M. M. Meerschaert and E. Nane, Distributed order fractional diffusion on bounded domains, *J. Math. Anal. Appl.*, **379** (2011), 216–228.

[24] M. M. Meerschaert, E. Nane, and P. Vellaisamy, The fractional Poisson process and the inverse stable subordinator, *Electron. J. Probab.*, **16** (2011), 59, 1600–1620.

[25] M. M. Meerschaert and H.-P. Scheffler, Triangular array limits for continuous time random walks, *Stoch. Process. Appl.*, **118** (2008), 1606–1633. **120** (2010), 2520–2521.

[26] R. Metzler and J. Klafter, The random walk's guide to anomalous diffusion: a fractional dynamics approach, *Phys. Rep.*, **339** (2000), 1–77.

[27] R. Metzler and J. Klafter, The restaurant at the end of the random walk: recent developments in the description of anomalous transport by fractional dynamics, *J. Phys. A*, **37** (2004), R161–R208.

[28] M. Naber, Distributed order fractional subdiffusion, *Fractals*, **12** (2004), 23–32.

[29] M. Reed and B. Simon, *Methods of Modern Mathematical Physics. II. Fourier Analysis, Self-Adjointness*, Academic Press, New York, 1975.

[30] S. G. Samko, A. A. Kilbas, and O. I. Marichev, *Fractional Integrals and Derivatives: Theory and Applications*, Gordon and Breach, New York, 1993.

[31] T. Sandev, A. Chechkin, H. Kantz, and R. Metzler, Diffusion and Fokker–Planck equations with generalized memory kernels, *Fract. Calc. Appl. Anal.*, **18** (2015), 1006–1038.

[32] T. Sandev, I. M. Sokolov, R. Metzler, and A. Chechkin, Beyond monofractional kinetics, *Chaos Solitons Fractals*, **102** (2017), 210–217.

[33] R. L. Schilling, R. Song, and Z. Vondraček, *Bernstein Functions. Theory and Applications*, Walter de Gruyter, Berlin, 2010.

[34] W. R. Schneider and W. Wyss, Fractional diffusion and wave equations, *J. Math. Phys.*, **30** (1989), 134–144.

[35] I. M. Sokolov, A. V. Chechkin, and J. Klafter, Distributed-order fractional kinetics, *Acta Geophys. Pol.*, **35** (2004), 1323–1341.

Pablo Raúl Stinga

User's guide to the fractional Laplacian and the method of semigroups

Abstract: The *method of semigroups* is a unifying, widely applicable, general technique to formulate and analyze fundamental aspects of fractional powers of operators L and their regularity properties in related functional spaces. The approach was introduced by the author and José L. Torrea in 2009 (arXiv:0910.2569v1). The aim of this chapter is to show how the method works in the particular case of the fractional Laplacian $L^s = (-\Delta)^s$, $0 < s < 1$. The starting point is the semigroup formula for the fractional Laplacian. From here, the classical heat kernel permits us to obtain the pointwise formula for $(-\Delta)^s u(x)$. One of the key advantages is that our technique relies on the use of heat kernels, which allows for applications in settings where the Fourier transform is not the most suitable tool. In addition, it provides explicit constants that are key to prove, under minimal conditions on u, the validity of the pointwise limits

$$\lim_{s\to 1^-}(-\Delta)^s u(x) = -\Delta u(x) \quad \text{and} \quad \lim_{s\to 0^+}(-\Delta)^s u(x) = u(x).$$

The formula for the solution to the Poisson problem $(-\Delta)^s u = f$ is found through the semigroup approach as the inverse of the fractional Laplacian $u(x) = (-\Delta)^{-s} f(x)$ (fundamental solution). We then present the Caffarelli–Silvestre extension problem, whose explicit solution is given by the semigroup formulas that were first discovered by the author and Torrea. With the extension technique, an interior Harnack inequality and derivative estimates for fractional harmonic functions can be obtained. The classical Hölder and Schauder estimates

$$(-\Delta)^{\pm s} : C^\alpha \to C^{\alpha \mp 2s}$$

are proved with the method of semigroups in a rather quick, elegant way. The crucial point for this will be the characterization of Hölder and Zygmund spaces with heat semigroups.

Keywords: Method of semigroups, fractional Laplacian, extension problem, regularity estimates

MSC 2010: 35R11, 26A33, 58J35

1 Introduction

Fractional powers, both positive and negative, as well as complex, of linear operators appear in many areas of mathematics. In particular, the fractional powers of the

Pablo Raúl Stinga, Department of Mathematics, Iowa State University, 396 Carver Hall, Ames, IA 50011, United States of America, e-mail: stinga@iastate.edu

https://doi.org/10.1515/9783110571660-012

Laplacian are nowadays classical objects. Fractional operators arise in potential theory [10, 18, 63, 87], probability [3, 8, 12, 15–17, 19, 37, 72], fractional calculus and hypersingular integrals [49, 77–80], harmonic analysis [5, 16, 49, 81, 88, 89], functional analysis [6, 53, 60, 62, 67, 99], and pseudo-differential operators [43, 52, 54–56, 58].

In recent years, the fractional Laplacian or, more generally, nonlocal equations of fractional order, gained a lot of attention from the partial differential equations research community. It can be said that the main driving force for this has been the fundamental work of Luis A. Caffarelli and his collaborators; see [23–27, 30, 84, 85], just to mention a few.

To introduce the notion of fractional Laplacian, let u be a function in the Schwartz class $\mathcal{S} = \mathcal{S}(\mathbb{R}^n)$, $n \geq 1$. The Fourier transform of u, denoted by $\widehat{u}$, is also in $\mathcal{S}$. For the Laplacian $-\Delta$ on $\mathbb{R}^n$, we have

$$\widehat{(-\Delta)u}(\xi) = |\xi|^2 \widehat{u}(\xi) \quad \text{for every } \xi \in \mathbb{R}^n.$$

The fractional Laplacian $(-\Delta)^s$, $0 < s < 1$, is then defined in a natural way as

$$\widehat{(-\Delta)^s u}(\xi) = |\xi|^{2s} \widehat{u}(\xi). \tag{1}$$

1.1 A few applications

Let us begin by briefly describing problems in probability, financial mathematics, elasticity, and biology where fractional powers of differential operators appear.

1.1.1 Lévy processes

Let $(X_t : t \geq 0)$ be a symmetric $2s$-stable $(0 < 2s \leq 2)$ $\mathbb{R}^n$-valued Lévy process starting at 0. By the Lévy–Khintchine formula [3, 12] the characteristic function of X_t is $\mathbb{E}(e^{i\xi\cdot X_t}) = e^{-t\kappa^{2s}|\xi|^{2s}}$, $\xi \in \mathbb{R}^n$, $t \geq 0$, for some positive constant κ that for simplicity we take equal to 1. For $u \in \mathcal{S}$, set $T_t u(x) = \mathbb{E}(u(X_t + x))$, $x \in \mathbb{R}^n$, $t \geq 0$. Then, by Fubini's theorem, $\widehat{T_t u}(\xi) = e^{-t|\xi|^{2s}} \widehat{u}(\xi)$. Therefore, the function $v(x,t) = T_t u(x)$ solves the fractional diffusion equation

$$\begin{cases} \partial_t v = -(-\Delta)^s v & \text{in } \mathbb{R}^n \times (0,\infty) \\ v(x,0) = u(x) & \text{on } \mathbb{R}^n. \end{cases}$$

There is a Markov process corresponding to the fractional powers of the Dirichlet Laplacian $-\Delta_D$ in a smooth bounded domain Ω. The process can be obtained as follows: we first kill a Wiener process W at τ_Ω, the first exit time of W from Ω, and then we subordinate the killed Wiener process using an s-stable subordinator T_t. This subordinated process has generator $(-\Delta_D)^s$; see [87].

1.1.2 Financial mathematics

For a symmetric $2s$-stable Lévy process X_t with $X_0 = x$, consider the optimal stopping time τ to maximize the function

$$u(x) = \sup_{\tau} \mathbb{E}[\varphi(X_\tau) : \tau < \infty]$$

where $\varphi \in C_0(\mathbb{R}^n)$. Then u is a solution to the free boundary problem

$$\begin{cases} u(x) \geq \varphi(x) & \text{in } \mathbb{R}^n \\ (-\Delta)^s u(x) \geq 0 & \text{in } \mathbb{R}^n \\ (-\Delta)^s u(x) = 0 & \text{in } \{u(x) > \varphi(x)\}. \end{cases} \tag{2}$$

This obstacle problem arises as a pricing model for American options [36, 84, 85].

1.1.3 Elasticity, biology

An equivalent formulation of the problem Antonio Signorini posed in [83] consists in finding the configuration of an elastic membrane in equilibrium that stays above some given *thin* obstacle. In mathematical terms, given $\varphi \in C_0(\mathbb{R}^n)$, the solution to the Signorini problem is the function $U = U(x, y)$, $x \in \mathbb{R}^n$, $y \geq 0$, that satisfies

$$\begin{cases} \partial_{yy} U + \Delta_x U = 0 & \text{in } \mathbb{R}^n \times (0, \infty) \\ U(x, 0) \geq \varphi(x) & \text{on } \mathbb{R}^n \\ \partial_y U(x, 0) \leq 0 & \text{on } \mathbb{R}^n \\ \partial_y U(x, 0) = 0 & \text{in } \{U(x, 0) > \varphi(x)\}; \end{cases}$$

see, for example, [23, 42]. A simple observation gives an equivalent description of the Signorini problem as an obstacle problem for the fractional Laplacian. The solution to $\partial_{yy} U + \Delta_x U = 0$ with boundary data $u(x) := U(x, 0)$ is given by convolution with the Poisson kernel in the upper half space:

$$U(x, y) = e^{-y(-\Delta_x)^{1/2}} u(x). \tag{3}$$

Taking the derivative of U with respect to y and evaluating it at $y = 0$ gives

$$\partial_y U(x, y)|_{y=0} = -(-\Delta_x)^{1/2} u(x).$$

Hence, the Signorini problem can be rewritten as

$$\begin{cases} \partial_{yy} U + \Delta_x U = 0 & \text{in } \mathbb{R}^n \times (0, \infty) \\ U(x, 0) \geq \varphi(x) & \text{on } \mathbb{R}^n \\ (-\Delta_x)^{1/2} U(x, 0) \geq 0 & \text{on } \mathbb{R}^n \\ (-\Delta_x)^{1/2} U(x, 0) = 0 & \text{in } \{U(x, 0) > \varphi(x)\}. \end{cases}$$

In other words, the Signorini problem is equivalent to the obstacle problem (2) for $s = 1/2$ through the relation $u(x) = U(x, 0)$ given by (3).

Consider next a Signorini problem where the Laplacian $-\Delta_x$ is replaced by another partial differential operator L in a domain $\Omega \subseteq \mathbb{R}^n$. For example, L can be the Dirichlet Laplacian $-\Delta_D$ (meaning the elastic membrane is kept at zero level on $\partial\Omega$) or the heat operator $\partial_t - \Delta$ (this becomes a model for semipermeable walls, like a cell membrane on $y = 0$, see [42]). The associated Poisson semigroup

$$U(x, y) = e^{-yL^{1/2}} u(x)$$

is the solution to

$$\begin{cases} \partial_{yy} U - LU = 0 & \text{in } \Omega \times (0, \infty) \\ U|_{y=0} = u & \text{on } \Omega \end{cases}$$

and satisfies

$$\partial_y U|_{y=0} = -L^{1/2} u.$$

Then the Signorini problem for L in place of $-\Delta_x$ can be formulated for $u = U|_{y=0}$ in an equivalent way as an obstacle problem for $L^{1/2}$:

$$\begin{cases} u \geq \varphi & \text{in } \Omega \\ L^{1/2} u \geq 0 & \text{in } \Omega \\ L^{1/2} u = 0 & \text{in } \{u > \varphi\}; \end{cases}$$

see [4, 29, 90, 93].

Our list of problems above does not pretend to be exhaustive at all. Just to mention some more, there are applications in fluid mechanics [30, 35], fractional kinetics and anomalous diffusion [71, 86, 101], strange kinetics [82], fractional quantum mechanics [64, 65], Lévy processes in quantum mechanics [75], plasmas [2], electrical propagation in cardiac tissue [20], and biological invasions [9].

1.2 The method of semigroups

Consider the situation where we have derived a model (usually a nonlinear PDE problem) that involves a fractional power of some partial differential operator L. As we saw before, L can be a Laplacian or a heat operator, or even an operator on a manifold [7, 39] or a lattice in the case of discrete models [34]. Then we are faced at least with the following basic questions.

1.2.1 Definition and pointwise formula for fractional operators

For a general operator L, classical functional analysis gives several ways to define L^s according to its analytical properties. Nevertheless, a pure abstract formula is not useful to treat concrete PDE problems and a more or less explicit pointwise expression for $L^s u(x)$ is needed in many cases. The starting point for the *method of semigroups* is the formula

$$L^s u = \frac{1}{\Gamma(-s)} \int_0^\infty (e^{-tL}u - u)\, \frac{dt}{t^{1+s}} \qquad 0 < s < 1$$

where $\Gamma(-s) = \frac{\Gamma(1-s)}{-s}$ is the Gamma function evaluated at $-s$. Here, $v = e^{-tL}u$ is the *heat diffusion semigroup* generated by L acting on u, namely, v is the solution to the heat equation for L with initial temperature u:

$$\begin{cases} \partial_t v = -Lv & \text{for } t > 0 \\ v|_{t=0} = u. \end{cases}$$

The semigroup formula for L^s is classical; see [6, 60, 62, 99]. The definition is motivated by the numerical identity

$$\lambda^s = \frac{1}{\Gamma(-s)} \int_0^\infty (e^{-t\lambda} - 1)\, \frac{dt}{t^{1+s}} \qquad \text{for any } \lambda \geq 0 \tag{4}$$

that can be easily checked with a simple change of variables. In a similar way, starting from the numerical identity

$$\lambda^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty e^{-t\lambda}\, \frac{dt}{t^{1-s}} \qquad \text{for any } \lambda > 0,\ s > 0 \tag{5}$$

we can write down the solution to $L^s u = f$ as

$$u = L^{-s} f = \frac{1}{\Gamma(s)} \int_0^\infty e^{-tL} f\, \frac{dt}{t^{1-s}}.$$

Again, this semigroup formula for L^{-s} is classical; see [6, 60, 62, 99]. It turns out these are quite concrete and useful ways of defining and understanding fractional operators. Indeed, when a heat kernel is available for the semigroup e^{-tL}, then pointwise formulas for both positive and negative powers of L can be obtained; see [13, 14, 29, 32–34, 38, 39, 45, 70, 76, 90–94]. For degenerate cases like the usual derivative or discrete derivatives, see [1, 11].

In this chapter, we will explain how these formulas work only for the case of $L = -\Delta$, as developed in [90, 91]. Sections 2 and 3 are devoted to show how the semigroup definitions of $(-\Delta)^s$ and $(-\Delta)^{-s}$ follow from the above-mentioned numerical formulas and how, with the help of the classical heat semigroup kernel, one can obtain the well-known nonlocal pointwise formula

$$(-\Delta)^s u(x) = c_{n,s} \mathrm{P.V.} \int_{\mathbb{R}^n} \frac{u(x) - u(z)}{|x - z|^{n+2s}} \, dz$$

and similarly for $(-\Delta)^{-s} f(x)$. Obviously, these formulas are very well known [63, 89] and can be deduced through several other techniques. Nevertheless, we present the semigroup ideas in this simple case so the reader can use them in other applications.

1.2.2 The nonlocal nature

The fractional Laplacian is a nonlocal operator. Indeed, the value of $(-\Delta)^s u(x)$ for a given $x \in \mathbb{R}^n$ depends on the values of u at infinity. Also, in general, if u has compact support then $(-\Delta)^s u$ has noncompact support. This basic property may create some issues. For example, the classical local PDE methods from the calculus of variations based on integration by parts and localization using test functions cannot be directly applied to the study of nonlinear problems for $(-\Delta)^s$. Even the notion of viscosity solution needs to be redefined to take into account the values of solutions at infinity [25]. L. A. Caffarelli and L. Silvestre showed in [24] that any fractional power of the Laplacian can be characterized as an operator that maps a Dirichlet boundary condition to a Neumann-type condition via an extension PDE problem. From a probabilistic point of view, the extension problem corresponds to the property that all symmetric stable processes can be obtained as traces of degenerate Bessel diffusion processes; see [72]. Consider the function $U = U(x, y) : \mathbb{R}^n \times [0, \infty) \to \mathbb{R}$ that solves the degenerate elliptic boundary value problem

$$\begin{cases} \Delta_x U + \frac{a}{y} U_y + U_{yy} = 0 & x \in \mathbb{R}^n,\ y > 0 \\ U(x, 0) = u(x) & x \in \mathbb{R}^n \end{cases}$$

where $a = 1 - 2s$. Then, for any $x \in \mathbb{R}^n$,

$$-\lim_{y \to 0^+} y^a U_y(x, y) = c_s (-\Delta)^s u(x);$$

see [24]. The constant $c_s > 0$ was computed explicitly for the first time in [90, 91]. We can interpret this result by saying that the new variable y added to extend u to the upper half space through U encodes the values of u at infinity needed to compute $(-\Delta)^s u$. The extension problem localizes the fractional Laplacian: it is enough to know U in some upper half ball around $(x, 0)$ to already get $(-\Delta)^s u(x)$. The nonlinear problems for the nonlocal fractional Laplacian can then be localized by adding a new variable. Now one can exploit the classical PDE tools and ideas that are available for these equations [44]. The work of Caffarelli and Silvestre [24] presented applications to Harnack inequalities and monotonicity formulas for $(-\Delta)^s$ by applying such local PDE techniques in the extension problem. Since then, [24] has created an explosion of results on problems with fractional Laplacians; see [23, 30] for a couple of important examples.

In general, fractional power operators L^s are nonlocal operators. It would be very useful in applications to have an analogous to the Caffarelli–Silvestre characterization for L^s as a Dirichlet-to-Neumann map via an extension problem. This problem was solved in [90, 91]. The author and Torrea discovered in [91] an extension problem for fractional operators on Hilbert spaces. Later on, J. E. Galé, P. J. Miana and the author found an extension problem characterization for fractional powers of operators in Banach spaces and, more generally, generators of integrated semigroups; see [47]. In addition, [47] included the case of complex fractional power operators. The semigroup point of view turned out to be fundamental. As a matter of fact, when $L = -\Delta$ in [47, 90, 91] then one recovers the extension PDE of [24]. Some of the main novelties of [47, 90, 91] were the analysis of the extension PDE by means of Bessel functions and the explicit semigroup formulas for the solution

$$\begin{aligned} U(y) &= \frac{y^{2s}}{4^s\Gamma(s)} \int_0^\infty e^{-y^2/(4t)} e^{-tL} u \, \frac{dt}{t^{1+s}} \\ &= \frac{1}{\Gamma(s)} \int_0^\infty e^{-y^2/(4t)} e^{-tL} (L^s u) \, \frac{dt}{t^{1-s}}. \end{aligned}$$

These formulas were new even for the case of the fractional Laplacian, where $L = -\Delta$. As it could be expected, these general extension problems found many applications such as free boundary problems [2, 4], fractional derivatives [11], master equations [14, 38, 93], fractional elliptic PDEs [29, 95, 100], fractional Laplacians on manifolds [7, 28, 31, 39, 46] and in infinite dimensions [74], symmetrization [40, 45], nonlocal Monge–Ampère equations [70], numerical analysis [73], biology [94], and inverse problems [48, 51].

On the other hand, an extension problem for higher powers of fractional operators in Hilbert spaces using heat semigroups was proved in [76]; see also [98] for the particular case of the fractional Laplacian on $\mathbb{R}^n$. The fractional powers of the Laplacian can also be characterized by means of a wave extension problem, see [61]. In such scenario, the wave and Schrödinger groups (instead of the heat semigroup) play a key, fundamental role.

We will not go into more details about all these general cases here, but we will only show how the semigroup ideas, techniques and formulas of [47, 90, 91] work for the extension problem in the fractional Laplacian case; see Section 4. Applications to Harnack inequalities and derivative estimates for s-harmonic functions are given in Section 5 by following [23, 24].

1.2.3 Regularity theory for fractional operators

Clearly, the Fourier transform definition of the fractional Laplacian does not seem to be the most useful formulation to prove regularity estimates in Hölder and Zygmund spaces. One strategy that has been followed for this problem is to make heavy use of the pointwise formulas for $(-\Delta)^s$ and $(-\Delta)^{-s}$; see, for example, [84, 85]. This is natural because pointwise formulas clearly allow us to handle differences of the form $|(-\Delta)^s u(x_1) - (-\Delta)^s u(x_2)|$.

We present here a semigroup method toward proving regularity estimates, where only the semigroup formulas for the fractional operators are needed. We will first show that Hölder and Zygmund spaces are characterized by means of the growth of time derivatives of the heat semigroup $\partial_t^k e^{t\Delta}$; see Section 6. The proof of such characterization is obviously nontrivial. It will be shown in Section 7 how the semigroup descriptions of Hölder–Zygmund spaces and fractional Laplacians allow for a quick, elegant proof of Hölder and Schauder estimates. We believe this is the first time these results have been presented and proved in such a systematic, complete way for the case of the fractional Laplacian.

If we now think about fractional powers of other differential operators L, we may ask for the "right" Schauder estimates for L^s. More precisely, what is the proper/adapted Hölder space to look for regularity properties of L^s? The semigroup approach then comes at hand: one can define regularity spaces associated to L in terms of the growth of heat semigroups $\partial_t^k e^{-tL}$ in complete analogy to the case of the classical Hölder–Zygmund spaces. As we mentioned, passing from a semigroup formulation to a pointwise description of such spaces is a nontrivial task that must be carefully handled in each particular situation. Despite this, the great advantage is that the regularity properties of fractional powers L^s on these spaces will follow at once using the semigroup representations. See, for example, [29] for the fractional Laplacian, [69, 92] for Schrödinger operators $L = -\Delta + V$, [50, 68] for the Ornstein–Uhlenbeck operator $L = -\Delta + \nabla \cdot x$, [14, 38, 93] for fractional powers of parabolic operators, [76] for the fractional Laplacian on the torus, and [13] for Bessel operators and radial solutions to the fractional Laplacian.

As we said before, the fractional Laplacian is a classical object in mathematics, and many of the results we will present here can be proved in several different ways and with other techniques. An exhaustive list of classical and modern references dealing with them is out of the scope of this chapter and the reader is invited to explore the references mentioned at the beginning of this section as well as those contained in other chapters of this volume.

2 Fractional Laplacian: semigroups, pointwise formulas, and limits

Recall the Fourier transform definition of the fractional Laplacian given in (1). It is obvious that $(-\Delta)^0 u = u$, $(-\Delta)^1 u = -\Delta u$ and, for any s_1, s_2 we have $(-\Delta)^{s_1} \circ (-\Delta)^{s_2} u = (-\Delta)^{s_1+s_2} u$. Even though $|\xi|^{2s}\widehat{u}(\xi)$ is a well-defined function of $\xi \in \mathbb{R}^n$, we still have

$$(-\Delta)^s u \notin \mathcal{S}$$

because $|\xi|^{2s}$ creates a singularity at $\xi = 0$. On the other hand, (1) implies that for any multi-index $\gamma \in \mathbb{N}_0^n$,

$$D_x^\gamma (-\Delta)^s = (-\Delta)^s D_x^\gamma. \tag{6}$$

In particular, if $u \in \mathcal{S}$ then $(-\Delta)^s u \in C^\infty(\mathbb{R}^n)$.

To compute $(-\Delta)^s u(x)$ for each point $x \in \mathbb{R}^n$, one could try to take the inverse Fourier transform in (1). In fact, since $|\xi|^{2s}\widehat{u}(\xi) \in L^1(\mathbb{R}^n)$, one can make sense to $(-\Delta)^s u(x) = \mathcal{F}^{-1}(|\xi|^{2s}\widehat{u}(\xi))(x)$. But here we are going to avoid this and, instead, apply the method of semigroups. If we choose $\lambda = |\xi|^2$, for $\xi \in \mathbb{R}^n$, in the numerical formula (4), multiply it by $\widehat{u}(\xi)$ and recall (1), then

$$\widehat{(-\Delta)^s u}(\xi) = |\xi|^{2s}\widehat{u}(\xi) = \frac{1}{\Gamma(-s)} \int_0^\infty \left(e^{-t|\xi|^2}\widehat{u}(\xi) - \widehat{u}(\xi)\right) \frac{dt}{t^{1+s}}.$$

Thus, by inverting the Fourier transform, we obtain the *semigroup formula for the fractional Laplacian* (see [6, 60, 62, 99], also [47, 90, 91])

$$(-\Delta)^s u(x) = \frac{1}{\Gamma(-s)} \int_0^\infty \left(e^{t\Delta} u(x) - u(x)\right) \frac{dt}{t^{1+s}}. \tag{7}$$

The family of operators $\{e^{t\Delta}\}_{t\geq 0}$ is the classical heat diffusion semigroup generated by Δ. Consider the solution $v = v(x,t)$, for $x \in \mathbb{R}^n$ and $t \geq 0$, of the heat equation on the whole space $\mathbb{R}^n$ with initial temperature u:

$$\begin{cases} \partial_t v = \Delta v & \text{for } x \in \mathbb{R}^n,\ t > 0 \\ v(x,0) = u(x) & \text{for } x \in \mathbb{R}^n. \end{cases}$$

If we compute the Fourier transform of v in the variable x for each fixed t, then

$$\widehat{v}(\xi, t) = e^{-t|\xi|^2}\widehat{u}(\xi) = \widehat{e^{t\Delta} u}(\xi) \tag{8}$$

so that $u \longmapsto e^{t\Delta} u$ is the solution operator. It is well known that

$$v(x,t) \equiv e^{t\Delta} u(x) = G_t * u(x) = \int_{\mathbb{R}^n} G_t(x-z) u(z)\, dz$$

where $G_t(x)$ is the *Gauss–Weierstrass heat kernel*:

$$G_t(x) = \frac{1}{(4\pi t)^{n/2}} e^{-|x|^2/(4t)}. \tag{9}$$

Observe that G_t defines an approximation of the identity. Moreover,

$$e^{t\Delta}1(x) = \int_{\mathbb{R}^n} G_t(x)\,dx \equiv 1 \quad \text{for any } x \in \mathbb{R}^n,\ t > 0. \tag{10}$$

Remark 1 (Maximum principle). The semigroup formula (7) and the positivity of the heat kernel (9) easily imply the maximum principle for the fractional Laplacian. Indeed, if $u \geq 0$ and $u(x_0) = 0$ at some point $x_0 \in \mathbb{R}^n$, then

$$(-\Delta)^s u(x_0) = \frac{1}{\Gamma(-s)} \int_0^\infty e^{t\Delta} u(x_0) \frac{dt}{t^{1+s}} \leq 0.$$

Moreover, $(-\Delta)^s u(x_0) = 0$ if and only if $e^{t\Delta}u(x_0) = 0$, that is, only when $u \equiv 0$. For another proof using pointwise formulas, see [84, 85]. For maximum principles for fractional powers of elliptic operators using semigroups, see [95].

The semigroup formula (7) and the heat kernel (9) permit us to compute the pointwise formula for the fractional Laplacian. The technique avoids the inverse Fourier transform in (1) and gives the constants explicitly.

Theorem 1 (Pointwise formulas). *Let $u \in \mathcal{S}$, $x \in \mathbb{R}^n$ and $0 < s < 1$.*

(i) *If $0 < s < 1/2$, then*

$$(-\Delta)^s u(x) = c_{n,s} \int_{\mathbb{R}^n} \frac{u(x) - u(z)}{|x - z|^{n+2s}}\,dz$$

and the integral is absolutely convergent.

(ii) *If $1/2 \leq s < 1$ then, for any $\delta > 0$,*

$$\begin{aligned}(-\Delta)^s u(x) &= c_{n,s} \lim_{\varepsilon \to 0^+} \int_{|x-z|>\varepsilon} \frac{u(x) - u(z)}{|x - z|^{n+2s}}\,dz \\ &= c_{n,s} \int_{\mathbb{R}^n} \frac{u(x) - u(z) - \nabla u(x) \cdot (x - z)\chi_{|x-z|<\delta}(z)}{|x - z|^{n+2s}}\,dz\end{aligned}$$

where the second integral is absolutely convergent.

The constant $c_{n,s} > 0$ in the formulas above is explicitly given by

$$c_{n,s} = \frac{4^s \Gamma(n/2 + s)}{|\Gamma(-s)|\pi^{n/2}} = \frac{s(1 - s)4^s \Gamma(n/2 + s)}{|\Gamma(2 - s)|\pi^{n/2}}. \tag{11}$$

In particular, $c_{n,s} \sim s(1 - s)$ as $s \to 0^+$ and $s \to 1^-$.

Sketch of proof. The detailed proof using the heat kernel can be found in [90, 91]. We write down the formula for $e^{t\Delta}u(x)$ in (7) and use (10) to get

$$(-\Delta)^s u(x) = \frac{1}{|\Gamma(-s)|} \int_0^\infty \int_{\mathbb{R}^n} G_t(x-z)\big(u(x)-u(z)\big)\,dz\,\frac{dt}{t^{1+s}}. \tag{12}$$

By (9) and the change of variables $r = |x-z|^2/(4t)$,

$$\frac{1}{|\Gamma(-s)|} \int_0^\infty G_t(x-z)\,\frac{dt}{t^{1+s}} = c_{n,s} \cdot \frac{1}{|x-z|^{n+2s}}. \tag{13}$$

When $0 < s < 1/2$, the double integral in (12) is absolutely convergent and Fubini's theorem gives (i). If $1/2 \le s < 1$, then one needs to use the fact that, for any $i = 1,\dots,n$ and $0 < \varepsilon < \delta$,

$$\int_{|z|<\delta} z_i G_t(z)\,dz = \int_{\varepsilon<|z|<\delta} z_i G_t(z)\,dz = 0$$

for all $t > 0$, for the gradient term to appear in (ii). □

The pointwise formulas in Theorem 1 are valid for $u \in \mathcal{S}$. However, the integrals are well defined for less regular functions. As a matter of fact, we can relax the requirement on u at infinity by asking that

$$\|u\|_{L_s} := \int_{\mathbb{R}^n} \frac{|u(x)|}{1+|x|^{n+2s}}\,dx < \infty. \tag{14}$$

When $0 < s < 1/2$, we only need u to be $C^{2s+\varepsilon}$ at x, for $2s+\varepsilon \le 1$, for the singular part of the integral (i. e., when z is near x) to be finite:

$$\int_{|x-z|<1} \frac{|u(x)-u(z)|}{|x-z|^{n+2s}}\,dz \le [u]_{C^{2s+\varepsilon}(x)} \int_{|x-z|<1} \frac{|x-z|^{2s+\varepsilon}}{|x-z|^{n+2s}}\,dz < \infty.$$

Similarly, when $1/2 \le s < 1$, the requirement $u \in C^{1,2s+\varepsilon-1}$ at x, for $2s+\varepsilon-1 \le 1$ would suffice as well.

To define the fractional Laplacian for less regular functions, we need to understand what is $(-\Delta)^s$ in the sense of distributions. The fractional Laplacian is a symmetric operator on $L^2(\mathbb{R}^n)$: when $u, f \in \mathcal{S}$,

$$\int_{\mathbb{R}^n} (-\Delta)^s u(x) f(x)\,dx = \int_{\mathbb{R}^n} |\xi|^{2s}\widehat{u}(\xi)\widehat{f}(\xi)\,d\xi = \int_{\mathbb{R}^n} u(x)(-\Delta)^s f(x)\,dx.$$

We may then think in the following way. For $u \in \mathcal{S}'$ (a tempered distribution) and $f \in \mathcal{S}$, one could define the distribution $(-\Delta)^s u$ as $\langle(-\Delta)^s u, f\rangle = \langle u, (-\Delta)^s f\rangle$. The problem here is that $(-\Delta)^s f \notin \mathcal{S}$, so this identity makes no sense for $u \in \mathcal{S}'$. First, we need to characterize the set $(-\Delta)^s(\mathcal{S})$; see [34, 84, 85].

Lemma 1. *Let $f \in \mathcal{S}$. Then $(-\Delta)^s f$ belongs to the class $\mathcal{S}_s$ defined by*

$$\mathcal{S}_s = \{\psi \in C^\infty(\mathbb{R}^n) : (1 + |x|^{n+2s})D^\gamma \psi(x) \in L^\infty(\mathbb{R}^n), \text{ for every } \gamma \in \mathbb{N}_0^n\}.$$

The class $\mathcal{S}_s$ of Lemma 1 is endowed with the topology induced by the countable family of seminorms

$$\rho_\gamma(\psi) = \sup_{x \in \mathbb{R}^n} \left|(1 + |x|^{n+2s})D^\gamma \psi(x)\right|, \quad \gamma \in \mathbb{N}_0^n.$$

Denote by $\mathcal{S}_s'$ the dual space of $\mathcal{S}_s$. Observe that $\mathcal{S} \subset \mathcal{S}_s$, so that $\mathcal{S}_s' \subset \mathcal{S}'$. The suitable space for the distributional definition of the fractional Laplacian is $\mathcal{S}_s'$.

Definition 1. Let $u \in \mathcal{S}_s'$. We define $(-\Delta)^s u \in \mathcal{S}'$ as

$$((-\Delta)^s u)(f) = u((-\Delta)^s f) \quad \text{for every } f \in \mathcal{S}.$$

In terms of pairings, we write $\langle (-\Delta)^s u, f \rangle_{\mathcal{S}', \mathcal{S}} = \langle u, (-\Delta)^s f \rangle_{\mathcal{S}_s', \mathcal{S}_s}$.

If $u \in \mathcal{S}$, then this distributional definition coincides with the one given in terms of the Fourier transform. Also, $(-\Delta)^s$ maps $\mathcal{S}_s'$ into $\mathcal{S}'$ continuously. Recall the definition of the space L_s given in (14). We have $L_s = L^1_{\mathrm{loc}}(\mathbb{R}^n) \cap \mathcal{S}_s'$. The proof of the following result is based on an approximation argument and the details can be found in [84, 85].

Theorem 2 (Pointwise formula for less regular functions). *Let Ω be an open subset of $\mathbb{R}^n$ and let $u \in L_s$, $0 < s < 1$. If $u \in C^{2s+\varepsilon}(\Omega)$ (or $C^{1,2s+\varepsilon-1}(\Omega)$ if $s \geq 1/2$) for some $\varepsilon > 0$, then $(-\Delta)^s u$ is a continuous function in Ω and $(-\Delta)^s u(x)$ is given by the pointwise formulas of Theorem 1, for every $x \in \Omega$.*

Remark 2 (Semigroup formula for less regular functions). It is an exercise to verify that the semigroup formula (7), that was initially derived for functions $u \in \mathcal{S}$, also holds for the class of less regular functions u considered in Theorem 2, for all $x \in \Omega$. Indeed, it is easy to work out the computations in the proof of Theorem 1 in a reverse order, namely, by starting with the pointwise integro-differential formula for $(-\Delta)^s u(x)$ and using the heat kernel identity (13) to end up with (7).

We turn our attention to the pointwise limits for $s \to 1^-$ and $s \to 0^+$ in the case of less regular functions. The explicit value of the constant $c_{n,s}$ in (11), that we found through the method of semigroups, plays a crucial role. Observe as well that we require minimal regularity on u for $(-\Delta)^s u(x)$ to be defined through the integral formula and for the *pointwise* limits to have sense.

Theorem 3. *If $u \in C^2(B_2(x)) \cap L^\infty(\mathbb{R}^n)$ at some $x \in \mathbb{R}^n$, then*

$$\lim_{s \to 1^-} (-\Delta)^s u(x) = -\Delta u(x).$$

Sketch of proof. The full details of the proof can be found in [90, 91]. We can write $(-\Delta)^s u(x)$ as

$$c_{n,s} \int\limits_{|x-z|>\delta} \frac{u(x)-u(z)}{|x-z|^{n+2s}}\,dz + c_{n,s} \int\limits_{|x-z|<\delta} \frac{u(x)-u(z)-\nabla u(x)\cdot(x-z)}{|x-z|^{n+2s}}\,dz.$$

Since u is bounded the first term above converges to zero as $s \to 1^-$. The second term can be written as

$$\begin{aligned} I - II := c_{n,s} \int_0^\delta r^{-1-2s} \int\limits_{|z'|=1} \left[\frac{r^2}{2}\langle D^2u(x)z', z'\rangle - R_1u(x, rz')\right] dS_{z'}\,dr \\ - c_{n,s} \int_0^\delta r^{-1-2s} \int\limits_{|z'|=1} \frac{r^2}{2}\langle D^2u(x)z', z'\rangle\, dS_{z'}\,dr \end{aligned}$$

where $R_1u(x, rz') = u(x - rz') - u(x) + \nabla u(x)\cdot(rz')$. By the regularity of u, $|I| \le C\varepsilon$ and using that

$$\int\limits_{|z'|=1} \langle D^2u(x)z', z'\rangle\, dz' = \frac{(n/2+1)\pi^{n/2}}{\Gamma(n/2+2)}\Delta u(x)$$

the conclusion follows. □

Recall the definition of the space L_s from (14), for $0 \le s \le 1$.

Theorem 4. *If $u \in C^\alpha(B_2(x)) \cap L_0$ for some $x \in \mathbb{R}^n$ and $0 < \alpha < 1$, then*

$$\lim_{s\to 0^+} (-\Delta)^s u(x) = u(x).$$

Sketch of proof. The complete details of the proof are found in [90]. We have

$$(-\Delta)^s u(x) = c_{n,s} \int\limits_{|x-z|<R} \frac{u(x)-u(z)}{|x-z|^{n+2s}}\,dz + c_{n,s} \int\limits_{|x-z|>R} \frac{u(x)-u(z)}{|x-z|^{n+2s}}\,dz$$

for $R = 1 + |x|$. Since u is regular, the first term above converges to zero as $s \to 0^+$. The second term is split into two: one with $u(x)$ (that will converge to $u(x)$ as $s \to 0^+$) and the other one with $u(y)$ (that will converge to zero as $s \to 0^+$). □

3 Inverse fractional Laplacian: semigroups, pointwise formula, and the Poisson problem

If we apply the Fourier transform to solve the Poisson equation

$$(-\Delta)^s u = f \quad \text{in } \mathbb{R}^n$$

we find that $|\xi|^{2s}\widehat{u}(\xi) = \widehat{f}(\xi)$. The inverse of the fractional Laplacian, or *negative power of the Laplacian* $(-\Delta)^{-s}$, $s > 0$, is defined for $f \in \mathcal{S}$ as

$$\widehat{(-\Delta)^{-s}f}(\xi) = |\xi|^{-2s}\widehat{f}(\xi) \quad \text{for } \xi \neq 0. \tag{15}$$

In principle, we need the restriction $0 < s < n/2$ because when $s \geq n/2$ the multiplier $|\xi|^{-2s}$ does not define a tempered distribution; see [84, 85, 89]. This operator is also known as the *fractional integral operator* in the harmonic analysis literature [41, 89].

To find a pointwise expression for $(-\Delta)^{-s}f(x)$ at a point $x \in \mathbb{R}^n$ one could try to compute the inverse Fourier transform in (15). This is a delicate task as the Fourier multiplier $|\xi|^{-2s}$ is not in $L^2(\mathbb{R}^n)$; see [89]. Instead, we apply the method of semigroups. We start by choosing $\lambda = |\xi|^2$, $\xi \neq 0$, in the numerical formula (5) to find that, for a.e. $\xi \in \mathbb{R}^n$,

$$\widehat{(-\Delta)^{-s}f}(\xi) = \frac{1}{\Gamma(s)}\int_0^\infty e^{-t|\xi|^2}\widehat{f}(\xi)\,\frac{dt}{t^{1-s}}.$$

Therefore, by inverting the Fourier transform, we obtain the *semigroup formula for the inverse fractional Laplacian* (see [6, 60, 62, 89, 99], also [90])

$$(-\Delta)^{-s}f(x) = \frac{1}{\Gamma(s)}\int_0^\infty e^{t\Delta}f(x)\,\frac{dt}{t^{1-s}}. \tag{16}$$

The proof of the following result using heat kernels when $0 < s < n/2$ is contained in [90]. The proof of the case $s = n/2$ is an original contribution of the author for this volume. Indeed, it has not been published elsewhere yet.

Theorem 5 (Fundamental solution). *Let $f \in \mathcal{S}$, $x \in \mathbb{R}^n$ and $0 < s \leq n/2$. In the case when $s = n/2$, assume in addition that $\int_{\mathbb{R}^n} f\,dx = 0$. Then*

$$(-\Delta)^{-s}f(x) = \int_{\mathbb{R}^n} K_{-s}(x-z)f(z)\,dz. \tag{17}$$

Here,

$$K_{-s}(x) = \begin{cases} c_{n,-s}\frac{1}{|x-z|^{n-2s}} & \text{if } 0 < s < n/2 \\ \frac{1}{\Gamma(n/2)(4\pi)^{n/2}}(-2\log|x| - \gamma) & \text{if } s = n/2 \end{cases}$$

where

$$\gamma = -\int_0^\infty e^{-r}\log r\,dr \approx 0.577215$$

is the Euler–Mascheroni constant and

$$c_{n,-s} = \frac{\Gamma(n/2-s)}{4^s\Gamma(s)\pi^{n/2}} \quad \text{for } 0 < s < n/2.$$

The reader should compare the explicit constant $c_{n,-s}$ in Theorem 5 (that we found through the method of semigroups, see [90]) with the constant $c_{n,s}$ for the fractional Laplacian in (11).

Proof of Theorem 5 *in the case* $2s = n$. As we mentioned above, this proof is an original contribution of the author for this volume. From (16) and (9), the change of variables $r = 1/(4t)$ and the fact that f has zero mean,

$$\begin{aligned}(-\Delta)^{-s} f(x) &= \frac{1}{\Gamma(n/2)(4\pi)^{n/2}} \int_0^\infty \int_{\mathbb{R}^n} e^{-r|z|^2} f(x-z)\,dz\,\frac{dr}{r} \\ &= \frac{1}{\Gamma(n/2)(4\pi)^{n/2}} \int_0^\infty \int_{\mathbb{R}^n} (e^{-r|z|^2} - \chi_{(0,1)}(r)) f(x-z)\,dz\,\frac{dr}{r}.\end{aligned} \tag{18}$$

The second double integral in (18) is absolutely convergent. Indeed,

$$\begin{aligned}&\int_0^\infty |e^{-r|z|^2} - \chi_{(0,1)}(r)|\,\frac{dr}{r} \\ &= \int_0^{|z|^2} (1-e^{-r})\,\frac{dr}{r} + \int_{|z|^2}^\infty e^{-r}\,\frac{dr}{r} \\ &= \int_0^{|z|^2} (1-e^{-r}) \frac{d}{dr}(\log r)\,dr + \int_{|z|^2}^\infty e^{-r} \frac{d}{dr}(\log r)\,dr \\ &= (1-e^{-|z|^2})\log(|z|^2) - \int_0^{|z|^2} e^{-r}\log r\,dr - e^{-|z|^2}\log(|z|^2) + \int_{|z|^2}^\infty e^{-r}\log r\,dr \\ &\le C + C\big|\log|z|\big| \in L^1_{\mathrm{loc}}(\mathbb{R}^n).\end{aligned}$$

Thus we can apply Fubini's theorem in (18). By following the computation we just did above, we obtain the formula for the kernel:

$$\begin{aligned}K_{-n/2}(z) &= \frac{1}{\Gamma(n/2)(4\pi)^{n/2}} \left[\int_0^1 (e^{-r|z|^2} - 1)\,\frac{dr}{r} + \int_1^\infty e^{-r|z|^2}\,\frac{dr}{r}\right] \\ &= \frac{1}{\Gamma(n/2)(4\pi)^{n/2}} \left[-\log(|z|^2) + \int_0^\infty e^{-r}\log r\,dr\right].\end{aligned}$$

□

Theorem 6 (Distributional solvability). *Let* $f \in L^\infty(\mathbb{R}^n)$ *with compact support and* $0 < s < \min\{1, n/2\}$. *Define*

$$u(x) = (-\Delta)^{-s} f(x) = \frac{1}{\Gamma(s)} \int_0^\infty e^{t\Delta} f(x)\,\frac{dt}{t^{1-s}}.$$

Then u is given by the pointwise formula (17), $u \in L^\infty(\mathbb{R}^n)$ *with*

$$\|u\|_{L^\infty(\mathbb{R}^n)} \le C_{n,s}\|f\|_{L^\infty(\mathbb{R}^n)}$$

for some constant $C_{n,s} > 0$, *and*

$$|u(x)| \to 0 \quad \text{as } |x| \to \infty.$$

In addition, $(-\Delta)^s u = f$ *in the sense of distributions.*

The proof of Theorem 6 follows the exact same lines as the proof of the one dimensional case presented in [34, Theorem 9.9].

4 Extension problem: semigroup approach, weak formulation

As we mentioned in the Introduction, the extension problem for the fractional Laplacian is a characterization of $(-\Delta)^s$ as the Dirichlet-to-Neumann map for a local degenerate elliptic PDE. This localization technique was introduced and exploited in the PDE context by Caffarelli and Silvestre [24]. The method of semigroups for this problem that the author and Torrea developed in [90, 91] (see also [47]) provided new insights, explicit formulas for the solution, the useful Bessel functions analysis and, ultimately, a unified approach. To present the extension problem, for $0 < s < 1$, we let

$$a = 1 - 2s \in (-1, 1).$$

Theorem 7 (Extension problem for positive powers). *Let* $u \in \mathcal{S}$. *The unique solution* $U = U(x,y) : \mathbb{R}^n \times [0,\infty) \to \mathbb{R}$ *to*

$$\begin{cases} \Delta U + \frac{a}{y}U_y + U_{yy} = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ U(x,0) = u(x) & \text{on } \mathbb{R}^n \end{cases} \tag{19}$$

that weakly vanishes as $y \to \infty$ *is given by the following formulas:*

$$\begin{aligned} U(x,y) &= \frac{y^{2s}}{4^s\Gamma(s)} \int_0^\infty e^{-y^2/(4t)} e^{t\Delta} u(x) \, \frac{dt}{t^{1+s}} \\ &= \frac{1}{\Gamma(s)} \int_0^\infty e^{-t} e^{-\frac{y^2}{4t}\Delta} u(x) \, \frac{dt}{t^{1-s}} \\ &= \frac{1}{\Gamma(s)} \int_0^\infty e^{-y^2/(4t)} e^{t\Delta} \big((-\Delta)^s u\big)(x) \, \frac{dt}{t^{1-s}} \\ &= \frac{\Gamma(n/2+s)}{\Gamma(s)\pi^{n/2}} \int_{\mathbb{R}^n} \frac{y^{2s}}{(y^2 + |x-z|^2)^{(n+2s)/2}} u(z) \, dz. \end{aligned} \tag{20}$$

Moreover, $U \in C^\infty(\mathbb{R}^n \times (0,\infty)) \cap C(\mathbb{R}^n \times [0,\infty))$ *satisfies*

$$-\lim_{y\to 0^+} y^a U_y(x,y) = \frac{\Gamma(1-s)}{4^{s-1/2}\Gamma(s)}(-\Delta)^s u(x) \tag{21}$$

and

$$-\lim_{y\to 0^+} \frac{U(x,y)-U(x,0)}{y^{2s}} = \frac{\Gamma(1-s)}{4^s\Gamma(1+s)}(-\Delta)^s u(x). \tag{22}$$

The first three formulas for U in (20) are due to [90, 91], while the last one was found in [24]. The explicit constants appearing in the limits (21) and (22) were first discovered in [90, 91]. Observe that when $s = 1/2$ the first two formulas in (20) reduce to the so-called *Bochner subordination formula*; see [15, 16], also [57, 88]. Nevertheless, the third formula in (20) is original from [90, 91] even for the case $s = 1/2$.

The idea for solving explicitly the extension problem by using Bessel functions was introduced in [90, 91]. For each $y > 0$, we apply the Fourier transform in the variable x to (19) to get an ODE of the form

$$\begin{cases} f_\xi''(y) + \frac{a}{y} f_\xi'(y) = \lambda f_\xi(y) & \text{for } y > 0 \\ f_\xi(0) = \widehat{u}(\xi) \end{cases}$$

where $\lambda = |\xi|^2$ and $f_\xi(y) = \widehat{U}(\xi, y)$. This is a Bessel differential equation. The condition that U weakly vanishes as $y \to \infty$ translates in the fact that $f_\xi(y) \to 0$ as $y \to \infty$. Therefore, the unique solution is (see [90, 91])

$$\widehat{U}(\xi,y) = \frac{2^{1-s}}{\Gamma(s)}(y|\xi|)^s \mathcal{K}_s(y|\xi|)\widehat{u}(\xi) \tag{23}$$

where $\mathcal{K}_s$ is the modified Bessel function of the second kind or Macdonald's function (see [66]). Notice that $\mathcal{K}_{1/2}(z) = (\frac{\pi}{2z})^{1/2}e^{-z}$ so, when $s = 1/2$, $\widehat{U}(\xi,y) = e^{-y|\xi|}\widehat{u}(\xi)$, which is the classical Poisson semigroup for the harmonic extension of u to the upper half space (3). By inverting the Fourier transform in (23), we obtain the functional calculus identity

$$U(x,y) = \frac{2^{1-s}}{\Gamma(s)}(y(-\Delta)^{1/2})^s \mathcal{K}_s(y(-\Delta)^{1/2})u(x).$$

Let us recall the following integral formula for the Bessel function (see [66]):

$$\mathcal{K}_s(z) = \frac{1}{2}\left(\frac{z}{2}\right)^s \int_0^\infty e^{-t}e^{-z^2/(4t)}\,\frac{dt}{t^{1+s}}.$$

We choose $z = y|\xi|$ and apply the change of variables $y^2/(4t) \to t$ to get

$$\widehat{U}(\xi,y) = \frac{1}{\Gamma(s)}\int_0^\infty e^{-y^2/(4t)}e^{-t|\xi|^2}(|\xi|^{2s}\widehat{u}(\xi))\,\frac{dt}{t^{1-s}}.$$

Because of (8), this is in fact the second to last formula for U in (20).

This Bessel function analysis was applied in the extension problem for other fractional operators in [11, 14, 29, 39, 73, 90].

The second identity in (20) follows from the first one by the change of variables $y^2/(4t) \to t$. The third one is obtained by computing the Fourier transform of U in x in the first formula

$$\widehat{U}(\xi,y) = \frac{y^{2s}}{4^s\Gamma(s)} \int_0^\infty e^{-y^2/(4t)} e^{-t|\xi|^2} \widehat{u}(\xi)\, \frac{dt}{t^{1+s}}$$

and performing the change of variables $y^2/(4t|\xi|^2) \to t$. The last convolution formula in (20), found for the first time in [24], follows immediately from the first one; see [90, 91] for the details.

Now that several identities for U were given, all the properties established in Theorem 7 are easy to verify. For example, using that $\Delta e^{t\Delta}u = \partial_t e^{t\Delta}u$ together with an integration by parts,

$$\begin{aligned}\Delta U(x,y) &= \frac{-y^{2s}}{4^s\Gamma(s)} \int_0^\infty \partial_t\Big(\frac{e^{-y^2/(4t)}}{t^{1+s}}\Big) e^{t\Delta}u(x)\, dt \\ &= -\frac{a}{y} U_y(x,y) - U_{yy}(x,y).\end{aligned}$$

The second identity in (20) immediately gives that $\lim_{y\to 0^+} U(x,y) = u(x)$.

Estimates for U in terms of u are easy to obtain by using the semigroup formulas and the fact that, for any $y > 0$,

$$\frac{y^{2s}}{4^s\Gamma(s)} \int_0^\infty e^{-y^2/(4t)}\, \frac{dt}{t^{1+s}} = 1. \tag{24}$$

For example, it is readily seen that $\|U(\cdot,y)\|_{L^p(\mathbb{R}^n)} \le \|u\|_{L^p(\mathbb{R}^n)}$, for all $y \ge 0$, for any $1 \le p \le \infty$.

For (21), by using (24) and the semigroup formula for the fractional Laplacian (7),

$$\begin{aligned}y^a U_y(x,y) &= \frac{y^{1-2s}}{4^s\Gamma(s)} \int_0^\infty \partial_y\big(y^{2s} e^{-y^2/(4t)}\big)\big(e^{t\Delta}u(x) - u(x)\big)\, \frac{dt}{t^{1+s}} \\ &\longrightarrow \frac{2s}{4^s\Gamma(s)} \int_0^\infty \big(e^{t\Delta}u(x) - u(x)\big)\, \frac{dt}{t^{1+s}} = -\frac{\Gamma(1-s)}{4^{s-1/2}\Gamma(s)} (-\Delta)^s u(x)\end{aligned}$$

as $y \to 0^+$. Similarly, one can check that (22) holds.

At this point, the reader can observe that the semigroup approach gives not only clear proofs, but can also avoid the use of the Fourier transform and the special symmetries of the Laplacian. Indeed, it relies only on heat semigroups and kernels. In addition, as we have already mentioned, the methods have a wide applicability in a variety of different contexts.

The extension problem can be written in an equivalent way as an extension problem for the negative powers of the fractional Laplacian $(-\Delta)^{-s}$. This is an immediate consequence of the third formula for U in (20) and the results of Theorem 7; see [90, 91]. For such explicit statement for negative powers L^{-s} in other contexts like manifolds and discrete settings, see [34, 39, 45, 70].

Theorem 8 (Extension problem for negative powers). *Let $f \in \mathcal{S}$. The unique smooth solution $U = U(x,y) : \mathbb{R}^n \times [0,\infty) \to \mathbb{R}$ to the Neumann extension problem*

$$\begin{cases} \Delta U + \frac{a}{y} U_y + U_{yy} = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ -y^a U_y(x,y)|_{y=0} = f(x) & \text{on } \mathbb{R}^n \end{cases}$$

that weakly vanishes as $y \to \infty$ is given by the formula

$$U(x,y) = \frac{1}{\Gamma(s)} \int_0^\infty e^{-y^2/(4t)} e^{t\Delta} f(x) \, \frac{dt}{t^{1-s}}.$$

Moreover,

$$\lim_{y\to 0^+} U(x,y) = \frac{4^{s-1/2}\Gamma(s)}{\Gamma(1-s)} (-\Delta)^{-s} f(x).$$

We consider next weak solutions to the extension problem. It is easy to check that if $u \in \mathcal{S}$ then

$$[u]^2_{H^s(\mathbb{R}^n)} \equiv \|(-\Delta)^{s/2} u\|^2_{L^2(\mathbb{R}^n)} = \frac{c_{n,s}}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(u(x)-u(z))^2}{|x-z|^{n+2s}} \, dx \, dz.$$

The fractional Sobolev space $H^s(\mathbb{R}^n)$, $0 < s < 1$, is defined as the completion of $C_c^\infty(\mathbb{R}^n)$ under the norm

$$\|u\|^2_{H^s(\mathbb{R}^n)} = \|u\|^2_{L^2(\mathbb{R}^n)} + [u]^2_{H^s(\mathbb{R}^n)}.$$

Then $H^s(\mathbb{R}^n)$ is a Hilbert space with inner product

$$\langle u, v\rangle_{H^s(\mathbb{R}^n)} = \langle u, v\rangle_{L^2(\mathbb{R}^n)} + \frac{c_{n,s}}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(u(x)-u(z))(v(x)-v(z))}{|x-z|^{n+2s}} \, dx \, dz.$$

The dual of $H^s(\mathbb{R}^n)$ is denoted by $H^{-s}(\mathbb{R}^n)$. The definition of the fractional Laplacian can be extended to functions in $H^s(\mathbb{R}^n)$. For any $u \in H^s(\mathbb{R}^n)$, we define $(-\Delta)^s u$ as the element on $H^{-s}(\mathbb{R}^n)$ that acts on $v \in H^s(\mathbb{R}^n)$ via

$$\begin{aligned} \langle (-\Delta)^s u, v\rangle_{H^{-s},H^s} &= \langle (-\Delta)^{s/2} u, (-\Delta)^{s/2} v\rangle_{L^2(\mathbb{R}^n)} \\ &= \frac{c_{n,s}}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(u(x)-u(z))(v(x)-v(z))}{|x-z|^{n+2s}} \, dx \, dz. \end{aligned}$$

The extension problem can be posed in this L^2 setting. Notice that

$$\Delta U + \frac{a}{y}U_y + U_{yy} = y^{-a}\operatorname{div}_{x,y}(y^a\nabla_{x,y}U).$$

The weighted Sobolev space

$$H_a^1 \equiv H^1(\mathbb{R}^n \times (0,\infty), y^a\,dxdy) \quad \text{where } a = 1-2s$$

is defined as the completion of $C_c^\infty(\mathbb{R}^n \times [0,\infty))$ under the norm

$$\|U\|_{H_a^1}^2 = \|U\|_{L^2(\mathbb{R}^n\times(0,\infty),y^a dxdy)}^2 + \|\nabla_{x,y}U\|_{L^2(\mathbb{R}^n\times(0,\infty),y^a dxdy)}^2.$$

Since $a \in (-1,1)$, the weight $\omega(x,y) = y^a$ belongs to the Muckenhoupt class $A_2(\mathbb{R}^n \times (0,\infty))$; see [41] for details about these weights and [44, 97] for weighted Sobolev spaces. Then H_a^1 is a Hilbert space with inner product

$$\langle U, V\rangle_{H_a^1} = \int_0^\infty\int_{\mathbb{R}^n} y^a UV\,dx\,dy + \int_0^\infty\int_{\mathbb{R}^n} y^a\nabla_{x,y}U\cdot\nabla_{x,y}V\,dx\,dy.$$

Given $u \in L^2(\mathbb{R}^n)$, we say that $U \in H_a^1$ is a weak solution to the extension problem (19) if for every $V \in C_c^\infty(\mathbb{R}^n \times (0,\infty))$

$$\int_0^\infty\int_{\mathbb{R}^n} y^a\nabla_{x,y}U\cdot\nabla_{x,y}V\,dx\,dy = 0$$

and $\lim_{y\to0^+} U(x,y) = u(x)$ in $L^2(\mathbb{R}^n)$. The proof of the following extension theorem in weak form is just a simple verification.

Theorem 9. *Let $u \in H^s(\mathbb{R}^n)$. The unique weak solution $U \in H_a^1$ to the extension problem* (19) *is given by* (20). *Moreover, $U(\cdot,y) \in C^\infty(\mathbb{R}^n \times (0,\infty)) \cap C([0,\infty); L^2(\mathbb{R}^n))$ satisfies* (21) *and* (22) *in the sense of $H^{-s}(\mathbb{R}^n)$. In addition, for any $V \in C_c^\infty(\mathbb{R}^n \times [0,\infty))$,*

$$\int_0^\infty\int_{\mathbb{R}^n} y^a\nabla_{x,y}U\cdot\nabla_{x,y}V\,dx\,dy = \frac{\Gamma(1-s)}{4^{s-1/2}\Gamma(s)}\langle(-\Delta)^s u, V(\cdot,0)\rangle_{H^{-s},H^s}$$

and also

$$\mathcal{I}_s(U) \equiv \int_0^\infty\int_{\mathbb{R}^n} y^a|\nabla_{x,y}U|^2\,dx\,dy = \frac{\Gamma(1-s)}{4^{s-1/2}\Gamma(s)}\int_{\mathbb{R}^n}|(-\Delta)^{s/2}u|^2\,dx.$$

Moreover, U is the unique minimizer of the energy functional $\mathcal{I}_s(V)$ among all functions $V \in H_a^1$ such that $\lim_{y\to0^+} V(x,y) = u(x)$ *in $L^2(\mathbb{R}^n)$.*

5 Applications of the extension problem: Harnack inequality and derivative estimates

This section is devoted to show how the extension problem can be used to prove an interior Harnack inequality and derivative estimates for fractional harmonic functions. These original ideas are due to Caffarelli–Silvestre [24] and Caffarelli–Salsa–Silvestre [23]. Obviously, such results for the fractional Laplacian are classical [63]; see also [8, 18, 25, 59, 80, 87] for other formulations, techniques, and nonlocal operators. Harnack inequalities using the extension technique for fractional powers of operators in divergence form were systematically developed in [95]. For applications of the extension method to other fractional operators, see [14, 29, 39, 46, 70, 76, 90–94]. Harnack inequalities for degenerate elliptic equations like the extension equation (19) were first proved by Fabes, Kenig, and Serapioni in [44].

The following reflection lemma will be needed in the proof of Harnack inequality; see [24], also [95]. In what follows, Ω denotes a domain in $\mathbb{R}^n$ that can be unbounded.

Lemma 2. *Fix $Y > 0$. Suppose that a function $U = U(x,y) : \Omega \times (0,Y) \to \mathbb{R}$ satisfies*

$$\operatorname{div}(y^a \nabla_{x,y} U) = 0 \quad \text{in } \Omega \times (0,Y)$$

in the weak sense, with

$$-\lim_{y\to 0^+} y^a U_y(x,y) = 0 \quad \text{on } \Omega.$$

Namely, suppose that U and $\nabla_{x,y}U$ belong to $L^2(\Omega \times (0,Y), y^a dxdy)$ and that for every test function $V \in C_c^\infty(\Omega \times [0,Y))$ we have

$$\int_0^\infty \int_\Omega y^a \nabla_{x,y} U \cdot \nabla_{x,y} V \, dx \, dy = 0$$

and

$$-\lim_{y\to 0^+} \int_\Omega y^a U_y(x,y) V(x,y) \, dx = 0.$$

Then the even reflection of U in the variable y defined as $\widetilde{U}(x,y) = U(x,|y|)$, for $y \in (-Y,Y)$, is a weak solution to

$$\operatorname{div}(|y|^a \nabla_{x,y} \widetilde{U}) = 0 \quad \text{in } \Omega \times (-Y,Y).$$

Theorem 10 (Interior Harnack inequality). *Let Ω' be a domain such that $\Omega' \Subset \Omega$. There exists a constant $c = c(\Omega,\Omega',s) > 0$ such that for any solution $u \in H^s(\mathbb{R}^n)$ to*

$$\begin{cases} (-\Delta)^s u = 0 & \text{in } \Omega \\ u \geq 0 & \text{in } \mathbb{R}^n \end{cases}$$

we have

$$\sup_{\Omega'} u \le c \inf_{\Omega'} u.$$

Moreover, solutions $u \in H^s(\mathbb{R}^n)$ *to* $(-\Delta)^s u = 0$ *in* Ω *are locally bounded and locally* α*-Hölder continuous in* Ω*, for some exponent* $0 < \alpha < 1$ *that depends only on n and s. More precisely, for any compact set* $K \subset \Omega$ *there exists* $C = C(c, K, \Omega) > 0$ *such that*

$$\|u\|_{C^{0,\alpha}(K)} \le C\|u\|_{L^2(\mathbb{R}^n)}.$$

If, in addition, $u \in L^\infty(\mathbb{R}^n)$ *then*

$$[u]_{C^\alpha(K)} \le C\|u\|_{L^\infty(\mathbb{R}^n)}.$$

Sketch of proof. Let U be the extension of u given by Theorem 9. If $u \ge 0$ in $\mathbb{R}^n$ then, by (20), $U \ge 0$. Moreover, U verifies the hypotheses of Lemma 2. Hence the reflection $\widetilde{U}$ is a nonnegative weak solution to

$$\operatorname{div}(|y|^a \nabla_{x,y} \widetilde{U}) = 0 \quad \text{in } \Omega \times (-2, 2).$$

The interior Harnack inequality for divergence form degenerate elliptic equations with A_2 weights applies to $\widetilde{U}$ (see [44]), and hence, to u. The estimates follow from (20) and (24). □

The extension equation in (19) is translation invariant in the variable x. Thus, an argument based on Caffarelli's incremental quotients lemma [22, Lemma 5.6] can be used to prove interior derivative estimates for fractional harmonic functions. Details of this argument are found in [23]. For similar techniques applied to fractional powers of the heat operator, see [93].

Corollary 1 (Interior derivative estimates). *Let* $u \in H^s(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ *be a solution to*

$$(-\Delta)^s u = 0 \quad \text{in } B_2.$$

Then u is smooth in the interior of B_2 *and, for any multi-index* $\gamma \in \mathbb{N}_0^n$ *there is a constant* $C = C(|\gamma|, n, s) > 0$ *such that*

$$\sup_{x \in B_1} |D^\gamma u(x)| \le C\|u\|_{L^\infty(\mathbb{R}^n)}.$$

6 Semigroup characterization of Hölder and Zygmund spaces

We explained in the Introduction that Hölder and Schauder estimates for the fractional Laplacian can be proved in a rather quick and elegant way by means of a characterization of Hölder and Zygmund spaces in terms of heat semigroups. In this manner, one can avoid the use of pointwise formulas or the Schauder estimates for the Laplacian as done by Silvestre in [84, 85]. Moreover, the semigroup method allows us to reach the endpoint case of $\alpha + 2s$ being an integer (where the appropriate regularity spaces turn out to be different than the rather "natural" endpoint Lipschitz or C^k spaces), and also the case of L^∞ right hand side. See [29, 84, 85, 89, 102] for considerations about endpoint spaces.

Let $\alpha > 0$ and take *any* $k \geq \lfloor \alpha/2 \rfloor + 1$. Define

$$\Lambda^\alpha = \Big\{ u \in L^\infty(\mathbb{R}^n) : [u]_{\Lambda^\alpha} = \sup_{x \in \mathbb{R}^n, t>0} \big| t^{k-\alpha/2} \partial_t^k e^{t\Delta} u(x) \big| < \infty \Big\}$$

under the norm $\|u\|_{\Lambda^\alpha} = \|u\|_{L^\infty(\mathbb{R}^n)} + [u]_{\Lambda^\alpha}$. It can be seen that this definition is independent of k and that the norms for different k are all equivalent.

The Zygmund space $\Lambda_* = \Lambda_*^1$ is the set of functions $u \in L^\infty(\mathbb{R}^n)$ such that

$$[u]_{\Lambda_*} = \sup_{x,h \in \mathbb{R}^n} \frac{|u(x+h) + u(x-h) - 2u(x)|}{|h|} < \infty$$

under the norm $\|u\|_{\Lambda_*} = \|u\|_{L^\infty(\mathbb{R}^n)} + [u]_{\Lambda_*}$; see [102]. Note that $C^{0,1}(\mathbb{R}^n) \subsetneq \Lambda_*$ continuously. For any integer $k \geq 2$, we denote

$$\Lambda_*^k = \{ u \in C^{k-1}(\mathbb{R}^n) : D^\gamma u \in \Lambda_* \text{ for all } |\gamma| = k-1 \}$$

with norm $\|u\|_{\Lambda_*^k} = \|u\|_{C^{k-1}(\mathbb{R}^n)} + \max_{|\gamma|=k-1} [D^\gamma u]_{\Lambda_*}$. Then $C^{k-1,1}(\mathbb{R}^n) \subsetneq \Lambda_*^k$.

The spaces Λ^α, given in terms of the rate of growth of the heat semigroup, coincide with the classical Hölder and Zygmund spaces. The following result for $0 < \alpha < 2$ can be found in [96, Theorem 4*] and [21]. For any $\alpha > 0$ and when $\mathbb{R}^n$ is replaced by the torus $\mathbb{T}^n$ see, for example, [76]. In [89, 96], a similar characterization is proved by using Poisson semigroups, and [93] contains the case of parabolic Hölder–Zygmund spaces. See [68] for the regularity spaces associated with the Ornstein–Uhlenbeck operator, and [38] for those related to the fractional powers of the parabolic harmonic oscillator.

Theorem 11. *If $\alpha > 0$, then*

$$\Lambda^\alpha = \begin{cases} C^{\lfloor \alpha \rfloor, \alpha - \lfloor \alpha \rfloor}(\mathbb{R}^n) & \text{if } \alpha \text{ is not an integer} \\ \Lambda_*^k & \text{if } \alpha = k \text{ is an integer} \end{cases}$$

with equivalent norms.

7 Hölder and Schauder estimates with the method of semigroups

This section is devoted to show how the fractional Laplacian interacts with Hölder and Zygmund spaces. For this, we apply the method of semigroups in combination with Theorem 11. We sketch part of the (rather simple) proofs here. In particular, the technique avoids the use of pointwise formulas, or the boundedness of Riesz transforms on Hölder spaces, or the Hölder and Schauder estimates for the Laplacian as in [84, 85]; see also [92]. Similar proofs but in different contexts can be found in [29, 38, 50, 69, 76, 93].

The first result establishes that the fractional Laplacian $(-\Delta)^s$ behaves as an operator of order $2s$ in the scale of Hölder spaces.

Theorem 12 (Hölder estimates). *Let $u \in C^{k,\alpha}(\mathbb{R}^n)$, for $k \geq 0$ and $0 < \alpha \leq 1$.*
(i) *If $0 < 2s < \alpha$, then $(-\Delta)^s u \in C^{k,\alpha-2s}(\mathbb{R}^n)$ and*

$$\|(-\Delta)^s u\|_{C^{k,\alpha-2s}(\mathbb{R}^n)} \leq C\|u\|_{C^{k,\alpha}(\mathbb{R}^n)}.$$

(ii) *If $0 < \alpha < 2s$ and $k \geq 1$, then $(-\Delta)^s u \in C^{k-1,\alpha-2s+1}(\mathbb{R}^n)$ and*

$$\|(-\Delta)^s u\|_{C^{k-1,\alpha-2s+1}(\mathbb{R}^n)} \leq C\|u\|_{C^{k,\alpha}(\mathbb{R}^n)}.$$

The constants $C > 0$ above depend only on n, s, k, and α.

The idea for (i) is as follows. In view of (6), it is enough to prove (i) for $k = 0$. By Theorem 11, we only have to show that $(-\Delta)^s u \in \Lambda^{\alpha-2s}$ for $u \in \Lambda^\alpha$ and $0 < 2s < \alpha \leq 1$. For this, by using (7), one can write

$$t\partial_t e^{t\Delta}[(-\Delta)^s u](x) = \frac{1}{\Gamma(-s)} \int_0^\infty t\partial_t e^{t\Delta}(e^{r\Delta}u(x) - u(x)) \,\frac{dr}{r^{1+s}}.$$

On one hand, since $\{e^{t\Delta}\}_{t\geq 0}$ is a semigroup of operators and $u \in \Lambda^\alpha$,

$$\begin{aligned}
\left|\int_0^t t\partial_t e^{t\Delta}(e^{r\Delta}u(x) - u(x)) \,\frac{dr}{r^{1+s}}\right| &= \left|\int_0^t t\partial_t e^{t\Delta}\left[\int_0^r \partial_\rho e^{\rho\Delta}u(x)\,d\rho\right] \frac{dr}{r^{1+s}}\right| \\
&= t\left|\int_0^t \int_0^r \partial_w^2 e^{w\Delta}u(x)\Big|_{w=t+\rho} \,d\rho\,\frac{dr}{r^{1+s}}\right| \\
&\leq C[u]_{\Lambda^\alpha} t^{(\alpha-2s)/2}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\left|\int_t^\infty t\partial_t e^{t\Delta}(e^{r\Delta}u(x) - u(x)) \,\frac{dr}{r^{1+s}}\right| \\
&\quad\leq t\int_t^\infty \left|\partial_w e^{w\Delta}u(x)\big|_{w=t+r}\right| \frac{dr}{r^{1+s}} + \left|t\partial_t e^{t\Delta}u(x)\right| \int_t^\infty \frac{dr}{r^{1+s}} \\
&\quad\leq C[u]_{\Lambda^\alpha} t^{(\alpha-2s)/2}.
\end{aligned}$$

If $k \geq 1$, then Theorem 11 shows that (ii) is a consequence of the fact that $(-\Delta)^s : \Lambda^{k+\alpha} \to \Lambda^{k+\alpha-2s}$. The latter can be accomplished with parallel arguments to those used for (i).

The semigroup formula from Theorem 6 and the characterization in Theorem 11 permit us to prove the Schauder estimates $(-\Delta)^{-s} : C^{\alpha} \to C^{\alpha+2s}$ in a rather simple way. This same result, only for the case when $\alpha + 2s$ is not an integer, was obtained using pointwise formulas and the Schauder estimates for the Laplacian in [84, 85]. In our case, with the semigroup method we are able to include the case when $\alpha + 2s \in \mathbb{N}$. See [29, 38, 50, 69, 76, 93] for similar proofs in different contexts.

Theorem 13 (Schauder–Zygmund estimates). *Let $f \in C^{0,\alpha}(\mathbb{R}^n)$ with compact support, for some $0 < \alpha \leq 1$, and define u as in Theorem 6.*

(i) *If $\alpha + 2s < 1$, then $u \in C^{0,\alpha+2s}(\mathbb{R}^n)$ and*

$$\|u\|_{C^{0,\alpha+2s}(\mathbb{R}^n)} \leq C\big(\|u\|_{L^\infty(\mathbb{R}^n)} + \|f\|_{C^{0,\alpha}(\mathbb{R}^n)}\big).$$

(ii) *If $1 < \alpha + 2s < 2$, then $u \in C^{1,\alpha+2s-1}(\mathbb{R}^n)$ and*

$$\|u\|_{C^{1,\alpha+2s-1}(\mathbb{R}^n)} \leq C\big(\|u\|_{L^\infty(\mathbb{R}^n)} + \|f\|_{C^{0,\alpha}(\mathbb{R}^n)}\big).$$

(iii) *If $2 < \alpha + 2s < 3$, then $u \in C^{2,\alpha+2s-2}(\mathbb{R}^n)$ and*

$$\|u\|_{C^{2,\alpha+2s-2}(\mathbb{R}^n)} \leq C\big(\|u\|_{L^\infty(\mathbb{R}^n)} + \|f\|_{C^{0,\alpha}(\mathbb{R}^n)}\big).$$

(iv) *If $\alpha + 2s = k$, $k = 1, 2$, then $u \in \Lambda_*^k$ and*

$$\|u\|_{\Lambda_*^k} \leq C\big(\|u\|_{L^\infty(\mathbb{R}^n)} + \|f\|_{C^{0,\alpha}(\mathbb{R}^n)}\big).$$

The constants $C > 0$ above depend only on n, s, and α.

As a direct consequence of the solvability result in Theorem 6, u is the unique bounded classical solution to $(-\Delta)^s u = f$ that vanishes at infinity.

In view of Theorem 11, the statement of Theorem 13 reduces to show that $(-\Delta)^{-s} : \Lambda^{\alpha} \to \Lambda^{\alpha+2s}$, and this is very easy to prove. Indeed, for any $k \geq \lfloor(\alpha + 2s)/2\rfloor + 1$,

$$\begin{aligned} \big|t^k \partial_t^k e^{t\Delta}[(-\Delta)^{-s} f](x)\big| &= C_s t^k \left| \int_0^\infty \partial_w^k e^{w\Delta} f(x)\Big|_{w=t+r} \frac{dr}{r^{1-s}} \right| \\ &\leq C[f]_{\Lambda^\alpha} t^{(\alpha+2s)/2}. \end{aligned}$$

With the semigroup method, we can prove the Schauder estimates in Hölder–Zygmund spaces for the case when the right-hand side is just bounded; see the details in [29].

Theorem 14 (Schauder–Hölder–Zygmund estimates). *Let $f \in L^\infty(\mathbb{R}^n)$ with compact support and define u as in Theorem 6.*

(i) *If* $2s < 1$, *then* $u \in C^{0,2s}(\mathbb{R}^n)$ *and*

$$\|u\|_{C^{0,2s}(\mathbb{R}^n)} \le C\big(\|u\|_{L^\infty(\mathbb{R}^n)} + \|f\|_{L^\infty(\mathbb{R}^n)}\big).$$

(ii) *If* $2s = 1$, *then* $u \in \Lambda_*$ *and*

$$\|u\|_{\Lambda_*} \le C\big(\|u\|_{L^\infty(\mathbb{R}^n)} + \|f\|_{L^\infty(\mathbb{R}^n)}\big).$$

(iii) *If* $2s > 1$, *then* $u \in C^{1,2s-1}(\mathbb{R}^n)$ *and*

$$\|u\|_{C^{1,2s-1}(\mathbb{R}^n)} \le C\big(\|u\|_{L^\infty(\mathbb{R}^n)} + \|f\|_{L^\infty(\mathbb{R}^n)}\big).$$

The constants $C > 0$ *above depend only on n and s.*

As a direct consequence of the solvability result in Theorem 6, *u is the unique bounded solution to* $(-\Delta)^s u(x) = f(x)$, *for a.e.* $x \in \mathbb{R}^n$, *that vanishes at infinity.*

For the next result, we do not assume that the right-hand side has compact support. The idea for the proof (see [84, 85], also [93] for the parabolic case) is to choose $\eta \in C_c^\infty(B_2)$ such that $0 \le \eta \le 1$, $\eta = 1$ in B_1, $|\nabla\eta| \le C$ in $\mathbb{R}^n$, and write $f = \eta f + (1-\eta)f = f_1 + f_2$ and $u = u_1 + u_2$, where u_1 is the solution to $(-\Delta)^s u_1 = f_1$ in $\mathbb{R}^n$. Then, since f_1 has compact support, u_1 is given as in Theorem 6 and, therefore, Theorems 13 and 14 apply to it. On the other hand, $(-\Delta)^s u_2 = 0$ in B_1, so by Corollary 1 we can bound, for any k and α,

$$\|u_2\|_{C^{k,\alpha}(B_{1/2})} \le C\|u - u_1\|_{L^\infty(\mathbb{R}^n)} \le C\big(\|u\|_{L^\infty(\mathbb{R}^n)} + \|f\|_{L^\infty(\mathbb{R}^n)}\big)$$

where $C > 0$ depends only on n, k and α. For a proof of part (a) of the following result in the case when $\alpha + 2s$ is not an integer and by using pointwise formulas, see [84, 85].

Theorem 15 (Schauder–Zygmund estimates). *Let* $u \in L^\infty(\mathbb{R}^n)$.

(a) *Assume that* $(-\Delta)^s u = f \in C^{0,\alpha}(\mathbb{R}^n)$ *for some* $0 < \alpha \le 1$. *Then u satisfies the estimates* (i)–(iv) *from Theorem* 13.

(b) *Assume that* $(-\Delta)^s u = f \in L^\infty(\mathbb{R}^n)$. *Then u satisfies the estimates* (i)–(iii) *from Theorem* 14.

Bibliography

[1] L. Abadias, M. De León-Contreras, and J. L. Torrea, Non-local fractional derivatives. Discrete and continuous, *J. Math. Anal. Appl.*, **449** (2017), 734–755.

[2] M. Allen, A fractional free boundary problem related to a plasma problem, *Commun. Anal. Geom.* (to appear).

[3] D. Applebaum, *Lévy Processes and Stochastic Calculus*, 2nd Ed., Cambridge Studies in Advanced Mathematics, vol. 116, Cambridge University Press, Cambridge, UK, 2009.

[4] I. Athanasopoulos, L. A. Caffarelli, and E. Milakis, On the regularity of the non-dynamic parabolic fractional obstacle problem, *J. Differ. Equ.*, **265** (2018), 2614–2647.

[5] P. Auscher, S. Hofmann, M. Lacey, A. McIntosh, and Ph. Tchamitchian, The solution of the Kato square root problem for second order elliptic operators on R^n, *Ann. Math. (2)*, **156** (2002), 633–654.

[6] A. V. Balakrishnan, Fractional powers of closed operators and the semigroups generated by them, *Pac. J. Math.*, **10** (1960), 419–437.

[7] V. Banica, M. d. M. González, and M. Sáez, Some constructions for the fractional Laplacian on noncompact manifolds, *Rev. Mat. Iberoam.*, **31** (2015), 681–712.

[8] R. F. Bass and D. A. Levin, Harnack inequalities for jump processes, *Potential Anal.*, **17** (2002), 375–388.

[9] H. Berestycki, J.-M. Roquejoffre, and L. Rossi, The influence of a line with fast diffusion on Fisher-KPP propagation, *J. Math. Biol.*, **66** (2013), 743–766.

[10] C. Berg and G. Forst, *Potential Theory on Locally Compact Abelian Groups*, Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 87, Springer-Verlag, New York–Heidelberg, 1975.

[11] A. Bernardis, F. J. Martín-Reyes, P. R. Stinga, and J. L. Torrea, Maximum principles, extension problem and inversion for nonlocal one-sided equations, *J. Differ. Equ.*, **260** (2016), 6333–6362.

[12] J. Bertoin, *Lévy Processes*, Cambridge Tracts in Mathematics, vol. 121, Cambridge Univ. Press, Cambridge, 1996.

[13] J. J. Betancor, A. J. Castro, and P. R. Stinga, The fractional Bessel equation in Hölder spaces, *J. Approx. Theory*, **184** (2014), 55–99.

[14] A. Biswas, M. De León-Contreras, and P. R. Stinga, Harnack inequalities and Hölder estimates for master equations, arXiv:1806.10072 (2018), 25 pages.

[15] S. Bochner, Diffusion equation and stochastic processes, *Proc. Natl. Acad. Sci. USA*, **35** (1949), 368–370.

[16] S. Bochner, *Harmonic Analysis and The Theory of Probability*, University of California Press, Berkeley and Los Angeles, 1955.

[17] K. Bogdan and T. Byczkowsi, Probabilistic proof of boundary Harnack principle for α-harmonic functions, *Potential Anal.*, **11** (1999), 135–156.

[18] K. Bogdan, T. Byczkowski, T. Kulczycki, M. Ryznar, R. Song, and Z. Vondraček, *Potential Analysis of Stable Symmetric Processes and Its Extensions*, Lecture Notes in Mathematics, vol. 1980, Springer-Verlag, Berlin, 2009.

[19] B. Böttcher, R. Schilling, and J. Wang, *Lévy Matters. III. Lévy-type Processes: Construction, Approximation and Sample Path Properties*, Lecture Notes in Mathematics, vol. 2099, Springer, Cham, 2013.

[20] A. Bueno-Orovio, D. Kay, V. Grau, B. Rodriguez, and K. Burrage, Fractional diffusion models of cardiac electrical propagation: role of structural heterogeneity in dispersion of repolarization, *J. R. Soc. Interface*, **11**(97) (2014), 20140352.

[21] P. L. Butzer and H. Berens, *Semi-Groups of Operators and Approximation*, Die Grundlehren der mathematischen Wissenschaften, vol. 145, Springer-Verlag New York Inc., New York, 1967.

[22] L. A. Caffarelli and X. Cabré, *Fully Nonlinear Elliptic Equations*, American Mathematical Society Colloquium Publications, vol. 43, American Mathematical Society, Providence, RI, 1995.

[23] L. A. Caffarelli, S. Salsa, and L. Silvestre, Regularity estimates for the solution and the free boundary of the obstacle problem for the fractional Laplacian, *Invent. Math.*, **171** (2008), 425–461.

[24] L. A. Caffarelli and L. Silvestre, An extension problem related to the fractional Laplacian, *Commun. Partial Differ. Equ.*, **32** (2007), 1245–1260.

[25] L. A. Caffarelli and L. Silvestre, Regularity theory for fully nonlinear integro-differential equations, *Commun. Pure Appl. Math.*, **62** (2009), 597–638.

[26] L. A. Caffarelli and L. Silvestre, Regularity results for nonlocal equations by approximation, *Arch. Ration. Mech. Anal.*, **200** (2011), 59–88.

[27] L. A. Caffarelli and L. Silvestre, The Evans–Krylov theorem for nonlocal fully nonlinear equations, *Ann. Math. (2)*, **174** (2011), 1163–1187.

[28] L. A. Caffarelli and Y. Sire, On some pointwise inequalities involving nonlocal operators, in *Harmonic Analysis, Partial Differential Equations and Applications*, Appl. Numer. Harmon. Anal., pp. 1–18, Birkhäuser/Springer, Cham, 2017.

[29] L. A. Caffarelli and P. R. Stinga, Fractional elliptic equations, Caccioppoli estimates and regularity, *Ann. Inst. Henri Poincaré, Anal. Non Linéaire*, **33** (2016), 767–807.

[30] L. A. Caffarelli and A. Vasseur, Drift diffusion equations with fractional diffusion and the quasi-geostrophic equation, *Ann. Math. (2)*, **171** (2010), 1903–1930.

[31] D. Chamorro and O. Jarrín, Fractional Laplacians, extension problems and Lie groups, *C. R. Math. Acad. Sci. Paris*, **353** (2015), 517–522.

[32] Ó. Ciaurri, T. A. Gillespie, L. Roncal, J. L. Torrea, and J. L. Varona, Harmonic analysis associated with a discrete Laplacian, *J. Anal. Math.*, **132** (2017), 109–131.

[33] Ó. Ciaurri, L. Roncal, and P. R. Stinga, Fractional integrals on compact Riemannian symmetric spaces of rank one, *Adv. Math.*, **235** (2013), 627–647.

[34] Ó. Ciaurri, L. Roncal, P. R. Stinga, J. L. Torrea, and J. L. Varona, Nonlocal discrete diffusion equations and the fractional discrete Laplacian, regularity and applications, *Adv. Math.*, **330** (2018), 688–738.

[35] P. Constantin and M. Ignatova, Critical SQG in bounded domains, *Ann. PDE*, **2** (2016), 8, 42 pages.

[36] R. Cont and P. Tankov, *Financial Modelling with Jump Processes*, Chapman & Hall/CRC Financial Mathematics Series, Chapman & Hall/CRC, Boca Raton, FL, 2004.

[37] P. Courrège, Générateur infinitésimal d'un semi-groupe de convolution sur $\mathbb{R}^n$, et formule de Lévy–Khinchine, *Bull. Sci. Math. (2)*, **88** (1964), 3–30.

[38] M. De León-Contreras and J. L. Torrea, Fractional powers of the parabolic Hermite operator. Regularity properties, arXiv:1708.02788 (2017), 26 pages.

[39] P. L. De Nápoli and P. R. Stinga, Fractional Laplacians on the sphere, the Minakshisundaram zeta function and semigroups, *Contemp. Math.* (to appear).

[40] G. Di Blasio and B. Volzone, Comparison and regularity results for the fractional Laplacian via symmetrization methods, *J. Differ. Equ.*, **253** (2012), 2593–2615.

[41] J. Duoandikoetxea, *Fourier Analysis*, Graduate Studies in Mathematics, vol. 29, American Mathematical Society, Providence, RI, 2001.

[42] G. Duvaut and J.-L. Lions, *Inequalities in Mechanics and Physics*, Grundlehren der Mathematischen Wissenschaften, vol. 219, Springer-Verlag, Berlin–New York, 1976, translated from the French by C. W. John.

[43] S. Eidelman, S. Ivasyshen, and A. N. Kochubei, *Analytic Methods in the Theory of Differential and Pseudo-differential Equations of Parabolic Type*, Operator Theory: Advances and Applications, vol. 152, Birkhäuser Verlag, Basel, 2004.

[44] E. B. Fabes, C. E. Kenig, and R. Serapioni, The local regularity of solutions of degenerate elliptic equations, *Commun. Partial Differ. Equ.*, **7** (1982), 77–116.

[45] F. Feo, P. R. Stinga, and B. Volzone, The fractional nonlocal Ornstein–Uhlenbeck equation, Gaussian symmetrization and regularity, *Discrete Contin. Dyn. Syst.*, **38** (2018), 3269–3298.

[46] F. Ferrari and B. Franchi, Harnack inequality for fractional sub-Laplacians in Carnot groups, *Math. Z.*, **279** (2015), 435–458.

[47] J. E. Galé, P. J. Miana, and P. R. Stinga, Extension problem and fractional operators: semigroups and wave equations, *J. Evol. Equ.*, **13** (2013), 343–368.

[48] N. García Trillos and D. Sanz-Alonso, The Bayesian formulation and well-posedness of fractional elliptic inverse problems, *Inverse Probl.*, **33** (2017), 065006, 23 pages.

[49] A. E. Gatto, C. Segovia, and S. Vági, On fractional differentiation and integration on spaces of homogeneous type, *Rev. Mat. Iberoam.*, **12** (1996), 111–145.

[50] A. E. Gatto and W. Urbina, On Gaussian Lipschitz spaces and the boundedness of fractional integrals and fractional derivatives on them, *Quaest. Math.*, **38** (2015), 1–25.

[51] T. Ghosh, Y.-H. Lin, and J. Xiao, The Calderón problem for variable coefficients nonlocal elliptic operators, *Commun. Partial Differ. Equ.*, **42** (2017), 1923–1961.

[52] G. Grubb, Fractional Laplacians on domains, a development of Hörmander's theory of μ-transmission pseudodifferential operators, *Adv. Math.*, **268** (2015), 478–528.

[53] G. Grubb, Regularity of spectral fractional Dirichlet and Neumann problems, *Math. Nachr.*, **289** (2016), 831–844.

[54] N. Jacob, *Pseudo Differential Operators and Markov Processes. Vol. I. Fourier Analysis and Semigroups*, Imperial College Press, London, 2001.

[55] N. Jacob, *Pseudo Differential Operators and Markov Processes. Vol. II. Generators and Their Potential Theory*, Imperial College Press, London, 2002.

[56] N. Jacob, *Pseudo Differential Operators and Markov Processes. Vol. III. Markov Processes and Applications*, Imperial College Press, London, 2005.

[57] N. Jacob and R. Schilling, Subordination in the sense of Bochner—an approach through pseudo differential operators, *Math. Nachr.*, **178** (1996), 199–231.

[58] N. Jacob and R. Schilling, Lévy-type processes and pseudo differential operators, in *Lévy Processes*, pp. 139–168, Birkhäuser Boston, Boston, MA, 2001.

[59] M. Kassmann, A new formulation of Harnack's inequality for nonlocal operators, *C. R. Math. Acad. Sci. Paris*, **349** (2011), 637–640.

[60] T. Kato, Fractional powers of dissipative operators, *J. Math. Soc. Jpn.*, **13** (1961), 246–274.

[61] M. Kemppainen, P. Sjögren, and J. L. Torrea, Wave extension problem for the fractional Laplacian, *Discrete Contin. Dyn. Syst.*, **35** (2015), 4905–4929.

[62] H. Komatsu, Fractional powers of operators, *Pac. J. Math.*, **19** (1966), 285–346.

[63] N. S. Landkof, *Foundations of Modern Potential Theory*, Die Grundlehren der mathematischen Wissenschaften, vol. 180, Springer-Verlag, New York–Heidelberg, 1972, translated from the Russian by A. P. Doohovskoy.

[64] N. Laskin, Fractional quantum mechanics, *Phys. Rev. E*, **62** (2000), 3135–3145.

[65] N. Laskin, Fractional Schrödinger equation, *Phys. Rev. E*, **66** (2002), 056108, 1–7.

[66] N. N. Lebedev, *Special Functions and Their Applications*, Prentice-Hall, Inc., Englewood Cliffs, N. J., 1965.

[67] J.-L. Lions and E. Magenes, *Problèmes aux Limites Non Homogènes et Applications. Vol. 1*, Travaux et Recherches Mathématiques, vol. 17, Dunod, Paris, 1968.

[68] L. Liu and P. Sjögren, A characterization of the Gaussian Lipschitz space and sharp estimates for the Ornstein–Uhlenbeck Poisson kernel, *Rev. Mat. Iberoam.*, **32** (2016), 1189–1210.

[69] T. Ma, P. R. Stinga, J. L. Torrea, and C. Zhang, Regularity properties of Schrödinger operators, *J. Math. Anal. Appl.*, **388** (2012), 817–837.

[70] D. Maldonado and P. R. Stinga, Harnack inequality for the fractional nonlocal linearized Monge–Ampère equation, *Calc. Var. Partial Differ. Equ.*, **56** (2017), 103, 45 pages.

[71] R. Metzler and J. Klafter, The random walk's guide to anomalous diffusion: a fractional dynamics approach, *Phys. Rep.*, **339** (2000), 1–77.

[72] S. A. Molčanov and E. Ostrovskiĭ, Symmetric stable processes as traces of degenerate diffusion processes, *Teor. Veroâtn. Primen.*, **14** (1969), 127–130 (in Russian).

[73] R. H. Nochetto, E. Otárola, and A. Salgado, A PDE approach to fractional diffusion in general domains: a priori error analysis, *Found. Comput. Math.*, **15** (2015), 733–791.

[74] M. Novaga, D. Pallara, and Y. Sire, A symmetry result for degenerate elliptic equations on the Wiener space with nonlinear boundary conditions and applications, *Discrete Contin. Dyn. Syst., Ser. S*, **9** (2016), 815–831.

[75] N. C. Petroni and M. Pusterla, Lévy processes and Schrödinger equation, *Physica A*, **388** (2009), 824–836.

[76] L. Roncal and P. R. Stinga, Fractional Laplacian on the torus, *Commun. Contemp. Math.*, **18** (2016), 1550033, 26 pages.

[77] S. G. Samko, *Hypersingular Integrals and Their Applications*, Analytical Methods and Special Functions, vol. 5, Taylor & Francis, Ltd., London, 2002.

[78] S. G. Samko, A. A. Kilbas, and O. I. Marichev, *Fractional Integrals and Derivatives: Theory and Applications*, Gordon and Breach Sci. Pub., Yverdon, 1993, translated from the 1987 Russian original.

[79] R. Schilling, Subordination in the sense of Bochner and a related functional calculus, *J. Aust. Math. Soc. A*, **64** (1998), 368–396.

[80] R. Schilling, R. Song, and Z. Vondraček, *Bernstein Functions. Theory and Applications*, 2nd Ed., De Gruyter Studies in Mathematics, vol. 37, Walter de Gruyter & Co., Berlin, 2012.

[81] R. T. Seeley, Complex powers of an elliptic operator, in *Singular Integrals*, pp. 288–307, Amer. Math. Soc., Providence, R. I., 1967.

[82] M. F. Shlesinger, G. M. Zaslavsky, and J. Klafter, Strange kinetics, *Nature*, **363** (1993), 31–37.

[83] A. Signorini, Questioni di elasticità non linearizzata e semilinearizzata, *Rend. Mat. Appl.*, **18** (1959), 95–139.

[84] L. Silvestre, PhD thesis, The University of Texas at Austin, 2005.

[85] L. Silvestre, Regularity of the obstacle problem for a fractional power of the Laplace operator, *Commun. Pure Appl. Math.*, **60** (2007), 67–112.

[86] I. M. Sokolov, J. Klafter, and A. Blumen, Fractional kinetics, *Phys. Today*, (2002), 48–54.

[87] R. Song and Z. Vondraček, Potential theory of subordinate killed Brownian motion in a domain, *Probab. Theory Relat. Fields*, **125** (2003), 578–592.

[88] E. M. Stein, *Topics in Harmonic Analysis Related to the Littlewood–Paley Theory*, Annals of Mathematics Studies, vol. 63, Princeton University Press, Princeton, N. J.; University of Tokyo Press, Tokyo, 1970.

[89] E. M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton Mathematical Series, vol. 30, Princeton Univ. Press, Princeton, NJ, 1970.

[90] P. R. Stinga, PhD thesis, Universidad Autónoma de Madrid, 2010.

[91] P. R. Stinga and J. L. Torrea, Extension problem and Harnack's inequality for some fractional operators, *Commun. Partial Differ. Equ.*, **35** (2010), 2092–2122.

[92] P. R. Stinga and J. L. Torrea, Regularity theory for the fractional harmonic oscillator, *J. Funct. Anal.*, **260** (2011), 3097–3131.

[93] P. R. Stinga and J. L. Torrea, Regularity theory and extension problem for fractional nonlocal parabolic equations and the master equation, *SIAM J. Math. Anal.*, **49** (2017), 3893–3924.

[94] P. R. Stinga and B. Volzone, Fractional semilinear Neumann problems arising from a fractional Keller–Segel model, *Calc. Var. Partial Differ. Equ.*, **54** (2015), 1009–1042.

[95] P. R. Stinga and C. Zhang, Harnack's inequality for fractional nonlocal equations, *Discrete Contin. Dyn. Syst.*, **33** (2013), 3153–3170.

[96] M. H. Taibleson, On the theory of Lipschitz spaces of distributions on Euclidean n-space. I. Principal properties, *J. Math. Mech.*, **13** (1964), 407–479.

[97] B. O. Turesson, *Nonlinear Potential Theory and Weighted Sobolev Spaces*, Lecture Notes in Mathematics, vol. 1736, Springer-Verlag, Berlin, 2000.

[98] R. Yang, On higher order extensions for the fractional Laplacian, arXiv:1302.4413 (2013), 14 pages.
[99] K. Yosida, *Functional Analysis*, Reprint of the sixth (1980) edition, Classics in Mathematics, Springer-Verlag, Berlin, 1995.
[100] H. Yu, Unique continuation for fractional orders of elliptic equations, *Ann. PDE*, **3** (2017), 16, 21 pages.
[101] G. M. Zaslavsky, Chaos, fractional kinetics, and anomalous transport, *Phys. Rep.*, **371** (2002), 461–580.
[102] A. Zygmund, Smooth functions, *Duke Math. J.*, **12** (1945), 47–76.

Victoria P. Knopova, Anatoly N. Kochubei, and Alexei M. Kulik

Parametrix methods for equations with fractional Laplacians

Abstract: We give a survey of parametrix methods and their applications for pseudo-differential parabolic equations with homogeneous symbols and their perturbations.

Keywords: Pseudo-differential operator, Cauchy problem, fundamental solution, parametrix method, Markov process

MSC 2010: 35S10, 35K99, 60J35, 60J75

1 Introduction

A model example of problems considered in this survey is the Cauchy problem

$$\partial_t u(t,x) + (Au)(t,x) = 0, \quad t > 0, x \in \mathbb{R}^n; \quad u(0,x) = u_0(x), \tag{1}$$

where A is a pseudo-differential operator with a symbol $a(\xi)$, homogeneous of a degree $\alpha > 0$, smooth outside the origin $\xi = 0$ and satisfying the parabolicity condition

$$\operatorname{Re} a(\xi) \geq a_0 > 0, \quad \xi \in \mathbb{R}^n, \quad |\xi| = 1.$$

The case $a(\xi) = |\xi|^\alpha$ corresponds to the fractional Laplacian $A = (-\Delta)^{\alpha/2}$.

The operator A is defined initially using the Fourier transform. Then it is rewritten as a hypersingular integral operator, which makes sense for wider classes of functions. The latter definition can be the initial one for more general operators, not necessarily admitting a pseudo-differential representation.

The main task is to construct a fundamental solution (FS) to (1), as well as to perturbations and generalizations of the problem, and to find the upper and lower estimates for the solution. In many interesting cases, this FS is a transition density of a jump-type Markov process, so that such a study has a deep probabilistic meaning. On the other hand, if we use the FS $\Gamma(t,x,y)$ corresponding to the problem (1) or its

Victoria P. Knopova, Technical University of Dresden, 01062 Zellescher Weg 12–14, Dresden, Germany, e-mail: victoria.knopova@tu-dresden.de
Anatoly N. Kochubei, Institute of Mathematics, National Academy of Sciences of Ukraine, Tereshchenkivska 3, Kyiv, 01004, Ukraine, e-mail: kochubei@imath.kiev.ua
Alexei M. Kulik, Institute of Mathematics, National Academy of Sciences of Ukraine, Tereshchenkivska 3, Kyiv, 01004, Ukraine; and Wroclaw University of Science and Technology, Faculty of Pure and Applied Mathematics, Wybrzeże Wyspiańskiego Str. 27, Wroclaw, 50-370, Poland, e-mail: oleksii.kulyk@pwr.edu.pl

https://doi.org/10.1515/9783110571660-013

generalizations, in order to construct the solution

$$u(t,x) = \int_{\mathbb{R}^n} \Gamma(t,x,y)u_0(y)\,dy, \tag{2}$$

it is important to find out, in which sense the function (2) satisfies (1).

In Section 2, we describe the results from [22, 50] where the classical parametrix method was carried out for pseudo-differential parabolic equations with variable quasi-homogeneous symbols. The idea, just as for parabolic differential equations [21, 25, 68], is to study first the case of constant symbols (depending only on the “dual” variable ξ), then the case of “frozen” symbols depending on parameters, and then to obtain FS as a solution of a certain integral equation. Note that the method is based on a series of limit constructions requiring precise estimates at every stage.

In Section 3, we show how the above construction is used to obtain a solution of the Cauchy problem. In particular, the constructed solution appears to be a transition probability density of a conservative Markov process with càdlàg trajectories.

Section 4 is devoted to a modification of the parametrix method in a super-critical regime, that is, if the principal part of the operator is the drift operator. We also discuss the concept of the “approximate FS” and the martingale problem approach in order to show that the constructed solution is unique. The relation to the problem of the uniqueness of a weak solution to the respective SDE is discussed.

Finally, in Section 5 we give an extensive overview of the already existing results on the parametrix method and related topics.

2 Pseudo-differential parabolic equations with quasi-homogeneous symbols

2.1 Main assumptions

Consider the Cauchy problem

$$\partial_t u(t,x) + (Au)(t,x) + \sum_{k=1}^{m}(A_k u)(t,x) = f(t,x), \quad t \in (0,T], x \in \mathbb{R}^n, \tag{3}$$

$$u(0,x) = \varphi(x), \tag{4}$$

where $A, A_1, \dots, A_m$ are pseudo-differential operators with the symbols $a(t,x,\xi)$, $a_1(t,x,\xi), \dots, a_m(t,x,\xi)$, that is,

$$(Au)(t,x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i(x,\xi)} a(t,x,\xi)\widehat{u}(t,\xi)\,d\xi, \tag{5}$$

where

$$\widehat{u}(t,\xi) = \int_{\mathbb{R}^n} e^{-i(y,\xi)} u(t,y)\,dy.$$

The principal symbol $a(t,x,\xi)$ is assumed to be homogeneous of the degree $\alpha \geq 1$ with respect to the variable ξ, and elliptic, the symbols $a_1(t,x,\xi),\dots,a_m(t,x,\xi)$ have the degrees of homogeneity $\alpha_k \in (0,\alpha)$, f and φ are bounded continuous functions. In addition, f is Hölder continuous in x, uniformly with respect to t:

$$\big|f(t,x) - f(t,y)\big| \leq C|x-y|^{\lambda},$$

where C, λ do not depend on t.

We assume the following conditions regarding the symbols of the pseudo-differential operators involved in (3). The principal symbol is such that

(A_1) $\operatorname{Re} a(t,x,\xi) \geq a_0 > 0$, $(t,x) \in \Pi_{(0,T]}$, $|\xi| = 1$;

(A_2) $a(t,x,\xi)$ has N continuous derivatives in ξ for $\xi \neq 0$, and

$$\big|\partial_\xi^{\mathbf{m}} a(t,x,\xi)\big| \leq C_N|\xi|^{\alpha-|\mathbf{m}|},$$
$$\big|\partial_\xi^{\mathbf{m}}[a(t,x,\xi) - a(\tau,y,\xi)]\big| \leq C_N\big(|x-y|^{\lambda} + |t-\tau|^{\lambda/\alpha}\big)|\xi|^{\alpha-|\mathbf{m}|},$$

where $\mathbf{m} \in \mathbb{R}^n$ is a multi-index, $|\mathbf{m}| := m_1 + m_2 + \cdots + m_n$, such that $|\mathbf{m}| \leq N$, $x,y,\xi \in \mathbb{R}^n$ ($\xi \neq 0$), $t,\tau \in [0,T]$. Here, $\lambda \in (0,1)$ is a constant, N is a natural number such that $N \geq 2n + 2[\alpha] + 1$.

Note that for $\lambda' \in (0,\lambda)$ we have $|x-y|^{\lambda'} \wedge 1 \leq |x-y|^{\lambda}$. Changing if necessary the particular value of λ, we furthermore assume that

$$\max_{1\leq k\leq m} \alpha_k < \alpha - \lambda.$$

We assume that the symbols $a_k(t,x,\xi)$ satisfy the condition

(A_3) If $\xi \neq 0$, $|\mathbf{m}| \leq N$, $x,y \in \mathbb{R}^n$, $t,\tau \in [0,T]$, then

$$\big|\partial_\xi^{\mathbf{m}} a_k(t,x,\xi)\big| \leq C|\xi|^{\alpha_k-|\mathbf{m}|},$$
$$\big|\partial_\xi^{\mathbf{m}}[a_k(t,x,\xi) - a_k(\tau,y,\xi)]\big| \leq C\big(|x-y|^{\lambda} + |t-\tau|^{\lambda/\alpha}\big)|\xi|^{\alpha_k-|\mathbf{m}|},$$

where C does not depend on x, y, ξ, t, τ.

Some additional requirements are imposed on the symbols of an integer order. Namely, a symbol of an even order is assumed to be a polynomial in ξ, and a symbol of an odd order is assumed to be either a polynomial or an even function of ξ. Finally, if α is an odd integer, we assume that in the expansion of the function $[a(t,x,\xi)]^{-1}$, $|\xi| = 1$, in spherical harmonics (see below) the coefficients at $Y_{2\nu,\mu}$ with $\alpha = n + 2\nu + 2k$,

$k = 0, 1, 2, \ldots$, are equal to zero. This condition is automatically satisfied if $\alpha < n$, or if n is even.

The expression (5) can be used only for smooth rapidly decreasing functions, and our first task is to reformulate it in terms of hyper-singular integrals.

Our brief exposition of the theory of hyper-singular integrals will be based on properties of spherical harmonics.

2.2 Spherical harmonics

A spherical harmonic of a degree $\nu \geq 0$ is a restriction to the sphere S^{n-1} of a homogeneous harmonic polynomial on $\mathbb{R}^n$ of the degree ν. The set of all spherical harmonics of a degree ν is a finite-dimensional subspace $H_\nu \subset L_2(S^{n-1})$. Let $\delta_\nu = \dim H_\nu$. Then

$$\delta_\nu \leq C\nu^{n-2}. \tag{6}$$

For each ν, we choose in H_ν an orthonormal basis $\{Y_{\nu\mu}\}$, $\mu = 1, \ldots, \delta_\nu$. The functions $Y_{\nu\mu}$ are even for an even ν, and odd otherwise. The system $\{Y_{\nu\mu}\}$ is an orthonormal basis of $L_2(S^{n-1})$. We use the estimates

$$\left|Y_{\nu\mu}(\sigma)\right| \leq C\nu^{(n-2)/2}, \quad \sigma \in S^{n-1}; \tag{7}$$

$$\left|\partial_x^{\mathbf{m}} Y_{\nu\mu}\left(\frac{x}{|x|}\right)\right| \leq C\nu^{|\mathbf{m}|+(n-2)/2}|x|^{-|\mathbf{m}|}, \quad x \in \mathbb{R}^n. \tag{8}$$

If $\varphi \in C^{2k}(S^{n-1})$, $k = 0, 1, \ldots$, then its Fourier coefficients

$$\varphi_{\nu\mu} = \int_{S^{n-1}} \varphi(\sigma)\overline{Y_{\nu\mu}(\sigma)}\, d\sigma$$

satisfy the estimate

$$|\varphi_{\nu\mu}| \leq C_{n,k}\nu^{-2k} \sup_{\sigma\in S^{n-1}, |\mathbf{m}|\leq 2k} \left|D^{\mathbf{m}}\varphi(\sigma)\right|. \tag{9}$$

If $\operatorname{Re}\lambda > -n$, then the function $x \mapsto |x|^\lambda Y_{\nu\mu}(\frac{x}{|x|})$ is locally integrable, and we can define an analytic family of tempered distributions

$$\left\langle |x|^\lambda Y_{\nu\mu}\left(\frac{x}{|x|}\right), \Phi(x)\right\rangle = \int_0^\infty r^{\lambda+n-1}\, dr \int_{S^{n-1}} Y_{\nu\mu}(\sigma)\Phi(r\sigma)\, d\sigma,$$

where $d\sigma$ is the surface Lebesgue measure. In fact, this family admits an analytic continuation to the whole complex plane except the points $\lambda = -\nu - n, -\nu - n - 2, \ldots$. The regularization of the function $r^\lambda Y_{\nu\mu}(\frac{x}{|x|})$, $r = |x|$, with $\operatorname{Re}\lambda \leq -n$ (and λ being outside the exceptional set) as a distribution from $\mathcal{S}'(\mathbb{R}^n)$ is carried out by subtracting a partial sum of the Taylor series of a test function.

The Fourier transform of the above distribution is given by

$$(r^\lambda Y_{\nu\mu})\hat{} = (-i)^\nu \pi^{n/2} 2^{n+\lambda} \frac{\Gamma(\frac{\lambda+\nu+n}{2})}{\Gamma(\frac{\nu-\lambda}{2})} r^{n-\lambda} Y_{\nu\mu} \tag{10}$$

for $\lambda \neq -\nu - n, -\nu - n - 2, \dots$.

The proofs of the above results, as well as additional information about spherical harmonics, can be found in [69, 84, 94].

2.3 Hyper-singular integrals

Consider the hyper-singular integral (HSI)

$$(\mathbf{D}^\alpha_\Omega f)(x) = \frac{1}{d_{n,l}(\alpha)} \int_{\mathbb{R}^n} \Omega\left(x, \frac{h}{|h|}\right) \frac{(\mathbf{\Delta}^l_h f)(x)}{|h|^{n+\alpha}}\, dh, \quad x \in \mathbb{R}^n, \tag{11}$$

$$(\mathbf{\Delta}^l_h f)(x) = \sum_{k=0}^{l} (-1)^k \binom{l}{k} f(x - kh),$$

of order $\alpha > 0$ with a characteristic Ω. We assume that $f(x)$ and $\Omega(x, \sigma)$ are bounded continuous complex-valued functions. Below we give a summary of some results by Samko [93–95] regarding properties of HSIs. Note that in expositions of the HSI theory the characteristic Ω usually does not depend on x; however the appropriate results carry over to our more general situation.

Suppose first that the number α is not an integer. The integral (11) is absolutely convergent if $l > \alpha$ and the function f has bounded derivatives up to the order $[\alpha] + 1$. The absolute convergence over the ball $\{|h| < \varepsilon\}$ is a consequence of the formula

$$(\mathbf{\Delta}^l_h f)(x) = \sum_{|\mathbf{m}|=m} \sum_{k=0}^{l} \frac{(-kh)^{\mathbf{m}}(-1)^k}{\mathbf{m}!} \binom{l}{k} (D^{\mathbf{m}} f)(x - \theta_k kh), \tag{12}$$

where $0 < \theta_k < 1$, $l \geq \mu$, and the usual notation for operations with multi-indices is employed. Formula (12) follows from the Taylor series expansion.

The restricted hyper-singular integral

$$(\mathbf{D}^\alpha_{\Omega,\varepsilon} f)(x) = \frac{1}{d_{n,l}(\alpha)} \int_{|h|>\varepsilon} \Omega\left(x, \frac{h}{|h|}\right) \frac{(\mathbf{\Delta}^l_h f)(x)}{|h|^{n+\alpha}}\, dh, \quad x \in \mathbb{R}^n,$$

is absolutely convergent if, for example, f is bounded. The above convergence conditions can be easily extended to the case when $f(x)$ and its derivatives grow not too rapidly as $|x| \to \infty$. We use also conditionally convergent hyper-singular integrals of bounded functions:

$$(\mathbf{D}^{\alpha}_{\Omega}f)(x) = \lim_{\varepsilon\to 0}(\mathbf{D}^{\alpha}_{\Omega,\varepsilon}f)(x), \tag{13}$$

if the limit in (13) exists for all $x \in \mathbb{R}^n$.

If the characteristic $\Omega(x,\sigma)$ is even in σ, that is, $\Omega(x,-\sigma) = \Omega(x,\sigma)$, $x \in \mathbb{R}^n$, $\sigma \in S^{n-1}$, then the hyper-singular integral (11) makes sense also for $l > 2[\alpha/2]$, by the formula

$$\lim_{\varepsilon\to 0}(\mathbf{D}^{\alpha}_{\Omega,\varepsilon}f)(x) = -\frac{1}{d_{n,l}(\alpha)}\sum_{k=1}^{(l-1)/2}\int_{\mathbb{R}^n}\Omega\Big(x,\frac{h}{|h|}\Big)\frac{(\boldsymbol{\Delta}_h^{l+1}f)(x+kh)}{|h|^{n+\alpha}}\,dh$$
$$-\frac{1}{2d_{n,l}(\alpha)}\int_{\mathbb{R}^n}\Omega\Big(x,\frac{h}{|h|}\Big)\frac{(\boldsymbol{\Delta}_h^{l+1}f)(x+\frac{l+1}{2}h)}{|h|^{n+\alpha}}\,dh, \tag{14}$$

where this time $l+1 > \alpha$.

Let α be an integer. If α is even, then $(\mathbf{D}^{\alpha}_{\Omega}f)$ can be defined as before. In this case the hyper-singular integral operator $\mathbf{D}^{\alpha}_{\Omega}$ is actually a differential operator of the order α. If α is odd, then for $l > \alpha$ the integral in (11) vanishes identically for any function f. In this case, $\mathbf{D}^{\alpha}_{\Omega}$ can be defined only for an even characteristic Ω by the formula (14) with $l = \alpha$.

Let $f \in C^l(\mathbb{R}^n)$, $|f(x)| \le C(1+|x|)^{-N_1}$, $|(D^{\mathbf{m}}f)(x)| \le C(1+|x|)^{-N_2}$, $|\mathbf{m}| = l$, where $N_1 > \alpha$, $N_2 > n$. Then

$$\big|(\mathbf{D}^{\alpha}_{\Omega}f)(x)\big| \le C\big(1+|x|\big)^{-\min(\alpha+N_1,N_2,\alpha+n)}. \tag{15}$$

Moreover, if Ω depends on a parameter t and is uniformly bounded with respect to t, then the constant in (15) does not depend on t. The same estimate, with a constant independent of ε, holds for the restricted hyper-singular integral $\mathbf{D}^{\alpha}_{\Omega,\varepsilon}f$.

Take $f_\xi(x) = e^{i\xi\cdot x}$. Obviously,

$$(\boldsymbol{\Delta}_h^l f_\xi)(x) = e^{i(\xi,x)}\sum_{k=0}^{l}(-1)^k\binom{l}{k}e^{-ik(\xi,h)} = e^{i(\xi,x)}\big(1-e^{-ik(\xi,h)}\big)^l,$$

hence,

$$(\mathbf{D}^{\alpha}_{\Omega}f_\xi)(x) = \widetilde{\Omega}(x,\xi)f_\xi(x), \tag{16}$$

where

$$\widetilde{\Omega}(x,\xi) = \frac{1}{d_{n,l}(\alpha)}\int_{\mathbb{R}^n}\frac{(1-e^{-i(\xi,h)})^l}{|h|^{n+\alpha}}\Omega\Big(x,\frac{h}{|h|}\Big)\,dh.$$

The function $\widetilde{\Omega}$ is called *the symbol* of the hyper-singular integral $\mathbf{D}_\Omega^\alpha f$. The normalization constants $d_{n,l}(\alpha)$ are chosen in such a way that the symbol (hence, the hyper-singular integral) does not depend on l. The explicit form of the constants can be found in [94]; here we only note that they do not depend on Ω, and $d_{n,l}(\alpha) > 0$ if $0 < \alpha < 2$. The symbol $\widetilde{\Omega}(x,\xi)$ is a homogeneous function of the degree α; for example, if $\Omega(x,\sigma) \equiv 1$, then $\widetilde{\Omega}(x,\xi) = |\xi|^\alpha$. The corresponding hyper-singular integral operator is often called the fractional Laplacian or the Riesz derivative. The symbol can be represented as

$$\widetilde{\Omega}(x,\xi) = C(n,\alpha) \int\limits_{S^{n-1}} \Omega(x,\sigma)\big(i(\xi,\sigma)\big)^\alpha \, d\sigma. \tag{17}$$

Note also that the symbol $\widetilde{\Omega}$ is even if and only if the characteristic Ω is even.

It follows from the formula (16) that on functions from the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ (and more generally, on smooth functions, which decrease rapidly enough) the operator $\mathbf{D}_\Omega^\alpha$ coincides with the pseudo-differential operator (ΨDO) with the symbol $\widetilde{\Omega}$. Indeed, if $f \in \mathcal{S}(\mathbb{R}^n)$, then

$$f(x) = (2\pi)^{-n} \int\limits_{\mathbb{R}^n} e^{i(x,\xi)} \hat{f}(\xi)\, d\xi,$$

and by the Fubini theorem

$$(\mathbf{D}_\Omega^\alpha f)(x) = (2\pi)^{-n} \int\limits_{\mathbb{R}^n} e^{i(x,\xi)} \widetilde{\Omega}(x,\xi) \hat{f}(\xi)\, d\xi.$$

Conversely, consider a ΨDO of the form (5), where the symbol $a(t,x,\xi)$ is continuous and homogeneous in ξ of the degree α, and either α is not an integer, or the symbol is even in ξ and α is an odd integer. Suppose that

$$\big|\partial_\xi^{\mathbf{m}} a(t,x,\xi)\big| \le C, \quad |\xi| = 1, x \in \mathbb{R}^n, t \ge 0, |\mathbf{m}| \le N,$$

where $N > 2n + \alpha - 1$. Expanding the symbol into spherical harmonics,

$$a(t,x,\xi) = \sum_{\nu=0}^{\infty} \sum_{\mu=1}^{\delta_\nu} c_{\nu\mu}(t,x) Y_{\nu\mu}(\xi), \quad |\xi| = 1,$$

we set

$$\Omega(t,x,\xi) = \sum_{\nu=0}^{\infty} \sum_{\mu=1}^{\delta_\nu} \frac{c_{\nu\mu}(t,x)}{\lambda(\nu,\alpha)} Y_{\nu\mu}(\xi). \tag{18}$$

It can be proved [22, 50] that the symbol of the HSI $\mathbf{D}_\Omega^\alpha$ coincides with $a(t,x,\xi)$.

2.4 Parametrix

The parametrix of the problem (3)–(4) corresponding to “freezing” the symbol at the point $x = \zeta$, $t = \tau$ is defined by

$$Z_0(t-s, x-y; \zeta, \tau) = (2\pi)^{-n} \int_{\mathbb{R}^n} \exp\{i(x-y)\xi - a(\tau, \zeta, \xi)(t-s)\}\, d\xi, \tag{19}$$

$x, y \in \mathbb{R}^n$, $t > s$, $\tau \geq 0$.

Estimates of Z_0, its differences, and partial derivatives are based on the following estimates of oscillatory integrals [22, 50].

Let $k \geq 0$ and $P_k(\xi)$ be a homogeneous polynomial of the degree k,

$$\Phi_k(z, \tau, \zeta) = \int_{\mathbb{R}^n} P_k(\xi) \exp\{i(z, \xi) - a(\tau, \zeta, \xi)\}\, d\xi.$$

Lemma 2.1. *If $N \geq 2n + k + [\alpha]$, then*

$$\left|\Phi_k(z, \tau, \zeta)\right| \leq C\left(1 + |z|\right)^{-n-\alpha-k},$$

where C does not depend on ζ and τ.

More generally, let $Q_\beta(r, \xi)$ be a homogeneous function in ξ of the degree $\beta > 0$ depending on a parameter r of an arbitrary nature, which has, as $\xi \neq 0$, N continuous derivatives in ξ, and

$$\left|\partial_\xi^{\mathbf{m}} Q_\beta(r, \xi)\right| \leq C B_N(r) |\xi|^{\beta - |\mathbf{m}|},$$

whenever $|\mathbf{m}| \leq N$, $\xi \neq 0$, for all values of r. Let us consider the function

$$\Psi_\beta(z, \tau, \zeta, r) = \int_{\mathbb{R}^n} Q_\beta(r, \xi) \exp\{i(z, \xi) - a(\tau, \zeta, \xi)\}\, d\xi.$$

Lemma 2.2. *If $N \geq 2n + [\alpha] + [\beta] + 1$, then*

$$\left|\Psi_\beta(z, \tau, \zeta, r)\right| \leq C B_N(r)\left(1 + |z|\right)^{-n-\alpha},$$

$$\left|\Psi_\beta(z, \tau, \zeta_1, r) - \Psi_\beta(z, \tau, \zeta_2, r)\right| \leq C B_N(r) |\zeta_1 - \zeta_2|^\lambda \left(1 + |z|\right)^{-n-\min(\alpha, \beta)},$$

where C does not depend on ζ_1, ζ_2, τ, and r.

Another lemma deals with the difference of the function

$$\Phi_0(z, \tau, \zeta) = \int_{\mathbb{R}^n} \exp\{i(z, \xi) - a(\tau, \zeta, \xi)\}\, d\xi.$$

Lemma 2.3. *If $N \geq 2n + [\alpha]$, then for any $z, \zeta_1, \zeta_2 \in \mathbb{R}^n$, $\tau > 0$,*

$$\left|\Phi_0(z, \tau, \zeta_1) - \Phi_0(z, \tau, \zeta_2)\right| \leq C|\zeta_1 - \zeta_2|^\lambda \left(1 + |z|\right)^{-n-\alpha}.$$

The proofs are based on the detailed analysis of various homogeneous functions using the theory of spherical harmonics.

Under the above conditions (A_1), (A_2), we obtain (cf. [50] for details) from Lemmae 2.1–2.3 the following estimates on the parametrized fundamental solution Γ (uniformly with respect to the parameters):

$$\left|\partial_x^{\mathbf{m}} Z_0(t-s, x-y, \tau, \zeta)\right| \le C_{\mathbf{m}}(t-s)\left[(t-s)^{1/\alpha} + |x-y|\right]^{-n-\alpha-|\mathbf{m}|}, \tag{20}$$

where $|\mathbf{m}| \le N - 2n - [\alpha]$,

$$\left|\partial_t Z_0(t-s, x-y, \tau, \zeta)\right| \le C\left[(t-s)^{1/\alpha} + |x-y|\right]^{-n-\alpha}; \tag{21}$$

$$\begin{aligned} &\left|Z_0(t-s, x-y, \tau, \zeta_1) - Z_0(t-s, x-y, \tau, \zeta_2)\right| \\ &\quad \le C(t-s)|\zeta_1 - \zeta_2|^{\lambda}\left[(t-s)^{1/\alpha} + |x-y|\right]^{-n-\alpha}; \end{aligned} \tag{22}$$

$$\begin{aligned} &\left|\partial_t Z_0(t-s, x-y, \tau, \zeta_1) - \partial_t Z_0(t-s, x-y, \tau, \zeta_2)\right| \\ &\quad \le C(t-s)|\zeta_1 - \zeta_2|^{\lambda}\left[(t-s)^{1/\alpha} + |x-y|\right]^{-n-\alpha}. \end{aligned} \tag{23}$$

From (19) and the Fourier inversion, we get

$$\int_{\mathbb{R}^n} Z_0(t-s, x-y, \tau, \zeta)\, dy = 1, \quad z, \zeta \in \mathbb{R}^n,\ t > s,\ \tau \ge 0. \tag{24}$$

Using (21), we derive from (24) that

$$\int_{\mathbb{R}^n} \partial_t Z_0(t-s, x-y, \tau, \zeta)\, dy = 0. \tag{25}$$

The inequality

$$\left|\int_{\mathbb{R}^n} \partial_t Z_0(t-s, x-y, \tau, y)\, dy\right| \le C(t-\tau)^{-1+\lambda/\alpha} \tag{26}$$

follows from (25) and (23).

As an example, consider the case of the fractional Laplacian, that is, where $a(t, x, \xi) = |\xi|^{\alpha}$. In this case, $Z_0(t, x) = t^{-n/\alpha} Z_0(1, t^{-1/\alpha}x)$. Blumenthal and Getoor [6] proved the asymptotic relation

$$\lim_{|x|\to\infty} |x|^{n+\alpha} Z_0(1, x) = \alpha \cdot 2^{\alpha-1}\left(\frac{1}{\pi}\right)^{\frac{n}{2}+1} \sin\frac{\alpha\pi}{2}\Gamma\left(\frac{n+\alpha}{2}\right)\Gamma\left(\frac{\alpha}{2}\right),$$

which shows the precise character of the above general estimates, in particular, the power-like decay of the FS at infinity, in contrast to the exponential decay known for classical parabolic equations.

2.5 Heat potential

Consider the heat potential

$$u(t,x,\tau) = \int_\tau^t ds \int_{\mathbb{R}^n} Z_0(t-s,x-y;s,y)f(s,y)\,dy. \tag{27}$$

We assume the above conditions (A_1), (A_2) with $N \geq 2n + 2[\alpha] + 1$. Suppose also that

$$|f(t,y)| \leq C(t-\tau)^{-\rho}, \quad |f(t,x) - f(t,y)| \leq C|x-y|^\lambda (t-\tau)^{-\rho},$$

where $0 \leq \rho < 1$, $0 < \lambda \leq 1$. The integral in (27) is absolutely convergent due to (20). It also follows from (20) that the function $u(x,t,\tau)$ has continuous derivatives in x of any order less than α, and they can be obtained by differentiating under the sign of integral.

Just as in the case of parabolic partial differential equations, we obtain

$$\begin{aligned}\partial_t u(t,x,\tau) = f(t,x) &+ \int_\tau^t ds \int_{\mathbb{R}^n} \partial_t Z_0(t-s,x-y;s,y)[f(s,y) - f(s,x)]\,dy \\ &+ \int_\tau^t f(s,x)\,ds \int_{\mathbb{R}^n} \partial_t Z_0(t-s,x-y;s,y)\,dy.\end{aligned}$$

The proof is based on the estimates (21), (23), and (26).

Now we consider the action of HSI operators $\mathbf{D}_\Omega^\beta$, $\beta \leq \alpha$, of the form (11) on the potential (27). For brevity, we do not indicate the dependence on the parameter t in the characteristics and symbols of HSIs.

Let us begin with the simplest case $\beta < \alpha$. Take β not an integer, and let $l \geq [\beta] + 1$. It follows from (12) and (20) that for $|h| \leq (t-s)^{1/\alpha}$

$$\begin{aligned}&|(\boldsymbol{\Delta}_h^l Z_0)(t-s,x-y;s,y)| \\ &\quad \leq C|h|^{[\beta]+1}(t-s) \sum_{k=0}^{l} \left[(t-s)^{1/\alpha} + |x - \theta_k kh - y|\right]^{-n-\alpha-[\beta]-1},\end{aligned}$$

where $0 < \theta_k < 1$. By (20), for $|h| \geq (t-s)^{1/\alpha}$

$$|(\boldsymbol{\Delta}_h^l Z_0)(t-s,x-y;s,y)| \leq C(t-s) \sum_{k=0}^{l} \left[(t-s)^{1/\alpha} + |x-y-kh|\right]^{-n-\alpha}.$$

In the case $\beta < \alpha$, we get from these estimates and the Fubini theorem that the HSI $\mathbf{D}_\Omega^\beta u$ converges absolutely, and

$$(\mathbf{D}_\Omega^\beta u)(t,x,\tau) = \int_\tau^t ds \int_{\mathbb{R}^n} Z_\Omega(t-s,x,x-y,s,y)f(s,y)\,dy \tag{28}$$

where $Z_\Omega = \mathbf{D}_\Omega^\beta Z_0$, that is,

$$Z_\Omega(t,x,z,s,y) = (2\pi)^{-n} \int\limits_{\mathbb{R}^n} \widetilde{\Omega}(x,\xi) \exp\{i(z,\xi) - ta(s,y,\xi)\}\, d\xi,$$

with $\widetilde{\Omega}$ being the symbol of the HSI under consideration. Using (14), one can check that (28) remains true also in the case of an odd integer $\beta < \alpha$ and an even characteristic Ω.

The case $\beta = \alpha$ is much more complicated. Assume that the symbol $\widetilde{\Omega}(x,\xi)$ has $N \geq 2n + 2[\alpha] + 1$ continuous derivatives in $\xi \neq 0$, and that

$$\begin{aligned} \left|\partial_\xi^{\mathbf{m}} \widetilde{\Omega}(x,\xi)\right| &\leq C_N |\xi|^{\alpha-|\mathbf{m}|}, \\ \left|\partial_\xi^{\mathbf{m}} [\widetilde{\Omega}(x,\xi) - \widetilde{\Omega}(y,\xi)]\right| &\leq C_N |x-y|^\lambda |\xi|^{\alpha-|\mathbf{m}|}, \quad |\mathbf{m}| \leq N. \end{aligned}$$

Let also $\widetilde{\Omega}(x,\xi) \neq 0$ if $\xi \neq 0$. If α is an integer, and the symbol $\widetilde{\Omega}(x,\xi)$ is not a polynomial in ξ (which is possible only for an odd α and an even characteristic Ω), we assume in addition the expansion

$$\left[\widetilde{\Omega}(x,\sigma)\right]^{-1} = \sum_{\nu=0}^{\infty} \sum_{\mu=1}^{\delta_{2\nu}} c_{2\nu,\mu}(x) Y_{2\nu,\mu}(\sigma),$$

where $c_{2\nu,\mu}(x) = 0$ if $\alpha = n + 2\nu + 2k$, $k = 0,1,2,\ldots$. We do not consider the case when $\widetilde{\Omega}$ is a polynomial in $\sigma = \xi/|\xi|$ because it is covered by the classical theory of parabolic differential equations.

Lemma 2.4. *Under the above conditions, the hyper-singular integral $\mathbf{D}_\Omega^\alpha u$ exists in the sense of the conditional convergence* (13)*, and*

$$\begin{aligned} (\mathbf{D}_\Omega^\alpha u)(t,x,\tau) = &\int\limits_\tau^t ds \int\limits_{\mathbb{R}^n} Z_\Omega(t-s,x,x-y,s,y)[f(s,y) - f(s,x)]\, dy \\ &+ \int\limits_\tau^t f(s,x)\, ds \int\limits_{\mathbb{R}^n} [Z_\Omega(t-s,x,x-y,s,y) - Z_\Omega(t-s,x,x-y,s,x)]\, dy, \end{aligned}$$

where $Z_\Omega = \mathbf{D}_\Omega^\alpha Z_0$, that is,

$$Z_\Omega(t,x,z,\tau,\zeta) = (2\pi)^{-n} \int\limits_{\mathbb{R}^n} \widetilde{\Omega}(x,\xi) \exp\{i(z,\xi) - ta(\tau,\zeta,\xi)\}\, d\xi,$$

and $\widetilde{\Omega}(x,\xi)$ is the symbol of the hyper-singular integral operator $\mathbf{D}_\Omega^\alpha$.

The proof (see [50]) is based on the techniques of the theory of hyper-singular integrals developed by Samko (see [94]) and the theory of homogeneous functions and spherical harmonics [69].

3 Cauchy problem

3.1 Existence of a solution

By a solution to the problem (3), (4) we mean a bounded function $u(t,x)$, jointly continuous on $\Pi_{[0,T]}$, which satisfies the initial condition (4), and solves the equation (3) with A and A_k replaced by the appropriate conditionally convergent hyper-singular integral operators $\mathbf{D}_{\Omega}^{\alpha}$ and $\mathbf{D}_{\Omega_k}^{\alpha_k}$, or by differential operators understood in the classical sense, if corresponding symbols are polynomials in ξ.

Theorem 3.1. *The Cauchy problem* (3), (4) *possesses the solution*

$$u(t,x) = \int_0^t d\tau \int_{\mathbb{R}^n} \Gamma(t,x;\tau,y) f(\tau,y)\,dy + \int_{\mathbb{R}^n} \Gamma(t,x;0,y)\varphi(y)\,dy,$$

where the fundamental solution $\Gamma(t,x;\tau,y)$, $x,y \in \mathbb{R}^n$, $0 \le \tau < t \le T$, *is of the form*

$$\Gamma(t,x;\tau,y) = Z_0(t-\tau, x-y;\tau,y) + W(t,x;\tau,y), \tag{29}$$

$$\begin{aligned} \big|W(t,x;\tau,y)\big| \le C\Big\{ &(t-\tau)^{(\alpha+\lambda)/\alpha}\big[(t-\tau)^{1/\alpha} + |x-y|\big]^{-n-\alpha} \\ &+ (t-\tau)\sum_{k=1}^{n+1}\big[(t-\tau)^{1/\alpha} + |x-y|\big]^{-n-\alpha_k}\Big\}, \quad \alpha_{m+1} = \alpha - \lambda, \end{aligned} \tag{30}$$

and Z_0 *satisfies the estimates* (20)–(23).

The proof (see [22, 50]) is based on the results of Section 2 and contains the justification of the Levi method in the present situation. In the Levi method, the FSCP is assumed to admit the representation (29), where

$$W(t,x;\tau,y) = \int_\tau^t ds \int_{\mathbb{R}^n} Z_0(t-s, x-z; s,z)\Phi(s,z;\tau,y)\,dz,$$

and Φ is determined from the integral equation

$$\Phi(t,x;\tau,y) = \Delta(t,x;\tau,y) + \int_\tau^t ds \int_{\mathbb{R}^n} \Delta(t,x;s,z)\Phi(s,z;\tau,y)\,dz,$$

in which

$$\begin{aligned} \Delta(t,x;\tau,y) = (2\pi)^{-n} \int_{\mathbb{R}^n} &\Big[a(\tau,y,\xi) - a(t,x,\xi) - \sum_{k=1}^{m} a_k(t,x,\xi)\Big] \\ &\cdot \exp\big\{i(x-y,\xi) - a(\tau,y,\xi)(t-\tau)\big\}\,d\xi. \end{aligned}$$

For an extension of these results to the systems of equations, see [19, 73].

3.2 Nonnegative solutions

Consider a subclass of equations of the form (3) whose solutions have properties similar to those of the solutions to the second order parabolic differential equations.

Consider the equation (3) with $f(t,x) = 0$ and $1 \le \alpha \le 2$. Assume the conditions of Section 2.1. In particular, if $\alpha = 2$, then

$$-A = \sum_{j,l=1}^{n} a_{jl}(t,x)\partial_{x_j}\partial_{x_l}$$

is a second-order elliptic differential operator whose coefficients are assumed to be real-valued. Representing all the ΨDOs as HSIs, we write (3) in a form, which encompasses both the case $\alpha = 2$ and the case $\alpha < 2$. Namely, we define

$$\begin{aligned} Lu(t,x) = {} & 1_{\{2\}}(\alpha) \sum_{j,l=1}^{n} a_{jl}(t,x)\partial_{x_j}\partial_{x_l} u(t,x) \\ & - 1_{(0,2)}(\alpha)\frac{1}{d_{n,l}(\alpha)} \int_{\mathbb{R}^n} \Omega\left(t,x,\frac{h}{|h|}\right)\frac{(\Delta_h^1 u)(t,x)}{|h|^{n+\alpha}}\,dh \\ & - \sum_{k=1}^{m} \frac{1}{d_{n,l}(\alpha_k)} \int_{\mathbb{R}^n} \Omega_k\left(t,x,\frac{h}{|h|}\right)\frac{(\Delta_h^1 u)(t,x)}{|h|^{n+\alpha_k}}\,dh + \sum_{j=1}^{n} b_j(t,x)\partial_{x_j} u(t,x), \end{aligned} \tag{31}$$

where the coefficients $b_j(t,x)$ are real-valued, $1_A(\alpha) = 1$ when $\alpha \in A$ and 0 otherwise, and the last sum is absent if $\alpha = 1$.

We also assume the following additional condition.

(A_4) The characteristics Ω and Ω_k are even (except, possibly, those Ω_k, for which $\alpha_k < 1$) and nonnegative.

Let $u(t,x)$ be a solution of the equation

$$\partial_t u(t,x) - Lu(t,x) = 0, \tag{32}$$

jointly continuous on $[\tau,T] \times \mathbb{R}^n$ and satisfying the initial condition

$$u(\tau,x) = \varphi(x), \tag{33}$$

where φ is a continuous bounded function. As before, HSIs are understood in the sense of the conditional convergence (13). The function $u(t,x)$ is assumed to be continuously differentiable with respect to x (if $\alpha > 1$) and t for $t > \tau$, and if $\alpha = 2$, also twice continuously differentiable with respect to x.

Lemma 3.1. *If $\varphi(x) \ge 0$ for all $x \in \mathbb{R}^n$ and $u(t,x) \to 0$ as $|x| \to \infty$, $t > \tau$, then $u(t,x) \ge 0$ for all $x \in \mathbb{R}^n$ and $t \in [\tau,T]$.*

Let $\Gamma(t,x;\tau,y)$ be a FS to the equation (31), and let

$$u(t,x,\tau) = \int_{\mathbb{R}^n} \Gamma(t,x;\tau,y)\varphi(y)\,dy. \tag{34}$$

Fix τ and denote $u(t,x) := u(t,x,\tau)$. Assume that the function φ has a compact support. It follows from (20) and (30) that $u(t,x) \to 0$ as $|x| \to \infty$, $t > \tau$. The function $u(t,x)$ is a solution to the Cauchy problem (31)–(33). The next result follows now from Lemma 3.1.

Theorem 3.2. *If the condition* (A_4) *is satisfied, then* $\Gamma(t,x;\tau,y) \geq 0$ *for all* $x,y \in \mathbb{R}^n$, $t > \tau$.

As a corollary to Lemma 3.1, we also have the following fact.

Lemma 3.2. *If the condition* (A_4) *is satisfied, then the problem* (31)–(33) *has at most one solution tending to zero as* $|x| \to \infty$.

It is deduced in the standard way from Lemma 3.2 and the estimates (20) and (30), that for $\tau < s < t \leq T$

$$\Gamma(t,x;\tau,y) = \int_{\mathbb{R}^n} \Gamma(t,x;s,z)\Gamma(s,z;\tau,y)\,dz. \tag{35}$$

Lemma 3.3. *If the condition* (A_4) *is satisfied, then for all* $x \in \mathbb{R}^n$, $t > \tau$

$$\int_{\mathbb{R}^n} \Gamma(t,x;\tau,y)\,dy = 1. \tag{36}$$

Let B be a compact subset of $\mathbb{R}^n$.

Lemma 3.4. *The following relations hold true:*

(1)

$$\lim_{R\to\infty} \sup_{|x|\geq R, 0\leq\tau<t\leq T} \int_B \Gamma(t,x;\tau,y)\,dy = 0.$$

(2) *For any* $\varepsilon > 0$

$$\lim_{\delta\to 0} \sup_{x\in B, 0\leq\tau<t<t+\delta} \int_{|y-x|>\varepsilon} \Gamma(t,x;\tau,y)\,dy = 0.$$

The proof follows easily from (20) and (30).

Suppose now that all the conditions of Sections 3.1 and 3.2 are satisfied for all $T > 0$. Using Theorems 3.1 and 3.2, the identity (35), Lemmas 3.1–3.3, and general theorems about Markov processes [20], we arrive at the following result.

Theorem 3.3. *The FS* $\Gamma(t,x;\tau,y)$ *is a transition density of a conservative Markov process with càdlàg trajectories.*

The uniqueness result of Lemma 3.2 is not quite satisfactory: in the classical theory of parabolic differential equations, the uniqueness is proved in the classes of growing functions. Results of this kind (where some power-like growth is permitted) are known for the equation $\partial_t u + (-\Delta)^{\alpha/2} u = 0$; see [2, 10].

4 Modified parametrix method in the super-critical regime

4.1 Motivation and outline

Consider for $\alpha < 2$ a particular, but important case of the operator (31):

$$Lu(t,x) = a(x) \int_{\mathbb{R}^n} \frac{u(t,x+h) - u(t,x)}{|h|^{n+\alpha}}\, dh + \sum_{j=1}^{n} b_j(x) \partial_{x_j} u(t,x), \tag{37}$$

where $a(\cdot)$, $b_j(\cdot)$, $j = 1, \dots, n$ are real-valued functions. Assume the following:

(B_1) the functions $a : \mathbb{R}^n \to \mathbb{R}$ and $b = (b_j)_{j=1}^n : \mathbb{R}^n \to \mathbb{R}^n$ are bounded;

(B_2) the functions a, b are Hölder continuous with the indices $\eta > 0$ and $\gamma > 0$, respectively; that is,

$$\big|a(x) - a(y)\big| \le C|x-y|^{\eta}, \quad \big|b(x) - b(y)\big| \le C|x-y|^{\gamma}, \quad x, y \in \mathbb{R}^n;$$

(B_3) there exists $c > 0$ such that

$$a(x) \ge c, \quad x \in \mathbb{R}^n.$$

It is easy to see that the operator (37) has the form (31) with $\Omega(t,x,h/|h|) = a(x)$ and $\Omega_k \equiv 0$, $k \ge 1$. If, in addition, $\alpha > 1$, then (B_1)–(B_3) provide (A_1)–(A_4), and Theorem 3.3 holds true. The Markov process given by this theorem has a natural interpretation (see more details in Section 4.4 below) as a solution to the Stochastic Differential Equation (SDE)

$$dX_t = b(X_t)\, dt + \sigma(X_{t-}) dZ_t^{(\alpha)}, \tag{38}$$

where $\sigma(x) = a(x)^{1/\alpha}$, and $Z^{(\alpha)}$ is an α-stable process with the characteristic function

$$Ee^{i\xi \cdot Z_t^{(\alpha)}} = \exp\bigg(t \int_{\mathbb{R}^n} \frac{e^{i\xi\cdot h} - 1}{|h|^{n+\alpha}}\, dh \bigg) = \exp(-t d_{n,1}(\alpha)|\xi|^{\alpha}), \quad \xi \in \mathbb{R}^n.$$

Stochastic equations driven by a stable noise naturally extend the class of stochastic equations driven by the Brownian noise. Theorem 3.3 gives an efficient tool for a detailed study of such equations, interpreting the transition probability density of the Markov process X as a FS to the Cauchy problem for the parabolic operator $\partial_t - L$, which by Theorem 3.1 admits further detailed analysis. Substantial obstacle in this line of arguments is the assumption $\alpha > 1$ made in previous sections. From the analytical point of view, this assumption is a domination-type condition on the stochastic part of the operator L (which now is a ΨDO of the order α) to dominate the gradient part. This essentially means that in the *subcritical regime* $\alpha > 1$ the Levi method is naturally related to the classical perturbation theory [41].

In the context of the SDE (38), assumption $\alpha > 1$ does neither have heuristic nor intuitive background, hence a natural question is how (38) and the associated Cauchy problem can be analyzed in the *critical* and *super-critical regimes* $\alpha = 1$ and $\alpha < 1$, respectively. In this section, we give a modification of the Levi method, which is applicable beyond the scopes of the standard perturbation theory. The exposition below is mainly based on [49, 62].

4.2 Modified parametrix method in the super-critical regime

In what follows, we denote by $C_\infty(\mathbb{R}^n)$ the space of continuous functions vanishing at ∞, and by $C^2_\infty(\mathbb{R}^n)$ the class of twice continuously differentiable functions vanishing at ∞ together with their derivatives up to order 2. Denote by L the operator on $C_\infty(\mathbb{R}^n)$ with the domain $D(L) = C^2_\infty(\mathbb{R}^n)$, defined by

$$L\phi(x) = a(x) \int_{\mathbb{R}^n} \frac{\phi(x+h) - \phi(x)}{|h|^{n+\alpha}}\, dh + \sum_{j=1}^n b_j(x) \partial_{x_j} \phi(x). \tag{39}$$

The transition density of the Markov process solution to the SDE (38) will be obtained in the form

$$\Gamma(t,x,y) = \Gamma_0(t,x,y) + W(t,x,y), \tag{40}$$

where $\Gamma_0(t,x,y)$ is a properly chosen zero-order approximation, and the residual term is defined by the integral equation

$$W(t,x,y) = \int_0^t \int_{\mathbb{R}^n} (\Gamma_0(t-s,x,z) + W(t-s,x,z))\Delta(s,z,y)\, dzds, \tag{41}$$

with

$$\Delta(t,x,y) = (L_x - \partial_t)\Gamma_0(t,x,y),$$

here and below L_x means that the operator L is applied with respect to the variable x. The structure of this representation is identical to that given in Section 3.1; that is, this is essentially the Levi method, with the principal part $\Gamma_0(t,x,y)$ in the representation (40) yet to be specified. A slight difference in the notation appears because now we restrict ourselves to the time-homogeneous case.

The crucial point in the entire construction is the choice of the principal part $\Gamma_0(t,x,y)$: since $\Delta(t,x,y)$ is determined by $\Gamma_0(t,x,y)$, a nonproper choice of the zero order term may cause that the kernel $\Delta(t,x,y)$ is not integrable, and thus the integral equation (41) is nonsolvable. Below we describe one of the possible choices of the zero order term $\Gamma_0(t,x,y)$ which leads to a version of the Levi method, well applicable for all values $\alpha \in (0,2)$.

In what follows, we assume that the Hölder index γ for the drift coefficient and the stability index α for the noise satisfy the following *balance condition*:

$$\alpha + \gamma > 1. \tag{42}$$

This condition is close to the necessary one for the process X to be properly defined by (38); see [100] for the example of an SDE (38) where this condition fails and the minimal and the maximal (weak) solutions to (38) are different. Define the following "mollified" version of the drift coefficient b:

$$b_t(x) = (2\pi)^{-n/2} t^{-n/\alpha} \int_{\mathbb{R}^n} b(z) e^{-|z-x|^2/2t^{2/\alpha}}\, dz, \quad t > 0,$$

and consider the Cauchy problem

$$d\kappa_t = -b_t(\kappa_t)dt, \quad \kappa_0 = y \in \mathbb{R}^n. \tag{43}$$

The function b_t is Lipschitz continuous with the constant $L(t) \le Ct^{\gamma/\alpha - 1/\alpha}$, hence the balance condition (42) yields that the function $L(\cdot)$ is locally integrable. Thus the problem (43) has the unique solution $\kappa_t = \kappa_t(y)$.

Let

$$Z_0(t, x-y; z) = (2\pi)^{-n} \int_{\mathbb{R}^n} \exp\big(i(x-y)\xi - t d_{n,1}(\alpha) a(z) |\xi|^\alpha\big)\, d\xi,$$

which is just the function defined in (19). Take the zero order term in (40) equal to

$$\Gamma_0(t,x,y) = Z_0\big(t, x - \kappa_t(y); y\big). \tag{44}$$

Then

$$\Delta(t,x,y) = L_x \Gamma_0(t,x,y) + \partial_t \kappa_t(y) \nabla_x \Gamma_0(t,x,y) - a(y) L_x^{(\alpha)} \Gamma_0(t,x,y),$$

where $L_x^{(\alpha)}$ denotes the operator

$$L^{(\alpha)} \phi(x) = \int_{\mathbb{R}^n} \frac{\phi(x+h) - \phi(x)}{|h|^{n+\alpha}}\, dh$$

applied with respect to the variable x. That is,

$$\Delta(t,x,y) = \big(a(x) - a(y)\big)L_x\Gamma_0(t,x,y) + \big(b(x) - b_t(y)\big)\cdot\nabla_x\Gamma_0(t,x,y). \tag{45}$$

By the choice of the mollified drift coefficient b_t, we have

$$\big|b_t(y) - b(y)\big| \le Ct^{\gamma/\alpha},$$

which combined with the estimates from Lemmae 2.1 and 2.2 provide the following: for any positive $\chi < \alpha \wedge \eta$ and any T, there exists a constant $C > 0$ such that for $t \le T$,

$$\big|\Delta(t,x,y)\big| \le Ct^{-1-n/\alpha}\big(t^\delta + \big|x - \kappa_t(y)\big|^\chi \wedge 1\big)\bigg(1 + \frac{|x-\kappa_t(y)|}{t^{1/\alpha}}\bigg)^{-n-\alpha}, \tag{46}$$

where $\delta = \gamma/\alpha - 1/\alpha + 1$. Note that the balance condition (42) means exactly that $\delta > 0$. Then the following sequence of kernels is well-defined iteratively:

$$\Delta^{\circledast 1}(t,x,y) = \Delta(t,x,y),$$

$$\Delta^{\circledast(k+1)}(t,x,y) = \int_0^t\int_{\mathbb{R}^n}\Delta^{\circledast k}(t-s,x,z)\Delta(s,z,y)\,dzds, \quad k \ge 1.$$

In addition, the following bound holds [62]: there exist C_1, C_2 such that for $t \le T$,

$$\begin{aligned}\big|\Delta^{\circledast k}(t,x,y)\big| \le{}& \frac{C_1C_2^k}{\Gamma(k\zeta)}t^{-1-n/\alpha+(k-1)\zeta}\\ &\times\big(t^\delta + \big|x-\kappa_t(y)\big|^\chi \wedge 1\big)\bigg(1 + \frac{|x-\kappa_t(y)|}{t^{1/\alpha}}\bigg)^{-n-\alpha},\end{aligned} \tag{47}$$

where

$$\zeta = \min\Big\{\delta, \chi, \frac{\chi}{\alpha}\Big\} > 0.$$

This yields that the kernel

$$\Phi(t,x,y) = \sum_{k=1}^\infty \Delta^{\circledast k}(t,x,y)$$

is well-defined and the solution to (41) can be given in the form

$$W(t,x,y) = \int_0^t\int_{\mathbb{R}^n}\Gamma_0(t-s,x,z)\Phi(s,z,y)\,dzds. \tag{48}$$

In addition, the kernel $\Phi(t,x,y)$ satisfies a bound similar to (46):

$$\big|\Phi(t,x,y)\big| \le Ct^{-1-n/\alpha}\big(t^\delta + \big|x-\kappa_t(y)\big|^\chi \wedge 1\big)\bigg(1 + \frac{|x-\kappa_t(y)|}{t^{1/\alpha}}\bigg)^{-n-\alpha}, \tag{49}$$

and $\Gamma_0(t,x,y)$, which is evidently non-negative, satisfies

$$\Gamma_0(t,x,y) \le Ct^{-n/\alpha}\left(1 + \frac{|x-\kappa_t(y)|}{t^{1/\alpha}}\right)^{-n-\alpha}. \tag{50}$$

The solution to (41) satisfies

$$\left|W(t,x,y)\right| \le Ct^{-n/\alpha}\left(t^\delta + \left|x-\kappa_t(y)\right|^\chi \wedge 1\right)\left(1 + \frac{|x-\kappa_t(y)|}{t^{1/\alpha}}\right)^{-n-\alpha}. \tag{51}$$

4.3 Approximate FS and the semigroup properties

It is an important problem to find out, in which sense the function $\Gamma(t,x,y)$ in (2), given by (40) and (48), solves the Cauchy problem for the operator $\partial_t - L$. In the sub-critical regime $\alpha > 1$, the results of Section 3.1 remain true, and $\Gamma(t,x,y)$ is known to be a FS in the classical sense; the changes caused by the current choice of $\Gamma_0(t,x,y)$ are not relevant. In the super-critical regime the situation changes drastically, and a one-to-one analogue of the above result can hardly be given. The crucial difficulty here is that, in order to justify that $W(t,x,y)$ is differentiable with respect to x, one can not use the Fubini-based arguments from Section 2.5 if $\alpha < 1$; that is, if the gradient part of the operator is not dominated by the stochastic part. In this section, we explain one natural way to avoid this analytic complication, which is technically quite simple, but on the other hand is completely sufficient for the further analysis of the probabilistic structure of the model.

Define the family of kernels

$$\Gamma^\varepsilon(t,x,y) = \Gamma_0(t+\varepsilon,x,y) + \int_0^t \int_{\mathbb{R}^n} \Gamma_0(t-s+\varepsilon,x,z)\Phi(s,z,y)\,dzds, \quad \varepsilon > 0, \tag{52}$$

and the corresponding family of operators in $C_\infty(\mathbb{R}^n)$

$$P_t^\varepsilon\varphi(x) = \int_{\mathbb{R}^n} \Gamma^\varepsilon(t,x,y)\varphi(y)\,dy, \quad \varphi \in C_\infty(\mathbb{R}^n), \quad t \ge 0, \quad \varepsilon > 0.$$

Define also for $\varphi \in C_\infty(\mathbb{R}^n)$

$$P_t\varphi(x) = \int_{\mathbb{R}^n} \Gamma(t,x,y)\varphi(y)\,dy, \quad t > 0, \quad P_0\phi(x) = \phi(x).$$

By (49) and (50), for a given $\varphi \in C_\infty(\mathbb{R}^n)$ and any $T > 0$,

$$P_t^\varepsilon\varphi(x) \to P_t\varphi(x), \quad \varepsilon \to 0+, \tag{53}$$

uniformly in $x \in \mathbb{R}^n$, $t \in [0,T]$.

The main obstacle which does not allow one to prove the differentiability of $W(t,x,y)$ with respect to x, is that for $\alpha < 1$ the kernel $\nabla_x\Gamma_0(t,x,y)$ is strongly singular in the neighbourhood of $t = 0$. Due to the shift by $\varepsilon > 0$ of the time variable in (52), for any $\varepsilon > 0$ such a difficulty does not occur. Namely, using the same Fubini-based argument as in Section 2.5, one can easily show that for $\phi \in \mathbb{R}^n$ and $\varepsilon > 0$, the function $P_t^\varepsilon\phi(x)$ belongs to C^1 in t, $C^2_\infty(\mathbb{R}^n)$ in x, and satisfies

$$L_xP_t^\varepsilon\phi(x) = \int_{\mathbb{R}^n} L_x\Gamma_0(t+\varepsilon,x,y)\phi(y)\,dy + \int_0^t\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} L_x\Gamma_0(t-s+\varepsilon,x,z)\Phi(s,z,y)\phi(y)\,dydzds. \tag{54}$$

The following statement is the key in the argument.

Lemma 4.1. *Let $\varphi \in C_\infty(\mathbb{R}^n)$ and $T > 0$, $\tau \in (0,T]$ be given. Then*

$$(\partial_t - L_x)P_t^\varepsilon\phi(x) \to 0, \quad \varepsilon \to 0+, \tag{55}$$

uniformly in $t \in [\tau,T]$, $x \in \mathbb{R}^n$.

The proof of this lemma relies only on the integral equation (41) and does not require complicated analytical results such as Lemma 2.4; see [49, Lemma 4.2].

The natural interpretation of (53), (55) is that $\Gamma(t,x,y)$ provides an *approximate* FS for the operator $\partial_t - L$. Namely, given $\phi \in C_\infty(\mathbb{R}^n)$, we have constructed a family of functions $u^\varepsilon(t,x), \varepsilon > 0$, such that this family approximates

$$u(t,x) = \int_{\mathbb{R}^n} \Gamma(t,x,y)\varphi(y)\,dy \tag{56}$$

as $\varepsilon \to 0$, and satisfies $(\partial_t - L_x)u^\varepsilon(t,x) \to 0$. Such a nonclassical notion of the FS appears however to be strong enough in order to be applicable for the classical Positive Maximum Principle (PMP). Namely, the following general statement holds true.

Lemma 4.2. *Suppose that an operator L on $C_\infty(\mathbb{R}^n)$ satisfies the PMP; that is, for any $\phi \in D(L)$, if x_0 is such that*

$$\phi(x_0) = \max_x \phi(x) > 0,$$

then

$$L\phi(x_0) \le 0.$$

Let $u(t,x)$, $t \ge 0$, $x \in \mathbb{R}^n$, be a continuous function and suppose that there exist a family of functions $u^\varepsilon(t,x), \varepsilon > 0$ such that:

(i) *for any* $T > 0$, $\varepsilon_0 > 0$

$$u^{\varepsilon}(t,x) \to 0, \quad |x| \to \infty,$$

uniformly in $\varepsilon < \varepsilon_0$, $t \in [0,T]$;

(ii) *for any* $T > 0$,

$$u^{\varepsilon}(t,x) \to u(t,x), \quad \varepsilon \to 0,$$

uniformly in $x \in \mathbb{R}^n$, $t \in [0,T]$;

(iii) *for any* $T > 0$, $\tau \in (0,T]$,

$$(\partial_t - L_x)u^{\varepsilon}(t,x) \to u(t,x), \quad \varepsilon \to 0,$$

uniformly in $x \in \mathbb{R}^n$, $t \in [\tau,T]$.

If in addition $u(0,x) \geq 0$, *then* $u(x,t) \geq 0$, $t \geq 0$.

A direct calculation shows that the operator (39) satisfies the PMP. Applying Lemma 4.2 to the function (56), we obtain the following corollary, which is a one-to-one analogue to Lemma 3.2.

Corollary 4.1. *Let* $\phi \in C_\infty(\mathbb{R}^n)$ *satisfy* $\phi(x) \geq 0$, $x \in \mathbb{R}^n$. *Then*

$$u(t,x) = P_t\phi(x) \geq 0, \quad t \geq 0, \quad x \in \mathbb{R}^n.$$

From Lemma 4.2, one can also derive in a standard way the following corollary.

Corollary 4.2. *For an arbitrary* $\phi \in C_\infty(\mathbb{R}^n)$,

$$P_{t+s}\phi(x) = P_tP_s\phi(x), \quad t,s \geq 0.$$

If $\phi \in C^2_\infty(\mathbb{R}^n)$, *then*

$$P_t\phi(x) = \phi(x) + \int_0^t P_sL\phi(x)\,ds.$$

The above construction and estimates are summarised in the following theorem.

Theorem 4.1. *Suppose that the assumptions* (B_1)–(B_3) *are satisfied and* $\Gamma_0(t,x,y)$ *is defined by* (44). *Then the following statements hold true.*

I. *The integral equation* (41) *possesses the unique solution* (48). *The function* $\Gamma(t,x,y)$, *defined by* (40), *is the kernel of a Feller semigroup; that is, a strongly continuous semigroup of nonnegative operators on* $C_\infty(\mathbb{R}^n)$:

$$P_t\phi(x) = \int_{\mathbb{R}^n} \phi(y)\Gamma(t,x,y)\,dy, \quad t > 0, \quad P_0\phi(x) = \phi(x), \quad \phi \in C_\infty(\mathbb{R}^n).$$

The generator $(A, D(A))$ *of this semigroup is an extension of the operator* $(L, C^2_\infty(\mathbb{R}^n))$ *defined by* (39).

II. *The function* $\Gamma(t,x,y)$ *is the transition probability density of a conservative Markov process with càdlàg trajectories.*
III. *For every* $T > 0$*, there exist constants* $0 < c_T < C_T < \infty$ *such that for* $t \in (0,T]$

$$\Gamma(t,x,y) \le C_T t^{-n/\alpha}\left(1 + \frac{|x-\kappa_t(y)|}{t^{1/\alpha}}\right)^{-n-\alpha}, \tag{57}$$

$$\Gamma(t,x,y) \ge c_T t^{-n/\alpha}\left(1 + \frac{|x-\kappa_t(y)|}{t^{1/\alpha}}\right)^{-n-\alpha}. \tag{58}$$

The upper bound (57) follows directly by (50) and (51). The proof of the lower bound (58), which combines properly (50) and (51) with the semigroup property of the kernel $\Gamma(t,x,y)$, is given in [62]; see also [49].

4.4 Probabilistic aspects: Martingale problem and the weak solution to an SDE

Statements I and II of Theorem 4.1 provide actually the *existence* result: for the operator L given by (39), a Markov process X is defined, which corresponds to this operator in the sense that X generates a Feller semigroup, and the generator of this semigroup is an extension of L. The *uniqueness* problem in this context can be solved in several related (yet different) ways. One of these ways, which is essentially analytical, is developed in [49]. Namely, Lemma 4.1 properly combined with the analysis of the derivatives in t of the terms $\Delta(t,x,y)$, $\Phi(t,x,y)$, $W(t,x,y)$ in the above construction, provides that the generator $(A, D(A))$ of the Feller semigroup specified in Theorem 4.1 coincides with the *closure* of the operator L; recall that the domain of L is $D(L) = C^2_\infty(\mathbb{R}^n)$. This evidently specifies uniquely the operator A, the corresponding semigroup, and thus the law of the Markov process X.

In this section, we explain another, more probabilistic, way to treat the uniqueness problem, which is simpler and more straightforward. It is based on the *martingale problem* (*MP*), which was developed originally for parabolic equations, see the original papers [97, 98], and also the monograph [99, Chapter 7], and appeared to be a perfect connection between these equations, diffusion processes, and the SDEs driven by Brownian noise. Below we explain how to apply this theory in the context of equations with fractional Laplacians. For that, we outline briefly the relevant part of the theory; see [23, Section 4.4] for the detailed exposition.

In what follows, all processes are assumed to have càdlàg trajectories, and by Law(X) we denote the law of a process X in the Skorokhod space $\mathbb{D}([0,\infty),\mathbb{R}^n)$. Let an operator L with the domain $D(L)$ in $C_\infty(\mathbb{R}^n)$ be given. The law of the process X (or, with some ambiguity of terminology, the process X itself) is called *a solution to the* *martingale problem* for $(L, D(L))$ if for any $\phi \in D(L)$ the process

$$\phi(X_t) - \int_0^t L\phi(X_s)\,ds$$

is a *martingale* with respect to the natural filtration $\{\mathcal{F}_t\}$.

Let X be a Markov process, which gives rise to a Feller semigroup, and let A be the generator of its semigroup; then X is a solution to the MP $(A, D(A))$. This simple observation is often called *the Dynkin formula*; see [20, Section 5.1] for a general theory.

On the other hand, let X be a (weak) solution to the SDE (38); note that the existence of such a solution follows by the usual compactness argument; see [49]. Then by the Itô formula (e. g., [32, Chapter II.5]), it is a solution to the MP $(L, C^2_\infty(\mathbb{R}^n))$ with L defined by (39). Note that by the Dynkin formula, the Markov process constructed in Theorem 4.1 solves the same MP. With these two observations in mind, the following uniqueness statement dramatically simplifies and unifies the entire picture.

Proposition 4.1. *Let operator L be given by* (39) *and* (B_1)–(B_3) *hold true. Then any two solutions X, Y to the MP $(L, C^2_\infty(\mathbb{R}^n))$ with càdlàg trajectories and the same initial value $X_0 = Y_0 = x \in \mathbb{R}^n$ have the same law in $\mathbb{D}([0,\infty), \mathbb{R}^n)$.*

The proof is based on an approximation argument which is in the same spirit as that from the preamble to [99, Chapter 7]. It is remarkably short and simple, which is a benefit of the parametrix construction. Here, we explain the core of the proof, referring to [62] for the technical details and further discussion.

By [23, Corollary 4.4.3], the required uniqueness holds true if, for any solutions to the MP with the same initial distribution μ, the corresponding one-dimensional distributions coincide. In what follows, we fix *some* solution Y and prove that

$$E\phi(Y_T) = \int_{\mathbb{R}^n} P_T\phi(x)\mu(dx), \quad \phi \in C_\infty, \quad T > 0. \tag{59}$$

It is easy to prove that Y_t, $t \geq 0$, is stochastically continuous. Then by [23, Lemma 4.3.4(a)] the process

$$h(t, Y_t) - \int_0^t \big(\partial_s h(s, Y_s) + L_x h(s, Y_s)\big)\,ds$$

is a martingale for any function $h(t,x)$ which is differentiable in t, belongs to $D(L) = C^2_\infty(\mathbb{R}^n)$ as a function of x, and $\partial_t h(t,x)$, $L_x h(t,x)$ are continuous bounded. For $\alpha > 1$, this observation gives (59) immediately. Namely, in this case $\Gamma(t,x,y)$ is a FS for the operator $\partial_t - L$, and for given $\phi \in C_\infty(\mathbb{R}^n)$ and $T > 0$ the function

$$h^{T,\phi}(t,x) = P_{T-t}\phi(x), \quad t \in [0,T],\ x \in \mathbb{R},$$

satisfies $\partial_t h^{T,\phi} + L_x h^{T,\phi} = 0$ for $t \in [0, T]$. Then $h^{T,\phi}(t, Y_t)$, $t \in [0, T]$, is a martingale, which yields (59) because

$$E\phi(Y_T) = Eh^{T,\phi}(T, Y_T) = Eh^{T,\phi}(0, Y_0) = \int_{\mathbb{R}^n} P_T\phi(x)\mu(dx).$$

If $\alpha < 1$, the argument remains essentially the same, with the only minor modification caused by the fact that now $\Gamma(t, x, y)$ provides a FS for the operator $\partial_t - L$ in the approximate sense. Define

$$h_\varepsilon^{T,\phi}(t, x) = P_{T-t,\varepsilon}\phi(x), \quad t \in [0, T], x \in \mathbb{R}, \quad \varepsilon > 0.$$

Then for any $\tau < T$

$$h_\varepsilon^{T,\phi}(t, x) \to h^{T,\phi}(t, x), \quad (\partial_t + L_x)h_\varepsilon^{T,\phi}(t, x) \to 0, \quad \varepsilon \to 0,$$

uniformly in $t \in [0, t]$, $x \in \mathbb{R}^n$; see Lemma 4.1. Then

$$\begin{aligned} EP_{T-\tau}(Y_\tau) - \int_{\mathbb{R}^n} P_T\phi(x)\mu(dx) \\ &= \lim_{\varepsilon\to 0}\big(Eh_\varepsilon^{T,\phi}(\tau, Y_\tau) - Eh_\varepsilon^{T,\phi}(0, Y_0)\big) \\ &= \lim_{\varepsilon\to 0} E\int_0^T (\partial_s + L_x)h_\varepsilon^{T,\phi}(s, Y_s)\, ds = 0. \end{aligned}$$

Taking $\tau \nearrow T$, we obtain (59) and complete the proof.

Summarizing the above considerations, we conclude that the Markov process X constructed in Theorem 4.1 can be uniquely specified in each of the following three ways:

- X is a unique Markov process such that on $C_\infty^2(\mathbb{R}^n)$ the generator of its Feller semigroup coincides with the operator L given by (39);
- X is the unique weak solution to the SDE (38);
- X is the unique solution to the MP $(L, C_\infty^2(\mathbb{R}^n))$.

5 Overview

The parametrix method first appears in the work of E. Levi [70] in the elliptic setting. Independently, this idea was proposed by Feller [24] in the probabilistic setting for an operator which is a sum of an elliptic operator and a bounded non-local operator. The idea of Levi was developed (independently from Feller) in the work of Dressel [16] for a parabolic setting and in the work of Ilyin, Kalashnikov, and Oleinik [33]. It was extended in the monographs of Friedman [25], Eidelman [21], and Ladyzhenskaya, Solonnikov, Uraltseva [68]. See also McKean, Singer [74], and Minakshisundaram [83] for a different modification of the Levi's method in the parabolic setting.

For the fractional Laplacian, the FS exists and is the transition probability density of a symmetric stable process; its asymptotic behavior was first investigated by Blumenthal and Getoor [6]. Their probabilistic paper and the earlier paper by Polya [90], in which the case $n = 1$ was considered, were noticed by the specialists in partial differential equations only much later; see the monograph of Eidelman, Ivasyshen, and Kochubei [22] for references to other early publications (Ya. M. Drin', S. D. Eidelman, M. V. Fedoryuk) containing the asymptotics or estimates of this FS. For the related fractional calculus needed in the method, see the monograph of Samko, Kilbas, and Marichev [95], and Samko [93, 94]. See also Caffarelli, Silvestre [13] where another representation of the fractional Laplacian is introduced, based on an extension of the state space and the fractional Poisson formula, and the paper by Kwaśnicki [67] for a survey of different definitions of the fractional Laplacian.

Drin', Eidelman [17, 18], construct the parametrix for the Cauchy problem for a pseudo-differential operator with a smooth enough symbol, which is homogeneous of some order in the variable ξ. However, the problem in which sense the constructed function is the solution to the initial equation was not properly addressed. This problem was discussed in details in Kochubei [50] for HSI. In the probabilistic context, a Cauchy problem with a stable-like operator was investigated in Kolokoltsov [51].

The parametrix approach from [22, 50] was further developed in several directions. Singular delta-function type gradient perturbation of the principal operator of order α are studied in Portenko [91, 92] (for $d = 1$) and further in Podplynny [88] and Podolynny, Portenko [89] (for $d \geq 1$). Under the assumption on the drift $b(x)$ that $\|b\|_p < \infty$, where $p > d/(\alpha - 1)$, the existence and uniqueness of a continuous solution are proved. The results from [89, 91, 92] are extended recently in the works of Osypchuk and Portenko [86, 87]. The effect of p-dependence (already in a different form appearing in [53]) was rediscovered in Bogdan, Jakubowski [7], see also Bogdan, Jakubowski, Sydor [8], and Bogdan, Sydor [9]. In Kim, Song [43], a symmetric stable generator perturbed by a singular drift which belongs to the Kato class with respect to the symmetric α-stable transition probability density is considered. See also Chen, Wang [14], for the approach which relies on [7] and the martingale problem method from Bass, Chen [5].

Further advances are made in Knopova, Kulik [49], and Ganychenko, Knopova, Kulik [26], Kulik [62], where the SDE of the form (38) are studied in the super-critical regime; that is, with nontrivial drift (gradient) term and the index of the fractional Laplacian $\alpha < 1$. In a greater generality, this problem for integrodifferential operators with nonhomogeneous symbol was studied in Knopova, Kulik [47, 48]. This approach requires a different type of estimates on the parametrix, and relies on the earlier works [45, 46] in this direction. On the other hand, in Chen, Zhang [15], Kim, Song, Vondraček [44] the parametrix construction from [50] was developed and generalized, along a different line, for perturbation of (isotropic) α-stable operators. In the recent works of Kühn [59–61], the parametrix solution was constructed by employing the complex analysis technique, which provides the kernel estimates. Quite recently, the Cauchy problem with homogeneous symbols of positive order was studied in Litovchenko [72, 73] by analytic methods in a bit different context: it is shown that this Cauchy problem is correctly solvable in the class of generalized functions, and belongs to a certain function space.

Completely different approach to the construction of the parametrix solution was proposed in the works of Ch. Iwasaki (Tsutsumi), N. Iwasaki [34–36, 103], and Kumano-go [63–65]; see the monograph by Kumano-go [66] for the details and more references, and the later papers by Negoro [85] and Kikuchi, Negoro [42]. The approach relies on the symbolic calculus technique, which allows one to prove the existence of the fundamental solution to the Cauchy problem for an operator L, provided that its symbol is smooth enough (i. e., belongs to the so-called "Hörmander class of symbols"). Hoh [30, 31] constructed the generalized Hörmander classes of symbols and developed the respective symbolic calculus ("Hoh's symbolic calculus," see also the monograph of Jacob [39], and the earlier works of Jacob [37, 38]). This calculus was further used to develop a modification of the parametrix method; this method was further extended to evolution equations in the papers of Böttcher [11, 12].

For the second-order elliptic differential operators, the well-posedness of the martingale problem was studied in the papers of Stroock and Varadhan [97, 98]. It was Grigelionis [27] first to give the martingale formulation of a Markov process associated with certain integrodifferential operator. Well-posedness of the martingale problem for a Lévy-type generator with a diffusion part was proved in Stroock [96] by using a purely probabilistic approach. Perturbations of α-stable operator were first studied by Tsuchiya in [101, 102] by constructing the perturbation of the resolvent operator in L_p and then showing the well-posedness of the martingale problem. The ideas from [101, 102] were developed and improved by Komatsu in [52, 53], where he investigated space-dependent perturbations of a (non-isotropic) α-stable operator, and proved that the parametrix expansion for the resolvent is well-defined and provides the unique solution to the Cauchy problem. In Bass [3, 4] similar resolvent approach as in [53] is used in a more general situation; the resolvent is also represented in the form of convergent series, but in contrast to [53] the series are not constructed starting from the resolvent of the nonperturbed process. Mikulevicius and Pragarauskas [75–79] studied the existence and uniqueness in Hölder and Sobolev spaces of classical solutions to the Cauchy problem (in $\mathbb{R}^d$) to parabolic integrodifferential equation of the order $\alpha \in (0, 2)$, where the kernel of the principal part of the operator is a perturbation of a radially symmetric α-stable Lévy measure. Then the analytical results are applied for the proof of the uniqueness of the corresponding martingale problem. The resolvent approach for the well-posedness of the martingale problem for Lévy-type pseudodifferential operators in the weak L_2-setting was developed in Hoh [28, 29]; see also the monograph of Jacob [40].

The parametrix method has several perspective applications. Bally, Kohatsu-Higa [1] provide in the diffusion case a probabilistic interpretation of the asymptotic expansion of the transition probability density provided by the parametrix method. This probabilistic representation naturally leads to the Monte Carlo type simulation methods. This work was developed further in Li, Kohazu-Higa [71], where an SDE with Hölder coefficients driven by a stable-type subordinator is considered. Another version of the parametrix, relied on the parametrix representation of McKean, Singer [74], was developed in Konakov, Mammen [54, 55] in order to derive the rate of convergence of the transition probability density of the respective Markov chain to the transition probability density of a diffusion process; see also the earlier paper [58]. This methodology was developed in further publications [57] in order to get the estimate on the error in the diffusion approximation, and further in [56] for finding the approximation of the transition probability density and the respective error in the model where the SDE is driven by a symmetric stable process without a drift and a Gaussian component. In the recent papers of Mikulevicius and Zhang [81, 82], the earlier results on the uniqueness of the solution to the Cauchy problem in Hölder spaces [76, 77, 80] are used to show that the Euler scheme for a Lévy-driven SDE yields a positive weak order of convergence.

Bibliography

[1] V. Bally and A. Kohatsu-Higa, A probabilistic interpretation of the parametrix method, *Ann. Appl. Probab.*, **25**(6) (2015), 3095–3138.

[2] B. Barrios, I. Peral, F. Soria, and E. Valdinoci, A Widder's type theorem for the heat equation with nonlocal diffusion, *Arch. Ration. Mech. Anal.*, **213** (2014), 629–650.

[3] R. F. Bass, Uniqueness in law for pure jump Markov processes, *Probab. Theory Relat. Fields*, **79**(2) (1988), 271–287.

[4] R. F. Bass, Stochastic differential equations with jumps, *Probab. Surv.*, **1** (2004), 1–19.

[5] R. F. Bass and Z.-Q. Chen, Brownian motion with singular drift, *Ann. Probab.*, **31**(2) (2003), 279–817.

[6] R. M. Blumenthal and R. K. Getoor, Some theorems on stable processes, *Trans. Am. Math. Soc.*, **95** (1960), 263–273.

[7] K. Bogdan and T. Jakubowski, Estimates of heat kernel of fractional Laplacian perturbed by gradient operators, *Commun. Math. Phys.*, **271** (2007), 179–198.

[8] K. Bogdan, T. Jakubowski, and S. Sydor, Estimates of perturbation series for kernels, *J. Evol. Equ.*, **12** (2012), 973–984.

[9] K. Bogdan and S. Sydor, On nonlocal perturbations of integral kernels, in *Semigroups of Operators: Theory and Applications*, Springer Proceedings in Math. & Stat., vol. 113, pp. 7–42, 2015.

[10] M. Bonforte, Y. Sire, and J. L. Vázquez, Optimal existence and uniqueness theory for the fractional heat equation, *Nonlinear Anal.*, **153** (2017), 142–168.

[11] B. Böttcher, A parametrix construction for the fundamental solution of the evolution equation associated with a pseudo-differential operator generating a Markov process, *Math. Nachr.*, **278** (2005), 1235–1241.

[12] B. Böttcher, Construction of time inhomogeneous Markov processes via evolution equations using pseudo-differential operators, *J. Lond. Math. Soc.*, **78** (2008), 605–621.

[13] L. Caffarelli and L. Silvestre, An extension problem related to the fractional Laplacian, *Commun. Partial Differ. Equ.*, **32**(7–9) (2007), 1245–1260.

[14] Z.-Q. Chen and L. Wang, Uniqueness of stable processes with drift, *Proc. Am. Math. Soc.*, **144** (2016), 2661–2675.

[15] Z.-Q. Chen and X. Zhang, Heat kernels and analyticity of non-symmetric jump diffusion semigroups, *Probab. Theory Relat. Fields*, **165** (2016), 267–312.

[16] F. G. Dressel, The fundamental solution to the parabolic equation, *Duke Math. J.*, **7** (1940), 186–203.

[17] Ja. M. Drin', Fundamental solution of the Cauchy problem for a class of parabolic pseudodifferential equations, *Dopov. Akad. Nauk Ukr. RSR, Ser. a Fiz.-Teh. Mat. Nauki* (1977), 198–203 (in Ukrainian).

[18] Ja. M. Drin' and S. D. Eidelman, Construction and investigation of classical fundamental solution of the Cauchy problem for uniformly parabolic pseudo-differential equations, *Mat. Issled.*, **63** (1981), 18–33 (in Russian).

[19] Ya. M. Drin' and S. D. Eidelman, On the theory of systems of parabolic pseudo-differential equations, *Dokl. Akad. Nauk Ukr. SSR, Ser. A. Fiz.-Mat. Teh. Nauki*, **4** (1989), 35–37.

[20] E. B. Dynkin, *Foundations of the Theory of Markov Processes*, Prentice-Hall, Englewood Cliffs, NJ, 1961.

[21] S. D. Eidelman, *Parabolic Systems*, North-Holland, Amsterdam, 1969.

[22] S. D. Eidelman, S. D. Ivasyshen, and A. N. Kochubei, *Analytic Methods in the Theory of Differential and Pseudo-Differential Equations of Parabolic Type*, Birkhäuser, Basel, 2004.

[23] S. N. Ethier and T. G. Kurtz, *Markov Processes: Characterization and Convergence*, Wiley, New York, 1986.

[24] W. Feller, Zur Theorie der stochastischen Prozesse. (Existenz- und Eindeutigkeitssätze), *Math. Ann.*, **113** (1936), 113–160.

[25] A. Friedman, *Partial Differential Equations of Parabolic Type*, Prentice-Hall, Englewood Cliffs, NJ, 1964.

[26] Iu. Ganychenko, V. Knopova, and A. Kulik, Accuracy of discrete approximation for integral functionals of Markov processes, *Modern Stochastics: Theory and Applications*, **2**(4) (2015), 401–420.

[27] B. Grigelionis, On a Markov property of Markov processes, *Liet. Mat. Rink.*, **8**(3) (1968), 489–502 (in Russian).

[28] W. Hoh, The martingale problem for a class of pseudo-differential operators, *Math. Ann.*, **300**(1) (1994), 121–147.

[29] W. Hoh, Pseudodifferential operators with negative definite symbols and the martingale problem, *Stoch. Stoch. Rep.*, **55**(3–4) (1995), 225–252.

[30] W. Hoh, *Pseudo differential operators generating Markov processes*, Habilitationsschrift, Bielefeld, 1998.

[31] W. Hoh, A symbolic calculus for pseudo differential operators generating Feller semigroups, *Osaka Math. J.*, **35** (1998), 789–820.

[32] N. Ikeda and S. Watanabe, *Stochastic Differential Equations and Diffusion Processes*, North-Holland, Amsterdam, 1981.

[33] A. Ilyin, A. Kalashnikov, and O. Oleinik, Linear equations of second order of parabolic type, *Russ. Math. Surv.*, **17**(3) (1962), 1–143.

[34] Ch. Iwasaki (Tsutsumi), The fundamental solution for pseudo-differential operators of parabolic type, *Osaka Math. J.*, **14**(3) (1977), 569–592.

[35] Ch. Iwasaki and N. Iwasaki, Parametrix for a degenerate parabolic equation, *Proc. Jpn. Acad.*, **55** (1979), 237–240.

[36] Ch. Iwasaki and N. Iwasaki, Parametrix for a Degenerate parabolic equation and its application to the asymptotic behavior of spectral functions for stationary problems, *Publ. RIMS, Kyoto Univ.*, **17** (1981), 577–655.

[37] N. Jacob, Further pseudo differential operators generating Feller semigroups and Dirichlet forms, *Rev. Mat. Iberoam.*, **9**(2) (1993), 373–407.

[38] N. Jacob, A class of Feller semigroups generated by pseudo-differential operators, *Math. Z.*, **215** (1994), 151–166.

[39] N. Jacob, *Pseudo Differential Operators and Markov Processes, II: Generators and Their Potential Theory*, Imperial College Press, London, 2002.

[40] N. Jacob, *Pseudo Differential Operators and Markov Processes, III: Markov Processes and Applications*, Imperial College Press, London, 2005.

[41] T. Kato, *Perturbation Theory for Linear Operators*, Springer, 1980.

[42] J. Kikuchi and A. Negoro, On Markov processes generated by pseudo- differential operator of variable order, *Osaka J. Math.*, **34** (1997), 319–335.

[43] P. Kim and R. Song, Stable process with singular drift, *Stoch. Process. Appl.*, **124** (2014), 2479–2516.

[44] P. Kim, R. Song, and Z. Vondracek, Heat kernels of non-symmetric jump processes: beyond the stable case, *Potential Anal.*, (2017), 1–54. https://doi.org/10.1007/s11118-017-9648-4.

[45] V. Knopova, Compound kernel estimates for the transition probability density of a Lévy process in $\mathbb{R}^n$, *Theory Probab. Math. Stat.*, **89** (2014), 57–70.

[46] V. Knopova and A. Kulik, Intrinsic small time estimates for distribution densities of Lévy processes, *Random Oper. Stoch. Equ.*, **21**(4) (2013), 321–344.

[47] V. Knopova and A. Kulik, Parametrix construction for certain Levy-type processes, *Random Oper. Stoch. Equ.*, **23**(2) (2015), 111–136.

[48] V. Knopova and A. Kulik, Intrinsic compound kernel estimates for the transition probability density of Lévy type processes and their applications, *Probab. Math. Stat.*, **37**(1) (2017), 53–100.

[49] V. Knopova and A. Kulik, Parametrix construction of the transition probability density of the solution to an SDE driven by α-stable noise, *Ann. Inst. Henri Poincaré*, **54**(1) (2018), 100–140.

[50] A. N. Kochubei, Parabolic pseudodifferential equations, hypersingular integrals, and Markov processes, *Math. USSR, Izv.*, **33** (1989), 233–259.

[51] V. Kolokoltsov, Symmetric stable laws and stable-like jump-diffusions, *Proc. Lond. Math. Soc.*, **80** (2000), 725–768.

[52] T. Komatsu, Markov processes associated with certain integro-differential operators, *Osaka J. Math.*, **10** (1973), 271–303.

[53] T. Komatsu, On the martingale problem for generators of stable processes with perturbations, *Osaka J. Math.*, **21**(1) (1984), 113–132.

[54] V. Konakov and E. Mammen, Local limit theorems for transition densities of Markov chains converging to diffusions, *Probab. Theory Relat. Fields*, **117** (2000), 551–587.

[55] V. Konakov and E. Mammen, Edgeworth type expansions for Euler schemes for stochastic differential equations, *Monte Carlo Methods Appl.*, **8**(3) (2002), 271–285.

[56] V. Konakov and S. Menozzi, Weak error for stable driven stochastic differential equations: expansion of the densities, *J. Theor. Probab.*, **24** (2011), 454–478.

[57] V. Konakov, S. Menozzi, and S. Molchanov, Explicit parametrix and local limit theorems for some degenerate diffusion processes, *Ann. Inst. Henri Poincaré Probab. Stat.*, **46**(4) (2004), 908–992.

[58] V. D. Konakov and S. A Molchanov, On the convergence of Markov chains to diffusion processes, *Teor. Veroyatn. Mat. Statist.* **31** (1984), 51–64 (in Russian). English translation *Theory Probab. Math. Statist.*, **31** (1985), 59–73.

[59] F. Kühn, *Probability and heat kernel estimates for Lévy(-type) processes*, PhD thesis, Technische Universität Dresden, 2016.

[60] F. Kühn, Transition probabilities of Lévy-type processes: parametrix construction, Preprint, 2017, available at arXiv:1702.00778v2.

[61] F. Kühn, *Lévy-Type Processes: Moments, Construction and Heat Kernel Estimates*, Lévy Matters, vol. 5, Springer, 2017.

[62] A. Kulik, On weak uniqueness and distributional properties of a solution to an SDE with α-stable noise, *Stoch. Process. Appl.* To appear. https://doi.org/10.1016/j.spa.2018.03.010.

[63] H. Kumano-go, Factorizations and fundamental solutions for differential operators of elliptic-hyperbolic type, *Proc. Jpn. Acad.*, **52**(9) (1976), 480–483.

[64] H. Kumano-go, A calculus of Fourier integral operators on $\mathbb{R}^n$ and the fundamental solution for an operator of hyperbolic type, *Commun. Partial Differ. Equ.*, **1**(1) (1976), 1–44.

[65] H. Kumano-go, Fundamental solutions for operators of regularly hyperbolic type, *J. Math. Soc. Jpn.*, **29**(3) (1977), 399–406.

[66] H. Kumano-go, *Pseudo-Differential Operators*, MIT Press, Cambridge, Mass., 1981.

[67] M. Kwaśnicki, Ten equivalent definitions of the fractional Laplace operator, *Fract. Calc. Appl. Anal.*, **20** (2017), 5–57.

[68] O. A. Ladyzhenskaya, V. A. Solonnikov, and N. N. Uraltseva, *Linear and Quasilinear Equations of Parabolic Type*, American Mathematical Society, Providence, 1968.

[69] C. Lemoine, Fourier transforms of homogeneous distributions, *Ann. Sc. Norm. Super. Pisa*, **26** (1972), 117–149.

[70] E. E. Levi, Sulle equazioni lineari totalmente ellittiche alle derivate parziali, *Rend. Circ. Mat. Palermo*, **24**(1) (1907), 275–317.

[71] L. Li and A. Kohatsu-Higa, Regularity of the density of a stable-like driven SDE with Holder continuous coefficients, *Stoch. Anal. Appl.*, **34**(6) (2016), 979–1024.

[72] V. A. Litovchenko, Cauchy problem with Riesz operator of fractional differentiation, *Ukr. Math. J.*, **37** (2005), 1937–1956.

[73] V. A. Litovchenko, The Cauchy problem for one class of parabolic pseudo-differential systems with nonsmooth symbols, *Sib. Math. J.*, **49** (2008), 300–316.

[74] H. P. McKean and I. M. Singer, Curvature and the eigenvalues of the Laplacian, *J. Differ. Geom.*, **1** (1967), 43–69.

[75] R. Mikulevicius and H. Pragarauskas, On the Cauchy problem for certain integro-differential operators in Sobolev and Holder Spaces, *Lith. Math. J.*, **32**(2) (1992), 238–264.

[76] R. Mikulevicius and H. Pragarauskas, On the martingale problem associated with nondegenerate Levy operators, *Lith. Math. J.*, **32**(3) (1992), 297–311.

[77] R. Mikulevicius and H. Pragarauskas, On Hölder solutions of the integro-differential Zaka equation, *Stoch. Process. Appl.*, **119**(10) (2009), 3319–3355.

[78] R. Mikulevicius and H. Pragarauskas, On the Cauchy problem for integro-differential operators in Holder classes and the uniqueness of the martingale problem, *Potential Anal.*, **40**(4) (2014), 539–563.

[79] R. Mikulevicius and H. Pragarauskas, On the Cauchy problem for integro-differential operators in Sobolev classes and the martingale problem, *J. Differ. Equ.*, **256**(4) (2014), 1581–1626.

[80] R. Mikulevicius and H. Pragarauskas, On the Cauchy problem for integro-differential operators in Holder classes and the uniqueness of the martingale problem, *Potential Anal.*, **40**(4) (2014), 539–563.

[81] R. Mikulevicius and C. Zhang, On the rate of convergence of weak Euler approximation for nondegenerate SDEs driven by Lévy processes, *Stoch. Process. Appl.*, **121** (2011), 1720–1748.
[82] R. Mikulevicius and C. Zhang, Weak Euler approximation for Ito diffusion and jump processes, *Stoch. Anal. Appl.*, **33** (2015), 549–571.
[83] S. Minakshisundaram, Eigenfunctions on Riemannian manifolds, *J. Indian Math. Soc.*, **17** (1953), 158–165.
[84] U. Neri, *Singular Integrals*, Lecture Notes Math., vol. 200, 1971, pp. 1–272.
[85] A. Negoro, Stable-like processes: construction of the transition density and the behaviour of sample paths near $t = 0$, *Osaka J. Math.*, **31** (1994), 189–214.
[86] M. Osypchuk and M. Portenko, One type of singular perturbations of a multidimensional stable process, *Theory Stoch. Process.*, **19**(2) (2014), 42–51.
[87] M. Osypchuk and M. Portenko, On simple-layer potentials for one class of pseudo-differential operators, *Ukr. Math. J.*, **67**(11) (2016), 1704–1720.
[88] S. I. Podolynny, Perturbation of homogeneous parabolic pseudo-differential equations by locally unbounded vector fields, *Ukr. Math. J.*, **49** (1997), 1756–1762.
[89] S. I. Podolynny and N. I. Portenko, On multidimensional stable processes with locally unbounded drift, *Random Oper. Stoch. Equ.*, **3**(2) (1995), 113–124.
[90] G. Polya, On the zeros of an integral function represented by Fourier's integral, *Messenger of Math.*, **52** (1923), 185–188.
[91] N. I. Portenko, Some perturbations of drift-type for symmetric stable processes, *Random Oper. Stoch. Equ.*, **2**(3) (1994), 211–224.
[92] N. I. Portenko, On some perturbations of symmetric stable processes, in *Probability Theory and Mathematical Statistics. Proceedings of the Seventh Japan–Russia Symposium*, pp. 414–422, World Scientific, Singapore, 1996.
[93] S. G. Samko, Generalized Riesz potentials and hypersingular integrals with homogeneous characteristics, their symbols and inversion, *Proc. Steklov Inst. Math.*, **156** (1983), 173–243.
[94] S. G. Samko, *Hypersingular Integrals and Their Applications*, Taylor and Francis, London, 2001.
[95] S. G. Samko, A. A. Kilbas, and O. I. Marichev, *Fractional Integrals and Derivatives: Theory and Applications*, Gordon and Breach, New York, 1993.
[96] D. W. Stroock, Diffusion processes associated with Lévy generators, *Z. Wahrscheinlichkeitstheor. Verw. Geb.*, **32** (1975), 209–244.
[97] D. W. Stroock and S. R. S. Varadhan, Diffusion processes with continuous coefficients. Part I, *Commun. Pure Appl. Math.*, **22** (1969), 345–400.
[98] D. W. Stroock and S. R. S. Varadhan, Diffusion processes with continuous coefficients. Part II, *Commun. Pure Appl. Math.*, **22** (1969), 479–530.
[99] D. W. Stroock and S. R. S. Varadhan, *Multidimensional Diffusion Processes*, Springer, Berlin, 1979.
[100] H. Tanaka, M. Tsuchiya, and S. Watanabe, Perturbation of drift-type for Lévy processes, *J. Math. Kyoto Univ.*, **14** (1974), 73–92.
[101] M. Tsuchiya, On a small drift of Cauchy process, *J. Math. Kyoto Univ.*, **10** (1970), 475–492.
[102] M. Tsuchiya, On some perturbations of stable processes, in *Lecture Notes in Math., Proceedings of the Second Japan–USSR Symposium on Probability Theory, Kyoto*, pp. 490–497, 1973.
[103] Ch. Tsutsumi, The fundamental solution for a degenerate parabolic pseudo-differential operator, *Proc. Jpn. Acad.*, **50** (1974), 11–15.

Yuri Luchko and Masahiro Yamamoto

Maximum principle for the time-fractional PDEs

Abstract: This chapter is devoted to an in-depth discussion of the maximum principle for the time-fractional partial differential equations. Some of its applications including uniqueness of solutions to the initial-boundary-value problems for the time-fractional partial differential equations and their a priori norm estimates are discussed.

Keywords: fractional partial differential equations, extremum principle for fractional derivatives, maximum principle, positivity of solutions, a priori norm estimates, uniqueness of solutions

MSC 2010: 26A33, 35A05, 35B30, 35B50, 35C05, 35E05, 35L05, 45K05, 60E99

1 Introduction

For boundary and initial-boundary-value problems for PDEs of elliptic and parabolic types, the maximum principle in the weak and strong formulations and its implications are known to play a very essential role (see, e. g., [28] and [29]).

Because the fractional diffusion equations can be interpreted as a kind of interpolation between the PDEs of elliptic and parabolic type, it is natural to expect that an appropriate maximum principle is valid also for these equations. Indeed, already in one of the first publications [12] devoted to analysis of the multidimensional fractional diffusion equation, some arguments related to a kind of a maximum principle have been employed. In [6], a maximum principle was formulated and proved for a one-dimensional fractional telegraph equation with the constant coefficients. In [33], general linear and quasilinear evolutionary partial integro-differential equations were investigated. In particular, the global boundedness of appropriately defined weak solutions and a maximum principle for the weak solutions of such equations were established in [33]. In [19] and [21], a maximum principle for the strong solutions to the initial-boundary value problems for the multi-dimensional time-fractional diffusion equation was derived and applied for analysis of solutions to these problems. Later on, the maximum principle was extended to the case of both multi-term fractional diffusion equations [22] and fractional diffusion equations of distributed order [20]. In the publications mentioned above, the maximum principle was established for the

Yuri Luchko, Beuth University of Applied Sciences Berlin, Department of Mathematics, Physics, and Chemistry, Luxemburger Str. 10, 13353 Berlin, Germany, e-mail: luchko@beuth-hochschule.de
Masahiro Yamamoto, The University of Tokyo, Graduate School of Mathematical Sciences, 3-8-1 Komaba, Meguro-ku, Tokyo 153-8914, Japan, e-mail: myama@ms.u-tokyo.ac.jp

https://doi.org/10.1515/9783110571660-014

time-fractional diffusion equations with the fractional derivatives in the Caputo sense. The case of the fractional diffusion equations with the time-fractional derivatives in the Riemann–Liouville sense was treated in [2–4].

A maximum principle for the evolution equations with a general fractional derivative, which contains the single and the multi-term time-fractional diffusion equations as well as the time-fractional diffusion equations of the distributed order was derived in [33] for the weak solutions and recently in [25] for the strong solutions and the weak solutions in the Vladimirov sense [31].

The publications mentioned above dealt with the weak maximum principles for the time-fractional diffusion equations. A positivity property for the single-term time-fractional diffusion equation was proved in [16]. In a recent paper [26], the positivity for this equation was proved under weaker conditions compared to those formulated in [16]. Strong positivity of solutions to the time-fractional diffusion equations with multiple Caputo derivatives was established in [15]. As an application, uniqueness of solution to an inverse problem of determining the temporal component of the equation source term was proved in [15].

In this chapter, a survey of some important results regarding the maximum principles for the partial fractional differential equations is presented. The focus is on the maximum principle for the strong solutions to the initial-boundary-value problems for the time-fractional diffusion equations. As to the results regarding the maximum principle for the weak solutions of these equations that were derived in [33–35], they are discussed in the chapter [36] of this book and we do not repeat them here. The rest of the chapter is organized as follows: In the second section, the maximum principles for the time-fractional diffusion equations with the Caputo fractional derivative are introduced and applied for analysis of the initial-boundary-value problems for these equations. In the third section, a discussion of positivity and comparison properties of the weak solutions to these equations are discussed. The fourth section is devoted to the maximum principle for the time-fractional diffusion equations with the Riemann–Liouville fractional derivatives and its applications. In the fifth section, we give an overview of some recent results regarding the maximum principle for the time-fractional diffusion equation with the general fractional derivative. Finally, in the last section, some conclusions and results for other types of fractional partial differential equations are discussed.

2 Time-fractional diffusion equations with the Caputo fractional derivatives

This section is devoted to the maximum principles for solutions to the initial-boundary value problems for the time-fractional diffusion equations with the Caputo fractional derivatives. For details and proofs, we refer the reader to the papers [19–27].

We start with a single-term time-fractional diffusion equation. This equation is obtained from the conventional diffusion equation by replacing the first-order time derivative by the fractional Caputo derivative of the order α ($0 < \alpha \le 1$):

$$(D_t^\alpha u)(t) = L_x(u) + F(x,t), \tag{1}$$
$$0 < \alpha \le 1,\ (x,t) \in \Omega_T := \Omega \times (0,T),\ \Omega \subset \mathbb{R}^n,$$

where

$$L_x(u) = \operatorname{div}(p(x)\operatorname{grad} u) - q(x)u, \tag{2}$$
$$p \in C^1(\overline{\Omega}),\ q \in C(\overline{\Omega}),\ p(x) > 0,\ q(x) \ge 0,\ x \in \overline{\Omega}, \tag{3}$$

the fractional derivative D_t^α is defined in the Caputo sense

$$(D_t^\alpha f)(t) = (I^{1-\alpha} f')(t), \quad 0 < \alpha \le 1, \tag{4}$$

I^α being the fractional Riemann–Liouville integral:

$$(I^\alpha f)(t) := \begin{cases} \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau, & 0 < \alpha < 1, \\ f(t), & \alpha = 0, \end{cases} \tag{5}$$

the domain Ω with the boundary $\partial\Omega$ is open and bounded in $\mathbb{R}^n$, and by $\overline{\Omega}$ we denote the closure of Ω.

If $\alpha = 1$, then (1) is the conventional diffusion equation, so that in the further discussions we focus on the case $0 < \alpha < 1$. We look for solutions of (1) that satisfy the initial condition

$$u|_{t=0} = u_0(x), \quad x \in \overline{\Omega} \tag{6}$$

and the Dirichlet boundary condition

$$u|_{\partial\Omega} = v(x,t), \quad (x,t) \in \partial\Omega \times (0,T). \tag{7}$$

The maximum principle for the equation (1) will be formulated for a solution $u = u(x,t)$ to the initial-boundary-value problem (1), (6), (7) that is defined in the domain $\Omega_T := \Omega \times (0,T)$ and belongs to the space $CW_T(\Omega) := C(\overline{\Omega}_T) \cap W_t^1(0,T) \cap C_x^2(\Omega)$. Here, $u \in C_x^2(\Omega)$ means that $u(\cdot,t) \in C^2(\Omega)$ for each $t \in [0,T]$ and by $W_t^1(0,T)$ we denote the space of functions f that satisfy $f \in C^1(0,T]$ and $f' \in L^1(0,T)$.

In the proof of the maximum principle, an extremum principle for the Caputo fractional derivative (4) plays a very essential role.

Theorem 1 ([19]). *If a function $f \in W_t^1(0,T) \cap C[0,T]$ attains its maximum over the interval $[0,T]$ at the point $\tau = t_0 \in (0,T]$, then the Caputo fractional derivative of f is nonnegative at the point t_0 for any α, $0 < \alpha < 1$, that is,*

$$(D_t^\alpha f)(t_0) \ge 0, \quad 0 < \alpha < 1. \tag{8}$$

A stronger inequality compared to one given in Theorem 1, namely,

$$(D_t^\alpha f)(t_0) \geq \frac{t_0^{-\alpha}}{\Gamma(1-\alpha)}(f(t_0) - f(0)), \quad 0 < \alpha < 1 \tag{9}$$

was derived in [1] (see also [2]). However, in order to prove the maximum principle for the time-fractional diffusion equation (1), the inequality (8) is sufficient.

In [19], the following weak maximum principle for the generalized time-fractional diffusion equation (1) was proved.

Theorem 2 ([19]). *If a function $u \in CW_T(\Omega)$ is a solution of the time-fractional diffusion equation* (1) *and $F(x,t) \leq 0$, $(x,t) \in \Omega_T$, then either $u(x,t) \leq 0$, $(x,t) \in \overline{\Omega}_T$ or u attains its positive maximum on the bottom or side parts $S_\Omega^T := (\overline{\Omega} \times \{0\}) \cup (\partial\Omega \times [0,T])$, that is,*

$$u(x,t) \leq \max\Big\{0, \max_{(x,t)\in S_\Omega^T} u(x,t)\Big\}, \quad \forall (x,t) \in \overline{\Omega}_T. \tag{10}$$

An appropriate minimum principle is correspondingly valid, also.

Theorem 3 ([19]). *If a function $u \in CW_T(\Omega)$ is a solution of the time-fractional diffusion equation* (1) *and $F(x,t) \geq 0$, $(x,t) \in \Omega_T$, then either $u(x,t) \geq 0$, $(x,t) \in \overline{\Omega}_T$ or u attains its negative minimum on the part S_Ω^T, that is,*

$$u(x,t) \geq \min\Big\{0, \min_{(x,t)\in S_\Omega^T} u(x,t)\Big\}, \quad \forall (x,t) \in \overline{\Omega}_T. \tag{11}$$

Like in the case of PDEs, the maximum principle can be applied to prove uniqueness of solution to the initial-boundary-value problem (1), (6)–(7). Indeed, we can prove some a priori estimate of the solution norm, which directly yields the uniqueness of solution.

Theorem 4 ([19]). *Let u be a solution to the initial-boundary-value problem* (1), (6)–(7) *and F belong to the space $C(\overline{\Omega}_T)$. Then*

$$\|u\|_{C(\overline{\Omega}_T)} \leq \max\{M_0, M_1\} + \frac{T^\alpha}{\Gamma(1+\alpha)} M, \tag{12}$$

$$M := \|F\|_{C(\overline{\Omega}_T)}, \; M_0 := \|u_0\|_{C(\overline{\Omega})}, \; M_1 := \|v\|_{C(\partial\Omega\times[0,T])}. \tag{13}$$

The results formulated in Theorem 4 are then used to prove the following important theorem.

Theorem 5 ([19]). *The initial-boundary-value problem* (1), (6)–(7) *possesses at most one solution. This solution continuously depends on the problem data in the sense that if*

$$\|F - \widetilde{F}\|_{C(\overline{\Omega}_T)} \leq \epsilon,$$

$$\|u_0 - \widetilde{u}_0\|_{C(\overline{\Omega})} \leq \epsilon_0, \quad \|v - \widetilde{v}\|_{C(\partial\Omega\times[0,T])} \leq \epsilon_1,$$

then the estimate

$$\|u - \tilde{u}\|_{C(\overline{\Omega}_T)} \le \max\{\epsilon_0, \epsilon_1\} + \frac{T^\alpha}{\Gamma(1+\alpha)}\,\epsilon \tag{14}$$

holds true for the corresponding solutions u and $\tilde{u}$.

The uniqueness of the solution immediately follows from the fact that the homogeneous problem (1), (6)–(7), that is, the problem with $F \equiv 0$, $u_0 \equiv 0$, and $v \equiv 0$ has only one strong solution, namely, $u(x,t) \equiv 0$, $(x,t) \in \overline{\Omega}_T$. The last statement is a simple consequence from the norm estimate (12) established in Theorem 4. The same estimate is used to prove the inequality (14). This time, it is applied to the function $u - \tilde{u}$ that is a solution to the initial-boundary-value problem (1), (6)–(7) with $F - \tilde{F}$, $u_0 - \tilde{u}_0$, and $v - \tilde{v}$ instead of F, u_0, and v, respectively.

As to the question regarding the existence of a solution to the initial-boundary-value problem (1), (6)–(7), it was positively answered in [21] for the generalized solution in Vladimirov sense [31] and in [24] for the strong solution.

As mentioned in the Introduction, the maximum principle for both multi-term fractional diffusion equation and the fractional diffusion equation of distributed order have been also derived. In [22], a weak maximum principle of the type formulated in Theorem 2 was proved for the multi-term fractional diffusion equation

$$P(D_t)u = L_x(u) + F(x,t), \quad (x,t) \in \Omega_T := \Omega \times (0,T),\ \Omega \subset \mathbb{R}^n, \tag{15}$$

where the operator L_x is defined by (2)–(3) and the multi-term fractional derivative $P(D_t)$ by the equation

$$P(D_t) = D_t^\alpha + \sum_{i=1}^{m} \lambda_i D_t^{\alpha_i}, \tag{16}$$

$$0 < \alpha_m < \cdots < \alpha_1 < \alpha \le 1,\ 0 \le \lambda_i,\ i = 1, \dots, m,\ m \in \mathbb{N}$$

with the fractional derivatives in the Caputo sense. In [15], a kind of positivity property was derived for the multi-term fractional diffusion equation

$$P(D_t)u = \sum_{i,j=1}^{n} \partial_i\big(a_{ij}(x)\partial_j u(x,t)\big) + c(x)u(x,t) + F(x,t), \quad x \in \Omega \subset \mathbb{R}^n,\ t > 0 \tag{17}$$

with the homogeneous boundary condition

$$u(x,t) = 0, \quad x \in \partial\Omega,\ t > 0, \tag{18}$$

and an inhomogeneous initial condition

$$u(x,0) = a(x), \quad x \in \Omega. \tag{19}$$

In the equation (17), the operator $P(D_t)$ is defined by (16), $0 < \alpha < 1$, Ω is an open bounded domain with a smooth boundary $\partial\Omega$ and the spatial differential operator in equation (17) is uniformly elliptic one. Moreover, the condition $c(x) \le 0$, $x \in \overline{\Omega}$ is supposed to be valid.

Finally, let us mention that a maximum principle of the type formulated in Theorem 2 was proved in [20] for the fractional diffusion equation of the distributed order (see also [7] for the one-dimensional case)

$$\mathbb{D}_t^{w(\alpha)} u = L_x(u) + F(x,t), \quad (x,t) \in \Omega_T := \Omega \times (0,T),\ \Omega \subset \mathbb{R}^n, \tag{20}$$

where the operator L_x is given by (2)–(3). The distributed order derivative $\mathbb{D}_t^{w(\alpha)}$ is defined by the equation

$$\mathbb{D}_t^{w(\alpha)} f(t) = \int_0^1 (D_t^\alpha f)(t)\, w(\alpha)\, d\alpha, \tag{21}$$

where D_t^α denotes the Caputo fractional derivative (4), and $w : [0,1] \longrightarrow \mathbb{R}$ is a continuous function, not identically equal to zero on the interval $[0,1]$, the conditions

$$0 \le w(\alpha),\ w \not\equiv 0,\ \alpha \in [0,1], \qquad \int_0^1 w(\alpha)\, d\,\alpha > 0 \tag{22}$$

hold true, and the domain Ω with the boundary $\partial\Omega$ is open and bounded in $\mathbb{R}^n$.

3 Positivity and comparison properties for solutions of the time-fractional diffusion equation

In this section, we discuss the weak positivity, the strong positivity, and some comparison properties for solutions to the following initial-boundary-value problem for the time-fractional diffusion equation with the Caputo fractional derivative D_t^α of order $\alpha \in (0,1)$:

$$(D_t^\alpha u)(t) = \sum_{i,j=1}^n \partial_i\big(a_{ij}(x)\partial_j u\big) + c(x)u + F(x,t), \quad x \in \Omega \subset \mathbb{R}^n,\ t > 0, \tag{23}$$

$$u(x,t) = 0, \quad x \in \partial\Omega,\ t > 0, \tag{24}$$

$$u(x,0) = a(x), \quad x \in \Omega, \tag{25}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with a smooth boundary $\partial\Omega$.

We also suppose that $c \in C(\overline{\Omega})$ and the spatial differential operator of the second order from the right-hand side of (23) is uniformly elliptic, that is, $a_{ij} \equiv a_{ji} \in C^1(\overline{\Omega})$,

$1 \le i, j \le n$ and there exists a constant $\mu_0 > 0$ such that $\sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \ge \mu_0 \sum_{i=1}^n \xi_i^2$ for all $x \in \overline{\Omega}$ and $\xi_1, \dots, \xi_n \in \mathbb{R}$. The fractional derivative D_t^α in (23) is understood as an extension of the conventional Caputo derivative to an operator defined on a subspace $H_\alpha(0,T)$ of the fractional Sobolev space $H^\alpha(0,T)$ (see [10] for details). Thus we interpret the initial-boundary-value problem (23)–(25) as the fractional diffusion equation (23) subject to the inclusions

$$\begin{cases} u(\cdot,t) \in H_0^1(\Omega), & t > 0, \\ u(x,\cdot) - a(x) \in H_\alpha(0,T), & x \in \Omega. \end{cases} \tag{26}$$

Here, we define

$$H_\alpha(0,T) = \begin{cases} H^\alpha(0,T), & 0 < \alpha < \frac{1}{2}, \\ \{v \in H^{\frac{1}{2}}(0,T);\ \int_0^T \frac{|v(t)|^2}{t} dt < \infty\}, & \alpha = \frac{1}{2}, \\ \{v \in H^\alpha(0,T);\ v(0) = 0\}, & \frac{1}{2} < \alpha < 1, \end{cases}$$

where for $\frac{1}{2} < \alpha$, the equation $v(0) = 0$ is understood in the sense of the trace of v in the Sobolev space. The space $H_\alpha(0,T)$ is a Banach space equipped with the norm

$$\|v\|_{H_\alpha(0,T)} = \begin{cases} \|v\|_{H^\alpha(0,T)}, & 0 < \alpha < 1,\ \alpha \ne \frac{1}{2}, \\ (\|v\|^2_{H^{\frac{1}{2}}(0,T)} + \int_0^T \frac{|v(t)|^2}{t} dt)^{\frac{1}{2}}, & \alpha = \frac{1}{2}. \end{cases}$$

According to the results presented in [10], for any initial condition $a \in H_0^1(\Omega)$ and any source function $F \in L^2(\Omega \times (0,T))$, there exists a unique solution $u_{a,F} \in L^2(0,T;H^2(\Omega) \cap H_0^1(\Omega)) \cap H_\alpha(0,T;L^2(\Omega))$ to (23) subject to the inclusions (26). In this section, if we do not specify, we always consider $u_{a,F}$ as a solution to (23) and (26) from the space $L^2(0,T;H^2(\Omega) \cap H_0^1(\Omega)) \cap H_\alpha(0,T;L^2(\Omega))$. For other definitions including weak solutions to the problem under consideration, we refer the reader to [14] and [34].

For $\frac{1}{2} < \alpha < 1$, in view of the Sobolev embedding, the solution $u_{a,F}$ belongs to the function space $C([0,T];L^2(\Omega))$ and satisfies the initial condition (25) in the sense that $\lim_{t\to 0} \|u(\cdot,t) - a\|_{L^2(\Omega)} = 0$.

The following positivity of $u_{a,F}$ follows from the maximum principle proved in [19] and the results presented in [30].

Theorem 6. *Let $a \in L^2(\Omega)$, $F \in L^2(\Omega \times (0,T))$, and $c \le 0$ on $\overline{\Omega}$.*

If $a \ge 0$ a. e. in Ω and $F \ge 0$ a. e. in $\Omega \times (0,T)$, then $u_{a,F} \ge 0$ a. e. in $\Omega \times (0,T)$.

In [19], more regular solutions were considered and the statement of Theorem 6 was proved for all points $(x,t) \in \overline{\Omega} \times [0,T]$. In this section, we deal with solutions from the function space $L^2(0,T;H^2(\Omega) \cap H_0^1(\Omega)) \cap H_\alpha(0,T;L^2(\Omega))$.

For the sake of technical simplicity, in the further discussions we assume that

$$u = 0 \quad \text{on } \partial\Omega \times (0,T),$$

that is, $u(\cdot,t) \in H_0^1(\Omega)$ for almost all $t > 0$.

Let us recall that the statement of Theorem 6 for the equation (23) with $\alpha = 1$ (the conventional diffusion equation) is valid without any conditions on the coefficient c. As was shown in [26], in the case of the fractional diffusion equation (23), Theorem 6 is also valid without the condition $c \le 0$ on $\overline{\Omega}$:

Theorem 7 ([26]). *Let $a \in H_0^1(\Omega)$ and $F \in L^2(\Omega \times (0,T))$.*
If $F \ge 0$ a. e. in $\Omega \times (0,T)$ and $a \ge 0$ a. e. in Ω, then $u_{a,F} \ge 0$ a. e. in $\Omega \times (0,T)$.

The proof of Theorem 7 presented in [26] was based on the fixed-point theorem and the property that the solution mapping $\{a, F\} \longrightarrow u_{a,F}$ preserves its sign on the set of the solutions, and thus the nonnegativity of the solution is valid almost everywhere and not pointwise.

Theorem 7 immediately yields important comparison properties formulated in two corollaries below.

Corollary 1. *Let $a_1, a_2 \in H_0^1(\Omega)$ and $F_1, F_2 \in L^2(\Omega\times(0,T))$. If $a_1 \ge a_2$ a. e. in Ω and $F_1 \ge F_2$ a. e. in $\Omega \times (0,T)$, then $u_{a_1,F_1} \ge u_{a_2,F_2}$ a. e. in $\Omega \times (0,T)$.*

For a fixed source function $F \ge 0$ and a fix initial value $a \ge 0$, by $u_c = u_c(x,t)$ we denote the weak solution to the initial-boundary-value problem (23) and (26) with the coefficient $c = c(x)$ in (23).

Corollary 2. *Let $c_1, c_2 \in C(\overline{\Omega})$. If $c_1 \ge c_2$ in Ω, then $u_{c_1} \ge u_{c_2}$ a. e. in $\Omega \times (0,T)$.*

In the rest of this section, we discuss the strong positivity property: Let the solution u to the initial-boundary-value problem (23) and (26) with $F \equiv 0$, be also in $C((0,T]; C(\overline{\Omega}))$. Assuming that $a \in L^2(\Omega)$, $a \ge 0$, $a \not\equiv 0$ a. e. in Ω, can we conclude that

$$u > 0 \quad \text{in } \Omega \times (0,T]?$$

In [16], the following result was proved.

Theorem 8 ([16]). *Let $c \le 0$ on $\overline{\Omega}$, $a \in L^2(\Omega)$, and $n \le 3$. If $a \ge 0$, $a \not\equiv 0$ a. e. in Ω and $F \equiv 0$, then the solution u to* (23) *and* (26) *belongs to the space $C((0,T]; C(\overline{\Omega}))$ and the set $E_x := \{t > 0;\ u(x,t) \le 0\}$ is at most a finite set for each $x \in \Omega$.*

In [26], the assumption $c \le 0$ on $\overline{\Omega}$ was removed from the formulation of Theorem 8 and its statement was proved for any $c \in C(\overline{\Omega})$.

The statement of Theorem 8 remains true for the initial-boundary-value problem (23) and (26) with less regularity posed on the coefficients of the second order spatial differential operator in (23), for the equation of type (23) with an additional first-order differential operator $\sum_{j=1}^n b_j(x)\partial_j u(x,t)$ under suitably regularity of b_j, and for the equation of type (23) with the time dependent coefficients (see [26] for details).

For the equation (23) with $\alpha = 1$ (the conventional diffusion equation), the statement of Theorem 8 is valid with $E_x = \emptyset$. It turns out that the same holds true for the fractional diffusion equation, also. For the sake of technical simplicity, we restrict ourselves to the case of equation (23) with $c \equiv 0$.

Theorem 9. *Let $F \equiv 0$, $a \in L^2(\Omega)$, $a \geq 0$, $a \not\equiv 0$ a. e. in Ω, and the solution u to the initial-boundary-value problem* (23) *and* (26) *belong to the function space $C((0,T];C(\overline{\Omega}))$.*

Then the solution is strictly positive:

$$u(x,t) > 0, \quad x \in \Omega,\ 0 < t \leq T.$$

In what follows, we present a short sketch of a proof of Theorem 9. Let us first note that the inclusion $u \in C((0,T];C(\overline{\Omega}))$ does not automatically follow from the condition $a \in L^2(\Omega)$ and we need more regularity conditions posed on the function a from the initial condition (26). Say, one of the possibilities is the inclusion $a \in \mathcal{D}(L^\theta)$ with $\theta > \frac{n}{4}$. By L^γ, the fractional power of the operator L

$$Lv(x) = -\sum_{i,j=1}^{n} \partial_i(a_{ij}(x)\partial_j v(x)), \quad x \in \Omega$$

in $L^2(\Omega)$ with $\mathcal{D}(L) = H^2(\Omega) \cap H_0^1(\Omega)$ is denoted. If $\gamma \geq 0$, the inclusion $\mathcal{D}(L^\gamma) \subset H^{2\gamma}(\Omega)$ is valid. Then the solution u belongs to the function space $C((0,T];C(\overline{\Omega}))$ by the eigenfunction expansion of u (see, e. g., [30]) and the Sobolev embedding $H^\gamma(\Omega) \subset C(\overline{\Omega})$ if $2\gamma > n$.

For simplicity, we assume in the further discussions that $a \in \mathcal{D}(L^\theta)$ with $\theta > \frac{n}{4}$.

The key component of our proof is the weak Harnack inequality proved in [35]. In order to formulate it, some notations are needed. For arbitrarily fixed $\delta \in (0,1)$, $t_0 \geq 0$, $r > 0$, $\tau > 0$, and $x_0 \in \Omega$, we set $B(x_0,r) := \{x \in \mathbb{R}^n : |x - x_0| < r\}$, and

$$Q_-(t_0,x_0,r) := B(x_0,\delta r) \times (t_0, t_0 + \delta\tau r^{\frac{2}{\alpha}})$$
$$Q_+(t_0,x_0,r) := B(x_0,\delta r) \times (t_0 + (2-\delta)\tau r^{\frac{2}{\alpha}}, t_0 + 2\tau r^{\frac{2}{\alpha}}).$$

By $|Q_-(t_0,x_0,r)|$, we denote the measure in $\mathbb{R}^n \times \mathbb{R}$.

Theorem 10 ([35]). *Let $0 < \delta < 1$, $\tau > 0$ be fixed. For any $t_0 \geq 0$, $0 < p < \frac{2+n\alpha}{2+(n-2)\alpha}$ and $r > 0$ with $t_0 + 2\tau r^{\frac{2}{\alpha}} \leq T$ and $B(x_0,2r) \subset \Omega$, we have*

$$\left(\frac{1}{|Q_-(t_0,x_0,r)|} \int_{Q_-(t_0,x_0,r)} u(x,t)^p \,dxdt\right)^{\frac{1}{p}} \leq C \inf_{Q_+(t_0,x_0,r)} u.$$

Here, $C > 0$ is a constant depending on a_{ij}, δ, τ, α, n, p, r.

We proceed to the proof of Theorem 9 and apply a proof by contradiction. First, let us assume that there exist $x_0 \in \Omega$ and $\tilde{t} \in (0,T]$ such that $u(x_0,\tilde{t}) = 0$. We choose $r > 0$ sufficiently small such that $B(x_0,2r) \subset \Omega$. Now set $t_0 = \tilde{t} - s > 0$ and

$$\tau = sr^{-\frac{2}{\alpha}}\left(2 - \frac{\delta}{2}\right)^{-1}.$$

We further take $s > 0$ sufficiently small. Then $\tau r^{\frac{2}{\alpha}} = s(2 - \frac{\delta}{2})^{-1}$ is sufficiently small, so that $t_0 + 2\tau r^{\frac{2}{\alpha}} \le T$. Since

$$(2-\delta)\tau r^{\frac{2}{\alpha}} = (2-\delta)s\left(2 - \frac{\delta}{2}\right)^{-1} < s$$

and

$$2\tau r^{\frac{2}{\alpha}} = 2s\left(2 - \frac{\delta}{2}\right)^{-1} > s,$$

we can verify that $\widetilde{t} \in (t_0 + (2-\delta)\tau r^{\frac{2}{\alpha}}, t_0 + 2\tau r^{\frac{2}{\alpha}})$. Hence $(x_0, \widetilde{t}) \in Q_+(t_0, x_0, r)$. By Theorem 6, for $a \ge 0$ the inequality $u \ge 0$ holds true on $\overline{\Omega} \times [0, T]$. Therefore, $\inf_{Q_+(x_0,t_0,r)} u = 0$. The weak Harnack inequality yields that there exists $t_1 > 0$ such that $u(x,t) = 0$ for $x \in B(x_0, \delta r)$ and $t_0 < t < t_0 + t_1$. By the uniqueness property of u (Theorem 4.2 in [30]), we then obtain $u(x,t) = 0$ for $x \in \Omega$ and $0 \le t \le T$. This contradicts the condition $a \not\equiv 0$ of Theorem 9 and the proof is completed.

4 Time-fractional diffusion equations with the Riemann–Liouville fractional derivatives

The maximum principles for the single and multi-term time-fractional diffusion equations as well as for the time-fractional equation of the distributed order with the Riemann–Liouville fractional derivatives were derived in [2–4]. In this section, we present a short overview of these results. For the details and complete proofs, we refer the readers to the original papers [2–4].

The Riemann–Liouville fractional derivative is defined as follows:

$$(D_{0+}^{\alpha} f)(t) = \frac{d}{dt}(I^{1-\alpha} f)(t), \quad 0 < \alpha \le 1, \tag{27}$$

I^{α} being the Riemann–Liouville fractional integral (5). For $0 < \alpha < 1$ and $f \in C^1[0,T]$, the Riemann–Liouville fractional derivative (5) is connected with the Caputo fractional derivative (4) by the relation (see, e. g., [18])

$$(D_{0+}^{\alpha} f)(t) = \frac{t^{-\alpha}}{\Gamma(1-\alpha)} f(0) + (D_t^{\alpha} f)(t). \tag{28}$$

The theory of the fractional differential equations with the Caputo fractional derivatives is very different from one for the equations with the Riemann–Liouville fractional derivatives and it is not always possible to “translate” the results obtained for the equations with the Caputo derivatives to the case of the equations with the Riemann–Liouville derivatives and vice versa. In particular, the maximum principles for the fractional differential equations with the Riemann–Liouville fractional derivatives were derived independently from the results known for the equations with the Caputo fractional derivatives.

As in the case of the equations with the Caputo derivatives, the main component of the proofs of the maximum principle for the equations with the Riemann–Liouville fractional derivatives is an extremum principle for the Riemann–Liouville fractional derivative, that is, an estimate of the fractional derivative of a function at its maximum point. This result was first established in [1] (see also [2–4]).

Theorem 11 ([1]). *If a function* $f \in C^1[0,T]$ *attains its maximum at a point* $t_0 \in (0,T]$, *then the inequality*

$$(D^{\alpha}_{0+}f)(t_0) \geq \frac{t_0^{-\alpha}}{\Gamma(1-\alpha)} f(t_0) \tag{29}$$

holds true for $0 < \alpha < 1$.

We start with some results for a family of the nonlinear one-dimensional time-fractional diffusion equations

$$(D^{\alpha}_{0+}u)(t) = L(u) + f(x,t,u), \quad 0 < \alpha \leq 1,\ (x,t) \in \Omega_T = (0,\ell) \times (0,T), \tag{30}$$

where L is a second-order differential operator defined by

$$L(u) = a(x,t)u_{xx} + b(x,t)u_x + c(x,t)u \tag{31}$$

and the fractional derivative D^{α}_{0+} is the Riemann–Liouville derivative (27). In what follows, we assume that the functions $a = a(x,t)$, $b = b(x,t)$, and $c = c(x,t)$ from (31) are continuous functions on $\overline{\Omega_T} = [0,\ell] \times [0,T]$ and $a(x,t) > 0$, $(x,t) \in \overline{\Omega_T}$. The function $f = f(x,t,u)$ is supposed to be continuous with respect to the variables x and t and smooth with respect to the variable u.

With the notation

$$P_{\alpha}(u) = (D^{\alpha}_{0+}u)(t) - L(u), \quad 0 < \alpha < 1, \tag{32}$$

the equation (30) can be rewritten in the form

$$P_{\alpha}(u) = f(x,t,u), \quad (x,t) \in \Omega_T. \tag{33}$$

A weak maximum principle for the operator P_{α} is formulated in the following theorem.

Theorem 12 ([2]). *Let a function* $u \in C([0,T]; C^2[0,\ell]) \cap C^1([0,T]; C[0,\ell])$ *satisfy the inequality* $P_{\alpha}(u(x,t)) \leq 0$, $(x,t) \in \Omega_T$ *and* $c(x,t) \leq 0$, $(x,t) \in \Omega_T$.

Then the inequality

$$\max_{(x,t)\in\overline{\Omega_T}} u(x,t) \leq \max_{(x,t)\in S_1\cup S_2\cup S_3} \{u(x,t),\ 0\} \tag{34}$$

holds true, where

$$S_1 = \{x = 0,\ 0 \le t \le T\}, \quad S_2 = \{0 \le x \le \ell,\ t = 0\}, \tag{35}$$

$$S_3 = \{x = \ell,\ 0 \le t \le T\}. \tag{36}$$

A kind of a strong maximum principle for the fractional differential operator P_α was formulated and proved in [2].

Theorem 13 ([2]). *Let a function $u \in C([0,T]; C^2[0,\ell]) \cap C^1([0,T]; C[0,\ell])$ satisfy the equation $P_\alpha(u) = 0$, $(x,t) \in \Omega_T$ and $c(x,t) \le 0$, $(x,t) \in \Omega_T$.*

If u attains its maximum and its minimum at some points from Ω_T, then the function u is a constant, more precisely $u(x,t) = 0$, $(x,t) \in \overline{\Omega_T}$.

The maximum principles formulated above can be used for analysis of the initial-boundary-value problems for linear and nonlinear fractional diffusion equations with the Riemann–Liouville fractional derivatives. Let us first consider an initial-boundary-value problem for the linear fractional diffusion equation

$$P_\alpha(u) = (D^\alpha_{0+}u)(t) - L(u) = 0, \quad 0 < \alpha < 1, \quad (x,t) \in \Omega_T \tag{37}$$

along with the initial and boundary conditions

$$u(x,0) = g(x), \quad 0 \le x \le \ell, \tag{38}$$

$$u(x,t) = h(x,t), \quad (x,t) \in S_1 \cup S_3, \tag{39}$$

where S_1 and S_3 are defined as in (35)–(36).

The following result is a direct consequence of the weak maximum principle.

Theorem 14 ([2]). *Let $u \in C([0,T]; C^2[0,\ell]) \cap C^1([0,T]; C[0,\ell])$ fulfill the inequality $P_\alpha(u) \le 0$, $(x,t) \in \Omega_T$, and $c(x,t) \le 0$, $(x,t) \in \Omega_T$. If u satisfies the initial and boundary conditions* (38) *and* (39)*, respectively, $g(x) \le 0$, $0 \le x \le \ell$, and $h(x,t) \le 0$, $(x,t) \in S_1 \cup S_3$, then $u(x,t) \le 0$, $(x,t) \in \overline{\Omega_T}$.*

Let us now consider a nonlinear fractional diffusion equation

$$Q_\alpha(u) = (D^\alpha_{0+}u)(t) - \big(a(x,t)u_{xx} + b(x,t)u_x\big) = f(x,t,u), \quad 0 < \alpha < 1, \tag{40}$$

where $a(x,t) > 0$, $(x,t) \in \Omega_T$ and D^α_{0+} stands for the Riemann–Liouville fractional derivative.

Under some suitable conditions on the nonlinear term f, the maximum principle for the operator P_α leads to an uniqueness result for the initial-boundary-value problem (40), (38)–(39).

Theorem 15 ([2]). *Let $f = f(x,t,u)$ be a smooth and nonincreasing function with respect to the variable u. Then the initial-boundary-value problem* (40), (38)–(39) *possesses at most one solution $u = u(x,t)$, $(x,t) \in \overline{\Omega_T}$ in the function space $C([0,T]; C^2[0,\ell]) \cap C^1([0,T]; C[0,\ell])$.*

A uniqueness result for the initial-boundary-value problem (37), (38)–(39) for the linear fractional diffusion equation is a simple consequence from Theorem 15. If we consider the initial-boundary-value problem for the equation,

$$P_\alpha(u) = (D^\alpha_{0+}u)(t) - (a(x,t)u_{xx} + b(x,t)u_x + c(x,t)u) = g(x,t) \tag{41}$$

with the initial and boundary conditions (38)–(39), then the right-hand side function $f(x,t,u) = c(x,t)u + g(x,t)$ is a smooth and nonincreasing function with respect to the variable u. Thus we can apply Theorem 15 and obtain the following result.

Corollary 3. *In the function space $C([0,T];C^2[0,\ell]) \cap C^1([0,T];C[0,\ell])$ the initial-boundary-value problem* (41), (38)–(39) *possesses at most one solution.*

The weak maximum principle can be also applied to derive some stability results for the solutions of the initial-boundary-value problems (41), (38)–(39).

Theorem 16 ([2]). *Let u_1, $u_2 \in C([0,T];C^2[0,\ell]) \cap C^1([0,T];C[0,\ell])$ be two solutions of the fractional diffusion equation* (41) *that satisfy the same boundary condition* (39) *and the initial conditions $u_1(x,0) = g_1(x)$, $u_2(x,0) = g_2(x)$, $0 \le x \le \ell$, respectively, and $c(x,t) \le 0$, $(x,t) \in \Omega_T$. Then the inequality*

$$\max_{(x,t)\in\overline{\Omega_T}} |u_1(x,t) - u_2(x,t)| \le \max_{x\in[0,\ell]} |g_1(x) - g_2(x)|$$

holds true.

Now we consider a family of the multi-term time-fractional diffusion equations

$$\aleph(D_t)u = L_x(u) + f(x,t,u), \quad (x,t) \in \Omega \times (0,T), \tag{42}$$

where Ω is a bounded open domain in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$ (e. g., of C^2 class), L_x is a uniformly elliptic operator defined by

$$L_x(u) = \sum_{i,j=1}^n a_{ij}(x,t)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i(x,t)\frac{\partial u}{\partial x_i}, \tag{43}$$

$$\aleph(D_t) = D^\alpha_{0+} + \sum_{i=1}^m \lambda_i D^{\alpha_i}_{0+}, \tag{44}$$

$$0 < \alpha_m < \cdots < \alpha_1 < \alpha \le 1,\ 0 \le \lambda_i,\ i = 1,\dots,m,\ m \in \mathbb{N},$$

where D^α_{0+} is the Riemann–Liouville fractional derivative (27). Henceforth we always assume that $a_{ij} = a_{ji} \in C^1(\overline{\Omega_T})$, $b_i \in C(\overline{\Omega_T})$, $1 \le i,j \le n$, and there exists a constant $\mu_0 > 0$ such that $\sum_{i,j=1}^n a_{ij}(x,t)\xi_i\xi_j \ge \mu_0 \sum_{i=1}^n \xi_i^2$ for all $(x,t) \in \overline{\Omega_T}$ and $\xi_1,\dots,\xi_n \in \mathbb{R}$. The function f is assumed to be continuous with respect to the variables x and t and smooth with respect to the variable u.

For a proof of the maximum principle for the equation (42), an estimate for the Riemann–Liouville fractional derivative of a function f at its maximum point similar to one stated in Theorem 11 is needed. However, the condition $f \in C^1[0,T]$ from Theorem 11 is too restrictive and in fact not necessary for existence of the Riemann–Liouville fractional derivative. Say, the function $f(t) = \sqrt{t} - t$ attains its maximum at the point $t_0 = \frac{1}{4} \in (0,1]$ and is α-differentiable on the interval $(0,1]$ for $0 < \alpha < 1$, but f does not belong to the space $C^1[0,1]$. A natural question is whether the inequality from Theorem 11 holds true for such functions, that is, for the functions from the space $CW^1[0,T] := C[0,T] \cap W_t^1(0,T)$. We recall that by $W_t^1(0,T)$ the space of functions $f \in C^1(0,T]$ such that $f' \in L^1(0,T)$ is denoted.

Theorem 17 ([3]). *If a function $f \in CW^1[0,T]$ attains its maximum at a point $t_0 \in (0,T]$, then the inequality*

$$(D_{0+}^{\alpha} f)(t_0) \geq \frac{t_0^{-\alpha}}{\Gamma(1-\alpha)} f(t_0) \tag{45}$$

holds true for $0 < \alpha < 1$.

The inequality (45) is a key component in derivation of the maximum principle for the fractional differential operator of the parabolic type

$$\begin{aligned} P_\alpha(u) &= \aleph(D_t)u(x,t) - L_x(u) - c(x,t)u \\ &= \aleph(D_t)u(x,t) - \sum_{i,j=1}^{n} a_{ij}(x,t)\frac{\partial^2 u}{\partial x_i \partial x_j} - \sum_{i=1}^{n} b_i(x,t)\frac{\partial u}{\partial x_i} - c(x,t)u, \end{aligned} \tag{46}$$

where the function $c = c(x,t)$ is bounded on $\overline{\Omega} \times [0,T]$.

We start with a weak maximum principle that is formulated in the following theorem.

Theorem 18 ([3]). *If a function $u \in C([0,T]; C^2(\overline{\Omega})) \cap W^1(0,T; L^2(\Omega))$ satisfies the inequality $P_\alpha(u(x,t)) \leq 0$, $(x,t) \in \Omega \times (0,T)$ and $c(x,t) \leq 0$, $(x,t) \in \Omega \times (0,T)$, then the inequality*

$$\max_{(x,t)\in \overline{\Omega}\times[0,T]} u(x,t) \leq \max\left\{\max_{x\in\overline{\Omega}} u(x,0),\ \max_{(x,t)\in \partial\Omega\times[0,T]} u(x,t),\ 0\right\} \tag{47}$$

holds true.

A kind of a strong maximum principle for the fractional differential operator P_α is given in the next theorem.

Theorem 19 ([3]). *Let a function $u \in C([0,T]; C^2(\overline{\Omega})) \cap W^1(0,T; L^2(\Omega))$ satisfy the equation $P_\alpha(u) = 0$, $(x,t) \in \Omega \times (0,T)$ and $c(x,t) \leq 0$, $(x,t) \in \Omega \times (0,T)$. If u attains its maximum and its minimum at some points from $\Omega \times (0,T]$, then u is a constant, more precisely, $u(x,t) = 0$, $(x,t) \in \Omega \times (0,T]$.*

The maximum principles for the multi-term fractional differential operator P_α defined by (46) can be applied for analysis of the initial-boundary-value problems formulated in terms of P_α. Let us start with the initial-boundary value problem for the linear equation

$$P_\alpha(u) = \aleph(D_t)u(x,t) - L_x(u) - c(x,t)u = 0, \quad (x,t) \in \Omega \times (0,T), \tag{48}$$

$$u(x,0) = g(x), \quad x \in \Omega, \tag{49}$$

$$u(x,t) = h(x,t), \quad (x,t) \in \partial\Omega \times (0,T), \tag{50}$$

where the function $c = c(x,t)$ is bounded on $\overline{\Omega} \times [0,T]$ and the operators L_x and $\aleph(D_t)$ are defined by (43) and (44), respectively.

The weak maximum principle formulated in Theorem 18 leads to the following result.

Theorem 20 ([3]). *Let a function $u \in C([0,T]; C^2(\overline{\Omega})) \cap W^1(0,T; L^2(\Omega))$ fulfill the inequality $P_\alpha(u) \le 0$, $(x,t) \in \Omega \times (0,T)$ and $c(x,t) \le 0$, $(x,t) \in \Omega \times (0,T)$. If u satisfies the initial and boundary conditions* (49) *and* (50)*, respectively, $g(x) \le 0$, $x \in \Omega$ and $h(x,t) \le 0$, $(x,t) \in \partial\Omega \times (0,T)$, then $u(x,t) \le 0$, $(x,t) \in \overline{\Omega} \times [0,T]$.*

Theorem 20 can be used, among other things, for analysis of a nonlinear multi-term time-fractional reaction-diffusion equation in the form

$$Q_\alpha(u) = \aleph(D_t)u(x,t) - L_x(u) = f(x,t,u), \quad (x,t) \in \Omega \times (0,T). \tag{51}$$

Indeed, under some suitable conditions on the nonlinear term f, the maximum principle for the operator P_α or, more precisely, its implication given in Theorem 20, leads to a uniqueness result for the initial-boundary-value problem (51), (49)–(50) for the multi-term fractional diffusion equation that is formulated in the following theorem.

Theorem 21 ([3]). *Let $f = f(x,t,u)$ be a smooth and nonincreasing function with respect to the variable u. Then the initial-boundary-value problem* (51), (49)–(50) *for the multi-term fractional diffusion equation possesses at most one solution $u = u(x,t)$, $(x,t) \in \overline{\Omega} \times [0,T]$ in the function space $C([0,T]; C^2(\overline{\Omega})) \cap W^1(0,T; L^2(\Omega))$.*

Theorem 21 applied to the function $f(x,t,u) = c(x,t)u + g(x,t)$ (if $c(x,t) \le 0$, $(x,t) \in \Omega \times (0,T)$, then the function f is a smooth and nonincreasing function with respect to the variable u) immediately leads to the following result.

Theorem 22 ([3]). *Let $c(x,t) \le 0$, $(x,t) \in \Omega \times (0,T]$. The initial-boundary-value problem* (51), (49)–(50) *for the linear multi-term fractional diffusion equation*

$$\aleph(D_t)u(x,t) - L_x(u) - c(x,t)u = g(x,t) \tag{52}$$

possesses at most one solution in $C([0,T]; C^2(\overline{\Omega})) \cap W^1(0,T; L^2(\Omega))$.

A stability result of the type formulated in Theorem 16 is valid also for the initial-boundary-value problem (51), (49)–(50) for the linear multi-term fractional diffusion equation (see [3] for details).

Finally, let us briefly mention some results for the fractional diffusion equations of the distributed order with the Riemann–Liouville fractional derivatives in the form (see [4] for details):

$$D_t^{\omega(\alpha)}u = L_x(u) - q(x)u + g(x,t), \quad (x,t) \in \Omega \times (0,T), \tag{53}$$

where Ω is a bounded open domain in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$, L_x is a uniformly elliptic operator defined by (43), and $q \in C(\overline{\Omega})$, $q(x) \geq 0$, $x \in \overline{\Omega}$.

The distributed order derivative $D_t^{\omega(\alpha)}$ is defined by the relation

$$(D_t^{\omega(\alpha)}f)(t) = \int_0^1 (D_{0+}^{\alpha}f)(t)\omega(\alpha)d\alpha, \quad t > 0, \tag{54}$$

where D_{0+}^{α} is the Riemann–Liouville fractional derivative (27) and $\omega = \omega(\alpha)$ is a non-negative weight function that is continuous and not identically equal to zero on the interval $[0,1]$.

As in the case of the multi-term fractional diffusion equation, the estimate (45) plays a key role in derivation of the maximum principle for the fractional differential operator of parabolic type that is defined by the formula

$$\begin{aligned} P_{\omega(\alpha)}(u) &= D_t^{\omega(\alpha)}u - L_x(u) + q(x)u \\ &= D_t^{\omega(\alpha)}u - \sum_{i,j=1}^{n} a_{ij}(x,t)\frac{\partial^2 u}{\partial x_i \partial x_j} - \sum_{i=1}^{n} b_i(x,t)\frac{\partial u}{\partial x_i} + q(x)u. \end{aligned} \tag{55}$$

Theorem 23 ([4]). *If a function $u \in C([0,T];C^2(\overline{\Omega})) \cap W^1(0,T;L^2(\Omega))$ satisfies the inequality*

$$P_{\omega(\alpha)}\big(u(x,t)\big) \leq 0, \quad (x,t) \in \Omega \times (0,T),$$

then the estimate

$$\max_{(x,t)\in\,\overline{\Omega}\times[0,T]} u(x,t) \leq \max\Big\{\max_{x\in\overline{\Omega}} u(x,0), \max_{(x,t)\in\,\partial\Omega\times[0,T]} u(x,t),\ 0\Big\} \tag{56}$$

holds true.

Theorem 24 ([4]). *Let a function $u \in C([0,T];C^2(\overline{\Omega})) \cap W^1(0,T;L^2(\Omega))$ satisfy the equation $P_{\omega(\alpha)}(u(x,t)) = 0$, $(x,t) \in \Omega \times (0,T)$. If u attains its maximum and its minimum at some points that belong to $\Omega \times (0,T]$ then u is a constant, more precisely $u(x,t) \equiv 0$, $(x,t) \in \Omega \times (0,T)$.*

The maximum principles stated in Theorems 23 and 24 can be used among other things for analysis of the initial-boundary-value problems for linear and nonlinear time-fractional diffusion equations of distributed order. Let us start with the linear equation in the form

$$Q_{\omega(\alpha)}(u) = D_t^{\omega(\alpha)}u - L_x(u) + q(x)u - g(x,t) = 0,$$
$$(x,t) \in \Omega \times (0,T), \tag{57}$$
$$u(x,0) = h(x), \quad x \in \Omega, \tag{58}$$
$$u(x,t) = r(x,t), \quad (x,t) \in \partial\Omega \times (0,T). \tag{59}$$

In what follows, we suppose that the functions $g = g(x,t)$, $h = h(x)$, and $r = r(x,t)$ are continuous on $\overline{\Omega}_T = \overline{\Omega} \times [0,T]$, $\overline{\Omega}$, and $\partial\Omega \times [0,T]$, respectively.

The weak maximum principle allows to obtain some a priori estimates for the solution u.

Theorem 25 ([4]). *Let a function $u \in C([0,T]; C^2(\overline{\Omega})) \cap W^1(0,T; L^2(\Omega))$ be a solution to the initial-boundary-value problem* (57)–(59). *Then the solution norm estimate*

$$\|u\|_{C(\overline{\Omega}_T)} \le \max\{R, H\} + \frac{G\,T_M}{W_\Gamma}$$

holds true with $R = \|r\|_{C(\partial\Omega\times[0,T])}$, $H = \|h\|_{C(\overline{\Omega})}$, $G = \|g\|_{C(\overline{\Omega}_T)}$, $W_\Gamma = \int_0^1 \frac{w(\alpha)}{\Gamma(1-\alpha)}d\alpha > 0$, and $T_M = \max\{T, 1\}$.

The following important uniqueness and stability results are direct consequences of Theorem 25.

Theorem 26 ([4]). *The initial-boundary-value problem* (57)–(59) *for the distributed order fractional diffusion equation possesses at most one solution in $C([0,T]; C^2(\overline{\Omega})) \cap W^1(0,T; L^2(\Omega))$.*

Theorem 27 ([4]). *Let u_1, $u_2 \in C([0,T]; C^2(\overline{\Omega})) \cap W^1(0,T; L^2(\Omega))$ be two solutions to the distributed order fractional diffusion equation* (57) *that satisfy the same boundary condition* (59) *and the initial conditions $u_1(x,0) = h_1(x)$, $u_2(x,0) = h_2(x)$, $x \in \Omega$, respectively. Then the norm inequality*

$$\|u_1 - u_2\|_{C(\overline{\Omega}_T)} \le \|h_1 - h_2\|_{C(\overline{\Omega})} \tag{60}$$

holds true.

Finally, let us consider an initial-boundary-value problem for the nonlinear fractional diffusion equation of the distributed order

$$NQ_{\omega(\alpha)}(u) := D_t^{\omega(\alpha)}u - L_x(u) + q(x)u - F(x,t,u) = 0 \tag{61}$$

with the initial and boundary conditions (58)–(59). The operator $NQ_{\omega(\alpha)}$ can be represented in the form

$$NQ_{\omega(\alpha)}(u) = P_{\omega(\alpha)}(u) - F(x,t,u), \tag{62}$$

where $P_{\omega(\alpha)}$ is given by (55). A weak comparison principle for the operator $NQ_{\omega(\alpha)}$ is formulated in the following theorem.

Theorem 28 ([4]). *Let a function $u \in C([0,T];C^2(\overline{\Omega})) \cap W^1(0,T;L^2(\Omega))$ fulfill the inequality $NQ_{\omega(\alpha)}(u) \le 0$, $(x,t) \in \Omega \times (0,T)$ and $F(x,t,u) \le 0$, $(x,t) \in \Omega \times (0,T)$. If u satisfies the initial and boundary conditions* (58) *and* (59)*, respectively, with $h(x) \le 0$, $x \in \Omega$, and $r(x,t) \le 0$, $(x,t) \in \Omega \times (0,T]$, then $u(x,t) \le 0$, $(x,t) \in \overline{\Omega}_T$.*

The next results are concerned with stability, uniqueness, and comparison of the solutions to the initial-boundary-value problem (61), (58)–(59) under some suitable conditions posed on the nonlinear part F.

Theorem 29 ([4]). *Let u_1, $u_2 \in C([0,T];C^2(\overline{\Omega})) \cap W^1(0,T;L^2(\Omega))$ be two solutions to the distributed order fractional diffusion equation* (61) *that satisfy the same boundary condition* (59) *and the initial conditions $u_1(x,0) = h_1(x)$, $u_2(x,0) = h_2(x)$, $x \in \Omega$, respectively. If $F = F(x,t,u)$ is a smooth and nonincreasing function with respect to the variable u, then the inequality*

$$\|u_1 - u_2\|_{C(\overline{\Omega}_T)} \le \|h_1 - h_2\|_{C(\overline{\Omega})} \tag{63}$$

holds true.

Theorem 30 ([4]). *Let $F = F(x,t,u)$ be a smooth and nonincreasing function with respect to the variable u. Then the initial-boundary value problem* (61), (58)–(59) *possesses at most one solution $u = u(x,t)$, $(x,t) \in \overline{\Omega}_T$ in $C([0,T];C^2(\overline{\Omega})) \cap W^1(0,T;L^2(\Omega))$.*

Theorem 31 ([4]). *Let u_1, $u_2 \in C([0,T];C^2(\overline{\Omega})) \cap W^1(0,T;L^2(\Omega))$ be two solutions to the initial-boundary-value problem* (61), (58)–(59) *with the nonlinear parts $F = F_1(x,t,u)$ and $F = F_2(x,t,u)$, respectively.*

If $F_1 = F_1(x,t,u)$ is a smooth and nonincreasing function with respect to the variable u and the inequality $F_1(x,t,u) \le F_2(x,t,u)$ holds true for all $(x,t) \in \overline{\Omega}_T$ and $u \in C([0,T];C^2(\overline{\Omega})) \cap W^1(0,T;L^2(\Omega))$, then:
1. $u_1(x,t) \le u_2(x,t)$, $(x,t) \in \overline{\Omega}_T$.
2. *In the case $T_M \le W_\Gamma$, the norm inequality*

$$\|u_1 - u_2\|_{C(\overline{\Omega}_T)} \le \|F_1 - F_2\|_{C(\overline{\Omega})} \tag{64}$$

holds true, where $T_M = \max\{T,1\}$ and $W_\Gamma = \int_0^1 \frac{w(\alpha)}{\Gamma(1-\alpha)} d\alpha > 0$.

5 Time-fractional diffusion equation with the general fractional derivative

In this section, we present some results regarding the maximum principle for a time-fractional diffusion equation that generalizes the single and the multi-term time-fractional diffusion equations as well as the time-fractional diffusion equation of the distributed order (see the very recent paper [25] for details).

Let k be a nonnegative locally integrable function on $\mathbb{R}_+$. Following [13], the general fractional derivatives of the Caputo and the Riemann–Liouville type are defined as

$$(\mathbb{D}_{(k)}^{C} f)(t) = \int_0^t k(t-\tau) f'(\tau)\, d\tau, \tag{65}$$

$$(\mathbb{D}_{(k)}^{\mathrm{RL}} f)(t) = \frac{d}{dt} \int_0^t k(t-\tau) f(\tau)\, d\tau, \tag{66}$$

respectively. For an absolutely continuous function f satisfying the inclusion $f' \in L^1_{\mathrm{loc}}(0,\infty)$, we have the relation

$$(\mathbb{D}_{(k)}^{C} f)(t) = \frac{d}{dt} \int_0^t k(t-\tau) f(\tau)\, d\tau \; - \; k(t) f(0) \; = \; (\mathbb{D}_{(k)}^{\mathrm{RL}} f)(t) - k(t) f(0) \tag{67}$$

between the Caputo and Riemann–Liouville types of the general fractional derivatives. Let us mention that in [13] the general fractional derivative was introduced in form (67) that is valid for a wider class of functions (in particular, for absolutely continuous functions) compared to the definition (65) that additionally requires the inclusion $f' \in L^1_{\mathrm{loc}}(0,\infty)$.

The conventional Caputo and Riemann–Liouville fractional derivatives are particular cases of the general fractional derivative with the kernel function

$$k(\tau) = \frac{\tau^{-\alpha}}{\Gamma(1-\alpha)}, \quad 0 < \alpha < 1 \tag{68}$$

in the formulas (65) and (66), respectively. Other important particular cases of (65) and (66) are given by

$$k(\tau) = \sum_{k=1}^{n} a_k \frac{\tau^{-\alpha_k}}{\Gamma(1-\alpha_k)}, \quad 0 < \alpha_1 < \cdots < \alpha_n < 1 \tag{69}$$

and

$$k(\tau) = \int_0^1 \frac{\tau^{-\alpha}}{\Gamma(1-\alpha)}\, d\rho(\alpha), \tag{70}$$

where ρ is a Borel measure on $[0, 1]$. They correspond to the multi-term derivative and derivative of the distributed order, respectively.

In [13], Kochubei introduced a special class of general fractional derivatives in the form (67) with the kernel functions k satisfying the following conditions:

(C1) The Laplace transform:

$$\tilde{k}(p) := \int_0^\infty k(t)\, e^{-pt}\, dt$$

of k exists for all $p > 0$,

(C2) $\tilde{k}(p)$ is a Stiltjes function,

(C3) $\tilde{k}(p) \to 0$ and $p\tilde{k}(p) \to \infty$ as $p \to \infty$,

(C4) $\tilde{k}(p) \to \infty$ and $p\tilde{k}(p) \to 0$ as $p \to 0$.

For the general fractional derivatives with the kernels satisfying the conditions (C1)–(C4), the following results are valid [13]:

(A) For any $\lambda > 0$, the initial value problem for the relaxation equation

$$(\mathbb{D}^C_{(k)} u)(t) = -\lambda u(t), \quad t > 0 \tag{71}$$

with the initial condition

$$u(0) = 1 \tag{72}$$

has a unique solution $u_\lambda = u_\lambda(t)$ that belongs to the class $C^\infty(0,\infty)$ and is a completely monotone function, that is,

$$(-1)^n u_\lambda^{(n)}(t) \geq 0, \quad t > 0,\ n = 0, 1, 2 \dots. \tag{73}$$

(B) There exists a completely monotone function $\kappa = \kappa(t)$ with the property

$$(k * \kappa)(t) = \int_0^t k(t-\tau)\kappa(\tau)\, d\tau = 1 \quad \text{for all } t > 0. \tag{74}$$

(C) For $f \in L^1_{\mathrm{loc}}(0,\infty)$, the relation

$$(\mathbb{D}^C_{(k)} \mathbb{I}_{(k)} f)(t) = f(t) \tag{75}$$

holds true, where the general fractional integral $\mathbb{I}_{(k)}$ is defined by the formula

$$(\mathbb{I}_{(k)} f)(t) = \int_0^t \kappa(t-\tau) f(\tau)\, d\tau. \tag{76}$$

In what follows, we deal with the maximum principle for the fractional diffusion equation with the general fractional derivative of the Caputo type in the form

$$(\mathbb{D}^C_{(k)} u)(t) = D_2(u) + D_1(u) - q(x)u + F(x,t), \quad (x,t) \in \Omega \times (0,T), \tag{77}$$

where $q \in C(\overline{\Omega})$, $q(x) \geq 0$ for $x \in \overline{\Omega}$, and

$$D_1(u) = \sum_{i=1}^{n} b_i(x) \frac{\partial u}{\partial x_i}, \quad D_2(u) = \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j}. \tag{78}$$

Moreover we assume that D_2 is a uniformly elliptic differential operator.

In this section, we consider the initial-boundary-value problems for the equation (77) with the initial condition

$$u(x,t)|_{t=0} = u_0(x), \quad x \in \overline{\Omega} \tag{79}$$

and the boundary conditions

$$u(x,t)|_{(x,t)\in\partial\Omega\times(0,T]} = v(x,t), \quad (x,t) \in \partial\Omega \times (0,T). \tag{80}$$

By $S(\Omega, T)$, we denote the space of functions $u = u(x,t)$, $(x,t) \in \overline{\Omega} \times [0,T]$ that satisfy the inclusions $u \in C(\overline{\Omega} \times [0,T])$, $u(\cdot,t) \in C^2(\Omega)$ for any $t > 0$, and $\partial_t u(x,\cdot) \in C(0,T] \cap L^1(0,T)$ for any $x \in \Omega$. A function $u \in S(\Omega,T)$ is called a strong solution to the initial-boundary-value problem (77), (79), (80) if it satisfies both the equation (77) and the initial and boundary conditions (79)–(80).

In order to proceed with the maximum principle, we assume the following conditions on the kernel k in (65) and (66):

(K1) $k \in C^1(0,\infty) \cap L^1_{\mathrm{loc}}(0,\infty)$,

(K2) $k(\tau) > 0$ and $k'(\tau) < 0$ for $\tau > 0$,

(K3) $k(\tau) = o(\tau^{-1})$, $\tau \to 0$.

Let us note that the conditions (C1)–(C4) stated before are not needed for the maximum principle for the general diffusion equation (77). However, if the condition (C3) is fulfilled, then it follows from the Feller-Karamata Tauberian theorem for the Laplace transform [9] that the condition (K3) is satisfied, also.

As in the previous sections, the maximum principle for the general diffusion equation (77) is based on appropriate estimates of the general fractional derivative of a function f at its maximum point. They are given in the following theorem.

Theorem 32 ([25]). *Let the conditions* (K1)–(K3) *be fulfilled, a function* $f \in C[0,T]$ *attain its maximum over the interval* $[0,T]$ *at the point* t_0, $t_0 \in (0,T]$, *and* $f' \in C(0,T] \cap L^1(0,T)$.

Then the following inequalities hold true:

$$(\mathbb{D}^{\mathrm{RL}}_{(k)} f)(t_0) \geq k(t_0) f(t_0), \tag{81}$$

$$(\mathbb{D}^{C}_{(k)} f)(t_0) \geq k(t_0)(f(t_0) - f(0)) \geq 0. \tag{82}$$

Let us note here that for the Riemann–Liouville and Caputo fractional derivatives, the inequalities (81) and (82) take the form

$$(\mathbb{D}^{\alpha}_{0+} f)(t_0) \geq \frac{t_0^{-\alpha}}{\Gamma(1-\alpha)} f(t_0) \tag{83}$$

and

$$(\mathbb{D}^{\alpha}f)(t_0) \geq \frac{t_0^{-\alpha}}{\Gamma(1-\alpha)}(f(t_0) - f(0)) \geq 0. \tag{84}$$

In what follows, we consider the operator

$$\mathbb{P}_{(k)}(u) := (\mathbb{D}_{(k)}^{C}u)(t) - D_2(u) - D_1(u) + q(x)u(x,t). \tag{85}$$

Theorem 33 ([25]). *Let the conditions* (K1)–(K3) *be fulfilled and a function* $u \in S(\Omega, T)$ *satisfy the inequality*

$$\mathbb{P}_{(k)}(u) \leq 0, \quad (x,t) \in \Omega \times (0,T). \tag{86}$$

Then the following maximum principle holds true:

$$\max_{(x,t)\in\,\overline{\Omega}\times[0,T]} u(x,t) \leq \max\Big\{\max_{x\in\overline{\Omega}} u(x,0),\ \max_{(x,t)\in\,\partial\Omega\times[0,T]} u(x,t),\ 0\Big\}. \tag{87}$$

The maximum principle formulated in Theorem 33 can be applied, among other things, for derivation of a priori estimates for the strong solutions of the initial-boundary problem (77), (79)–(80).

Theorem 34 ([25]). *Let the conditions* (C1)–(C4) *and* (K1)–(K3) *be fulfilled and u be a strong solution of the initial-boundary-value problem* (77), (79)–(80).

Then the solution norm estimate

$$\|u\|_{C(\overline{\Omega}\times[0,T])} \leq \max\{M_0,\ M_1\} + Mf(T), \tag{88}$$

holds true, where

$$M_0 = \|u_0\|_{C(\overline{\Omega})}, \quad M_1 = \|v\|_{C(\partial\Omega\times[0,T])}, \quad M = \|F\|_{C(\Omega\times[0,T])}, \tag{89}$$

and

$$f(t) = \int_0^t \kappa(\tau)\,d\tau, \tag{90}$$

the function κ *being defined by* (74).

The uniqueness of the strong solution to the initial-boundary-value problem (77) and (79)–(80) and its continuous dependence on problem data easily follow from estimate (88).

Theorem 35 ([25]). *The initial-boundary-value problem* (77) *and* (79)–(80) *possesses at most one strong solution.*

This solution continuously depends on the data in the sense that if u and ũ are strong solutions to the problems with the sources functions F and F̃ and the initial and boundary conditions u_0 and $\tilde{u}_0$ and v and ṽ, respectively, and

$$\|F - \tilde{F}\|_{C(\overline{\Omega}\times[0,T])} \le \epsilon,$$
$$\|u_0 - \tilde{u}_0\|_{C(\overline{\Omega})} \le \epsilon_0, \quad \|v - \tilde{v}\|_{C(\partial\Omega\times[0,T])} \le \epsilon_1,$$

then the norm estimate

$$\|u - \tilde{u}\|_{C(\overline{\Omega}\times[0,T])} \le \max\{\epsilon_0, \epsilon_1\} + \epsilon f(T) \tag{91}$$

holds true, where the function f is defined by (90).

In the rest of this section, we deal with a weak solution to the initial-boundary-value problem (77) and (79)–(80) in the sense of Vladimirov [31].

Definition 1. Let $F_k \in C(\overline{\Omega} \times [0,T])$, $u_{0k} \in C(\overline{\Omega})$ and $v_k \in C(\partial\Omega \times [0,T])$, $k = 1,2,\dots$ satisfy the following conditions:

(V1) There exist functions F, u_0, and v, such that

$$\|F_k - F\|_{C(\overline{\Omega}\times[0,T])} \to 0 \quad \text{as } k \to \infty, \tag{92}$$
$$\|u_{0k} - u_0\|_{C(\overline{\Omega})} \to 0 \quad \text{as } k \to \infty, \tag{93}$$
$$\|v_k - v\|_{C(\partial\Omega\times[0,T])} \to 0 \quad \text{as } k \to \infty. \tag{94}$$

(V2) For any $k = 1,2,\dots$ there exists a strong solution $u_k = u_k(x,t)$ to the initial-boundary-value problem for the general time-fractional diffusion equation

$$u_k|_{t=0} = u_{0k}(x), \quad x \in \overline{\Omega}, \tag{95}$$
$$u_k|_{\partial\Omega\times(0,T]} = v_k(x,t), \quad (x,t) \in \partial\Omega \times (0,T) \tag{96}$$
$$(\mathbb{D}^C_{(k)} u_k)(t) = D_2(u_k) + D_1(u_k) - q(x)u_k + F_k(x,t), \quad (x,t) \in \Omega \times (0,T). \tag{97}$$

Then the function $u \in C(\overline{\Omega} \times [0,T])$ defined by

$$\|u_k - u\|_{C(\overline{\Omega}\times[0,T])} \to 0 \quad \text{as } k \to \infty \tag{98}$$

is called a weak solution to the initial-boundary-value problem (77), (79)–(80) in the sense of Vladimirov.

In [25], the correctness of Definition 1 was shown. A weak solution to the problem (77), (79)–(80) in the sense of Vladimirov is a continuous function, not a distribution. However, the weak solutions are not required to be smooth.

On the other hand, any strong solution is evidently also a weak solution. According to Theorem 35, the strong solution—if it exists—is unique. It turned out that under some additional conditions the weak solution is unique, also.

If the problem (77), (79)–(80) possesses a weak solution, then the functions F, u_0 and v from the problem formulation have to belong to the spaces $C(\overline{\Omega} \times [0,T])$, $C(\overline{\Omega})$, and $C(\partial\Omega \times [0,T])$, respectively, as the limits of sequences of the continuous functions in the uniform norm.

Moreover, the estimate (88) for the strong solutions of the problem (77), (79)–(80) holds true for the weak solutions, also. To show this, we let $k \to +\infty$ in the inequality,

$$\|u_k\|_{C(\overline{\Omega}_T)} \le \max\{M_{0k}, M_{1k}\} + M_k f(T), \tag{99}$$

$$M_{0k} := \|u_{0k}\|_{C(\overline{\Omega})},\ M_{1k} := \|v_k\|_{C(\partial\Omega\times[0,T])},\ M_k := \|F_k\|_{C(\overline{\Omega}\times[0,T])}$$

that is valid for $k = 1, 2 \ldots$.

The estimate (88) for the weak solutions is then employed to prove the following important theorem.

Theorem 36 ([25]). *The initial-boundary-value problem* (77), (79)–(80) *possesses at most one weak solution. The weak solution—if it exists—continuously depends on the data given in the problem in the sense of the estimate* (91).

For further results regarding the properties of the weak solutions to the initial-boundary-value problem (77), (79)–(80), we refer the interested reader to [25].

6 Conclusions and other results

In this chapter, we provided a survey of some results mainly devoted to the maximum principle for the time-fractional differential equations including the single-term and the multi-term equations as well as the equations of distributed order and the equations with the general fractional derivatives. We considered the equations with the time-fractional derivatives in the Caputo and in the Riemann–Liouville sense. For discussions on the maximum principle for the weak solutions to some of the above mentioned equations, we can refer to [33–35].

In the meantime, several papers devoted to the maximum principle for the space-time-fractional partial differential equations of different kinds were published. A maximum principle for the multi-term space-time-fractional diffusion equations with the modified Riesz space-fractional derivative in the Caputo sense was introduced and employed in [32]. In [17], a maximum principle for the multi-term space-time-fractional variable-order differential equations with the Riesz–Caputo fractional derivatives was proved and applied for analysis of these equations. In [5], a maximum principle for the space-time-fractional diffusion equation with the fractional Laplace operator and the Riemann–Liouville time-fractional derivative was derived. A maximum principle for a space-time-fractional diffusion equation with the Riemann–Liouville time-fractional derivative and the fractional Laplace operator was derived in [11] both for the strong and for the weak solutions. For results regarding the maximum principles for the space-fractional elliptic and parabolic partial differential equations with the fractional Laplace operator, we refer the reader to [8] and the literature mentioned there.

We conclude this section with a nonnegativity property of solutions to a class of space-time-fractional partial differential equations. More precisely, let X be a Hilbert space over $\mathbb{R}$. For $0 < \alpha, \beta < 1$, we consider the following initial-value problem for an abstract evolution equation in X:

$$D_t^\alpha u(t) = -(-A)^\beta u \quad \text{in } X, t > 0, \tag{100}$$

$$u(0) = a \in X. \tag{101}$$

Here, for simplicity, we assume that the operator A is self-adjoint, has a compact resolvent, and $(-\infty, 0] \subset \rho(-A)$, $\rho(-A)$ being the resolvent of $-A$. For example, these conditions are fulfilled for $X = L^2(\Omega)$, $A = \Delta$ and $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$ with a bounded smooth domain $\Omega \subset \mathbb{R}^n$, $n \geq 1$.

We note that the equation (100) implies that $u(\cdot, t) := u(t) \in D((-A)^\beta)$ for $t > 0$, and thus a boundary condition is incorporated into the equation (100). For example, in the case $\Omega \subset \mathbb{R}^n$ is a bounded domain and $A = \Delta$ with $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$, the equation (100) yields $u(t) \in H_0^1(\Omega)$ for $t > 0$ if $\beta > \frac{1}{4}$.

For solutions $u_{\alpha,\beta}(x, t)$ to the initial-boundary-value problem (100)–(101), the following weak nonnegativity property is valid.

Theorem 37 ([27]). *Let $0 < \alpha, \beta < 1$. If $a \geq 0$ in Ω, then $u_{\alpha,\beta}(\cdot, t) \geq 0$ in Ω for $t \geq 0$.*

The proof of this theorem consists of two steps: (1) derivation of nonnegativity of solutions to the corresponding space-fractional equation ($\alpha = 1$ in (100)), (2) extension of this property to the general case by means of a subordination formula. For details, we refer the readers to [27].

Bibliography

[1] M. Al-Refai, On the fractional derivative at extreme points, *Electron. J. Qual. Theory Differ. Equ.*, **55** (2012), 1–5.

[2] M. Al-Refai and Yu. Luchko, Maximum principles for the fractional diffusion equations with the Riemann–Liouville fractional derivative and their applications, *Fract. Calc. Appl. Anal.*, **17** (2014), 483–498.

[3] M. Al-Refai and Yu. Luchko, Maximum principle for the multi-term time-fractional diffusion equations with the Riemann–Liouville fractional derivatives, *Appl. Math. Comput.*, **257** (2015), 40–51.

[4] M. Al-Refai and Yu. Luchko, Analysis of fractional diffusion equations of distributed order: Maximum principles and their applications, *Analysis*, **36** (2016), 123–133.

[5] A. Alsaedi, B. Ahmad, and M. Kirane, Maximum principle for certain generalized time and space fractional diffusion equations, *Q. Appl. Math.*, **73** (2015), 163–175.

[6] T. M. Atanackovic, S. Pilipovic, and D. Zorica, A diffusion wave equation with two fractional derivatives of different order, *J. Phys. A*, **40** (2007), 5319–5333.
[7] T. M. Atanackovic, S. Pilipovic, and D. Zorica, Time distributed-order diffusion-wave equation. Applications of Laplace and Fourier transformations, *Proc. R. Soc. A*, **465** (2009), 1893–1917.
[8] B. Barrios and M. Medina, Strong maximum principles for fractional elliptic and parabolic problems with mixed boundary conditions, arXiv:1607.01505, Analysis of PDEs (math.AP), 2016.
[9] W. Feller, *An Introduction to Probability Theory and its Applications, Vol. 2*, Wiley, New York, 1966.
[10] R. Gorenflo, Yu. Luchko, and M. Yamamoto, Time-fractional diffusion equation in the fractional Sobolev spaces, *Fract. Calc. Appl. Anal.*, **18** (2015), 799–820.
[11] J. Jia and K. Li, Maximum principles for a time-space fractional diffusion equation, *Appl. Math. Lett.*, **62** (2016), 23–28.
[12] A. N. Kochubei, Fractional-order diffusion, *Differ. Equ.*, **26** (1990), 485–492.
[13] A. N. Kochubei, General fractional calculus, evolution equations, and renewal processes, *Integral Equ. Oper. Theory*, **71** (2011), 583–600.
[14] A. Kubica and M. Yamamoto, Initial-boundary value problems for fractional diffusion equations with time-dependent coefficients, *Fract. Calc. Appl. Anal.*, **21** (2018), 276–311.
[15] Y. Liu, Strong maximum principle for multi-term time-fractional diffusion equations and its application to an inverse source problem, *Comput. Math. Appl.*, **73** (2017), 96–108.
[16] Y. Liu, W. Rundell, and M. Yamamoto, Strong maximum principle for fractional diffusion equations and an application to an inverse source problem, *Fract. Calc. Appl. Anal.*, **19** (2016), 888–906.
[17] Z. Liu, Sh. Zeng, and Y. Bai, Maximum principles for multi-term space-time variable-order fractional diffusion equations and their applications, *Fract. Calc. Appl. Anal.*, **19** (2016), 188–211.
[18] Yu. Luchko, Operational method in fractional calculus, *Fract. Calc. Appl. Anal.*, **2** (1999), 463–489.
[19] Yu. Luchko, Maximum principle for the generalized time-fractional diffusion equation, *J. Math. Anal. Appl.*, **351** (2009), 218–223.
[20] Yu. Luchko, Boundary value problems for the generalized time-fractional diffusion equation of distributed order, *Fract. Calc. Appl. Anal.*, **12** (2009), 409–422.
[21] Yu. Luchko, Some uniqueness and existence results for the initial-boundary value problems for the generalized time-fractional diffusion equation, *Comput. Math. Appl.*, **59** (2010), 1766–1772.
[22] Yu. Luchko, Initial-boundary-value problems for the generalized multi-term time-fractional diffusion equation, *J. Math. Anal. Appl.*, **374** (2011), 538–548.
[23] Yu. Luchko, Maximum principle and its application for the time-fractional diffusion equations, *Fract. Calc. Appl. Anal.*, **14** (2011), 110–124.
[24] Yu. Luchko, Initial-boundary-value problems for the one-dimensional time-fractional diffusion equation, *Fract. Calc. Appl. Anal.*, **15** (2012), 141–160.
[25] Yu. Luchko and M. Yamamoto, General time-fractional diffusion equation: Some uniqueness and existence results for the initial-boundary-value problems, *Fract. Calc. Appl. Anal.*, **19** (2016), 676–695.
[26] Yu. Luchko and M. Yamamoto, On the maximum principle for a time-fractional diffusion equation, *Fract. Calc. Appl. Anal.*, **20** (2017), 1131–1145.
[27] Yu. Luchko and M. Yamamoto, A survey on the recent results regarding maximum principles for the time-fractional diffusion equations, in S. Bhalekar (ed.) *Frontiers in Fractional Calculus*, pp. 33–69, Bentham Science Publishers, Sharjah, 2018.

[28] M. H. Protter and H. F. Weinberger, *Maximum Principles in Differential Equations*, Springer, Berlin, 1999.
[29] P. Pucci and J. B. Serrin, *The Maximum Principle*, Birkhäuser, Basel, 2007.
[30] K. Sakamoto and M. Yamamoto, Initial value/boundary value problems for fractional diffusion-wave equations and applications to some inverse problems, *J. Math. Anal. Appl.*, **382** (2011), 426–447.
[31] V. S. Vladimirov, *Equations of Mathematical Physics*, Nauka, Moscow, 1971.
[32] H. Ye, F. Liu, V. Anh, and I. Turner, Maximum principle and numerical method for the multi-term time-space Riesz–Caputo fractional differential equations, *Appl. Math. Comput.*, **227** (2014), 531–540.
[33] R. Zacher, Boundedness of weak solutions to evolutionary partial integro-differential equations with discontinuous coefficients, *J. Math. Anal. Appl.*, **348** (2008), 137–149.
[34] R. Zacher, Weak solutions of abstract evolutionary integro-differential equations in Hilbert spaces, *Funkc. Ekvacioj*, **52** (2009), 1–18.
[35] R. Zacher, A weak Harnack inequality for fractional evolution equations with discontinuous coefficients, *Ann. Sc. Norm. Super. Pisa, Cl. Sci.*, **12** (2013), 903–940.
[36] R. Zacher, Time fractional diffusion equations: solutions concepts, regularity and long-time behaviour, in *Handbook of Fractional Calculus with Applications*, vol. 2, 2019.

Teodor M. Atanacković, Sanja Konjik, and Stevan Pilipović

Wave equation involving fractional derivatives of real and complex fractional order

Abstract: Waves in nonlocal and viscoelastic media are investigated through the constitutive equations containing fractional derivatives of real order in the time and space variables. The distributional framework of the transferred equation enable us to find out distributional solutions for which we show the continuity. The classical solutions can be obtained in special cases which are not considered. The central part of the paper is the Zener model, for a generalized wave equation with the real and complex fractional derivatives. It is formulated so to satisfy dissipation inequality implying thermodynamical restrictions. Moreover, with suitable additional assumptions and initial data, we solve $u_{tt}(x,t) = L_c(t) * u_{xx}(x,t)$, $t > 0$, $x \in (0,\infty)$, where L_c comes from the Zener model, in the frame of distributions. The last part of the paper is an overview of nonlocal Zener-type models.

Keywords: Wave equation, fractional derivatives of real and complex order, constitutive equations, thermodynamical restrictions

MSC 2010: 26A33, 74D05, 35R11

1 Introduction

The classical wave equation has been generalized in the framework of fractional calculus in several ways. Rather popular is the direct fractionalization of the wave equation. In this approach, the second derivative with respect to time is changed by a real order Riemann–Liouville fractional derivative of the order $\alpha \in (0,2)$. This leads, in the case of one-dimensional wave equation, to

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \lambda \frac{\partial^2 u(x,t)}{\partial x^2}, \quad 0 < \alpha < 2, \tag{1}$$

with corresponding initial and boundary conditions. More often, the Caputo fractional derivative with respect to $t > 0$ (for any x in a suitable domain) is considered in (1) un-

Acknowledgement: This work is supported by Projects 174005 and 174024 of the Serbian Ministry of Science and Project 142-451-2384 of the Provincial Secretariat for Higher Education and Scientific Research.

Teodor M. Atanacković, Faculty of Technical Sciences, University of Novi Sad, Trg D. Obradovića 6, 21000 Novi Sad, Serbia, e-mail: atanackovic@uns.ac.rs
Sanja Konjik, Stevan Pilipović, Faculty of Sciences, University of Novi Sad, Trg D. Obradovića 4, 21000 Novi Sad, Serbia, e-mails: sanja.konjik@dmi.uns.ac.rs, stevan.pilipovic@dmi.uns.ac.rs

https://doi.org/10.1515/9783110571660-015

der the appropriate assumptions for u. There is a huge literature related to equation (1). We refer to [31, 42], and references given therein as well as to [32, 33, 45] for higher order dimensional problems. The time partial derivative can also be replaced by either two fractional derivatives of different order as in [10], or by a distributed order fractional derivative, that is, the integral over a given domain for the order α of fractional derivatives; we refer to [38, 46] for the domain $\alpha \in (0, 1)$, and to [11, 12] for the domain $\alpha \in [0, 2]$. All the quoted authors noted that the general form of (1) gives possibilities for the unified treatment of the wave and the diffusion equation since both equations can be obtained as special cases of it.

Contrary to the direct approach, our first goal in this paper is to present in Section 2 a fractionalization of the wave equation so that the Newton equations of motion remain valid. Therefore, the changes that we will propose will be done in the constitutive equation and in the strain measure. We will discuss the transition from the classical wave equation to the fractional type wave equation, through the Zener model for a one-dimensional body. We note that the fractional Zener model was introduced by M. Caputo and F. Mainardi [15, 16] although it was also considered in [47] but without the use of the fractional calculus. Waves in fractional type viscoelastic materials are studied in [7, 19, 39, 40, 44]. Similar problems, related to the stress waves in viscoelastic media, were investigated in [30] by means of the inversion formula for the Laplace transform.

We present in Section 3 the generalization of Hooke's law in order to include the nonlocal effects (cf. our paper [13]). The solution of the problem is given and some of its properties are examined. The analysis is presented for a spatially one dimensional body (rod). Next, in our main Section 4, we analyze the Zener-type generalization of the constitutive equation for a viscoelastic body following our paper [3]. Our generalization is done in a way that the classical strain was used and that the first-order time derivative is replaced by a combination of fractional derivatives of real and complex order. This leads to a locally integrable function L_c describing a constitutive connection between stress and strain and plays the main role in the equation $u_{tt}(x, t) = L_c(t)u_{xx}(x, t)$ under the consideration. The restrictions on the coefficients that follow from the dissipation condition as well as an additional condition on coefficients enable us to solve the corresponding problem in the frame of distributions. Finally, in the last Section 5, we give an overview concerning the nonlocal Zener type models. We formulate relevant equations for such models and present results of our paper [5] where we have given the existence and the properties of solutions of such equations. The results presented in Section 3 for complex order Zener model can be used for the generalization of results from Section 5. Thus, an interesting problem would be a study of waves in nonlocal Zener-type viscoelastic body, as proposed in Section 5, with real and complex order derivatives, as proposed in Section 4.

2 Wave equation with fractional derivatives

2.1 Introductory remarks

In order to show the intrinsic point of our transfer to fractional derivatives, we present the classical derivation of the wave equation for a linearly elastic three-dimensional body. Three types of equations will be presented. The first one is the equation of motion and it reads

$$\partial_{x_j}\sigma_{ij}(x,t) = \rho\,\partial_t^2 u_i(x,t), \quad x_i \in (a_i,b_i), t > 0, i,j \in \{1,2,3\}, \tag{2}$$

where $x = (x_1,x_2,x_3)$, σ_{ij} is the Cauchy stress tensor, ρ is the density of a material, while x_i are the spatial coordinates and u_i are components of the displacement vector; t is time. The second one is the constitutive equation connecting stress and deformation measure (strain). For linearly elastic body, we have (see [2])

$$\sigma_{ij}(x,t) = \lambda\theta(x,t)\,\delta_{ij} + 2\mu\,\varepsilon_{ij}(x,t), \quad x_i \in (a_i,b_i), t > 0, i,j \in \{1,2,3\},$$

where $\theta = \operatorname{tr}\varepsilon = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}$, λ and μ are Lamé constants, while δ_{ij} is the Kronecker delta ($\delta_{ij} = 0$ if $i \neq j$, and $\delta_{ij} = 1$ if $i = j$), and ε_{ij} are components of the (linear) strain tensor. Also, $a_i, b_i \in \mathbb{R}_+$ with $a_i \leq b_i$. The third group of equations represents the geometrical conditions, that is, connections between the displacement vector and the strain tensor. In the linear version, they are of the form

$$\varepsilon_{ij}(x,t) = \frac{1}{2}[\partial_{x_j}u_i(x,t) + \partial_{x_i}u_j(x,t)], \quad x_i \in (a_i,b_i), t > 0, i,j \in \{1,2,3\}.$$

Let us consider the one-dimensional case on $(a,b) \times (0,\infty)$, so, $\sigma_{11} = \sigma \neq 0$, $\sigma_{22} = \sigma_{33} = 0$. Then we have

$$\begin{aligned} \partial_x\sigma(x,t) &= \rho\,\partial_t^2 u(x,t),\\ \sigma(x,t) &= \lambda\theta(x,t)\,\varepsilon_{11}(x,t) + 2\mu\,\varepsilon_{11}(x,t) = E\varepsilon_{11},\\ \varepsilon_{11}(x,t) &= \partial_x u(x,t), \quad x \in (a,b), t > 0, \end{aligned} \tag{3}$$

where E is the modulus of elasticity $E = \frac{\mu(3\lambda+2\mu)}{\lambda+\mu}$. Combining equations in system (2)–(3), we obtain the classical wave equation

$$c^2\partial_x^2 u(x,t) = \partial_t^2 u(x,t), \quad c = \sqrt{\frac{E}{\rho}}, x \in (a,b), t > 0. \tag{4}$$

If $(a,b) = \mathbb{R}$, then

$$u(x,t) = c_1\delta(x-ct) + c_2\delta(x+ct), \quad x \in \mathbb{R}, t > 0,$$

where c_1 and c_2 are arbitrary constants and δ is the Dirac delta distribution (see next subsection). If we consider u in $[0,\infty)\times(0,\infty)$ and $u(x,0)=c_1\delta(x)$, then the solution of (4) is

$$u(x,t)=c_1\delta(x-ct). \tag{5}$$

Finally, we point out that the generalizations of the wave equation, that we are considering, are consequences of the generalization of the constitutive equation $(3)_2$ and generalization of the strain measure $(3)_3$.

2.2 Preliminaries from the distribution theory

One of the main characterizations of our approach to the fractional calculus is the distributional framework, which enables us to use a strong theory of Schwartz- type spaces. We will present in this subsection some of the basic notions of this theory which will be used throughout the paper. We recall the definition of the left Riemann–Liouville fractional derivative operator of order $\alpha\in(0,1)$:

$${}_0D_t^{\alpha}u(t)=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t\frac{u(\tau)}{(t-\tau)^{\alpha}}\,d\tau,\qquad t\geq 0.$$

Usually, we assume that u is an absolutely continuous function in $[0,a]$, $u\in AC([0,a])$ for any $a>0$, which means that u is continuous and have the first derivative u' belonging to the space of integrable functions on $[0,a]$ for any $a>0$, that is, $u'\in L^1_{\mathrm{loc}}([0,\infty))$.

We refer to [49, 52] for the Schwartz test spaces of smooth, compactly supported functions $\mathcal{D}(\mathbb{R}^n)$ and of smooth rapidly decreasing functions $\mathcal{S}(\mathbb{R}^n)$ and their duals, Schwartz distribution spaces $\mathcal{D}'(\mathbb{R}^n)$ and the space of tempered distributions $\mathcal{S}'(\mathbb{R}^n)$; $\mathcal{S}'(\mathbb{R}^n)\subset\mathcal{D}'(\mathbb{R}^n)$. A closed set $Z\subset\mathbb{R}^n$ is the support of $u\in\mathcal{S}'(\mathbb{R}^n)$ if $\langle u,\varphi\rangle=0$, for any rapidly decreasing function φ different from zero in $\mathbb{R}^n\setminus Z$. Then we write $\operatorname{supp}u=Z$. In the same way, one defines the support of $u\in\mathcal{D}'(\mathbb{R}^n)$. In case $n=1$, if $\operatorname{supp}u\subset[0,\infty)$ we say that $u\in\mathcal{S}'_+(\mathbb{R})$ or $\mathcal{D}'_+(\mathbb{R})$. Usually, we will skip $\mathbb{R}$ and write simply $\mathcal{D}'$, $\mathcal{S}'$, $\mathcal{D}'_+$, $\mathcal{S}'_+$ in all respective cases. We will consider functions and distributions depending on two variables, $u=u(x,t)$. Then $\mathcal{S}'(\mathbb{R}\times\mathbb{R}_+)$ is the space of all distributions $u\in\mathcal{S}'(\mathbb{R}^2)$, which vanish for $t<0$, that is, $\langle u(x,t),\theta(x)\varphi(t)\rangle=0$ for all $\theta,\varphi\in\mathcal{S}(\mathbb{R})$, so that $\varphi=0$ in $[0,\infty)$ (φ is supported by $(-\infty,0)$). The convolution of two locally integrable functions f and g is defined by $f*g(x)=\int_{\mathbb{R}^n}f(x-t)g(t)\,dt$ for all $x\in\mathbb{R}^n$ for which this integral exists. If f and g are locally integrable on $\mathbb{R}$, equal zero on $(-\infty,0)$, then their convolution exists and it is a locally integrable function on $\mathbb{R}$ and equals zero in $(-\infty,0]$. Moreover, the convolution exists in the sense of distributions if $f,g\in\mathcal{D}'_+$. Then $f*g\in\mathcal{D}'_+$ (or in $\mathcal{S}'_+$, if $f,g\in\mathcal{S}'_+$). If one of f or g (say g) is compactly supported, then their convolution exists, as well. It belongs to that space to which f belongs, to $\mathcal{D}'$ or $\mathcal{S}'$.

If f is a locally integrable function of polynomial growth, that is, $|f(t)| \le p(t)$, for almost all $t \in \mathbb{R}$, for some positive polynomial p, then f defines a regular tempered distribution, denoted again by f, as

$$\langle f(t), \varphi(t)\rangle = \int_{\mathbb{R}} f(t)\varphi(t)dt, \quad \varphi \in \mathcal{S}.$$

In the case of $\mathcal{D}'$, if f is a locally integrable function on $\mathbb{R}$, then again by this integral and $\varphi \in \mathcal{D}$, one defines a regular distribution in $\mathcal{D}'$.

In the distributional setting, one introduces a family $\{f_\alpha\}_{\alpha\in\mathbb{R}} \in \mathcal{S}'_+ \subset \mathcal{D}'_+$ as

$$f_\alpha(t) = \begin{cases} H(t)\frac{t^{\alpha-1}}{\Gamma(\alpha)}, & \alpha > 0, \\ \frac{d^N}{dt^N} f_{\alpha+N}(t), & \alpha \le 0, \alpha + N > 0, N \in \mathbb{N}, t \in \mathbb{R}, \end{cases}$$

where H is the Heaviside function (it is the characteristic function of $[0,\infty)$). Then $f_\alpha *$ is a convolution operator acting on $\mathcal{D}'_+$ (also, $f_\alpha * : \mathcal{S}'_+ \to \mathcal{S}'_+$). For $\alpha > 0$, $f_{-\alpha} *$ is called the operator of fractional differentiation. Moreover, for $y \in AC([0,a])$ it coincides with the left Riemann–Liouville fractional derivative, that is, ${}_0D_t^\alpha y = f_{-\alpha} * y$, $\alpha > 0$. Recall that for $y \in \mathcal{S}'$ the Fourier transform is defined as

$$\langle \mathcal{F}y, \varphi\rangle = \langle y, \mathcal{F}\varphi\rangle, \quad \varphi \in \mathcal{S},$$

where for $\varphi \in \mathcal{S}$, $\mathcal{F}\varphi(\xi) = \hat{\varphi}(\xi) = \int_{-\infty}^{\infty} \varphi(x)e^{-i\xi x}\,dx$, $\xi \in \mathbb{R}$. Note that $\mathcal{F}$ is an isomorphism of $\mathcal{S}$ onto $\mathcal{S}$.

Let $y \in \mathcal{D}'_+(\mathbb{R})$ such that $e^{-\xi t}y \in \mathcal{S}'(\mathbb{R})$, for all $\xi > a > 0$. Then the Laplace transform of y is defined by

$$\mathcal{L}y(s) = \tilde{y}(s) = \mathcal{F}(e^{-\xi t}y)(\eta), s = \xi + i\eta, \quad \xi > a, \eta \in \mathbb{R}.$$

The set $S = \{s = \xi + i\eta, \xi > a, \eta \in \mathbb{R}\}$ is its range of convergence; $\mathcal{L}y$ is an analytic function in S. For tempered distributions in $\mathcal{S}'_+$, one has $a = 0$. In order to introduce the inverse Laplace transform, let $Y = \mathcal{L}y$, as above, that is, $e^{-\xi t}y \in \mathcal{S}'(\mathbb{R})$, with the range of convergence S. Then its inverse is y given by

$$y(t) = \mathcal{L}^{-1}Y(t) = \lim_{p\to\infty} \frac{1}{2\pi i}\int_{b-ip}^{b+ip} Y(s)e^{st}\,ds, \quad \text{with } b > a,$$

in the sense of distributions.

We use in the sequel the notation $\mathbb{C}_+ = \{s = \xi + i\eta \in \mathbb{C} : \xi > 0, \eta \in \mathbb{R}\}$.

Recall that $H'(t) = \delta(t)$, $H''(t) = \delta'(t)$, $t \in \mathbb{R}$, where δ is the Dirac delta distribution concentrated at zero.

Consider a function $u \in C^2([0,\infty))$ (which means that all the derivatives up to order 2 of u are continuous on $[0,\infty)$). It is a regular distribution in $\mathcal{D}'_+$. Then $H(t)u(t)$, $t \in \mathbb{R}$, is an element of $\mathcal{D}'(\mathbb{R})$ and satisfies, in the sense of distributions,

$$\big(H(t)u(t)\big)'' = H(t)u''(t) + u(0)\delta'(t) + u'(0)\delta(t).$$

We will use this identity in Subsection 3.1 when we transfer our classical problem into the distributional setting.

3 Wave equation for nonlocal materials

Our aim in this section is to formulate a generalized one-dimensional wave equation for nonlocal materials which involves fractional derivatives both in $t \geq 0$ and $x \in [a,l]$. Then in Subsection 3.1, we consider $l = \infty$ and transfer the classical problem to a distributional one. Even in the distributional setting we have to assume additional properties on the initial data which imply the distributional solution being continuous in t (cf. Subsection 3.2). Our exposition is based on [13].

Our model deals with a one-dimensional body, namely a rod lying on the x axis of our reference frame. In our analysis, this rod is a part of the x axes $x \in [a,l]$ with $a < l \leq \infty$. We need the definitions of the left ${}^C D^\alpha_{a+}y$, and right, ${}^C D^\alpha_{l-}y$, Caputo fractional derivatives, $\alpha \in (0,1)$, written as regularized Riemann–Liouville derivative

$$ {}^C D^\alpha_{a+}y(x) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_a^x \frac{y(\tau)-y(a)}{(x-\tau)^\alpha}d\tau, \quad x \in [a,l], \tag{6}$$

and

$$ {}^C D^\alpha_{l-}y(x) = -\frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_x^l \frac{y(\tau)-y(l)}{(\tau-x)^\alpha}d\tau, \quad x \in [a,l], \tag{7}$$

respectively. Note that we assume the absolute continuity of y in (6) and (7) over $[a,l]$, and we perform differentiation with respect to the variable x. We recall the definition of the linearized strain tensor,

$$\varepsilon(x,t) = \partial_x u(x,t), \quad x \in [a,l], t > 0, \tag{8}$$

with u being the single component of the displacement vector and x being oriented along the rod axis. Here, we use a generalization of (8) defined as

$$\mathcal{E}^\alpha(x,t) = \frac{1}{2}\big[{}^C D^\alpha_{a+}u(x,t) - {}^C D^\alpha_{l-}u(x,t)\big], \quad x \in [a,l], t \geq 0, \tag{9}$$

where $\alpha \in (0,1)$ and derivatives are taken with respect to x. In this considerations, we assume that $u(\cdot,t)$ is absolutely continuous on $[a,l]$ for every $t \geq 0$, which means that $\partial_x u(\cdot,t) \in L^1([a,l])$ for every $t \geq 0$. Then, for $x \in [a,l]$, $t \geq 0$,

$$\mathcal{E}^\alpha(x,t) = \frac{1}{2}\frac{1}{\Gamma(1-\alpha)}\left[\int_a^x (x-\xi)^{-\alpha}\partial_\xi u(\xi,t)\,d\xi + \int_x^l (\xi-x)^{-\alpha}\partial_\xi u(\xi,t)\,d\xi\right].$$

Note that the strain measure $\mathcal{E}^\alpha$ is nonlocal in the sense that it takes into account the weighted gradient of the deformation, presented as $\partial_x u(x,t)$, at all points of a body. Also

$$\mathcal{E}^1(x,t) = \varepsilon_{11}(x,t), \quad \mathcal{E}^0(x,t) = 0, \quad x \in [a,l], t \geq 0,$$

see (8). Suppose now that $u(x,t) = c(t)$, with $c(t)$, $t \geq 0$, being an arbitrary function (not depending on x). Clearly, in this case

$$\mathcal{E}^\alpha(x,t) = 0, \quad x \in [a,l], t \geq 0.$$

In other words we see that the translatory motion of a rod, that is, $u(x,t) = c(t)$, with $c(t)$, $t \geq 0$, being arbitrary, leads to zero strain.

Next, we examine the solution to the equation $\mathcal{E}^\alpha(x,t) = 0$, $x \in (a,l)$, $t \geq 0$. This leads to

$$\frac{1}{2}\left[{}^C D^\alpha_{a+} u(x,t) - {}^C D^\alpha_{l-} u(x,t)\right] = 0. \tag{10}$$

Since $u(\cdot,t)$ is absolutely continuous on $[a,l]$, for every fixed $t \geq 0$, equation (10) leads to

$$\int_a^x (x-\xi)^{-\alpha}\partial_\xi u(\xi,t)d\xi + \int_x^l (\xi-x)^{-\alpha}\partial_\xi u(\xi,t)d\xi = 0, \tag{11}$$

or

$$\int_a^l |x-\xi|^{-\alpha}(\partial_\xi u(\xi,t))d\xi = 0. \tag{12}$$

We prove now the following lemma from [13] adding some details.

Lemma 3.1. *With the quoted assumption on u, the only solution to* (11) *for $x \in [a,l]$, $t \geq 0$, is $u(x,t) = c(t)$, with an arbitrary function c.*

Proof. Let U be defined as

$$U(x,t) = \begin{cases} \partial_x u(x,t), & a < x < l \\ 0, & x < a \text{ and } x > l, \end{cases} \qquad t \geq 0,$$

that is, U extends u out of $[a,l]$ by zero, for every $t \geq 0$. Then, for $t \geq 0$,

$$\int_a^l |x-\xi|^{-\alpha}\partial_\xi u(\xi,t)\,d\xi = \int_{-\infty}^{+\infty} |x-\xi|^{-\alpha}U(\xi,t)\,d\xi = \big(|\xi|^{-\alpha} * U\big)(x,t), \quad x \in \mathbb{R},$$

where $*$ is the convolution which exists since $U(\cdot,t)$ is compactly supported and $\xi \mapsto |\xi|^{-\alpha}$ is locally integrable on $\mathbb{R}$, and thus, belongs to $\mathcal{S}'(\mathbb{R})$. We rewrite equation (12) in the form

$$\big(|\xi|^{-\alpha} * U(\xi,t)\big)(x,t) = 0, \quad x \in \mathbb{R}, t \geq 0,$$

and apply the Fourier transform. Since

$$\mathcal{F}\big(|\xi|^{-\alpha}\big)(\omega) = 2\Gamma(1-\alpha)\sin\left(\frac{\alpha\pi}{2}\right)|\omega|^{\alpha-1}, \quad \omega \in \mathbb{R},$$

we have

$$\mathcal{F}\big([|\xi|^{-\alpha}] * [U(\xi,t)]\big)(\omega) = h|\omega|^{\alpha-1}\mathcal{F}\big([U(\cdot,t)]\big)(\omega) = 0, \quad t > 0.$$

We conclude that $\mathcal{F}(u) = 0$. Thus $U = 0$ on $\mathbb{R}$, for any $t \geq 0$. This implies that $U(x,t) = 0$, for $x \in [a,l]$, $t \geq 0$. Now we have $\partial_x u(x,t) = 0$ in $[a,l]$ it follows that u does not depend on $x \in [a,l]$, and we obtain the proof of the assertion. □

Note that the expression similar to (9) was used in [37], where it was called symmetric fractional derivative. In [41], the authors introduces

$$\frac{\partial u(x,t)}{\partial x}\left({}^C D^\alpha_{a+}\frac{\partial u(x,t)}{\partial x} + {}^C D^\alpha_{l-}\frac{\partial u(x,t)}{\partial x}\right), \quad x \in [a,l], t > 0,$$

into the strain energy function. This term gives the possibility for the modeling of a phases coexistence in solids. We refer to [41] for further results in this domain.

3.1 Case $l = \infty$

We now consider a rod laying in $[a,\infty)$. We extend a function u defined in $[a,\infty) \times [0,\infty)$ with $u(x,t) = 0$, $x < a$, $t > 0$. In this way we consider a function or embedded distribution in $\mathcal{S}'(\mathbb{R})$ supported by $[a,\infty]$ in x for any fixed $t > 0$. Moreover, we assume $u(a,t) = u(\infty,t) = 0$, $t > 0$.

With the deformation measure (9), we propose the linear stress-deformation measure, in the form of generalized Hooke's law, and obtain

$$\sigma(x,t) = E_0\,\mathcal{E}^\alpha(x,t), \quad x \in [a,\infty), t > 0, \alpha \in (0,1), \tag{13}$$

where σ is the stress and E_0 is a constant called generalized modulus of elasticity. With (13), equation of motion $\partial_x\sigma(x,t) = \rho\,\partial_t^2 u(x,t)$ becomes (on $(a,\infty)\times\mathbb{R}_+$),

$$\rho\,\partial_t^2 u(x,t) = E_0\,\partial_x\big(\mathcal{E}^\alpha(x,t)\big) = \frac{1}{2}E_0\,\partial_x\big({}^C D_{a+}^\alpha u(t,x) - {}^C D_{l-}^\alpha u(t,x)\big), \tag{14}$$

where ρ is the density of the material. Recall that for $x\in[a,\infty)$, $t\geq 0$,

$${}^C D_{a+}^\alpha u(x,t) = D_{a+}^\alpha\big(u - u(a,t)\big)(x,t), \quad x\in[a,\infty), t\geq 0,$$

and

$${}^C D_{\infty-}^\alpha u(x,t) = D_{l-}^\alpha\big(u - u(\infty,t)\big)(x,t), \quad x\in[a,\infty), t\geq 0.$$

Since $u(a,t) = u(\infty,t) = 0$, $t\geq 0$, we have

$${}^C D_{a+}^\alpha u(x,t) = D_{a+}^\alpha u(x,t) \quad\text{and}\quad {}^C D_{\infty-}^\alpha u(x,t) = D_{\infty-}^\alpha u(x,t).$$

and use notation $D_{a+} = D_+$ and $D_{\infty-} = D_-$. Equation (14), in the present case of a semi infinite rod becomes

$$\partial_t^2 u(x,t) = \frac{1}{2}B\,\partial_x\big(D_+^\alpha u(t,x) - D_-^\alpha u(t,x)\big), \quad x\in(a,\infty), t>0, \tag{15}$$

where $B = \frac{1}{\rho}E_0$. We prescribe to (15) the initial conditions

$$u(x,0) = C_1(x), \quad \frac{\partial u}{\partial t}(x,0) = C_2(x), \tag{16}$$

where we assume that C_i, $i = 1,2$, are continuous functions on $[a,\infty)$. Note also that ${}^C D_+^\alpha u(x,t) - {}^C D_-^\alpha u(x,t) = \frac{\partial}{\partial x}R(u)(x,t)$, where

$$R(u)(x,t) = \frac{1}{\Gamma(1-\alpha)}\int_{-\infty}^{\infty}\frac{u(\xi,t)}{|x-\xi|^\alpha}d\xi, \quad x\in(a,\infty), t>0,$$

is the Riesz potential. Since $D_+^1 u(x,t) = \partial_x u(x,t)$, $D_-^1 u(x,t) = -\partial_x u(x,t)$, equation (15) for $\alpha = 1$ becomes the classical wave equation

$$\partial_t^2 u(x,t) = B\,\partial_x^2 u(x,t).$$

Another special case corresponds to $\alpha = 0$. From (15), it follows

$$D_+^0 u(x,t) = u(x,t), \quad D_-^0 u(x,t) = u(x,t)$$

and equation (15) becomes

$$\partial_t^2 u(x,t) = 0. \tag{17}$$

Equation (17) shows that there is no interaction between the material points of the rod. It is clear from (13) that fractional model used here takes into account nonlocal action, since the displacements of all points $u(x,t)$, $a<x<\infty$, influence the stress at the point x.

3.2 Solutions of the generalized Cauchy problem (15), (16)

We transfer equations (15), (16) into the distributional form as follows. If u would be a classical solution, that is, u is of the C^2-class in $t \in (0,\infty)$ for any x, C^1-class in $t \in [0,\infty)$ for any x, and of AC^1-class with respect to $x \in [0,M]$ for any t and every $M > 0$, we can perform the same calculation as in the end of Subsection 2.2. Multiplying $u(x,t)$ with the Heaviside function $H(t)$, we can calculate the second derivative with respect to t. Actually, we did it in the end of Subsection 2.2. Inserting $\partial_t^2(H(t)u(x,t))$ into equation (15) we obtain the form of this equation in the setting of distributions. The equation in $\mathcal{S}'(\mathbb{R}_+ \times \mathbb{R})$ which corresponds to (15), (16) becomes

$$D_t^2 u = B\frac{1}{2}D_x(D_+^\alpha - D_-^\alpha)u + C_1(x) \times \delta^{(1)}(t) + C_2(x) \times \delta(t). \tag{18}$$

This is the distributional transfer of 15), (16). From now on, we consider this equation with $0 < \alpha < 1$, $u \in \mathcal{S}'(\mathbb{R} \times \mathbb{R}_+)$, $\operatorname{supp} u \subset \mathbb{R} \times [0,\infty)$, and $C_1, C_2 \in \mathcal{S}'_+(\mathbb{R})$. We note that we return in (18) to notation $u(x,t)$ instead of $H(t)u(x,t)$.

Remark 3.2. We will solve (18) in the framework of tempered distributions since we are not able to find out whether this distributional solution satisfies the quoted regularity properties in order to be the classical solution. Our aim is to show that $u(x,\cdot)$ is continuous on $[0,\infty)$, for every $x \in \mathbb{R}$. ($t \mapsto \langle u(t,x), \varphi(x)\rangle$ is continuous.)

In order to solve (18), we assume additionally that

$$C_k, \quad k = 1,2, \text{ are continuous (cf. (16)) and of polynomial growth.} \tag{19}$$

With (19), we have that C_1 and C_2 are tempered distributions. Moreover, for the calculations which are to follow, we also assume that the Fourier transforms

$$\mathcal{F}(C_k)(\omega) = \hat{C}_k(\omega), \quad \omega \in \mathbb{R}, k = 1,2, \text{ are regular at zero,} \tag{20}$$

which means that around zero (i. e., in a neighborhood of zero) they are continuous so that the multiplication with a smooth bounded function in $\mathbb{R} \setminus \{0\}$ and continuous at zero is well-defined in $\mathcal{S}'(\mathbb{R})$.

Applying the Fourier and Laplace transforms to (18), we obtain ($i = \sqrt{-1}$)

$$s^2(\mathcal{L}\mathcal{F}u)(\omega,s) = \frac{B}{2}(-i\omega)\big((-i\omega)^\alpha - (i\omega)^\alpha\big)\mathcal{L}\mathcal{F}u + \hat{C}_1(\omega)s + \hat{C}_2(\omega), \quad \omega \in \mathbb{R}, \operatorname{Re} s > 0.$$

Consequently,

$$\mathcal{L}\mathcal{F}u(\omega,s) = \frac{\hat{C}_1(\omega)s + \hat{C}_2(\omega)}{s^2 - A(-i\omega)^{1+\alpha}}, \quad \omega \in \mathbb{R}, \operatorname{Re} s > 0, \tag{21}$$

where $A = \frac{B}{2}(1 - \exp(\alpha\pi i)\operatorname{sgn}(\omega))$ and $(-i\omega)^{1+\alpha} = |\omega| \exp(\frac{1+\alpha}{2}\pi i)\operatorname{sgn}\omega$. Also

$$A(-i\omega)^{1+\alpha} = -B|\omega|^{1+\alpha}\sin\frac{\alpha\pi}{2}, \quad \omega \in \mathbb{R},$$

where

$$B = \begin{cases} |\omega|^{1+\alpha}(-i)(-\sin\frac{\alpha\pi}{2})(i), & \omega > 0 \\ |\omega|^{1+\alpha}(i)(\sin\frac{\alpha\pi}{2})(i), & \omega < 0. \end{cases}$$

Now, (21) can be written as

$$\mathcal{L}\mathcal{F}u(\omega, s) = \frac{\hat{C}_1(\omega)s + \hat{C}_2(\omega)}{s^2 + |\omega|^{1+\alpha}B\sin\frac{\alpha\pi}{2}}, \quad \omega \in \mathbb{R}, \operatorname{Re} s > 0. \tag{22}$$

The inverse Laplace transform gives

$$\mathcal{F}u(\omega, t) = \hat{C}_1(\omega)[\cos(|\omega|^\gamma at)] + \hat{C}_2(\omega)\left[\frac{\sin(|\omega|^\gamma at)}{a|\omega|^\gamma}\right], \quad \omega \in \mathbb{R}, t \ge 0, \tag{23}$$

where $a^2 = B\sin\frac{\alpha\pi}{2}$ and $\gamma = \frac{1+\alpha}{2}$ (cf. [43, p. 171]). Note that the notation $t \ge 0$ for a distribution means that it is supported by $[0, \infty)$. (Recall that we started with Hu.)

From (20), it follows that the products in (22) and (23) are well-defined in $\mathcal{S}'(\mathbb{R})$. Finally, for $t \ge 0$,

$$u(x, t) = \mathcal{F}^{-1}(\hat{C}_1(\omega)[\cos(|\omega|^\gamma at)])(x) + \mathcal{F}^{-1}\left(C_2(y)\left[\frac{\sin(|\omega|^\gamma at)}{a|\omega|^\gamma}\right]\right)(x), \quad x \in \mathbb{R}. \tag{24}$$

Now u is a solution to (18) if and only if $(-i\omega)^\alpha(\mathcal{F}u)(\omega) \in \mathcal{S}'$. This follows from the fact that the functions $\cos(|\omega|^\gamma at)$ and $\sin(|\omega|^\gamma at)(a|\omega|^\gamma)^{-1}$ on $\mathbb{R}$ are bounded.

Moreover, by the use of distribution theory we have that

$$\langle u(x, t), \varphi(x)\rangle = \left\langle \hat{C}_1(\omega)(\cos(|\omega|)^\gamma at)) + C_2(y)\frac{\sin(|\omega|)^\gamma at)}{a|\omega|^\gamma}, \mathcal{F}^{-1}\varphi \right\rangle, \quad t \ge 0,$$

for every $\varphi \in \mathcal{S}(\mathbb{R})$. We can easily see that $t \mapsto \langle u(x, t), \varphi(x)\rangle$ is continuous on $[0, \infty)$. Therefore, we have proved the following theorem.

Theorem 3.3. *Let* $0 < \alpha < 1$ *and* C_1 *and* C_2 *satisfy* (19) *and* (20). *Then the distribution* u *given by* (24) *is a solution to* (18). *Moreover, for* $\varphi \in \mathcal{S}(\mathbb{R})$ *the function* $t \mapsto \langle u(x, t), \varphi(x)\rangle$, $t \ge 0$, *is continuous.*

Remark 3.4. The classical solution involves much more restrictions on C_1, C_2, and their Fourier transforms.

4 Zener model with complex order fractional derivatives

Our focus in this section are waves in a specific viscoelastic material, that is, a generalized Zener model with real and complex order fractional derivatives. Our exposition follows [3]. The system of equations that corresponds to such a motion is defined for $t > 0$, $x \in [0, l]$, where l is the length of a rod. We also consider the case when $l = \infty$, then $[0, l] = [0, \infty)$.

The system of equations that corresponds to the isothermal motion of a viscoelastic rod that we shall study reads: $t > 0$, $x \in [0, l]$,

$$\begin{aligned}
&\partial_x \sigma(x,t) = \rho\, \partial_t^2 u(x,t),\\
&\sigma(x,t) + a_1\, {}_0D_t^{\alpha}\sigma(x,t) + b_1\, {}_0\bar{D}_t^{\alpha,\beta}\sigma(x,t)\\
&\quad = E\left(\varepsilon(x,t) + a_2\, {}_0D_t^{\alpha}\varepsilon(x,t) + b_2\, {}_0\bar{D}_t^{\alpha,\beta}\varepsilon(x,t)\right),\\
&\varepsilon(x,t) = \partial_x u(x,t), \quad x \in [0,l], t > 0,
\end{aligned} \tag{25}$$

together with the initial conditions

$$u(x,0) = 0, \quad \partial_t u(x,0) = 0, \quad \sigma(x,0) = 0, \quad \varepsilon(x,0) = 0, \tag{26}$$

and boundary conditions

$$u(0,t) = U(t), \quad u(l,t) = 0, \quad \text{if } l < \infty. \tag{27}$$

In the case $l = \infty$ the condition $(27)_2$ is replaced by

$$\lim_{x \to \infty} u(x,t) = 0. \tag{28}$$

As before, here we use u, σ and ε to denote the displacement, stress, and strain, respectively. The spatial coordinate x is oriented along the axis of the rod. Constants $\rho, E, a_1, a_2, b_1, b_2 \in \mathbb{R}_+$ characterize the material; E represents the modulus of elasticity and ρ is the density of the material. The term ${}_0D_t^{\alpha}$ is already introduced, but here it is applied to a function of two variables. This will mean in the sequel that for any $x \in [0, l]$ it is given by

$${}_0D_t^{\alpha}u(x,t) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t \frac{u(x,\tau)}{(t-\tau)^{\alpha}}\, d\tau, \quad t > 0, x \in [0,l], 0 < \alpha < 1,$$

so that this definition has a sense within distribution theory. For example, if $u(x, t)$ is absolutely continuous in t for any fixed $x \in [0, l]$ then this integral defines the fractional derivative. Also in (25) we use the following fractional operator of complex order:

$${}_0\bar{D}_t^{\alpha,\beta} := \frac{1}{2}\left(\hat{b}_1\, {}_0D_t^{\alpha+i\beta} + \hat{b}_2\, {}_0D_t^{\alpha-i\beta}\right), \tag{29}$$

where

$$i = \sqrt{-1}, \quad \hat{b}_1 = T^{i\beta}, \quad \hat{b}_2 = T^{-i\beta}, \quad |\hat{b}_1| = |\hat{b}_2|$$

(T is a constant having the dimension of time). The conditions for the existence of ${}_0\bar{D}_t^{\alpha,\beta}$ are the same as for the case of real fractional derivative.

The form (29) of the symmetrized fractional derivative of complex order was introduced in [4, 8]. Recall that the form of ${}_0\bar{D}_t^{\alpha,\beta}$ is obtained from the real valued compatibility constraints which require that when a fractional derivative of complex order is applied to a real-valued function the resulting output is again real-valued. In the second equation of (25), the coefficients a_1, a_2, b_1, b_2 satisfy restrictions that will be determined in the next section. Those restrictions follow from the second law of thermodynamics. Recall that in the case of the classical wave equation $u_{tt} = c\,u_{xx}$, which describes wave propagation in an elastic media, the corresponding constitutive equation is given by the Hooke law $\sigma = E\,\varepsilon$. The last equation in (25) is the strain measure for the small local deformations. The generalization of the present problem would be if we take the strain measure (9) instead of $(25)_3$.

The system is subjected to the initial conditions (26), from which we see that there is no initial displacement, velocity, stress, and strain, as well as to the boundary conditions, prescribing displacement at the point $x = 0$ and at $x = l$ or infinity, if $l = \infty$.

We introduce the dimensionless parameters. Let

$$\bar{x} = \frac{x}{L}, \quad \bar{t} = \frac{t}{T}, \quad \bar{u} = \frac{u}{L}, \quad \bar{\sigma} = \frac{\sigma}{E}, \quad \bar{a}_i = \frac{a_i}{T^\alpha}, \quad \bar{b}_i = \frac{b_i}{T^\alpha} \quad (i = 1, 2), \quad \bar{U} = \frac{U}{L}, \quad \bar{l} = \frac{l}{L},$$

where $T = (a_2)^{1/\alpha}$, $L = (a_2)^{1/\alpha}\sqrt{\frac{E}{\rho}}$. Note that ε is already dimensionless quantity. By inserting dimensionless quantities (and by dropping the bar sign) into (25), we obtain the same equation with $\rho = 1$, $E = 1$.

Applying formally the Laplace transform with respect to t with values in a domain $A = \{s \in \mathbb{C} : \Re s > a\} \subset \mathbb{C}$,

$$\mathcal{L}[u(x,t)](x,s) = \tilde{u}(x,s) = \int_0^\infty e^{-ts} u(x,t)\,dt, \quad s \in A \subset \mathbb{C},$$

to the constitutive equation $(25)_2$, we obtain

$$(1 + a_1 s^\alpha + b_1 (s^{\alpha+i\beta} + s^{\alpha-i\beta}))\tilde{\sigma}(x,s) = (1 + a_2 s^\alpha + b_2 (s^{\alpha+i\beta} + s^{\alpha-i\beta}))\tilde{\varepsilon}(x,s), \tag{30}$$

from which we express the stress σ (after applying the inverse Laplace transform, that is,

$$\mathcal{L}^{-1}[\tilde{\sigma}(x,s)](x,t) = \frac{1}{2\pi i}\int_{s_0-i\infty}^{s_0+i\infty} e^{ts}\tilde{\sigma}(x,s)\,ds,$$

for some $s_0 > a$ as

$$\sigma(x,t) = \left(\mathcal{L}^{-1}\left[\frac{1 + a_2 s^\alpha + b_2 (s^{\alpha+i\beta} + s^{\alpha-i\beta})}{1 + a_1 s^\alpha + b_1 (s^{\alpha+i\beta} + s^{\alpha-i\beta})}\right] *_t \varepsilon\right)(x,t), \quad x \in [a, l], t > 0,$$

where $*_t$ denotes convolution with respect to t. If we now replace ε from $(25)_3$ into the last equation, and then insert the result into $(25)_1$, we obtain

$$\partial_t^2 u(x,t) = L_c(t) *_t \partial_x^2 u(x,t), \quad x \in [0,l], t > 0, \tag{31}$$

with the initial and boundary conditions

$$\begin{aligned} &u(x,0) = 0, \quad \frac{\partial}{\partial t}u(x,0) = 0, \\ &u(0,t) = U(t), \quad u(l,t) = 0 \quad \Big(\text{or } \lim_{x\to\infty} u(x,t) = 0\Big), \end{aligned} \tag{32}$$

where

$$L_c(t) = \mathcal{L}^{-1}\left[\frac{1 + a_2 s^\alpha + b_2 (s^{\alpha+i\beta} + s^{\alpha-i\beta})}{1 + a_1 s^\alpha + b_1 (s^{\alpha+i\beta} + s^{\alpha-i\beta})}\right](t), \quad t \ge 0, \alpha \in (0,1), \beta > 0. \tag{33}$$

In the sequel, we will analyze problem (31)–(32). Note that it includes several wave equations analyzed earlier; see [9]. For example, if the rod is elastic we have $a_1 = b_1 = a_2 = b_2 = 0$, so that $L_c(t) = \delta(t)$ and we obtain the classical wave equation $\partial_x^2 u(x,t) = \partial_t^2 u(x,t)$. If the rod is described by a fractional Zener model with derivatives of real order, we have $b_1 = b_2 = 0$ so that

$$L_c(t) = \mathcal{L}^{-1}\left[\frac{1 + a_2 s^\alpha}{1 + a_1 s^\alpha}\right](t), \quad t \ge 0,$$

and problem (31)–(32) simplifies. Restrictions on parameters a_1, a_2, b_1, b_2, α, β will be determined in the following section, in such a way that the physical meaning of the problem remains preserved.

Remark 4.1. The function $L_c(t)$, $t \ge 0$, given by (33), is a real valued function. To see this, recall the theorem of Doetsch [27, p. 293, Satz 2], which states that a function L_c is real-valued (almost everywhere) if its Laplace transform is real-valued for all real s in the half-plane of convergence right from some real x_0. Function

$$s \mapsto \frac{1 + a_2 s^\alpha + b_2 (s^{\alpha+i\beta} + s^{\alpha-i\beta})}{1 + a_1 s^\alpha + b_1 (s^{\alpha+i\beta} + s^{\alpha-i\beta})},$$

clearly satisfies this condition (cf. [8]).

4.1 Thermodynamical restrictions

Beside the application of the Laplace transform, we also consider the Fourier transform with respect to time variable t. Consider the constitutive equation $(25)_2$ for $t > 0$,

$x \in \mathbb{R}_+$, $\alpha \in (0,1)$ and $\beta > 0$. As it is done often, we omit the variable x in order to simplify notation. Then

$$\sigma(t) + a_1\, {}_0D_t^{\alpha}\sigma(t) + b_1\, {}_0\bar{D}_t^{\alpha,\beta}\sigma(t) = \varepsilon(t) + a_2\, {}_0D_t^{\alpha}\varepsilon(t) + b_2\, {}_0\bar{D}_t^{\alpha,\beta}\varepsilon(t). \tag{34}$$

Recall that we assumed $a_i, b_i \geq 0$, $i = 1,2$, and follow the method proposed in [14]. Applying the Fourier transform to (34), with respect to t considering $\sigma(t)$ as $\sigma(t)H(t)$ and $\varepsilon(t)$ as $\varepsilon(t)H(t)$, $t \in \mathbb{R}$, where H is the Heaviside function, we obtain the complex modulus of elasticity $\hat{E}(\omega)$ from $\hat{\sigma}(\omega) = \hat{E}(\omega)\hat{\varepsilon}(\omega)$, $\omega \in \mathbb{R}$, as

$$\hat{E}(\omega) = \frac{\hat{P}(\omega)}{\hat{Q}(\omega)} = \frac{\operatorname{Re}\hat{P}(\omega) + i\operatorname{Im}\hat{P}(\omega)}{\operatorname{Re}\hat{Q}(\omega) + i\operatorname{Im}\hat{Q}(\omega)} = E'(\omega) + iE''(\omega), \quad \omega \in \mathbb{R}, \tag{35}$$

where E' and E'' are the loss and the storage modulus. Also, for $\omega \in \mathbb{R}$,

$$\hat{P}(\omega) = 1 + a_2\,(i\omega)^{\alpha} + b_2\,\omega^{\alpha}\big(e^{-\frac{\beta\pi}{2}}e^{i(\frac{\alpha\pi}{2}+\ln\omega^{\beta})} + e^{\frac{\beta\pi}{2}}e^{i(\frac{\alpha\pi}{2}-\ln\omega^{\beta})}\big),$$

and

$$\hat{Q}(\omega) = 1 + a_1\,(i\omega)^{\alpha} + b_1\,\omega^{\alpha}\big(e^{-\frac{\beta\pi}{2}}e^{i(\frac{\alpha\pi}{2}+\ln\omega^{\beta})} + e^{\frac{\beta\pi}{2}}e^{i(\frac{\alpha\pi}{2}-\ln\omega^{\beta})}\big).$$

We consider functions $\hat{P}(\omega)$ and $\hat{Q}(\omega)$ for $\omega > 0$.

The dissipativity condition (the second law of thermodynamics under isothermal conditions) holds if $\operatorname{Re}\hat{E}(\omega) = E'(\omega) \geq 0$ and $\operatorname{Im}\hat{E}(\omega) = E''(\omega) \geq 0$, for $\omega > 0$, see [1, 14]. These conditions are equivalent to

$$\operatorname{Re}\hat{P}(\omega)\operatorname{Re}\hat{Q}(\omega) + \operatorname{Im}\hat{P}(\omega)\operatorname{Im}\hat{Q}(\omega) \geq 0$$

and

$$\operatorname{Im}\hat{P}(\omega)\operatorname{Re}\hat{Q}(\omega) - \operatorname{Re}\hat{P}(\omega)\operatorname{Im}\hat{Q}(\omega) \geq 0,$$

respectively. By the calculation given in [8] and [3] we have that $\operatorname{Re}\hat{E}(\omega) \geq 0$ and $\operatorname{Im}\hat{E}(\omega) \geq 0$ if

$$a_2b_1 - a_1b_2 = 0, \tag{36}$$

$$(a_2 - a_1)\left(1 - 2\frac{b_1}{a_1}\cosh\frac{\beta\pi}{2}\sqrt{1 + \left(\operatorname{ctg}\frac{\alpha\pi}{2}\operatorname{tgh}\frac{\beta\pi}{2}\right)^2}\right) \geq 0, \tag{37}$$

and

$$(a_1 + a_2) \geq 2(b_1 + b_2)\cosh\frac{\beta\pi}{2}\sqrt{1 + \left(\operatorname{tg}\frac{\alpha\pi}{2}\operatorname{tgh}\frac{\beta\pi}{2}\right)^2}. \tag{38}$$

Summing up, we obtain that inequalities (36), (37), and (38) represent the restrictions following from the second law of thermodynamics, that is, they are sufficient conditions for $E'(\omega) > 0$, $E''(\omega) \geq 0$, for $\omega \in (0,\infty)$.

Remark 4.2. Inserting condition (36) into (38) and using the fact that $a_i, b_i > 0$, $i = 1, 2$, we further conclude that the following two relations hold:

$$a_1 \geq 2b_1 \cosh \frac{\beta\pi}{2} \sqrt{1 + \left(\operatorname{tg} \frac{\alpha\pi}{2} \operatorname{tgh} \frac{\beta\pi}{2}\right)^2},$$
$$a_2 \geq 2b_2 \cosh \frac{\beta\pi}{2} \sqrt{1 + \left(\operatorname{tg} \frac{\alpha\pi}{2} \operatorname{tgh} \frac{\beta\pi}{2}\right)^2},$$

which will be used in the next subsection. Moreover, from the condition $\operatorname{Im} \hat{E}(\omega) = E''(\omega) \geq 0$ it follows (see [3], p. 547)

$$a_2 - a_1 > 2(b_2 - b_1) \cosh \frac{\beta\pi}{2} \sqrt{1 + \left(\operatorname{tg} \frac{\alpha\pi}{2} \operatorname{tgh} \frac{\beta\pi}{2}\right)^2}. \tag{39}$$

Thus, this condition and (37) imply the previous two, but the converse does not hold.

The complex modulus $\hat{E}(\omega)$ may be treated as a special value of the complex function $\tilde{E}(s)$ obtained from the Laplace transform of the constitutive equation $\tilde{\sigma}(x, s) = \tilde{E}(s)\tilde{\varepsilon}(x, s)$ (see (30)), calculated for $\operatorname{Re} s = 0$, $\operatorname{Im} s = \omega \in \mathbb{R}_+$. In the sequel, we examine properties of the Laplace transform $\tilde{E}(s)$. Recall that $\tilde{F}(iw) = \hat{F}(\omega)$, $\omega > 0$. Also, from (30) we obtain

$$\tilde{E}(s) = \frac{1 + a_2 s^\alpha + b_2 \left[s^{\alpha+i\beta} + s^{\alpha-i\beta}\right]}{1 + a_1 s^\alpha + b_1 \left[s^{\alpha+i\beta} + s^{\alpha-i\beta}\right]} = \frac{\tilde{P}(s)}{\tilde{Q}(s)} = \operatorname{Re} \tilde{E}(s) + i \operatorname{Im} \tilde{E}(s), \quad s \in \mathbb{C}_+.$$

For $\tilde{E}$, we have the following result; see [3].

Proposition 4.3. *Suppose that the thermodynamical restrictions* (36), (37), *and* (38) *are satisfied. Let us denote* $s = s_0 + ip$, *for* $p \in \mathbb{R}$, *and fixed* $s_0 \geq 0$. *Then* $\operatorname{Re} \tilde{E}(s) > 0$, *for all* $p > 0$. *Also, there exist* $p_0 > 0$, *such that for each* $p > p_0$ *we have* $\operatorname{Im} \tilde{E}(s) \geq 0$.

Here, we present sketch of the proof. As in [3], we show that $\operatorname{Re} \tilde{E}(s) > 0$.
In order to show that $\operatorname{Im} \tilde{E}(s) \geq 0$, we have to show

$$(a_2 - a_1) > 2(b_2 - b_1) \cosh \beta\varphi \sqrt{1 + (\operatorname{ctg} \alpha\varphi \operatorname{tgh} \beta\varphi)^2}. \tag{40}$$

Since $\varphi \to \pi/2$ when $p \to \infty$, and the right-hand side of (40) is a continuous function of φ, we conclude that there is $p_0 > 0$ such that (40) is satisfied and, therefore, $\operatorname{Im} \tilde{E}(s) \geq 0$. The main point is (39).

Remark 4.4. Using the symmetry properties of trigonometric and hyperbolic functions, and the fact that $f(x, -\varphi) = f(x, \varphi)$ and $g(x, -\varphi) = -g(x, \varphi)$, we conclude that $\tilde{E}(\bar{s}) = \overline{\tilde{E}(s)}$. Thus, $\operatorname{Re} \tilde{E}(s) > 0$, for all $s = s_0 - ip$ satisfying $s_0 \geq 0$, $p > 0$. Also, $\operatorname{Im} \tilde{E}(s) \leq 0$, for all $s = s_0 - ip$ satisfying $s_0 \geq 0$, $p > p_0$.

4.2 Solution to (31)–(32)

We return to the initial-boundary-value problem (31)–(32). Note that we assume suitable conditions on coefficients a_i, b_i, $i = 1, 2$, which we will mention in the sequel but stress out condition (39) since it was not explicitly indicated in our paper [3]. Applying formally the Laplace transform to (31), we obtain

$$\partial_x^2 \tilde{u}(x,s) - s^2 M^2(s)\tilde{u}(x,s) = 0, \quad s \in \mathbb{C}_+, \tag{41}$$

where for $s \in \mathbb{C}_+$

$$M^2(s) = \frac{1}{\tilde{E}(s)} = \frac{1 + a_1 s^\alpha + b_1 [s^{\alpha+i\beta} + s^{\alpha-i\beta}]}{1 + a_2 s^\alpha + b_2 [s^{\alpha+i\beta} + s^{\alpha-i\beta}]} =: \frac{\tilde{Q}(s)}{\tilde{P}(s)}, \tag{42}$$

$$\tilde{u}(0,s) = \tilde{U}(s), \quad \hat{u}(l,s) = 0, \tag{43}$$

if l is finite, and

$$\tilde{u}(0,s) = \tilde{U}(s), \quad \lim_{x\to\infty} \tilde{u}(x,s) = 0, \tag{44}$$

if $l = \infty$. We assume that $U \in \mathcal{S}'_+$. We will consider the domain $\{s \in \mathbb{C}_+ : \operatorname{Re} s > 0\}$. Solutions to (41), (43), and (41), (44) are

$$\tilde{u}(x,s) = \tilde{U}(s)\left[\frac{e^{sM(s)x}}{1 - e^{2sM(s)l}} + \frac{e^{-sM(s)x}}{1 - e^{-2sM(s)l}}\right],$$

and

$$\tilde{u}(x,s) = \tilde{U}(s)e^{-sM(s)x},$$

respectively. The following result about M are given in [3].

Proposition 4.5. *M has no singular points with positive real part.*

The proof of the Proposition 4.5 is based on the standard application of the argument principle, and it is omitted.

Corollary 4.6. *Functions $1 - e^{2sM(s)l}$ and $1 - e^{-2sM(s)l}$, $s \in \mathbb{C}_+$, have no zeros.*

Remark 4.7. Since by (35) and (42),

$$M^2(i\omega) = \frac{1}{\hat{E}(\omega)} = \frac{E'(\omega) - iE''(\omega)}{|\hat{E}(\omega)|^2}, \quad \omega \in \mathbb{R},$$

we conclude that $\operatorname{Re} M^2(i\omega) > 0$ and $\operatorname{Im} M^2(i\omega) \le 0$ for $\omega > 0$ if the thermodynamical restrictions (36), (37), and (38) are satisfied, and hence $\operatorname{Re} M(i\omega) > 0$ and $\operatorname{Im} M(i\omega) \le 0$.

Also, from Remark 4.4, it follows $\operatorname{Re} M(-i\omega) > 0$ and $\operatorname{Im} M(-i\omega) \geq 0$ for $\omega > 0$. Similarly, from

$$M^2(s) = \frac{1}{\tilde{E}(s)} = \frac{E'(s) - iE''(s)}{|\tilde{E}(s)|^2}, \quad s \in \mathbb{C}_+,$$

and Proposition 4.3, we conclude that $\operatorname{Re} M^2(s) > 0$, for all $s = s_0 + ip$, with $s_0, p > 0$, and also $\operatorname{Im} M^2(s) \leq 0$, for all $s = s_0 + ip$, with $s_0 > 0$, $p > p_0$. Consequently, it holds $\operatorname{Re} M(s) > 0$, for all $s = s_0 + ip$, with $s_0, p > 0$, and also $\operatorname{Im} M(s) \leq 0$, for all $s = s_0 + ip$, with $s_0 > 0$, $p > p_0$.

We now state the main result of this section.

Theorem 4.8 ([3]). *Problem* (31)–(32) *has a distributional solution given by*

$$u(x,t) = (U *_t K)(x,t) = \int_0^t U(t-\tau)K(x,\tau)\,d\tau, \quad x \geq 0, t > 0, \tag{45}$$

where

$$K(x,t) = \frac{1}{2\pi i} \int_{s_0 - i\infty}^{s_0 + i\infty} \exp(ts)\left[\frac{\exp(sM(s)x)}{1 - exp(2sM(s)l)} + \frac{\exp(-sM(s)x)}{1 - exp(-2sM(s)l)}\right] ds, \quad x \geq 0, t > 0,$$

or

$$K(x,t) = \frac{1}{2\pi i} \int_{s_0 - i\infty}^{s_0 + i\infty} \exp(ts)\exp(-sM(s)x)\,ds, \quad x \geq 0, t > 0 \tag{46}$$

if $l = \infty$. Here, $s_0 > 0$.

In both cases, K is a continuous function in $[0,\infty) \times [0,\infty)$.

In the case $l = \infty$, solution (45) can be computed more explicitly, as it is given in the following statement.

Theorem 4.9 ([3]). *Let $l = \infty$. Then the solution kernel* (46) *takes the form*

$$K(x,t) = \frac{1}{\pi} \int_0^\infty \exp[\tau \operatorname{Im} M(i\tau)x] \cos[\tau(t - \operatorname{Re} M(i\tau)x)]\,d\tau, \quad x \geq 0, t > 0. \tag{47}$$

We will give only some details of the proof. Set $s = s_0 + i\tau$ ($ds = i\,d\tau$) and $p > 0$. Then (46) becomes

$$K(x,t) = \frac{1}{2\pi} \lim_{p\to\infty} \int_{-p}^{p} \exp[(s_0 + i\tau)(t - M(s_0 + i\tau)x)]\,d\tau. \tag{48}$$

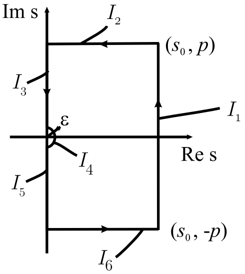


Figure 1: Integration path I.

Consider the contour shown in Figure 1. In all calculations, we will always have $x \geq 0$, $t > 0$. Then $K(x,t) = \frac{1}{2\pi} \lim_{p\to\infty} I_1(x,t,p)$, where from the Cauchy integral theorem we have

$$I_1 = -(I_2 + I_3 + I_4 + I_5 + I_6).$$

By assumptions, one can prove that

$$\lim_{p\to\infty} |I_2 + I_6| = 0.$$

Similarly, for I_4 we have $s = \varepsilon \exp(i\varphi)$, $ds = i\varepsilon \exp(i\varphi)\, d\varphi$ so that

$$\lim_{\varepsilon\to 0} I_4 = \lim_{\varepsilon\to 0} \int_{-\pi/2}^{\pi/2} \exp[\varepsilon \exp(i\varphi)(t - M(\varepsilon \exp(i\varphi))x)] i\varepsilon \exp(i\varphi)\, d\varphi = 0.$$

Therefore, $I_1 = -(I_3 + I_5)$ so that with $s = i\tau$ we get

$$\begin{aligned} I_1 &= -i\left[\int_p^{\varepsilon} \exp[i\tau(t - M(i\tau)x)]\, d\tau + \int_{-\varepsilon}^{-p} \exp[i\tau(t - M(i\tau)x)]\, d\tau\right] \\ &= -2i\left[\int_p^{\varepsilon} \exp[\tau \operatorname{Im} M(i\tau)x] \cos[\tau(t - \operatorname{Re} M(i\tau)x)]\, d\tau\right], \end{aligned}$$

where we used $\operatorname{Im} M(-ip) = -\operatorname{Im} M(ip)$, $\operatorname{Re} M(ip) = \operatorname{Re} M(-ip)$. From (48), we finally obtain

$$K(x,t) = \frac{1}{\pi} \lim_{\varepsilon\to 0}\left(\lim_{p\to\infty} \int_{\varepsilon}^{p} \exp[\tau \operatorname{Im} M(i\tau)x] \cos[\tau(t - \operatorname{Re} M(i\tau)x)]\, d\tau\right),$$

which proves the claim.

Remark 4.10. Choosing parameters in system (31)–(32):

$$a_1 = 1, \quad a_2 = 20, \quad b_1 = 0.1, \quad l = \infty$$

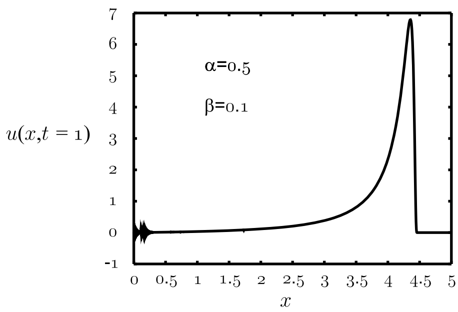


Figure 2: Displacement $u(x,t)$ for $\alpha = 0.5$, $\beta = 0.1$, and $t = 1$.

we presented in [3] a numerical experiment supposing that $U(t) = \delta(t)$, the Dirac distribution. Then combining (45) and (47), the solution reads

$$u(x,t) = \frac{1}{\pi}\int_0^\infty \exp[\tau \operatorname{Im} M(i\tau)x]\cos[\tau(t - \operatorname{Re} M(i\tau)x)]\,d\tau, \quad x \in (0,\infty), t \geq 0. \tag{49}$$

In Figure 2, we present $u(x,t)$ given by (49) for $\alpha = 0.5$, $\beta = 0.1$, and $t = 1$.

5 Space-time fractional Zener wave equation

Our aim in this section is to explain the nonlocal effects in a material of Zener type. Let

$$\mathcal{E}^\beta(x,t) = \frac{1}{2\ell^{1-\beta}\Gamma(1-\beta)}|x|^{-\beta} *_x \varepsilon_{\mathrm{cl}}(x,t), \quad x \in \mathbb{R}, t > 0, \tag{50}$$

where ℓ (measured in m) denotes the length-scale parameter, $\mathcal{E}^\beta$ is the symmetrized Caputo fractional derivative of order $\beta \in [0,1)$, and $\varepsilon_{\mathrm{cl}}$ is ε_{11} from $(3)_3$. Both, strain measure $\mathcal{E}^\beta$ and classical strain $\varepsilon_{\mathrm{cl}}$ are dimensionless quantities. Note also that (50) reduces to (9) when $a = -\infty$, $l = \infty$. Thus, (50) is a generalization of (9).

5.1 Model

We consider a rod with a finite cross-sectional area A so that the length-scale parameter is given as $\ell = \sqrt{A}$. We refer to [50] and references therein for the derivation of the multidimensional fractional strain measure through the standard continuum mechanics approach and its geometrical interpretation. For the discussion of various strain measures, other than the classical strain, that are acceptable in the continuum field theories; see [51, p. 268]. In Section 3, it was shown that (50) can be used as a strain measure because a displacement field is only a function of time (corresponding to the rigid body motion), if and only if $\mathcal{E}^\beta$ from (50) equals zero. The Fourier transform of the strain measure (50) reads

$$\hat{\varepsilon}(\xi,t) = \frac{1}{\ell^{1-\beta}} i \frac{\xi}{|\xi|^{1-\beta}} \sin\frac{\beta\pi}{2} \hat{u}(\xi,t), \quad \xi \in \mathbb{R}, t > 0,$$

which for $\beta = 1$ becomes $\hat{\varepsilon}(\xi,t) = i\xi\hat{u}(\xi,t)$. It is the Fourier transform of the classical strain $\hat{\varepsilon}_{\mathrm{cl}}$. Regarding the fractionalization of the strain measure, we follow the approach presented in [13]. In that paper, the symmetrized fractional derivative is introduced in order to describe the nonlocal effects of the material. Note that in [6], the same type of fractional derivative is used in the framework of the heat conduction problem of the space-time fractional Cattaneo-type equation.

One may also treat the non-locality in viscoelastic media by different approaches. Contrary to (50), one may retain the classical strain measure and introduce the non-locality in the constitutive equation. In the classical setting it was done by Eringen, [28]. This approach was followed by the use of fractional calculus in [17, 18, 22, 23, 26], while the wave equation, obtained from a system consisting of the equation of motion, fractional Eringen-type constitutive equation and classical strain measure, was studied in [20, 48]. We refer to [36] for the construction of spatially fractional viscoelastic wave equations in three dimensions based on the corresponding fractional constitutive equations in the spirit of Kunin's stress-strain relations for crystalline solids, or Edelen's and Eringen's approach to non-locality. Moreover, we refer to [34, 35] for the applications of an anisotropic multi-dimensional operator in diffusion in biological tissues. Some recent mechanical models of fractional-order viscoelasticity can be found in [21, 24, 25, 41].

We propose the time-fractional Zener model in the form

$$\sigma(x,t) + \tau_\sigma^\alpha \, {}_0D_t^\alpha \sigma(x,t) = E(\mathcal{E}^\beta(x,t) + \tau_\varepsilon^\alpha \, {}_0D_t^\alpha \mathcal{E}^\beta(x,t)), \quad x \in \mathbb{R}, t > 0, \tag{51}$$

with $\mathcal{E}^\beta$ given by (50). Following the approach where the classical strain measure is retained and nonlocality is introduced in the constitutive equation, the constitutive equation (51), solved with respect to σ, reads

$$\sigma(x,t) = E\left(\left(\frac{\tau_\varepsilon}{\tau_\sigma}\right)^\alpha \delta(t) + \left(\left(\frac{\tau_\varepsilon}{\tau_\sigma}\right)^\alpha - 1\right) e_\alpha'(t)\right) *_t \mathcal{E}^\beta(x,t), \quad x \in \mathbb{R}, t > 0. \tag{52}$$

Here, δ is the Dirac distribution, $e_\alpha' = \frac{d}{dt} e_\alpha$, with e_α being the Mittag-Leffler function

$$e_\alpha(t) = E_\alpha\left(-\left(\frac{t}{\tau_\sigma}\right)^\alpha\right), \quad t > 0.$$

When constitutive equation, written in the form (52), is combined with the strain measure (50), a single constitutive relation between stress σ and classical strain $\varepsilon_{\mathrm{cl}}$, nonlocal in space and time, is obtained as

$$\sigma(x,t) = \frac{E}{2\ell^{1-\beta}\Gamma(1-\beta)}\left(\left(\frac{\tau_\varepsilon}{\tau_\sigma}\right)^{\alpha}|x|^{-\beta}\delta(t) + \left(\left(\frac{\tau_\varepsilon}{\tau_\sigma}\right)^{\alpha} - 1\right)|x|^{-\beta}e'_\alpha(t)\right)$$
$$*_{x,t}\,\varepsilon_{\mathrm{cl}}(x,t), \quad x \in \mathbb{R}, t > 0. \tag{53}$$

Constitutive equation (53) corresponds to a constitutive modeling of materials that show both: history dependent properties ([29]), as well as length-scale, or nonlocal properties ([28]). System of equations, equivalent to (25)–(28), consisting of equation of motion, space-time nonlocal constitutive equation, and classical strain $\varepsilon_{\mathrm{cl}}$ reads

$$\begin{aligned}
\partial_x\sigma(x,t) &= \rho\,\partial_t^2 u(x,t),\\
\sigma(x,t) &= \frac{E}{2\ell^{1-\beta}\Gamma(1-\beta)}\left(\left(\frac{\tau_\varepsilon}{\tau_\sigma}\right)^{\alpha}|x|^{-\beta}\delta(t) + \left(\left(\frac{\tau_\varepsilon}{\tau_\sigma}\right)^{\alpha} - 1\right)|x|^{-\beta}e'_\alpha(t)\right)\\
&\quad *_{x,t}\,\varepsilon_{\mathrm{cl}}(x,t),\\
\varepsilon_{\mathrm{cl}}(x,t) &= \partial_x u(x,t), \quad x \in \mathbb{R}, t > 0.
\end{aligned} \tag{54}$$

The initial conditions corresponding to system of equations (54) are

$$u(x,0) = u_0(x), \quad \partial_t u(x,0) = v_0(x), \quad \sigma(x,0) = 0, \quad \varepsilon(x,0) = 0, \quad x \in \mathbb{R},$$

where u_0 and v_0 are initial displacement and velocity, while the boundary conditions are

$$\lim_{x\to\pm\infty} u(x,t) = 0, \quad \lim_{x\to\pm\infty} \sigma(x,t) = 0, \quad t > 0. \tag{55}$$

Note that boundary conditions (55) are the natural choice in the case of the unbounded domain, while in the case of the bounded domain there can be a large variety of different boundary conditions depending on the type of a problem. We conclude the paper with the claim that the existence and uniqueness of the solution to the distributional version of the wave propagation problem (54)–(55) is studied in [5].

Bibliography

[1] G. Amendola, M. Fabrizio, and J. M. Golden, *Thermodynamics of Materials with Memory*, Springer, New York, 2012.

[2] T. M. Atanacković and A. Guran, *Theory of Elasticity for Scientists and Engineers*, Birkhäuser, Boston, 2000.

[3] T. M. Atanacković, M. Janev, S. Konjik, and S. Pilipović, Wave equation for generalized Zener model containing complex order fractional derivatives, *Contin. Mech. Thermodyn.*, **29**(2) (2017), 569–583.

[4] T. M. Atanacković, M. Janev, S. Konjik, S. Pilipović, and D. Zorica, Vibrations of an elastic rod on a viscoelastic foundation of complex fractional Kelvin–Voigt type, *Meccanica*, **50**(7) (2015), 685–704.

[5] T. M. Atanacković, M. Janev, Lj. Oparnica, S. Pilipović, and D. Zorica, Complementary variational principles with fractional derivatives, *Proc. R. Soc. Lond., Ser. A, Math. Phys. Eng. Sci.*, **471**(2174) (2015), 20140614.

[6] T. M. Atanacković, S. Konjik, Lj. Oparnica, and D. Zorica, The Cattaneo type space-time fractional heat conduction equation, *Contin. Mech. Thermodyn.*, **24**(4–6) (2011), 293–311.

[7] T. M. Atanacković, S. Konjik, Lj. Oparnica, and D. Zorica, Thermodynamical restrictions and wave propagation for a class of fractional order viscoelastic rods, *Abstr. Appl. Anal.*, **2011** (2011), 975694, 32 pages.

[8] T. M. Atanacković, S. Konjik, S. Pilipović, and D. Zorica, Complex order fractional derivatives in viscoelasticity, *Mech. Time-Depend. Mater.*, **20**(2) (2016), 175–195.

[9] T. M. Atanacković, S. Pilipović, B. Stanković, and D. Zorica, *Fractional Calculus with Applications in Mechanics: Wave Propagation, Impact and Variational Principles*, Wiley-ISTE, London, 2014.

[10] T. M. Atanacković, S. Pilipović, and D. Zorica, Diffusion wave equation with two fractional derivatives of different order, *J. Phys. A, Math. Theor.*, **40** (2007), 5319–5333.

[11] T. M. Atanacković, S. Pilipović, and D. Zorica, Time distributed-order diffusion-wave equation. I. Volterra-type equation, *Proc. R. Soc. Lond., Ser. A, Math. Phys. Eng. Sci.*, **465** (2009), 1869–1891.

[12] T. M. Atanacković, S. Pilipović, and D. Zorica, Time distributed-order diffusion-wave equation. II. Applications of Laplace and Fourier transformations, *Proc. R. Soc. Lond., Ser. A, Math. Phys. Eng. Sci.*, **465** (2009), 1893–1917.

[13] T. M. Atanacković and B. Stanković, Generalized wave equation in nonlocal elasticity, *Acta Mech.*, **208** (2009), 1–10.

[14] R. L. Bagley and P. J. Torvik, On the fractional calculus model of viscoelastic behavior, *J. Rheol.*, **30**(1) (1986), 133–155.

[15] M. Caputo and F. Mainardi, Linear models of dissipation in anelastic solids, *Riv. Nuovo Cimento (Ser. II)*, **1** (1971) 161–198.

[16] M. Caputo and F. Mainardi, A new dissipation model based on memory mechanism, *Pure Appl. Geophys.*, **91**(1) (1971), 134–147.

[17] A. Carpinteri, P. Cornetti, and A. Sapora, Static-kinematic fractional operators for fractal and non-local solids, *Z. Angew. Math. Mech.*, (2009), 207–217.

[18] N. Challamel, D. Zorica, T. M. Atanacković, and D. Spasić, On the fractional generalization of Eringen's nonlocal elasticity for wave propagation, *C. R. Mecanique*, **341** (2013), 296–303.

[19] I. Colombaro, A. Giusti, and F. Mainardi, On transient waves in linear viscoelasticity, *Wave Motion*, **74** (2017), 191–212.

[20] G. Cottone, M. Di Paola, and M. Zingales, Elastic waves propagation in 1D fractional non-local continuum, *Physica E, Low-Dimens. Syst. Nanostruct.*, **42** (2009), 95–103.

[21] M. Di Paola, Exact mechanical models of fractional hereditary materials, *J. Rheol.*, **56** (2012), 983–1004.

[22] M. Di Paola, G. Failla, A. Pirrotta, A. Sofi, and M. Zingales, The mechanically based non-local elasticity: an overview of main results and future challenges, *Philos. Trans. R. Soc. Lond. A*, **371** (2013), 20120433.

[23] M. Di Paola, G. Failla, and M. Zingales, Physically-based approach to the mechanics of strong non-local linear elasticity theory, *J. Elast.*, **97**(2) (2009), 103–130.

[24] M. Di Paola, F. P. Pinnola, and M. Zingales, A discrete mechanical model of fractional hereditary materials, *Meccanica*, **48** (2013), 1573–1586.

[25] M. Di Paola, F. P. Pinnola, and M. Zingales, Fractional differential equations and related exact mechanical models, *Comput. Math. Appl.*, **86** (2013), 608–620.

[26] M. Di Paola and M. Zingales, Long-range cohesive interactions of non-local continuum faced by fractional calculus, *Int. J. Solids Struct.*, **45** (2008), 5642–5659.
[27] G. Doetsch, *Handbuch der Laplace-Transformationen I*, Birkhäuser, Basel, 1950.
[28] A. C. Eringen, *Nonlocal Continuum Field Theories*, Vols. 1, 2, Springer, New York, 2002.
[29] M. Fabrizio and A. Morro, *Mathematical Problems in Linear Viscoelasticity*, SIAM Studies in Applied Mathematics, SIAM, Philadelphia, 1992.
[30] V. L. Gonsovski and Yu. A. Rossikhin, Stress waves in a viscoelastic medium with a singular hereditary kernel, *J. Appl. Mech. Tech. Phys.*, **14** (1973), 595–597.
[31] D. N. Goos and G. F. Reyero, Mathematical analysis of a Cauchy problem for the time-fractional diffusion-wave equation with $\alpha \in (0,2)$, *J. Fourier Anal. Appl.*, (2017). https://doi.org/10.1007/s00041-017-9527-9.
[32] A. Hanyga, Multi-dimensional solutions of space-time-fractional diffusion equations, *Proc. R. Soc. Lond., Ser. A, Math. Phys. Eng. Sci.*, **458**(2018) (2002), 429–450.
[33] A. Hanyga, Multidimensional solutions of time-fractional diffusion-wave equations, *Proc. R. Soc. Lond., Ser. A, Math. Phys. Eng. Sci.*, **458**(2020) (2002), 933–957.
[34] A. Hanyga and R. L. Magin, A new anisotropic fractional model of diffusion suitable for applications of diffusion tensor imaging in biological tissues, *Proc. R. Soc. Lond., Ser. A, Math. Phys. Eng. Sci.*, **470** (2014), 20140319.
[35] A. Hanyga and M. Seredyńska, Anisotropy in high-resolution diffusion-weighted MRI and anomalous diffusion, *J. Magn. Reson.*, **220** (2012), 85–93.
[36] A. Hanyga and M. Seredyńska, Spatially fractional-order viscoelasticity, non-locality, and a new kind of anisotropy, *J. Math. Phys.*, **53**(5) (2012) 052902, 1–21.
[37] M. Klimek, Fractional sequential mechanics—model with symmetric fractional derivative, *Czechoslov. J. Phys.*, **51** (2001), 1348–1354.
[38] A. N. Kochubei, Distributed order calculus and equations of ultraslow diffusion, *J. Math. Anal. Appl.*, **340**(1) (2008), 252–281.
[39] S. Konjik, Lj. Oparnica, and D. Zorica, Waves in fractional Zener type viscoelastic media, *J. Math. Anal. Appl.*, **365**(1) (2010), 259–268.
[40] S. Konjik, Lj. Oparnica, and D. Zorica, Waves in viscoelastic media described by a linear fractional model, *Integral Transforms Spec. Funct.*, **22**(4–5) (2011), 283–291.
[41] K. A. Lazopoulos, Non-local continuum mechanics and fractional calculus, *Mech. Res. Commun.*, **33** (2006), 753–757.
[42] Y. Luchko, F. Mainardi, and Y. Povstenko, Propagation speed of the maximum of the fundamental solution to the fractional diffusion–wave equation, *Comput. Math. Appl.*, **66**(5) (2013), 774–784.
[43] W. Magnus and F. Oberhettinger, *Formeln und Sätze für die speziellen Funktionen der mathematischen Physik*, Springer-Verlag, Berlin, 1948.
[44] F. Mainardi, *Fractional Calculus and Waves in Linear Viscoelasticity*, Imperial College Press, London, 2010.
[45] F. Mainardi, Fractional calculus: some basic problems in continuum and statistical mechanics, in F. Mainardi and A. Carpinteri, eds, *Fractals and Fractional Calculus in Continuum Mechanics*, CISM Courses and Lectures, vol. 378, pp. 291–348, Springer-Verlag, Wien and New York, 1997.
[46] F. Mainardi, G. Pagnini, and R. Gorenflo, Some aspects of fractional diffusion equations of single and distributed order, *Appl. Math. Comput.*, **187**(1) (2007), 295–305.
[47] Yu. N. Rabotnov, Equilibrium of an elastic medium with after effect, *Prikl. Mat. Meh.*, **12** (1948), 81–91 (in Russian).
[48] A. Sapora, P. Cornetti, and A. Carpinteri, Wave propagation in nonlocal elastic continua modelled by a fractional calculus approach, *Commun. Nonlinear Sci. Numer. Simul.*, **18** (2013), 63–74.

[49] L. Schwartz, *Théorie des Distributions*, Hermann, Paris, 1966.
[50] W. Sumelka and T. Blaszczyk, Fractional continua for linear elasticity, *Arch. Mech.*, **66** (2014), 147–172.
[51] C. Truesdell and R. Toupin, The classical field theories, in S. Flügge, ed. *Principles of Classical Mechanics and Field Theory*, Encyclopedia of Physics, vol. III/1, Springer-Verlag, Berlin–Heidelberg–New York, 1960.
[52] V. S. Vladimirov, *Equations of Mathematical Physics*, Mir Publishers, Moscow, 1984.

Rafail K. Gazizov, Alexey A. Kasatkin, and Stanislav Yu. Lukashchuk

Symmetries, conservation laws and group invariant solutions of fractional PDEs

Abstract: We give the extensions of several basic techniques of Lie group analysis which are intended for symmetry properties investigation of fractional partial differential equations. The presented formulas and algorithms give ones the opportunity to calculate symmetries, invariant solutions, and conservation laws for such equations and their systems. All described techniques are illustrated by simple examples. The chapter also includes brief summaries of known symmetry analysis results for fractional partial differential equations.

Keywords: Partial fractional differential equation, Lie group of point transformations, symmetry, symmetry reduction, invariant solution, conservation law, nonlinear self-adjointness

MSC 2010: 35R11, 70G65, 70S10

1 Introduction

During the last two decades, fractional partial differential equations (FPDEs) with fractional derivatives of different types [15, 50, 72] are being increasingly used in practice for modeling systems, processes, and media with memory and spatial nonlocality [5, 32, 51, 52, 79, 93, 94]. Finding exact solutions of such equations is a challenging problem, especially for nonlinear problems.

Lie group analysis of differential equations [10, 28, 38, 40, 57, 68, 69] provides efficient techniques for investigating symmetry properties of nonlinear differential and integro-differential equations and for constructing their exact solutions. In this chapter, extensions of some basic algorithms of symmetry analysis to FPDEs are discussed. We restrict our attention to FPDEs with the Riemann–Liouville and Caputo fractional derivatives [50] since these derivatives are the most commonly used in practice. We focus on the formulas and algorithms which allow one to calculate symmetries, invariant solutions, and conservation laws for such FPDEs and their systems. All presented techniques are illustrated by simple examples.

The chapter will be useful to researchers concerned with fractional calculus and Lie group analysis. We assume that the reader is familiar with methods of Lie group analysis for integer-order differential equations. Some basic facts of this theory are given in the chapter “Symmetries and group invariant solutions of fractional ordinary

Rafail K. Gazizov, Alexey A. Kasatkin, Stanislav Yu. Lukashchuk, Ufa State Aviation Technical University, Karl Marx str. 12, Ufa 450008, Russia, e-mail: gazizovrk@gmail.com

https://doi.org/10.1515/9783110571660-016

differential equations” of this handbook. Also, the mentioned chapter contains a discussion of some peculiarities of Lie group analysis application to fractional differential equations.

This chapter is organized as follows. Section 2 contains the extensions of basic Lie group analysis methods to FPDEs and is divided into three subsections. In Subsection 2.1, the prolongation of point transformations group to partial fractional derivatives of the Riemann–Liouville and Caputo types is considered, and corresponding prolongation formulas are presented. Peculiarities of symmetry construction for FPDEs are discussed in Subsection 2.2. Techniques of the symmetry reduction and invariant solutions construction are illustrated in Subsection 2.3. Section 3 is devoted to algorithms of conservation laws constructing for FPDEs both with and without Lagrangians (Subsection 3.1 and 3.2, respectively). A possibility to find exact solutions of FPDEs using their conservation laws is illustrated in Subsection 3.3. Finally, Section 4 contains brief summary of known symmetry analysis results for FPDEs.

2 Symmetry analysis of FPDEs

2.1 Prolongation of group transformations to partial fractional derivatives

Let $u = (u^1, \dots, u^m)$ be a vector-function of dependent variables u^μ ($\mu = 1, \dots, m$) which are the functions of independent variables $x = (x^1, \dots, x^n)$. We consider invertible transformations

$$T_a : \begin{array}{lll} \bar{x}^i = \varphi^i(x,u,a), & \varphi^i|_{a=0} = x^i, & i = 1, \dots, n; \\ \bar{u}^\mu = \psi^\mu(x,u,a), & \psi^\mu|_{a=0} = u^\mu, & \mu = 1, \dots, m, \end{array} \tag{1}$$

where a belongs to some neighborhood Δ of the point $a = 0$: $a \in \Delta \subset \mathbb{R}$.

Definition 1. The transformations (1) form a one-parameter local Lie group G if for all a, b, $a + b \in \Delta$ the following conditions hold:

$$T_0 = I \in G, \quad T_a T_b = T_{a+b} \in G, \quad T_a^{-1} = T_{-a} \in G,$$

where I is the identity transformation.

Definition 2. A first-order linear differential operator

$$X = \xi^i(x,u)\frac{\partial}{\partial x^i} + \eta^\mu(x,u)\frac{\partial}{\partial u^\mu}, \tag{2}$$

where

$$\xi^i(x,u) = \left.\frac{\partial \varphi^i(x,u,a)}{\partial a}\right|_{a=0}, \quad \eta^\mu(x,u) = \left.\frac{\partial \psi^\mu(x,u,a)}{\partial a}\right|_{a=0},$$

is called the infinitesimal generator (or infinitesimal operator) of the one-parameter group (1).

Remark 1. In (2) and hereafter, the following convention of summation over repeated indices is implied:

$$\xi^i \frac{\partial}{\partial x^i} \equiv \sum_{i=1}^{n} \xi^i \frac{\partial}{\partial x^i}, \quad \eta^\mu \frac{\partial}{\partial u^\mu} \equiv \sum_{\mu=1}^{m} \eta^\mu \frac{\partial}{\partial u^\mu}. \tag{3}$$

The infinitesimal generator (2) can be rewritten in the *canonical form*

$$X = W^\mu \frac{\partial}{\partial u^\mu}, \quad W^\mu = \eta^\mu - \xi^i \frac{\partial u^\mu}{\partial x^i}.$$

Expanding the functions $\varphi^i(x,u,a)$ and $\psi^\mu(x,u,a)$ into the Taylor series with respect to the group parameter a in a neighborhood of $a = 0$ gives the following *infinitesimal transformations* of the group G:

$$\begin{aligned} \bar{x}^i &= x^i + a\xi^i(x,u) + o(a), \quad i = 1,\dots,n; \\ \bar{u}^\mu &= u^\mu + a\eta^\mu(x,u) + o(a), \quad \mu = 1,\dots,m. \end{aligned} \tag{4}$$

It is proved in classical Lie group analysis that one-parameter local groups are determined by their infinitesimal transformations [38, 40, 69].

Theorem 1. *For a given generator* (2), *group transformations* (1) *can be found by solving the Lie equations*

$$\begin{aligned} \frac{d\bar{x}^i}{da} &= \xi^i(\bar{x},\bar{u}), \quad \bar{x}^i|_{a=0} = x^i, \quad i = 1,\dots,n; \\ \frac{d\bar{u}^\mu}{da} &= \eta^\mu(\bar{x},\bar{u}), \quad \bar{u}^\mu|_{a=0} = u^\mu, \quad \mu = 1,\dots,m. \end{aligned} \tag{5}$$

Using (4), the infinitesimal prolongations of the group G to fractional integrals and partial fractional derivatives can be constructed. Assuming that all functions $u^\mu(x)$ are defined on

$$\Omega = (c^1, d^1) \times \cdots \times (c^n, d^n), \quad -\infty < c^i < d^i < \infty, \quad i = 1,\dots,n,$$

one can define the left-sided and right-sided fractional integrals of order α as

$$(I^\alpha_{c^i+} u^\mu)(x) \equiv \frac{1}{\Gamma(\alpha)} \int_{c^i}^{x^i} \frac{u^\mu(x^1,\dots,x^{i-1},s,x^{i+1},\dots,x^n)}{(x^i - s)^{1-\alpha}} ds \tag{6}$$

and

$$(I^\alpha_{d^i-} u^\mu)(x) \equiv \frac{1}{\Gamma(\alpha)} \int_{x^i}^{d^i} \frac{u^\mu(x^1,\dots,x^{i-1},s,x^{i+1},\dots,x^n)}{(s - x^i)^{1-\alpha}} ds, \tag{7}$$

respectively. Here, $\Gamma(z)$ is the gamma function.

Only the Riemann–Liouville partial fractional derivatives (see, e. g., [50])

$$(D^{\alpha}_{c^i+}u^{\mu})(x) = \frac{\partial^k}{(\partial x^i)^k} I^{k-\alpha}_{c^i+}u^{\mu}, \quad (D^{\alpha}_{c^i-}u^{\mu})(x) = (-1)^k \frac{\partial^k}{(\partial x^i)^k} I^{k-\alpha}_{c^i-}u^{\mu}, \tag{8}$$

where $k = 1 + [\alpha]$, and the Caputo partial fractional derivatives (see, e. g., [50])

$$({}^{C}D^{\alpha}_{c^i+}u^{\mu})(x) = I^{k-\alpha}_{c^i+} \frac{\partial^k u^{\mu}}{(\partial x^i)^k}, \quad ({}^{C}D^{\alpha}_{c^i-}u^{\mu})(x) = (-1)^k I^{k-\alpha}_{c^i-} \frac{\partial^k u^{\mu}}{(\partial x^i)^k}, \tag{9}$$

will be considered in this chapter. For these derivatives, the following equivalent notations will also be used:

$$\begin{aligned} D^{\alpha}_{c^i+} &\equiv {}_{c^i}D^{\alpha}_{x^i} \equiv D^{\alpha}_{i+}, \quad {}^{C}D^{\alpha}_{c^i+} \equiv {}^{C}_{c^i}D^{\alpha}_{x^i} \equiv {}^{C}D^{\alpha}_{i+}, \\ D^{\alpha}_{d^i-} &\equiv {}_{x^i}D^{\alpha}_{d^i} \equiv D^{\alpha}_{i-}, \quad {}^{C}D^{\alpha}_{d^i-} \equiv {}^{C}_{x^i}D^{\alpha}_{d^i} \equiv {}^{C}D^{\alpha}_{i-}. \end{aligned}$$

The theorem below gives the infinitesimal prolongation of the group G to fractional integrals (see [25, 63]).

Theorem 2. *Let $\alpha > 0$, the left-sided fractional integral (6) exists and belongs to $C^1(\Omega)$, $I^{\alpha}_{c^i+}W^{\mu}$ exists, and the condition*

$$\xi^i(x, u(x))\big|_{x^i \to c^i+} = 0 \tag{10}$$

holds. Then the infinitesimal transformation of fractional integral (6) has the form

$$(I^{\alpha}_{c^i+}\bar{u}^{\mu})(\bar{x}) = (I^{\alpha}_{c^i+}u^{\mu})(x) + a\zeta^{\mu}_{(-\alpha),i+} + o(a), \tag{11}$$

where $\zeta^{\mu}_{(-\alpha),i+}$ is defined by the prolongation formula

$$\zeta^{\mu}_{(-\alpha),i+} = I^{\alpha}_{c^i+}W^{\mu} + \xi^j \frac{\partial}{\partial x^j} I^{\alpha}_{c^i+}u^{\mu}. \tag{12}$$

Similarly, for the right-sided fractional integral (7) belonging to $C^1(\Omega)$, in view of the condition

$$\xi^i(x, u(x))\big|_{x^i \to d^i-} = 0, \tag{13}$$

the infinitesimal transformation has the form

$$(I^{\alpha}_{d^i-}\bar{u}^{\mu})(\bar{x}) = (I^{\alpha}_{d^i-}u^{\mu})(x) + a\zeta^{\mu}_{(-\alpha),i-} + o(a), \tag{14}$$

where $\zeta^{\mu}_{(-\alpha),i-}$ is defined by the prolongation formula

$$\zeta^{\mu}_{(-\alpha),i-} = I^{\alpha}_{d^i-}W^{\mu} + \xi^j \frac{\partial}{\partial x^j} I^{\alpha}_{d^i-}u^{\mu}. \tag{15}$$

Theorem 2 allows to construct the prolongations of the group G to partial fractional derivatives (8) and (9). For brevity, we denote by $\mathcal{D}^{\alpha}_{i\pm}u^{\mu}$ both these derivatives with respect to variable x^i ("+" for the left-sided and "–" for the right-sided).

Theorem 3. *Let $\alpha > 0$, $\mathcal{D}^{\alpha}_{i\pm}u^{\mu} \in C^1(\Omega)$, $\mathcal{D}^{\alpha}_{i\pm}W^{\mu}$ exist, and the condition* (10) *or* (13) *holds for the left-sided or the right-sided fractional derivatives, respectively. Then the infinitesimal transformation of fractional derivative has the form*

$$(\mathcal{D}^{\alpha}_{i\pm}\bar{u}^{\mu})(\bar{x}) = (\mathcal{D}^{\alpha}_{i\pm}u^{\mu})(x) + a\zeta^{\mu}_{(\alpha),i\pm} + o(a), \tag{16}$$

where $\zeta^{\mu}_{(\alpha),i\pm}$ is defined by the prolongation formula

$$\zeta^{\mu}_{(\alpha),i\pm} = \mathcal{D}^{\alpha}_{i\pm}W^{\mu} + \xi^j \frac{\partial}{\partial x^j}\mathcal{D}^{\alpha}_{i\pm}u^{\mu}. \tag{17}$$

Remark 2. In a limiting case $\alpha \in \mathbb{N}$, the prolongation formulas from Theorem 3 coincide with the well-known (see, e. g., [37, 69]) prolongation formulas for integer-order derivatives.

Other forms and particular cases of the prolongation formula (17) can be found in [22–24, 27] (see also the chapter "Symmetries and group invariant solutions of fractional ordinary differential equations" in this handbook).

Example 1. Let $u = u(x, y)$ be a function of two independent variables x and y. Consider the group of scaling transformations with the generator

$$X = x\frac{\partial}{\partial x} + \beta y\frac{\partial}{\partial y} + \gamma u\frac{\partial}{\partial u}.$$

Then we have $W = \gamma u - xu_x - \beta y u_y$.

Using (16) and (17), we find the infinitesimal transformation of ${}_0D^{\alpha}_x u$ in the form

$$({}_0D^{\alpha}_{\bar{x}}\bar{u})(\bar{x}) = ({}_0D^{\alpha}_x u)(x) + a\zeta_{(\alpha),1+} + o(a)$$

with

$$\zeta_{(\alpha),1+} = {}_0D^{\alpha}_x(\gamma u - xu_x - \beta y u_y) + x\,{}_0D^{\alpha+1}_x u + \beta y\frac{\partial}{\partial y}{}_0D^{\alpha}_x u.$$

Assuming that operators ${}_0D^{\alpha}_x$ and $\frac{\partial}{\partial y}$ commute, using the relation $xu_x = \frac{\partial}{\partial x}(xu) - u$, and taking into account that ${}_0D^{\alpha}_x D_x(xu) = {}_0D^{\alpha+1}_x(xu)$ since $(xu)|_{x\to 0+} = 0$, after simple calculations we obtain

$$\zeta_{(\alpha),1+} = (\gamma + 1){}_0D^{\alpha}_x u - {}_0D^{\alpha+1}_x(xu) + x{}_0D^{\alpha+1}_x u.$$

The second term in the right-hand side of this expression can be expanded using the generalized Leibniz rule [86] as ${}_0D^{\alpha+1}_x(xu) = x\,{}_0D^{\alpha+1}_x u + (\alpha + 1){}_0D^{\alpha}_x u$. Then

$$\zeta_{(\alpha),1+} = (\gamma - \alpha){}_0D^{\alpha}_x u$$

and the generator of prolonged group has the form

$$\tilde{X} = x\frac{\partial}{\partial x} + \beta y\frac{\partial}{\partial y} + \gamma u\frac{\partial}{\partial u} + (\gamma - \alpha)_0 D_x^\alpha u \frac{\partial}{\partial\, {}_0D_x^\alpha u}.$$

Since the coefficients of this generator are defined on the closed set of variables $\{x, y, u, {}_0D_x^\alpha u\}$, one can obtain the prolonged transformations solving Lie's equation (5):

$$\bar{x} = e^a x, \quad \bar{y} = e^{\beta a} y, \quad \bar{u} = e^{\gamma a} u, \quad {}_0D_{\bar{x}}^\alpha \bar{u} = e^{(\gamma-\alpha)a}\, {}_0D_x^\alpha u.$$

Remark 3. It can be proved that the infinitesimal transformation (16) and the prolongation formula (17) are also applicable for the Liouville fractional derivatives [50, 86] defined on the whole axis $\mathbb{R}$ (i. e., when $c^i \to -\infty$ and $d^i \to +\infty$). Moreover, these formulas are valid in a more general case when the operator $\mathcal{D}_{i\pm}^\alpha$ is replaced by any compositions of operators of integer-order differentiation and fractional integration with respect to any (possible mixed) variables x^i $(i = 1, \dots, n)$.

Example 2. Let $u = u(t,x)$ be a function of two independent variables t and x. Consider the infinitesimal prolongation of the group G with the generator

$$X = \xi^0(t,x,u)\frac{\partial}{\partial t} + \xi^1(t,x,u)\frac{\partial}{\partial x} + \eta(t,x,u)\frac{\partial}{\partial u} \tag{18}$$

on the fractional variable ${}_xD_1^\beta\,({}_0^CD_x^\alpha u_t)$. One has

$$\big({}_{\bar{x}}D_1^\beta\,({}_0^CD_{\bar{x}}^\alpha \bar{u}_{\bar{t}})\big)(\bar{t},\bar{x}) = \big({}_xD_1^\beta\,({}_0^CD_x^\alpha u_t)\big)(t,x) + a\zeta_{(*)} + o(a),$$

where

$$\zeta_{(*)} = {}_xD_1^\beta\,({}_0^CD_x^\alpha\, D_t W) + \xi^0\frac{\partial}{\partial t}\big({}_xD_1^\beta\,({}_0^CD_x^\alpha u_t)\big) + \xi^1\frac{\partial}{\partial x}\big({}_xD_1^\beta\,({}_0^CD_x^\alpha u_t)\big)$$

with $W = \eta - \xi^0 u_t - \xi^1 u_x$.

2.2 Symmetries of FPDEs

We restrict our attention by FPDEs without the mixed partial fractional derivatives (such as $\mathcal{D}_{i+}^\alpha \mathcal{D}_{j+}^\beta u$, $i \neq j$, e. g.). A more general case with such mixed derivatives can be described using the so-called linear fractional integro-differential variables, which were first introduced in [60].

We introduce a finite set of the left-sided and right-sided fractional derivatives (both the Riemann–Liouville and Caputo types)

$$\mathbb{D}^\alpha u = \{\mathcal{D}_{i\pm}^{\alpha_j} u^\mu : \alpha_j \in \mathbb{R}_+ \backslash \mathbb{N};\ \ i = 1, \dots, n;\ \ \mu = 1, \dots, m;\ \ j = 1, \dots, l\}, \tag{19}$$

and consider a system of FPDEs in the form

$$F_\sigma(x, u, u_{(1)}, \dots, u_{(k)}, \mathbb{D}^\alpha u) = 0, \quad \sigma = 1, \dots, m. \tag{20}$$

Here (see, e. g., [40]),

$$\begin{aligned} &u_{(1)} = \{u_i^\mu\}, \quad u_{(2)} = \{u_{i_1 i_2}^\mu\}, \quad \dots, \quad u_{(k)} = \{u_{i_1 \dots i_k}^\mu\}, \\ &\mu = 1, \dots, m, \quad i, i_1, \dots, i_k = 1, \dots, n, \end{aligned}$$

where $u_i^\mu = \frac{\partial u^\mu(x)}{\partial x^i}$, $u_{i_1 i_2}^\mu = \frac{\partial^2 u^\mu(x)}{\partial x^{i_1} \partial x^{i_2}}$, and so on.

Definition 3. A group G of transformations (1) is called a symmetry group for a system of FPDEs (20) if G converts every solution of the system into a solution of the same system and preserves the form of the system.

As usual (see, e. g., [40]), for a symmetry group G we will also say that the system of FPDEs (20) is *invariant* under the group G, or that G is *admitted* by the system (20). The generator X of a symmetry group G is called a *symmetry*, or an *admitted operator* for the system (20).

Similar to ordinary fractional differential equations (see the chapter "Symmetries and group invariant solutions of fractional ordinary differential equations" of this handbook), there are no constructive algorithms for finding all symmetries of FPDEs. In practice, only the so-called *linearly autonomous symmetries* [27] can be found. These symmetries have the form

$$\xi^i = \xi^i(x) \quad (i = 1, \dots, n), \quad \eta^\mu = \eta_0^\mu(x) + \eta_\nu^\mu(x) u^\nu \quad (\mu, \nu = 1, \dots, m). \tag{21}$$

In this case, all coordinates of prolonged generators will be linear functions with respect to u^μ.

Using the generalized Leibnitz rules [86], fractional derivatives of products of two sufficiently smooth functions can be represented as infinite series. For example, for the left-sided Riemann–Liouville fractional derivatives such rule reads

$$D_{c^i+}^\alpha(fg) = \sum_{k=0}^{\infty} \binom{\alpha}{k} D_{c^i+}^{\alpha-k} f D_{x^i}^k g, \tag{22}$$

where $D_{c^i+}^{\alpha-k} f = I_{c^i+}^{k-\alpha} f$ for all $k > \alpha$, and $\binom{\alpha}{k} = \frac{\Gamma(\alpha+1)}{k!\Gamma(\alpha-k+1)}$ is a binomial coefficient. There are similar expansions for the right-sided Riemann–Liouville fractional derivatives and for the Caputo derivatives.

Using (22), the coefficient $\zeta_{\alpha i+}^\mu$ for the linearly autonomous symmetry can be written as

$$\begin{aligned} \zeta_{\alpha,i+}^\mu = D_{c_i+}^\alpha \eta_0^\mu &+ \sum_{\nu=1}^{m} \sum_{k=0}^{\infty} \binom{\alpha}{k} D_{c_i+}^{\alpha-k} u^\mu \frac{\partial^k}{\partial (x^i)^k} \left[\eta_\nu^\mu - \frac{k-\alpha}{k+1} \frac{\partial \xi^i}{\partial x^i} \right] \\ &- \sum_{\substack{j=1 \\ j \neq i}}^{n} \sum_{k=1}^{\infty} \binom{\alpha}{k} D_{c_i+}^{\alpha-k} \left(\frac{\partial u^\mu}{\partial x^j} \right) \frac{\partial^k \xi^j}{\partial (x^i)^k} \end{aligned} \tag{23}$$

(there is no summation here over repeated index i). It can be seen that in (23) all functions u^μ are involved as linear fractional-order variables $D^{\alpha-k}_{c_i+}u^\mu$ $(k = 0, 1, \ldots)$ and their integer-order derivatives. Such variables will be called the *"natural" variables*.

The necessary condition for G being a symmetry group is the equality $\tilde{X}F_\sigma = 0$, where $\tilde{X}$ is the generator of the group prolonged to all integer-order and fractional-order partial derivatives involved in the system (20). This equality should be satisfied in virtue of (20), that is,

$$\tilde{X}F_\sigma|_{(20)} = 0, \quad \sigma = 1, \ldots, m. \tag{24}$$

Equations (24) are known as *determining equations* [40].

Example 3. Consider a one-dimensional linear diffusion-wave equation with the time-fractional Riemann–Lioville fractional derivative:

$${}_0D^\alpha_t u - u_{xx} = 0, \quad u = u(t,x), \quad t > 0, \quad \alpha \in (1,2). \tag{25}$$

For this equation, one should seek the generator of symmetry group in the form (18). Then (24) leads to the determining equation

$$(\zeta_\alpha - \zeta_2)|_{(25)} = 0, \tag{26}$$

where

$$\begin{aligned} \zeta_\alpha &= {}_0D^\alpha_t(\eta - \xi^0 u_t - \xi^1 u_x) + \xi^0\, {}_0D^{\alpha+1}_t u + \xi^1\, {}_0D^\alpha_t u_x, \\ \zeta_2 &= D^2_x(\eta) - D^2_x(\xi^0)u_t - D^2_x(\xi^1)u_x - 2D_x(\xi^0)u_{tx} - 2D_x(\xi^1)u_{xx}. \end{aligned} \tag{27}$$

Substituting (27) into (26), and eliminating u_{xx} by setting $u_{xx} = {}_0D^\alpha_t u$, we get the equation for unknown coefficients ξ^0, ξ^1, and η. This equation contains the variables t, x, u, u_x, u_t, u_{tx}, and fractional derivatives of orders α and $\alpha + 1$ with respect to t. At the first step, we isolate the terms containing u_x, u_{tx}, and ${}_0D^{\alpha-k}_t u_x$ (the obtained determining equation should holds identically in this variable):

$$\begin{array}{llll} u_x^3: & \xi^1_{uu} = 0, & u_x u_t: & \xi^0_{xu} = 0, \\ u_x^2: & \eta_{uu} - 2\xi^1_{xu} = 0, & u_x\, {}_0D^\alpha_t u: & \xi^1_u = 0, \\ u_x^2 u_t: & \xi^0_{uu} = 0, & u_x u_{tx}: & \xi^0_u = 0, \\ u_x: & 2\eta_{xu} - \xi^1_{xx} = 0, & u_{tx}: & \xi^0_x = 0, \\ {}_0D^{\alpha-k}_t u_x: & D^k_t \xi^1 = 0 \quad (k = 1, 2, \ldots). & & \end{array}$$

Here, we have used the generalized Leibnitz rule (22) for ${}_0D^\alpha_t(\xi^1 u_x)$. Solving the obtained system, we find

$$\xi^0 = \xi^0(t), \quad \xi^1 = \xi^1(x), \quad \eta = \eta^0(t,x) + \left(\frac{1}{2}\xi^1_x + p(t)\right)u, \tag{28}$$

where $p(t)$ is an arbitrary function. Thus, the equation (25) has only linearly autonomous symmetries.

Substituting (28) into the remaining part of the determining equation, and applying the generalized Leibnitz rule, we get the equation

$$
\begin{aligned}
&{}_0D_t^\alpha \eta^0 - \eta^0_{xx} + \left(2\xi^1_x - \alpha\xi^0_t - \frac{1}{2}\xi^1_{xxx}u\right){}_0D_t^\alpha u \\
&+ \sum_{k=1}^{\infty}\binom{\alpha}{k}{}_0D_t^{\alpha-k}u\left(p^{(k)} - \frac{k-\alpha}{k+1}(\xi^0)^{(k+1)}\right) = 0.
\end{aligned}
$$

Splitting this equation with respect to u, ${}_0D_t^\alpha u$, and ${}_0D_t^{\alpha-k}u$ $(k = 1, 2, \ldots)$, we obtain the overdetermined system

$$
{}_0D_t^\alpha \eta^0 = \eta^0_{xx}, \quad 2\xi^1_x = C, \quad \alpha\xi^0_t = C, \quad p^{(k)} = \frac{k-\alpha}{k+1}(\xi^0)^{(k+1)}, \quad (k = 1, 2, \ldots),
$$

where C is an arbitrary constant. The solution of this system satisfying to the condition $\xi^0(0) = 0$ (see (10)) has the form

$$
\xi^0 = 2C_2 t, \quad \xi^1 = \alpha C_2 x + C_1, \quad p(t) = C_3, \quad \eta^0 = g(t, x), \tag{29}
$$

where $C_1, C_2 = C/(2\alpha)$, and C_3 are arbitrary constants, and $g(t, x)$ is an arbitrary solution of the equation ${}_0D_t^\alpha g = g_{xx}$.

Substituting (29) in (18), we obtain the following linearly independent symmetries for the equation (25):

$$
X_1 = \frac{\partial}{\partial x}, \quad X_2 = 2t\frac{\partial}{\partial t} + \alpha x\frac{\partial}{\partial x}, \quad X_3 = u\frac{\partial}{\partial u}, \quad X_g = g(t, x)\frac{\partial}{\partial u}. \tag{30}
$$

Thus, one obtain the following algorithm.

Algorithm of the symmetry group construction.

1. Using the prolongation formulas (17), the coefficients of group generator prolonged to all integer-order and fractional-order derivatives included in (20) are calculated.
2. Substituting all obtained coefficients into (24) and eliminating m higher derivatives (of the integer orders, if possible) using the equations (20), the determining equations are written.
3. The determining equations are split with respect to *integer-order* derivatives of the functions u^μ. The obtained system with respect to desired coefficients ξ^i and η^μ should be solved. If a solution of the system is obtained and has the form (21), then one proves that the considered FPDE (system) has only linear autonomous symmetries. If the solution of the system cannot be obtained, then the assumption that we seek only linear autonomous symmetries should be done.

4. The generalized Leibnitz rules (like (22)) are applied to all derivatives of products of two functions, and the determining equations are rewritten in “natural” variables.
5. The determining equations are split with respect to the “natural” variables into an overdetermined system. Solving this system, one obtain the linear autonomous symmetries of the considered FPDE (system of FPDEs).

2.3 Symmetry reduction and invariant solutions

A knowledge of symmetry groups for FPDEs gives the opportunity to construct partial solutions of these equations.

Definition 4. If a group transformation maps a solution into itself, then such solution is called a group invariant solution.

The algorithms of invariant solutions constructing for FPDEs are exact the same as ones for partial integer-order differential equations. For more details see, for example, books [40, 69].

Here, we only point out some peculiarities that arise for FPDEs during invariant solutions construction using one-parameter symmetry groups. If a symmetry of partial differential equation (integer or fractional order) with n independent variables is known, then the equation is reduced to an equation with $n-1$ independent variables. Such technique is usually called the *symmetry reduction* and resulting equation is called the *reduced equation* (or *factor-equation*). A solution of the reduced equation provides an invariant solution of FPDE under consideration. Note that a symmetry (2) can be used for symmetry reduction of the equation if and only if there is a j such that $\xi^j \neq 0$ (for multiparameter groups there is a more complex condition [69]).

Example 4. Consider the symmetry group (30) of the equation (25). The symmetries X_3 and X_g cannot be used for symmetry reduction. A simplest invariant solution can be obtained using the symmetry X_1. It has two invariants: $I_1 = t$ and $I_2 = u$. Hence, invariant solution can be sought in the form $u = v(t)$. Substituting this equality into the equation (25), we get a reduced equation $D_t^\alpha v = 0$, $\alpha \in (1, 2)$. The solution of this equation gives the invariant solution of the equation (30) in the form $u = A_1 t^\alpha + A_2 t^{\alpha-1}$, where A_1 and A_2 are arbitrary constants.

FPDEs usually admit scaling transformation groups and these groups can be used for finding so-called self-similar invariant solutions. Apparently, the paper [12] was the first work devoted to symmetry reduction of FPDE on a such group.

Example 5. The generator X_2 from (30) has two invariants: $I_1 = xt^{-\alpha/2}$ and $I_2 = u$. Then the corresponding invariant solution of the equation (25) has the form $u = v(z)$ with $z = xt^{-\alpha/2}$. In [12], it was proved that after substituting this function into the equation

(25), one gets the reduced equation in the form

$$(P_{2/\alpha}^{1-\alpha,\alpha}v)(z) = v''(z), \quad z > 0, \quad \alpha \in (1,2), \tag{31}$$

where $P_{2/\alpha}^{1-\alpha,\alpha}$ is the special case of the Erdelyi–Kober fractional differential operator of order $\alpha > 0$, $\alpha \notin \mathbb{N}$, defined by

$$(P_\beta^{\tau,\alpha}g)(z) = \prod_{j=0}^{[\alpha]}\left(\tau + j - \frac{1}{\beta}z\frac{d}{dz}\right)(K_\beta^{\tau+\alpha,1-\{\alpha\}}g)(z),$$

and

$$(K_\beta^{\tau,\alpha}g)(z) = \frac{1}{\Gamma(\alpha)}\int_1^\infty (s-1)^{\alpha-1}s^{-\tau+\alpha}g(zs^{1/\beta})ds, \quad \alpha > 0$$

is the Erdelyi–Kober fractional integral operator.

In [12], it was also proved that the common solution of the equation (31) has the form

$$v(z) = C_1\phi(-\alpha/2;1;z) + C_2\phi(-\alpha/2;1;-z), \tag{32}$$

where C_1, C_2 are arbitrary constants and

$$\phi(\rho,\lambda;z) = \sum_{k=0}^\infty \frac{1}{\Gamma(\rho k + \lambda)}\frac{z^k}{k!}$$

is the Wright function.

Equation (31) is not a unique reduced equation that can be obtained using the symmetry X_2. Indeed, the function $\tau = tx^{-2/\alpha}$ is also an invariant of X_2. Therefore, one can seek the invariant solution in the form $u = w(\tau)$. It is easy to prove that in this case ${}_0D_t^\alpha u = x^{-2}\,{}_0D_\tau^\alpha w(\tau)$, and the corresponding reduced equation can be written as

$$\alpha^2\,{}_0D_\tau^\alpha w = 4\tau^2 w'' + 2(2+\alpha)\tau w'. \tag{33}$$

The equation (33) is an ordinary fractional differential equation with the Riemann–Liouville fractional derivative. It has the solution $w(\tau) = v(\tau^{-\alpha/2})$, where the function v is defined by (32). Hence, both presented approaches lead to the same invariant solution of the equation (25).

Generally, a reduced equation can be written using the same types of fractional differential operators that the considered FPDE involves only if the solution of FPDE is sought in the certain form (usually called ansatz). For FPDEs with the Riemann–Liouville fractional derivative $({}_0D_t^\alpha u)(t,x)$, the following two forms are known [62].

Ansatz 1. Let $u(t,x) = f(x)w(tg(x))$, where $f(x)$ and $g(x)$ are known functions, and $\tau = tg(x)$ is the new independent variable. Then

$$({}_0D_t^\alpha u)(t,x) = f(x)g^\alpha(x)\,({}_0D_\tau^\alpha w)(\tau).$$

Ansatz 2. Let $u(t,x) = f(x)w(\frac{t}{1+tg(x)})$, where $f(x)$ and $g(x)$ are known functions, and $\tau = \frac{t}{1+tg(x)}$ is the new independent variable. Then

$$({}_0D_t^\alpha u)(t,x) = f(x)[1+\tau g^\alpha(x)]^{1-\alpha}\,({}_0D_\tau^\alpha w)(\tau).$$

Note that, in particular, Ansatz 1 arises if the generator of scaling transformations is used for symmetry reduction, and Ansatz 2 arises during the symmetry reduction with the generator of projective group.

Since any linear combination of symmetries is also a symmetry of the equation in question, such linear combinations can also be used for constructing invariant solutions.

Example 6. Consider the symmetry reduction of equation (25) using the symmetry $X_2 + \gamma X_3$, $\gamma \in \mathbb{R}$. This generator has two invariants $I_1 = tx^{-2/\alpha}$ and $I_2 = x^\gamma u$. Then one can seek the invariant solution in the form $u(t,x) = x^{-\gamma}w(\tau)$, $\tau = tx^{-2/\alpha}$. Corresponding reduced equation can be written as

$$\alpha^2\,{}_0D_\tau^\alpha w = 4\tau^2 w'' + 2(2\alpha\gamma + \alpha + 2)\tau w' + \alpha^2\gamma(\gamma+1)w.$$

The general solution of this equation has the form [62]

$$w(\tau) = \tau^{-\alpha\gamma/2}[C_1\phi(-\alpha/2; 1-\alpha\gamma/2; \tau^{-\alpha/2}) + C_2\phi(-\alpha/2; 1-\alpha\gamma/2; -\tau^{-\alpha/2})].$$

It is necessary to note that different linear combinations of symmetries can lead to the same invariant solutions, or to the solutions that can be transformed to each other by changing of variables. To construct all essentially different invariant solutions, the concept of optimal systems of subalgebras for a Lie algebra of symmetries of a given equation can be used [28, 68, 69]. An example of using this approach for FPDE is given in [62].

3 Conservation laws for FPDEs

Conservation law is a fundamental concept of modern natural science.

Definition 5. The equality

$$D_i C^i = 0 \tag{34}$$

is called a conservation law for the FPDE (20) if it is satisfied identically on every solution of (20). The n-dimensional vector $C = (C^1, \dots, C^n)$ is called a conserved vector for the equation (20).

If a FPDE has a Lagrangian, then appropriate conservation laws can be constructed by the variational symmetries of this equation using different fractional generalizations of Noether's theorem (see, e. g., [2, 11, 19, 66] and references therein). Nevertheless, conservation laws can also be found for FPDEs that do not have Lagrangians. A constructive algorithm for finding conservation laws for FPDEs both with and without Lagrangians is based on a fractional generalization of the fundamental operator identity [37, 69].

3.1 Conservation laws for FPDEs with Lagrangians

A system of FPDEs is said to have a fractional Lagrangian

$$\mathcal{L} = \mathcal{L}(x, u, u_{(1)}, \dots, u_{(k)}, \mathbb{D}^{\alpha} u) \tag{35}$$

with $\mathbb{D}^{\alpha} u$ defined by (19), if this system can be written as a system of the Euler–Lagrange equations having the form

$$\frac{\delta \mathcal{L}}{\delta u^{\mu}} = 0, \quad \mu = 1, \dots, m. \tag{36}$$

Here,

$$\frac{\delta}{\delta u^{\mu}} = \frac{\partial}{\partial u^{\mu}} + \sum_{p=1}^{k} (-1)^{p} D_{i_1} \cdots D_{i_p} \frac{\partial}{\partial u^{\mu}_{i_1 \dots i_p}} + \left(\mathcal{D}^{\alpha_j}_{i\pm}\right)^{*} \frac{\partial}{\partial \mathcal{D}^{\alpha_j}_{i\pm} u^{\mu}} \tag{37}$$

is the variational (or the Euler–Lagrange) derivative operator. The summation convention like (3) is used in (37) for indexes $i, i_1, \dots, i_p, j$, and $(\mathcal{D}^{\alpha_j}_{i\pm})^{*}$ denote the adjoint fractional operators corresponding to operators of fractional differentiation $\mathcal{D}^{\alpha_j}_{i\pm}$. It can be shown (see, e. g., [60]) that

$$\left(D^{\alpha}_{i\pm}\right)^{*} = {}^{C}D^{\alpha}_{i\mp}, \quad \left({}^{C}D^{\alpha}_{i\pm}\right)^{*} = D^{\alpha}_{i\mp}.$$

The Euler–Lagrange equations (36) provide an extremum of the fractional generalization of the variational integral

$$\Phi[u] = \int_{\Omega} \mathcal{L}(x, u, u_{(1)}, \dots, u_{(k)}, \mathbb{D}^{\alpha} u) dx. \tag{38}$$

In [60], a fractional generalization of the fundamental operator identity was proved for a wide class of functions depending on fractional-order derivatives. This identity relates the infinitesimal generator $\tilde{X}$ of the group prolonged to all arguments of such functions, the Euler–Lagrange operator $\frac{\delta}{\delta u^{\mu}}$, and the so-called Noether operators $\mathcal{N}^{i}$. In particular, for the function (35) this identity has the form

$$\tilde{X}\mathcal{L} + \mathcal{L} D_i(\xi^i) = W^{\mu} \frac{\delta \mathcal{L}}{\delta u^{\mu}} + D_i(\mathcal{N}^{i} \mathcal{L}), \tag{39}$$

where (see [60])

$$
\begin{aligned}
\mathcal{N}^i = \xi^i &+ W^\mu\left(\frac{\partial}{\partial u_i^\mu} + \sum_{p=1}^{k-1}(-1)^p D_{j_1}\cdots D_{j_p}\frac{\partial}{\partial u^\mu_{ij_1\dots j_p}}\right)\\
&+ \sum_{r=1}^{k-1} D_{l_1}\cdots D_{l_r}(W^\mu)\left[\frac{\partial}{\partial u^\mu_{il_1\dots l_r}} + \sum_{p=1}^{k-1-r}(-1)^p D_{j_1}\cdots D_{j_p}\frac{\partial}{\partial u^\mu_{il_1\dots l_r j_1\dots j_r}}\right]\\
&- \sum_j \sum_{r=1}^{[\alpha_j]+1}(-1)^r\left[D_i^{[\alpha_j]+1-r}(I_{i\pm}^{1-\{\alpha_j\}}W^\mu)\,D_i^{r-1}\frac{\partial}{\partial(D_{i\pm}^{\alpha_j}u^\mu)}\right.\\
&\left.+ D_i^{r-1}(W^\mu)\,D_i^{[\alpha_j]+1-r}\left(I_{i\mp}^{1-\{\alpha_j\}}\frac{\partial}{\partial({}^C D_{i\pm}^{\alpha_j}u^\mu)}\right)\right]\\
&+ \sum_j\left[(-1)^{[\alpha_j]+1}J_{i+}^{\alpha_j}\left\{W^\mu, D_i^{[\alpha_j]+1}\frac{\partial}{\partial(D_{i+}^{\alpha_j}u^\mu)}\right\}\right.\\
&+ J_{i-}^{\alpha_j}\left\{W^\mu, D_i^{[\alpha_j]+1}\frac{\partial}{\partial(D_{i-}^{\alpha_j}u^\mu)}\right\} + J_{i+}^{\alpha_j}\left\{D_i^{[\alpha_j]+1}(W^\mu), \frac{\partial}{\partial({}^C D_{i+}^{\alpha_j}u^\mu)}\right\}\\
&\left.+ (-1)^{[\alpha_j]+1}J_{i-}^{\alpha_j}\left\{D_i^{[\alpha_j]+1}(W^\mu), \frac{\partial}{\partial({}^C D_{i-}^{\alpha_j}u^\mu)}\right\}\right]
\end{aligned}
\tag{40}
$$

(here there is no summation over repeated index i). In (40), the operators $J_{i\pm}^{\alpha_j}\{\cdot,\cdot\}$ act on an ordered pair of functions $\{f(x), g(x)\}$ ($x \in \Omega$) by the following rules:

$$
J_{i+}^{\alpha_j}\{f,g\} = -\frac{1}{\Gamma(\alpha_j)}\int_{c_i}^{x_i}\int_{x^i}^{d^i}\frac{f(x_i(t))g(x_i(s))}{(s-t)^{1-\alpha_j}}dsdt,
$$

$$
J_{i-}^{\alpha_j}\{f,g\} = \frac{1}{\Gamma(\alpha_j)}\int_{c_i}^{x_i}\int_{x^i}^{d^i}\frac{f(x_i(s))g(x_i(t))}{(s-t)^{1-\alpha_j}}dsdt,
$$

where $x_i(s) = (x^1,\dots,x^{i-1},s,x^{i+1},\dots,x^n)$.

Similar to the integer-order case [37], we give the following definition.

Definition 6. An action integral (38) is invariant under the group G of point transformations (1) if

$$
\int_{\bar\Omega}\mathcal{L}(\bar x,\bar u,\bar u_{(1)},\dots,\bar u_{(k)},\mathbb{D}^\alpha\bar u)d\bar x = \int_{\Omega}\mathcal{L}(x,u,u_{(1)},\dots,u_{(k)},\mathbb{D}^\alpha u)dx. \tag{41}
$$

In [36], it was proved that the equality (41) leads to the condition

$$
\tilde X\mathcal{L} + \mathcal{L}D_i(\xi^i) = 0. \tag{42}
$$

Remark 4. Similar to the integer-order differential equations, the condition (42) can be changed to a so-called «divergence» condition

$$\tilde{X}\mathcal{L} + \mathcal{L}D_i(\xi^i) = D_i(H^i), \tag{43}$$

where H^i $(i = 1, \dots, n)$ are some known functions.

Using (39) and (42), one can easily prove the following fractional generalization of Noether's theorem.

Theorem 4. *Let the action integral* (38) *be invariant with respect to a group G, i. e. the condition* (42) *is valid. Then a vector C with components*

$$C^i = \mathcal{N}^i\mathcal{L}, \quad i = 1, \dots, n, \tag{44}$$

where $\mathcal{N}^i$ $(i = 1, \dots, n)$ *are defined by* (40), *is a conserved vector for the fractional Euler–Lagrange equations* (36).

Remark 5. If the condition (43) holds, then the components of the conserved vector can be calculated by the formulas

$$C^i = \mathcal{N}^i\mathcal{L} - H^i, \quad i = 1, \dots, n. \tag{45}$$

Formulas (42), (44), and (40) provide a simple *algorithm* for conservation laws construction. On the first step, one should check that the condition (42) (or (43)) is satisfied for a given symmetry of the FPDE under consideration. On the second step, the components of the conserved vectors are calculated using (44) and (40).

Example 7. Let us consider a fractional analogue of the wave equation with space-fractional derivatives:

$$u_{tt} = -{}_x^C D_1^\alpha({}_0D_x^\alpha u), \quad x \in (0, 1), \quad \alpha \in (0, 1). \tag{46}$$

This equation is the Euler-Lagrange equation (36) with the Lagrangian

$$\mathcal{L} = \frac{1}{2}[({}_0D_x^\alpha u)^2 - u_t^2] \tag{47}$$

and, in particular, has the symmetries

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = u\frac{\partial}{\partial u}, \quad X_3 = x^{\alpha-1}\frac{\partial}{\partial u}, \quad X_4 = x^\alpha\frac{\partial}{\partial u}.$$

The conservation law (34) takes the form

$$D_tC^t + D_xC^x = 0. \tag{48}$$

It can be easily verified that with this Lagrangian, the operators X_1 and X_3 satisfy the condition (42), and the operator X_4 satisfies the divergence condition (43) with

$H^x = \Gamma(\alpha+1)_0 I_x^{1-\alpha} u$. The operator X_2 does not satisfy these conditions and, therefore, it cannot be used for finding conservation laws. For a Lagrangian of the form $\mathcal{L} = \mathcal{L}(u_t, {}_0D_x^\alpha u)$, the components of the conserved vector defined by (45), (40) take the forms

$$C^t = \mathcal{N}^t \mathcal{L} = \xi^t \mathcal{L} + W \frac{\partial \mathcal{L}}{\partial u_t},$$
$$C^x = \mathcal{N}^x \mathcal{L} - H^x = \xi^x \mathcal{L} + {}_0I_x^{1-\alpha}(W) \frac{\partial \mathcal{L}}{\partial({}_0D_x^\alpha u)} - J_{x+}^{1-\alpha}\left\{W, D_x \frac{\partial \mathcal{L}}{\partial({}_0D_x^\alpha u)}\right\} - H^x$$

with $W = \eta - \xi^0 u_t - \xi^1 u_x$.

The operator X_1 gives $W = -u_t$ and provides a conserved vector with the components

$$C^t = \frac{u_t^2}{2} + \frac{({}_0D_x^\alpha)^2 u}{2}, \quad C^x = -({}_0D_x^\alpha u)({}_0I_x^{1-\alpha} u_t) + J_{x+}^{1-\alpha}\{u_t, {}_0D_x^{\alpha+1} u\}.$$

The corresponding conservation law is the energy conservation law.

For the operator X_3, one has $W = x^{\alpha-1}$, and the components of a conserved vector can be written as

$$C^t = -x^{\alpha-1} u_t, \quad C^x = \Gamma(\alpha)_0 D_x^\alpha u - J_{x+}^{1-\alpha}\{x^{\alpha-1}, {}_0D_x^{\alpha+1} u\}.$$

The corresponding conservation law is a fractional differential analogue of the momentum conservation law.

The operator X_4 gives $W = x^\alpha$ and in view of Remark 2 provides a conserved vector with the components

$$C^t = -x^\alpha u_t, \quad C^x = \Gamma(1+\alpha)[x\, {}_0D_x^\alpha u - {}_0I_x^{1-\alpha} u] - J_{x+}^{1-\alpha}\{x^\alpha, {}_0D_x^{\alpha+1} u\}.$$

The corresponding conservation law is a consequence of the momentum conservation law.

Direct differentiation allows one to easily verify that all the conserved vectors found above satisfy conservation law (48) by virtue of (46).

3.2 Conservation laws for FPDEs without Lagrangians

Most FPDEs have no Lagrangians and, therefore, classical Noether's theorem cannot be used for constructing appropriate conservation laws. For integer-order differential equations without Lagrangians, the method of nonlinear self-adjointness was developed in [39, 41] for finding conservation laws. In [21, 61], it was shown that this method is also applicable to FPDEs.

For the system (20), it is possible to introduce a *formal Lagrangian* [39, 41]

$$\mathcal{L} = v^\sigma F_\sigma(x, u, u_{(1)}, \dots, u_{(k)}, \mathbb{D}^\alpha u), \tag{49}$$

where $v = (v^1, \dots, v^m)$ are new dependent variables. Then the equations

$$F_\mu^*(x, u, v, \dots) \equiv \frac{\delta \mathcal{L}}{\delta u^\mu} = 0, \quad \mu = 1, \dots, m \tag{50}$$

are the *adjoint equations* to equations (20). In (50), the variational derivatives are defined by (37).

Definition 7. The system (20) is said to be nonlinearly self-adjoint if the adjoint system (50) is satisfied for all solutions $u(x)$ of equations (20) upon a substitution

$$v^\mu = \varphi^\mu(x, u), \quad \mu = 1, \dots, m, \tag{51}$$

such that $\varphi(x, u) \neq 0$, where φ is the m-dimensional vector $\varphi = (\varphi^1, \dots, \varphi^m)$.

For nonlinear self-adjoint systems, the following equations hold (see [39, 41]):

$$F_\mu^*(x, u, \varphi(x, u), \dots) = \lambda_\mu^\sigma(x, u) F_\sigma(x, u, u_{(1)}, \dots, u_{(k)}, \mathbb{D}^\alpha u), \tag{52}$$

where λ_μ^σ are undetermined coefficients.

If a system of FPDEs is nonlinearly self-adjoint, then the components of conserved vectors can be found from the equations (44) considered for the formal Lagrangian (49) upon a substitution (51).

Remark 6. The formal Lagrangian (49) is identically equal to zero for all solutions of the system (20) and the condition (42) is satisfied on such solutions. Therefore, any symmetry of the system (20) can be used for conservation laws construction using the formal Lagrangian.

Example 8. Consider a fractional diffusion-wave generalization of the Kompaneets equation [21]

$${}_0D_t^\alpha u_t = \frac{1}{x^2} \frac{\partial}{\partial x} \left[x^4 \left(\frac{\partial u}{\partial x} + u^2 \right) \right], \quad \alpha \in (0, 1), \quad t \in (0, T). \tag{53}$$

This equation has a single symmetry [21]

$$X = -x \frac{\partial}{\partial x} + u \frac{\partial}{\partial u}. \tag{54}$$

The adjoint equation (50) for the equation (53) is written as

$${}_tD_T^\alpha v_t + x^2 v_{xx} - 2x^2 u v_x + 4xuv - 2v = 0. \tag{55}$$

In the considered case, the substitution (51) has the form $v = \varphi(t, x, u)$, and (52) takes the form

$$\begin{aligned} &{}_tD_T^\alpha \varphi_t + {}_tD_T^\alpha(\varphi_u u_t) + x^2(\varphi_x + 2\varphi_{xu} u_x + \varphi_{uu} u_x^2 + \varphi_u u_{xx}) - 2x^2 u(\varphi_x + \varphi_u u_x) \\ &\quad + 2(2xu - 1)\varphi = \lambda\big[{}_0D_t^\alpha u_t - 4x(u_x + u^2) - x^2(u_{xx} + 2uu_x)\big]. \end{aligned}$$

The left-hand side of this equation contains the right-sided fractional derivatives, and the right-hand side of the equation contains the left-sided derivative. Therefore, one can conclude that this equation can be satisfied only for $\lambda = 0$ and ${}_tD_T^\alpha \varphi_t + {}_tD_T^\alpha(\varphi_u u_t) = 0$. Equating to zero the coefficient of u_{xx}, one finds $\varphi_u = 0$. Hence, $\varphi = \phi_1(x)+\phi_2(x)(T-t)^\alpha$. From the remaining part of the equation, one gets $\phi_1(x) = A_1 x^2$ and $\phi_2(x) = A_2 x^2$, where A_1 and A_2 are arbitrary constants. Thus, we prove that the equation (53) is nonlinearly self-adjoint with the substitution $v = x^2(A_1 + A_2(T-t)^\alpha)$. Then the formal Lagrangian (49) can be written as

$$\mathcal{L} = x^2(A_1 + A_2(T-t)^\alpha)[{}_0D_t^\alpha u_t - 4x(u_x + u^2) - x^2(u_{xx} + 2uu_x)], \tag{56}$$

and from (40) we get formulas for the components of a conserved vector $C = (C^t, C^x)$ in the form

$$\begin{aligned} C^t &= {}_0I_t^{1-\alpha} D_t(W) \frac{\partial \mathcal{L}}{\partial({}_0D_t^\alpha u_t)} - W\, {}_tI_T^{1-\alpha} D_t \frac{\partial \mathcal{L}}{\partial({}_0D_t^\alpha u_t)} \\ &\quad + J_{t+}^{1-\alpha}\left\{D_t(W), D_t \frac{\partial \mathcal{L}}{\partial({}_0D_t^\alpha u_t)}\right\}, \\ C^x &= W\left(\frac{\partial \mathcal{L}}{\partial u_x} - D_x \frac{\partial \mathcal{L}}{\partial u_{xx}}\right) + D_x(W) \frac{\partial \mathcal{L}}{\partial u_{xx}}. \end{aligned}$$

For the symmetry (54), we have $W = u + xu_x$, and using (56) two different conservation laws can be found:

$$\begin{aligned} &C_1^t = x^2\, {}_0^CD_t^\alpha u, \quad C_1^x = -x^4(u_x + u^2); \\ &C_2^t = x^2[(T-t)^\alpha\, {}_0I_t^{1-\alpha} u_t + \Gamma(1+\alpha)u - \alpha J_{t+}^{1-\alpha}\{u_t, (T-t)^{\alpha-1}\}], \\ &C_2^x = -x^4(T-t)^\alpha(u_x + u^2). \end{aligned} \tag{57}$$

The vector C_1 corresponds to the constant A_1, and C_2 corresponds to A_2.

3.3 Conservation laws and partial solutions

Conservation laws can be used for finding exact partial solutions by the "method of conservation laws" [41]. The main idea of this method is very simple: if we have a conservation law of the form (34), then some partial solutions of the corresponding equation are solutions of the system

$$D_{x^1} C^1 = 0, \quad \dots, \quad D_{x^n} C^n = 0. \tag{58}$$

Note that solutions of this system are not necessary invariant solutions.

Example 9. One can find exact solutions of the equation (53) by using the conservation laws (57). For the conserved vector C_1, the system (58) can be written as

$${}_0^CD_t^\alpha u = f(x), \quad x^4(u_x + u^2) = g(t), \tag{59}$$

where $f(t)$ and $g(t)$ are unknown functions. It can be proved that the conserved vector C_2 provides exactly the same system (59).

Solving the system (59), one gets the following families of solutions:

$$\begin{aligned} &u(t,x) = x^{-2}[x + c_1 \tanh(c_2 - c_1 x^{-1})], \quad u(t,x) = x^{-2}(x + c_1 t^{\alpha} + c_2), \\ &u(t,x) = x^{-2}[x - c_3 \tan(c_4 - c_3 x^{-1})], \quad u(t,x) = (x + c_5)^{-1}. \end{aligned} \tag{60}$$

These solutions are not invariant solutions. Indeed, the generator (54) has two invariants $I_1 = t$ and $I_2 = xu$. Therefore, the invariant solution has the form $u(t,x) = x^{-1}h(t)$, where $h(t)$ is a solution of the reduced equation

$${}_0D_t^{\alpha} h_t = 2(h^2 - h).$$

The solutions (60) have another form.

4 Brief summary of symmetry analysis results for FPDEs and their systems

At the present time, there are nearly one hundred papers devoted to symmetry analysis and conservation laws for different FPDEs and their systems. Therefore, it is not possible to present here all the obtained results. Nevertheless, in Table 1, we collect (without claim to completeness) the investigated FPDEs with the Riemann–Liouville and Caputo fractional derivatives, and give corresponding references to the published papers containing symmetries, invariant solutions, and conservation laws for these equations. Note that some of equations presented in Table 1 can be considered as special cases of another ones also presented in this table.

It is interesting to note that most of equations from Table 1 has the form

$${}_0D_t^{\alpha} u = F(t, x, u, u_x, u_{xx}, u_{xxx}, \dots)$$

with the Riemann–Liouville time-fractional derivative. The number of studied equations with the Caputo fractional derivatives and spatial (especially right-sided) fractional derivatives is not large enough.

An additional remark is necessary. Since the techniques of finding symmetries and conservation laws for FPDEs are more complex than ones for integer-order equations, some of the papers citing in Table 1 contain errors. A most frequent error is that a FPDE containing fractional derivative with respect to independent variable x^i has the symmetry $X = \frac{\partial}{\partial x^i}$ which is inconsistent with the conditions (10) and (13). The reader should be attentive to this fact.

Table 1: Symmetry analysis results for FPDEs.

No	Equation or system	Symmetries	Symmetry reduction, invariant solutions	Conservation laws
1	${}_0D_t^\alpha u + auu_x + bu(1-u) = 0$	[30, 67]	[30, 67]	–
${}_0D_t^\alpha u = F(t,x,u,u_x,u_{xx})$				
2	${}_0D_t^\alpha u = au_{xx}$, $\alpha \in (0,1) \cup (1,2)$	[23, 67]	[12, 23, 67]	[61]
3	${}_0D_t^\alpha u = c^2 u_{xx}$	[3]	[3]	–
4	${}_0D_t^\alpha u = (k(u)u_x)_x$, $\alpha \in (0,1) \cup (1,2)$	[23]	[12, 23]	[61]
5	${}_0^C D_t^\alpha u = (k(u)u_x)_x$, $\alpha \in (0,1) \cup (1,2)$	[23]	[23]	[61]
6	${}_0D_t^\alpha u + au_{xx} + bu(1-u) = 0$	[67]	[67]	–
7	${}_0D_t^\alpha u = u_{xx} - u^3 + u$	[42]	[42]	–
8	${}_0D_t^\alpha u = u_{xx} + au + bu^2 + cu^3$	[42]	[42]	–
9	${}_0D_t^\alpha u = u_{xx} + u(1-u^n)$, $n = 0, 1, 2, \ldots$	[31]	[31]	–
10	${}_0D_t^\alpha u = au_{xx} + bu_x + f(t,x,u)$	[47, 48]	[47, 48]	–
11	${}_0D_t^\alpha u = au_{xx} + buu_x$	[67]	[67]	–
12	${}_0D_t^\alpha u = au_{xx} + bu^m u_x$	[81, 104]	[81, 104]	[104]
13	${}_0D_t^\alpha u = u_{xx} - au^\delta u_x - bu(1-u^\delta)(u^\delta - \gamma) = 0$	[43]	[43]	–
14	$\mathcal{D}_t^{\gamma(\alpha)} u = x^{-2}[x^4(u_x + u + u^2)]_x$	[21]	[21]	[21]
15	${}_0^C D_t^\alpha u = a(u^n)_{xx} + bu^m u_x$	[98]	–	[98]
16	${}_0D_t^\alpha u = (\frac{1}{2}\sqrt{u}u_x - u^2)_x$, $\alpha \in (0,1) \cup (1,2)$	[77]	[77]	[77]
17	${}_0D_t^\alpha u = auu_{xx} + bu^2 u_x + cu_x^2$	[103]	[103]	[103]
18	${}_0D_t^\alpha u = u(u^v)_{xx} + u(u^k)_x$	[29]	[29]	–
19	${}_0D_t^\alpha u = (k(u)u_x)_x + f(u)$, $\alpha \in (0,1) \cup (1,2)$	[64]	[62]	–
20	${}_0D_t^\alpha u = (A(u)u_x^n)_x + B(u)u_x^m + C(u)$	[13]	[13]	–
21	${}_0^C D_t^\alpha u = (kt^{\alpha+2})^{-1} u_{xx}$	–	[17]	–
22	${}_0^C D_t^\alpha u = (kt^{\alpha+2})^{-1}(x^4 u_{xx} + 2x^3 u_x)$	–	[17]	–
23	${}_0D_t^\alpha u = auu_{xx} + a(u_x)^2 - buu_x$	[14]	[14]	–
24	${}_0D_t^\alpha u = (au^2 + bu)u_{xx} + (2au + b)(u_x)^2$	[14]	[14]	–
25	$\mathcal{D}_t^{\gamma(\alpha)} u + uu_x + u_{xx} = 0$	[74]	–	[74]
26	${}_0D_t^\alpha u + uu_x + g(t)u_{xx} = 0$	[49]	[49]	–
27	$i\,{}_0D_t^\alpha u + u_{xx} + f(u)u = 0$	[53]	[53]	–
28	$iD_t^\alpha u + a_1 u_{xx} + V(x)u + a_2 u\lvert u\rvert^2 = 0$	[109]	[109]	–
${}_0D_t^\alpha u = F(t,x,u,u_x,u_{xx},u_{xxx})$				
29	${}_0D_t^\alpha u + au_{xxx} = 0$	[67]	[67]	–
30	${}_0D_t^\alpha u + auu_x + bu_{xxx} = 0$	[67, 96]	[67, 96]	–
31	$\tilde{D}_t^\alpha u + u^2 u_x + u_{xxx} = 0$	[1]	[1]	[1]
32	${}_0D_t^\alpha u - au^p u_x - u_{xxx} = 0$	[81]	[81]	–
33	${}_0D_t^\alpha u + 6uu_{xx} + u_{xxx} + \frac{1}{2}t^{-\alpha}u = 0$	[4]	[4]	–
34	${}_0D_t^\alpha u + auu_x + bu_{xx} + cu_{xxx} + du(1-u) = 0$	[67]	[67]	–
35	${}_0D_t^\alpha u + 3au_x^2 + 3au^2 u_x + 3auu_{xx} + au_{xxx} = 0$	[100]	[100]	–
36	${}_0D_t^\alpha u - (au^2 - bu)u_x - e(u_x)^2 + (c - du)u_{xx} - ku_{xxx} = 0$	[97]	[97]	–
37	${}_0D_t^\alpha u - \frac{1}{3}u^4 u_x - 3uu_x^2 - u^2 u_{xx} - u_{xxx} = 0$	[7]	[7]	[7]
38	${}_0D_t^\alpha u + au^{2p} u_x + bu^p u_{xx} + cu_{xxx} = 0$, $p > 0$	[110]	[110]	–
39	${}_0D_t^\alpha u + uu_x + 3u_x u_{xx} + uu_{xxx} = 0$	[73]	[73]	[73]

Table 1: (continued).

No	Equation or system	Symmetries	Symmetry reduction, invariant solutions	Conser-vation laws
40	${}_0D_t^\alpha u + au^{p+1}u_x + bu_x u_{xx} + cuu_{xxx} = 0$, $p > 0$	[110]	[110]	–
41	${}_0D_t^\alpha u - u^2 u_x - \frac{2}{9}u_x^3 + uu_x u_{xx} + u^2 u_{xxx} = 0$	[87]	[87]	–
42	${}_0D_t^\alpha u - u^3 u_{xxx} = 0$	[35]	[35]	–
43	${}_0D_t^\alpha u - a(u^n)_x + b(u^m)_{xxx} = 0$	[101]	[101]	–
44	${}_0D_t^\alpha u + a(u^m)_x + b^{-1}(u^c(u^b)_{xx})_x = 0$, $\alpha \in (0,1) \cup (1,2)$	[95]	[95]	[95]
	${}_0D_t^\alpha u = F(t,x,u,u_x,u_{xx},u_{xxx},u_{xxxx})$			
45	$\mathcal{D}_t^{\gamma(\alpha)} u + uu_x + u_{xx} + u_{xxxx} = 0$	[75]	–	[75]
46	${}_0D_t^\alpha u + au_{xx} + bu_{xx}^2 + cu_{xxxx} = 0$, $\alpha \in (0,1) \cup (1,2)$	[75]	[75]	[59]
47	${}_0D_t^{\alpha+1} u - u_{xx} + (u^2)_{xx} + (uu_{xx})_{xx} = 0$	[108]	[108]	–
48	${}_0D_t^\alpha u + (u^2)_{xx} - (u^2)_{xxxx} = 0$	[46]	[46]	–
49	$D_t^\alpha u + (u(\ln u)_{xx})_{xx} = 0$, $\alpha \in (0,1) \cup (1,2)$	[78]	[78]	[78]
	${}_0D_t^\alpha u = F(t,x,u,u_x,u_{xx},u_{xxx},u_{4x},u_{5x})$			
50	${}_0D_t^\alpha u + au^p u_x - u_{5x} = 0$	[99]	[99]	–
51	${}_0D_t^\alpha u + au^2 u_x + bu_x u_{xx} - u_{5x} = 0$	[102]	[102]	–
52	$\mathcal{D}_t^{\gamma(\alpha)} u + bu_{xxx} + cu_{5x} = 0$, $b > 0$	[75]	–	[75]
53	$\mathcal{D}_t^{\gamma(\alpha)} u + auu_x + bu_{xxx} + cu_{5x} = 0$, $b > 0$	[75]	–	[75]
54	$\mathcal{D}_t^{\gamma(\alpha)} u + (1+u)u_x + bu_{xx} + u_{xxx} + bu_{xxxx} + cu_{5x} = 0$, $b > 0$	[75]	[75]	[75]
55	${}_0D_t^\alpha u + 5u^2 u_x + 5u_x u_{xx} + 5uu_{xxx} + u_{5x} = 0$, $\alpha > 0$	[106]	[106]	[106]
56	${}_0D_t^\alpha u - 20u^2 u_x - 25u_x u_{xx} - 10uu_{xxx} - u_{5x} = 0$	[45]	[45]	–
57	${}_0D_t^\alpha u + 45u^2 u_x - 15au_x u_{xx} - 15uu_{xxx} + u_{5x} = 0$	[112]	[112]	[112]
58	${}_0D_t^\alpha u + 180u^2 u_x + 30u_x u_{xx} + 30uu_{xxx} + u_{5x} = 0$	[8]	[8]	[8]
59	${}_0D_t^\alpha u + au^2 u_x + bu_x u_{xx} + cuu_{xxx} + du_{5x} = 0$	[111]	[111]	–
60	${}_0D_t^\alpha u + 5u^4 u_x - 5u_x^3 - 20uu_x u_{xx} + 5u_{xx}^2 - 5u^2 u_{xxx} + 5u_x u_{xxx} + u_{5x} = 0$, $\alpha \in (0,1) \cup (1,2)$	[18]	[18]	[18]
61	${}_0D_t^\alpha u + au^p u_x + bu_x u_{xx} + cuu_{xxx} + du_{5x} = 0$	[58]	[58]	–
	${}_0D_t^\alpha u = F(t,x,u,u_x,\dots,u_{nx})$, $n \in \mathbb{N}$			
62	$D_t^\alpha u + 252u^3 u_x + 63u_x^3 + 378uu_x u_{xx} + 126u^2 u_{xxx} + 63u_{xx}u_{xxx} + 42u_x u_{xxxx} + 21uu_{5x} + u_{7x} = 0$	[107]	[107]	[107]
63	${}_0D_t^\alpha u + au^m u_x + bu_{nx} = 0$, $n \in \mathbb{N}$	[33, 105]	[33, 105]	[105]
64	${}_0D_t^\alpha u - \sum_{k=0}^n A_k(x)\frac{\partial^k u}{\partial x^n} = 0$	[13]	[13]	–
	${}_0D_t^\alpha u = F(t,x,u,u_x,{}_0D_x^\beta u,\dots)$			
65	$u_t = c^2\,{}_0D_x^\beta u$, $\beta \in (1,2)$	[3]	[3]	–
66	${}_0D_t^\alpha u - {}_0D_x^\beta u - u^2 u_{xx} - u^3 - u = 0$, $\beta \in (1,2)$	[44]	[44]	[44]
67	${}_0D_t^\alpha u - k\,{}_0D_x^\beta u - au + bu^q = 0$, $\beta \in (1,2)$	[44]	[44]	[44]
68	${}_0D_t^\alpha u + u\,{}_0D_x^\beta u + 3u_x u_{xx} + uu_{xxx} = 0$, $\alpha \in (0,1]$, $\beta \in (0,2)$	[9]	[9]	[9]
69	${}_0D_t^\alpha u + a_0D_x^\beta u - bu_x u_{xx} - uu_{xxx} = 0$, $\alpha, \beta \in (0,2)$	[89]	[89]	[89]
70	${}_0D_t^\alpha u + {}_0D_x^\beta u + 6au_x^3 + 18auu_x u_{xx} + 3au^2 uu_{xxx} = 0$, $\alpha, \beta \in (0,2)$	[89]	[89]	[89]
71	${}_0D_t^\alpha u + u_0D_x^\beta u + {}_0D_x^\gamma u = 0$, $\alpha,\beta,\gamma \in \mathbb{R}^+$	[55]	[55]	–

Table 1: (continued).

No	Equation or system	Symmetries	Symmetry reduction, invariant solutions	Conservation laws
72	$u_t = \sum_{i=1}^{n} a_i u^{b_i}\,{}_0D_{x^i}^{\alpha_i} u$, $\alpha_i \in \mathbb{R}\backslash\mathbb{N}$, $a_i, b_i \in \mathbb{R}\backslash\{0\}$	[56]	[56]	–
73	$f(u)_0D_t^\alpha u = [k(u)(a_0D_x^\alpha u + b_x D_1^\alpha u)]_x$	[26]	[26]	–
${}_0D_t^\alpha u = F(t,x,y,z,u,\dots)$				
74	$i\tilde{D}_t^\alpha u + a_1\tilde{D}_x^{2\beta} u - a_2 D_y^{2\gamma} u + a_3 u\|u\|^2 = 0$, α, β, $\gamma \in (0,1)$	[20]	–	[20]
75	${}_0D_t^\alpha u + u^2 u_x + u_{xxx} + u_{xyy} = 0$	[6]	[6]	[6]
76	${}_0D_t^\alpha u = (f(u)u_x)_x + (g(u)u_y)_y + (h(u)u_z)_z$, $\alpha \in (0,2)$	[54]	–	[54]
77	${}_0D_t^\alpha u + u_x + 2auu_x + b(u_{xt} + u_{yy})_x = 0$	[76]	[76]	[76]
78	${}_0D_t^\alpha u + au^2 u_x + u_{xxx} + u_{xyy} + u_{xzz} = 0$	[83]	[83]	[83]
79	$u_y u_{xy} - u_x u_{yy} = v_0 D_y^{\alpha+2} u$	[70, 71]	[70, 71]	[70]
80	$i_0D_t^\alpha u + u_{xx} + u_{yy} + f(u)u = 0$	[53]	[53]	–
Systems of FPDEs				
81	${}_0D_t^\alpha u = C^2(x)v_x$, ${}_0D_t^\alpha v = u_x$	[34, 90]	[34]	–
82	${}_0D_t^\alpha u = v_x$, ${}_0D_t^\alpha v = -uu_x$	[49]	[49]	–
83	${}_0D_t^\alpha u = v_x$, ${}_0D_t^\alpha v = b^2(u)u_x$	[16]	[16]	–
84	${}_0D_t^\alpha u + 2v(u_x^2 + v_x^2) = 0$, ${}_0D_t^\alpha v - 2u(u_x^2 + v_x^2) = 0$	[90]	–	–
85	${}_0D_t^\alpha u + v_x + 2v(u^2 + v^2) = 0$, ${}_0D_t^\alpha v - u_x - 2u(u^2 + v^2) = 0$	[90]	–	–
86	${}_0D_t^\alpha u = u_{xx} + 2uu_x - (uv)_x$, ${}_0D_t^\alpha v = v_{xx} + 2vv_x - (uv)_x$, $\alpha \in (0,2)$	[88]	[88]	[91]
87	${}_0D_t^\alpha u = u_{xx} + \rho(vv_x)_x + vv^2$, ${}_0D_t^\alpha v = v_{xx} + \beta u_{xx} + \gamma u + \delta v$	[82]	[82]	–
88	${}_0D_t^\alpha u - 2u\,{}_0D_y^{\beta_2} u - 2v\,{}_0D_x^{\beta_1} u - 2w\,{}_0D_z^{\beta_3} u - \Delta u = 0$, ${}_0D_x^{\beta_1} u - {}_0D_y^{\beta_2} v = 0$, ${}_0D_z^{\beta_3} u - {}_0D_y^{\beta_2} w = 0$, $\beta_i \in (0,1)$	[92]	[92]	–
89	${}_0D_t^\alpha u + Lu + {}_0D_x^{\beta_1} p - \mu\Delta u = 0$, ${}_0D_t^\alpha v + Lv + {}_0D_y^{\beta_2} p - \mu\Delta v = 0$, ${}_0D_t^\alpha w + Lw + {}_0D_z^{\beta_3} p - \mu\Delta w = 0$, ${}_0D_x^{\beta_1} zu + {}_0D_y^{\beta_2} v + {}_0D_z^{\beta_3} w = 0$, $L = u\,{}_0D_x^{\beta_1} + v\,{}_0D_y^{\beta_2} + w\,{}_0D_z^{\beta_3}$, $\beta_i \in (0,1)$	[92]	[92]	–
90	${}_0D_t^\alpha u = -vuu_x - \gamma v_x - \beta u_{xx}$, ${}_0D_t^\alpha v = -(uv)_x + \beta v_{xx} - \mu u_{xxx}$	[82]	[82]	–
91	${}_0D_t^\alpha u - 6uu_x - 6v_x + u_{xxx} = 0$, ${}_0D_t^\alpha v + 6uv_x - 2v_{xxx} = 0$	[84]	[84]	[84]
92	${}_0D_t^\alpha u - 3uu_x - vu_x - u_{xxx} = 0$, ${}_0D_t^\alpha v - uv_x - vu_x = 0$, $\alpha \in (0,2)$	[88]	[88]	[91]
93	${}_0D_t^\alpha u + 6auu_x - 6vv_x + au_{xxx} = 0$, ${}_0D_t^\alpha v + 3auv_x + av_{xxx} = 0$, $\alpha \in (0,2)$	[88]	[88]	[91]
94	${}_0D_t^\alpha u = a_1 uu_x + a_2 vv_x + a_3 u_{xxx}$, ${}_0D_t^\alpha v = b_1 uv_x + b_2 vu_x + b_3 v_{xxx}$	[82]	[82]	–
95	${}_0D_t^\alpha u - 6uu_x - \frac{3}{2}v^2 u_x - 6uvv_x + \frac{9}{2}v_x v_{xx} + u_{xxx} + \frac{3}{2}vv_{xxx} = 0$, ${}_0D_t^\alpha v - 6vu_x - 6uv_x - \frac{15}{2}v^2 v_x + v_{xxx} = 0$	[65]	[65]	[65]
96	${}_0D_t^\alpha u + 2uu_x - 3wv_x - 3vw_x - \frac{1}{2}u_{xxx} = 0$, ${}_0D_t^\alpha v - 3uv_x + v_{xxx} = 0$, ${}_0D_t^\alpha w - 3uw_x + w_{xxx} = 0$	[80]	[80]	[80]
97	${}_0D_t^\alpha u - 3uu_x + 3(v^2 - w)_x - \frac{1}{4}u_{xxx} = 0$, ${}_0D_t^\alpha v + 3uv_x + \frac{1}{2}v_{xxx} = 0$, ${}_0D_t^\alpha w + 3uw_x + \frac{1}{2}w_{xxx} = 0$, $\alpha \in (0,2)$	[88]	[88]	[91]

Table 1: (continued).

No	Equation or system	Symmetries	Symmetry reduction, invariant solutions	Conservation laws
98	${}_0D_t^\alpha u + 6(u^2+v^2+w^2+z^2)u_x + u_{xxx} = 0$, ${}_0D_t^\alpha v + 6(u^2+v^2+w^2+z^2)v_x + v_{xxx} = 0$, ${}_0D_t^\alpha w + 6(u^2+v^2+w^2+z^2)w_x + w_{xxx} = 0$, ${}_0D_t^\alpha z + 6(u^2+v^2+w^2+z^2)z_x + z_{xxx} = 0$, $\alpha \in (0,2)$	[88]	[88]	[91]
99	${}_0D_t^\alpha u + \frac{3}{2}(uv)_x - \frac{3}{4}u^2u_x + \frac{1}{2}u_{xxx} = 0$, ${}_0D_t^\alpha v - \frac{3}{2}vv_x + \frac{3}{4}u^2v_x + 2ww_x - \frac{3}{2}(u_xw)_x - \frac{1}{4}v_{xxx} = 0$, ${}_0D_t^\alpha w + \frac{3}{4}u_xv_x + \frac{3}{2}vw_x - \frac{3}{4}(u^2w)_x + \frac{3}{4}uv_{xx} + \frac{1}{2}w_{xxx} = 0$	[85]	[85]	[85]
100	${}_0D_t^\alpha u + a(u\,{}_0D_x^\beta + v\,{}_0D_x^\beta u) + cu_{xxx} + du_{yyy} = 0$, ${}_0D_x^\beta u - 2v\,{}_0D_y^\gamma v = 0$, $\alpha,\beta,\gamma \in (0,1)$	[92]	[92]	–

Here, $\alpha \in (0,1)$, unless otherwise indicated; a, b, c, d, e, a_i, b_i are arbitrary real constants; $\tilde{D}_{x^i}^\alpha$ is the modified Riemann–Liouville fractional derivative (see, e. g., [1]); $\mathcal{D}_t^{\gamma(\alpha)} = \{{}_0D_t^\alpha,\ {}_0^CD_t^\alpha,\ {}_0D_t^{1+\alpha},\ {}_0D_t^\alpha D_t\}$.

Bibliography

[1] A. Akbulut and F. Taşcan, Lie symmetries, symmetry reductions and conservation laws of time fractional modified Korteweg–de Vries (mkdv) equation, *Chaos, Solitons and Fractals*, **100** (2017), 1–6.

[2] T. M. Atanackovic, S. Konjik, S. Pilipovic, and S. Simic, Variational problems with fractional derivatives: invariance conditions and Nöther's theorem, *Nonlinear Analysis*, **71**(5–6) (2009), 1504–1517.

[3] F. Bahrami, R. Najafi, and M. S. Hashemi, On the invariant solutions of space/time-fractional diffusion equations, *Indian Journal of Physics*, **91**(12) (2017), 1571–1579.

[4] T. Bakkyaraj and R. Sahadevan, Group formalism of Lie transformations to time-fractional partial differential equations, *Pramana—Journal of Physics*, **85**(5) (2015), 849–860.

[5] D. Baleanu, K. Diethelm, E. Scalas, and J. J. Trujillo, *Fractional Calculus: Models and Numerical Methods*, World Scientific, Singapore, 2012.

[6] D. Baleanu, M. Inc, A. Yusuf, and A. I. Aliyu, Lie symmetry analysis, exact solutions and conservation laws for the time fractional modified Zakharov–Kuznetsov equation, *Nonlinear Analysis: Modelling and Control*, **22**(6) (2017), 861–876.

[7] D. Baleanu, M. Inc, A. Yusuf, and A. I. Aliyu, Time fractional third-order evolution equation: symmetry analysis, explicit solutions, and conservation laws, *Journal of Computational and Nonlinear Dynamics*, **13**(2) (2018), 021011.

[8] D. Baleanu, M. Inc, A. Yusuf, and A. I. Aliyu, Lie symmetry analysis, exact solutions and conservation laws for the time fractional Caudrey–Dodd–Gibbon–Sawada–Kotera equation, *Communications in Nonlinear Science and Numerical Simulation*, **59** (2018), 222–234.

[9] D. Baleanu, M. Inc, A. Yusuf, and A. I. Aliyu, Space-time fractional Rosenou–Haynam equation: Lie symmetry analysis, explicit solutions and conservation laws, *Advances in Difference Equations*, **2018**(1) (2018), 46.

[10] G. Bluman and S. Kumei, *Symmetries and Differential Equations*, Springer-Verlag, New York, Heidelberg, Berlin, 1989.

[11] L. Bourdin, J. Cresson, and I. Greff, A continuous/discrete fractional Noether's theorem, *Communications in Nonlinear Science and Numerical Simulation*, **18**(4) (2013), 878–887.

[12] E. Buckwar and Yu. Luchko, Invariance of a partial differential equation of fractional order under the Lie group of scaling transformations, *Journal of Mathematical Analysis and Applications*, **227**(1) (1998), 81–97.

[13] C. Chen and Y.-L. Jiang, Lie group analysis method for two classes of fractional partial differential equations, *Communications in Nonlinear Science and Numerical Simulation*, **26**(1–3) (2015), 24–35.

[14] C. Chen and Y.-L. Jiang, Lie group analysis and invariant solutions for nonlinear time-fractional diffusion-convection equations, *Communications in Theoretical Physics*, **68**(3) (2017), 295–300.

[15] K. Diethelm, *The Analysis of Fractional Differential Equations. An Application-Oriented Exposition Using Differential Operators of Caputo Type*, Springer, Heidelberg, 2010.

[16] K. Dorjgotov, H. Ochiai, and U. Zunderiya, Lie symmetry analysis of a class of time fractional nonlinear evolution systems, *Applied Mathematics and Computation*, **329** (2018), 105–117.

[17] A. Elsaid, M. S. Abdel Latif, and M. Maneea, Similarity solutions of fractional order heat equations with variable coefficients, *Miskolc Mathematical Notes*, **17**(1) (2016), 245–254.

[18] L.-L. Feng, S.-F. Tian, X.-B. Wang, and T.-T. Zhang, Lie symmetry analysis, conservation laws and exact power series solutions for time-fractional Fordy–Gibbons equation, *Communications in Theoretical Physics*, **66**(3) (2016), 321–329.

[19] G. S. F. Frederico and D. F. M. Torres, Fractional Noether's theorem with classical and Riemann–Liouville derivative, in *Decision and Control (CDC), Proc. of 51st IEEE Conference on Decision and Control*, Maui, Hawaii, pp. 6885–6890, IEEE, 2012.

[20] C. Fu, C. N. Lu, and H. W. Yang, Time-space fractional $(2+1)$ dimensional nonlinear Schrödinger equation for envelope gravity waves in baroclinic atmosphere and conservation laws as well as exact solutions, *Advances in Difference Equations*, **2018**(1) (2018), 56.

[21] R. K. Gazizov, N. H. Ibragimov, and S. Yu. Lukashchuk, Nonlinear self-adjointness, conservation laws and exact solutions of time-fractional Kompaneets equations, *Communications in Nonlinear Science and Numerical Simulation*, **23**(1–3) (2015), 153–163.

[22] R. K. Gazizov, A. A. Kasatkin, and S. Yu. Lukashchuk, Continuous transformation groups of fractional-order differential equations, *Vestnik USATU*, **3**(21) (2007), 125–135 (in Russian).

[23] R. K. Gazizov, A. A. Kasatkin, and S. Yu. Lukashchuk, Symmetry properties of fractional diffusion equations, *Physica Scripta*, **T136** (2009), 014016.

[24] R. K. Gazizov, A. A. Kasatkin, and S. Yu. Lukashchuk, Group-invariant solutions of fractional differential equations, in J. Machado, A. Luo, R. Barbosa, M. Silva, and L. Figueiredo (eds) *Nonlinear Science and Complexity*, pp. 51–59, Springer, Dordrecht, 2011.

[25] R. K. Gazizov, A. A. Kasatkin, and S. Yu. Lukashchuk, Fractional differential equations: change of variables and nonlocal symmetries, *Ufa Mathematical Journal*, **4**(4) (2012), 54–67.

[26] R. K. Gazizov, A. A. Kasatkin, and S. Yu. Lukashchuk, Symmetries and exact solutions of fractional filtration equations, *AIP Conference Proceedings*, **1907** (2017), 020010.

[27] R. K. Gazizov, A. A. Kasatkin, and S. Yu. Lukashchuk, Linearly autonomous symmetries of the ordinary fractional differential equations, in *IEEE: Proc. of 2014 International Conference on Fractional Differentiation and Its Applications (ICFDA 2014)*, pp. 1–6.

[28] Y. N. Grigoriev, N. H. Ibragimov, V. F. Kovalev, and S. V. Meleshko, *Symmetries of Integro-Differential Equations with Applications in Mechanics and Plasma Physics*, Springer Science & Business Media, 2010.

[29] M. S. Hashemi, Group analysis and exact solutions of the time fractional Fokker–Planck equation, *Physica A: Statistical Mechanics and Its Applications*, **417** (2015), 141–149.

[30] M. S. Hashemi and D. Baleanu, Lie symmetry analysis and exact solutions of the time fractional gas dynamics equation, *Journal of Optoelectronics and Advanced Materials*, **18**(3–4) (2016), 383–388.
[31] M. S. Hashemi and D. Baleanu, On the time fractional generalized fisher equation: group similarities and analytical solutions, *Communications in Theoretical Physics*, **65**(1) (2016), 11–16.
[32] R. Hilfer, *Applications of Fractional Calculus in Physics*, World Scientific, Singapore, 2000.
[33] J. Hu, Y. Ye, S. Shen, and J. Zhang, Lie symmetry analysis of the time fractional KdV-type equation, *Applied Mathematics and Computation*, **233** (2014), 439–444.
[34] Q. Huang and S. Shen, Lie symmetries and group classification of a class of time fractional evolution systems, *Journal of Mathematical Physics*, **56**(12) (2015), 123504.
[35] Q. Huang and R. Zhdanov, Symmetries and exact solutions of the time fractional Harry–Dym equation with Riemann–Liouville derivative, *Physica A: Statistical Mechanics and Its Applications*, **409** (2014), 110–118.
[36] N. H. Ibragimov, Invariant variational problems and conservation laws, *Theoretical and Mathematical Physics*, **1**(3) (1969), 267–276.
[37] N. H. Ibragimov, *Transformation Groups Applied to Mathematical Physics*, Reidel, Dordrecht, 1985.
[38] N. H. Ibragimov, *Elementary Lie Group Analysis and Ordinary Differential Equations*, Wiley, 1999.
[39] N. H. Ibragimov, Nonlinear self-adjointness and conservation laws, *Journal of Physics A: Mathematical and Theoretical*, **44** (2011), 432002.
[40] N. H. Ibragimov, *Transformation Groups and Lie Algebras*, World Scientific, 2013.
[41] N. Kh. Ibragimov and E. D. Avdonina, Nonlinear self-adjointness, conservation laws, and the construction of solutions of partial differential equations using conservation laws, *Russian Mathematcal Surveys*, **68**(5) (2013), 889–921.
[42] M. Inc, A. Yusuf, A. I. Aliyu, and D. Baleanu, Time-fractional Cahn–Allen and time-fractional Klein–Gordon equations: Lie symmetry analysis, explicit solutions and convergence analysis, *Physica A: Statistical Mechanics and Its Applications*, **493** (2018), 94–106.
[43] M. Inc, A. Yusuf, A. I. Aliyu, and D. Baleanu, Lie symmetry analysis and explicit solutions for the time fractional generalized Burgers–Huxley equation, *Optical and Quantum Electronics*, **50**(2) (2018), 94.
[44] M. Inc, A. Yusuf, A. I. Aliyu, and D. Baleanu, Lie symmetry analysis, explicit solutions and conservation laws for the space-time fractional nonlinear evolution equations, *Physica A: Statistical Mechanics and Its Applications*, **496** (2018), 371–383.
[45] H. Jafari, N. Kadkhoda, M. Azadi, and M. Yaghoubi, Group classification of the time-fractional Kaup–Kupershmidt equation, *Scientia Iranica*, **24**(1) (2017), 302–307.
[46] H. Jafari, N. Kadkhoda, and D. Baleanu, Fractional Lie group method of the time-fractional Boussinesq equation, *Nonlinear Dynamics*, **81**(3) (2015), 1569–1574.
[47] A. Jannelli, M. Ruggieri, and M. P. Speciale, Analytical and numerical solutions of fractional type advection-diffusion equation, *AIP Conference Proceedings*, **1863** (2017), 530005.
[48] A. Jannelli, M. Ruggieri, and M. P. Speciale, Exact and numerical solutions of time-fractional advection-diffusion equation with a nonlinear source term by means of the Lie symmetries, *Nonlinear Dynamics*, **92**(2) (2018), 543–555.
[49] G. F. Jefferson and J. Carminati, FracSym: automated symbolic computation of Lie symmetries of fractional differential equations, *Computer Physics Communications*, **185**(1) (2014), 430–441.
[50] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier, Amsterdam, 2006.

[51] J. Klafter, S. C. Lim, and R. Metzler (eds), *Fractional Dynamics: Recent Advances*, World Scientific, Singapore, 2011.

[52] R. Klages, G. Radons, and I. M. Sokolov (eds), *Anomalous Transport: Foundations and Applications*, Willey-VCH, Berlin, 2008.

[53] E. Lashkarian and S. R. Hejazi, Exact solutions of the time fractional nonlinear Schrödinger equation with two different methods, *Mathematical Methods in the Applied Sciences*, **41**(7) (2018), 2664–2672.

[54] E. Lashkarian, S. R. Hejazi, and E. Dastranj, Conservation laws of $(3 + \alpha)$-dimensional time-fractional diffusion equation, *Computers and Mathematics with Applications*, **75**(3) (2018), 740–754.

[55] R. A. Leo, G. Sicuro, and P. Tempesta, A theorem on the existence of symmetries of fractional PDEs, *Comptes Rendus Mathematique*, **352**(3) (2014), 219–222.

[56] R. A. Leo, G. Sicuro, and P. Tempesta, A foundational approach to the lie theory for fractional order partial differential equations, *Fractional Calculus and Applied Analysis*, **20**(1) (2017), 212–231.

[57] S. Lie and G. Scheffers, *Lectures on Differential Equations with Known Infinitesimal Transformations*, Teubner, Leipzig, 1891.

[58] H. Liu, Complete group classifications and symmetry reductions of the fractional fifth-order kdv types of equations, *Studies in Applied Mathematics*, **131**(4) (2013), 317–330.

[59] C. Lu, C. Fu, and H. Yang, Time-fractional generalized Boussinesq equation for Rossby solitary waves with dissipation effect in stratified fluid and conservation laws as well as exact solutions, *Applied Mathematics and Computation*, **327** (2018), 104–116.

[60] S. Yu. Lukashchuk, Constructing conservation laws for fractional-order integro-differential equations, *Theoretical and Mathematical Physics*, **184**(2) (2015), 1049–1066.

[61] S. Yu. Lukashchuk, Conservation laws for time-fractional subdiffusion and diffusion-wave equations, *Nonlinear Dynamics*, **80**(1–2) (2015), 791–802.

[62] S. Yu. Lukashchuk, Symmetry reduction and invariant solutions for nonlinear fractional diffusion equation with a source term, *Ufa Mathematical Journal*, **8**(4) (2016), 111–122.

[63] S. Yu. Lukashchuk, *Analytical methods and numerical algorithms for investigating diffusion-type mathematical models with fractional derivatives*, Thesis (Dr. Phys.-Mathem. Science), 2017 (in Russian).

[64] S. Yu. Lukashchuk and A. V. Makunin, Group classification of nonlinear time-fractional diffusion equation with a source term, *Applied Mathematics and Computation*, **257** (2015), 335–343.

[65] A. Majlesi, H. Roohani Ghehsareh, and A. Zaghian, On the fractional Jaulent–Miodek equation associated with energy-dependent Schrödinger potential: Lie symmetry reductions, explicit exact solutions and conservation laws, *European Physical Journal Plus*, **132**(12) (2017), 516.

[66] A. B. Malinowska, A formulation of the fractional Noether-type theorem for multidimensional Lagrangians, *Applied Mathematics Letters*, **25**(11) (2012), 1941–1946.

[67] R. Najafi, F. Bahrami, and M. S. Hashemi, Classical and nonclassical Lie symmetry analysis to a class of nonlinear time-fractional differential equations, *Nonlinear Dynamics*, **87**(3) (2017), 1785–1796.

[68] P. J. Olver, *Applications of Lie Groups to Differential Equations*, Springer Science & Business Media, 2000.

[69] L. V. Ovsyannikov, *Group Analysis of Differential Equations*, Academic Press, New York, 1982.

[70] M. Pan, L. Zheng, C. Liu, and F. Liu, Symmetry analysis and conservation laws to the space-fractional Prandtl equation, *Nonlinear Dynamics*, **90**(2) (2017), 1343–1351.

[71] M. Pan, L. Zheng, F. Liu, and X. Zhang, Lie group analysis and similarity solution for fractional Blasius flow, *Communications in Nonlinear Science and Numerical Simulation*, **37** (2016), 90–101.

[72] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, 1999.

[73] C.-Y. Qin, S.-F. Tian, X.-B. Wang, and T.-T. Zhang, Lie symmetries, conservation laws and explicit solutions for time fractional Rosenau–Haynam equation, *Communications in Theoretical Physics*, **67**(2) (2017), 157–165.

[74] S. Rashidi and S. R. Hejazi, Symmetry properties, similarity reduction and exact solutions of fractional Boussinesq equation, *International Journal of Geometric Methods in Modern Physics*, **14**(6) (2017), 1750083.

[75] S. Rashidi and S. R. Hejazi, Analyzing Lie symmetry and constructing conservation laws for time-fractional Benny–Lin equation, *International Journal of Geometric Methods in Modern Physics*, **14**(12) (2017), 1750170.

[76] S. S. Ray and S. Sahoo, Invariant analysis and conservation laws of $(2+1)$ dimensional time-fractional ZK-BBM equation in gravity water waves, *Computers and Mathematics with Applications*, **75**(7) (2018), 2271–2279.

[77] W. Rui and X. Zhang, Invariant analysis and conservation laws for the time fractional foam drainage equation, *European Physical Journal Plus*, **130**(10) (2015), 192.

[78] W. Rui and X. Zhang, Lie symmetries and conservation laws for the time fractional Derrida–Lebowitz–Speer–Spohn equation, *Communications in Nonlinear Science and Numerical Simulation*, **34** (2016), 38–44.

[79] J. Sabatier, O. P. Agrawal, and J. A. T. Machado (eds), *Advances in Fractional Calculus. Theoretical Developments and Applications in Physics and Engineering*, Springer, Dordrecht, 2007.

[80] E. Saberi and S. Reza Hejazi, Lie symmetry analysis, conservation laws and exact solutions of the time-fractional generalized Hirota–Satsuma coupled KdV system, *Physica A: Statistical Mechanics and Its Applications*, **492** (2018), 296–307.

[81] R. Sahadevan and T. Bakkyaraj, Invariant analysis of time fractional generalized Burgers and Korteweg–de Vries equations, *Journal of Mathematical Analysis and Applications*, **393**(2) (2012), 341–347.

[82] R. Sahadevan and P. Prakash, On Lie symmetry analysis and invariant subspace methods of coupled time fractional partial differential equations, *Chaos, Solitons and Fractals*, **104** (2017), 107–120.

[83] S. Sahoo and S. S. Ray, Analysis of Lie symmetries with conservation laws for the $(3+1)$ dimensional time-fractional mKdV-ZK equation in ion-acoustic waves, *Nonlinear Dynamics*, **90**(2) (2017), 1105–1113.

[84] S. Sahoo and S. S. Ray, Invariant analysis with conservation laws for the time fractional Drinfeld–Sokolov–Satsuma–Hirota equations, *Chaos, Solitons and Fractals*, **104** (2017), 725–733.

[85] S. Sahoo and S. S. Ray, The conservation laws with Lie symmetry analysis for time fractional integrable coupled KdV-mKdV system, *International Journal of Non-Linear Mechanics*, **98** (2018), 114–121.

[86] S. G. Samko, A. A. Kilbas, and O. I. Marichev, *Fractional Integrals and Derivatives: Theory and Applications*, Gordon and Breach, New York, 1993.

[87] S. San, Invariant analysis of nonlinear time fractional Qiao equation, *Nonlinear Dynamics*, **85**(4) (2016), 2127–2132.

[88] K. Singla and R. K. Gupta, On invariant analysis of some time fractional nonlinear systems of partial differential equations I, *Journal of Mathematical Physics*, **57** (2016), 101504.

[89] K. Singla and R. K. Gupta, Space-time fractional nonlinear partial differential equations: symmetry analysis and conservation laws, *Nonlinear Dynamics*, **89**(1) (2017), 321–331.

[90] K. Singla and R. K. Gupta, Comment on “Lie symmetries and group classification of a class of time fractional evolution systems” [J. Math. Phys. 56, 123504 (2015)], *Journal of Mathematical Physics*, **58**(5) (2017), 054101.

[91] K. Singla and R. K. Gupta, Conservation laws for certain time fractional nonlinear systems of partial differential equations, *Communications in Nonlinear Science and Numerical Simulation*, **53** (2017), 10–21.

[92] K. Singla and R. K. Gupta, Generalized Lie symmetry approach for fractional order systems of differential equations III, *Journal of Mathematical Physics*, **58**(6) (2017), 061501.

[93] V. E. Tarasov, *Fractional Dynamics: Application of Fractional Calculus to Dynamics of Particles, Fields and Media*, Springer, Heidelberg, 2011.

[94] V. Uchaikin and R. Sibatov (eds), *Fractional Kinetics in Solids: Anomalous Charge Transport in Semiconductors, Dielectrics and Nanosystems*, World Scientific, Singapore, 2013.

[95] G. Wang, A. H. Kara, and K. Fakhar, Symmetry analysis and conservation laws for the class of time-fractional nonlinear dispersive equation, *Nonlinear Dynamics*, **82**(1–2) (2015), 281–287.

[96] G. Wang and T. Xu, Symmetry properties and explicit solutions of the nonlinear time fractional KdV equation, *Boundary Value Problems*, **2013** (2013), 232.

[97] G. Wang and T. Xu, Invariant analysis and explicit solutions of the time fractional nonlinear perturbed Burgers equation, *Nonlinear Analysis: Modelling and Control*, **20**(4) (2015), 570–584.

[98] G.-W. Wang and A. H. Kara, Group analysis, fractional explicit solutions and conservation laws of time fractional generalized burgers equation, *Communications in Theoretical Physics*, **69**(1) (2018), 5–8.

[99] G.-W. Wang, X.-Q. Liu, and Y.-Y. Zhang, Lie symmetry analysis to the time fractional generalized fifth-order KdV equation, *Communications in Nonlinear Science and Numerical Simulation*, **18**(9) (2013), 2321–2326.

[100] G.-W. Wang and T.-Z. Xu, Invariant analysis and exact solutions of nonlinear time fractional Sharma–Tasso–Olver equation by Lie group analysis, *Nonlinear Dynamics*, **76**(1) (2014), 571–580.

[101] G. W. Wang and M. S. Hashemi, Lie symmetry analysis and soliton solutions of time-fractional $K(m,n)$ equation, *Pramana—Journal of Physics*, **88**(1) (2017), 7.

[102] G. W. Wang, T. Z. Xu, and T. Feng, Lie symmetry analysis and explicit solutions of the time fractional fifth-order KdV equation, *PLoS ONE*, **9**(2) (2014), e88336.

[103] L. Wang, S.-F. Tian, Z.-T. Zhao, and X.-Q. Song, Lie symmetry analysis and conservation laws of a generalized time fractional foam drainage equation, *Communications in Theoretical Physics*, **66**(1) (2016), 35–40.

[104] X.-B. Wang, S.-F. Tian, C.-Y. Qin, and T.-T. Zhang, Lie symmetry analysis, conservation laws and exact solutions of the generalized time fractional Burgers equation, *Europhysics Letters*, **114**(2) (2016), 20003.

[105] X.-B. Wang, S.-F. Tian, C.-Y. Qin, and T.-T. Zhang, Lie symmetry analysis, conservation laws and analytical solutions of a time-fractional generalized KdV-type equation, *Journal of Nonlinear Mathematical Physics*, **24**(4) (2017), 516–530.

[106] Z. Xiao and L. Wei, Symmetry analysis, conservation laws of a time fractional fifth-order Sawada–Kotera equation, *Journal of Applied Analysis and Computation*, **7**(4) (2017), 1275–1284.

[107] E. Yaşar, Y. Yildirim, and C. M. Khalique, Lie symmetry analysis, conservation laws and exact solutions of the seventh-order time fractional Sawada–Kotera–Ito equation, *Results in Physics*, **6** (2016), 322–328.

[108] X.-J. Yu and X.-Q. Liu, Lie symmetry analysis of the time fractional Boussinesq equation, *Wuli Xuebao/Acta Physica Sinica*, **62**(23) (2013), 230201.

[109] H. A. Zedan, S. Shapll, and A. Abdel-Malek, Invariance of the nonlinear generalized NLS equation under the Lie group of scaling transformations, *Nonlinear Dynamics*, **82**(4) (2015), 2001–2005.

[110] Y. Zhang, Lie symmetry analysis of time-fractional generalised Korteweg–de Vries equations, *International Journal of Dynamical Systems and Differential Equations*, **5**(1) (2015), 27–41.

[111] Y. Zhang, Lie symmetry analysis and exact solutions of general time-fractional fifth-order Korteweg–de Vries equation, *IAENG International Journal of Applied Mathematics*, **47**(1) (2017), 1–9.

[112] Z. Zhao and B. Han, Symmetry analysis and conservation laws of the time fractional Kaup–Kupershmidt equation from capillary gravity waves, *Mathematical Modelling of Natural Phenomena*, **13**(2) (2018), 24.

Sabir Umarov

Fractional Duhamel principle

Abstract: The chapter discusses fractional generalizations of the well-known Duhamel principle. The Duhamel principle, introduced nearly 200 years ago, reduces the Cauchy problem for the inhomogeneous differential equation to the Cauchy problem for a corresponding homogeneous differential equation. Unlike the classical case, fractional versions of the Duhamel principle require a fractional derivative of the inhomogeneous term in the initial condition of the reduced equation. We present generalizations of the Duhamel principle to wide classes of single time-fractional and distributed order pseudo-differential equations, both containing Caputo–Djrbashian and Riemann–Liouville derivatives. The abstract case also presented to capture initial-boundary value problems for equations given on bounded domains. Note that the fractional Duhamel principle for equations containing Caputo–Djrbashian derivatives and Riemann–Liouville derivatives significantly differ. A number of applications of the fractional Duhamel principle have been appeared recent years. Here, we discuss some of these applications, as well.

Keywords: Fractional Duhamel principle, pseudo-differential operator with singular symbol, fractional stochastic differential equation

MSC 2010: 34A08, 35R11, 35S10, 60G22

1 Introduction

1.1 The Duhamel principle

Beauty is the first test; there is no permanent place in the world for ugly mathematics
Godfrey Harold Hardy

The well-known classical Duhamel principle was first introduced by the French mathematician and physicist, Jean-Marie-Constant Duhamel,[1] nearly 200 years ago. The main idea of the Duhamel principle is to reduce the Cauchy problem for an inhomogeneous partial differential equation considered on a time interval with the initial time $t = 0$ to the Cauchy problem for the corresponding homogeneous equation, but considered on the time interval with the variable initial time $t = \tau$. Then the solution to the Cauchy problem for the inhomogeneous equation can be obtained through the

1 J.-M.-C. Duhamel, born February 5, 1797, Saint-Malo, France—died April 29, 1872, Paris, France.

Sabir Umarov, University of New Haven, West Haven, USA, e-mail: sumarov@newhaven.edu

https://doi.org/10.1515/9783110571660-017

Duhamel integral composed via the solution of the Cauchy problem for the homogeneous equation. Duhamel demonstrated this principle for the heat equation and for the wave equation. The exact formulation of the Duhamel principle in these two cases is given in Section 2.1.

In this chapter, we present various generalizations of this famous principle to a wide classes of fractional-order differential, pseudo-differential, and differential-operator equations. We note that unlike the classical case, fractional versions of the Duhamel principle require a fractional derivative of the source function in the initial condition of the reduced Cauchy problem. Moreover, the fractional version of the Duhamel principle for equations containing fractional-order Riemann–Liouville derivatives significantly differs from the case when equations contain fractional order Caputo–Djrbashian derivatives. Thus, the fractional versions of the Duhamel principle are nontrivial generalizations of the classical Duhamel principle.

What concerns the history of fractional Duhamel principle, in the particular case of fractional-order partial differential equations with a single "fractional" term in the equation, a fractional analog of the Duhamel principle was obtained in [32, 33]. The general case of distributed order fractional differential-operator equations is studied in [28] and in Chapter 6 of the book [29]. The fractional Duhamel principle plays an important role in the theory of modern fractional differential equations with nonhomogeneous terms. We note that a nonhomogeneous term may emerge in fractional-order differential equations as a memory effect even without external sources (see Section 4.3). Currently, the fractional Duhamel principle is widely used by many researchers for the solution of theoretical and applied problems in mathematics and applied sciences; see, for example, papers [2, 3, 6, 7, 10, 14, 18, 21, 22, 25, 29, 35] and references therein.

In Section 1.2 we expound the classical Duhamel principle formally. However, its informal presentation requires some appropriate function spaces. For this reason and for exhaustive exposition of the fractional Duhamel principle we introduce in Section 2 a class of pseudo-differential operators with singular symbols and the spaces where these operators are meaningfully defined. Therefore, the general integer order case of the Duhamel principle is presented in Section 2.3. Section 3 exhibits fractional generalizations of the Duhamel principle. First, we formulate the fractional Duhamel principle for equations with a single fractional derivative. Section 3.2 contains fractional generalizations of the Duhamel principle for general equations containing fractional derivatives in the sense of Caputo–Djrbashian. The case of equations containing fractional derivatives in the sense of Riemann–Liouville is discussed in Section 3.3. Lastly, some important applications of the fractional Duhamel principle are discussed in Section 4.

1.2 Classical Duhamel principle

Suppose that $B = B(x, \frac{\partial}{\partial t}, D_x)$, where $D_x = (\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n})$, is a linear differential operator with coefficients not depending on t, and containing temporal derivatives of order not higher than 1. Consider the Cauchy problem

$$\frac{\partial^2 u}{\partial t^2}(t,x) + Bu(t,x) = h(t,x), \quad t > 0, \ x \in \mathbb{R}^n, \tag{1}$$

with homogeneous initial conditions

$$u(0,x) = 0, \quad \frac{\partial u}{\partial t}(0,x) = 0, \ x \in \mathbb{R}^n. \tag{2}$$

Let a sufficiently smooth function $v(t,\tau,x)$, $t \geq \tau$, $\tau \geq 0$, $x \in R^n$, (with the dummy variable τ) be a solution of the homogeneous equation

$$\frac{\partial^2 v}{\partial t^2}(t,\tau,x) + Bv(t,\tau,x) = 0,$$

for all $t > \tau$ and $x \in \mathbb{R}^n$, which satisfies the following conditions:

$$v(t,\tau,x)|_{t=\tau} = 0, \quad \frac{\partial v}{\partial t}(t,\tau,x)\bigg|_{t=\tau} = h(\tau,x).$$

Then a solution of the Cauchy problem (1), (2) is given by means of the integral

$$u(t,x) = \int_0^t v(t,\tau,x)d\tau. \tag{3}$$

The formulated statement is known as *Duhamel's principle*, and the integral in (3) as *Duhamel's integral*.

A similar statement is valid in the case of the Cauchy problem with a homogeneous initial condition for a first-order inhomogeneous partial differential equation

$$\frac{\partial u}{\partial t}(t,x) + Cu(t,x) = h(t,x), \quad t > 0, \ x \in R^n, \tag{4}$$

where $C = C(x, D_x)$ is a linear differential operator containing only spatial derivatives, and with coefficients not depending on t (see [5]). Namely, let a sufficiently smooth function $v(t,\tau,x)$, $t \geq \tau$, $\tau \geq 0$, $x \in R^n$, solve the homogeneous equation

$$\frac{\partial v}{\partial t}(t,\tau,x) + Cv(t,\tau,x) = 0, \quad t > \tau, \ x \in \mathbb{R}^n,$$

for all $t > \tau$ and $x \in \mathbb{R}^n$, and satisfy the following condition:

$$v(t,\tau,x)|_{t=\tau} = h(\tau,x), \quad x \in \mathbb{R}^n.$$

Then the function $u(t,x)$ defined by means of the Duhamel integral

$$u(t,x) = \int_0^t v(t,\tau,x)d\tau, \tag{5}$$

solves the Cauchy problem for nonhomogeneous equation (4) with the homogeneous initial condition $u(0,x) = 0$, $x \in \mathbb{R}^n$.

We note that the Duhamel principle is valid for differential equations on a bounded domain as well with an appropriate boundary conditions. In Section 3.3, we will discuss the case of fractional Duhamel principle for equations given on a bounded domain in the operator language.

2 Pseudo-differential operators with singular symbols

Formulations of fractional versions of the Duhamel principle in the general case are given for time-fractional pseudo-differential equations with singular symbols. Such pseudo-differential operators contain in particular space-fractional derivatives and operators, as well (see [29]). Therefore, in this section we present a brief depiction of pseudo-differential operators with singular symbols and spaces of functions and distributions where these operators are well-defined.

2.1 The space of test functions and corresponding ψ-distributions

Let $\Psi_G(\mathbb{R}^n)$ be the set of tempered distributions φ on $\mathbb{R}^n$ whose Fourier transform $\Phi(\xi) = F[\varphi](\xi)$ has a compact support on a set $G \in \mathbb{R}^n$. It follows from the general theory of tempered distributions that Φ is also a tempered distribution. In accordance with the Paley–Wiener–Schwartz theorem ([29], p. 38), $\varphi(x) = F^{-1}[\Phi] \in \Psi_G(\mathbb{R}^n)$ has an analytic extension $\varphi(x+iy)$ to the whole complex space $\mathbb{C}^n$ which satisfies the estimate

$$\left|\varphi(x+iy)\right| \le C_m\big(1+|x|\big)^m e^{R|y|}, \quad x+iy \in \mathbb{C}^n,$$

with some integer m, constant C_m, and real $R > 0$, such that the ball $|x| \le R$ contains the support of $\Phi = F[\varphi]$. Denote by $\Psi_{G,p}(\mathbb{R}^n)$, $p \ge 1$, the subset of $\Psi_G(\mathbb{R}^n)$ containing functions $\varphi \in L_p(\mathbb{R}^n) \cap \Psi_G(\mathbb{R}^n)$. A sequence $\varphi_k \in \Psi_{G,p}(\mathbb{R}^n)$, $k = 1,2,\dots$, is said to converge to $\varphi \in \Psi_{G,p}(\mathbb{R}^n)$, if:

1. there exist a compact $K \subset G$, such that supports $\operatorname{supp} F[\varphi_k] \subseteq K$, for all $k = 1,2,\dots$, and
2. $\varphi_k \to \varphi$ as $k \to \infty$ in the norm of $L_p(\mathbb{R}^n)$.

The set $\Psi_{G,p}(\mathbb{R}^n)$ with the introduced convergence forms a space of test functions for ψ-distributions. Namely, a linear continuous functional defined on $\Psi_{G,p}(\mathbb{R}^n)$ is called a ψ-distribution. The space of ψ-distributions is denoted by $\Psi'_{-G,p'}(\mathbb{R}^n)$, where $p' = p/(p-1)$. The ψ-distributions are studied in detail in [29].

The denseness of the space of test functions $\Psi_{G,p}(\mathbb{R}^n)$ in $L_p(\mathbb{R}^n)$ and other classical function spaces depends on G. The proposition below shows denseness of $\Psi_{G,p}(\mathbb{R}^n)$ in Besov spaces $B^s_{p,q}(\mathbb{R}^n)$ with $1 < p, q < \infty$ and $-\infty < s < \infty$.

Proposition 2.1 ([29], p. 57). *Let* $1 < p, q < \infty$, $-\infty < s < +\infty$.

1. *The embeddings*

$$\Psi_{G,p}(\mathbb{R}^n) \hookrightarrow B^s_{pq}(\mathbb{R}^n) \hookrightarrow \Psi'_{-G,p'}(\mathbb{R}^n), \tag{6}$$

are continuous.

2. *Moreover, if* $\mathbb{R}^n \setminus G$ *has the n-dimensional zero measure and* $2 \le p < \infty$, *then the left embedding in* (6) *is dense. Conversely, the denseness of the left embedding in* (6) *implies that* $\mathbb{R}^n \setminus G$ *has the n-dimensional zero measure, that is,* $\mu_n(\mathbb{R}^n \setminus G) = 0$.

2.2 Pseudo-differential operators with singular symbols

Pseudo-differential operators with singular symbols are a wide generalization of classical pseudo-differential operators introduced in the 1960th by Kon, Nirenberg, and Hörmander. As is known, in many boundary value problems of mathematical physics symbols of solution operators have different types of singularities: strong singularities on a finite part of the space, singularities due to increase or nonsufficiently fast decrease at infinity, or singularities due to irregular points of symbols. These kind of operators have two distinctions from the standard pseudo-differential ones. First, their symbols contain singularities with respect to dual variable, and second, their orders, generally speaking, are not bounded. This section briefly discusses the definition and some mapping properties of such operators. We refer the reader to [8, 29] for a detailed discussion of properties of pseudo-differential operators and equations with singular symbols.

Definition 2.2. Let $A(\xi) \in C^\infty(G)$, $G \subset R^n$. We determine an operator $A(D)$ by the formula

$$A(D)f(x) = \frac{1}{(2\pi)^n} \int\limits_G A(\xi)F[f](\xi)e^{-ix\xi}d\xi, \tag{7}$$

provided the integral on the right-hand side exists. The function $A(\xi)$ is called a symbol of $A(D)$.

In Definition 2.2, the function $A(\xi)$ may have an arbitrary type of singularities outside G or on its boundary. Generally speaking, operators $A(D)$ with symbols $A \in$

$C^\infty(G)$, may not be meaningful even for functions in the space $C_0^\infty(\mathbb{R}^n)$. Indeed, let $\xi_0 \in \mathbb{R}^n$ be a nonintegrable singular point of $A(\xi)$ and denote by $O(\xi_0)$ some neighborhood of ξ_0. Let us take a function $f_0 \in C_0^\infty(\mathbb{R}^n)$ with $F[f_0](\xi) > 0$ for $\xi \in O(\xi_0)$ and $F[f_0](\xi_0) = 1$. Then it is easy to verify that $A(D)f_0(x) = \infty$. However, for functions $f \in \Psi_{G,p}(\mathbb{R}^n)$ the integral in (7) is convergent due to the compactness of $\operatorname{supp} F[f] \subset G$ and, therefore, $A(D)f$ is well-defined. In this sense, the space $\Psi_{G,p}(\mathbb{R}^n)$ serves as a domain of pseudo-differential operators with symbols singular in the dual variable. We use the abbreviation ΨDOSS for *pseudo-differential operators with singular symbols*.

Proposition 2.3. *The space $\Psi_{G,p}(\mathbb{R}^n)$, $1 \le p \le \infty$, is invariant with respect to any operator $A(D)$ with the symbol $A(\xi) \in C^\infty(G)$. Moreover, the mapping $A(D) : \Psi_{G,p}(\mathbb{R}^n) \to \Psi_{G,p}(\mathbb{R}^n)$ is continuous.*

Similar statements hold also for the space $\Psi'_{-G,p'}(\mathbb{R}^n)$. This fact and the continuity of closures of pseudo-differential operators with singular symbols in Besov spaces see in [29].

2.3 Duhamel principle for integer order equations: general case

Consider the Cauchy problem for a nonhomogeneous pseudo-differential equation with homogeneous initial conditions

$$\frac{\partial^m u(t,x)}{\partial t^m} + \sum_{k=0}^{m-1} A_k(t,D)\frac{\partial^k u(t,x)}{\partial t^k} = h(t,x), \quad t > 0,\ x \in \mathbb{R}^n, \tag{8}$$

$$\frac{\partial^k u(0,x)}{\partial t^k} = 0, \quad x \in \mathbb{R}^n,\ k = 0,\dots,m-1. \tag{9}$$

Here, the operators $A_k(t,D)$, $k = 0,\dots,m-1$, are pseudo-differential operators, generally speaking, with singular symbols in $S^\infty(G)$, and $h(t,x)$ is a continuous function in the variable t and belongs to the space $\Psi_{G,p}(\mathbb{R}^n)$. Alternatively, $h(t,x)$ also can be a distribution in $\Psi'_{-G,p'}(\mathbb{R}^n)$ continuous in the variable t. The Duhamel principle establishes a connection between the solution of the Cauchy problem (8)–(9) and the solution of the following Cauchy problem for homogeneous equation with nonhomogeneous initial conditions:

$$\frac{\partial^m U(t,\tau,x)}{\partial t^m} + \sum_{k=0}^{m-1} A_k(t,D)\frac{\partial^k U(t,\tau,x)}{\partial t^k} = 0, \quad t > \tau,\ x \in \mathbb{R}^n, \tag{10}$$

$$\left.\frac{\partial^k U(t,\tau,x)}{\partial t^k}\right|_{t=\tau+0} = 0, \quad x \in \mathbb{R}^n,\ k = 0,\dots,m-2, \tag{11}$$

$$\left.\frac{\partial^{m-1} U(t,\tau,x)}{\partial t^{m-1}}\right|_{t=\tau+0} = h(\tau,x), \quad x \in \mathbb{R}^n. \tag{12}$$

In the general case, the classical Duhamel principle is formulated as follows.

Theorem 2.4. *Let $U(t,\tau,x)$ be a solution of the Cauchy problem* (10)–(12). *Then a solution of the Cauchy problem* (8), (9) *is represented via the Duhamel integral*

$$u(t,x) = \int_0^t U(t,\tau,x)d\tau, \quad t > 0,\ x \in \mathbb{R}^n. \tag{13}$$

Here is a brief sketch why this statement is true. Obviously, $u(t,x)$ defined by (13) satisfies the condition $u(0,x) = 0$. Further, for the first-order derivative of $u(t,x)$ in the variable t, one has

$$\frac{\partial u(t,x)}{\partial t} = U(t,t,x) + \int_0^t \frac{\partial U(t,\tau,x)}{\partial t} d\tau. \tag{14}$$

By virtue of (11), $k = 0$, the first term on the right-hand side of equation (14) is zero. This implies $\frac{\partial u(0,x)}{\partial t} = 0$. Further, consequently differentiating and taking into account conditions (11) with $k = 1,\dots,m-2$, we have

$$\begin{aligned}\frac{\partial^k u(t,x)}{\partial t^k} &= \frac{\partial^{k-1} U(t,t,x)}{\partial t^{k-1}} + \int_0^t \frac{\partial^k U(t,\tau,x)}{\partial t^k} d\tau \\ &= \int_0^t \frac{\partial^k U(t,\tau,x)}{\partial t^k} d\tau.\end{aligned}$$

The latter implies that $\frac{\partial^k u(0,x)}{\partial t^k} = 0$, for $k = 2,\dots,m-1$. Therefore, the function $u(t,x)$ in (13) satisfies all the initial conditions (9). Moreover, substituting (13) to (8), and taking into account condition (12), we have

$$\begin{aligned}&\frac{\partial^m u(t,x)}{\partial t^m} + \sum_{k=0}^{m-1} A_k(t,D) \frac{\partial^k u(t,x)}{\partial t^k} \\ &= \frac{\partial^m}{\partial t^m} \int_0^t U(t,\tau,x)d\tau + \sum_{k=0}^{m-1} A_k(t,D) \frac{\partial^k}{\partial t^k} \int_0^t U(t,\tau,x)d\tau \\ &= \frac{\partial^{m-1} U(t,\tau,x)}{\partial t^{m-1}} + \int_0^t \frac{\partial^m U(t,\tau,x)}{\partial t^m} d\tau + \sum_{k=0}^{m-1} A_k(t,D) \int_0^t \frac{\partial^k U(t,\tau,x)}{\partial t^k} d\tau \\ &= h(t,x) + \int_0^t \left[\frac{\partial^m U(t,\tau,x)}{\partial t^m} + \sum_{k=0}^{m-1} A_k(t,D) \frac{\partial^k U(t,\tau,x)}{\partial t^k} \right] d\tau \\ &= h(t,x), \quad t > 0,\ x \in \mathbb{R}^n.\end{aligned} \tag{15}$$

The expression in brackets on the right-hand side of (15) vanishes since $U(t,\tau,x)$ is a solution to equation (10). Hence, $u(t,x)$ defined by (13) satisfies also equation (8).

The reader can follow this idea in the case of fractional Duhamel principle as well.

3 Fractional Duhamel principle

3.1 Fractional Duhamel principle: single fractional derivative

We start with a useful heuristic observation for understanding the fractional Duhamel principle. Assume $0 < \beta < 1$. Consider the Cauchy problem for the nonhomogeneous fractional heat equation with the Caputo–Djrbashian FD of order β:

$$^{\mathrm{C}}D^{\beta}_{0+}u(t,x) = k_\beta \Delta u(t,x) + h(t,x), \quad t > 0,\ x \in \mathbb{R}^n, \tag{16}$$

with the initial condition $u(0,x) = 0$, $x \in \mathbb{R}^n$. Then, due to the Duhamel principle, the solution of this Cauchy problem can be represented as the Duhamel integral

$$u(t,x) = \int_0^t V(t,\tau,x)d\tau, \tag{17}$$

where $V(t,\tau,x)$ is a solution of the Cauchy problem for the following homogeneous equation:

$$^{\mathrm{C}}D^{\beta}_{\tau+}V(t,\tau,x) = k_\beta \Delta\, V(t,\tau,x), \quad t > \tau,\ x \in R^n, \tag{18}$$

with an initial condition $V(\tau,\tau,x) = H(\tau,x)$. Here, $^{\mathrm{C}}D^{\beta}_{\tau+}$ is the Caputo–Djrbashian fractional derivative of order β with the initial point τ, and $H(\tau,x)$ is a function related to $h(t,x)$ in a certain way. In order to see this relationship between $H(t,x)$ and $h(t,x)$, suppose that t is small and the initial temperature is zero. Then ignoring the heat flow during the time interval $(0,t)$, that is, $k_\beta \Delta u \sim 0$, it follows from equation (16) that $^{\mathrm{C}}D^{\beta}_{0+}u(t,x) \sim h(t,x)$. Further, taking into account the well-known relation (see, e. g., [29])

$$^{\mathrm{C}}D^{\beta}_{0+} = J^{1-\beta}_{0+}D,$$

we have

$$Du(t,x) \sim {}^{\mathrm{RL}}D^{1-\beta}_{0+}h(t,x).$$

For small t, this implies

$$V(0,0,x) \sim Du(0,x) \sim {}^{\mathrm{RL}}D^{1-\beta}_{0+}h(0,x),$$

which can easily be verified differentiating (17). Repeating these heuristic calculations for small time interval $(\tau, \tau + \varepsilon)$, we obtain the relationship $H(\tau,x) \sim {}^{\mathrm{RL}}D^{1-\beta}_{0+}h(\tau,x)$. Hence, one can expect that the initial condition for $V(t,\tau,x)$ in equation (18) has the form

$$V(t,\tau,x)|_{t=\tau} = {}^{\mathrm{RL}}D^{1-\beta}_{0+}h(\tau,x),$$

where $h(t,x)$ is the function on the right-hand side of equation (16). As the assertion below shows, the above heuristic observation is always true.

Theorem 3.1. *Suppose that $V(t,\tau,x)$, $0 \le \tau \le t$, $x \in \mathbb{R}^n$, is a solution of the Cauchy problem for a homogeneous equation of fractional-order $\beta \in (0,1)$*

$$ {}^{\mathrm{C}}D^{\beta}_{\tau+}V(t,\tau,x) - A(x,D_x)V(t,\tau,x) = 0, \quad t > \tau,\ x \in \mathbb{R}^n, \tag{19}$$

$$ V(t,\tau,x)\Big|_{t=\tau} = {}^{\mathrm{C}}D^{1-\beta}_{0+}h(\tau,x), \quad x \in \mathbb{R}^n, \tag{20}$$

where $h(t,x) \in C^1[t \ge 0; \mathcal{D}(A(D_x))]$, and satisfies the condition $h(0,x) = 0$. Then

$$ v(t,x) = \int_0^t V(t,\tau,x)d\tau \tag{21}$$

is a solution of the inhomogeneous Cauchy problem

$$ {}^{\mathrm{C}}D^{\beta}_{0+}v(t,x) - A(x,D_x)v(t,x) = h(t,x), \quad t > 0,\ x \in \mathbb{R}^n, \tag{22}$$

$$ v(0,x) = 0, \quad x \in \mathbb{R}^n. \tag{23}$$

Remark 3.2.
1. In the Theorem 3.1 $\mathcal{D}(A(x,D_x))$ means a domain of the operator $A(D_x)$. For example, if $A(x,D_x)$ is a pseudo-differential operator with a symbol $a(x,\xi) \in S^{\infty}(\mathbb{R}^n \times G)$, then as $\mathcal{D}(A(x,D_x))$ can be taken $\Psi_{G,p}(\mathbb{R}^n)$, or $\Psi'_{-G,p'}(\mathbb{R}^n)$.
2. The condition $h(0,x) = 0$ in the theorem is not essential. If this condition is not verified, then the initial condition (20) takes the form

$$ V(t,\tau,x)\Big|_{t=\tau} = {}^{\mathrm{RL}}D^{1-\beta}h(\tau,x), \quad x \in \mathbb{R}^n.$$

The time-fractional derivative in (pseudo-) differential equation (19) is in the sense of Caputo–Djrbashian. However, if the time fractional derivative in the equation is in the sense of Rieman–Liouville, then formulation of the Duhamel principle changes.

Theorem 3.3. *Suppose that $V(t,\tau,x)$, $t \ge \tau \ge 0$, is a solution of the Cauchy type problem for the homogeneous equation*

$$ {}^{\mathrm{RL}}D^{\alpha}_{\tau+}V(t,\tau,x) + A(x,D)V(t,\tau,x) = 0, \quad t > \tau,\ x \in \mathbb{R}^n, \tag{24}$$

$$ J^{1-\alpha}_{\tau+}V(t,\tau,x)|_{t=\tau+} = h(\tau,x), \quad x \in \mathbb{R}^n, \tag{25}$$

where $0 < \alpha < 1$ and $h(\tau,x)$, $\tau \ge 0$, is a function continuous in τ and for each fixed τ belongs to $\mathcal{D}(A(x,D))$. Then the Duhamel integral

$$ u(t,x) = \int_0^t V(t,\tau,x)d\tau, \quad t > 0,\ x \in \mathbb{R}^n, \tag{26}$$

solves the Cauchy-type problem for the inhomogeneous equation

$$ {}^{\mathrm{RL}}D^{\alpha}_{0+}u(t) + A(x,D)u(t,x) = h(t,x), \quad t > 0,\ x \in \mathbb{R}^n, \tag{27}$$

with the homogeneous initial condition

$$J_{0+}^{1-\alpha} u(0+, x) = 0, \quad x \in \mathbb{R}^n.$$

For equations of higher order $\beta > 1$, the above theorems take the following forms.

Theorem 3.4. *Assume $m \geq 1$, $m - 1 < \beta \leq m$, and $V(t, \tau, x)$ is a solution of the Cauchy problem for the homogeneous equation*

$$^{\mathrm{C}}D_{\tau+}^{\beta} V(t, \tau, x) - A(x, D_x) V(t, \tau, x) = 0, \quad t > \tau, \ x \in \mathbb{R}^n, \tag{28}$$

with the Cauchy conditions

$$\frac{\partial^k V}{\partial t^k}(t, \tau, x)\bigg|_{t=\tau} = 0, \quad x \in \mathbb{R}^n, \quad k = 0, \ldots, m-2, \tag{29}$$

$$\frac{\partial^{m-1} V}{\partial t^{m-1}}(t, \tau, x)\bigg|_{t=\tau} = {}^{\mathrm{C}}D_{0+}^{m-\beta} h(\tau, x), \quad x \in \mathbb{R}^n, \tag{30}$$

for all $\tau > t$, where $h(t, x)$, $t > 0$, $x \in \mathbb{R}^n$, is a given function satisfying the conditions in Theorem (3.1). *Then $u(t, x)$ defined by the Duhamel integral (26) is a solution of the following Cauchy problem for the inhomogeneous equation:*

$$^{\mathrm{C}}D_{0+}^{\beta} u(t, x) - A(x, D_x) u(t, x) = h(t, x), \quad t > 0, x \in \mathbb{R}^n,$$

$$\frac{\partial^k u}{\partial t^k}(0, x) = 0, \quad x \in \mathbb{R}^n, k = 0, \ldots, m-1.$$

Remark 3.5. Note that if the condition $h(0, x) = 0$ is not verified, then the initial condition (30) takes the form

$$\frac{\partial^{m-1} V}{\partial t^{m-1}}(t, \tau, x)\bigg|_{t=\tau} = {}^{\mathrm{RL}}D_{0+}^{m-\beta} h(\tau, x), \quad x \in \mathbb{R}^n.$$

Moreover, if $h(t, x) = J_{0+}^{1-\beta} f(t, x)$, then due to the well-known relation

$$^{\mathrm{RL}}D_{0+}^{m-\beta} J_{0+}^{m-\beta} f(t, x) = f(t, x),$$

the initial condition (30) obviously reduces to the form

$$\frac{\partial^{m-1} V}{\partial t^{m-1}}(t, \tau, x)\bigg|_{t=\tau} = f(\tau, x), \quad x \in \mathbb{R}^n.$$

The theorem below generalizes the Duhamel principle for the case of fractional differential equations with a single Riemann–Liouville derivative.

Theorem 3.6 ([28, 29]). *Suppose that $m - 1 < \mu < m$ and $V(t, \tau)$ is a solution of the following Cauchy-type problem:*

$$^{\mathrm{RL}}D_{\tau+}^{\mu} V(t, \tau, x) + A(x, D_x) V(t, \tau, x) = 0, \quad t > \tau, \ x \in \mathbb{R}^n, \tag{31}$$

$$ {}^{\mathrm{RL}}D_{\tau+}^{\mu-1} V(t,\tau,x)|_{t=\tau+} = h(\tau,x), \quad x \in \mathbb{R}^n, \tag{32} $$

$$ {}^{\mathrm{RL}}D_{\tau+}^{\mu-j} V(t,\tau,x)|_{t=\tau+} = 0, \quad x \in \mathbb{R}^n,\ j = 2,\dots,m-1, \tag{33} $$

$$ J_{\tau+}^{m-\mu} V(t,\tau,x)|_{t=\tau+} = 0, \quad x \in \mathbb{R}^n. \tag{34} $$

Then the Duhamel integral

$$ u(t,x) = \int_0^t V(t,\tau,x)d\tau, \quad t \ge 0,\ x \in \mathbb{R}^n, \tag{35} $$

solves the Cauchy-type problem

$$ {}^{\mathrm{RL}}D_{0+}^{\mu} u(t,x) + A(x,D_x)u(t,x) = h(t,x), \quad t > 0,\ x \in \mathbb{R}^n, \tag{36} $$

$$ {}^{\mathrm{RL}}D_{0+}^{\mu-1} u(t,x)|_{t=0+} = 0, \quad x \in \mathbb{R}^n, \tag{37} $$

$$ {}^{\mathrm{RL}}D_{0+}^{\mu-j} u(t,x)|_{t=0+} = 0, \quad x \in \mathbb{R}^n,\ j = 2,\dots,m-1, \tag{38} $$

$$ J_{0+}^{m-\mu} u(t,x)|_{t=0+} = 0, \quad x \in \mathbb{R}^n. \tag{39} $$

3.2 Fractional Duhamel principle for general distributed fractional-order differential equations

In this section, we formulate the fractional Duhamel principle for distributed fractional order (pseudo)-differential and differential-operator equations. Thus theorems in this section represent the general case covering in particular the Duhamel principle discussed in previous sections. We distinguish two different cases:

1. fractional distributed order differential equations defined through the Caputo–Djrbashian fractional derivative, and
2. fractional distributed order differential equations defined through the Riemann–Liouville fractional derivative.

We note that formulations of the fractional Duhamel principle are different in these cases.

Let λ be a finite Borel measure with $\operatorname{supp}\lambda \subset [0, m-1]$. We define a measure $\Lambda = \delta_\mu + \lambda$, where μ is a number such that $m-1 < \mu < m$. Introduce the following distributed order differential operator [17, 20, 30]

$$ {}^{\mathrm{C}}L_{\tau+}(\mu,\lambda)[u(t,x)] \equiv {}^{\mathrm{C}}D_{\tau+}^{\mu} u(t,x) + \int_0^{m-1} f(\alpha, D_x)\, {}^{\mathrm{C}}D_{\tau+}^{\alpha} u(t,x)\lambda(d\alpha), \tag{40} $$

acting on m-times differentiable $\Psi_{p,G}(\mathbb{R}^n)$- or $\Psi'_{-G,p'}(\mathbb{R}^n)$-valued vector-functions $u(t,x)$, $t \ge \tau \ge 0$, and $f(\alpha, A)$ is a family of pseudo-differential operators defined on $\Psi_{p,G}(\mathbb{R}^n)$ or $\Psi'_{-G,p'}(\mathbb{R}^n)$, respectively, with some $G \subset \mathbb{R}^n$. If $\tau = 0$, then instead of ${}^{\mathrm{C}}L_\tau(\mu, \lambda)$ we write ${}^{\mathrm{C}}L(\mu, \lambda)$.

Consider the Cauchy problem for the inhomogeneous equation

$$ {}^{\mathrm{C}}L(\mu, \lambda)[u(t,x)] = h(t,x), \quad t > 0,\ x \in \mathbb{R}^n, \tag{41}$$

with the homogeneous Cauchy conditions

$$\frac{\partial^k u}{\partial t^k}(0,x) = 0, \quad k = 0, \dots, m-1,\ x \in \mathbb{R}^n. \tag{42}$$

For simplicity, we assume that $h(t,x)$ on the right-hand side of equation (41) is a vector-function differentiable in the variable t and with values in $\Psi_{p,G}(\mathbb{R}^n)$ or $\Psi'_{-G,p'}(\mathbb{R}^n)$. The fractional Duhamel principle establishes a connection between the solutions of this problem and the Cauchy problem for the homogeneous equation

$$ {}^{\mathrm{C}}L_{\tau+}(\mu, \lambda)[V(t,\tau,x)] = 0, \quad t > \tau,\ x \in \mathbb{R}^n, \tag{43}$$

$$\left.\frac{\partial^k V}{\partial t^k}(t,\tau,x)\right|_{t=\tau+0} = 0, \quad k = 0, \dots, m-2,\ x \in \mathbb{R}^n, \tag{44}$$

$$\left.\frac{\partial^{m-1} V}{\partial t^{m-1}}(t,\tau,x)\right|_{t=\tau+0} = {}^{\mathrm{RL}}D_{0+}^{m-\mu} h(\tau,x), \quad x \in \mathbb{R}^n. \tag{45}$$

Theorem 3.7. *Suppose that $\Psi_{p,G}(\mathbb{R}^n)$- or $\Psi'_{-G,p'}(\mathbb{R}^n)$-valued vector-function $V(t,\tau,x)$, defined for $t \geq \tau$ and $x \in \mathbb{R}^n$, is a solution of the Cauchy problem* (43)–(45). *Then the Duhamel integral*

$$u(t,x) = \int_0^t V(t,\tau,x)\,d\tau, \quad t > 0,\ x \in \mathbb{R}^n, \tag{46}$$

solves the Cauchy problem (41), (42).

If $h(t,x)$ satisfies the additional condition $h(0,x) = 0$ for all $x \in \mathbb{R}^n$, then condition (45) can be replaced by

$$\left.\frac{\partial^{m-1} V}{\partial t^{m-1}}(t,\tau)\right|_{t=\tau} = {}^{\mathrm{C}}D_{0+}^{m-\mu} h(\tau,x),$$

due to well-known relationship (see, e. g., [24, 29])

$$ {}^{\mathrm{RL}}D_{a+}^{\alpha} f(t) = {}^{\mathrm{C}}D_{a+}^{\alpha} f(t) - \frac{f(0)}{\Gamma(1-\alpha)(t-a)^\alpha}, \quad t > a,$$

between the Riemann–Liouville and Caputo–Djrbashian fractional derivatives, valid for any $\alpha \in (0,1)$ and $a \in \mathbb{R}$. As a consequence, the formulation of the fractional Duhamel's principle takes the form.

Theorem 3.8. *Suppose that for all $\tau : 0 < \tau < t$ a function $V(t,\tau,x)$, is a solution to the Cauchy problem for the homogeneous equation*

$$
\begin{aligned}
{}^{\mathrm{C}}L^{(\mu,\lambda)}_{\tau+}[V(t,\tau,x)] &= 0, \quad t > \tau,\ x \in \mathbb{R}^n,\\
\frac{\partial^k V}{\partial t^k}(t,\tau,x)\bigg|_{t=\tau+0} &= 0, \quad k = 0,\dots,m-2, x \in \mathbb{R}^n,\\
\frac{\partial^{m-1} V}{\partial t^{m-1}}(t,\tau,x)\bigg|_{t=\tau+0} &= {}^{\mathrm{C}}D^{m-\mu}_{0+}h(\tau,x), \quad x \in \mathbb{R}^n,
\end{aligned}
$$

where $h(t,x)$ is a given differentiable $\Psi_{p,G}(\mathbb{R}^n)$- or $\Psi'_{-G,p'}(\mathbb{R}^n)$-valued vector-function such that $h(0,x) = 0$ for all $x \in \mathbb{R}^n$. Then the Duhamel integral

$$
u(t,x) = \int_0^t V(t,\tau,x)d\tau
$$

solves the Cauchy problem for the inhomogeneous equation (41), (42).

To formulate the Duhamel principle for fractional-order differential operator equations, we need to introduce abstract versions of the spaces $\Psi_{p,G}(\mathbb{R}^n)$ and $\Psi'_{-G,p'}(\mathbb{R}^n)$.

Let X be a reflexive Banach space with a norm $\|v\|$, $v \in X$. Let A be a closed linear operator with a domain $\mathcal{D}(A)$ dense in X and a spectrum $\sigma(A) \subset \mathbb{C}$ is not empty. Assume that G is an open set in $\mathbb{C}$ not necessarily containing $\sigma(A)$. Let $0 < r \le +\infty$ and $\nu < r$. Denote by $\mathrm{Exp}_{A,\nu}(\mathrm{X})$ the set of elements $v \in \bigcap_{k\ge 1}\mathcal{D}(A^k)$ satisfying the inequalities $\|A^k v\| \le C\nu^k\|v\|$ for all $k = 1,2,\dots$, with a constant $C > 0$ not depending on k. An element $v \in \mathrm{Exp}_{A,\nu}(X)$ is said to be a vector of exponential type ν [29]. A sequence of elements v_n, $n = 1,2,\dots$, is said to converge to an element $v_0 \in \mathrm{Exp}_{A,\nu}(\mathrm{X})$ iff:

(1) All the vectors v_n are of exponential type $\nu < r$, and
(2) $\|v_n - v_0\| \to 0$, $n \to \infty$.

Obviously, $\mathrm{Exp}_{A,\nu_1}(\mathrm{X}) \subset \mathrm{Exp}_{A,\nu_2}(\mathrm{X})$, if $\nu_1 < \nu_2$. Let $\mathrm{Exp}_{A,r}(\mathrm{X})$ be the inductive limit of spaces $\mathrm{Exp}_{A,\nu}(\mathrm{X})$ when $\nu \to r$. For basic notions of topological vector spaces including inductive and projective limits we refer the reader to [29]. Set $A_\lambda = A - \lambda I$, where $\lambda \in G$, and denote $\mathrm{Exp}_{A,r,\lambda}(\mathrm{X}) = \{u_\lambda \in X : u_\lambda \in \mathrm{Exp}_{A_\lambda,r}(\mathrm{X})\}$, with the induced topology. Finally, for arbitrary $G \subset \sigma(A)$, denote by $\mathrm{Exp}_{A,G}(\mathrm{X})$ the space whose elements are the locally finite sums of elements in $\mathrm{Exp}_{A,r,\lambda}(\mathrm{X})$, $\lambda \in G$, $r < \mathrm{dist}(\lambda,\partial G)$, with the corresponding topology. Namely, any $u \in \mathrm{Exp}_{A,G}(\mathrm{X})$, by definition, has a representation $u = \sum_\lambda u_\lambda$ with a finite sum. It is clear, that $\mathrm{Exp}_{A,G}(\mathrm{X})$ is a subspace of the space of vectors of exponential type if $r < +\infty$, and coincides with it if $r = +\infty$. $\mathrm{Exp}_{A,G}(\mathrm{X})$ is an abstract analog of the space $\Psi_{G,p}(\mathbb{R}^n)$, where $A = (-i\frac{\partial}{\partial x_1},\dots,-i\frac{\partial}{\partial x_n})$, $G \subseteq \mathbb{R}^n$, $\mathrm{X} = L_p(\mathbb{R}^n)$, $1 < p < \infty$.

Further, let $f(\lambda)$ be an analytic function on G. represented as a finite sum. Then for $u \in \mathrm{Exp}_{A,G}(\mathrm{X})$ with the representation $u = \sum_{\lambda \in G} u_\lambda$, $u_\lambda \in \mathrm{Exp}_{A,r,\lambda}(\mathrm{X})$, the operator $f(A)$ is defined by the formula

$$f(A)u = \sum_{\lambda \in G} f_\lambda(A)u_\lambda, \quad \text{where } f_\lambda(A)u_\lambda = \sum_{n=0}^{\infty} \frac{f^{(n)}(\lambda)}{n!}(A - \lambda I)^n u_\lambda. \tag{47}$$

In other words, each f_λ represents f locally in a neighborhood of $\lambda \in G$, and for u_λ the operator $f_\lambda(A)$ is well-defined.

Let X^* denote the dual of X, and $A^* : X^* \to X^*$ be the operator adjoint to A. Further, denote by $\mathrm{Exp}'_{A^*,G^*}(\mathrm{X}^*)$ the space of linear continuous functionals defined on $\mathrm{Exp}_{A,G}(\mathrm{X})$, with respect to weak convergence. Specifically, a sequence $u_m^* \in \mathrm{Exp}'_{A^*,G^*}(\mathrm{X}^*)$ converges to an element $u^* \in \mathrm{Exp}'_{A^*,G^*}(\mathrm{X}^*)$ if for all $v \in \mathrm{Exp}_{A,G}(\mathrm{X})$ the convergence $\langle u_m^* - u, v\rangle \to 0$ holds as $m \to \infty$. For an analytic function f^w defined on $G^* = \{z \in \mathbb{C} : \bar{z} \in G\}$, we define *a weak extension* of $f(A)$ as follows:

$$\langle f^w(A^*)u^*, v\rangle = \langle u^*, f(A)v\rangle, \quad \forall v \in \mathrm{Exp}_{A,G}(\mathrm{X}),$$

where $u^* \in \mathrm{Exp}'_{A^*,G^*}(\mathrm{X}^*)$. It is known (see [29]) that the following mappings are well-defined and continuous:

(a) $f(A) : \mathrm{Exp}_{A,G}(\mathrm{X}) \to \mathrm{Exp}_{A,G}(\mathrm{X})$,

(b) $f^w(A^*) : \mathrm{Exp}'_{A^*,G^*}(\mathrm{X}^*) \to \mathrm{Exp}'_{A^*,G^*}(\mathrm{X}^*)$.

Remark 3.9. Note that if $\sigma(A)$ is discrete then the space $\mathrm{Exp}_{A,G}(\mathrm{X})$ consists of the root lineals of eigenvectors corresponding to the part of $\sigma(A)$ with nonempty intersection with G. If the spectrum $\sigma(A)$ is empty, then an additional investigation is required for solution spaces to be nontrivial (for details see, [9]).

In the next theorem (Theorem 3.10), we assume that the vector-functions $h(t)$, $t \geq 0$, and $V(t,\tau)$, $t \geq \tau \geq 0$, are $\mathrm{Exp}_{A_\lambda,G}(\mathrm{X})$-, or $\mathrm{Exp}'_{A^*,G^*}(\mathrm{X}^*)$-valued, $h(t)$ is differentiable, $V(t,\tau)$ is an m times differentiable with respect to the variable t, and the derivatives $\frac{\partial^j V(t,\tau)}{\partial t^j}$, $0 \leq j \leq k-1$, are jointly continuous in the topology of $\mathrm{Exp}_{A_\lambda,G}(\mathrm{X})$, or of $\mathrm{Exp}'_{A^*,G^*}(\mathrm{X}^*)$, respectively.

Theorem 3.10. *Suppose that for all $\tau : 0 < \tau < t$ a function $V(t,\tau)$, is a solution to the Cauchy problem for the homogeneous equation*

$${}^{\mathrm{C}}D_{\tau+}^{\mu} V(t,\tau) + \int_0^{m-1} f(\alpha, D_x)\, {}^{\mathrm{C}}D_{\tau+}^{\alpha} V(t,\tau)\lambda(d\alpha) = 0, \quad t > \tau,$$

$$\frac{\partial^k V}{\partial t^k}(t,\tau)\Big|_{t=\tau+0} = 0, \quad k = 0, \dots, m-2,$$

$$\frac{\partial^{m-1} V}{\partial t^{m-1}}(t,\tau)\Big|_{t=\tau+0} = {}^{\mathrm{RL}}D_{0+}^{m-\mu} h(\tau).$$

Then Duhamel's integral $u(t) = \int_0^t V(t,\tau)d\tau$ *solves the Cauchy problem for the inhomogeneous equation*

$$ {}^{\mathrm{C}}D^{\mu}_{0+}u(t) + \int\limits_0^{m-1} f(\alpha, D_x)\,{}^{\mathrm{C}}D^{\alpha}_{0+}u(t)\lambda(d\alpha) = h(t), \quad t > 0, $$

$$ u^k(0) = 0, \quad k - 0, \dots, m-1. $$

Remark 3.11. Theorems 3.7, 3.8, and 3.10 can be extended to absolutely continuous in the variable t source functions $h(t,x)$ (or $h(t)$ in the abstract case) with an appropriate meaning of the solutions. It is also known [24] that the fractional derivative ${}^{\mathrm{RL}}D^{m-\mu}_{0+}h(t)$ exists a.e., if $h(t)$ is an absolutely continuous function on $[0,T]$ for all $T > 0$. These facts imply that the fractional generalizations of the Duhamel principle proved above hold true for absolutely continuous functions $h(t)$.

3.3 Riemann–Liouville case

In this section, we formulate a fractional generalization of the Duhamel principle for distributed order differential operators defined via the Riemann–Liouville fractional derivative. However, we will do this in the particular case of the measure Λ, namely $\Lambda(d\alpha) = [\delta_\mu(\alpha) + \sum_{k=1}^{m-1}\delta_{\mu-k}(\alpha)]d\alpha$, where $m-1 < \mu \le m$. In this case, the corresponding family of distributed order differential operators for $\tau \ge 0$ has the form

$$ {}^{\mathrm{RL}}L^{\Lambda}_{\tau+}[u] = {}^{\mathrm{RL}}D^{\mu}_{\tau+}u(t) + \sum_{k=1}^{m-1} B_k\,{}^{\mathrm{RL}}D^{\mu-k}_{\tau+}u(t) + B_0u(t), \tag{48} $$

where B_k, $k = 0,\dots,m-1$, are linear closed operators independent of the variable t, and with domains dense in X.

Consider the following Cauchy type problem for a nonhomogeneous differential-operator equation:

$$ {}^{\mathrm{RL}}L^{\Lambda}_{0+}[u] = h(t), \quad t > 0, \tag{49} $$

$$ {}^{\mathrm{RL}}D^{\mu-j}_{0+}u(t)|_{t=0+} = 0, \quad j = 1,\dots,m-1, \tag{50} $$

$$ J^{m-\mu}_{0+}u(t)|_{t=0+} = 0. \tag{51} $$

Theorem 3.12. *Suppose that* $V(t,\tau)$ *is a solution of the following Cauchy type problem:*

$$ {}^{\mathrm{RL}}L^{\Lambda}_{\tau+}[V(t,\tau)] = 0, \quad t > \tau, \tag{52} $$

$$ {}^{\mathrm{RL}}D^{\mu-1}_{\tau+}V(t,\tau)|_{t=\tau+} = h(\tau), \tag{53} $$

$$ {}^{\mathrm{RL}}D^{\mu-j}_{\tau+}V(t,\tau)|_{t=\tau+} = 0, \quad j = 2,\dots,m-1, \tag{54} $$

$$ J^{m-\mu}_{\tau+}V(t,\tau)|_{t=\tau+} = 0. \tag{55} $$

Then the Duhamel integral

$$u(t) = \int_0^t V(t,\tau)d\tau \tag{56}$$

solves the Cauchy type problem (49)–(51).

Remark 3.13. Theorems 3.7, 3.10, and 3.12 represent fractional generalizations of the Duhamel principle for integer order differential-operator equations proved in Theorem 2.4 in Section 2.3.

Theorem 3.7 (and Theorem 3.8 as well) represents the particular case of Theorem 3.10 when the operator $A = D \equiv (D_{x_1}, \dots, D_{x_n})$ is acting on functions (or distributions) defined on $\mathbb{R}^n$. Consider an example of the case when operator A is acting on functions defined on a bounded domain Ω with a smooth boundary $\partial\Omega$.

Let $A \equiv A(x, D)$ be an elliptic self-adjoint operator of order 2ℓ of the form

$$A(x, D) = \sum_{|\alpha| \le 2\ell} a_\alpha(x) D^\alpha,$$

where $a_\alpha(x)$ are smooth functions on $\overline{\Omega}$. The domain $D(A) = W_2^{2\ell}(\Omega) \cap \overset{\circ}{W}{}_2^{\ell}(\Omega)$. Here, $W_2^{2\ell}(\Omega)$ and $\overset{\circ}{W}{}_2^{\ell}(\Omega)$ are Sobolev spaces of functions defined on Ω and vanishing on the boundary $\partial\Omega$, respectively [29]. Consider the equation

$$b_0\, \partial^\mu u(t,x) + \sum_{k=1}^{m-1} b_k\, \partial^{\mu-k} u(t,x) + A(x,D)u(t,x) = h(t,x), \quad t > 0,\ x \in \Omega, \tag{57}$$

where ∂^β is a fractional derivative either in the Caputo–Djrbashian sense or Riemann–Liouville sense, $m - 1 < \mu \le m$, $b_k \in \mathbb{C}$, $k = 0, \dots, m-1$, are some numbers.

In order to apply Theorem 3.10 to the initial-boundary value problem for equation (57) with boundary conditions,

$$\left.\frac{\partial^j u(t,x)}{\partial \vec{n}^j}\right|_{\partial\Omega} = 0, \quad t > 0,\ j = 0, \dots, m-1, \tag{58}$$

and with an appropriate initial conditions, one should introduce the abstract operator A defined as

$$A = A(x, D) \quad \text{and} \quad X = \left\{\varphi \in D(A) : \left.\frac{\partial^j \varphi(x)}{\partial \vec{n}^j}\right|_{\partial\Omega} = 0,\ j = 0, \dots, m-1\right\}.$$

The set X with respect to the norm induced from $W^{2\ell}$ is a Hilbert space. The operator A is a closed operator defined on a Hilbert space X, has a discrete spectrum $\lambda_1, \lambda_2, \dots \in \mathbb{R}_+$, with corresponding eigenvectors $\phi_1, \phi_2, \dots \in X$, which form a base in X.

Now, for simplicity, assume that $m = 2$, $b_0 = 1$ and $b_1 = b$, that is,

$$\partial^\alpha u(t) + b\partial^\beta u(t) + Au(t) = h(t,x), \quad t > 0, \tag{59}$$

with $1 < \alpha \le 2$ and $0 < \beta \le 1$. The Cauchy problem for this equation with Caputo–Drbashian derivatives and $A = -\Delta$, the Laplace operator, in the homogeneous case was studied in [25, 29]; the case $\alpha = 2$, $\beta = 1/2$, and $A \equiv d^4/dx^4$ is studied by Agraval [1], and the case $\alpha = 2$, $\beta = 3/2$, and A is constant, by Bagley and Torvik in [4].

For the solution of the initial value problem,

$$^{\mathrm{C}}D^\alpha_{0+}u(t) + b\,^{\mathrm{C}}D^\beta_{0+}u(t) + Au(t) = h(t), \quad t > 0, \tag{60}$$

$$u(0) = \varphi_0, \tag{61}$$

$$u'(0) = \varphi_1, \tag{62}$$

in the case $h(t) = 0$, the following representation holds (see [29] p. 252, [25] p. 32)

$$u(t) = S_0(A,t)\varphi_0 + S_1(A,t)\varphi_1, \tag{63}$$

where

$$S_0(A;t) = S_*(A;t) + bt^{\mu-\beta}S_{**}(A;t), \quad S_1(A;t) = JS_*(A;t),$$

with

$$S_*(A;t) = \sum_{n=0}^{\infty} \frac{(At^\mu)^n}{n!} E^{(n)}_{\mu-\beta,\, n\beta+1}(-bt^{\mu-\beta}); \tag{64}$$

$$S_{**}(A;t) = \sum_{n=0}^{\infty} \frac{(At^\mu)^n}{n!} E^{(n)}_{\mu-\beta,\, n\beta-\beta+\mu+1}(-bt^{\mu-\beta}). \tag{65}$$

Here, $E_{p,q}(z)$ is the double-indexed Mittag-Leffler function. The right-hand side of (63) is meaningful if $\varphi_0, \varphi_1 \in \Psi_{p,G}(\mathbb{R}^n)$ or $\varphi_0, \varphi_1 \in \Psi'_{-G,p'}(\mathbb{R}^n)$. Further, applying the fractional Duhamel principle, we can find a solution in the nonhomogeneous case (with homogeneous initial conditions) in the form

$$v(t) = \int_0^t S_1(A, t-\tau)\,^{\mathrm{RL}}D^{2-\alpha}_{0+}h(\tau)d\tau.$$

Now consider the case of equation (59) with the Riemann–Liouville derivatives with orders $\alpha = 3/2$ and $\beta = 1/2$. Namely, consider the Cauchy type problem

$$^{\mathrm{RL}}D^{3/2}_{0+}u(t) + b\,^{\mathrm{RL}}D^{1/2}_{0+}u(t) + Au(t) = h(t), \quad t > 0, \tag{66}$$

$$(^{\mathrm{RL}}D^{1/2}_{0+}u)(0+) = \psi_1, \tag{67}$$

$$(J^{1/2}_{0+}u)(0+) = \psi_2. \tag{68}$$

Let us first solve this problem in the homogeneous case:

$$ {}^{\mathrm{RL}}D_{0+}^{3/2}u(t) + b\,{}^{\mathrm{RL}}D_{0+}^{1/2}u(t) + Au(t) = 0, \quad t > 0, \tag{69} $$

$$ ({}^{\mathrm{RL}}D_{0+}^{1/2}u)(0+) = \psi_1, \tag{70} $$

$$ (J_{0+}^{1/2}u)(0+) = \psi_2. \tag{71} $$

Applying the Fourier transform to (69), we have

$$ M(s)L[u](s) = \psi_1 + (s+b)\psi_2, \quad s > \eta_0, $$

where $M(s) = s^{3/2} + bs^{1/2} + A$, and $\eta_0 > 0$ is a number such that $M(s) \neq 0$ for all $s > \eta_0$.[2] Let $\mathcal{S}_1(t,A) = L^{-1}[1/M(s)]$ and $\mathcal{S}_2(t,A) = L^{-1}[(s+b)/M(s)]$. The power series representations of operators $\mathcal{S}_j(t,A)$ can be obtained similar to (64), (65). Then the solution to problem (69)–(71) has the representation

$$ u(t) = \mathcal{S}_1(t,A)\psi_1 + \mathcal{S}_2(t,A)\psi_2, $$

where

$$ \mathcal{S}_k(t,A)\psi_k = \sum_{n=1}^{\infty} \mathcal{S}_k(t,\lambda_n)\psi_{k,n}\phi_n, \quad k = 1,2, $$

Here, $\psi_{k,n} = (\psi_k, \phi_n)$, Fourier coefficients of elements ψ_k, $k = 1,2$.

Now, we return to the non homogeneous problem (66)–(68), however, we assume now that $\psi_k = 0$, $k = 1,2$. To solve this we apply the fractional Duhamel principle. In accordance with Theorem 3.12, the solution is given by the Duhamel integral $\int_0^t V(t,\tau)d\tau$, where $V(t,\tau)$ is a solution to the following problem:

$$ {}^{\mathrm{RL}}D_{\tau+}^{3/2}V(t,\tau) + b\,{}^{\mathrm{RL}}D_{\tau+}^{1/2}V(t,\tau) + Au(t) = 0, \quad t > \tau, \tag{72} $$

$$ {}^{\mathrm{RL}}D_{\tau+}^{1/2}V(\tau+,\tau) = h(\tau), \tag{73} $$

$$ J_{\tau+}^{1/2}V(\tau+,\tau) = 0. \tag{74} $$

It is not hard to verify that the solution of the Cauchy type problem (72)–(74) can be obtained from the solution of problem (69)–(71) by translation $t \to t+\tau$. Namely, $V(t,\tau) = S_1(t-\tau,A)h(\tau)$. Hence, the solution to the given problem in (66)–(68) has the representation

$$ u(t) = S_1(t,A)\varphi_1 + S_2(t,A)\varphi_2 + \int_0^t S_1(t-\tau,\ A)h(\tau)d\tau. $$

2 We assume that such η_0 exists.

4 Some applications of fractional Duhamel principle

In this section, we consider some applications of the Duhamel principle from different fields actively using in the recent research publications. Namely, applications to existence and uniqueness theorems, to fractional generalizations of stochastic partial differential equations, and to the study of variable fractional order differential equations. There are many other applications of the fractional Duhamel principle including applications to weakly nonlinear fractional differential equations, fractional Nave–Stocks equations, fractional Schrödinger equation with potentials, etc; see [2, 3, 6, 7, 10, 14, 18, 26, 29, 35] and references therein.

4.1 Existence and uniqueness theorems

Consider the following Cauchy problem for fractional distributed order differential equations:

$$ {}^{C}L_{t_0+}(\mu,\lambda)[u(t,x)] = h(t,x), \quad t > t_0,\ x \in \mathbb{R}^n, \tag{75} $$

$$ \frac{\partial^k u}{\partial t^k}(t,x)\Big|_{t=t_0+} = \varphi_k(x), \quad k = 0,\dots,m-1,\ x \in \mathbb{R}^n, \tag{76} $$

where the operator ${}^{C}L_{t_0+}(\mu,\lambda)$ is defined in (40), $m-1 < \mu \le m$, $t_0 \in \mathbb{R}$ is an initial time, $h(t,x)$ and $\varphi_k(x)$, $k = 0,\dots,m-1$, are functions (or distributions) in the spaces specified below, λ is a Borel measure supported on $[0, m-1]$, and $\{A(\alpha, D_x), 0 < \alpha < m-1\}$ is a family of pseudo-differential operators. Without loss of generality, we can assume that $t_0 = 0$.

The fractional Duhamel principle can be applied for a wide class of initial-boundary value problems containing problems (75)–(76) as a particular case. In order to discuss, the general case we proceed to operator language. Define an abstract analog of the operator ${}^{C}L_{\tau+}(\mu,\lambda)$ introduced in equation (40) in the form

$$ {}^{C}L_{\tau+}(\mu,\lambda)[u] \equiv {}^{C}D^{\mu}_{\tau+}u(t) + \int_0^{m-1} f(\alpha, A)\, {}^{C}D^{\alpha}_{\tau+}u(t)\lambda(d\alpha), $$

where A is a closed linear operator with a dense domain in a reflexive Banach space X, and $f(\alpha,z)$ is a function continuous in $\alpha \in [0, m-1]$ and analytic in $z \in G \subset \mathbb{C}$. Introduce the characteristic function of the operator ${}^{C}L_{\tau+}(\mu,\lambda)$ as

$$ \Delta(s,z) = s^{\mu} + \int_0^{m-1} f(\alpha,z)s^{\alpha}d\lambda. $$

and denote by $\hat{v}(s) = \mathcal{L}[v](s)$ the Laplace transform of a vector-function $v(t)$, namely

$$\mathcal{L}[v](s) = \int_0^{\infty} e^{-st} v(t)dt, \quad s > s_0,$$

where $s_0 \geq 0$ is a real number. It is not hard to verify that if $v(t) \in \mathrm{Exp}_{A,G}(X)$ for each $t \geq 0$ and satisfies the condition $\|v(t)\| \leq Ce^{\gamma t}$, $t \geq 0$, with some constants $C > 0$ and γ, then $\hat{v}(s)$ exists and

$$\|A^k \hat{v}(s)\| \leq \frac{C_s}{s-\gamma} \nu^k, \quad s > \gamma,$$

implying $\hat{v}(s) \in \mathrm{Exp}_{A,G}(X)$ for each fixed $s > \gamma$. Further, let $c_\beta(t,z) = \mathcal{L}^{-1}[\frac{s^\beta}{\Delta(s,z)}](t)$, $z \in G \subset \mathbb{C}$, where $\mathcal{L}^{-1}$ stands for the inverse Laplace transform, and

$$S_k(t,z) = c_{\mu-k-1}(t,z) + \int_k^{m-1} f(\alpha,z) c_{\alpha-k-1}(t,z)\lambda(d\alpha), \quad k = 0,\dots,m-1. \tag{77}$$

Then, as is known (see [29]), $S_k(t,A)\varphi_k$ solves the Cauchy problem

$${}^{\mathrm{C}}L_{0+}(\mu,\lambda)[u] = 0, \quad u^{(j)}(0) = \delta_{j,k}\varphi_j, \quad j = 0,\dots,m-1.$$

where $\delta_{j,k}$ denote the Kronecker delta, that is $\delta_{j,k} = 1$ if $j = k$, and $\delta_{j,k} = 0$, if $j \neq k$. This fact implies that the solution of the Cauchy problem

$${}^{\mathrm{C}}L_{0+}(\mu,\lambda)[u] = 0, \quad t > 0, \quad u^{(j)}(0) = \varphi_j, \quad j = 0,\dots,m-1. \tag{78}$$

is given by the representation formula

$$u(t) = \sum_{k=0}^{m-1} S_k(t,A)\varphi_k. \tag{79}$$

Note that if the equation ${}^{\mathrm{C}}L_{t_0+}(\mu,\lambda)[u] = 0$ in (78) is considered for $t > t_0$ with the initial time t_0 and initial conditions $u^j(t_0) = \varphi_j, j = 0,\dots,m-1$, then symbols of solution operators depend on τ and have the form $S_k(t,t_0,z) = S_k(t-t_0,z)$, $k = 0,\dots,m-1$, where $S_k(t,z)$ is defined in (77).

Further, denote by $C^{(m)}[t > 0; \mathrm{Exp}_{A,G}(X)]$ and by $AC[t > 0; \mathrm{Exp}_{A,G}(X)]$ the space of $m-$ times continuously differentiable functions and the space of absolutely continuous functions on $(0;+\infty)$ with values in the space $\mathrm{Exp}_{A,G}(X)$, respectively. A vector-function $u(t) \in C^{(m)}[t > 0; \mathrm{Exp}_{A,G}(X)] \cap C^{(m-1)}[t \geq 0; \mathrm{Exp}_{A,G}(X)]$ is called a solution of the problem

$${}^{\mathrm{C}}L_{t_0+}(\mu,\lambda)[u] = h(t), \quad t > t_0, \tag{80}$$

$$u^{(j)}(t_0+) = \varphi_j, \quad j = 0, \dots, m-1. \tag{81}$$

if it satisfies the equation (80) and the initial conditions (81) in the topology of $\mathrm{Exp}_{A,G}(X)$.

In order to prove the existence of a unique solution of problem (80), (81), we split it into two Cauchy problems. Namely, to problem (78) (we again assume that $t_0 = 0$) and

$$^{\mathrm{C}}L_{0+}(\mu, \lambda)[v] = h(t), \quad t > 0, \tag{82}$$

$$v^{(k)}(0+) = 0, \quad k = 0, \dots, m-1. \tag{83}$$

The unique solution to problem (78) is given by (79), namely by

$$U(t) = \sum_{k=0}^{m-1} S_k(t, A)\varphi_k. \tag{84}$$

For the Cauchy problem (82), (83), in accordance with the fractional Duhamel's principle (Theorem 3.10), it suffices to solve the Cauchy problem for the homogeneous equation:

$$_{\tau}L^{\Lambda}[V(t,\tau)](t) = 0, \quad t > \tau, \tag{85}$$

$$\left.\frac{\partial^k V(t,\tau)}{\partial t^k}\right|_{t=\tau+} = 0, \quad k = 0, \dots, m-2, \tag{86}$$

$$\left.\frac{\partial^{m-1} V(t,\tau)}{\partial t^{m-1}}\right|_{t=\tau+} = {}^{\mathrm{RL}}D_{0+}^{m-\mu} h(\tau). \tag{87}$$

The solution of this problem has the representation

$$V(t,\tau) = S_{m-1}(t-\tau, A)\,{}^{\mathrm{RL}}D_{0+}^{m-\mu} h(\tau). \tag{88}$$

From the discussion above, it follows that $V(t,\tau) \in C^{(m)}[t > \tau; \mathrm{Exp}_{A,G}(X)]$ for all $\tau \geq 0$, as well as its Duhamel integral. Thus, the following theorem holds.

Theorem 4.1. *Let* $\varphi_k \in \mathrm{Exp}_{A,G}(X)$, $k = 0, \dots, m-1$, $h(t) \in AC[0 \leq t \leq T; \mathrm{Exp}_{A,G}(X)]$ *for any* $T > 0$, *and* ${}^{\mathrm{RL}}D_{0+}^{m-\mu} h(t) \in C[0 \leq t \leq T; \mathrm{Exp}_{A,G}(X)]$. *Then the Cauchy problem* (80), (81) *has a unique solution. This solution is given by*

$$u(t) = \sum_{k=0}^{m-1} S_k(t, A)\varphi_k + \int_0^t S_{m-1}(t-\tau, A)\,{}^{\mathrm{RL}}D_{0+}^{m-\mu} h(\tau)\,d\tau. \tag{89}$$

The duality immediately implies the following theorem.

Theorem 4.2. *Let* $\varphi_k^* \in \mathrm{Exp}'_{A^*,G^*}(X^*)$, $k = 0, \dots, m-1$, $h^*(t) \in AC[0 \leq t \leq T; \mathrm{Exp}'_{A^*,G^*}(X^*)]$ *and* ${}^{\mathrm{RL}}D_{0+}^{m-\mu} h^*(t) \in C[0 \leq t \leq T; \mathrm{Exp}'_{A^*,G^*}(X^*)]$. *Assume also that*

$\mathrm{Exp}_{A,G}(X)$ *is dense in* X. *Then the Cauchy problem* (80), (81) (*with* A *switched to* A^*) *is meaningful and has a unique weak solution. This solution is given by*

$$u^*(t) = \sum_{k=0}^{m-1} S_k(t, A^*)\varphi_k^* + \int_0^t S_{m-1}(t-\tau, A^*)^{\mathrm{RL}}D_{0+}^{m-\mu}h^*(\tau)d\tau.$$

Assume that $\mathrm{Exp}_{A,G}(X)$ is densely embedded into X. Besides, let the solution operators $S_k(t, A)$ for each $k = 0, \dots, m-1$, satisfy the estimates

$$\|S_k(t, A)\varphi\| \le C\|\varphi\|, \quad \forall\, t \in [0, T], \tag{90}$$

where $\varphi \in \mathrm{Exp}_{A,G}(X)$, and $C > 0$ does not depend on φ. Then there exists a unique closure $\bar{S}_k(t)$ to X of the operator $S_k(t, A)$ which satisfies the estimate $\|\bar{S}_k(t)u\| \le C\|u\|$ for all $u \in X$. Using the standard technique of closure (see [27, 29]), we can prove the following theorem.

Theorem 4.3. *Let* $\varphi_k \in X$, $k = 0, \dots, m-1$, $h(t) \in AC[0 \le t \le T; X]$ *for any* $T > 0$, *and* ${}^{\mathrm{RL}}D_{0+}^{m-\mu}h(t) \in C[0 \le t \le T; X]$. *Further, let* $\mathrm{Exp}_{A,G}(X)$ *be densely embedded into* X, *and the estimates* (90) *hold for solution operators* $S_k(t, A)$, $k = 0, \dots, m-1$. *Then the Cauchy problem* (80), (81) *has a unique solution* $u(t) \in C^m[0 < t \le T; X]$. *This solution is given by*

$$u(t) = \sum_{k=0}^{m-1} \bar{S}_k(t)\varphi_k + \int_0^t \bar{S}_{m-1}(t-\tau)^{\mathrm{RL}}D_{0+}^{m-\mu}h(\tau)d\tau.$$

To illustrate these theorems, consider the following example of the Cauchy problem for diffusion-wave equation. Let $1 < \beta < 2$. Consider the Cauchy problem

$$ {}^{\mathrm{C}}D_{0+}^{\beta}u(t,x) = c^2 \frac{\partial^2}{\partial x^2}u(t,x) + h(t,x), \quad t > 0,\ x \in \mathbb{R}, \tag{91}$$

$$u(0,x) = \varphi_0(x), \quad u_t(0,x) = \varphi_1(x), \quad x \in \mathbb{R}, \tag{92}$$

for a given constant $c > 0$, and appropriate functions $h(t,x)$, $\varphi_0(x)$, and $\varphi_1(x)$. In accordance with the fractional Duhamel principle (Theorem 3.4) the influence of the external force $h(t,x)$ appears in the form

$$ {}^{\mathrm{C}}D_{\tau+}^{\beta}V(t,\tau,x) = c^2 \frac{\partial^2}{\partial^2 x}V(t,\tau,x), \quad t > \tau,\ x \in \mathbb{R},$$

$$V(t,\tau,x)\Big|_{t=\tau+} = 0, \quad \frac{\partial V}{\partial t}(t,\tau,x)\Big|_{t=\tau+} = {}^{\mathrm{RL}}D_{0+}^{2-\beta}h(\tau,x).$$

The unique solution of the latter can be presented through the pseudo-differential operator

$$V(t,\tau,x) = JE_\beta\big(-c^2(t-\tau)^\beta \mathcal{D}^2\big)\ {}^{\mathrm{RL}}D_{0+}^{2-\beta}h(\tau,x),$$

where $\mathcal{D} = d/(idx)$, and $E_\beta(\cdot)$ is the Mittag-Leffler function. Hence, applying Theorem 4.3, the solution of the Cauchy problem (91), (92) can be written in the form

$$u(t,x) = E_\beta(-c^2 t^\beta \mathcal{D}^2)\varphi_0(x) + JE_\beta(-c^2 t^\beta \mathcal{D}^2)\varphi_1(x) + \int_0^t JE_\beta(-c^2(t-\tau)^\beta \mathcal{D}^2)\,{}^{\mathrm{RL}}D_{0+}^{2-\beta} h(\tau,x)d\tau. \tag{93}$$

Notice, that if $h(t,x) \equiv 0$ in equation (91), then (93) reduces to

$$u(t,x) = E_\beta(-c^2 t^\beta \mathcal{D}^2)\varphi_0(x) + JE_\beta(-c^2 t^\beta \mathcal{D}^2)\varphi_1(x), \tag{94}$$

which can be interpreted as the fractional version of famous D'Alambert's formula. We note that the bounded domain case was studied in [23]. Indeed, assuming that φ_0 is sufficiently smooth and making use of the fact that $E_2(-u^2) = \cos u$ (see, e. g. [11]), valid for any $u \in \mathbb{R}$, in the $\beta \to 2$ case for the first term of (94) we have

$$\begin{aligned} E_2(-c^2 t^2 \mathcal{D}^2)\varphi_0(x) &= \cos(ct\mathcal{D})\varphi_0(x) \\ &= \frac{e^{ct\frac{d}{dx}}\varphi_0(x) + e^{-ct\frac{d}{dx}}\varphi_0(x)}{2} = \frac{\varphi_0(x+ct) + \varphi_0(x-ct)}{2}. \end{aligned}$$

Similarly, one can show that the second term in (94) has the form

$$JE_2(-c^2 t^\beta \mathcal{D}^2)\varphi_1(x) = \frac{1}{2c}\int_{x-ct}^{x+ct} \varphi_1(\xi)d\xi.$$

Thus, representation (94) reduces to D'Alambert's formula if $\beta = 2$, generalizing it to values $1 < \beta < 2$.

4.2 Fractional-order stochastic differential equations

This section discusses applications of the fractional Duhamel principle to fractional order stochastic partial differential equations (SPDEs), not going into details; we refer the interested reader to books [15, 16] on SPDEs, and to papers [2, 3, 6, 12, 21, 22, 35] on fractional-order SPDEs. Stochastic partial differential equations model dynamical processes with random fluctuations. The effect of random fluctuation may arise from random sources (initial data, outer sources), measurement errors, errors arising from the use of devices for measurement, etc. The standard form of a linear SPDE is

$$du(t,x) = A(t,x,D_x)u(t,x)dt + \sigma(t,x,u)dB_t^x, \quad t > t_0, \tag{95}$$

with an initial condition

$$u(t_0,x) = u_0(x), \tag{96}$$

where $A(t, x, D)$ is a pseudo-differential operator (see Section 2.2) with the symbol $a(t, x, \xi)$ and $\sigma(t, x)$ is a function defined for all $t > t_0$, $x \in \mathbb{R}^n$. Both the symbol $a(t, x, \xi)$ and the function $\sigma(t, x)$ satisfy the Lipschitz and linear growth conditions for all $t > t_0$, $x \in \mathbb{R}^d$ and any fixed $\xi \in G$; B_t^x is Brownian motion; and $u_0(x)$ is, in general, a random quantity (variable). If one ignores random fluctuations, that is $\sigma(t, x) \equiv 0$, then one has a deterministic (or averaged) model of the underlying dynamical process:

$$\frac{d\bar{u}(t, x)}{dt} = A(t, x, D_x)\bar{u}(t, x), \quad \bar{u}(t_0, x) = \bar{u}_0(x). \tag{97}$$

Indeed, if one takes the mathematical expectation of both sides of equations (95), (96), then one obtains equation (97) with $\bar{u}(t, x)$ and $\bar{u}_0(x)$, the means of random vectors $u(t, x)$ (for any fixed $t > t_0$, $x \in \mathbb{R}^n$) and $u_0(x)$, respectively.

Evidently, SPDE (95) represents an improved model of (97) with random fluctuations taken into account. Now the natural question arises: what is a SPDE version of a fractional-order differential equation

$${}^{\mathrm{C}}D_{t_0+}^{\beta} v(t, x) = A(t, x, D_x)v(t, x), \quad v(t_0, x) = v_0(x)? \tag{98}$$

To answer this question, one can follow the approach used in papers [3, 6, 21, 22]. For further convenience, without loss of generality, we can assume $t_0 = 0$ and $v_0(x) = 0$. Suppose, a stochastically perturbed version of this equation is given by

$${}^{\mathrm{C}}D_{t_0+}^{\beta} v(t, x) = A(t, x, D_x)v(t, x) + W(t, x), \quad v(0, x) = 0, \tag{99}$$

where $W(t, x)$ is a white noise, that is $W = dB_t^x/dt$, with Brownian motion B_t^x. Then, in accordance with the Duhamel principle the solution is given by the stochastic process—Duhamel integral

$$v(t, x) = \int_0^t V(t, \tau, x)d\tau,$$

where $V(t, \tau, x)$ solves the following (random) initial value problem:

$$\begin{aligned} {}^{\mathrm{C}}D_{t_0+}^{\beta} V(t, \tau, x) &= A(t, x, D_x)V(t, \tau, x), \quad t > \tau, \\ V(t, \tau, x)\Big|_{t=\tau+} &= {}^{\mathrm{RL}}D_{0+}^{1-\beta} W(\tau, x). \end{aligned}$$

Thus, due to Theorems 3.7 and 4.2 the solution of (99) can be presented as

$$v(t, x) = \int_0^t S(t - \tau, A)\, {}^{\mathrm{RL}}D_{0+}^{1-\beta} W(\tau, x)d\tau, \tag{100}$$

where $S(t, A)$ is a solution operator of the initial-value problem (98). Here, the operator A is the same operator $A \equiv A(t, x, D)$ on the right of (98) with an appropriate domain.

The stochastic integral on the right of (100) cannot be interpreted as an Itô stochastic integral due to fractional derivative of order $1-\beta$ of the white noise term W. However, if one has a fractional integral term $J_{0+}^{1-\beta}W$, which is meaningful, then equation (100) takes the form

$$\begin{aligned} v(t,x) &= \int_0^t S(t-\tau,A)\,{}^{\mathrm{RL}}D_{0+}^{1-\beta}J_{0+}^{1-\beta}W(\tau,x)d\tau, \\ &= \int_0^t S(t-\tau,A)dB_\tau^x, \end{aligned} \tag{101}$$

The obtained stochastic integral is in the sense of Itô, thus is meaningful. Therefore, this approach applied to fractional equations of the form (98), in general, leads to the following form of the fractional SPDE:

$${}^{\mathrm{C}}D_{t_0+}^{\beta}v(t,x) = A(t,x,D_x)v(t,x) + J_{t_0+}^{1-\beta}\left[\sigma(t,x,v(t,x))\frac{dB_t^x}{dt}\right], \tag{102}$$

with the initial condition $v(t_0,x) = v_0(x)$. The meaning of this SDE is

$$v(t,x) = v_0(x) + \frac{1}{\Gamma(\beta)}\int_{t_0}^t \frac{A(s,x,D_x)v(s,x)}{(t-s)^{1-\beta}}ds + \int_{t_0}^t \sigma(s,x,v(s,x))dB_s^x. \tag{103}$$

The second term on the right is the fractional integral of order β of the expression $A(t,x,D_x)v(t,x)$ and the third term represents an Itô integral driven by Brownian motion. The latter is well-defined, for instance, if the function $\sigma(t,x,y)$, $(t,x,y) \in [t_0,\infty)\times\mathbb{R}^2$ satisfies the Lipschitz condition in variables (t,x) for all $y \in \mathbb{R}$. Thus the right-hand side of (103) is meaningful under usual conditions to the symbol of the operator $A(t,x,D_x)$ and $\sigma(t,x,u)$. The existence, uniqueness, continuity (Hölder continuity), stability theorems related to fractional stochastic partial differential equations of the form (103) are studied in [2, 3, 6, 18, 21, 22, 35].

4.3 Variable fractional-order differential equations

The fractional Duhamel principle play an important role in the theory of the Cauchy problem for variable fractional-order differential equations. Variable differential equations model diffusion processes with a random change of diffusion modes. An interesting phenomenon related to diffusion processes with changing modes is that a memory effect quantified as a natural source term emerges when diffusion mode changes. Hence, in the mathematical model of such diffusion processes an inhomogeneous term appears naturally, even in cases no external sources are assumed originally in the model.

For the definition of variable fractional-order differential operators and their various properties, we refer the reader to [13, 19, 29, 31, 34] and references therein. By definition, a variable order differential operator with a piecewise constant order function $\beta(t) = \sum_{k=0}^{N} \mathcal{I}_k \beta_k$, where $\mathcal{I}_k$ is the indicator function of $[T_k, T_{k+1})$, and $0 < \beta_k \leq 1$, $k = 0, \dots, N$, is

$$ {}^{\mathrm{C}}\mathcal{D}_{\mu,\nu}^{\beta(t)} f(t) = \int_0^t \mathcal{K}_{\mu,\nu}^{\beta(t)}(t,\tau) \frac{df(\tau)}{d\tau} d\tau, $$

with the kernel function

$$ \mathcal{K}_{\mu,\nu}^{\beta(t)}(t,\tau) = \frac{1}{\Gamma(1-\beta(\mu t + \nu\tau))(t-\tau)^{\beta(\mu t + \nu \tau)}}, \quad 0 < \tau < t, $$

and parameters μ and ν taking values in the, so called, causality LH-parallelogram: $\Pi = \{(\mu, \nu) \in \mathbb{R}^2 : 0 \leq \mu \leq 1, -1 \leq \nu \leq +1, 0 \leq \mu + \nu \leq 1\}$ (see [19, 29]).

The Cauchy problem for variable order differential equations has the form

$$ {}^{\mathrm{C}}\mathcal{D}_{\{\mu,\nu\}}^{\beta(t)} u(t,x) = \mathcal{A}(D) u(t,x), \quad t > 0,\ x \in \mathbb{R}^n, \tag{104} $$

$$ u(0,x) = \varphi(x), \quad x \in \mathbb{R}^n, \tag{105} $$

$$ u(T_k - 0, x) = u(T_k + 0, x), \quad k = 1, \dots, T_N,\ x \in \mathbb{R}^n. \tag{106} $$

To explain the intrinsic memory phenomenon effecting in the emergence of an inhomogeneous term in the equation, we will consider a simplest case of variable order differential equations with a single change of diffusion mode. Assume the function $\beta(t)$ takes only two values β_1, if $0 < t < T$ and β_2, if $t > T$. In other words, the diffusion mode changes at time $t = T$ from a subdiffusive mode β_1 to a subdiffusive mode β_2. Since the first mode is subdiffusive, a non-Markovian memory arises, which effects on the actual change of diffusion mode occurring at time $T_* \geq T$. Here, T_* depends on the parameters μ and ν; see [29, 34], where the value of T_* is calculated. For simplicity, suppose $\nu = 0$ and $\mu = 1$. In this case, $T_* = T$, and we assume the following continuity condition at the change of mode time $t = T$:

$$ u(T) = u(T-0). \tag{107} $$

For $0 < t < T$, equation (104) is a fractional equation of order β_1, so a solution to the Cauchy problem (104)–(105) can be found by the standard method discussed above. If $t > T$, then we have

$$ {}^{\mathrm{C}}\mathcal{D}_{0+}^{\beta(t)} u(t) = \int_0^T \mathcal{K}_{1,0}^{\beta_1}(t,\tau) \frac{du(\tau)}{d\tau} d\tau + \int_T^t \mathcal{K}_{1,0}^{\beta_2}(t,\tau) \frac{du(\tau)}{d\tau} d\tau. $$

Hence, for $t > T$ equation (104) takes the form

$$ {}^{\mathrm{C}}\mathcal{D}_{T+}^{\beta_2} u(t) = \mathcal{A} u(t) + h(t), \quad t > T, \tag{108} $$

with the initial condition (107). Equation (108) is no longer homogeneous, due to the nonhomogeneous term

$$h(t) = -\int_0^T \mathcal{K}_{1,0}^{\beta_1}(t,\tau)\frac{du(\tau)}{d\tau}d\tau.$$

To solve the Cauchy problem for equation (108) with initial condition (107), one can apply the fractional Duhamel principle (Theorem 3.1).

Bibliography

[1] O. Agraval, Analytical solution for stochastic response of a fractionally damped beam, *J. Vib. Acoust.*, **126** (2004), 561–566.

[2] M. Ali and S. Malik, An inverse problem for a family of time fractional diffusion equations, *Inverse Probl. Sci. Eng.*, **25**(9) (2017), 1299–1322.

[3] H. Alloubaa and Y. Xiaob, L-Kuramoto–Sivashinsky SPDEs vs. time-fractional SPIDEs: exact continuity and gradient moduli, 1/2-derivative criticality, and laws, *J. Differ. Equ.*, **263** (2017), 1552–1610.

[4] R. L. Bagley and P. J. Torvic, On the existence of the order domain and the solution of distributed order equations I, II, *Int. J. Appl. Math.*, **2** (2000), 865–882, 965–987.

[5] L. Bers, F. John, and M. Schechter, *Partial Differential Equations*, Interscience Publishers, New York–London–Sydney, 1964.

[6] L. Chen, Nonlinear stochastic time-fractional diffusion equations on $\mathbb{R}$: moments, Hölder regularity and intermittency, *Trans. Am. Math. Soc.*, **369** (2017), 8497–8535.

[7] P. de Carvalho-Neto and G. Planas, Mild solutions to the time fractional Navier–Stokes equations in $\mathbb{R}^N$, *J. Differ. Equ.*, **259**(7) (2015), 2948–2980.

[8] Yu. A. Dubinskii, The algebra of pseudo-differential operators with analytic symbols and its applications to mathematical physics, *Sov. Math. Surv.*, **37** (1982), 107–153.

[9] Yu. A. Dubinskii, Sobolev spaces of infinite order, *Russ. Math. Surv.*, **46**(6) (1991), 97–131.

[10] Zh. Gao and J. Wang, Hyers–Ulam stability and existence of solutions for Nigmatullin's fractional diffusion equation, *Adv. Math. Phys.*, **2017** (2017), Article ID 9692685, 6 pages.

[11] R. Gorenflo, A. Kilbas, F. Mainardi, and S. V. Rogozin, *Mittag-Leffler Functions, Related Topics, and Applications*, Springer Monographs in Mathematics, Springer, 2014.

[12] M. Hahn, K. Kobayashi, and S. Umarov, SDEs driven by a time-changed Lévy process and their associated time-fractional pseudo-differential equations, *J. Theor. Probab.*, **25**(1) (2012), 262–279.

[13] M. Hahn and S. Umarov, Fractional Fokker–Planck–Kolmogorov type equations and their associated stochastic differential equations, *Fract. Calc. Appl. Anal.*, **14**(1) (2011), 56–80.

[14] R. Ibrahim, Solutions to systems of arbitrary-order differential equations in complex domains, *Electron. J. Differ. Equ.*, **2014**(46) (2016), 1–13.

[15] N. Ikeda and Sh. Watanabe, *Stochastic Differential Equations and Diffusion Processes*, North-Holland Publishing Co., Amsterdam–Oxford–New York, 1981.

[16] I. Karatzas and S. E. Shreve, *Brownian Motion and Stochastic Calculus*, Springer, 1991.

[17] A. N. Kochubei, Distributed order calculus and equations of ultraslow diffusion, *J. Math. Anal. Appl.*, **340** (2008), 252–281.

[18] X. Li and X. Yang, Error estimates of finite element methods for stochastic fractional differential equations, *J. Comput. Math.*, **35**(3) (2017), 346–362.
[19] C. F. Lorenzo and T. T. Hartley, Variable order and distributed order fractional operators, *Nonlinear Dyn.*, **29** (2002), 57–98.
[20] M. M. Meerschaert and H.-P. Scheffler, Stochastic model for ultraslow diffusion, *Stoch. Process. Appl.*, **116** (2006), 1215–1235.
[21] J. Mijena and E. Nane, Space-time fractional stochastic partial differential equations, *Stoch. Process. Appl.*, **125** (2015), 3301–3326.
[22] J. Mijena and E. Nane, Intermittence and space-time fractional stochastic partial differential equations, *Potential Anal.*, **44** (2016), 295–312.
[23] A. B. Pskhu, D'Alambert's formula for fractional diffusion-wave equation, *Dokl. Adygea AcS*, **17**(4) (2015), 61–67 (in Russian).
[24] S. G. Samko, A. A. Kilbas, and O. I. Marichev, *Fractional Integrals and Derivatives: Theory and Applications*, Gordon and Breach Science Publishers, New York–London, 1993.
[25] M. Stojanović, Well-posedness of diffusion-wave problem with arbitrary finite number of time fractional derivatives in Sobolev spaces H^s, *Fract. Calc. Appl. Anal.*, **13**(1) (2010), 21–42.
[26] M. Stojanović, Singular fractional differential equations, *Proc. Appl. Math. Mech.*, **13** (2013), 367–368.
[27] S. Umarov, Nonlocal boundary value problems for pseudo-differential and differential operator equations I, *Differ. Equ.*, **33** (1997), 831–840.
[28] S. Umarov, On fractional Duhamel's principle and its applications, *J. Differ. Equ.*, **252**(10) (2012), 5217–5234.
[29] S. Umarov, *Introduction to Fractional and Pseudo-Differential Equations with Singular Symbols*, Springer, 2015.
[30] S. Umarov and R. Gorenflo, Cauchy and nonlocal multi-point problems for distributed order pseudo-differential equations, *Z. Anal. Anwend.*, **24** (2005), 449–466.
[31] S. Umarov, M. Hahn, and K. Kobayashi, *Beyond the Triangle: Brownian Motion, Itô Calculus, and Fokker–Planck Equation—Fractional Generalizations*, World Scientific, 2018.
[32] S. Umarov and E. Saydamatov, A fractional analog of the Duhamel principle, *Fract. Calc. Appl. Anal.*, **9**(1) (2006), 57–70.
[33] S. Umarov and E. Saydamatov, A generalization of the Duhamel principle for fractional order differential equations, *Dokl. Akad. Nauk, Ross. Akad. Nauk*, **412**(4) (2007), 463–465 (English Translation: *Doklady Mathematics*, **75**(1) (2007), 94–96).
[34] S. Umarov and S. Steinberg, Variable order differential equations with piecewise constant order-function and diffusion with changing modes, *Z. Anal. Anwend.*, **28** (2009), 431–450.
[35] Y. Wen, X.-F. Zhou, and J. Wang, Stability and boundedness of solutions of the initial value problem for a class of time-fractional diffusion equations, *Adv. Differ. Equ.*, **2017** (2017), 230, 1–10, DOI:10.1186/s13662-017-1271-6.

Yikan Liu, Zhiyuan Li, and Masahiro Yamamoto

Inverse problems of determining sources of the fractional partial differential equations

Abstract: In this chapter, we mainly review theoretical results on inverse source problems for diffusion equations with the Caputo time-fractional derivatives of order $\alpha \in (0, 1)$. Our survey covers the following types of inverse problems:
- determination of time-dependent functions in interior source terms
- determination of space-dependent functions in interior source terms
- determination of time-dependent functions appearing in boundary conditions

Keywords: Fractional diffusion equations, inverse source problems, uniqueness, stability

MSC 2010: 35R11, 26A33, 35R30, 65M32

1 Introduction

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with smooth boundary $\partial\Omega$, $T > 0$, $\alpha \in (0, 1)$ and $x = (x_1, \dots, x_d) \in \mathbb{R}^d$, $\partial_t = \frac{\partial}{\partial t}$, $\partial_j = \frac{\partial}{\partial x_j}$. By $\triangle := \sum_{j=1}^d \partial_j^2$, we denote the usual Laplacian in space, and ∂_t^α, $0 < \alpha < 1$ stands for the Caputo derivative in time:

$$\partial_t^\alpha f(t) := \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \frac{\mathrm{d}f}{\mathrm{d}s}(s)\,\mathrm{d}s,$$

where Γ denotes the gamma function.

Then we consider an initial-boundary value problem

$$\begin{cases} (\partial_t^\alpha - \triangle)u(x,t) = g(x)\rho(t), & x \in \Omega,\ 0 < t < T, \\ u(x,0) = 0, & x \in \Omega, \\ u(x,t) = 0, & x \in \partial\Omega,\ 0 < t < T. \end{cases} \tag{1}$$

The initial-boundary value problem governs the time evolution of density $u(x,t)$ at location x and time t of some substance such as contaminants. Here, $g(x)\rho(t)$ is an in-

Yikan Liu, Masahiro Yamamoto, Graduate School of Mathematical Sciences, The University of Tokyo, 3-8-1 Komaba, Meguro-ku, Tokyo 153-8914, Japan, e-mails: ykliu@ms.u-tokyo.ac.jp, myama@ms.u-tokyo.ac.jp
Zhiyuan Li, School of Mathematics and Statistics, Shandong University of Technology, Zibo, Shandong 255049, China, e-mail: zyli@sdut.edu.cn

https://doi.org/10.1515/9783110571660-018

terior source term producing the substance in Ω, and the source term can be often assumed to be modelled in the form of the separation of variables. Here, g and ρ describe the spatial distribution of the source and the time evolution pattern, respectively. In the typical inverse problems for (1), we are requested to determine $\rho(t)$ and/or $g(x)$ by extra data of solution $u(x,t)$ to (1).

Mainly, we survey the following two types of inverse source problems.

Inverse t-source problem
Let u satisfy (1). We fix $x_0 \in \Omega$ arbitrarily. Provided that the spatial component g in the source term is known, determine the temporal component ρ in $(0,T)$ by the single point observation of u in $\{x_0\} \times (0,T)$.

Inverse x-source problem
Let u satisfy (1). We suppose that the temporal component ρ in the source term is known. Determine the spatial component g by:
(a) the final time observation of u in $\Omega \times \{T\}$, or
(b) the partial interior observation of u in $\omega \times (0,T)$, where $\omega \subset \Omega$ is an arbitrarily chosen nonempty subdomain.

The above inverse t-source problem and the inverse x-source problem (a) with the final time observation have been well studied and many theoretical researches have been published for classical partial differential equations. As monograph, we should refer to Prilepko, Orlovsky, and Vasin [32]. Moreover see for example Choulli and Yamamoto [6–8], Isakov [13], Tikhonov and Eidelman [40] and the references therein for related inverse problems for usual partial differential equations.

As for the fractional differential equations, we know that we can construct theories parallel to [32], and the works are now made continuously. This chapter is nothing but an incomplete survey for these works in full progress.

The identification of $\rho(t)$ fits, for example, in the cases of disasters of nuclear power plants, in which the source location can be assumed to be known but the decay of the radiative strength in time is unknown and important to be estimated. On the other hand, one example of the identification of $g(x)$ can be illustrated by the detection of illegal discharge of sewage, which is a serious problem in some countries. Also by such practical demands, the inverse source problems have been strongly required to be studied both theoretically and numerically.

Some papers surveyed later discuss the case where $-\triangle$ is replaced by a general linear elliptic operator:

$$-\sum_{i,j=1}^{d} \partial_j\big(a_{ij}(x,t)\partial_i u\big) + \sum_{j=1}^{d} b_j(x,t)\partial_j u + c(x,t)u.$$

Moreover, we can similarly discuss the following generalized cases:
- Multi-term $\sum_{j=1}^{m} q_j \partial_t^{\alpha_j} u$ or distributed order $\int_0^t \mu(\alpha)\partial_t^{\alpha} u \, \mathrm{d}\alpha$ in time (see, e. g., the chapter on “Inverse problems of determining parameters of the fractional partial

differential equations" of this handbook). Here, $0 < \alpha_m < \cdots < \alpha_1 < 1$, $q_1, \ldots, q_m$ are constants, and $\mu \in L^1(0,1), \geq 0, \not\equiv 0$.
- $\rho(t)g(x,t)$ in the inverse t-source problem and $g(x)\rho(x,t)$ in the inverse x-source problem.
- Other kinds of boundary conditions and inhomogeneous boundary values, and the whole space $\mathbb{R}^d$.

However, in order to focus on the main topic, we choose a simple formulation (1), which definitely captures the essence of the inverse source problems.

We note that in the cases of $\rho(t)g(x,t)$ and $g(x)\rho(x,t)$, the inverse source problems are considered as linearized problems of inverse coefficient problems (e. g., see the chapter on "Inverse problems of determining coefficients of the fractional partial differential equations" of this handbook). For example, we discuss the determination of $p(x)$ in

$$(\partial_t^\alpha - \triangle)y(x,t) = p(x)y(x,t), \quad x \in \Omega,\ 0 < t < T,$$

by extra data. Let y and z be the corresponding solutions respectively with the coefficients $p(x)$ and $q(x)$. Then, setting $u = y - z$, $g(x) = p(x) - q(x)$ and $\rho(x,t) = z(x,t)$, we reduce the inverse problem of determining a coefficient to an inverse source problem for

$$(\partial_t^\alpha - \triangle - p(x))u(x,t) = g(x)\rho(x,t), \quad x \in \Omega,\ 0 < t < T.$$

Most of publications on inverse problems for fractional equations are concerned with the case of the Caputo derivatives in t of orders $\alpha \in (0,1)$. Mainly, we discuss the Caputo derivative ∂_t^α with $0 < \alpha < 1$, although other kinds of fractional derivatives and/or $1 < \alpha < 2$ are also meaningful.

In this chapter, we mainly review theoretical results in the existing literature for inverse source problems with slight improvements time by time. We will at most provide key ideas and sketches of the proofs instead of detailed arguments.

The remainder of this chapter is organized as follows. In Section 2, we prepare the necessary ingredients for dealing with the inverse problems including the basic facts on forward problems of (1). In Sections 3–4, we review the inverse t- and x-source problems, respectively. In Section 5, we survey related inverse problems of determining some functions in boundary conditions. Finally, we summarize the chapter with concluding remarks in Section 6.

2 Preliminaries

Henceforth, $L^2(\Omega)$, $H_0^1(\Omega)$, $H^2(\Omega)$ are the usual L^2-space and the Sobolev spaces of real-valued functions (e. g., Adams [1]): $L^2(\Omega) = \{f;\ \int_\Omega |f(x)|^2\,dx < \infty\}$, and let $(\,\cdot\,,\,\cdot\,)$ denote the scalar product: $(f,g) = \int_\Omega f(x)g(x)\,\mathrm{d}x$ for $f,g \in L^2(\Omega)$. We define the Laplace operator $-\triangle$ with the domain $\mathcal{D}(-\triangle) = H^2(\Omega) \cap H_0^1(\Omega)$. An eigensystem $\{(\lambda_n, \varphi_n)\}_{n=1}^\infty$ of $-\triangle$ is defined by $\{\varphi_n\}_{n=1}^\infty \subset \mathcal{D}(-\triangle)$ and $0 < \lambda_1 < \lambda_2 \le \dots$ such that $-\triangle\varphi_n = \lambda_n\varphi_n$ and $\{\varphi_n\}_{n=1}^\infty$ is a complete orthonormal system of $L^2(\Omega)$.

For $\gamma \ge 0$, the fractional Laplacian $(-\triangle)^\gamma$ is defined as

$$\mathcal{D}((-\triangle)^\gamma) := \left\{ f \in L^2(\Omega); \sum_{n=1}^\infty \left|\lambda_n^\gamma (f,\varphi_n)\right|^2 < \infty \right\},$$
$$(-\triangle)^\gamma f := \sum_{n=1}^\infty \lambda_n^\gamma (f,\varphi_n)\varphi_n.$$

We know that $\mathcal{D}((-\triangle)^\gamma)$ is a Hilbert space with the norm

$$\|f\|_{\mathcal{D}((-\triangle)^\gamma)} := \left(\sum_{n=1}^\infty \left|\lambda_n^\gamma (f,\varphi_n)\right|^2 \right)^{1/2}, \quad f \in \mathcal{D}((-\triangle)^\gamma).$$

For $1 \le p \le \infty$ and a Banach space X, we say that $f \in L^p(0,T;X)$ if

$$\|f\|_{L^p(0,T;X)} := \begin{Bmatrix} (\int_0^T \|f(\cdot,t)\|_X^p \mathrm{d}t)^{\frac{1}{p}} & \text{if } 1 \le p < \infty \\ \operatorname{ess\,sup}_{0<t<T} \|f(\cdot,t)\|_X & \text{if } p = \infty \end{Bmatrix} < \infty.$$

We define the forward Riemann–Liouville integral operator of order $\alpha \in (0,1)$:

$$J_{0+}^\alpha f(t) := \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s)}{(t-s)^{1-\alpha}}\,\mathrm{d}s.$$

Then it is easily seen that the Caputo derivative $\partial_t^\alpha f = J_{0+}^{1-\alpha}(\frac{\mathrm{d}f}{\mathrm{d}t})$. For the solution representation, we define the Mittag-Leffler function (e. g., Podlubny [31])

$$E_{\alpha,\beta}(z) := \sum_{k=0}^\infty \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad z \in \mathbb{C},\ \alpha > 0,\ \beta > 0.$$

The following estimate is later useful:

$$|E_{\alpha,\beta}(-\eta)| \le \frac{C}{1+\eta}, \quad \eta \ge 0,\ 0 < \alpha < 2,\ \beta > 0, \tag{2}$$

where $C > 0$ is a constant depending only on α and β.

For inverse problems, we have to study the unique existence of solution $u(x,t)$ to the initial-boundary value problem (1) and its properties, which is called the forward problem, contrasted with the inverse problems.

Now we collect the basic results on the forward problem

$$\begin{cases} (\partial_t^\alpha - \triangle)w(x,t) = F(x,t), & x \in \Omega,\ 0 < t < T, \\ w(x,0) = a(x), & x \in \Omega, \\ w(x,t) = 0, & x \in \partial\Omega,\ 0 < t < T. \end{cases} \tag{3}$$

Lemma 1. (a) *Let* $F = 0$ *and* $a \in \mathcal{D}((-\triangle)^{\gamma})$ *with* $\gamma \geq 0$. *Then there exists a unique solution* $w \in \bigcap_{1\leq p\leq\infty} L^p(0,T;\mathcal{D}((-\triangle)^{\gamma+\frac{1}{p}}))$ *to* (3), *where we interpret* $\frac{1}{p} = 0$ *for* $p = \infty$. *Moreover, the solution* w *allows the representation*

$$w(\,\cdot\,,t) = \sum_{n=1}^{\infty} E_{\alpha,1}(-\lambda_n t^{\alpha})(a,\varphi_n)\varphi_n$$

in $L^p(0,T;\mathcal{D}((-\triangle)^{\gamma+\frac{1}{p}}))$ *for any* $p \in [1,\infty]$, *and* $w : (0,T] \longrightarrow L^2(\Omega)$ *can be analytically extended to a sector* $\{z \in \mathbb{C};\ z \neq 0,\ |\arg z| < \frac{\pi}{2}\}$. *Furthermore, there exists a constant* $C > 0$ *such that for any* $t > 0$,

$$\|w(\,\cdot\,,t)\|_{\mathcal{D}((-\triangle)^{\gamma+1/p})} \leq C\,t^{-\frac{\alpha}{p}}\|a\|_{\mathcal{D}((-\triangle)^{\gamma})},$$
$$\|w(\,\cdot\,,t)\|_{\mathcal{D}((-\triangle)^{\gamma+1})} + \|\partial_t^{\alpha} w(\,\cdot\,,t)\|_{\mathcal{D}((-\triangle)^{\gamma})} \leq C\,t^{-\alpha}\|a\|_{\mathcal{D}((-\triangle)^{\gamma})}.$$

(b) *Let* $a = 0$ *and* $F \in L^p(0,T;\mathcal{D}((-\triangle)^{\gamma}))$ *with* $p \in [1,\infty]$ *and* $\gamma \geq 0$. *Then there exists a unique solution* $w \in \bigcap_{0<\varepsilon\leq 1} L^p(0,T;\mathcal{D}((-\triangle)^{\gamma+1-\varepsilon}))$ *to* (3). *Moreover, the solution* w *allows the representation*

$$w(\,\cdot\,,t) = \sum_{n=1}^{\infty}\left(\int_0^t s^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n s^{\alpha})\,(F(\,\cdot\,,t-s),\varphi_n)\,\mathrm{d}s\right)\varphi_n \tag{4}$$

in $L^p(0,T;\mathcal{D}((-\triangle)^{\gamma+1-\varepsilon}))$ *for any* $\varepsilon \in (0,1]$. *Furthermore, there exists a constant* $C > 0$ *such that for* $\forall\,\varepsilon \in (0,1]$,

$$\|w\|_{L^p(0,T;\mathcal{D}((-\triangle)^{\gamma+1-\varepsilon}))} + \|\partial_t^{\alpha} w\|_{L^p(0,T;\mathcal{D}((-\triangle)^{\gamma-\varepsilon}))} \leq \frac{C}{\varepsilon}\|F\|_{L^p(0,T;\mathcal{D}((-\triangle)^{\gamma}))}.$$

The above well-posedness results are refinements of that stated in [35, Theorems 2.1–2.2], which can be easily verified by the arguments in Li, Liu and Yamamoto [22].

Especially, Lemma 1 describes the limited smoothing property of time-fractional diffusion equations, that is, in the case of $F \equiv 0$, the improvement of the spatial regularity of solution is at most 2, compared with the initial value. More precisely, the regularity improvement of the homogeneous problem can exactly reach 2 at the cost of a weakened norm in time. On the other hand, with a source term of L^p regularity in time, the regularity improvement of the inhomogeneous one can never reach 2 except for the special case of $p = 2$, where the complete monotonicity property of Mittag-Leffler functions can be utilized.

Henceforth, we understand the class of solutions as described in Lemma 1.

We can refer to Eidelman and Kochubei [9] on fundamental solutions, Gorenflo, Luchko and Yamamoto [11] on the suitable function spaces for solutions, Kubica and Yamamoto [21] on the weak solution and improved regularity, Luchko [29] on a solution formula and the maximum principle, Sakamoto and Yamamoto [35] on the well-posedness including some inverse problems by the representation of solutions, Zacher

[59] on generalized treatments on weak solutions. Here we present very limited references and we can consult also other related chapters of this handbook.

We can represent the solution to (1) by a fractional Duhamel's principle (see Liu, Rundell, and Yamamoto [27]).

Lemma 2. *Let u be the solution to* (1), *where $\rho \in L^1(0,T)$ and $g \in \mathcal{D}((-\triangle)^\gamma)$ with $\gamma \geq 0$. Then u allows the representation*

$$J_{0+}^{1-\alpha} u(\,\cdot\,,t) = \int_0^t \rho(t-s) v(\,\cdot\,,s)\,\mathrm{d}s, \tag{5}$$

where $J_{0+}^{1-\alpha}$ is the $(1-\alpha)$-th order Riemann–Liouville integral, and v solves the homogeneous problem

$$\begin{cases} (\partial_t^\alpha - \triangle) v(x,t) = 0, & x \in \Omega,\ t > 0, \\ v(x,0) = g(x), & x \in \Omega, \\ v(x,t) = 0, & x \in \partial\Omega,\ t > 0. \end{cases} \tag{6}$$

Lemma 2 relates the inhomogeneous problem (1) with the homogeneous one (6). Therefore, it suffices to study (6) for the inverse source problems for (1). Later we will see that Lemma 2 acts as the starting point for discussing the inverse problems. For the fractional Duhamel's principle, see also Umarov and Saidamatov [42], Zhang and Xu [61].

3 Inverse t-source problems

3.1 Two-sided Lipschitz stability in the case of $x_0 \in \operatorname{supp} g$

Theorem 1. *Let $g \in \mathcal{D}((-\triangle)^\gamma)$ with $\gamma > \frac{d}{4}$ and u satisfy* (1) *for $\rho \in C[0,T]$. If $g(x_0) \neq 0$, then there exists a constant $C > 0$ such that*

$$C^{-1} \|\partial_t^\alpha u(x_0,\,\cdot\,)\|_{L^\infty(0,T)} \leq \|\rho\|_{C[0,T]} \leq C \|\partial_t^\alpha u(x_0,\,\cdot\,)\|_{L^\infty(0,T)}. \tag{7}$$

By the Sobolev embedding $\mathcal{D}((-\triangle)^\gamma) \subset C(\overline{\Omega})$ with $\gamma > \frac{d}{4}$, the regularity of the known spatial component g in the above theorem turns out to be sufficient for a pointwise definition of $g(x)$. Theorem 1 slightly improves the regularity $g \in \mathcal{D}((-\triangle)^{\gamma_0})$ with $\gamma_0 > \frac{3d}{4} + 1$ in [35, Theorem 4.4]. Here we briefly explain how to realize such a reduction in regularity.

Actually, the first inequality in (7) is a direct corollary of Lemma 1(b) and the Sobolev embedding theorem. In order to show the second inequality in (7), we substitute the representation (4) into the governing equation in (1) and substitute $x = x_0$ to write formally

$$\rho(t) = \frac{1}{g(x_0)}\left\{\partial_t^\alpha u(x_0,t) + \int_0^t Q(x_0,s)\rho(t-s)\,\mathrm{d}s\right\}, \tag{8}$$

where

$$Q(\,\cdot\,,t) := t^{\alpha-1}\sum_{n=1}^{\infty}\lambda_n E_{\alpha,\alpha}(-\lambda_n t^\alpha)(g,\varphi_n)\varphi_n.$$

Introducing $\varepsilon := \frac{1}{2}(\gamma - \frac{d}{4}) > 0$, we employ (2) to estimate

$$\begin{aligned}\|Q(\,\cdot\,,t)\|^2_{\mathcal{D}((-\triangle)^{d/4+\varepsilon})} &= t^{2(\alpha-1)}\sum_{n=1}^{\infty}\left|\lambda_n^{1-\varepsilon}E_{\alpha,\alpha}(-\lambda_n t^\alpha)\right|^2\left|\lambda_n^{2\varepsilon+\frac{d}{4}}(g,\varphi_n)\right|^2\\ &\le C^2\,t^{2(\alpha-1)}\sum_{n=1}^{\infty}\left(\frac{(\lambda_n t^\alpha)^{1-\varepsilon}}{1+\lambda_n t^\alpha}t^{\alpha(\varepsilon-1)}\right)^2|\lambda_n^\gamma(g,\varphi_n)|^2\\ &\le \left(C\|g\|_{\mathcal{D}((-\triangle)^\gamma)}t^{\alpha\varepsilon-1}\right)^2.\end{aligned}$$

Using the Sobolev embedding $\mathcal{D}((-\triangle)^{\frac{d}{4}+\varepsilon}) \subset H^{\frac{d}{2}+2\varepsilon}(\Omega) \subset C(\overline{\Omega})$, we obtain

$$|Q(x_0,t)| \le \|Q(\,\cdot\,,t)\|_{C(\overline{\Omega})} \le C\|Q(\,\cdot\,,t)\|_{\mathcal{D}((-\triangle)^{d/4+\varepsilon})} \le C\|g\|_{\mathcal{D}((-\triangle)^\gamma)}t^{\alpha\varepsilon-1}.$$

Therefore, the above estimate implies

$$|\rho(t)| \le C\|\partial_t^\alpha u(x_0,\,\cdot\,)\|_{L^\infty(0,T)} + C\|g\|_{\mathcal{D}((-\triangle)^\gamma)}\int_0^t s^{\alpha\varepsilon-1}|\rho(t-s)|\,\mathrm{d}s.$$

Applying a Gronwall-type inequality in [12, Lemma 7.1.1], we complete the proof of the second inequality in (7).

In the same direction, also several other papers obtained Lipschitz stability in slightly different formulations. By assuming the homogeneous Neumann condition instead of that in (1), the observation point x_0 can be placed on the boundary, and we have the following result.

Theorem 2 (Wei, Li, and Li [48]). *Let u be the solution to* (1) *with the homogeneous Neumann boundary condition, where ρ is absolutely continuous on* $[0,T]$ *and* $g \in \mathcal{D}((-\triangle+1)^\gamma)$ *with* $\gamma > \frac{d}{2}+1$. *If* $x_0 \in \partial\Omega$ *and* $g(x_0) \ne 0$, *then* (7) *still holds.*

By a similar argument as that of Theorem 1, we can also reduce the regularity assumption in the above theorem, and we skip the details here.

On the other hand, the following result reveals that Theorem 1 holds true with a more general source term.

Theorem 3 (Fujishiro and Kian [10]). *Let u be the solution to* (1) *with* $F(x,t) = \rho(t)g(x,t)$, *where* $g \in L^p(0,T;\mathcal{D}(-\triangle))$ *with* $\frac{8}{\alpha} < p \le \infty$ *and there exists a constant* $\delta > 0$ *such that* $|g(x_0,\cdot)| \ge \delta$ *a. e. in* $(0,T)$. *Then there exists a constant* $C > 0$ *such that*

$$\|\rho\|_{L^p(0,T)} \le C\|\partial_t^\alpha u(x_0,\cdot)\|_{L^p(0,T)}.$$

The key to the proof is an L^p estimate for (8) (see [10, Lemma 4]), and here we omit the details. In [10], Theorem 3 is applied for establishing the conditional stability for a corresponding inverse coefficient problem by the same observation data. For $1 < \alpha < 2$, Wu and Wu [54] obtained the uniqueness in a more general formulation under the same non-vanishing condition.

Ruan and Wang [33] adopts distributed observations: given $\sigma \in C_0^\infty(\Omega)$ satisfying $\sigma \ge 0$ and $\sigma \not\equiv 0$, one measures

$$\int_\Omega u(x,t)\sigma(x)\,\mathrm{d}x, \quad 0 < t < T.$$

Under the condition

$$(\sigma,g) = \int_\Omega \sigma(x)g(x)\,\mathrm{d}x \ne 0,$$

[33, Theorem 1] established a similar estimate to (7) in fractional Sobolev norms. In this direction, see also Aleroev, Kirane and Malik [2], which restricts $\sigma \equiv 1$ but generalizes $F(x,t) = \rho(t)g(x,t)$ in (1).

3.2 Uniqueness and stability with general observation point x_0

First, we state a uniqueness result for any $x_0 \in \Omega$ not necessarily satisfying $g(x_0) \ne 0$, which removes the restriction on the space dimensions and reduces the required regularity of ρ in the result in Liu, Rundell, and Yamamoto [27].

Theorem 4. *We assume that* $\rho \in L^1(0,T)$, $g \in \mathcal{D}((-\triangle)^\gamma)$ *with* $\gamma > \frac{d}{4} - 1$, $g \ge 0$ *and* $g \not\equiv 0$. *Then*

$$u(x_0,t) = 0, \quad 0 < t < T \quad \textit{implies} \quad \rho(t) = 0, \quad 0 < t < T.$$

The key to such an improvement in Theorem 4 is Lemma 2, that is, a weak form of the fractional Duhamel's principle. Taking $x = x_0$ in (5) of Lemma 2 and using $u(x_0,\cdot) = 0$ in $(0,T)$, we have

$$0 = J_{0+}^{1-\alpha}u(x_0,t) = \int_0^t \rho(t-s)v(x_0,s)\,\mathrm{d}s, \quad 0 < t < T.$$

After this step, one can follow the arguments in [27] to employ the Titchmarsh convolution theorem (see Titchmarsh [41]) and some strict positivity property of the solution to (6) (see [27]) to conclude the result.

Moreover Liu [26] generalized Theorem 4 for multi-term time-fractional diffusion equations.

Next, we continue to investigate the stability of the inverse t-source problem especially in the case of $x_0 \notin \operatorname{supp} g$. Only in this part, instead of the initial-boundary value problem (1), we consider the Cauchy problem in the whole space:

$$\begin{cases} (\partial_t^\alpha - \triangle)u(x,t) = g(x)\rho(t), & x \in \mathbb{R}^d,\ 0 < t < T, \\ u(x,0) = 0, & x \in \mathbb{R}^d. \end{cases} \tag{9}$$

In order to state the stability, for given constants $M > 0$ and $N \in \{0, 1, 2, \ldots\}$, we define the admissible set of unknown temporal components ρ by

$$\begin{aligned} \mathcal{U}_N := \{f \in C^1[0,T];\ \|f\|_{C^1[0,T]} \le M,& \\ f \text{ changes signs at most } N \text{ times on } (0,T)\}.& \end{aligned} \tag{10}$$

This admissible set was introduced in Saitoh, Tuan, and Yamamoto [34], where the same inverse t-source problem is discussed for $\alpha = 1$.

With the above preparations, now we can state the main stability result.

Theorem 5 (Liu and Zhang [28]). *Let u satisfy* (9), *where we assume that*

$$g \ge 0, \quad g \not\equiv 0, \quad \begin{cases} g\text{: bounded continuous} & \text{if } d = 1, \\ g\text{: locally Hölder continuous} & \text{if } d = 2, 3. \end{cases} \tag{11}$$

Let $\mathcal{U}_N$ be defined by (10) *with given constants $M > 0$ and $N \in \{0, 1, 2, \ldots\}$.*

(a) *Let $N = 0$. Then for any $\delta_1 \in (0, T)$, there exists a constant $B_{\delta_1} > 0$ depending only on x_0, g such that $B_{\delta_1} \to \infty$ as $\delta_1 \to 0$ and*

$$\|\rho\|_{L^1(0,T-\delta_1)} \le \frac{T^{1-\alpha} B_{\delta_1}}{\Gamma(2-\alpha)} \|u(x_0, \cdot)\|_{L^1(0,T)} \tag{12}$$

for all $\rho \in \mathcal{U}_0$.

(b) *Let $N \in \{1, 2, 3, \ldots\}$. We further assume that $x_0 \notin \operatorname{supp} g$ and $\|u(x_0, \cdot)\|_{L^1(0,T)}$ is sufficiently small. Then there exist a sufficiently small $\delta_2 > 0$ and a constant $C > 0$ such that*

$$\|\rho\|_{L^1(0,T-\delta_2)} \le C \big(\underbrace{\log(\cdots(\log|\log}_{N+1} \|u(x_0, \cdot)\|_{L^1(0,T)}|)\cdots)\big)^{-2(\frac{2}{\alpha}-1)} \tag{13}$$

for all $\rho \in \mathcal{U}_N$.

In the conditional stability (12) and (13), we need to take data over a longer observation time interval $(0,T)$ for estimating $\|\rho\|_{L^1(0,T-\delta_k)}$ with $k = 1,2$. Our stability estimate is weaker when the time N of changing signs is increasing.

The parameter $B_{\delta_1} > 0$ in Theorem 5(a) turns out to be

$$B_{\delta_1} := \frac{1}{\|v(x_0,\cdot)\|_{L^1(0,\delta_1)}} = \left(\int_0^{\delta_1} v(x_0,t)\,\mathrm{d}t\right)^{-1},$$

where v solves the homogeneous Cauchy problem

$$\begin{cases} (\partial_t^\alpha - \triangle)v(x,t) = 0, & x \in \mathbb{R}^d,\ t > 0, \\ v(x,0) = g(x) \ge 0, \not\equiv 0, & x \in \mathbb{R}^d. \end{cases} \tag{14}$$

For the details, we refer to [28, Section 2].

According to Eidelman and Kochubei [9], v is strictly positive at least for sufficiently small $t > 0$. Furthermore [9] gives a lower bound for v, which validates the quantitative analysis when ρ changes signs.

More precisely, the proof of Theorem 5 is based on the following lemma which one can prove also by using [9].

Lemma 3. (a) *Let u be the solution to* (9), *where we assume $\rho \in C[0,\infty)$ and* (11). *Then Lemma* 2 *still holds, that is, u allows the same representation* (5), *where v solves the initial value problem* (14) *for the homogeneous equation.*

(b) *Let g satisfy* (11). *Then there exists a classical solution to* (14), *which takes the form*

$$v(x,t) = \int_{\mathbb{R}^d} K_\alpha(x-\xi,t)g(\xi)\,\mathrm{d}\xi,$$

where the fundamental solution $K_\alpha(x,t)$ satisfies the following asymptotic behavior as $t \downarrow 0$: If $|x| > r$ for some fixed $r > 0$, then there exist a constant $C > 0$ depending on α, d, r such that

$$K_\alpha(x,t) \sim t^{-\frac{\alpha d}{2(2-\alpha)}} |x|^{-\frac{d(1-\alpha)}{2-\alpha}} \exp(-C\,t^{-\frac{\alpha}{2-\alpha}} |x|^{\frac{2}{2-\alpha}}).$$

For other related works on the inverse t-source problem, see also Jin and Rundell [15], Wang and Wu [43]. At the end of this section, we briefly mention the numerical reconstruction method for the inverse t-source problem developed in [28]. In order to specify the dependency on ρ, by $u(\rho)$ we denote the solution of (1), and let ρ_* be the true solution. Under the same non-negativity assumption of g as before, we propose the fixed-point iteration

$$\rho_m(t) = \begin{cases} 0, & m = 0, \\ \rho_{m-1}(t) + \frac{\partial_t^\alpha(u(\rho_*)-u(\rho_{m-1}))(x_0,t)}{K}, & m = 1,2,\dots, \end{cases}$$

where $K > 0$ is a constant such that $\|v(x_0, \cdot)\|_{C[0,T]} \le K$. Since v can be computed in advance, we can easily evaluate K, and the proposed iteration only involves one-dimensional computation in time by taking time derivatives in (5) of Lemma 2. See [28] for further details on the convergence analysis and numerical examples.

4 Inverse x-source problems

In this section, we investigate inverse x-source problems for (1).

4.1 Final observation data

For the inverse x-source problem with final time observation data, we review three relevant results. First, we introduce the conditional Hölder stability obtained in Wang, Zhou, and Wei [45]:

Theorem 6. *Let u be the solution to* (1), *where we assume $g \in \mathcal{D}((-\triangle)^\gamma)$ with some fixed $\gamma > 0$ and $\rho(t) \equiv 1$. If g satisfies an a priori estimate $\|g\|_{\mathcal{D}((-\triangle)^\gamma)} \le E$ with a constant $E > 0$, then there exists a constant $C > 0$ such that*

$$\|g\|_{L^2(\Omega)} \le C E^{\frac{1}{\gamma+1}} \|u(\cdot, T)\|_{L^2(\Omega)}^{\frac{\gamma}{\gamma+1}}.$$

Similarly to several previous theorems, the above theorem follows immediately from Lemmata 1 and 2 and the key estimate (2) in Section 2. Such results as Theorem 6 are called to be conditional stability, because in order to estimate the L^2 norm of g, one should assume its a priori bound with some norm.

In general, it is technically difficult to establish unconditional stability for the inverse x-source problem for general $\alpha \in (0,1)$. Only in the case of $\alpha = \frac{1}{2}$, Yamamoto and Zhang [57] proved such stability as a by-product for treating a corresponding inverse coefficient problem. For conciseness, here we only discuss a slightly simpler formulation than that in [57]:

$$\begin{cases} (\partial_t^{\frac{1}{2}} - \partial_x^2) u(x,t) = g(x)\rho(x,t), & 0 < x < \ell,\ 0 < t < T, \\ u(x,0) = 0, & 0 < x < \ell, \\ u(0,t) = \partial_x u(0,t) = 0, & 0 < t < T. \end{cases} \tag{15}$$

We state the local Hölder stability for the inverse x-source problem in this case.

Theorem 7. *Let u satisfy* (15) *and suppose that u, g and ρ are sufficiently smooth. Fix $t_0 \in (0,T)$ arbitrarily and choose $x_0 > \ell$ such that $x_0 - \ell > 0$ is sufficiently small. Define the level set $\Omega_\varepsilon := \{x \in (0,\ell);\ |x - x_0|^2 > \varepsilon\}$ for $\varepsilon > 0$. If $\rho(\cdot, t_0) \ne 0$ in $\overline{\Omega}$ and $g(0) = g'(0) = 0$, then there exist constants $C > 0$ and $\kappa \in (0,1)$ such that*

$$\|g\|_{H^2(\Omega_{3\varepsilon})} \le C\|u(\cdot, t_0)\|_{H^4(\Omega_\varepsilon)}^{\kappa}.$$

The key idea to prove the above theorem is a transform of (15) to an equation governed by the fourth-order differential operator $\partial_t - \partial_x^4$. For such an equation, one can apply a class of weighted L^2 estimates called Carleman estimates to prove the stability of inverse problems; see also Xu, Cheng, and Yamamoto [55] for the derivation of the Carleman estimate. Unfortunately, the idea of transforming to an equation of integer order works at most for $\alpha \in \mathbb{Q}$. Actually, even the case of $\alpha = \frac{1}{3}$ involves huge amounts of calculations in applying Carleman estimates. Also in Chapter "Inverse problems of determining coefficients of the fractional partial differential equations" of this handbook, we describe about Carleman estimates for fractional diffusion equations in general spatial dimensions. Here we refer to Cheng, Lin and Nakamura [4] for $\alpha = \frac{1}{2}$, Lin and Nakamura [23] for $\alpha \in (0,1)$, and Lin and Nakamura [24] for equations with multi-term time fractional derivatives of orders $\in (0,2) \setminus \{1\}$. Their Carleman estimates yield the unique continuation for Caputo time-fractional diffusion equations, but are not applicable to inverse problems.

Next, we discuss the structure of the inverse problem with final observation data by the Fredholm alternative. We consider a more general inverse source problem when ρ depends also on x:

$$\begin{cases} (\partial_t^\alpha - \Delta - p(x))u(x,t) = g(x)\rho(x,t), & x \in \Omega,\ 0 < t < T, \\ u(x,0) = 0, & x \in \Omega, \\ u(x,t) = 0, & x \in \partial\Omega,\ 0 < t < T. \end{cases}$$

With suitable regularity assumption on ρ, we reduce this inverse source problem with final data to a Fredholm equation of the second kind:

$$g + Kg = -\frac{(\Delta + p(x))u(x,T)}{\rho(x,T)}, \quad x \in \Omega,$$

where $K : L^2(\Omega) \longrightarrow L^2(\Omega)$ is a compact linear operator (e. g., Sakamoto and Yamamoto [36]). Therefore if -1 is not an eigenvalue of K, then the inverse source problem is well-posed in the sense of Hadamard. In the case where -1 is an eigenvalue of K, the non-uniqueness for the inverse problem is restricted only in a finite dimensional subspace of $L^2(\Omega)$. This property by the Fredholm alternative was originally proved for the inverse problem with the final observation for parabolic equations (e. g., [32]) and the same property holds also for the fractional differential equation (e. g., [36]).

Finally, we treat the problem from a different point of view. Let us consider the perturbation of the governing equation in (1) with a parameter $r > 0$:

$$\begin{cases} \partial_t^\alpha u(x,t) = r^\alpha(\Delta + p(x))u + g(x)\rho(x,t), & x \in \Omega,\ 0 < t < T, \\ u(x,0) = 0, & x \in \Omega, \\ u(x,t) = 0, & x \in \partial\Omega,\ 0 < t < T. \end{cases} \tag{16}$$

Specifying the dependency on r and g and given smooth ρ, by $u(r,g)$ we denote the unique solution to (16). We state the generic well-posedness result in the following theorem, which is a slight simplification of the main theorem in Sakamoto and Yamamoto [36].

Theorem 8. *Let $u(r,g)$ be the solution to* (16). *We assume that $p \in C(\overline{\Omega})$, $\rho \in C^1([0,T]; L^\infty(\Omega))$ and $\rho(\cdot,T) \neq 0$ on $\overline{\Omega}$. For any open interval $I \subset (0,\infty)$, there exists a finite set $E = E(\alpha,\rho,I) \subset I$ such that for any $r \in I \backslash E$ and $\psi \in \mathcal{D}(-\triangle)$, there exists a unique solution $\{u(r,g),g\} \in C([0,T];\mathcal{D}(-\triangle)) \times L^2(\Omega)$ to* (16) *satisfying $u(r,g)(\,\cdot\,,T) = \psi$. Moreover, there exists a constant $C > 0$ such that*

$$\|g\|_{L^2(\Omega)} + \|u(r,g)\|_{C([0,T];H^2(\Omega))} + \|\partial_t^\alpha u(r,g)\|_{C([0,T];L^2(\Omega))} \le C\|\psi\|_{H^2(\Omega)}.$$

We do not know whether $E = \emptyset$. We can understand Theorem 8 as follows. For an arbitrarily given target function $\psi \in \mathcal{D}(-\triangle)$, we attempt to find a pair $(r,g) \in (0,\infty) \times L^2(\Omega)$ such that $u(r,g)(\,\cdot\,,T) = \psi$. Unfortunately, with an arbitrarily fixed $r > 0$, for example, $r = 1$, we do not know whether there exists a unique $g \in L^2(\Omega)$ satisfying the above condition. However, by taking I as any open neighborhood of $r = 1$, Theorem 8 asserts that the problem may be ill-posed only for a finite set of I. This inverse problem is generically well-posed in the sense of Hadamard, although we do not know whether the original problem with $r = 1$ is well-posed. We refer to Choulli and Yamamoto [6, 8] for the generic well-posedness for inverse parabolic problems.

With the aid of the Fredholm alternative, the key to proving Theorem 8 is the analytic perturbation theory (see Kato [16]).

In this direction, Tatar and Ulusoy [38] discusses the same type of inverse x-source problem with final observation data for

$$(\partial_t^\alpha + r^\alpha(-\triangle)^{\frac{\beta}{2}})u(x,t) = g(x)\rho(t), \quad x \in \Omega,\ 0 < t < T$$

with given $\beta \in (0,2)$, and see Tatar, Tinaztepe and Ulusoy [37] as for a numerical method.

For other related works on the inverse x-source problem with final observation, see also Kawamoto [17], Kirane and Malik [18], Kirane, Malik and Al-Gwaiz [19].

4.2 Uniqueness by partial interior observation

Now we continue to study the inverse x-source problem with interior observation data. Regarding the same type of problems for classical partial differential equations such as wave and heat equations, we know that there are quite a lot of stability results based on Carleman estimates (e. g., Bellassoued and Yamamoto [3], Klibanov and Timonov [20], Yamamoto [56]). However, except for rather special cases like $\alpha = \frac{1}{2}$, such methodology does not work for fractional equations due to the absence of convenient formulae of integration by parts for fractional derivatives. As a result, to the best of our knowledge, the stability of the inverse x-source problem with general $\alpha \in (0,1)$ mostly keeps open, and here we can only review the uniqueness result in Jiang, Li, Liu and Yamamoto [14].

Theorem 9 (Jiang, Li, Liu, and Yamamoto [14]). *Let $g \in L^2(\Omega)$ and assume that $\rho \in C^1[0,T]$ with $\rho(0) \neq 0$. Let u be the solution to* (1) *and $\omega \subset \Omega$ be an arbitrary nonempty subdomain. Then*

$$u = 0 \textit{ in } \omega \times (0,T) \quad \textit{implies} \quad g = 0 \textit{ in } \Omega.$$

The keys to proving the above theorem are the fractional Duhamel's principle (5) and the following uniqueness for (6) in [14]:

Lemma 4. *Let v be the solution to* (6) *with $g \in L^2(\Omega)$, and $\omega \subset \Omega$ be an arbitrary nonempty subdomain. Then*

$$v = 0 \textit{ in } \omega \times (0,T) \quad \textit{implies} \quad v = 0 \textit{ in } \Omega \times (0,T).$$

We briefly introduce a numerical method for the inverse x-source problem developed in [14]. In order to specify the dependency on g, let us denote the solution of (1) by $u(g)$, and let g_* be the true solution. Then we propose the iterative thresholding algorithm

$$g_{m+1} = \frac{K}{K+\beta} g_m - \frac{1}{K+\beta} \int_0^T \rho\, z(g_m)\, \mathrm{d}t, \quad m = 0,1,2,\dots,$$

where $K > 0$ and $\beta > 0$ are suitably chosen parameters. Here $z(g_m)$ solves the backward problem

$$\begin{cases} -J_{T-}^{1-\alpha}(\partial_t z) - \triangle z = \chi_\omega(u(g_m) - u(g_*)) & \text{in } \Omega \times (0,T), \\ z = 0 & \text{in } \Omega \times \{T\}, \\ z = 0 & \text{on } \partial\Omega \times (0,T), \end{cases}$$

where χ_ω is the characteristic function of ω and $J_{T-}^{1-\alpha}$ denotes the backward Riemann–Liouville integral operator defined by

$$J_{T-}^{\alpha} f(t) := \frac{1}{\Gamma(\alpha)} \int_t^T \frac{f(s)}{(s-t)^{1-\alpha}}\, \mathrm{d}s.$$

See [14] for further details.

We mention that there are other kinds of observation data. For example, Zhang and Xu [61] discussed the determination of g in

$$\begin{cases} (\partial_t^\alpha - \partial_x^2) u(x,t) = g(x), & 0 < x < 1,\ 0 < t < T, \\ u(x,0) = 0, & 0 < x < 1, \\ \partial_x u(0,t) = \partial_x u(1,t) = 0, & 0 < t < T \end{cases}$$

by the boundary observation $u(0,t)$. The uniqueness can be easily shown by using Lemmata 1 and 2 and Laplace transform. The key is the t-analyticity of the solution $u(x,t)$, which is guaranteed by $\rho \equiv 1$, and their argument does not work for general $\rho \in C[0,T]$.

In higher spatial dimensions, Wei, Sun, and Li [49] studied an inverse x-source problem by extra boundary data for $0 < t < \infty$ and established the uniqueness in the inverse problem. In the case where we consider the initial-boundary value problem and data over the infinite time interval $0 < t < \infty$, similarly to [61] we can take the Laplace transforms, so that the uniqueness follows.

5 Related inverse source problems

In inverse problems, we are required to determine various quantities such as source terms, coefficients, and parameters describing the fractional derivatives and so there are naturally various types of inverse problems. Thus it is not reasonable to rigorously classify the inverse problems for fractional partial differential equations and in this section, we supplementarily survey inverse problems of determining boundary quantities, some of which is the determination of source term located on the boundary $\partial\Omega$.

5.1 Inverse problem of determining $\rho(t)$ in the boundary source

Let $\gamma \subset \partial\Omega$ be a relatively open subboundary. We consider

$$\begin{cases} \partial_t^\alpha u(x,t) = \triangle u(x,t), & x \in \Omega,\ 0 < t < T, \\ u(x,t) = \begin{cases} \rho(t)g(x), \\ 0, \end{cases} & \begin{matrix} x \in \gamma,\ 0 < t < T, \\ x \in \partial\Omega \setminus \overline{\gamma},\ 0 < t < T, \end{matrix} \\ u(x,0) = 0, & x \in \Omega. \end{cases}$$

Then, given g, determine $\rho(t)$, $0 < t < T$ by distributed measurement data

$$\int_\Omega u(x,t)\sigma(x)\,\mathrm{d}x, \quad 0 < t < T.$$

Here, $\sigma \in C_0^\infty(\Omega)$, ≥ 0, $\not\equiv 0$ in Ω, is an arbitrarily chosen function, which describes a weight. As an extremal case, setting $\sigma(x) = \delta(x - x_0)$ with fixed $x_0 \in \Omega$, we can reduce our data to the pointwise $u(x_0,t)$, $0 < t < T$. However, such pointwise data are not studied.

Liu, Yamamoto, and Yan [25] proves

Theorem 10. *Let $\frac{1}{2} < \alpha < 1$ and $g \in C_0^\infty(\gamma)$ satisfy $g \geq 0$, $\not\equiv 0$ on γ. Then there exists a constant $C > 0$ such that*

$$C^{-1}\left\|\int_\Omega u(x,\cdot)\sigma(x)\,\mathrm{d}x\right\|_{H^\alpha(0,T)} \le \|\rho\|_{H^\alpha(0,T)} \le C\left\|\int_\Omega u(x,\cdot)\sigma(x)\,\mathrm{d}x\right\|_{H^\alpha(0,T)}$$

for all $\rho \in H^\alpha(0,T)$ satisfying $\rho(0)=0$.

Here, $H^\alpha(0,T)$, $0<\alpha<1$, is the fractional Sobolev space defined by Sobolev–Slobodecki norm (e. g., Adams [1]). By the Sobolev embedding, we know that $H^\alpha(0,T)\subset C[0,T]$ for $\alpha>\frac{1}{2}$, and so $\rho(0)=0$ makes sense for $\rho\in H^\alpha(0,T)$ with $\alpha>\frac{1}{2}$. The paper [25] considers a more general elliptic operator instead of $-\triangle$, and gives numerical methods for reconstructing $\rho(t)$ by noisy data for $\int_\Omega u(x,t)\sigma(x)\,\mathrm{d}x$ for $0<t<T$.

5.2 Determination of *t*-coefficient in the Robin boundary condition

For

$$\begin{cases} \partial_t^\alpha u(x,t) = \partial_x^2 u(x,t), & 0<x<1,\ 0<t<T,\\ -\partial_x u(0,t)+\rho(t)u(0,t) = h_0(t), & 0<t<T,\\ \partial_x u(1,t)+\rho(t)u(1,t) = h_1(t), & 0<t<T,\\ u(x,0)=a(x), & 0<x<1, \end{cases}$$

the papers Wei and Wang [51] and Wei and Zhang [53] discuss an inverse problem of determining $\rho(t)$, $0<t<T$ by data $u(0,t)$, $0<t<T$, and study numerical methods by reducing the inverse problem to a Volterra equation in ρ.

6 Concluding remarks

Taking the simple formulation (1) as a model problem, in this chapter we mainly reviewed the theoretical results on the determination of temporal and spatial components in source terms by several kinds of observation data. It reveals that the fractional derivative in time results in essential difficulties in treating these problems more than classical partial differential equations, so that the available arguments are limited, for example, representation formulae of the solution to the initial-boundary value problem (1) by the Mittag-Leffler functions. Thus we should recognize wider varieties of works on inverse problems for fractional partial differential equations.

In this chapter, our survey concentrates on theoretical works, but by natural necessity various works on numerical methods for related inverse problems have been continuously published. Numerical researches for inverse problems for fractional differential equations are tremendously expanding and here we are restricted to making a partial list of related works: Chi, Li, and Jia [5], Jin and Rundell [15], Murio and Mejía

[30], Tian, Li, Deng and Wu [39], Wang and Wei [44], Wang, Yamamoto and Han [46], Wei, Chen, Sun and Li [47], Wei and Wang [50], Wei and Zhang [52], Yang, Fu and Li [58], Zhang, Li, Jia and Li [60], Zhang and Wei [62].

Finally, we mention several prospects on future topics. Compared with problems with order $\alpha \in (0,1)$, less has been done for the cases of $\alpha \in (1,2)$ which may also have practical significance. On the other hand, for realistic applications, other kinds of sources should also be taken into account, for example, the multiple point source $\sum_{i=1}^{N} \rho_i(t)\delta(x - x_i)$ and the moving source $g(x - \xi(t))$. Numerically, it is preferable to develop advanced regularization methods capturing the fractional essence instead of direct optimization techniques.

Bibliography

[1] R. A. Adams, *Sobolev Spaces*, Academic Press, New York, 1975.

[2] T. Aleroev, M. Kirane, and S. Malik, Determination of a source term for a time fractional diffusion equation with an integral type over-determining condition, *Electron. J. Differ. Equ.*, **270** (2013), 1–16.

[3] M. Bellassoued and M. Yamamoto, *Carleman Estimates and Applications to Inverse Problems for Hyperbolic Systems*, Springer Japan, Tokyo, 2017.

[4] J. Cheng, C.-L. Lin, and G. Nakamura, Unique continuation property for the anomalous diffusion and its application, *J. Differ. Equ.*, **254** (2013), 3715–3728.

[5] G. Chi, G. Li, and X. Jia, Numerical inversions of a source term in the FADE with a Dirichlet boundary condition using final observations, *Comput. Math. Appl.*, **62** (2011), 1619–1626.

[6] M. Choulli and M. Yamamoto, Generic well-posedness of an inverse parabolic problem–the Hölder-space approach, *Inverse Probl.*, **12** (1996), 195–205.

[7] M. Choulli and M. Yamamoto, An inverse parabolic problem with non-zero initial condition, *Inverse Probl.*, **13** (1997), 19–27.

[8] M. Choulli and M. Yamamoto, Generic well-posedness of a linear inverse parabolic problem with diffusion parameters, *J. Inverse Ill-Posed Probl.*, **7** (1999), 241–254.

[9] S. D. Eidelman and A. N. Kochubei, Cauchy problem for fractional diffusion equations, *J. Differ. Equ.*, **199** (2004), 211–255.

[10] K. Fujishiro and Y. Kian, Determination of time dependent factors of coefficients in fractional diffusion equations, *Math. Control Relat. Fields*, **6** (2016), 251–269.

[11] R. Gorenflo, Yu. Luchko, and M. Yamamoto, Time-fractional diffusion equation in the fractional Sobolev spaces, *Fract. Calc. Appl. Anal.*, **18** (2015), 799–820.

[12] D. Henry, *Geometric Theory of Semilinear Parabolic Equations*, Springer-Verlag, Berlin, 1981.

[13] V. Isakov, Inverse parabolic problems with the final overdetermination, *Commun. Pure Appl. Math.*, **44** (1991), 185–209.

[14] D. Jiang, Z. Li, Y. Liu, and M. Yamamoto, Weak unique continuation property and a related inverse source problem for time-fractional diffusion-advection equations, *Inverse Probl.*, **33** (2017), 055013.

[15] B. Jin and W. Rundell, A tutorial on inverse problems for anomalous diffusion processes, *Inverse Probl.*, **31** (2015), 035003.

[16] T. Kato, *Perturbation Theory for Linear Operators*, Springer-Verlag, Berlin, 1976.

[17] A. Kawamoto, Hölder stability estimate in an inverse source problem for a first and half-order time fractional diffusion equation, *Inverse Probl. Imaging*, **12** (2018), 315–330.

[18] M. Kirane and S. A. Malik, Determination of an unknown source term and the temperature distribution for the linear heat equation involving fractional derivative in time, *Appl. Math. Comput.*, **218** (2011), 163–170.

[19] M. Kirane, S. Malik, and M. Al-Gwaiz, An inverse source problem for a two dimensional time fractional diffusion equation with nonlocal boundary conditions, *Math. Methods Appl. Sci.*, **36** (2013), 1056–1069.

[20] M. V. Klibanov and A. A. Timonov, *Carleman Estimates for Coefficient Inverse Problems and Numerical Applications*, VSP, Utrech, 2004.

[21] A. Kubica and M. Yamamoto, Initial-boundary value problems for fractional diffusion equations with time-dependent coefficients, *Fract. Calc. Appl. Anal.*, **21** (2018), 276–311.

[22] Z. Li, Y. Liu, and M. Yamamoto, Initial-boundary value problems for multi-term time-fractional diffusion equations with positive constant coefficients, *Appl. Math. Comput.*, **257** (2015), 381–397.

[23] C.-L. Lin and G. Nakamura, Unique continuation property for anomalous slow diffusion equation, *Commun. Partial Differ. Equ.*, **41** (2016), 749–758.

[24] C.-L. Lin and G. Nakamura, Unique continuation property for multi-terms time fractional diffusion equations, *Math. Ann.* (2 July 2018), https://doi.org/10.1007/s00208-018-1710-z.

[25] J. J. Liu, M. Yamamoto, and L. L. Yan, On the reconstruction of unknown time-dependent boundary sources for time fractional diffusion process by distributing measurement, *Inverse Probl.*, **32** (2016), 015009.

[26] Y. Liu, Strong maximum principle for multi-term time-fractional diffusion equations and its application to an inverse source problem, *Comput. Math. Appl.*, **73** (2017), 96–108.

[27] Y. Liu, W. Rundell, and M. Yamamoto, Strong maximum principle for fractional diffusion equations and an application to an inverse source problem, *Fract. Calc. Appl. Anal.*, **19** (2016), 888–906.

[28] Y. Liu and Z. Zhang, Reconstruction of the temporal component in the source term of a (time-fractional) diffusion equation, *J. Phys. A*, **50** (2017), 305203.

[29] Yu. Luchko, Maximum principle for the generalized time-fractional diffusion equation, *J. Math. Anal. Appl.*, **351** (2009), 218–223.

[30] D. A. Murio and C. E. Majía, Source terms identification for time fractional diffusion equation, *Rev. Colomb. Mat.*, **42** (2008), 25–46.

[31] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, 1999.

[32] A. I. Prilepko, D. G. Orlovsky, and I. A. Vasin, *Methods for Solving Inverse Problems in Mathematical Physics*, Marcel Dekkers, New York, 2000.

[33] Z. Ruan and Z. Wang, Identification of a time-dependent source term for a time fractional diffusion problem, *Appl. Anal.*, **96** (2017), 1638–1655.

[34] S. Saitoh, V. K. Tuan, and M. Yamamoto, Reverse convolution inequalities and applications to inverse heat source problems, *J. Inequal. Pure Appl. Math.*, **3** (2002), 80.

[35] K. Sakamoto and M. Yamamoto, Initial value/boundary value problems for fractional diffusion-wave equations and applications to some inverse problems, *J. Math. Anal. Appl.*, **382** (2011), 426–447.

[36] K. Sakamoto and M. Yamamoto, Inverse source problem with a final overdetermination for a fractional diffusion equation, *Math. Control Relat. Fields*, **1** (2011), 509–518.

[37] S. Tatar, R. Tinaztepe, and S. Ulusoy, Determination of an unknown source term in a space-time fractional diffusion equation, *J. Fract. Calc. Appl.*, **6** (2015), 83–90.

[38] S. Tatar and S. Ulusoy, An inverse source problem for a one-dimensional space-time fractional diffusion equation, *Appl. Anal.*, **94** (2015), 2233–2244.

[39] W. Tian, C. Li, W. Deng, and Y. Wu, Regularization methods for unknown source in space fractional diffusion equation, *Math. Comput. Simul.*, **85** (2012), 45–56.

[40] I. V. Tikhonov and Yu. S. Eidelman, Problems on correctness of ordinary and inverse problems for evolutionary equations of a special form, *Math. Notes*, **56** (1994), 830–839.
[41] E. C. Titchmarsh, The zeros of certain integral functions, *Proc. Lond. Math. Soc.*, **2** (1926), 283–302.
[42] S. R. Umarov and E. M. Saidamatov, A generalization of Duhamel's principle for differential equations of fractional order, *Dokl. Math.*, **75** (2007), 94–96.
[43] H. Wang and B. Wu, On the well-posedness of determination of two coefficients in a fractional integrodifferential equation, *Chin. Ann. Math.*, **35B** (2014), 447–468.
[44] J.-G. Wang and T. Wei, Quasi-reversibility method to identify a space-dependent source for the time-fractional diffusion equation, *Appl. Math. Model.*, **39** (2015), 6139–6149.
[45] J.-G. Wang, Y.-B. Zhou and T. Wei, Two regularization methods to identify a space-dependent source for the time-fractional diffusion equation, *Appl. Numer. Math.*, **68** (2013), 39–57.
[46] W. Wang, M. Yamamoto, and B. Han, Numerical method in reproducing kernel space for an inverse source problem for the fractional diffusion equation, *Inverse Probl.*, **29** (2013), 095009.
[47] H. Wei, W. Chen, H. Sun, and X. Li, A coupled method for inverse source problem of spatial fractional anomalous diffusion equations, *Inverse Probl. Sci. Eng.*, **18** (2010), 945–956.
[48] T. Wei, X. L. Li, and Y. S. Li, An inverse time-dependent source problem for a time-fractional diffusion equation, *Inverse Probl.*, **32** (2016), 085003.
[49] T. Wei, L. Sun, and Y. Li, Uniqueness for an inverse space-dependent source term in a multi-dimensional time-fractional diffusion equation, *Appl. Math. Lett.*, **61** (2016), 108–113.
[50] T. Wei and J. Wang, A modified quasi-boundary value method for an inverse source problem of the time-fractional diffusion equation, *Appl. Numer. Math.*, **78** (2014), 95–111.
[51] T. Wei and J. Wang, Determination of Robin coefficient in a fractional diffusion problem, *Appl. Math. Model.*, **40** (2016), 7948–7961.
[52] T. Wei and Z. Q. Zhang, Reconstruction of a time-dependent source term in a time-fractional diffusion equation, *Eng. Anal. Bound. Elem.*, **37** (2013), 23–31.
[53] T. Wei and Z. Q. Zhang, Robin coefficient identification for a time-fractional diffusion equation, *Inverse Probl. Sci. Eng.*, **24** (2016), 647–666.
[54] B. Wu and S. Wu, Existence and uniqueness of an inverse source problem for a fractional integrodifferential equation, *Comput. Math. Appl.*, **68** (2014), 1123–1136.
[55] X. Xu, J. Cheng, and M. Yamamoto, Carleman estimate for a fractional diffusion equation with half order and application, *Appl. Anal.*, **90** (2011), 1355–1371.
[56] M. Yamamoto, Carleman estimates for parabolic equations and applications, *Inverse Probl.*, **25** (2009), 123013.
[57] M. Yamamoto and Y. Zhang, Conditional stability in determining a zeroth-order coefficient in a half-order fractional diffusion equation by a Carleman estimate, *Inverse Probl.*, **28** (2012), 105010.
[58] F. Yang, C. Fu, and X. Li, The inverse source problem for time-fractional diffusion equation: stability analysis and regularization, *Inverse Probl. Sci. Eng.*, **23** (2015), 969–996.
[59] R. Zacher, Weak solutions of abstract evolutionary integro-differential equations in Hilbert spaces, *Funkc. Ekvacioj*, **52** (2009), 1–18.
[60] D. Zhang, G. Li, X. Jia, and H. Li, Simultaneous inversion for space-dependent diffusion coefficient and source magnitude in the time fractional diffusion equation, *J. Math. Res.*, **5** (2013), 65–78.
[61] Y. Zhang and X. Xu, Inverse source problem for a fractional diffusion equation, *Inverse Probl.*, **27** (2011), 035010.
[62] Z. Q. Zhang and T. Wei, Identifying an unknown source in time-fractional diffusion equation by a truncation method, *Appl. Math. Comput.*, **219** (2013), 5972–5983.

Zhiyuan Li, Yikan Liu, and Masahiro Yamamoto

Inverse problems of determining parameters of the fractional partial differential equations

Abstract: When considering fractional diffusion equation as model equation in analyzing anomalous diffusion processes, some important parameters in the model related to orders of the fractional derivatives, are often unknown and difficult to be directly measured, which requires one to discuss inverse problems of identifying these physical quantities from some indirectly observed information of solutions. Inverse problems in determining these unknown parameters of the model are not only theoretically interesting, but also necessary for finding solutions to initial-boundary value problems and studying properties of solutions. This chapter surveys works on such inverse problems for fractional diffusion equations.

Keywords: Fractional diffusion equations, parameter inversion, uniqueness

MSC 2010: 35R11, 26A33, 35R30, 65M32

1 Introduction

The fractional equations have been playing important roles in various fields such as physics, chemistry, astrophysics during the last few decades. In particular, in heterogenous media, the diffusion often indicates anomalous profiles which cannot be simulated by the classical diffusion equation. Thus several model equations have been introduced, and a time-fractional diffusion equation is one of them. For a flexible type of time-fractional diffusion equation, we consider a fractional derivative of distributed order defined by

$$\mathbb{D}_t^{(\mu)} v(t) := \int_0^1 \mu(\alpha)\partial_t^\alpha v(t)\, d\alpha,$$

where ∂_t^α is the Caputo derivative given by

$$\partial_t^\alpha v(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \frac{dv}{ds}(s) ds.$$

Zhiyuan Li, School of Mathematics and Statistics, Shandong University of Technology, Zibo, Shandong 255049, China, e-mail: zyli@sdut.edu.cn
Yikan Liu, Masahiro Yamamoto, Graduate School of Mathematical Sciences, The University of Tokyo, 3-8-1 Komaba, Meguro-ku, Tokyo 153-8914, Japan, e-mails: ykliu@ms.u-tokyo.ac.jp, myama@ms.u-tokyo.ac.jp

https://doi.org/10.1515/9783110571660-019

Throughout this chapter, let Ω be an open bounded domain in $\mathbb{R}^d$ with smooth boundary $\partial\Omega$ of C^2-class, and $T > 0$ be arbitrarily fixed. We understand that $d \ge 1$, that is, we discuss general spatial dimensions, if we do not specify.

Then an equation with time-distributed order derivative is described by

$$\mathbb{D}_t^{(\mu)} u(x,t) = \Delta u(x,t), \quad x \in \Omega,\ 0 < t < T. \tag{1}$$

Here, $u(x,t)$ denotes the density of a substance such as contaminants at the location x and the time t.

In order to make (1) a model which can interpret observation data better, and well realize asymptotic behavior of the density $u(x,t)$, we have to choose a weight function $\mu(\alpha)$, $0 < \alpha < 1$. This determination problem is inverse problems, and we are requested to determine a function $\mu(\alpha)$ or related parameters, and this is the subject of the current chapter.

We mainly discuss an initial-boundary value problem:

$$\begin{cases} \mathbb{D}_t^{(\mu)} u(x,t) = \Delta u(x,t), & x \in \Omega,\ 0 < t < T, \\ u(x,0) = a(x), & x \in \Omega, \\ u(x,t) = g(x,t), & x \in \partial\Omega,\ 0 < t < T, \end{cases} \tag{2}$$

where $\mu \in C[0,1]$, ≥ 0, $\not\equiv 0$.

As for references on the forward problems (2), we are restricted to Kochubei [11, 12], Li, Luchko, and Yamamoto [18, 19], Luchko [22], Meerschaert, Nane, and Vellaisamy [25] and the references therein.

Henceforth, for simplicity, we consider only the Laplacian Δ in the fractional diffusion equation and a Dirichlet boundary condition.

Moreover, we consider also the following two special cases of (2).

1: Single time-fractional diffusion equation
We formally choose $\mu = \delta(\cdot - \alpha)$ where $0 < \alpha < 1$ is fixed and $\delta(\cdot - \alpha)$ is the Dirac delta function at α.

Then the distributed order fractional diffusion equation is reduced to a single-term fractional diffusion equation:

$$\begin{cases} \partial_t^{\alpha} u(x,t) = \Delta u(x,t), & x \in \Omega,\ 0 < t < T, \\ u(x,0) = a(x), & x \in \Omega, \\ u(x,t) = g(x,t), & x \in \partial\Omega,\ 0 < t < T. \end{cases} \tag{3}$$

The diffusion equation with a single-term time fractional derivative has attracted great attention in different areas, and here we refer to a restricted number of works on the direct problem such as the well-posedness of (3) and other qualitative properties, which are necessary for discussions of the inverse problems: Gorenflo, Luchko, and Yamamoto [5], Gorenflo, Luchko, and Zabrejko [6], Luchko [21], Sakamoto and Yamamoto [28], Zacher [34].

A natural extension for the single-term time-fractional diffusion equation is a multi-term time-fractional diffusion equation:

2: Multiterm time-fractional diffusion equation
We set $\mu = \sum_{j=1}^{\ell} p_j \delta(\cdot - \alpha_j)$ where $0 < \alpha_\ell < \cdots < \alpha_1 < 1$, $p_1, \ldots, p_\ell$ are positive constants:

$$\begin{cases} \sum_{j=1}^{\ell} p_j \partial_t^{\alpha_j} u(x,t) = \Delta u(x,t), & x \in \Omega,\ 0 < t < T, \\ u(x,0) = a(x), & x \in \Omega, \\ u(x,t) = g(x,t), & x \in \partial\Omega,\ 0 < t < T. \end{cases} \tag{4}$$

As minimum references on the well-posedness for (4), we list Li, Liu, and Yamamoto [17] and Li, Huang, and Yamamoto [15].

For nonstationary partial differential equations such as classical diffusion and wave equations, the inverse problems have been widely studied. More precisely assuming, for example, that $p(x)$ is unknown in the diffusion equation $\partial_t u(x,t) = \operatorname{div}(p(x)\nabla u(x,t))$, we are requested to determine $p(x)$, $x \in \Omega$ by some limited extra data on the boundary (e. g., Beilina and Klibanov [1], Bellassoued and Yamamoto [2], Isakov [8], Klibanov and Timonov [10]). See the chapter on "Inverse problems of determining coefficients of the fractional partial differential equations" of this handbook as for such inverse coefficient problems.

In this chapter, not for classical partial differential equations, we survey other important types of inverse problems of determining $\mu(\alpha)$ or related parameters which are considered to govern the anomaly of diffusion and describe an index of the inhomogeneity in heterogeneous media. It is not until such parameters are reasonably determined that we can start discussions on the fundamental issues like the existence of solutions to the initial-boundary value problem (2). In practice, one can usually identify or estimate such anomaly indices empirically. On the other hand, mathematical discussions for related inverse problems should be not only interesting but also helpful for determination of parameters by experiments. One of such inverse problems is the determination of the unknown parameters in order to match available data such as $u(x_0, t)$, $0 < t < T$ at a monitoring point $x_0 \in \Omega$.

We note that our survey is far from the perfect, because researches on inverse problems for fractional partial differential equations are very rapidly developed.

2 Preliminaries

In this section, we will set up notations and terminologies, review some of standard facts on the fractional calculus, and introduce several important results related to the forward problem for the time-fractional diffusion equations (1), which are the starting points for further researches concerning the theory of inverse problems.

We recall the Mittag-Leffler function $E_{\alpha,\beta}(z)$ defined by

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha+\beta)}, \quad z \in \mathbb{C},\ \alpha,\beta > 0$$

and the asymptotic behavior

Lemma 1. (i) *Let $\alpha \in (0,2)$, $\beta > 0$, and $\mu \in (\frac{\alpha\pi}{2}, \min\{\alpha\pi,\pi\})$. Then for $N \in \mathbb{N}$*

$$E_{\alpha,\beta}(z) = -\sum_{k=1}^{N} \frac{z^{-k}}{\Gamma(\beta-\alpha k)} + O\left(\frac{1}{|z|^{N+1}}\right) \quad \textit{with } |z| \to \infty,\ \mu \le |\arg z| \le \pi.$$

Moreover, there exists a positive constant $C = C(\mu,\alpha,\beta)$ such that

$$|E_{\alpha,\beta}(z)| \le \frac{C}{1+|z|} \quad \textit{with } \mu \le |\arg z| \le \pi.$$

See Podlubny [26] for example.

Henceforth, $L^2(\Omega)$, $H_0^1(\Omega)$, $H^2(\Omega)$ are the usual L^2-space and the Sobolev spaces of real-valued functions: $L^2(\Omega) = \{f;\ \int_\Omega |f(x)|^2 dx < \infty\}$, and let $(\cdot,\cdot)$ denote the scalar product: $(f,g) = \int_\Omega f(x)g(x)dx$ for $f,g \in L^2(\Omega)$. Moreover, we define the Laplacian $-\Delta$ with the domain $\mathcal{D}(-\Delta) = H^2(\Omega) \cap H_0^1(\Omega)$ and an eigensystem $\{(\lambda_n,\varphi_n)\}_{n=1}^{\infty}$ of $-\Delta$ is defined by $\{\varphi_n\}_{n\in\mathbb{N}} \subset \mathcal{D}(-\Delta)$ and $0 < \lambda_1 < \lambda_2 \le \ldots$ such that $-\Delta\varphi_n = \lambda_n\varphi_n$ and $\{\varphi_n\}_{n\in\mathbb{N}}$ is a complete orthonormal system of $L^2(\Omega)$.

For $\gamma \ge 0$, the fractional Laplacian $(-\Delta)^\gamma$ is defined as

$$\mathcal{D}((-\Delta)^\gamma) := \left\{ f \in L^2(\Omega);\ \sum_{n=1}^{\infty} \left|\lambda_n^\gamma (f,\varphi_n)\right|^2 < \infty \right\},$$

$$(-\Delta)^\gamma f := \sum_{n=1}^{\infty} \lambda_n^\gamma (f,\varphi_n)\varphi_n.$$

3 Single time-fractional diffusion equation

In this section, we are concerned with inversion for order in the fractional diffusion equation (3) with homogeneous Dirichlet boundary condition. Let us start with some observations mainly about the asymptotic behavior of the solution as $t \to 0$ and $t \to \infty$. By the eigenfunction expansion of $u(x,t)$ in terms of the Mittag-Leffler function $E_{\alpha,1}$, it was shown in [28] that the decay rate of the solution u to (3) is dominated by $t^{-\alpha}$ as $t \to \infty$, while $\|u(\cdot,t)\|_{H^2(\Omega)}$ is dominated by $t^{-\alpha}\|a\|_{L^2(\Omega)}$ as $t \to 0$. It turns out that the short (or long) asymptotic behavior heavily relies on the fractional orders of the derivatives. On the basis of the asymptotics of the solution as $t \to 0$ or $t \to \infty$, it is expected to show formulae of reconstructing the order of fractional derivative in time in the fractional diffusion equation by time history at one fixed spatial point.

Hatano, Nakagawa, Wang, and Yamamoto [7] proved

Theorem 1 (Hatano et al. [7]). (i) *Assume that*

$$a \in C_0^\infty(\Omega), \quad \Delta a(x_0) \neq 0.$$

Then

$$\alpha = \lim_{t\to 0} \frac{t\partial_t u(x_0,t)}{u(x_0,t) - a(x_0)}.$$

(ii) *Assume that*

$$a \in C_0^\infty(\Omega), \not\equiv 0, \quad a \geq 0 \text{ or } \leq 0, \text{ on } \overline{\Omega}.$$

Then

$$\alpha = -\lim_{t\to\infty} \frac{t\partial_t u(x_0,t)}{u(x_0,t)}.$$

In addition, let us mention the work from Meerschaert, Benson, Scheffler, and Baeumer [24] in which the space-time fractional diffusion equation

$$\partial_t^\alpha u = -(-\Delta)^{\frac{\gamma}{2}} u,$$

with $0 < \alpha < 1$ and $0 < \gamma < 2$ was considered. As is known, the fractional derivative in time here is usually used to describe particle trapping phenomena, while the fractional space derivative is used to model long particle jumps. These two effects combined together produce a concentration profile with a sharper peak, and heavier tails in the anomalous diffusion. We refer to Tatar and Ulusoy [31] for the uniqueness for an inverse problem of simultaneously determining the exponents of the fractional time and space derivatives by the interior point observation $u(x_0,\cdot)$ in $(0,T)$.

Theorem 2 (Tatar and Ulusoy [31]). *Let $x_0 \in (0,1)$ be any fixed point, and let $u[\alpha,\gamma]$ be the weak solution to*

$$\begin{cases} \partial_t^\alpha u = -(-\Delta)^{\frac{\gamma}{2}} u, & 0 < x < 1,\ 0 < t < T, \\ u(x,0) = a(x), & 0 < x < 1, \\ u(x,t) = 0, & x = 0,1,\ 0 < t < T. \end{cases} \tag{5}$$

Let $u[\beta,\eta]$ be the weak solution to (5) *where α,γ are replaced by $\beta \in (0,1)$ and $\eta \in (0,2)$. If $u[\alpha,\gamma](x_0,t) = u[\beta,\eta](x_0,t)$ for all $t \in (0,T)$ and $a \in L^2(0,1)$ satisfies*

$$(a,\varphi_n) > 0, \quad \text{for all } n \geq 1,$$

then $\alpha = \beta$ and $\gamma = \eta$.

In the proof of the above theorem, similar to Hatano et al. [7], the time-fractional order can be firstly determined by using a long-time asymptotic behavior of the solution to (5), which is a direct result from the property of Mittag-Leffler function in Lemma 1. The asymptotics of the eigenvalues λ_n of the one-dimensional Laplacian with the zero Dirichlet boundary condition is then used to show the uniqueness inversion for the spatial order. See also Tatar, Tinaztepe, and Ulusoy [30] as for a numerical reconstruction scheme for α and γ.

4 Multi-term time-fractional diffusion equation

In this section, we survey on multi-term time-fractional diffusion equations. In (4), we assume the boundary condition $g = 0$ and we investigate inverse problems of identifying fractional orders α_j and coefficients p_j.

Inverse Problem. Let $x_0 \in \Omega$ be fixed and let $I \subset (0,T)$ be a nonempty open interval. Determine the number ℓ of fractional orders α_j, fractional orders $\{\alpha_j\}_{j=1}^{\ell}$ and positive constant coefficients $\{p_j\}_{j=1}^{\ell}$ of the fractional derivatives by interior measurement $u(x_0,t)$, $t \in I$.

Remark 1. We should mention that the number ℓ of the fractional derivatives is also unknown in the above inverse problem.

For the statement of our main results, we introduce some notation. As an admissible set of unknown parameters, we set $\mathbb{R}_+ = \{p > 0\}$ and

$$\begin{aligned}\mathcal{U}_{\mathrm{noc}} = \{(\ell,\boldsymbol{\alpha},\boldsymbol{p});\ &\ell \in \mathbb{N},\ \boldsymbol{\alpha} = (\alpha_1,\dots,\alpha_\ell) \in (0,1)^\ell, \\ &0 < \alpha_\ell < \cdots < \alpha_1 < 1,\ \boldsymbol{p} = (p_1,\dots,p_\ell) \in \mathbb{R}_+^\ell\}.\end{aligned} \tag{6}$$

By means of the eigenfunction expansion of u, the unique determination of fractional orders is proved by using the t-analyticity of the solution and the strong maximum principle for elliptic equations.

Theorem 3 (Li and Yamamoto [20]). *Assuming that*

$$a \ge 0 \text{ in } \Omega,\ a \not\equiv 0 \text{ and } a \in H^{2\gamma}(\Omega) \cap H_0^1(\Omega) \text{ with some } \gamma > \max\left\{\frac{d}{2} - 1, 0\right\}.$$

Let $u[\ell,\boldsymbol{\alpha},\boldsymbol{p}]$, $u[m,\boldsymbol{\beta},\boldsymbol{q}]$ be the weak solutions to (4) with respect to $(\ell,\boldsymbol{\alpha},\boldsymbol{p}),(m,\boldsymbol{\beta},\boldsymbol{q}) \in \mathcal{U}_{\mathrm{noc}}$. Then for any fixed $x_0 \in \Omega$,

$$u[\ell,\boldsymbol{\alpha},\boldsymbol{p}](x_0,t) = u[m,\boldsymbol{\beta},\boldsymbol{q}](x_0,t), \quad t \in I$$

implies

$$\ell = m, \quad \boldsymbol{\alpha} = \boldsymbol{\beta}, \quad \boldsymbol{p} = \boldsymbol{q}.$$

We describe a numerical method for the reconstruction of orders. For simplicity, we fix the number ℓ of the fractional derivatives, and denote by $u[\boldsymbol{\alpha}, \boldsymbol{p}]$ the unique solution to (4) with $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_\ell)$ and $\boldsymbol{p} = (p_1, \dots, p_\ell)$. For the numerical treatment, in Li, Zhang, Jia, and Yamamoto [13], Sun, Li, and Jia [29], the authors reformulated the inverse problem into an optimization problem

$$\min_{(\boldsymbol{\alpha}, \boldsymbol{p}) \in \mathcal{U}_{\mathrm{oc}}} \| u[\boldsymbol{\alpha}, \boldsymbol{p}](x_0, \cdot) - h \|_{L^2(0,T)}, \tag{7}$$

where $\mathcal{U}_{\mathrm{oc}}$ is the admissible set (6) with fixed ℓ and $h \in L^2(0,T)$ stands for available observation data $u(x_0, t)$, $0 < t < T$. The feasibility of this optimization is guaranteed by the following result.

Proposition 1. *For any observation data $h \in L^2(0,T)$, there exists a minimizer to the optimization problem* (7).

The existence of a minimizer is easily verified by taking a minimizing sequence and a routine compactness argument. The key is the Lipschitz continuity of the solution with respect to $(\boldsymbol{\alpha}, \boldsymbol{p})$, namely,

$$\big\| (u[\boldsymbol{\alpha}, \boldsymbol{p}] - u[\boldsymbol{\beta}, \boldsymbol{q}])(x_0, \cdot) \big\|_{L^2(0,T)} \le C \sum_{j=1}^{\ell} (|\alpha_j - \beta_j| + |p_j - q_j|),$$

where $(\boldsymbol{\alpha}, \boldsymbol{p}), (\boldsymbol{\beta}, \boldsymbol{q}) \in \mathcal{U}_{\mathrm{oc}}$ satisfy certain additional assumptions.

Needless to say, this methodology also works for the single-term case. However, for $\ell \ge 3$ it becomes extremely difficult to reconstruct smaller orders α_j in $\boldsymbol{\alpha}$ because they contribute less to the solution. In order to restore the numerical stability, one can attach some regularization terms to (7), but here we omit the details.

5 Distributed order fractional diffusion equation

We focus on the determination of the weight function $\mu(\alpha)$ in (2), which is important for experimentally evaluating the characteristics of the diffusion process in heterogeneous medium.

Inverse Problem. Let $x_0 \in \Omega$ be fixed and $I \subset (0,T)$ be a nonempty open interval. Let u be the solution to the initial-boundary value problem (2). We will investigate whether u in $\{x_0\} \times I$ can determine μ in $(0,1)$.

In this section, we discuss the uniqueness as the fundamental theoretical topic for the inverse problem and attempt to establish results parallel to that for the multi-term case (4). We separately discuss two cases: the case of the homogeneous boundary condition and the case of nonhomogeneous boundary condition, and it is technically difficult to consider both simultaneously.

5.1 Case of the homogeneous boundary condition

In this subsection, we assume that $g = 0$ in (2), that is, the boundary value vanishes on $\partial\Omega \times (0, T)$. We introduce an admissible set of unknown weight function

$$\mathcal{U}_0 = \{\mu \in C[0,1];\ \mu \geq 0, \not\equiv 0\}.$$

From [19], the solution to the initial-boundary value problem with the homogeneous boundary condition is t-analytic. Therefore, after taking the Laplace transform of the solution u in t, we can reformulate the fractional diffusion equation into an elliptic equation with parameters in the Laplace frequency domain. The uniqueness of the inverse problem can be easily derived by noting the strong maximum principle for elliptic equations. We have the following.

Theorem 4 (Li, Luchko and Yamamoto [19]). *We assume that* $g = 0$ *in* (2). *Let* $\mu, \omega \in \mathcal{U}_0$ *and Let* $u[\mu], u[\omega]$ *be the solutions to the initial-boundary value problem* (2) *with respect to* $\mu, \omega \in \mathcal{U}_0$ *separately. Assume that* $a \geq 0$ *in* Ω, $a \not\equiv 0$ *and* $a \in H^{2\gamma}(\Omega) \cap H_0^1(\Omega)$ *with* $\gamma > \max\{\frac{d}{2} - 1, 0\}$, *Then* $\mu = \omega$ *if*

$$u[\mu](x_0, t) = u[\omega](x_0, t), \quad x_0 \in \Omega,\ t \in I.$$

The proof of the above theorem heavily relies on the t-analyticity of the solution. Let us point out that the proof of the uniqueness for the corresponding inverse problem for the initial-boundary value problem (2) with nonhomogeneous boundary condition is totally different. In the next subsection, we discuss the case of nonhomogeneous boundary condition.

Finally, as a direct application of well-posedness of the initial-boundary value problem discussed in [19], we can show the Lipschitz stability of the solutions with respect to weight functions:

$$\|(u[\mu] - u[\omega])(x_0, \cdot)\|_{L^2(0,T)} \leq C\|\mu - \omega\|_{L^\infty(0,1)},$$

where $\mu, \omega \in \mathcal{U}_0$ satisfy certain additional assumptions. We can formulate the minimization problem to prove the existence of a minimizer similarly to Section 4, but we omit the details.

5.2 Case of nonhomogeneous boundary condition

In this section, we continue the discussion of the inverse problem of determining the weight function in (2) but with nonhomogeneous boundary condition.

In the case of $\Omega = (0, 1)$, we first introduce Rundell and Zhang [27] which investigates an inverse problem of determining the weight function in (2) in terms of the value of the solution u in the time interval $(0, \infty)$.

Theorem 5 (Rundell and Zhang [27]). *In* (2), *we assume that* $\Omega = (0,1)$, $a(x) = 0$ *for* $0 < x < 1$ *and* $u(1,t) = 0$, $u(0,\cdot) \in L^\infty(0,\infty), \not\equiv 0$. *Let* $x_0 \in (0,1)$ *be arbitrarily chosen. For arbitrarily fixed* $c_0 > 0$ *and* $0 < \beta_0 < 1$, *let* $\mu, \omega \in C^1[0,1]$ *be positive unknown functions and satisfy*

$$\mu(\alpha), \omega(\alpha) \geq c_0 > 0 \quad \text{on } (\beta_0, 1).$$

Let $u[\mu]$ *and* $u[\omega]$ *be the solutions to the initial-boundary value problem* (2) *with respect to* μ *and* ω, *respectively, and let the Laplace transform of one of* $u[\mu](0,\cdot)$ *and* $u[\omega](0,\cdot)$ *satisfy*

$$\mathcal{L}\{u(0,t); s\} \neq 0, \quad s \in (0,\infty). \tag{8}$$

Then $u[\mu](x_0,t) = u[\omega](x_0,t)$ *for any* $t \in (0,\infty)$ *implies* $\mu = \omega$ *on* $[0,1]$.

Remark 2. It is not convenient to use the above result to recover the unknown weight function practically because we have to assume (8), which means that one of μ and ω is known.

How about using measurement over a finite time length? As mentioned above, for the homogeneous boundary case, analytic continuation of the solution ensures the equivalence between the overposed data $u(x_0,t)$, $t \in (0,T)$ and $u(x_0,t)$, $t \in (0,\infty)$, which is the key idea of the proof of Theorem 3. Our question is whether it is possible to prove the uniqueness of the inverse problem of determining the weight function without using the analytic continuation because we cannot rely on the t-analyticity of the solutions for the nonhomogeneous boundary counterpart.

We have the following lemma.

Lemma 2. *Let the weight function* $\mu \in C[0,1]$ *be nonnegative, and not vanish on* $[0,1]$. *Suppose that the nonnegative function* $u \in C_0^\infty((0,T_0); H^4(\Omega))$ *satisfies*

$$\mathbb{D}_t^{(\mu)} u - \Delta u \leq, \not\equiv 0 \quad \text{in } \Omega \times (0,T),$$

where $0 < T < T_0$. *Suppose also that* Ω *is connected, open and bounded. Then for any* $x \in \Omega$, *there exists* $t_x \in (0,T)$ *such that* $u(x,t_x) > 0$.

Li, Fujishiro, and Li [14] gave an affirmative answer by setting the boundary value g vanishing near the final time T, and restricting the weight function in the following admissible set:

$$\mathcal{U}_{\mathrm{f}} := \{\mu \in C[0,1];\ \mu \geq 0, \not\equiv 0;\ \mu \text{ is finitely oscillatory.}\}$$

Here, we call that a function μ is finitely oscillatory if for any $c \in \mathbb{R}$, the set $\{\alpha; \mu(\alpha) = c\}$ is finite.

Theorem 6 ([14]). *Let* $\Omega \subset \mathbb{R}^d$, $d \leq 3$, *be connected, open, and bounded, and let* $T > 0$, $x_0 \in \Omega$ *be arbitrarily chosen. Let* $g \in C_0^\infty((0,T); H^{\frac{7}{2}}(\partial\Omega))$ *satisfy* $g \geq, \not\equiv 0$ *on* $\partial\Omega \times (0,T)$. *We further suppose that* $u[\mu]$ *and* $u[\omega]$ *are the solutions to the problem* (2) *with respect to* μ, $\omega \in \mathcal{U}_{\mathrm{f}}$. *Then* $\mu = \omega$ *on* $[0,1]$ *if* $u[\mu](x_0,\cdot) = u[\omega](x_0,\cdot)$ *in* $(0,T)$.

6 Conclusions and open problems

In this chapter, we are mainly concerned with the uniqueness results on the determination of a weight function $\mu(\alpha)$ of distributed order derivatives in t and related parameters of the fractional diffusion equation by one interior point observation data. Our key ideas of proofs are based on the properties such as asymptotic behavior and t-analyticity of the solutions of the initial-boundary value problem (2).

The uniqueness in determining orders α_j, $j = 1, \ldots, \ell$ in (3) or (4) follows as by-product from the uniqueness results (e. g., Cheng, Nakagawa, Yamazaki, and Yamamoto [4], Li, Imanuvilov, and Yamamoto [16], Kian, Oksanen, Soccorsi, and Yamamoto [9]) for inverse coefficient problems which are surveyed in the chapter "Inverse problems of determining coefficients of the fractional partial differential equations" of this handbook. Here as a supplement to this chapter, we give a partial list of numerical researches on the reconstruction for fractional orders and related coefficients for fractional differential equations: Chen, Liu, Jiang, Turner, and Burrage [3], Lukashchuk [23], Sun, Li, and Jia [29], Tatar, Tinaztepe, and Ulusoy [30], Yu, Jiang, and Qi [32], Yu, Jiang, and Wang [33], Zhang, Li, Chi, Jia, and Li [35], Zheng and Wei [36] and the references therein.

On the other hand, the studies on inverse problems of the recovery of the fractional orders or the weight function $\mu(\alpha)$ in the model (2), are far from satisfactory since all the publications either assumed the homogeneous boundary condition (Theorems 1, 2, 3, and 4), or studied this inverse problem by the measurement on $t \in (0, \infty)$ (Theorem 5). Although Theorem 6 proved that the weight function μ in the problem (2) with nonhomogeneous boundary condition can be uniquely determined from one interior point overposed data but with an additional assumption that the unknown μ lies in the restrictive admissible set $\mathcal{U}_f$. It would be interesting to investigate inverse problems by the value of the solution at a fixed time as the observation data.

Finally, as far as the authors know, the stability of the inverse problem of determining the fractional orders remains open.

Bibliography

[1] L. Beilina and M. Klibanov, *Approximate Global Convergence and Adaptivity for Coefficient Inverse Problems*, Springer-Verlag, Berlin, 2012.

[2] M. Bellassoued and M. Yamamoto, *Carleman Estimates and Applications to Inverse Problems for Hyperbolic Systems*, Springer Japan, Tokyo, 2017.

[3] S. Chen, F. Liu, X. Jiang, I. Turner, and K. Burrage, Fast finite difference approximation for identifying parameters in a two-dimensional space-fractional nonlocal model with variable diffusivity coefficients, *SIAM J. Numer. Anal.*, **54** (2016), 606–624.

[4] J. Cheng, J. Nakagawa, M. Yamamoto, and T. Yamazaki, Uniqueness in an inverse problem for a one-dimensional fractional diffusion equation, *Inverse Probl.*, **25** (2009), 115002.

[5] R. Gorenflo, Y. Luchko, and M. Yamamoto, Time-fractional diffusion equation in the fractional Sobolev spaces, *Fract. Calc. Appl. Anal.*, **18** (2015), 799–820.
[6] R. Gorenflo, Y. Luchko, and P. P. Zabrejko, On solvability of linear fractional differential equations in Banach spaces, *Fract. Calc. Appl. Anal.*, **2** (1999), 163–177.
[7] Y. Hatano, J. Nakagawa, S. Wang, and M. Yamamoto, Determination of order in fractional diffusion equation, *J. Math-for-Ind.*, **5A** (2013), 51–57.
[8] V. Isakov, *Inverse Problems for Partial Differential Equations*, Springer-Verlag, Berlin, 2006.
[9] Y. Kian, L. Oksanen, E. Soccorsi, and M. Yamamoto, Global uniqueness in an inverse problem for time fractional diffusion equations, *J. Differ. Equ.*, **264** (2018), 1146–1170.
[10] M. Klibanov and A. Timonov, *Carleman Estimates for Coefficient Inverse Problems and Numerical Applications*, VSP, Utrecht, 2004.
[11] A. N. Kochubei, Distributed order calculus and equations of ultraslow diffusion, *J. Math. Anal. Appl.*, **340** (2008), 252–281.
[12] A. N. Kochubei, Distributed order derivatives and relaxation patterns, *J. Phys. A, Math. Theor.*, **42** (2009), 1943–1953.
[13] G. Li, D. Zhang, X. Jia, and M. Yamamoto, Simultaneous inversion for the space-dependent diffusion coefficient and the fractional order in the time-fractional diffusion equation, *Inverse Probl.*, **29** (2013), 065014, 36 pp.
[14] Z. Li, K. Fujishiro, and G. Li, An inverse problem for distributed order time-fractional diffusion equations, arXiv preprint, arXiv:1707.02556v2, 2018.
[15] Z. Li, X. Huang, and M. Yamamoto, Initial-boundary value problems for multi-term time-fractional diffusion equations with x-dependent coefficients, arXiv preprint, arXiv:1802.06269, 2018.
[16] Z. Li, O. Imanuvilov, and M. Yamamoto, Uniqueness in inverse boundary value problems for fractional diffusion equations, *Inverse Probl.*, **32** (2016), 015004.
[17] Z. Li, Y. Liu, and M. Yamamoto, Initial-boundary value problem for multi-term time-fractional diffusion equation with positive constants coefficients, *Appl. Math. Comput.*, **257** (2015), 381–397.
[18] Z. Li, Y. Luchko, and M. Yamamoto, Asymptotic estimates of solutions to initial-boundary-value problems for distributed order time-fractional diffusion equations, *Fract. Calc. Appl. Anal.*, **17** (2014), 1114–1136.
[19] Z. Li, Y. Luchko, and M. Yamamoto, Analyticity of solutions to a distributed order time-fractional diffusion equation and its application to an inverse problem, *Comput. Math. Appl.*, **73** (2017), 1041–1052.
[20] Z. Li and M. Yamamoto, Uniqueness for inverse problems of determining orders of multi-term time-fractional derivatives of diffusion equation, *Appl. Anal.*, **94** (2015), 570–579.
[21] Y. Luchko, Maximum principle for the generalized time-fractional diffusion equation, *J. Math. Anal. Appl.*, **351** (2009), 218–223.
[22] Y. Luchko, Boundary value problems for the generalized time-fractional diffusion equation of distributed order, *Fract. Calc. Appl. Anal.*, **12** (2009), 409–422.
[23] S. Yu. Lukashchuk, Estimation of parameters in fractional subdiffusion equations by the time integral characteristics method, *Comput. Math. Appl.*, **62** (2011), 834–844.
[24] M. M. Meerschaert, D. A. Benson, H. P. Scheffler, and B. Baeumer, Stochastic solution of space-time fractional diffusion equations, *Phys. Rev. E, Stat. Nonlinear Soft Matter Phys.*, **65** (2002), 041103.
[25] M. M. Meerschaert, E. Nane, and P. Vellaisamy, Distributed-order fractional diffusions on bounded domains, *J. Math. Anal. Appl.*, **379** (2011), 216–228.
[26] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, 1999.

[27] W. Rundell and Z. Zhang, Fractional diffusion: recovering the distributed fractional derivative from overposed data, *Inverse Probl.*, **33** (2017), 035008.
[28] K. Sakamoto and M. Yamamoto, Initial value/boundary value problems for fractional diffusion-wave equations and applications to some inverse problems, *J. Math. Anal. Appl.*, **382** (2011), 426–447.
[29] C. Sun, G. Li, and X. Jia, Numerical inversion for the multiple fractional orders in the multiterm TFDE, *Adv. Math. Phys.*, **2017** (2017), 1–7.
[30] S. Tatar, R. Tinaztepe, and S. Ulusoy, Simultaneous inversion for the exponents of the fractional time and space derivatives in the space-time fractional diffusion equation, *Appl. Anal.*, **95** (2016), 1–23.
[31] S. Tatar and S. Ulusoy, A uniqueness result for an inverse problem in a space-time fractional diffusion equation, *Electron. J. Differ. Equ.*, **258** (2013), 1–9.
[32] B. Yu, X. Jiang, and H. Qi, An inverse problem to estimate an unknown order of a Riemann–Liouville fractional derivative for a fractional Stokes' first problem for a heated generalized second grade fluid, *Acta Mech. Sin.*, **31** (2015), 153–161.
[33] B. Yu, X. Jiang, and C. Wang, Numerical algorithms to estimate relaxation parameters and Caputo fractional derivative for a fractional thermal wave model in spherical composite medium, *Appl. Math. Comput.*, **274** (2016), 106–118.
[34] R. Zacher, Weak solutions of abstract evolutionary integro-differential equations in Hilbert spaces, *Funkc. Ekvacioj*, **52** (2009), 1–18.
[35] D. Zhang, G. Li, G. Chi, X. Jia, and H. Li, Numerical identification of multiparameters in the space fractional advection dispersion equation by final observations, *J. Appl. Math.*, **2012** (2012), 1–14.
[36] G. Zheng and T. Wei, A new regularization method for the time fractional inverse advection-dispersion problem, *SIAM J. Numer. Anal.*, **49** (2011), 1972–1990.

Zhiyuan Li and Masahiro Yamamoto

Inverse problems of determining coefficients of the fractional partial differential equations

Abstract: When considering fractional diffusion equation as model equation in analyzing anomalous diffusion processes, some important functions in the model, for example, the source term are often unknown, which requires one to discuss inverse problems to identify these physical quantities from some additional information that can be observed or measured practically. This chapter investigates several kinds of inverse coefficient problems for the fractional diffusion equation.

Keywords: Inverse coefficient problem, time fractional partial differential equation, uniqueness, stability, lateral boundary data, Carleman estimate, Dirichlet-to-Neumann map

MSC 2010: 35R30, 35R11, 65M32

1 Introduction

In this chapter, let $\Omega \subset \mathbb{R}^d$ be a bounded domain with smooth boundary $\partial\Omega$ and let $0 < \alpha < 1$, if we do not specify. We consider fractional diffusion equations:

$$\partial_t^\alpha y(x,t) = \Delta y(x,t) + p(x)y(x,t), \quad x \in \Omega,\ 0 < t < T, \tag{1}$$

and

$$\partial_t^\alpha y(x,t) = \operatorname{div}\big(p(x)\nabla y(x,t)\big), \quad x \in \Omega,\ 0 < t < T. \tag{2}$$

Here and henceforth, $\partial_t^\alpha v$ denotes the Caputo derivative given by

$$\partial_t^\alpha v(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \frac{dv}{ds}(s)ds.$$

To each of (1) and (2), we attach boundary and initial conditions:

$$y(x,t) = h(x,t), \quad x \in \partial\Omega,\ 0 < t < T, \tag{3}$$

and

$$y(x,0) = a(x), \quad x \in \Omega. \tag{4}$$

Zhiyuan Li, School of Mathematics and Statistics, Shandong University of Technology, Zibo, Shandong 255049, China, e-mail: zyli@sdut.edu.cn
Masahiro Yamamoto, Graduate School of Mathematical Sciences, The University of Tokyo, 3-8-1 Komaba, Meguro-ku, Tokyo 153-8914, Japan, e-mail: myama@ms.u-tokyo.ac.jp

https://doi.org/10.1515/9783110571660-020

We can consider other type of boundary conditions but we mainly attach the Dirichlet boundary condition.

Given p, $\alpha \in (0,1)$, h and a in (1)–(4), it is a forward problem that we find y satisfying (1) or (2) with (3) and (4) in suitable function spaces. Then (1)–(3)–(4) and (2)–(4) are called initial-boundary value problems. On the other hand, for example, in (2), $p(x)$ describes a spatial diffusivity coefficient in the media under consideration and so it is physically important how to choose $p(x)$. When we make modeling, we have to determine $\alpha \in (0,1)$ and $p(x)$, $x \in \Omega$ in order that the solution y to the initial-boundary value problem (1) –(3)–(4) or (2)–(4) behaves appropriately: for instance, in the one-dimensional case $d = 1$, we are requested to determine $p(x)$, $x \in \Omega$ such that $y(x_0, t) = \mu(t)$ for $0 < t < T$, where $\mu(t)$ is a prescribed target function or observation data function and an $x_0 \in \Omega$ is a fixed monitoring point. This is one inverse coefficient problem for a fractional diffusion equation which we survey in this chapter.

For more precise formulations of the inverse coefficient problems, restricted to simple equations such as (1) and (2), we consider a solution $y(\alpha, p; h, a) = y(\alpha, p; h, a)(x, t)$ to

$$\begin{cases} \partial_t^\alpha y(x,t) = \Delta y(x,t) + p(x)y(x,t), & x \in \Omega,\ 0 < t < T, \\ y(x,t) = h(x,t), & x \in \partial\Omega,\ 0 < t < T, \\ y(x,0) = a(x), & x \in \Omega. \end{cases} \tag{5}$$

Here and henceforth, we understand the solution $y(\alpha, p; h, a)$ in suitable classes (e. g., strong solutions), on which we can consult Gorenflo, Luchko, and Yamamoto [12], Kubica and Yamamoto [26], Zacher [66], for example.

In this chapter, we mainly survey results on the determination of spatially varying coefficients $p(x)$. Let $\gamma \subset \partial\Omega$ be a suitable sub-boundary and $\nu = \nu(x)$ be the outward unit normal vector to $\partial\Omega$. We set $\partial_\nu u = \nabla u \cdot \nu$.

In Sections 2 and 3, we can survey the following two major formulations of the inverse coefficient problems.

Section 2: A single or a finite time of observations of lateral boundary data
Let h and a be appropriately chosen. Determine $p(x)$ and/or $\alpha \in (0,1)$ by

$$\partial_\nu y(\alpha, p; h, a)(x,t), \quad x \in \gamma,\ 0 < t < T.$$

Section 3: Infinitely many times of observations of lateral boundary data
We fix $a = 0$ in (5) for example. We define the Dirichlet-to-Neuman map by

$$\Lambda_{\alpha,p} : h \mapsto \partial_\nu y(\alpha, p; h, 0)(x,t)|_{\gamma\times(0,T)}$$

with suitable domain space (e. g., $C_0^\infty(\gamma \times (0,T))$) and range space. Then determine $p(x)$ and/or $\alpha \in (0,1)$ by $\Lambda_{\alpha,p}$.

Moreover, we can consider the following.

Final overdetermination
We fix h and a. Determine $p(x)$ and/or α by

$$y(\alpha, p; h, a)(x, T), \quad x \in \Omega.$$

However, to the best of the authors' knowledge, there are no publications on the inverse coefficient problems with final observation data. See Section 4.1 in the chapter "Inverse problems of determining sources of the fractional partial differential equations" of this handbook.

The mathematical subjects are: **Uniqueness** and **Stability** and we survey them.

The rest sections of this chapter are

Section 4: Other related inverse problems
Section 5: Numerical works

In the final section, we are not limited to inverse coefficient problems, and we review numerical works on inverse problems, which include the lateral Cauchy problem and the backward problem in time, because many numerical articles are published rapidly also by practical demands and we think that even a tentative review may be useful for overlooking future researches.

We close this section with descriptions of available methodologies for inverse problems for fractional partial differential equations. As long as the cases of $\alpha = 1$ and $\alpha = 2$ are concerned, a comprehensive method for inverse problems with the formulation by a finite time of observations of lateral boundary data is based on Carleman estimates, which was created by Bukhgeim and Klibanov [6] and yields the uniqueness and the stability for inverse coefficient problems. See also Beilina and Klibanov [2], Bellassoued and Yamamoto [3], Klibanov [25], Yamamoto [63], for example. Moreover, for the formulation with infinitely many times of observations of lateral boundary data, an essential step is to construct special forms of solutions to differential equations under consideration, which are called complex geometric optics solutions, and Carleman estimates give flexible constructions (e. g., Bukhgeim [5], Imanuvilov and Yamamoto [13]).

We describe a Carleman estimate in a simplified form as follows. Let $Q \subset \mathbb{R}^{d+1}$ be a domain in the (x,t)-space, $\varphi = \varphi(x,t)$ a suitably chosen weight function and let P be a partial differential operator. Then a Carleman estimate is stated by: there exist constants $C > 0$ and $s_0 > 0$ such that

$$\int_Q s|u(x,t)|^2 e^{2s\varphi(x,t)}dxdt \le C\int_Q |Pu(x,t)|^2 e^{2s\varphi(x,t)}dxdt$$

for all $s \ge s_0$ and $u \in C_0^\infty(Q)$.

Here, we note that the Carleman estimate holds uniformly for all sufficient large $s > 0$. In other words, the constant $C > 0$ is chosen independently of $s \ge s_0$ as well as u. The power of s on the left-hand side can change according to partial differential equations.

For proving Carleman estimates, the indispensable step is the integration by parts:

$$\partial(yz) = y\partial z + z\partial y, \tag{6}$$

where $\partial = \frac{\partial}{\partial x_i}$ or $= \frac{\partial}{\partial t}$. Carleman estimates have been established for various equations such as the parabolic equation ($\alpha = 1$) and the hyperbolic equation ($\alpha = 2$) and we refer, for example, to Bellassoued and Yamamoto [3], Isakov [14], Yamamoto [63]. Here, we do not intend any complete list of references on Carleman estimates.

However, for fractional derivatives, convenient formulae such as (6), do not hold, so that we cannot prove Carleman estimates in general for fractional partial differential equations.

Thus for inverse problems for fractional cases, we have no comprehensive methodologies and there are not many results on the uniqueness and the stability for the inverse problems in spite of the significance.

For the inverse coefficient problems, in Sections 2 and 3, we can mainly have the following strategies.

1. Representation of solutions to (1) by means of eigenfunctions
We extract information of the spectrum of the operator $\Delta + p(x)$ with the boundary condition. As for the representation of the solution see, for example, Sakamoto and Yamamoto [49]. In the case of $p = 0$, see also Lemma 1 of the chapter "Inverse problems of determining sources of the fractional partial differential equations" of this handbook. In particular, in the one-dimensional case $d = 1$, we can apply the Gel'fand–Levitan theory.

2. Transforms in t
We can apply integral transforms such as the Laplace transform, or the limits as $t \to \infty$.

3. Reduction to partial differential equations with integer-orders
In special cases (e. g., $\alpha = \frac{1}{2}$), the reduction is possible, so that we can prove a Carleman estimate.

2 A finite number of observations of lateral boundary data

In this section, we present several results regarding inverse problems of determining the coefficients in the fractional diffusion equations from lateral Cauchy data.

2.1 One-dimensional case by the Gel'fand–Levitan theory

We start with a one-dimensional fractional diffusion equation:

$$\begin{cases} \partial_t^\alpha y(x,t) = \partial_x(D(x)\partial_x y(x,t)), & 0 < x < 1,\ 0 < t < T, \\ y(x,0) = \delta(x), & 0 < x < 1, \\ \partial_x y(0,t) = \partial_x y(1,t) = 0, & 0 < t < T. \end{cases} \tag{7}$$

Here, δ is the Dirac delta function.

We discuss the following.

Inverse Problem. Determine the order $\alpha \in (0,1)$ of the time derivative and the diffusion coefficient $D(x)$ from boundary data $y(0,t)$, $0 < t \le T$.

We introduce the admissible set

$$\mathcal{U} := \{(\alpha, D);\ \alpha \in (0,1),\ D \in C^2[0,1],\ D > 0 \text{ on } [0,1]\},$$

and then have

Theorem 1 (Cheng, Nakagawa, Yamamoto, and Yamazaki [9]). *Let $y(\alpha_1, D_1)$, $y(\alpha_2, D_2)$ be the weak solutions to* (7) *with respect to $(\alpha_1, D_1), (\alpha_2, D_2) \in \mathcal{U}$, respectively. Then $y(\alpha_1, D_1)(0,t) = y(\alpha_2, D_2)(0,t)$, $0 < t \le T$ with some $T > 0$, implies $\alpha_1 = \alpha_2$ and $D_1(x) = D_2(x)$, $0 \le x \le 1$.*

By an argument similar to the above theorem, Li, Zhang, Jia, and Yamamoto [27] proved the uniqueness for determining the fractional order and the diffusion coefficient in the one-dimensional time-fractional diffusion equation with smooth initial functions by using boundary measurements. For $i = 1, 2$, let $\{\varphi_n^{(i)}\}_{n\in\mathbb{N}}$ be the set of all the orthonormal eigenfunctions of $\frac{d}{dx}(D_i(x)\frac{d}{dx})$ with the zero Neumann boundary condition $\frac{d\varphi}{dx}(0) = \frac{d\varphi}{dx}(1) = 0$.

Theorem 2 (Li, Zhang, Jia, and Yamamoto [27]). *Suppose that $(\alpha_i, D_i) \in \mathcal{U}$, and let $y(\alpha_i, D_i)$, $i = 1, 2$ satisfy*

$$\begin{cases} \partial_t^{\alpha_i} y(x,t) = \partial_x(D_i(x)\partial_x y(x,t)), & 0 < x < 1,\ 0 < t < T, \\ y(x,0) = a(x), & 0 < x < 1, \\ \partial_x y(0,t) = \partial_x y(1,t) = 0, & 0 < t < T. \end{cases}$$

Suppose

$$a \in H^4(0,1), \quad a''(0) \ne 0, \quad a'(0) = a'(1) = 0$$

and

$$\int_0^1 a(x)\varphi_n^{(1)}(x)dx \ne 0, \quad \forall n \in \mathbb{N} \quad \textit{or} \quad \int_0^1 a(x)\varphi_n^{(2)}(x)dx \ne 0, \quad \forall n \in \mathbb{N}. \tag{8}$$

If $y(\alpha_1, D_1)(0,t) = y(\alpha_2, D_2)(0,t)$ and $y(\alpha_1, D_1)(1,t) = y(\alpha_2, D_2)(1,t)$, $0 < t \le T$ with some $T > 0$, then $\alpha_1 = \alpha_2$ and $D_1(x) = D_2(x)$, $0 \le x \le 1$.

Their result corresponds to classical results for the case $\alpha = 1$ by Murayama [39], Murayama and Suzuki [40].

As initial values, in Theorem 1, we have to choose exactly the Dirac delta function, while in Theorem 2, the initial value must satisfy very restricted non-degeneracy condition (8). One natural conjecture is that the uniqueness holds if $a \not\equiv 0$. It has not been proved even in the case of $\alpha = 1$. However, Pierce [44] proved the uniqueness for the one-dimensional heat equation with not identically vanishing boundary data. We can prove a similar result for the one-dimensional fractional diffusion equation.

Theorem 3 (Rundell and Yamamoto [48]). *Suppose that $0 < \alpha < 1$, $p_1, p_2 \in C[0,1]$, $p_1, p_2 < 0$ on $[0,1]$. Let $y(p_k)$ be sufficiently smooth and satisfy*

$$\begin{cases} \partial_t^\alpha y(x,t) = \partial_x^2 y(x,t) + p_k(x) y(x,t), & 0 < x < 1,\ 0 < t < T, \\ y(x,0) = 0, & 0 < x < 1, \\ \partial_x y(0,t) = 0, \quad \partial_x y(1,t) = h(t), & 0 < t < T. \end{cases}$$

If $h \not\equiv 0$ in $(0,T)$, then $y(p_1)(1,t) = y(p_2)(1,t)$, $0 < t < T$ yields $p_1(x) = p_2(x)$, $0 \le x \le 1$.

Once that we establish the representation formula for the solution $y(p_k)$, the proof is similar to [44]. We can similarly prove also the uniqueness for α as well as $p(x)$ by the same data, but we omit the details.

For the one-dimensional time-fractional diffusion equation with spatially dependent source term:

$$\begin{cases} \partial_t^\alpha y(x,t) - \partial_x^2 y(x,t) + p(x) y(x,t) = f(x), & 0 < x < 1,\ t > 0, \\ y(0,t) = y(1,t) = 0, & t > 0, \\ y(x,0) = 0, & 0 < x < 1. \end{cases} \tag{9}$$

Jin and Rundell [18] investigated other type of inverse coefficient problem, and gave the following uniqueness of the inverse problem.

Theorem 4 (Jin and Rundell [18]). *Let $y(p_i, f_j)(x,t)$ be the solution of (9) with f_j, $j \in \mathbb{N}$ and p_i, $i = 1, 2$ in $\{p \in L^\infty(0,1);\ p \ge 0, \|p\| \le M\}$. Here, $M > 0$ is an arbitrarily fixed constant. We suppose that the set $\{f_j\}_{j\in\mathbb{N}}$ of input sources forms a complete basis in $L^2(0,1)$:*

1. *If further, $p_1 = p_2$ on the interval $[1-\delta, 1]$ for some $\delta \in (0,1)$ and $M < \delta\pi$, then there exists a time $t^* > 0$ such that*

   $$\partial_x y(p_1, f_j)(0,t) = \partial_x y(p_2, f_j)(0,t), \quad j \in \mathbb{N}$$

   for any $t > t^*$ implies $p_1 = p_2$ on the interval $[0,1]$.

2. *If $M < \pi$, then there exists a time $t^* > 0$ such that*

$$\partial_x y(p_1, f_j)(1,t) - \partial_x y(p_1, f_j)(0,t) = \partial_x y(p_2, f_j)(1,t) - \partial_x y(p_2, f_j)(0,t), \quad j \in \mathbb{N}$$

for any $t > t^*$ implies $p_1 = p_2$ on the interval $[0,1]$.

As for related results, see also Jin and Rundell [19].

2.2 Integral transform

For the general dimensional case, we refer to Miller and Yamamoto [38] which applied an integral transform which is given in terms of the Wright function (e. g., Bazhlekova [1]). For presenting the main result in [38], some notation and settings are needed.

Let $y = y(p, \alpha)$ be a smooth solution to

$$\begin{cases} \partial_t^\alpha y(x,t) = \Delta y(x,t) + p(x)y(x,t), & x \in \Omega,\ 0 < t < T, \\ y(x,0) = a(x), & x \in \Omega, \quad \text{if } 0 < \alpha < 1, \\ y(x,0) = a(x),\ \partial_t y(x,0) = 0, & x \in \Omega, \quad \text{if } 1 < \alpha < 2, \\ y(x,t) = 0, & x \in \partial\Omega,\ 0 < t < T. \end{cases}$$

Here, $\Omega \subset \mathbb{R}^d$, $d \geq 1$ is a bounded domain with smooth $\partial\Omega$. Let ω be a subdomain of Ω such that $\partial\omega \supset \partial\Omega$. We set

$$\mathcal{U}_M := \{p \in W^{1,\infty}(\Omega);\ p \leq 0 \text{ in } \Omega,\ \|p\|_{W^{1,\infty}(\Omega)} \leq M,\ p|_\omega = \eta\}$$

with a constant $M > 0$ and a smooth function η, which are arbitrarily chosen. Moreover, the initial value a of the diffusion system satisfies

$$a \in H^3(\Omega) \cap H_0^2(\Omega), \quad \Delta a \in H_0^1(\Omega) \tag{10}$$

and

$$a(x) > 0, \quad x \in \overline{\Omega \setminus \omega}. \tag{11}$$

Now we are ready to state the unique result from [38].

Theorem 5 (Miller and Yamamoto [38]). *Let $\alpha_1, \alpha_2 \in (0,1) \cap (1,2)$, and the initial value satisfy* (10) *and* (11)*, and let $p_1, p_2 \in \mathcal{U}_M$:*
1. *If $y(p_1, \alpha_1) = y(p_2, \alpha_1)$ in $\omega \times (0,T)$, then $p_1 = p_2$ in Ω.*
2. *Moreover, let*

$$a \leq 0 \text{ or } \geq 0, \not\equiv 0.$$

If $y(p_1, \alpha_1) = y(p_2, \alpha_2)$ in $\omega \times (0,T)$, then $\alpha_1 = \alpha_2$ and $p_1 = p_2$ in Ω.

Here, the natural numbers for α_1, α_2 are excluded. However, using the existing uniqueness result for the parabolic inverse problem (e. g., pp. 594–595 in [25]), Theorem 5 holds for all $0 < \alpha_1, \alpha_2 < 2$.

As other proof of Theorem 5, we can argue as follows. By the formula on the Laplace transform of the Caputo derivative,

$$(L\partial_t^\alpha v)(s) := \int_0^\infty e^{-st}\partial_t^\alpha v(t)dt = s^\alpha L(v)(s) - s^{\alpha-1}v(0)$$

for smooth v, we can apply also the Laplace transform of the fractional equations under consideration. We here omit the details.

2.3 Carleman estimates in restricted cases

In Section 1, we have described no general Carleman estimates for fractional partial differential equations, unlike the classical partial differential equations. Thus the stability results for the inverse problem for the fractional diffusion equation with general fractional order from the lateral Cauchy data are rather limited. However, for some special orders such as $\alpha = 1/2$, Carleman estimates were proved and/or applied to inverse problems in Ren and Xu [46], Xu, Cheng, and Yamamoto [62], and Yamamoto and Zhang [64].

Yamamoto and Zhang [64] considered the following fractional diffusion equation with an initial condition and Cauchy data in a spatially one-dimensional case:

$$\begin{cases} (\partial_t^{\frac{1}{2}} - \partial_x^2)y(x,t) = p(x)y(x,t), & 0 < x < 1,\ 0 < t < T, \\ y(x,0) = a(x), & 0 < x < 1, \\ y(0,t) = h_0(t), \quad \partial_x y(0,t) = h_1(t), & 0 < t < T, \end{cases} \tag{12}$$

where $a(x)$, $h_0(t)$, $h_1(t)$ are given, and $a \not\equiv 0$ in $(0,1) \times (0,T)$ or $h_0, h_1 \not\equiv 0$ in $(0,T)$, and established a Carleman estimate and proved the conditional stability in determining a coefficient $p(x)$ for a fractional diffusion equation.

Following [64], we let $t_0 \in (0,T)$ be arbitrarily fixed. We choose $x_0 > 1$ such that $(x_0 - 1)^2 < \frac{1}{3}$, and set

$$d(x) = |x - x_0|^2, \quad \psi(x,t) = d(x) - \beta(t-t_0)^2 > 0, \quad 0 \le x \le 1,\ 0 \le t \le T,$$

where $\beta > 0$ satisfies

$$\sqrt{\frac{x_0^2}{\beta}} < \min\{t_0, T - t_0\}.$$

For $x_0^2 > \varepsilon > (x_0-1)^2$, we set

$$Q_\varepsilon := \{(x,t) \in (0,1)\times(0,T);\ \psi(x,t) > \varepsilon\}, \quad \Omega_\varepsilon := Q_\varepsilon \cap \{t = t_0\}.$$

Then we have the following.

Theorem 6 (Yamamoto and Zhang [64]). *For $k = 1, 2$, let $y(p_k)$ satisfy* (12) *with p_k. Assume that there exist $M > 0$ and sufficiently small $\varepsilon_0 > 0$ such that*

$$\|y(p_k)\|_{C^2([\varepsilon_0,T-\varepsilon_0];W^{2,\infty}(\Omega)\cap H^4(\Omega))\cap C^3([\varepsilon_0,T-\varepsilon_0];L^2(\Omega))} + \|y(p_k)\|_{C([0,T];L^\infty(\Omega))} \le M,$$
$$\partial_x^j p_1(0) = \partial_x^j p_2(0), \quad j = 0, 1, \quad \|p_k\|_{W^{2,\infty}(\Omega)} \le M, \quad k = 1, 2.$$

Moreover, suppose

$$y(p_k)(x, t_0) \ne 0, \quad 0 \le x \le 1, \quad k = 1 \text{ or } k = 2.$$

Then, for any $\varepsilon \in ((x_0 - 1)^2, \frac{1}{3}x_0^2)$, there exist constants $C > 0$ and $\theta \in (0, 1)$ depending on M and ε such that

$$\|p_1 - p_2\|_{H^2(\Omega_{3\varepsilon})} \le C\big\|(y(p_1) - y(p_2))(\cdot, t_0)\big\|^{\theta}_{H^4(\Omega_\varepsilon)}.$$

Unlike inverse coefficient problems for equations with natural number orders, one needs two spatial data $y(\cdot, 0)$ and $y(\cdot, t_0)$ to construct the unknown coefficients. By $(x_0 - 1)^2 < \varepsilon < \frac{1}{3}x_0^2$, we note that $x_0 - \sqrt{3\varepsilon} < 1$, that is, $(0, x_0 - \sqrt{3\varepsilon}) \subset\subset (0, 1)$. Thus, in [64], only in a subinterval $0 < x < x_0 - \sqrt{3\varepsilon}$ of $[0, 1]$, we can estimate $p_1 - p_2$, which means that Theorem 6 gives the stability locally in $(0, 1)$

Next, let $0 < t_0 < T$. For a system

$$\begin{cases} (\partial_t^{\frac{1}{2}} - \partial_x^2)u(x, t) = g(x)\rho(x, t), & 0 < x < 1,\ 0 < t < T, \\ u(x, 0) = 0, & 0 < x < 1, \\ u(0, t) = u(1, t) = 0, & 0 < t < T, \end{cases} \tag{13}$$

Kawamoto [21] discussed an inverse source problem: given ρ, determine $g = g(x)$, $0 < x < 1$ by data $\partial_x u(0, t)$, $\partial_x u(1, t)$ for $0 < t < T$ and $u(x, t_0)$ for $0 < x < 1$ and estimated g over the whole interval $0 < x < 1$. We note that the system (13) is a linearization of (12) around $p = 0$.

For (12), his result easily yields the estimate $\|p_1 - p_2\|_{H^2(0,1)}$ for the inverse coefficient problem, which is in contrast with the local estimate of Theorem 6. We notice that [21] considered a more general elliptic operator but we omit the details.

It should be mentioned here that in general the regularity of the solution at $t = 0$ cannot be improved more than the assumptions in Theorem 6. For concentrating on the inverse problem, the discussion for the regularity of the solution to the initial-boundary value problem (12) is omitted.

The proof of the Carleman estimate for the one-dimensional case with order $\alpha = \frac{1}{2}$ is done by twice applying the Caputo derivative to convert the original fractional diffusion equation to a usual partial differential operator: $\partial_t - \partial_x^4$. Such arguments for general rational α is direct but extremely complicated, and so far it turns out not to work.

For fractional partial differential equations in general dimensions, a different method brings Carleman estimates: Cheng, Lin, and Nakamura [8] for $\alpha = \frac{1}{2}$, Lin and Nakamura [32] for $0 < \alpha < 1$, and Lin and Nakamura [33] for multi-term time fractional derivatives of orders $\in (0,1) \cap (1,2)$. The Carleman estimate in [32] produces the following uniqueness: if y with some regularity satisfies

$$\begin{cases} \partial_t^\alpha y = \Delta y(x,t) + \sum_{j=1}^d b_j(x,t)\partial_j y + c(x,t)y, & x \in \Omega,\ 0 < t < T, \\ y(x,0) = 0, & x \in \Omega, \end{cases}$$

and

$$y = 0 \quad \text{in } \omega \times (0,T)$$

with some subdomain $\omega \subset \Omega$, then

$$y = 0 \quad \text{in } \Omega \times (0,T).$$

Here, we assume $b_j, c \in L^\infty(\Omega \times (0,T))$.

In general, we need to assume that $y(\cdot,0) = 0$ in Ω, which is not requested for the unique continuation for the classical partial differential equations.

Li and Yamamoto [31] proved that if $y \in L^\infty(0,T;H^2(\Omega))$ satisfies

$$\begin{cases} \partial_t^\alpha y(x,t) = \partial_x^2 y(x,t), & 0 < x < 1,\ 0 < t < T, \\ y(x,t) = 0, & x \in I,\ 0 < t < T \end{cases}$$

with some nonempty open interval $I \subset (0,1)$, then

$$y = 0 \quad \text{in } (0,1) \times (0,T).$$

The applications of the Carleman estimates in [8, 32], and [33] require a transform of (x,t) called the Holmgren transform. Therefore, for the inverse coefficient problem, the Holmgren transform does not keep the structure of the equation in (5). Moreover, for the uniqueness, the zero initial condition is needed and so their Carleman estimates are not applicable to inverse source or inverse coefficient problems as we formulate.

We further review recent works on Carleman estimates. Kawamoto and Machida [22] considers a fractional transport equation:

$$\begin{aligned} &\partial_t^{\frac{1}{2}} y(x,v,t) + v\partial_x y(x,v,t) + \sigma_t(x,v)y(x,v,t) \\ &\quad = \sigma_s(x,v)\int_V r(x,v,v')y(x,v',t)dv', \quad x \in \Omega := (0,\ell),\ v \in V := \{v_0 \le |v| \le v_1\}, \end{aligned} \tag{14}$$

with

$$y(x,v,0)=a(x,v), \quad x\in\Omega,\ v\in V$$

and

$$y(x,v,t)=h(x,v,t), \quad (x,v)\in\Gamma_-,\ 0<t<T.$$

Here, $v_0, v_1, \ell > 0$ are constants and we set

$$\Gamma_{\pm} := \{(x,v)\in\partial\Omega\times V;\ \mp v<0 \text{ at } x=0,\ \pm v>0 \text{ at } x=\ell\},$$

where the double-signs correspond. We note that $y(x,v,t)$ describes the density of some particles at the point $x\in\Omega$ and the time t with the velocity v. As for some physical backgrounds (see, e. g., Machida [37]), and the equation where $\partial_t^{\frac{1}{2}}$ is replaced by ∂_t is called a radiative transport equation (e. g., Duderstadt and Martin [10]).

By an idea similar to Xu, Cheng, and Yamamoto [62], reducing (14) to $\partial_t y - v^2\partial_x^2 y$ with integral term, the article [22] established a Carleman estimate and proved the Lipschitz stability in determining $\sigma_t(x,v)$ and/or $\sigma_s(x,v)$ by $y(\cdot,\cdot,t_0)$ in $\Omega\times V$, $\partial_x y(\cdot,\cdot,0)$ in $V\times J$ and $y|_{\Gamma_+\times J}$, where $t_0\in(0,T)$ is arbitrarily fixed and $J\subset(0,T)$ is an arbitrary open subinterval including t_0.

Again let $\Omega\subset\mathbb{R}^d$ be a smooth bounded domain. Now we review Carleman estimates for a mixed type of fractional derivatives:

$$\partial_t y + \rho\partial_t^\alpha y = \sum_{i,j=1}^d \partial_i\big(a_{ij}(x)\partial_j y\big) + \sum_{j=1}^d b_j(x,t)\partial_j y + c(x,t)y.$$

Here, $\rho>0$ is a constant and we assume $a_{ij}=a_{ji}\in C^1(\overline{\Omega})$, $b_j, c\in L^\infty(\Omega\times(0,T))$ and the uniform ellipticity.

Kawamoto [20] proved a Carleman estimate for $\alpha=\frac{1}{2}$, and Li, Huang, and Yamamoto [29] established Carleman estimates for both cases:
1. $0<\alpha<\frac{1}{2}$.
2. α is a rational number such that $0<\alpha\le\frac{3}{4}$.

See also Kawamoto and Machida [72].

We note that the Carleman estimate in [29] holds also for a more general equation

$$\begin{aligned}\partial_t y + \sum_{k=1}^{\ell} q_k(x,t)\partial_t^{\alpha_k} y &= \sum_{i,j=1}^d \partial_i\big(a_{ij}(x,t)\partial_j y\big)\\ &+ \sum_{j=1}^d b_j(x,t)\partial_j y + c(x,t)y, \quad 0<\alpha_\ell<\cdots<\alpha_1\le\frac{1}{2},\end{aligned}$$

with suitable conditions on a_{ij}, b_j, c, and see also [29] for possible generalizations.

According to the method by Bukhgeim and Klibanov [6], their Carleman estimates yield the stability and the uniqueness for the inverse source problems under some suitable formulations, but we omit further details.

3 Infinite many times of observations of lateral boundary data

We fix $\lambda \in C^\infty[0,\infty)$ such that

$$\lambda(0) = \frac{d\lambda}{dt}(0) = 0$$

and there exist constants $C > 0$ and $\theta \in (0, \frac{\pi}{2})$ such that λ can be analytically extended to $\{z \in \mathbb{C} \setminus \{0\};\ |\arg z| < \theta\}$ and

$$\left|\frac{d^2\lambda}{dt^2}(t)\right| = O(e^{Ct})$$

as $t \to \infty$. First, letting boundary values take the form of separation of variables, we consider an inverse coefficient problem with infinitely many times of observations.

We discuss the following initial-boundary value problem:

$$\begin{cases} \sum_{k=1}^{\ell} p_k(x)\partial_t^{\alpha_k} y(x,t) = \Delta y(x,t) + p(x)y(x,t), & x \in \Omega,\ 0 < t < T, \\ y(x,0) = 0, & x \in \Omega, \\ y(x,t) = \lambda(t)h(x), & x \in \partial\Omega,\ 0 < t < T. \end{cases} \tag{15}$$

We recall that ν is the outward unit normal vector to $\partial\Omega$ and we denote $\partial_\nu v = \nabla v \cdot \nu$. For $\ell \in \mathbb{N}$, we set $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_\ell) \in (0,1)^\ell$ where $\alpha_\ell < \alpha_{\ell-1} < \cdots < \alpha_1$.

We study

Inverse Problem. For $h \in H^{\frac{3}{2}}(\partial\Omega)$, we define the Dirichlet-to-Neumann map by

$$\Lambda(\ell, \boldsymbol{\alpha}, p_k, p)h := \partial_\nu y|_{\partial\Omega\times(0,T)} \in L^2(0,T; H^{\frac{1}{2}}(\partial\Omega)).$$

Then we discuss whether $(\ell, \boldsymbol{\alpha}, p_k, p)$ is uniquely determined by the Dirichlet-to-Neumann map $\Lambda(\ell, \boldsymbol{\alpha}, p_k, p) : H^{\frac{3}{2}}(\partial\Omega) \longrightarrow L^2(0,T; H^{\frac{1}{2}}(\partial\Omega))$.

As an admissible set of unknowns including numbers and coefficients, we set

$$\begin{aligned} \mathcal{U} = \{&(\ell, \boldsymbol{\alpha}, p_k, p) \in \mathbb{N} \times (0,1)^\ell \times C^\infty(\overline{\Omega})^{\ell+1},\ p_1 > 0, \\ &p_k \ge 0, \not\equiv 0,\ 2 \le k \le \ell,\ p \le 0 \text{ on } \overline{\Omega}\}. \end{aligned}$$

The analyticity in t of the solution to (15) reduces the Dirichlet-to-Neumann map for (15) by letting $t \to \infty$ to an inverse coefficient problem by elliptic Dirichlet-to-Neumann map. Therefore, one can apply the existing uniqueness results.

Theorem 7 (Li, Imanuvilov and Yamamoto [30]). *Let $(\ell, \boldsymbol{\alpha}, p_k, p), (m, \boldsymbol{\beta}, q_k, q) \in \mathcal{U}$. Then*

$$\Lambda(\ell, \boldsymbol{\alpha}, p_k, p)h = \Lambda(m, \boldsymbol{\beta}, q_k, q)h, \quad \forall h \in H^{\frac{3}{2}}(\partial\Omega)$$

implies that

$$\ell = m, \quad \boldsymbol{\alpha} = \boldsymbol{\beta}, \quad p_k = q_k, \quad 1 \le k \le \ell, \quad p = q \quad \text{in } \Omega.$$

In $\mathcal{U}$, the regularity of $p, p_1, \ldots, p_\ell$ can be relaxed but we do not discuss here. In particular, the result by Dirichlet-to-Neumann map in the two-dimensional case $d = 2$ (e. g., Imanuvilov and Yamamoto [13]) yields a sharp uniqueness result where Dirichlet inputs and Neumann outputs can be restricted on an arbitrary subboundary and the required regularity of unknown coefficients is relaxed.

Corollary 1 (Li, Imanuvilov and Yamamoto [30]). *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary $\partial\Omega$ and $\gamma \subset \partial\Omega$ be an arbitrarily given sub-boundary and let $r > 2$ be arbitrarily fixed. Assume*

$$\begin{aligned}\widehat{\mathcal{U}} = \{&(\ell, \boldsymbol{\alpha}, p_k, p) \in \mathbb{N} \times (0,1)^\ell \times (W^{2,\infty}(\Omega))^\ell \times W^{1,r}(\Omega);\\ &p_1 > 0,\ p_k \ge 0, \not\equiv 0,\ 2 \le k \le \ell,\ p \le 0 \text{ on } \overline{\Omega}\}.\end{aligned}$$

If $(\ell, \boldsymbol{\alpha}, p_k, p)$ and $(m, \boldsymbol{\beta}, q_k, q) \in \widehat{\mathcal{U}}$ satisfy

$$\Lambda(\ell, \boldsymbol{\alpha}, p_k, p)h = \Lambda(m, \boldsymbol{\beta}, q_k, q)h \quad \text{on } \gamma$$

for all $h \in H^{\frac{3}{2}}(\partial\Omega)$ with supp $h \subset \gamma$, *then $\ell = m$, $\boldsymbol{\alpha} = \boldsymbol{\beta}$, $p_k = q_k$, $1 \le k \le \ell$ and $p = q$ in Ω.*

Next, we review an inverse coefficient problem by Dirichlet-to-Neumann map at one shot. Assuming that $0 < \alpha < 1$ or $1 < \alpha < 2$, we discuss

$$\begin{cases} \rho(x)\partial_t^\alpha y(x,t) = \operatorname{div}(a(x)\nabla y(x,t)) + V(x)y(x,t), & x \in \Omega,\ 0 < t < T,\\ y(x,t) = h(x,t), & x \in \partial\Omega,\ 0 < t < T,\\ y(x,0) = 0, & x \in \Omega \text{ if } 0 < \alpha < 1,\\ y(x,0) = \partial_t y(x,0) = 0, & x \in \Omega \text{ if } 1 < \alpha < 2. \end{cases} \tag{16}$$

Let $0 < T_0 < T$ be arbitrarily fixed. We assume

$$\rho_k, a_k \in C^2(\overline{\Omega}), > 0 \quad \text{on } \overline{\Omega}, \quad V_k \in L^\infty(\Omega)$$

with $k = 1, 2$. We define the Dirichlet-to-Neumann map Λ_{ρ,a,V,T_0} at one shot. Let $S_{\text{in}}, S_{\text{out}} \subset \partial\Omega$ be relatively open nonempty sub-boundaries and $S_{\text{in}} \cap S_{\text{out}} \ne \emptyset$, $\overline{S_{\text{in}} \cup S_{\text{out}}} = \partial\Omega$. We define Λ_{ρ,a,V,T_0} by

$$\Lambda_{\rho,a,V,T_0} h = a(\cdot)\partial_\nu y(\cdot, T_0)|_{S_{\text{out}}},$$

where y is the solution to (16). Then we have the following.

Theorem 8 (Kian, Oksanen, Soccorsi, and Yamamoto [23]). *We assume that either of the following conditions is fulfilled:*

1. $\rho_1 = \rho_2$ *in* Ω *and* $\nabla a_1 = \nabla a_2$ *on* $\partial\Omega$.
2. $a_1 = a_2$ *in* Ω *and there exists a constant* $C_0 > 0$ *such that* $|\rho_1(x) - \rho_2(x)| \le C_0|\operatorname{dist}(x, \partial\Omega)|^2$ *for* $x \in \Omega$.

3. *$V_1 = V_2$ in Ω, $\nabla a_1 = \nabla a_2$ on $\partial\Omega$, and there exists a constant $C_0 > 0$ such that $|\rho_1(x) - \rho_2(x)| \le C_0|\mathrm{dist}\,(x, \partial\Omega)|^2$ for $x \in \Omega$.*

Then

$$\Lambda_{a_1,\rho_1,V_1,T_0} h = \Lambda_{a_2,\rho_2,V_2,T_0} h \quad \text{on } S_{\mathrm{out}}$$

for all $h \in C_0(S_{\mathrm{in}} \times (0, T_0)) \cap C^2([0, T]; H^{\frac{3}{2}}(\partial\Omega))$ implies $(a_1, \rho_1, V_1) = (a_2, \rho_2, V_2)$ in Ω.

Thus the Dirichlet-to-Neumann map can identify at most two coefficients in the elliptic part under some additional information on ∇a_k on $\partial\Omega$ or $\rho_1 - \rho_2$ in Ω. See Canuto and Kavian [7] for the case of $\alpha = 1$. The proof of the theorem is similar to [7] and one essential step is the eigenfunction expansion of y and so it is very difficult to discuss the inverse coefficient problem of determining also $A(x)$ for a nonsymmetric equation $\rho\partial_t^\alpha y = \mathrm{div}\,(a\nabla y) + A(x) \cdot \nabla y + V(x)y$.

In view of degrees of freedom, we compare Theorems 7 and 8.

In Theorem 7

(a) Degree of freedom of unknowns: d
More precisely, each unknown function depends on d spatial variables $x_1, \dots, x_d$.

(b) Degree of freedom of data set: $(d - 1) + d = 2d - 1$
More precisely, inputs are a set of h's, each of which depends on $(d - 1)$-spatial variables, and the output is Neumann data on $\partial\Omega \times (0, T)$, that is, $(d - 1) + 1 = d$. Therefore, the degree of freedom is one of the product set of the inputs and the outputs sets, so that it is $2d - 1$.

Therefore, our inverse problem is overdetermining for $d \ge 2$ because $d < 2d - 1$.

In Theorem 8

(a) Degree of freedom of unknowns: d

(b) Degree of freedom of data: $d + (d - 1) = 2d - 1$.
More precisely, input is Dirichlet data on $\partial\Omega \times (0, T)$, that is, the degree is $(d-1)+1 = d$, while the output is the corresponding Neumann data at fixed time T_0, that is, $d - 1$.

Therefore, the formulation is equally overdetermining.

So far, we assume that the order α of the derivative is constant and is mainly in $(0, 1)$. However, in some complex media, the presence of heterogeneous regions may cause variations of the permeability in different spatial positions, and in this case, the variable order time-fractional model is more relevant for describing the diffusion process (e. g., Sun, Chen and Chen[51]).

Let us turn to considering an inverse problem of determining the variable order in x and related coefficients. We introduce several necessary notation and settings. We are given constants $0 < \alpha_0 < \alpha_M$, $0 < \rho_0 < \rho_M$ and two functions $\alpha \in L^\infty(\Omega)$ and

$\rho \in L^\infty(\Omega)$ satisfying

$$0 < \alpha_0 \leq \alpha(x) \leq \alpha_M < 1 \quad \text{and} \quad 0 < \rho_0 \leq \rho(x) \leq \rho_M < \infty, \quad x \in \Omega. \tag{17}$$

We consider an initial-boundary value problem for a space-dependent variable order fractional diffusion equation

$$\begin{cases} \rho(x)\partial_t^{\alpha(x)} y(x,t) = \Delta y(x,t) + q(x)y(x,t), & x \in \Omega,\ t > 0, \\ y(x,0) = a(x), & x \in \Omega, \\ y(x,t) = \lambda(t)h(x), & x \in \partial\Omega,\ t > 0 \end{cases} \tag{18}$$

with suitable h. Henceforth, by y_h we denote a unique solution in $C([0,\infty); H^2(\Omega))$ to (18). Given nonempty relatively open sub-boundaries S_{in} and S_{out} of $\partial\Omega$, we introduce the following boundary operator:

$$\Lambda(\alpha,\rho,q)(t) : h \in \mathcal{H}_{\text{in}} \mapsto \partial_\nu y_h(\cdot,t)|_{S_{\text{out}}}, \quad 0 < t < \infty,$$

where

$$\mathcal{H}_{\text{in}} := \{h \in H^{3/2}(\partial\Omega);\ \operatorname{supp} h \subset \overline{S_{\text{in}}}\}.$$

Kian, Soccorsi, and Yamamoto [24] analysed the uniqueness in the inverse problem of determining simultaneously the variable order α, coefficients ρ and q of the diffusion equation in (18) from the knowledge of the boundary operators $\{\Lambda(\alpha,\rho,q)(t_n);\ n \in \mathbb{N}\}$ associated with a time sequence t_n, $n \in \mathbb{N}$ fulfilling

$$\text{the set } \{t_n;\ n \in \mathbb{N}\} \text{ has an accumulation point in } (0,\infty). \tag{19}$$

Moreover, Ω, S_{in}, and S_{out} are assumed to satisfy the following conditions.

Case I: spatial dimension $d = 2$.
It is required that $\partial\Omega$ is composed of a finite number of smooth closed contours. In this case, choose $S_{\text{in}} = S_{\text{out}} =: \gamma$, where γ is any arbitrary nonempty relatively open subset of $\partial\Omega$, and the set of admissible unknown coefficients reads

$$\mathcal{U} := \{(\alpha,\rho,q);\ \alpha,\rho \in W^{1,r}(\Omega) \text{ fulfill (17) and } q > 0, \in W^{1,r}(\Omega)\},$$

where $r \in (2,\infty)$.

Case II: $d \geq 3$.
Let $x_0 \in \mathbb{R}^d$ outside the convex hull of $\overline{\Omega}$, and assume that

$$\{x \in \partial\Omega;\ (x - x_0)\cdot\nu \geq 0\} \subset S_{\text{in}} \quad \text{and} \quad \{x \in \partial\Omega;\ (x - x_0)\cdot\nu \leq 0\} \subset S_{\text{out}}$$

Furthermore, we define the set of admissible unknown coefficients by

$$\mathcal{V} := \{(\alpha,\rho,q);\ \alpha \in L^\infty(\Omega),\ \rho \in L^\infty(\Omega) \text{ fulfill (17) and } q > 0, \in L^\infty(\Omega)\}.$$

The uniqueness results for the inverse coefficients problem are as follows.

Theorem 9 (Kian, Soccorsi, and Yamamoto [24]). *Let t_n, $n \in \mathbb{N}$ fulfill* (19). *We assume Case I or II. Let*

$$(\alpha_i, \rho_i, q_i) \in \mathcal{U}, \quad i = 1, 2 \quad \textit{in Case I}$$

and

$$(\alpha_i, \rho_i, q_i) \in \mathcal{V}, \quad i = 1, 2 \quad \textit{in Case II.}$$

If

$$\Lambda(\alpha_1, \rho_1, q_1)(t_n) = \Lambda(\alpha_2, \rho_2, q_2)(t_n), \quad n \in \mathbb{N},$$

then $(\alpha_1, \rho_1, q_1) = (\alpha_2, \rho_2, q_2)$.

4 Other related inverse problems

4.1 Determination of time varying coefficients

For

$$\begin{cases} \partial_t^\alpha y(x,t) - a(t)\partial_x^2 y(x,t) = 0, & 0 < x < 1,\ 0 < t < T, \\ y(x,0) = 0, & 0 < x < 1, \\ y(0,t) = h(t),\ y(1,t) = 0, & 0 < t < T, \end{cases}$$

where $\alpha \in (0,1)$ and $T > 0$, h are given, the article Zhang [67] proved the uniqueness for an inverse problem of recovering $a(t)$ from additional boundary data

$$-a(t)\partial_x y(0,t), \quad 0 < t < T.$$

Moreover, we can refer to Fujishiro and Kian [11]. In the case where unknown coefficients are dependent only on t, we can expect that the pointwise data of the solution at monitoring points over a time interval can guarantee the stability as well as the uniqueness. Indeed we often reduce such an inverse problem to a Volterra equation of the second kind.

Janno [16] discussed an inverse problem of determining an order and a time varying function in an integral term, while Wang and Wu [55] studied an inverse problem of determining two time varying functions in an integral term and a source term.

4.2 Determination of nonlinear terms

Luchko, Rundell, Yamamoto, and Zuo [36] considers

$$\begin{cases} \partial_t^\alpha y(x,t) = \Delta y(x,t) + f(y(x,t)), & x \in \Omega \subset \mathbb{R}^d,\ 0 < t < T, \\ \partial_\nu y(x,t) = h(x,t), & x \in \partial\Omega,\ 0 < t < T, \\ y(x,0) = a_0, & x \in \Omega, \end{cases}$$

where a_0 is a constant. The article [36] discussed the determination of semilinear term f by extra data $y(x_0,t)$, $0 < t < T$ at a fixed point $x_0 \in \overline{\Omega}$ to establish a uniqueness result for the inverse problem within suitable admissible sets of f's and provided a numerical method for reconstructing f.

Rundell, Xu, and Zuo [47] studied a similar inverse problem of determining a non-linear term f in the boundary condition $-\partial_x y(1,t) = f(y(1,t))$, $0 < t < T$ for the one-dimensional fractional diffusion equation. See Tatar and Ulusoy [53] for an optimization method for reconstructing a in

$$\partial_t^\alpha y(x,t) = \operatorname{div}\big(a(y(x,t))\nabla y(x,t)\big) + F(x,t), \quad x \in \Omega \subset \mathbb{R}^d,\ 0 < t < T.$$

Finally, we note that Janno and Kasemets [17] considered an initial-boundary value problem for

$$\partial_t^\alpha y(x,t) = \sum_{i,j=1}^{d} \partial_i\big(a_{ij}(x)\partial_j y\big) + \sum_{j=1}^{d} b_j(x)\partial_j y + p\big(y(x,t),x,t\big)g(x) + q\big(y(x,t),x,t\big)$$

for $x \in \Omega$ and $0 < t < T$, and established the uniqueness in determining $g(x)$ by extra data

$$\int_0^T y(x,t)\mu(t)dt$$

with suitable weight $\mu(t)$.

Lopushanska and Rapita [35] discussed an inverse problem of determining $r(t)$ in a semilinear fractional telegraph equation

$$\partial_t^\alpha y + r(t)\partial_t^\beta y = \Delta y + F\big(x,t,y,\partial_t^\beta y\big)$$

where $1 < \alpha < 2$ and $0 < \beta < 1$ by means of data

$$\int_\Omega y(x,t)w(x)dx$$

with some weight function $w(x)$. See also Ismailov and Çiçek [15] for a similar inverse problem.

5 Numerical works

The literature on numerics increases up rapidly for inverse coefficient problems. Here, we refer to only a few papers: Bondarenko and Ivaschenko [4], Li, Zhang, Jia, and Yamamoto [27], Sun, Li, and Jia [50], Sun and Wei [52]. The publications so far are limited to individual inverse problems, and we can expect more comprehensive numerical works.

Next, we briefly survey numerical works for other types of inverse problems.

5.1 Lateral Cauchy problems

In the lateral Cauchy problem, we are requested to determine $y(x,t)$ satisfying

$$\begin{cases} \partial_t^\alpha y(x,t) = \Delta y(x,t), & x \in \Omega \subset \mathbb{R}^d,\ 0 < t < T, \\ y(x,t) = h_0(x,t), & \\ \partial_\nu y(x,t) = h_1(x,t), & x \in \Gamma \subset \partial\Omega,\ 0 < t < T \end{cases} \tag{20}$$

with given h_0, h_1. Here, $\Gamma \subset \partial\Omega$ is a sub-boundary and the initial values are also unknown. The lateral Cauchy problem is closely related to the unique continuation which is expected to be proved by Carleman estimates. Various numerical methods are available: Li, Xi, and Xiong [28], Murio [41–43], Qian [45], Xiong, Zhou, and Hon [61], Zheng and Wei [68–71].

We point out that theoretical researches for the uniqueness and the conditional stability in determining y satisfying (20) are not satisfactorily made.

5.2 Backward problems in time

We consider

$$\begin{cases} \partial_t^\alpha y(x,t) = \sum_{i,j=1}^d \partial_i(a_{ij}(x)\partial_j y), & x \in \Omega,\ 0 < t < T, \\ y(x,t) = 0, & x \in \partial\Omega,\ 0 < t < T. \end{cases} \tag{21}$$

Here, $a_{ij} = a_{ji} \in C^1(\overline{\Omega})$ and there exists a constant $\mu_0 > 0$ such that $\sum_{i,j=1}^d a_{ij}(x)\xi_i\xi_j \ge \mu_0 \sum_{j=1}^d \xi_j^2$ for all $x \in \overline{\Omega}$ and $\xi_1, \dots, \xi_d \in \mathbb{R}$.

We can discuss a more general elliptic operator, but it suffices to consider (21).

We consider the following.

Backward problem in time
Given $b \in L^2(\Omega)$, solve $y(x,t)$, $x \in \Omega$, $0 < t < T$ satisfying (21) and

$$y(x,T) = b(x), \quad x \in \Omega. \tag{22}$$

In the case of $\alpha = 1$, it is well known that this problem has no solutions in general, and is ill-posed. However, in the case of $0 < \alpha < 1$, the situation is completely different, and Sakamoto and Yamamoto [49] proves that for given $b \in H^2(\Omega) \cap H_0^1(\Omega)$ there exists a unique solution $y \in L^2(0,T;H^2(\Omega) \cap H_0^1(\Omega)) \cap C([0,T];L^2(\Omega))$ satisfying (21) and (22), and there exist constants $C_1, C_2 > 0$ such that

$$C_1 \|b\|_{H^2(\Omega)} \le \|y(\cdot,0)\|_{L^2(\Omega)} \le C_2 \|b\|_{H^2(\Omega)}$$

for all $b \in H^2(\Omega) \cap H_0^1(\Omega)$.

That is, the backward problem in time for fractional diffusion equations with order $\alpha \in (0,1)$ is well-posed in the sense of Hadamard if we strengthen the regularity of data $y(\cdot,T)$ in $H^2(\Omega) \cap H_0^1(\Omega)$, which means that the smoothing effect is with only 2 by order of Sobolev spaces. The smoothing in the case of $0 < \alpha < 1$ is much weaker than the case of $\alpha = 1$. Thus we can expect more stable reconstruction of initial value by means of the final value $y(\cdot,T)$. In the case of $\alpha = 1$, we can prove an inequality of Carleman type where the weight is given in the form $e^{\lambda t}$ with large constant $\lambda > 0$, and establish the conditional stability for more general parabolic equations. However, for fractional partial differential equations, again by the lack of convenient formulae of integration by parts, we cannot prove such a Carleman estimate, so that the theoretical researches for the backward problem are very restricted.

We refer to numerical works:

Liu and Yamamoto [34], Tuan, Long, and Tatar [54], Wang and Liu [57, 58], Wang, Wei, and Zhou [56], Wei and Wang [59], Xiong, Wang, and Li [60], Yang and Liu [65].

Bibliography

[1] E. Bazhlekova, *Fractional Evolution Equations in Banach Spaces*, Doctoral thesis, University Press Facilities, Eindhoven University of Technology, 2001. DOI:10.6100/IR549476.

[2] L. Beilina and M. V. Klibanov, *Approximate Global Convergence and Adaptivity for Coefficient Inverse Problems*, Springer-Verlag, Berlin, 2012.

[3] M. Bellassoued and M. Yamamoto, *Carleman Estimates and Applications to Inverse Problems for Hyperbolic Systems*, Springer Japan, Tokyo, 2017.

[4] A. N. Bondarenko and D. S. Ivaschenko, Numerical methods for solving inverse problems for time fractional diffusion equation with variable coefficient, *J. Inverse Ill-Posed Probl.*, **17** (2009), 419–440.

[5] A. Bukhgeim, Recovering the potential from Cauchy data in two dimensions, *J. Inverse Ill-Posed Probl.*, **16** (2008), 19–34.

[6] A. Bukhgeim and M. V. Klibanov, Global uniqueness of class of multidimensional inverse problems, *Sov. Math. Dokl.*, **24** (1981), 244–247.

[7] B. Canuto and O. Kavian, Determining coefficients in a class of heat equations via boundary measurements, *SIAM J. Math. Anal.*, **32** (2001), 963–986.

[8] J. Cheng, C.-L. Lin, and G. Nakamura, Unique continuation property for the anomalous diffusion and its application, *J. Differ. Equ.*, **254** (2013), 3715–3728.

[9] J. Cheng, J. Nakagawa, M. Yamamoto, and T. Yamazaki, Uniqueness in an inverse problem for a one dimensional fractional diffusion equation, *Inverse Probl.*, **25** (2009), 115002.

[10] J. J. Duderstadt and W. R. Martin, *Transport Theory*, Wiley, New York, 1979.

[11] K. Fujishiro and Y. Kian, Determination of time dependent factors of coefficients in fractional diffusion equations, *Math. Control Relat. Fields*, **6** (2017), 251–269.

[12] R. Gorenflo, Y. Luchko, and M. Yamamoto, Time-fractional diffusion equation in the fractional Sobolev spaces, *Fract. Calc. Appl. Anal.*, **18** (2015), 799–820.

[13] O. Imanuvilov and M. Yamamoto, Uniqueness for inverse boundary value problems by Dirichlet-to-Neumann map on subboundaries, *Milan J. Math.*, **81** (2013), 187–258.

[14] V. Isakov, *Inverse Problems for Partial Differential Equations*, Springer-Verlag, Berlin, 2006.

[15] M. I. Ismailov and M. Çiçek, Inverse source problem for a time-fractional diffusion equation with nonlocal boundary conditions, *Appl. Math. Model.*, **40** (2016), 4891–4899.

[16] J. Janno, Determination of the order of fractional derivative and a kernel in an inverse problem for a generalized time fractional diffusion equation, *Electr. J. Differ. Equ.*, **199** (2016), 1–28.

[17] J. Janno and K. Kasemets, Uniqueness for an inverse problem for a semilinear time-fractional diffusion equation, *Inverse Probl. Imaging*, **11** (2017), 125–149.

[18] B. Jin and W. Rundell, An inverse problem for a one-dimensional time-fractional diffusion problem, *Inverse Probl.*, **28** (2012), 75010–75028 (19 pp).

[19] B. Jin and W. Rundell, A tutorial on inverse problems for anomalous diffusion processes, *Inverse Probl.*, **31** (2015), 035003 (40 pp).

[20] A. Kawamoto, Hölder stability estimate in an inverse source problem for a first and half-order time fractional diffusion equation, *Inverse Probl. Imaging*, **12** (2018), 315–330.

[21] A. Kawamoto, Lipschitz stability estimates in inverse source problems for a fractional diffusion equation of half order in time by Carleman estimates, *J. Inverse Ill-Posed Probl.*, **26** (2018), 647–672.

[22] A. Kawamoto and M. Machida, Global Lipschitz stability for a fractional inverse transport problem by Carleman estimates, arXiv preprint arXiv:1608.07914, 2016.

[23] Y. Kian, L. Oksanen, E. Soccorsi, and M. Yamamoto, Global uniqueness in an inverse problem for time fractional diffusion equations, *J. Differ. Equ.*, **264** (2018), 1146–1170.

[24] Y. Kian, E. Soccorsi, and M. Yamamoto, On time-fractional diffusion equations with space-dependent variable order, *Ann. Henri Poincaré*, Online First (2018), https://doi.org/10.1007/s00023-018-0734-y.

[25] M. V. Klibanov, Inverse problems and Carleman estimates, *Inverse Probl.*, **8** (1992), 575–596.

[26] A. Kubica and M. Yamamoto, Initial-boundary value problems for fractional diffusion equations with time-dependent coefficients, *Fract. Calc. Appl. Anal.*, **21** (2018), 276–311.

[27] G. Li, D. Zhang, X. Jia, and M. Yamamoto, Simultaneous inversion for the space-dependent diffusion coefficient and the fractional order in the time-fractional diffusion equation, *Inverse Probl.*, **29** (2013), 065014 (36 pp).

[28] M. Li, X.-X. Xi, and X.-T. Xiong, Regularization for a fractional sideway heat equation, *J. Comput. Appl. Math.*, **255** (2014), 28–43.

[29] Z. Li, X. Huang, and M. Yamamoto, Carleman estimates for the time-fractional advection-diffusion equations and applications, arXiv preprint arXiv:1704.06011, 2017.

[30] Z. Li, O. Y. Imanuvilov, and M. Yamamoto, Uniqueness in inverse boundary value problems for fractional diffusion equations, *Inverse Probl.*, **32** (2016), 015004.

[31] Z. Li and M. Yamamoto, Unique continuation for a one-dimensional time-fractional diffusion equation, arXiv preprint arXiv:1806.06164, 2018.

[32] C.-L. Lin and G. Nakamura, Unique continuation property for anomalous slow diffusion equation, *Commun. Partial Differ. Equ.*, **41** (2016), 749–758.

[33] C.-L. Lin and G. Nakamura, Unique continuation property for multi-terms time fractional diffusion equations, *Math. Ann.* (2 July 2018), https://doi.org/10.1007/s00208-018-1710-z.

[34] J.-J. Liu and M. Yamamoto, A backward problem for the time-fractional diffusion equation, *Appl. Anal.*, **89** (2010), 1769–1788.

[35] H. Lopushanska and V. Rapita, Inverse coefficient problem for the semi-linear fractional telegraph equation, *Electr. J. Differ. Equ.*, **2015** (2015), 1–13.
[36] Y. Luchko, W. Rundell, M. Yamamoto, and L. Zuo, Uniqueness and reconstruction of an unknown semilinear term in a time-fractional reaction-diffusion equation, *Inverse Probl.*, **29** (2013), 065019.
[37] M. Machida, The time-fractional radiative transport equation-Continuous-time random walk, diffusion approximation, and Legendre-polynomial expansion, *J. Math. Phys.*, **58** (2017), 013301.
[38] L. Miller and M. Yamamoto, Coefficient inverse problem for a fractional diffusion equation, *Inverse Probl.*, **29** (2013), 075013.
[39] R. Murayama, The Gel'fand-Levitan theory and certain inverse problems for the parabolic equation, *J. Fac. Sci., Univ. Tokyo, Sect. 1A, Math.*, **28** (1981), 317–330.
[40] R. Murayama and T. Suzuki, A uniqueness theorem in an identification problem for coefficients of parabolic equations, *Proc. Jpn. Acad.*, **56** (1980), 259–263.
[41] D. A. Murio, Stable numerical solution of a fractional-diffusion inverse heat conduction problem, *Comput. Math. Appl.*, **53** (2007), 1492–1501.
[42] D. A. Murio, Time fractional IHCP with Caputo fractional derivative, *Comput. Math. Appl.*, **56** (2008), 2371–2381.
[43] D. A. Murio, Stable numerical evaluation of Grünwald-Letnikov fractional derivatives applied to a fractional IHCP, *Inverse Probl. Sci. Eng.*, **17** (2009), 229–243.
[44] A. Pierce, Unique identification of eigenvalues and coefficients in a parabolic problem, *SIAM J. Control Optim.*, **17** (1979), 494–499.
[45] Z. Qian, Optimal modified method for a fractional-diffusion inverse heat conduction problem, *Inverse Probl. Sci. Eng.*, **18** (2010), 521–533.
[46] C. Ren and X. Xu, Local stability for an inverse coefficient problem of a fractional diffusion equation, *Chin. Ann. Math.*, **35B** (2014), 429–446.
[47] W. Rundell, X. Xu, and L. Zuo, The determination of an unknown boundary condition in a fractional diffusion equation, *Appl. Anal.*, **92** (2013), 1511–1526.
[48] W. Rundell and M. Yamamoto, Recovery of a potential in a fractional diffusion equation, arXiv preprint arXiv:1811.05971.
[49] K. Sakamoto and M. Yamamoto, Initial value/boundary value problems for fractional diffusion-wave equations and applications to some inverse problems, *J. Math. Anal. Appl.*, **382** (2011), 426–447.
[50] C. Sun, G. Li, and X. Jia, Simultaneous inversion for the diffusion and source coefficients in the multi-term TFDE, *Inverse Probl. Sci. Eng.*, **25** (2017), 1618–1638.
[51] H. G. Sun, W. Chen, and Y. Q. Chen, Variable-order fractional differential operators in anomalous diffusion modeling, *Physica, A Stat. Mech. Appl.*, **388** (2009), 4586–4592.
[52] L. Sun and T. Wei, Identification of the zeroth-order coefficient in a time fractional diffusion equation, *Appl. Numer. Math.*, **111** (2017), 160–180.
[53] A. Tatar and S. Ulusoy, An inverse problem for a nonlinear diffusion equation with time-fractional derivative, *J. Inverse Ill-Posed Probl.*, **25** (2017), 185–193.
[54] N. H. Tuan, L. D. Long, and S. Tatar, Tikhonov regularization method for a backward problem for the inhomogeneous time-fractional diffusion equation, *Appl. Anal.*, **97** (2018), 842–863.
[55] H. Wang and B. Wu, On the well-posedness of determination of two coefficients in a fractional integrodifferential equation, *Chin. Ann. Math.*, **35B** (2014), 447–468.
[56] J.-G. Wang, T. Wei, and Y.-B. Zhou, Tikhonov regularization method for a backward problem for the time-fractional diffusion equation, *Appl. Math. Model.*, **37** (2013), 8518–8532.
[57] L. Wang and J.-J. Liu, Data regularization for a backward time-fractional diffusion problem, *Comput. Math. Appl.*, **64** (2012), 3613–3626.

[58] L. Wang and J.-J. Liu, Total variation regularization for a backward time-fractional diffusion problem, *Inverse Probl.*, **29** (2013), 115013.
[59] T. Wei and J.-G. Wang, A modified quasi-boundary value method for the backward time-fractional diffusion problem, *ESAIM: Math. Model. Numer. Anal.*, **48** (2014), 603–621.
[60] X.-T. Xiong, J.-X. Wang, and M. Li, An optimal method for fractional heat conduction problem backward in time, *Appl. Anal.*, **91** (2012), 823–840.
[61] X.-T. Xiong, Q. Zhou, and Y. C. Hon, An inverse problem for fractional diffusion equation in 2-dimensional case: Stability analysis and regularization, *J. Math. Anal. Appl.*, **393** (2012), 185–199.
[62] X. Xu, J. Cheng, and M. Yamamoto, Carleman estimate for a fractional diffusion equation with half order and application, *Appl. Anal.*, **90** (2011), 1355–1371.
[63] M. Yamamoto, Carleman estimates for parabolic equations and applications, *Inverse Probl.*, **25** (2009), 123013.
[64] M. Yamamoto and Y. Zhang, Conditional stability in determining a zeroth-order coefficient in a half-order fractional diffusion equation by a Carleman estimate, *Inverse Probl.*, **28** (2012), 105010.
[65] M. Yang and J.-J. Liu, Solving a final value fractional diffusion problem by boundary condition regularization, *Appl. Numer. Math.*, **66** (2013), 45–58.
[66] R. Zacher, Weak solutions of abstract evolutionary integro-differential equations in Hilbert spaces, *Funkc. Ekvacioj*, **52** (2009), 1–18.
[67] Z. Zhang, An undetermined coefficient problem for a fractional diffusion equation, *Inverse Probl.*, **32** (2016), 015011.
[68] G. Zheng and T. Wei, Spectral regularization method for a Cauchy problem of the time fractional advection-dispersion equation, *J. Comput. Appl. Math.*, **233** (2010), 2631–2640.
[69] G. Zheng and T. Wei, A New regularization method for the time fractional inverse advection-dispersion problem, *SIAM J. Numer. Anal.*, **49** (2011), 1972–1990.
[70] G. Zheng and T. Wei, A new regularization method for solving a time-fractional inverse diffusion problem, *J. Math. Anal. Appl.*, **378** (2011), 418–431.
[71] G. Zheng and T. Wei, A new regularization method for a Cauchy problem of the time fractional diffusion equation, *Adv. Comput. Math.*, **36** (2012), 377–398.
[72] A. Kawamoto and M. Machida, Lipschitz stability in inverse source and inverse coefficient problems for a first- and half-order time-fractional diffusion equation, arXiv preprint arXiv:1811.06223.

Carlos Lizama

Abstract linear fractional evolution equations

Abstract: We review the fundamental theory of solution operators associated to abstract linear fractional evolution equations. We provide their basic results concerning generation, analyticity, and inversion. We show the subordination principle as well as perturbation, approximation, ergodicity, and compactness properties. The immediate norm continuity is also analyzed. We show how this functional-operator machinery can be applied to solve the fractional-order Cauchy problem. We finish this review with some comments and open problems.

Keywords: Fractional resolvents, abstract Cauchy problem, subordination principle, generation, perturbation, approximation

MSC 2010: 35R11, 26A33, 47A58, 47B07

1 Fractional resolvents

1.1 The theory of α-resolvents

Let Y be a Banach space. For a vector-valued function $f : \mathbb{R}_+ \to Y$, we recall that the Riemann–Liouville fractional integral of order $\beta \geq 0$ is defined by

$$J_t^\beta f(t) = (g_\beta * f)(t) := \int_0^t g_\beta(t-s)f(s)ds,$$

where $g_0(t) := \delta(t)$, the Dirac delta, and $g_\beta(t) := \frac{t^{\beta-1}}{\Gamma(\beta)}$ for $t > 0$. We begin with a purely algebraic notion of the theory of α-resolvents of bounded and linear operators essentially due to Chen and Li [13, Definition 3.1].

Definition 1.1. Let X be a Banach space and $\alpha > 0$. A one parameter family $\{S_\alpha(t)\}_{t\geq 0}$ of bounded linear operators from X to X is called an α-resolvent family if the following conditions are satisfied:
(a) $S_\alpha(0) = I$;
(b) $S_\alpha(s)S_\alpha(t) = S_\alpha(t)S_\alpha(s)$ for all $s, t \geq 0$;
(c) The functional equation

$$S_\alpha(s)J_t^\alpha S_\alpha(t) - J_s^\alpha S_\alpha(s)S_\alpha(t) = J_t^\alpha S_\alpha(t) - J_s^\alpha S_\alpha(s),$$

holds for all $t, s \geq 0$.

Carlos Lizama, Universidad de Santiago de Chile, Departamento de Matemática y Ciencia de la Computación, Las Sophoras 173, Estación Central, Santiago, Chile, e-mail: carlos.lizama@usach.cl

https://doi.org/10.1515/9783110571660-021

The integrals in (c) are understood strongly in the sense of Bochner. We observe the remarkable fact that in the scalar case, that is, $X = \mathbb{C}$, we have that the Mittag-Leffler function $E_\alpha(zt^\alpha)$, $z \in \mathbb{C}$, satisfies the functional equation (c); see [52, Example 3.10]. In particular, it shows that the functional equation (c) is a proper generalization of Cauchy's functional equation (corresponding to the case $\alpha = 1$) and the D'Alembert functional equation (corresponding to the case $\alpha = 2$), because for $\alpha = 1$ we have $E_1(zt) = e^{zt}$, and for $\alpha = 2$ we have $E_2(zt^2) = \cosh(\sqrt{z}\,t)$, $z \in \mathbb{C}$.

Remark 1.2. For $0 < \alpha < 1$, an equivalent functional equation to (c) was proposed by Peng and Li [68, 69]; see [52, Remark 3.11]. Other equivalent representation involving the sum $S_\alpha(t+s)$ appears in [63]. A better comprehension of this functional equations in the context of the double Laplace transform and the connection with the problem of extension from local to global can be found in the reference [1].

Definition 1.3. An α-resolvent family $\{S_\alpha(t)\}_{t\geq 0}$ is called uniformly continuous if

$$\lim_{t\to s}\|S_\alpha(t) - S_\alpha(s)\| = 0, \tag{1.1}$$

for every $s \geq 0$.

The linear operator A defined by

$$D(A) := \left\{x \in X \,:\, \lim_{t\to 0^+} \frac{S_\alpha(t)x - x}{t^\alpha} \text{ exists}\right\} \tag{1.2}$$

and

$$Ax := \Gamma(\alpha+1) \lim_{t\to 0^+} \frac{S_\alpha(t)x - x}{t^\alpha} \quad \text{for } x \in D(A) \tag{1.3}$$

is called the *generator* of the α-resolvent family $\{S_\alpha(t)\}_{t\geq 0}$, $D(A)$ is the domain of A. This formula was proposed by the first time in [7, Proposition 3.1].

Theorem 1.4 ([55, Lemma 3.1 and Lemma 3.2]). *Let $\alpha > 0$. A linear operator A is the generator of a uniformly continuous α-resolvent family if and only if A is a bounded operator.*

We remark that in certain Banach spaces, like L^∞, the uniform continuity of an α-resolvent family is automatic [46, Theorem 3.2]. Observe that if A is a bounded operator, then

$$S_\alpha(t) := \sum_{n=0}^{\infty} g_{\alpha n+1}(t)A^n = \sum_{n=0}^{\infty} \frac{A^n t^{\alpha n}}{\Gamma(\alpha n+1)} = E_\alpha(At^\alpha), \quad t \geq 0, \tag{1.4}$$

defines a uniformly continuous α-resolvent family. Therefore, α-resolvent families are, in some sense, abstract versions of Mittag-Leffler functions.

Definition 1.5. An α-resolvent family of bounded linear operators $\{S_\alpha(t)\}_{t\geq 0}$ on X is called a strongly continuous α-resolvent family if $\lim_{t\to s}\|S_\alpha(t)x - S_\alpha(s)x\| = 0$ for all $s \geq 0$ and every $x \in X$.

The following characterization is sometimes used as starting point for their definition.

Theorem 1.6 ([13, Theorem 3.4], [52, Theorem 3.1 and Theorem 4.1]). *Let $\alpha > 0$. Let A be a linear operator in X with domain $D(A)$. A strongly continuous family $\{S_\alpha(t)\}_{t\geq 0} \subset \mathcal{B}(X)$ of bounded linear operators in X is an α-resolvent family generated by A if and only if the following conditions are satisfied:*
(i) $S_\alpha(0) = I$;
(ii) $S_\alpha(t)x \in D(A)$ *and* $S_\alpha(t)Ax = AS_\alpha(t)x$ *for all* $x \in D(A)$ *and* $t \geq 0$;
(iii) $S_\alpha(t)x = x + \int_0^t g_\alpha(t-s)AS_\alpha(s)xds$, $t \geq 0$, $x \in D(A)$.

This characterization shows that α-resolvent families are particular cases of the theory of resolvent families associated to Volterra integral equations, introduced by J. Prüss [71, Chapter 1, Section 1]. The following properties can be found in [49, Lemma 2.2], [52, Theorem 4.1] and [13, Proposition 3.3], for instance.

Proposition 1.7. *Let $\{S_\alpha(t)\}_{t\geq 0}$ be an strongly continuous α-resolvent family and let A be its generator. Then:*
(i) *For all* $x \in X$, $\int_0^t g_\alpha(t-s)S_\alpha(s)xds \in D(A)$ *and*

$$S_\alpha(t)x = x + A\int_0^t g_\alpha(t-s)S_\alpha(s)xds, \quad t \geq 0, \quad x \in X. \tag{1.5}$$

(ii) $D(A)$, *the domain of A, is dense in X and A is a closed operator.*

A simple application is the following generalization of the Kallmann–Rota inequality [51, Theorem 3.2].

Corollary 1.8. *Let A be the generator of a bounded and strongly continuous α-resolvent family $\{S_\alpha(t)\}_{t\geq 0}$, that is, satisfying $\|S_\alpha(t)\| \leq M$ for all $t \geq 0$. If $x \in D(A^2)$, then*

$$\|Ax\|^2 \leq 8M^2\frac{\Gamma(\alpha+1)^2}{\Gamma(2\alpha+1)}\|x\|\|A^2x\|.$$

1.2 Integrated α-resolvents

The concept of n-times integrated resolvents ($n \in \mathbb{N}_0$) was introduced by Arendt and Kellermann [5, Definition 1.1], after previous work of W. Arendt and M. Hieber on β-times integrated semigroups ($\beta > 0$). Following [52] (see also [13, Definition 3.7]), the definition of β-times integrated α-resolvent families can be introduced as follows.

Definition 1.9. Let X be a Banach space and $\alpha > 0$, $\beta > 0$. A one parameter family $\{R_{\alpha,\beta}(t)\}_{t\geq 0} \subset \mathcal{B}(X)$ is called an (α,β)-resolvent family if the following conditions are satisfied:

(a) $\lim_{t\to 0} t^{1-\beta}R_{\alpha,\beta}(t) = \frac{1}{\Gamma(\beta)}I$ if $0 < \beta < 1$, $R_{\alpha,1}(0) = I$ and $R_{\alpha,\beta}(0) = 0$ if $\beta > 1$.
(b) $R_{\alpha,\beta}(s)R_{\alpha,\beta}(t) = R_{\alpha,\beta}(t)R_{\alpha,\beta}(s)$ for all $s, t > 0$;
(c) The functional equation

$$R_{\alpha,\beta}(s)J_t^\alpha R_{\alpha,\beta}(t) - J_s^\alpha R_{\alpha,\beta}(s)R_{\alpha,\beta}(t) = g_\beta(s)J_t^\alpha R_{\alpha,\beta}(t) - g_\beta(t)J^\alpha R_{\alpha,\beta}(s),$$

holds for all $t, s > 0$.

For the development of many properties, it is important to observe that the notion of (α,β)-resolvent families is included into the theory of (a,k)-regularized families [49] with $a(t) = g_\alpha(t)$ and $k(t) = g_\beta(t)$.

Of course, $(\alpha, 1)$-resolvent families are α-resolvent families. For $0 < \alpha = \beta < 1$, the above definition was studied by Li and Peng [39]. See also Li, Peng and Jia [40]. We note that this concept was introduced earlier [4], but without reference to the condition near to zero given in (a).

Remark 1.10. An equivalent identity to (c) in the spirit of Remark 1.2 has been proved in [44, Theorem 5].

Remark 1.11 (Notation). In what follows, an (α,α)-resolvent family $\{R_{\alpha,\alpha}(t)\}_{t>0}$ will be simply denoted by $\{R_\alpha(t)\}_{t>0}$ and an $(\alpha,1)$-resolvent family by $\{S_\alpha(t)\}_{t\geq 0}$.

The linear operator A defined by

$$D(A) := \left\{x \in X \,:\, \lim_{t\to 0^+} \frac{R_{\alpha,\beta}(t)x - g_\beta(t)x}{g_{\alpha+\beta}(t)} \text{ exists}\right\} \tag{1.6}$$

and

$$Ax := \lim_{t\to 0^+} \frac{R_{\alpha,\beta}(t)x - g_\beta(t)x}{g_{\alpha+\beta}(t)} \quad \text{for } x \in D(A) \tag{1.7}$$

is called the *generator* of the (α,β)-resolvent family $\{R_{\alpha,\beta}(t)\}_{t>0}$.

If A is a bounded operator, then

$$R_{\alpha,\beta}(t) := \sum_{n=0}^{\infty} g_{\alpha n+\beta}(t)A^n = t^{\beta-1}\sum_{n=0}^{\infty} \frac{A^n t^{\alpha n}}{\Gamma(\alpha n+\beta)} = t^{\beta-1}E_{\alpha,\beta}(At^\alpha), \quad t > 0, \tag{1.8}$$

defines a uniformly continuous (α,β)-resolvent family. Given $\beta > 1$, observe that the family $\{R_{\alpha,\beta}(t)\}_{t>0}$ is $(\beta-1)$-times integrated with respect to $\{R_{\alpha,1}(t)\}_{t\geq 0}$ because the identity

$$R_{\alpha,\beta}(t) = g_{\beta-1} * R_{\alpha,1}(t) = J_t^{\beta-1}R_{\alpha,1}(t), \quad t > 0,$$

holds. More generally, if A is the generator of an (α,β)-resolvent family, then for all $\gamma \geq 0$ we have that A is the generator of an $(\alpha,\beta+\gamma)$-resolvent family [49, Remark 2.4 (4)]. The following characterization is often used as definition.

Theorem 1.12 ([52, Theorem 3.1 and Theorem 4.3]). *Let $\alpha > 0$ and $\beta > 0$ be given. A strongly continuous family $\{R_{\alpha,\beta}(t)\}_{t>0} \subset \mathcal{B}(X)$ of bounded linear operators in X is an (α,β)-resolvent family generated by A if and only if the following conditions are satisfied:*
(i) $\lim_{t\to 0} t^{1-\beta} R_{\alpha,\beta}(t) = \frac{1}{\Gamma(\beta)} I$ *if* $0 < \beta < 1$, $R_{\alpha,1}(0) = I$ *and* $R_{\alpha,\beta}(0) = 0$ *if* $\beta > 1$.
(ii) $R_{\alpha,\beta}(t)x \in D(A)$ *and* $R_{\alpha,\beta}(t)Ax = AR_{\alpha,\beta}(t)x$ *for all* $x \in D(A)$ *and* $t \geq 0$;
(iii) $R_{\alpha,\beta}(t)x = g_\beta(t)x + \int_0^t g_\alpha(t-s)AR_{\alpha,\beta}(s)x\,ds$, $t \geq 0$, $x \in D(A)$.

For $\beta > 1$, we have that $D(A)$ is closed, but not necessarily densely defined [13, Proposition 3.10]. In the diagonal case $\alpha = \beta$, this result appears by the first time in [4, Definition 2.3]. If $0 < \alpha = \beta < 1$, then A must be densely defined [39, Theorem 3.1]. We have the following criteria.

Theorem 1.13 ([4, Theorem 2.6]). *Let A be the generator of a strongly continuous cosine family $\{S_2(t)\}_{t\geq 0}$, then for any $\alpha \in [1,2)$, A generates an (α,α)-resolvent family.*

Roughly speaking, the notion of $(\alpha,1)$-resolvent families is associated with the Caputo fractional derivative, whereas the notion of (α,α)-resolvent family is linked with the Riemann–Liouville fractional derivative. Other relevant cases are $(\alpha, \gamma + (1-\gamma)\alpha)$-resolvent families with $0 < \alpha < 1$, $0 \leq \gamma \leq 1$ (see [26]), and $(\alpha, \alpha+\gamma(2-\alpha))$-resolvent families with $1 < \alpha < 2$, $0 \leq \gamma \leq 1$ (see [62]), because they are related with the notion of Hilfer fractional derivative that interpolates between the Caputo and Riemann–Liouville fractional derivative (for $0 < \alpha < 1$, take $\gamma = 1$ and $\gamma = 0$, respectively).

Assuming that A is the generator of an γ-times integrated semigroup ($\gamma \geq 0$), that is, an $(1, \gamma+1)$-resolvent family, then $(\alpha, \alpha\gamma + 1)$-resolvent families and $(\alpha, \alpha(\gamma + 1))$-resolvent families are important for $0 < \alpha < 1$ because these are the key for the treatment of existence, regularity, and representation of fractional diffusion equations; see [30]. The same happens with $(\alpha, \frac{\alpha\gamma}{2} + 1)$-resolvent families and $(\alpha, \alpha(\frac{\gamma}{2} + 1))$-resolvent families for $1 \leq \alpha \leq 2$ because these are present in the theoretical analysis of fractional wave equations; see [31].

Remark 1.14. There are weaker concepts of fractional resolvent operator functions in the literature. For instance, Chen and Li [13, Definition 3.6] introduced the notion of C-regularized resolvent functions. See also the section of comments.

1.3 Generation

A family of bounded and linear operators $\{S(t)\}_{t\geq 0}$ is called exponentially bounded (or of type (M,ω)) if there are constants $M > 0$ and $\omega \in \mathbb{R}$ such that

$$\|S(t)\| \leq Me^{\omega t}, \quad t \geq 0. \tag{1.9}$$

If A is the generator of an α-resolvent family $\{S_\alpha(t)\}_{t\geq 0}$ satisfying (1.9), we write $A \in \mathcal{C}^\alpha(M,\omega)$. Also, set $\mathcal{C}^\alpha(\omega) := \bigcup_{M\geq 0} \mathcal{C}^\alpha(M,\omega)$ and $\mathcal{C}^\alpha := \bigcup_{\omega\geq 0} \mathcal{C}^\alpha(\omega)$.

We observe that in the cases $\alpha = 1$ and $\alpha = 2$ the exponential boundedness is a consequence of the corresponding functional equation.

In the exponentially bounded case, we have the following remarkable result due to Bazhlekova.

Theorem 1.15 ([9, Theorem 2.1]). *Assume that $A \in \mathcal{C}^\alpha$ for some $\alpha > 2$, then $A \in \mathcal{B}(X)$.*

Another important consequence of exponential boundedness is the following useful characterization of α-resolvent families via the Laplace transform.

Theorem 1.16 ([49, Proposition 3.1]). *Let $\alpha > 0$. Let $\{S_\alpha(t)\}_{t\geq 0}$ be a strongly continuous α-resolvent family of type (M, ω). Then $\{S_\alpha(t)\}_{t\geq 0}$ is an α-resolvent family with generator A if and only if for every $\lambda > \omega$, $(\lambda^\alpha - A)^{-1}$ exists in $\mathcal{B}(X)$ and the identity*

$$\lambda^{\alpha-1}(\lambda^\alpha - A)^{-1}x = \int_0^\infty e^{-\lambda t} S_\alpha(t)x\,dt, \quad \text{for all } x \in X, \tag{1.10}$$

holds.

More generally, the same characterization is also true in the case of exponentially bounded (α, β)-resolvent families.

Theorem 1.17 ([13, Theorem 3.11]). *Let $\alpha > 0$ and $\beta \geq 1$. Let $\{R_{\alpha,\beta}(t)\}_{t\geq 0}$ be a strongly continuous (α, β)-resolvent family in $\mathcal{B}(X)$ of type (M, ω). Then $\{R_{\alpha,\beta}(t)\}_{t\geq 0}$ is an (α, β)-resolvent family with generator A if and only if for every $\lambda > \omega$, $(\lambda^\alpha - A)^{-1}$ exists in $\mathcal{B}(X)$ and the identity*

$$\lambda^{\alpha-\beta}(\lambda^\alpha - A)^{-1}x = \int_0^\infty e^{-\lambda t} R_{\alpha,\beta}(t)xdt, \quad \text{for all } x \in X, \tag{1.11}$$

holds.

For a generalization in the case $0 < \beta < 1$, see [2, Theorem 10]. The following result extends the Hille–Yosida theorem for C_0-semigroups to the context of α-resolvent families.

Theorem 1.18 ([49, Theorem 3.4] and [7, Theorem 3.1]). *A closed linear and densely defined operator A with domain $D(A)$ is the generator of a strongly continuous α-resolvent family of type (M, ω) if and only if the following conditions hold:*

(H1) $\lambda^\alpha \in \rho(A)$ *for all* $\operatorname{Re}\lambda > \omega$.

(H2) $H(\lambda) := \lambda^{\alpha-1}(\lambda^\alpha - A)^{-1}$ *satisfies the estimates*

$$\left\| \frac{d^n}{d\lambda^n} H(\lambda) \right\| \leq \frac{Mn!}{(\lambda - \omega)^{n+1}}, \quad \operatorname{Re}\lambda > \omega, \quad n \in \mathbb{N}_0.$$

For recent generation theorems for fractional resolvent families, see the paper [65] by Mu and Li. We also refer to the paper [37] for related results.

An analogous result holds in the case of (α,β)-resolvent families. It can be extracted from [49].

An important consequence is the following: If A is the generator of an (α,α)-resolvent family of type (M,ω), say $\{R_\alpha(t)\}_{t>0}$ for some $0<\alpha<1$, then A is the generator of an α-resolvent family $\{S_\alpha(t)\}_{t\geq 0}$ given by

$$S_\alpha(t) = (g_{1-\alpha} * R_\alpha)(t), \quad t \geq 0.$$

1.4 Analyticity

Given $\theta \in [0,\pi]$ and $\omega \in \mathbb{R}$, we denote $\Sigma_\theta(\omega) := \{z \in \mathbb{C} \setminus \{0\} \ : \ |\arg(z-\omega)| < \theta\}$. In the case $\omega = 0$, we simply denote $\Sigma_\theta = \Sigma_\theta(0)$.

Definition 1.19. Let $0 < \theta_0 \leq \pi/2$. An α-resolvent family (resp., an (α,β)-resolvent family) is called an analytic α-resolvent family of angle θ_0 (resp., an analytic (α,β)-resolvent family of angle θ_0) if it admits an analytic extension to the sector Σ_{θ_0} and the analytic extension is strongly continuous on Σ_{θ_0} for every $\theta \in (0,\theta_0)$.

Definition 1.20. An analytic α-resolvent family $S_\alpha(z)$ (resp., an analytic (α,β)-resolvent family $R_{\alpha,\beta}(z)$) is said to be of analyticity type (θ_0,ω_0) if for each $\theta \in (0,\theta_0)$ and each $\omega > \omega_0$ there exists a constant $M = M(\theta,\omega)$ such that

$$\|S_\alpha(z)\| \leq Me^{\omega \operatorname{Re}(z)}, \quad z \in \Sigma_\theta,$$

(resp., $\|R_{\alpha,\beta}(z)\| \leq Me^{\omega \operatorname{Re}(z)}$, $z \in \Sigma_\theta$). If $\omega_0 = 0$, an analytic α-resolvent family (resp. an analytic (α,β)-resolvent family) is called a bounded analytic α-resolvent family (resp., bounded analytic (α,β)-resolvent family).

If A generates an analytic α-resolvent family (resp., (α,β)-resolvent family) of type (θ_0,ω_0), we write $\mathcal{A}^\alpha(\theta_0,\omega_0)$ (resp. $\mathcal{A}^{\alpha,\beta}(\theta_0,\omega_0)$). In addition, we denote $\mathcal{A}^\alpha(\theta_0) := \bigcup_{\omega_0\in\mathbb{R}_+} \mathcal{A}^\alpha(\theta_0,\omega_0)$ and $\mathcal{A}^\alpha := \bigcup_{\theta_0\in(0,\pi/2)} \mathcal{A}^\alpha(\theta_0)$ (resp., $\mathcal{A}^{\alpha,\beta}(\theta_0) := \bigcup_{\omega_0\in\mathbb{R}_+} \mathcal{A}^{\alpha,\beta}(\theta_0,\omega_0)$ and $\mathcal{A}^{\alpha,\beta} := \bigcup_{\theta_0\in(0,\pi/2)} \mathcal{A}^{\alpha,\beta}(\theta_0)$). For $\alpha = 1$, we obtain the set of all generators of analytic semigroups.

Theorem 1.21 ([6, Theorem 2.14]). *Let* $0 < \alpha < 2$, $0 < \theta_0 < \min\{\frac{\pi}{2}, \frac{\pi}{\alpha} - \frac{\pi}{2}\}$ *and* $\omega_0 \geq 0$. *An operator A belongs to* $\mathcal{A}^\alpha(\theta_0,\omega_0)$ *if and only if:*

(i) $\lambda^\alpha \in \rho(A)$ *for each* $\lambda \in \Sigma_{\theta_0+\pi/2}(\omega_0)$,

(ii) *For any* $\omega > \omega_0$, $\theta < \theta_0$, *there is a constant* $C = C(\theta,\omega)$ *such that*

$$\|\lambda^{\alpha-1}(\lambda^\alpha - A)^{-1}\| \leq \frac{C}{|\lambda-\omega|}; \quad \lambda \in \Sigma_{\theta+\pi/2}(\omega).$$

For generators of analytic (α,β)-resolvent families, we have the following characterization.

Theorem 1.22 ([13, Theorem 4.6]). *Let $0 < \alpha < 2$, $\beta > 1$, $\theta_0 \in (0, \pi/2]$ and $\omega_0 \geq 0$. An operator A belongs to $\mathcal{A}^{\alpha,\beta}(\theta_0, \omega_0)$ if and only if:*
(i) $\lim_{\lambda \to \infty} \lambda^{\alpha-\beta+1} R(\lambda^\alpha, A)x = 0$ *for any* $x \in X$,
(ii) $\lambda^\alpha \in \rho(A)$ *for each* $\lambda \in \Sigma_{\theta_0+\pi/2}(\omega_0)$,
(iii) *For any $\omega > \omega_0$, $\theta < \theta_0$, there is a constant $C = C(\theta, \omega)$ such that*

$$\left\|\lambda^{\alpha-\beta}(\lambda^\alpha - A)^{-1}\right\| \leq \frac{C}{|\lambda - \omega|}; \quad \lambda \in \Sigma_{\theta+\pi/2}(\omega).$$

If the generator A is densely defined, then the condition (i) is automatically fulfilled. Other equivalent conditions can be found in [13, Section 4].

Recall that a closed linear operator A densely defined on a Banach space X is called *sectorial of angle* $\omega \in [0, \pi)$ ($A \in \mathrm{Sect}(\omega)$, in short) if:
(a) $\sigma(A) \subset \overline{\Sigma_\omega}$ for $\omega > 0$.
(b) For every $\omega' \in (\omega, \pi)$, $\sup\{\|z(z - A)^{-1}\| : z \in \mathbb{C} \setminus \overline{\Sigma_{\omega'}}\} < \infty$.

For bounded analytic α-resolvent families, we have the following important criteria.

Theorem 1.23 ([41, Lemma 2.7]). *Let $\alpha \in (0, 2)$ and $0 < \theta_0 < \min\{\frac{\pi}{2}, \frac{\pi}{\alpha} - \frac{\pi}{2}\}$. The following assertions are equivalent:*
(i) $A \in \mathcal{A}^\alpha(\theta_0, 0)$
(ii) $\Sigma_{\alpha(\pi/2+\theta_0)} \subset \rho(A)$ *and for all $0 < \theta < \theta_0$ there exists a constant M_θ such that*

$$\left\|\lambda(\lambda - A)^{-1}\right\| \leq M_\theta, \quad \lambda \in \Sigma_{\alpha(\pi/2+\theta)}.$$

(iii) $-A \in \mathrm{Sect}(\pi - \alpha(\frac{\pi}{2} + \theta_0))$.

A practical criteria is the following.

Theorem 1.24 ([7, Proposition 4.1]). *If $\{\lambda : Re(\lambda) > 0\} \subset \rho(A)$ and for some constant $C > 0$:*

$$\left\|(\lambda - A)^{-1}\right\| \leq \frac{C}{\mathrm{Re}(\lambda)}, \quad \mathrm{Re}(\lambda) > 0,$$

then for any $\alpha \in (0, 1)$, $A \in \mathcal{A}^\alpha(\min\{(\frac{1}{\alpha} - 1)\frac{\pi}{2}, \frac{\pi}{2}\}, 0)$.

1.5 Inversion

One of the fundamental problems in the theory of α-resolvent families, from the point of view of applications to fractional partial differential equations, is the relation between the α-resolvent family $\{S_\alpha(t)\}_{t\geq 0}$ and its generator. The reason is that for $x \in D(A)$, $S_\alpha(t)x$ is the solution of the fractional Cauchy problem

$$D_t^\alpha u(t) = Au(t),$$

with initial conditions $u(0) = x$, $u^{(k)}(0) = 0$, $k = 1, \ldots, n-1$; $n - 1 < \alpha \leq n$.

The following result corresponds to the vector-valued version of the complex inversion formula for the Laplace transform. The main difference with the finite dimensional case is that this is not true except in the case of the class of *UMD*-spaces. Recall that a Banach space X is *UMD* (or belongs to the class $\mathcal{HT}$) if the Hilbert transform is bounded on $L^2(\mathbb{R}, X)$. If X is not *UMD*, then we have to either handle integrated versions of α-resolvent families or restrict the validity of the formula for vectors belonging to the domain of powers of the generator A; see, for example, [67, Corollary 7.5] in the case $\alpha = 1$.

Theorem 1.25 ([20, Theorem 3.2]). *Let X be a* UMD *space. Let A be the generator of a strongly continuous α-resolvent family $\{S_\alpha(t)\}_{t\geq 0}$ of type (M, ω) and let $\gamma > \max(0, \omega)$. For every $x \in X$, we have*

$$S_\alpha(t)x = \frac{1}{2\pi i} \int\limits_{\gamma - i\infty}^{\gamma + i\infty} e^{\lambda t} \lambda^{\alpha-1} (\lambda^\alpha - A)^{-1} x d\lambda.$$

Remark 1.26. Checking the conditions in [20, Theorem 3.2], we note that the above theorem also holds for (α, β)- resolvent families, whenever $\alpha > 0$ and $0 < \beta \leq 1$.

Remark 1.27. If $\{S_\alpha(z)\}_{z\in\Sigma_\theta}$ is a bounded analytic α-resolvent family with generator A defined in a Banach space X, then for $t > 0$,

$$S_\alpha(t) = \frac{1}{2\pi i} \int\limits_{\Gamma_\theta} e^{\lambda t} \lambda^{\alpha-1} (\lambda^\alpha - A)^{-1} d\lambda,$$

where $\frac{\pi}{2} < \theta < \theta_0 + \frac{\pi}{2}$, with θ_0 the analytic angle of $S_\alpha(t)$, and the contour

$$\Gamma_\theta = \{\rho e^{i\theta} : \rho_0 < \rho < \infty\} \cup \{\rho e^{-i\theta} : \rho_0 < \rho < \infty\} \cup \{\rho_0 e^{i\varphi} : -\theta < \varphi < \theta\}$$

is oriented counterclockwise with $\rho_0 > 0$ any given constant; see [6].

An alternative representation of bounded analytic fractional resolvent families can be given by functional calculus via the Mittag-Leffler function $E_\alpha(z)$:

$$S_\alpha(t) = \frac{1}{2\pi i} \int\limits_{\Gamma_\theta} E_\alpha(\lambda t^\alpha)(\lambda - A)^{-1} d\lambda, \quad t > 0,$$

where $\frac{\pi}{2}\alpha < \theta < (\theta_0 + \frac{\pi}{2})\alpha$, with θ_0 the analytic angle of $S_\alpha(t)$, see [14, p. 183].

The next result is implicitly contained in [47, Theorem 2.1] and is a consequence of the Post–Widder inversion formula of the Laplace transform and [8, Lemma 4.1]. A formula for (α, β)-resolvent families have a more complicated writing and can be found in [47, Theorem 2.1].

Theorem 1.28. *Let X be a Banach space and A the generator of a strongly continuous α-resolvent family $\{S_\alpha(t)\}_{t\geq 0}$ of type (M,ω). Then*

$$S_\alpha(t)x = \lim_{n\to\infty} \frac{1}{n!} \sum_{k=1}^{n+1} b^\alpha_{k,n+1}\left(I - \frac{t^\alpha}{n^\alpha}A\right)^{-k} x, \quad x \in X,$$

where $b^\alpha_{k,n}$ are given by the recurrence relations

$$b^\alpha_{1,1} = 1, \quad b^\alpha_{k,n} = (n-1-k\alpha)b^\alpha_{k,n-1} + \alpha(k-1)b^\alpha_{k-1,n-1}, \quad 1 \leq k \leq n, \quad n = 2,3,\ldots$$

and $b^\alpha_{k,n} = 0$, $k > n$, $n = 1,2,\ldots$.

Some immediate consequences are characterizations of positivity of α-resolvent families in terms of positivity of the resolvent operator [47, Theorem 2.1].

In the case $\alpha = 1$, that is, A is the generator of a C_0-semigroup $\{S_1(t)\}_{t\geq 0}$. The formula reads

$$S_1(t)x = \lim_{n\to\infty}\left(I - \frac{t}{n}A\right)^{-n} x, \quad x \in X, \quad t \geq 0,$$

where the convergence is uniform in bounded t-intervals for each fixed $x \in X$. This formula has important implications for the numerical approximation of the trajectories of $\{S_1(t)\}_{t\geq 0}$, especially for implicit approximation schemes.

2 Properties of fractional resolvents

2.1 Spectral properties

Let $a \in \mathbb{C}$, $\alpha > 0$ be given, and define $s_\alpha(t) := t^{\alpha-1}E_{\alpha,\alpha}(at^\alpha)$ and $r_\alpha(t) := E_\alpha(at^\alpha)$. In order to study spectral mapping theorems, the following result due to Li and Zheng is fundamental.

Theorem 2.1 ([42, Lemma 3.1]). *Let A be a closed linear and densely defined operator A with domain $D(A)$ the generator of a strongly continuous α-resolvent family $\{S_\alpha(t)\}_{t\geq 0}$ where $\alpha \in (0,2]$, then*

$$(a - A)\int_0^t s_\alpha(t-s)S_\alpha(s)x\,ds = r_\alpha(t)x - S_\alpha(t)x, \quad x \in X.$$

and

$$\int_0^t s_\alpha(t-s)S_\alpha(s)(a-A)x\,ds = r_\alpha(t)x - S_\alpha(t)x, \quad x \in D(A).$$

We denote by $\sigma(A)$, $\sigma_p(A)$, $\sigma_r(A)$, $\sigma_a(A)$ the spectrum, point spectrum, residual spectrum, and approximate point spectrum of A, respectively. We have the following result.

Theorem 2.2 ([42, Theorem 3.2]). *Let A be the generator of a strongly continuous α-resolvent family $\{S_\alpha(t)\}_{t\geq 0}$ where $\alpha \in (0,2]$, then:*
(i) $E_\alpha(t^\alpha \sigma(A)) \subset \sigma(S_\alpha(t))$;
(ii) $E_\alpha(t^\alpha \sigma_p(A)) \subset \sigma_p(S_\alpha(t))$;
(iii) $E_\alpha(t^\alpha \sigma_a(A)) \subset \sigma_a(S_\alpha(t))$;
(iv) $E_\alpha(t^\alpha \sigma_r(A)) \subset \sigma_r(S_\alpha(t))$.

When $\alpha = 1$, this is the well-known spectral inclusions for C_0-semigroups [67, Section 2.2]. If $\alpha = 2$, since $E_2(z) = \cosh(z)$, so it gives the spectral inclusions for cosine operator functions [66].

A generalization of this spectral inclusions appear in [57] in the context of (a,k)-regularized families. The case of (α,β)-resolvent families read as follows.

Theorem 2.3 ([57]). *Let A be the generator of a strongly continuous (α,β)-resolvent family $\{R_{\alpha,\beta}(t)\}_{t\geq 0}$ where $\alpha \in (0,2]$ and $\beta > 0$. Then:*
(i) $t^{\beta-1}E_{\alpha,\beta}(t^\alpha \sigma(A)) \subset \sigma(R_{\alpha,\beta}(t))$;
(ii) $t^{\beta-1}E_{\alpha,\beta}(t^\alpha \sigma_p(A)) \subset \sigma_p(R_{\alpha,\beta}(t))$;
(iii) $t^{\beta-1}E_{\alpha,\beta}(t^\alpha \sigma_a(A)) \subset \sigma_a(R_{\alpha,\beta}(t))$;
(iv) $t^{\beta-1}E_{\alpha,\beta}(t^\alpha \sigma_r(A)) \subset \sigma_r(R_{\alpha,\beta}(t))$.

It is proved in [19] that contrary to the case of semigroups, where the spectral mapping theorem does not hold without further assumptions, such a theorem does hold for $(1,\beta)$-resolvent families ($\beta > 1$). Namely, if A is the generator of an $(1,\beta)$-resolvent family $R_{1,\beta}(t)$ then

$$\sigma_*(R_{1,\beta}(t)) \cup \{0\} = \left\{ \int_0^t g_\beta(t-s)e^{\lambda s} \;:\; \lambda \in \sigma_*(A) \right\} \cup \{0\}, \quad t > 0, \tag{2.1}$$

where $*$ stands for "p", "a" or "r".

2.2 Subordination principle

The subordination principle, presented earlier by Prüss [71] for general resolvent families, was studied in detail by Bajlekova for α-resolvent families in her Ph.D. thesis [6].

It is very important in this theory. We need the Wright-type function

$$
\begin{aligned}
\Phi_\gamma(z) &:= \sum_{n=0}^{\infty} \frac{(-z)^n}{n!\Gamma(-\gamma n + 1 - \gamma)} \\
&= \frac{1}{\pi\alpha} \sum_{n=1}^{\infty} (-z)^{n-1} \frac{\Gamma(n\alpha + 1)}{n!} \sin(n\pi\alpha) \\
&= \frac{1}{2\pi i} \int_\Gamma \mu^{\gamma-1} exp(\mu - z\mu^\gamma) d\mu,
\end{aligned}
\tag{2.2}
$$

where $0 < \gamma < 1$ and Γ is a contour which starts and ends at $-\infty$ and encircles the origin counterclockwise. This has sometimes also been called the Mainardi function. We note that the expressions above holds for all $z \in \mathbb{C}$ with the exception of (2.2) that holds only for all $z \in \mathbb{R}_+$. The Wright function connects to the Mittag-Leffler function through the Laplace transform:

$$
E_\gamma(z) = \int_0^\infty \Phi_\gamma(t) e^{zt} dt, \quad z \in \mathbb{C}, \quad 0 < \gamma < 1. \tag{2.3}
$$

Theorem 2.4 ([6, Theorem 3.1], [9, Theorem 3.1]). *Let $0 < \alpha < \beta \leq 2$, $\gamma = \alpha/\beta$, $\omega \geq 0$. If $A \in \mathcal{C}^\beta(\omega)$ then $A \in \mathcal{C}^\alpha(\omega^{|/\gamma})$ and the following representation holds:*

$$
S_\alpha(t) = \int_0^\infty t^{-\gamma} \Phi_\gamma(st^{-\gamma}) S_\beta(s) ds = \int_0^\infty \Phi_\gamma(\tau) S_\beta(t^\gamma \tau) d\tau, \quad t > 0. \tag{2.4}
$$

Moreover, $\{S_\alpha(t)\}_{t\geq 0}$ can be analytically extended to the sector Σ_θ with $\theta = \min\{(\frac{1}{\gamma} - 1)\frac{\pi}{2}, \frac{\pi}{2}\}$.

Since $\Phi_{1/2}(z) = \pi^{-1/2} e^{-z^2/4}$, the formula (2.4) coincides with the abstract Weierstrass formula

$$
S_1(t) = \frac{1}{\sqrt{\pi t}} \int_0^\infty e^{-s^2/4t} S_2(s) ds, \quad t > 0, \tag{2.5}
$$

relating the C_0-semigroup $\{S_1(t)\}_{t\geq 0}$ with the cosine operator function $\{S_2(t)\}_{t\geq 0}$. Other particular case, frequently used in applications, is $\beta = 1$. That is, assuming that A is the generator of a C_0-semigroup $\{S_1(t)\}_{t\geq 0}$ we have

$$
S_\alpha(t) = \int_0^\infty \Phi_\alpha(s) S_1(st^\alpha) ds, \quad t > 0, \quad 0 < \alpha < 1. \tag{2.6}
$$

Concerning (α, β)-resolvent families, an interesting and useful extension of the above theorem was proven by Abadías and Miana.

Theorem 2.5 ([2, Theorem 12]). *Let $0 < \eta_1 \leq 2$, $0 < \eta_2$. If A generates an exponentially bounded (η_1, η_2)-resolvent family $\{S_{\eta_1,\eta_2}(t)\}_{t>0}$, then A generates an exponentially bounded $(\alpha\eta_1, \alpha\eta_2 + \beta)$- resolvent family given by*

$$S_{\alpha\eta_1,\alpha\eta_2+\beta}(t) = \int_0^\infty \psi_{\alpha,\beta}(t,s) S_{\eta_1,\eta_2}(s)ds, \quad t > 0, \quad 0 < \alpha < 1, \quad \beta \geq 0. \tag{2.7}$$

Here,

$$\psi_{\alpha,\beta}(t,s) := t^{\beta-1} \sum_{n=0}^{\infty} \frac{(-st^{-\alpha})^n}{n!\Gamma(-\alpha n + \beta)}, \quad 0 < \alpha < 1, \quad \beta \geq 0,$$

is called scaled Wright function [2]. Note that if $0 < \eta_1 \leq 2$, $\eta_2 = 1$, $\beta = 1 - \alpha$ with $0 < \alpha < 1$ we retrieve Theorem 2.4. Also, choosing $\eta_1 = \eta_2 = 1$ and $\beta = 0$, we obtain the following useful consequence for applications: If A is the generator of a C_0-semigroup $\{S_1(t)\}_{t\geq 0}$, then A is the generator of an (α, α)-resolvent family given by

$$R_\alpha(t) = t^{\alpha-1} \int_0^\infty \alpha s \Phi_\alpha(s) S_1(st^\alpha)ds, \quad t > 0. \tag{2.8}$$

Concerning powers of the generator, the following generalized subordination principle was given in [13, Theorem 3.1].

Theorem 2.6. *Let $-A$ generate a bounded α-resolvent family $\{S_\alpha(t)\}_{t\geq 0}$ on X for some $\alpha \in (0,2]$ and let $0 < \gamma < 2$. Then $-A^\beta$ generates a bounded analytic γ-resolvent family $\{S_\gamma^\beta(t)\}_{t\geq 0}$ of angle $\varphi = \min\{\frac{\pi}{2}, \frac{\pi}{\gamma}(1-\beta) + \frac{\pi}{2}(\frac{\alpha}{\gamma}\beta - 1)\}$ on X for each $\beta \in (0, \frac{2-\gamma}{2-\alpha})$, and the following generalized subordination principle*

$$S_\gamma^\beta(t) = \int_0^\infty f_{\alpha,\gamma}^\beta(t,s) S_\alpha(s)ds, \quad t > 0,$$

holds in the strong sense, where

$$f_{\alpha,\gamma}^\beta(t,s) = \frac{-1}{2\pi i} \int_{\partial\Sigma_\omega} E_\gamma(-\mu^\beta t^\gamma)(-\mu)^{\frac{1}{\alpha}-1} e^{-(-\mu)^{1/\alpha}s} d\mu$$

with the path $\partial\Sigma_\omega$ oriented in the positive sense (from $\infty e^{i\omega}$ to $\infty e^{-i\omega}$) where $\omega \in (\pi - \frac{\pi}{2}\alpha, \min\{\pi, \frac{1}{\beta}(\pi - \frac{\pi}{2}\gamma)\})$ and $(-\rho e^{\pm i\omega})^{1/\alpha} = \rho^{1/\alpha} e^{\mp i(\pi-\omega)/\alpha}$.

We observe that $f_{1,1}^\beta$ is just the Bochner subordination kernel which is a probability density function [77, Chapter IX].

Let $0 < \gamma < \alpha \leq 2$. Then we can take $\beta = 1$ and obtain

$$\int_0^\infty e^{-\mu t} f_{\gamma,\alpha}^1(t,s)dt = \mu^{\gamma/\alpha-1} e^{-\mu^{\gamma/\alpha}s}, \quad \mathrm{Re}(\mu) > 0.$$

This yields $f^1_{\gamma,\alpha}(t,s) = t^{-\gamma/\alpha}\Phi_{\gamma/\alpha}(st^{-\gamma/\alpha})$. For other properties on this kernel, we refer the reader to [15, Section 3].

2.3 Immediate norm continuity

A family of bounded and linear operators $\{S(t)\}_{t\geq 0}$ on X is called immediately norm continuous (or equicontinuous) if the function $t \to S(t)$ is norm continuous from $(0,\infty)$ into $\mathcal{B}(X)$. In case of semigroups, this is a very important class that includes analytic semigroups, among others [22, Definition 4.17, Chapter II]. We have the following extension of this property for α-resolvent families.

Theorem 2.7 ([23, Lemma 3.8]). *Let $0 < \alpha < 1$. Suppose that $\{S_\alpha(t)\}_{t\geq 0}$ is an analytic α-resolvent family of analyticity type (ω,θ), then $\{S_\alpha(t)\}_{t\geq 0}$ is immediately norm continuous.*

A characterization in known only in case of Hilbert spaces, and is a consequence of a more general result in the context of resolvent operators due to Lizama [48].

Theorem 2.8 ([48]). *Let $0 < \alpha \leq 2$. Let A be the generator of an α-resolvent family $\{S_\alpha(t)\}_{t\geq 0}$ defined in a Hilbert space H and satisfying $\|S_\alpha(t)\| \leq Me^{\omega t}$ for some $M > 0$ and $\omega > 0$. The following assertions are equivalent:*
(i) *$\{S_\alpha(t)\}_{t\geq 0}$ is immediately norm continuous;*
(ii) *$\lim_{|\mu|\to\infty} \|(\mu_0 + i\mu)^{\alpha-1}((\mu_0 + i\mu)^\alpha - A)^{-1}\| = 0$ for some $\mu_0 > \omega$.*

We observe that an extension of the above result to (α,β)-resolvent families can be found in [54, Theorem 5.7]. In the case $\alpha = 1$, this result reduces to a characterization obtained by You [78]. Let us recall the following definition.

Definition 2.9. Let $-1 < \gamma < 0$ and $0 < \omega < \pi/2$. By $\Theta^\gamma_\omega(X)$ we denote the family of all linear closed operators $A : D(A) \subset X \to X$ which satisfy:
(i) $\sigma(A) \subset \overline{\Sigma_\omega} = \{z \in \mathbb{C} \setminus \{0\} : |\arg(z)| \leq \omega\} \cup \{0\}$ and
(ii) for every $\omega < \mu < \pi$ there exists a constant C_μ such that

$$\|R(z;A)\| \leq C_\mu |z|^\gamma \quad \text{for all } z \in \mathbb{C} \setminus \overline{\Sigma_\mu}.$$

A linear operator A will be called almost sectorial operator on X if $A \in \Theta^\gamma_\omega(X)$.

Observe that the limit case $\gamma = -1$ corresponds to the notion of sectorial operator. Let $0 < \alpha \leq 1$ and $A \in \Theta^\gamma_\omega(X)$, $-1 < \gamma < 0 < \omega < \pi/2$. We define the operator families

$$S_\alpha(t) := \frac{1}{2\pi i}\int_{\Gamma_\theta} E_\alpha(-zt^\alpha)R(z;A)dz, \quad t \in \Sigma_{\frac{\pi}{2}-\omega}$$

and

$$P_\alpha(t) := \frac{1}{2\pi i}\int_{\Gamma_\theta} E_{\alpha,\alpha}(-zt^\alpha)R(z;A)dz, \quad t \in \Sigma_{\frac{\pi}{2}-\omega}$$

where the integral contour $\Gamma_\theta := \{\mathbb{R}_+ e^{i\theta}\} \cup \{\mathbb{R}_+ e^{-i\theta}\}$ is oriented counter-clockwise and $\omega < \theta < \mu < \frac{\pi}{2} - |\arg(t)|$. By [76, Theorem 3.1], we have that for each $t \in \Sigma_{\frac{\pi}{2}-\omega}$, the operators $S_\alpha(t)$ and $P_\alpha(t)$ are linear and bounded in X. In the case $\alpha = 1$, we have that $S_1(t)$ defines an analytic semigroup for $t > 0$ that satisfies

$$\|S_1(t)\| \le Ct^{-\gamma-1}, \quad \|A^\beta S_1(t)\| \le Ct^{-\gamma-\mathrm{Re}\,\beta-1}, \quad t > 0, \quad Re(\beta) > 0.$$

From the subordination principle (Theorem 2.4), or the proof of [76, Theorem 3.1, formulae (3.2) and (3.3)], we have that the formula

$$S_\alpha(t) = \int_0^\infty \Phi_\alpha(s)S_1(st^\alpha)ds, \quad t \in \Sigma_{\frac{\pi}{2}-\omega} \tag{2.9}$$

defines an α-resolvent family for $t > 0$, and

$$P_\alpha(t) = \int_0^\infty \alpha s\Phi_\alpha(s)S_1(st^\alpha)ds, \quad t \in \Sigma_{\frac{\pi}{2}-\omega}, \tag{2.10}$$

defines an (α,α)-resolvent family given by $R_\alpha(t) := t^{\alpha-1}P_\alpha(t)$. Moreover, there exist constants $C_s = C(\alpha,\gamma) > 0$ and $C_p = C(\alpha,\gamma) > 0$ such that for all $t > 0$

$$\|S_\alpha(t)\| \le C_s t^{-\alpha(1+\gamma)}, \quad \|P_\alpha(t)\| \le C_p t^{-\alpha(1+\gamma)}.$$

See, for example, [80, Proposition 2.1 p. 42]. In particular,

$$\|R_\alpha(t)\| \le C_p t^{-\alpha\gamma-1}.$$

The limit case $\gamma = -1$ was proved in [45, Theorem 3.1] in the range $1 < \alpha < 2$. Also in [45, Theorem 3.2] it was proved that for $t > 0$ and $x \in X$, we have $R_\alpha(t)x \in D(A)$ and

$$\|AR_\alpha(t)\| \le \frac{C}{t}, \quad 1 < \alpha < 2,$$

for some positive constant C. By Theorem 2.4, [76, Theorem 3.2]), and [29, Theorem 5], we obtain the following.

Theorem 2.10. *Let $0 < \alpha < 1$ and $A \in \Theta^\gamma_\omega(X)$, $-1 < \gamma < 0 < \omega < \pi/2$ be given. Then A generates an immediately norm continuous α-resolvent family $S_\alpha(t)$ given by* (2.9), *and an (α,α)-resolvent family $R_\alpha(t) := t^{\alpha-1}P_\alpha(t)$ where $P_\alpha(t)$ given by* (2.10) *is immediately norm continuous.*

Concerning (α,β)-resolvent families, Ponce [70] proved the following result.

Theorem 2.11 ([70, Proposition 11]). *Let $\alpha > 0$ and $1 < \beta \le 2$. Assume that A is the generator of an exponentially bounded (α,β)-resolvent family $\{R_{\alpha,\beta}(t)\}_{t\ge0}$. Then $\{R_{\alpha,\beta}(t)\}_{t\ge0}$ is immediately norm continuous.*

2.4 Perturbation

A classical result for C_0-semigroups is the following: If A is the generator of a C_0-semigroup and $B \in \mathcal{B}(X)$, then $A + B$ is again the generator of a C_0-semigroup. This is not true in general for α-resolvent families with $0 < \alpha < 1$. See [8] for an example. However, in the case $1 \le \alpha \le 2$ perturbations by bounded operators are always possible. In the next theorem, proved by Bazhlekova, we show this even in the case of bounded time-dependent perturbations. For $\alpha = 2$, an analogous theorem was presented by Lutz in [59].

Theorem 2.12 ([8]). *Let $1 < \alpha < 2$ and A be the generator of a strongly continuous α-resolvent family $\{S_\alpha(t)\}_{t\ge 0}$ of type (M,ω) and for every $t > 0$, $B(t) \in \mathcal{B}(X)$. If the function $t \to B(t)$ is continuous in the uniform operator topology, then $A + B(t)$ generates a strongly continuous α-resolvent family $\{Q_\alpha(t)\}_{t\ge 0}$ given by the formula*

$$Q_\alpha(t) = \sum_{n=0}^{\infty} S_{\alpha,n}(t), \tag{2.11}$$

where

$$S_{\alpha,0}(t) := S_\alpha(t), \quad S_{\alpha,n}(t) := \int_0^t K_\alpha(t-s)B(s)S_{\alpha,n-1}(s)ds, \quad n \in \mathbb{N},$$

with

$$K_\alpha(t) := \int_0^t g_{\alpha-1}(t-s)S_\alpha(s)ds.$$

Moreover, if $K_T = \max_{t\in[0,T]} \|B(t)\|$, we have for all $t \in [0,T]$ the bounds

$$\|Q_\alpha(t)\| \le Me^{\omega t}E_\alpha(MK_T t^\alpha)$$

and

$$\|Q_\alpha(t) - S_\alpha(t)\| \le Me^{\omega t}(E_\alpha(MK_T t^\alpha) - 1).$$

A treatment of the perturbation problem from the point of view of the subjacent fractional abstract Cauchy problem in case $0 < \alpha \le 1$ was given by El-Borai [21].

In the case of (α,β)-resolvent families, we have the following alternative result which corresponds to a generalization of a perturbation result due to Miyadera and Voigt for C_0-semigroups.

Theorem 2.13 ([58, Theorem 3.1]). *Let A be the generator of an (α,β)-resolvent family $R_{\alpha,\beta}(t)$ of type (M,ω), where $\alpha \ge \beta$ and $\overline{D(A)} = X$. Let $B : D(B) \subseteq X \to X$ be a linear* *operator such that $D(A) \subseteq D(B)$. Suppose that there exists constants $\mu > \omega$ and $\gamma \in [0,1)$ such that*

$$\int_0^\infty e^{-\mu r}\|(g_{\alpha-\beta} * BR_{\alpha,\beta})(r)x\|dr \le \gamma\|x\|, \quad x \in D(A),$$

then $A + B$ generates an (α,β)-resolvent family $\{R^B_{\alpha,\beta}(t)\}_{t\ge 0}$ on X of type $(\frac{M}{1-\gamma},\mu)$ that satisfies

$$R^B_{\alpha,\beta}(t)x = R_{\alpha,\beta}(t)x + \int_0^t R^B_{\alpha,\beta}(t-r)(g_{\alpha-\beta} * BR_{\alpha,\beta})(r)xdr, \quad x \in D(A).$$

2.5 Approximation

We will consider two kinds of approximation. The first is for fixed $\alpha \in (0,2]$ the approximation of the generators, that is, the relations between the strong convergence of a sequence of α-resolvent families and that of the resolvents of their generators, as in the Trotter–Kato theorem for C_0-semigroups (cf. [67, Section 3.4]). The second is the approximation of the orders. As we know by the subordination principle, if $A \in C^\alpha$, then $A \in C^\beta$ for $\beta < \alpha$, so it is also natural to ask whether $S_\beta(t) \to S_\alpha(t)$ strongly as $\beta \to \alpha$, where $\{S_\beta(t)\}_{t\ge 0}$ is the β-resolvent family generated for A.

Recall that $R(\lambda, A) := (\lambda - A)^{-1}$ denotes the resolvent operator of A whenever it exists.

Theorem 2.14 ([42, Theorem 4.2]). *Let $\alpha \in (0,2]$ and let $\{S^0_\alpha(t)\}_{t\ge 0}$ and $\{S^n_\alpha(t)\}_{t\ge 0}$ be strongly continuous α-resolvent families generated by A_0 and A, respectively. Assume there are constants $M > 0$ and $\omega > 0$ such that $\|S^n_\alpha(t)\| \le Me^{\omega t}$ for all $t \ge 0$, $n \in \mathbb{N}_0$. The following assertions are equivalent:*
- (i) *$S^n_\alpha(t)x \to S^0_\alpha(t)x$ as $n \to \infty$ for all $x \in X$, uniformly for t on every bounded interval.*
- (ii) *$R(\lambda, A_n)x \to R(\lambda, A_0)x$ as $n \to \infty$ for all $x \in X$ and $\lambda > \omega^\alpha$.*

Remark 2.15. Using [50, Theorem 2.5], we observe that an analogous result holds for (α,β)-regularized families.

For the second kind of approximation, we have the following result.

Theorem 2.16 ([42, Theorem 4.5]). *Let A be the generator of a strongly continuous α-resolvent family $\{S_\alpha(t)\}_{t\ge 0}$. Let $\{S_\beta(t)\}_{t\ge 0}$ be the β-resolvent family generated by A for $\beta \le \alpha$. Then $S_\beta(t)x \to S_\alpha(t)x$ as $\beta \to \alpha$ for $t \ge 0$ and $x \in X$.*

2.6 Ergodicity

Let A be the generator of an (α,β)-resolvent family $\{R_{\alpha,\beta}(t)\}_{t>0}$. We analyze the behavior, as $t \to \infty$, of the following family of bounded and linear operators:

$$A_t^{\alpha,\beta}x := \frac{1}{g_{\alpha+\beta}(t)} \int_0^t g(t-s)R_{\alpha,\beta}(s)x\,ds, \quad t > 0, \quad x \in X.$$

Note that $A_t^{1,1}$ corresponds to the Cesáro mean of the semigroup $R_{1,1}(t)$. The family of operators $A_t^{1,\beta}$, $\beta > 0$ was studied by S. Y. Shaw [73, Theorem 5]. The family $A_t^{2,\beta}$ was considered by Lizama and Prado [56, Example 6]. The following result corresponds to a strong ergodic theorem with rates.

Theorem 2.17 ([56, Theorem 1]). *Let $\alpha,\beta > 0$ and A be the generator of an (α,β)-resolvent family $\{R_{\alpha,\beta}(t)\}_{t>0}$ such that*

$$\|R_{\alpha,\beta}(t)\| \le M_\beta t^{\beta-1}, \quad t > 0.$$

The following assertions hold:
(i) *The mapping $Px := \lim_{t\to\infty} A_t^{\alpha,\beta}x$ is a bounded linear projection with* $\mathrm{Ran}(P) = \mathrm{Ker}(A)$, $\mathrm{Ker}(P) = \overline{\mathrm{Ran}(A)}$ *and*

$$D(P) = \mathrm{Ker}(A) \oplus \overline{\mathrm{Ran}(A)}.$$

(ii) *For $0 < \gamma \le 1$ and $x \in \mathrm{Ker}(A) \oplus \overline{\mathrm{Ran}(A)}$, we have*

$$\|A_t^{\alpha,\beta}x - Px\| = O(t^{-\alpha\gamma}).$$

Other properties are given in [56]. The next result is the corresponding uniform ergodic theorem.

Theorem 2.18 ([56, Theorem 2]). *Let $\alpha,\beta > 0$ and A be the generator of an (α,β)-resolvent family $\{R_{\alpha,\beta}(t)\}_{t>0}$ such that*

$$\|R_{\alpha,\beta}(t)\| \le M_\beta t^{\beta-1}, \quad t > 0.$$

The following assertions are equivalent:
(i) *$D(P) = X$ and $\|A_t^{\alpha,\beta} - P\| \to 0$ as $t \to \infty$.*
(ii) $\mathrm{Ran}(A)$ *is closed.*
(iii) $\mathrm{Ran}(A^2)$ *is closed.*
(iv) $X = \mathrm{Ker}(A) \oplus \mathrm{Ran}(A)$.

Moreover, the convergence of the limit has order $O(t^{-\alpha})$.

Other properties in terms of the companion family $B_t x$ (see [56, Formula (11)]) are given in the reference [56]. We also observe, that there are analogous Abelian ergodic theorems with rates of approximation for the nets:

$$A_\lambda^\alpha = \lambda^\alpha(\lambda^\alpha - A)^{-1} \quad \text{and} \quad B_\lambda^\alpha = (\lambda^\alpha - A)^{-1}$$

where $\lambda^\alpha \in \rho(A)$ for $\lambda > 0$. See [56, Theorem 3 and Theorem 4] for details.

2.7 Compactness

A family $\{S(t)\}_{t\geq 0}$ of bounded and linear operators is called compact for $t > 0$ if for every $t > 0$, $S(t)$ is a compact operator.

The following theorem extends the compactness criteria for C_0-semigroups; see, for example, [22, Chapter II, Theorem 4.29].

Theorem 2.19 ([54, Theorem 3.4], [23, Theorem 3.6], [70, Proposition 16]). *Let $0 < \alpha < 2$ and A be the generator of an exponentially bounded α-resolvent family $\{S_\alpha(t)\}_{t\geq 0}$. Suppose that $\{S_\alpha(t)\}_{t\geq 0}$ is immediately norm continuous. Then the following assertions are equivalent:*
(i) *$S_\alpha(t)$ is a compact operator for all $t > 0$.*
(ii) *$(\mu - A)^{-1}$ is a compact operator for all (some) $\mu > \omega^{1/\alpha}$.*

Let A be the generator of an analytic semigroup $S_1(t)$ for $t > 0$ and $0 < \alpha < 1$. We consider the subordinated α-resolvent family generated by A

$$S_\alpha(t) = \int_0^\infty \Phi_\alpha(s) S_1(st^\alpha)\,ds, \quad t > 0.$$

We know that $S_\alpha(t)$ is analytic. We have the following result.

Theorem 2.20 ([23, Theorem 3.11]). *Let A be the generator of an analytic semigroup $S_1(t)$ for $t > 0$. If $(\mu - A)^{-1}$ is compact for every (some) $\mu > 0$, then for any $\alpha \in (0,1)$, $S_\alpha(t)$ is a compact analytic α-resolvent family for every $t > 0$.*

We now consider the (α,α)-resolvent family $R_\alpha(t) = t^{\alpha-1}P_\alpha(t)$, where

$$P_\alpha(t) = \int_0^\infty \alpha s \Phi_\alpha(s) S_1(st^\alpha)\,ds, \quad t > 0.$$

Theorem 2.21 ([76, Lemma 3.1 and Theorem 3.5]). *Let $0 < \alpha \leq 1$ and $A \in \Theta_\omega^\gamma(X)$, $-1 < \gamma < 0 < \omega < \pi/2$ be given. If $(\mu - A)^{-1}$ is compact for every (some) $\mu > 0$, then $S_\alpha(t)$ and $P_\alpha(t)$ are compact for every $t > 0$.*

For the case of (α,β)-resolvent families, we have the following result.

Theorem 2.22 ([70, Theorem 14]). *Let A be the generator of an exponentially bounded (α,β)-resolvent family $\{R_{\alpha,\beta}(t)\}_{t\geq 0}$ for some $\alpha > 0$ and $1 < \beta \leq 2$. Then $R_{\alpha,\beta}(t)$ is compact for all $t > 0$ if and only if $(\lambda - A)^{-1}$ is compact for every (some) $\lambda \in \rho(A)$.*

Two distinguished cases are the following.

Theorem 2.23 ([70, Proposition 17]). *Let $3/2 < \alpha < 2$ and A be the generator of an exponentially bounded and immediately norm continuous $(\alpha,\alpha-1)$-resolvent family $\{R_{\alpha,\alpha-1}(t)\}_{t\geq 0}$. Then $R_{\alpha,\alpha-1}(t)$ is compact for all $t > 0$ if and only if $(\lambda - A)^{-1}$ is compact for every (some) $\lambda \in \rho(A)$.*

Theorem 2.24 ([70, Proposition 18]). *Let $1/2 < \alpha < 1$ and A be the generator of an exponentially bounded and immediately norm continuous (α,α)-resolvent family $\{R_\alpha(t)\}_{t\geq 0}$. Then $R_\alpha(t)$ is compact for all $t > 0$ if and only if $(\lambda - A)^{-1}$ is compact for every (some) $\lambda \in \rho(A)$.*

Concerning compactness of the generator, we have the following criteria.

Theorem 2.25 ([46, Theorem 4.2] [3, Theorem 1]). *Let A be the generator of an α-resolvent family $\{S_\alpha(t)\}_{t\geq 0}$. The following assertions are equivalent:*
(i) *A is a compact operator.*
(ii) *$S_\alpha(t) - I$ is a compact operator for all $t > 0$.*
(iii) *$\lambda R(\lambda, A) - I$ is compact a compact operator for all (some) $\lambda \in \rho(A)$.*

2.8 Fractional powers of generators

If $0 \in \rho(A)$ for a sectorial operator A, then we can define its fractional powers as follows: For $b > 0$, define A^{-b} by

$$A^{-b} := \frac{1}{2\pi i} \int_{\Gamma(\xi)} \lambda^{-b}(\lambda - A)^{-1} d\lambda$$

where the path $\Gamma(\xi)$ runs in the resolvent set of A from $\infty e^{-i\xi}$ to $\infty e^{i\xi}$, while avoiding the negative real axis and the origin, and λ^b is taken as the principal branch. Noticing that $A^{-b} \in \mathcal{B}(X)$ is injective for all $b > 0$, we can define $A^b := (A^{-b})^{-1}$ and $A^0 := I$.

On the other hand, for a sectorial operator A without the assumption that $0 \in \rho(A)$, since $A + \epsilon$ is sectorial and $0 \in \rho(A + \epsilon)$ it makes sense to consider the operator $(A + \epsilon)^b$ and define the fractional powers of A by

$$A^b := s - \lim_{\epsilon \to 0^+} (A + \epsilon)^b,$$

for $b > 0$. For further information, we refer the reader to the monographs [27] and [60].

Theorem 2.26 ([41, Theorem 3.1]). *Let $\alpha \in (0,2]$, $\gamma \in (0,2)$ and $A \in \mathrm{Sect}(\pi - \frac{\alpha}{2}\pi)$.*

(i) *For each $\beta \in (0, \frac{2\pi-\pi\gamma}{2\pi-\pi\alpha})$, we have $-A^\beta \in \mathcal{A}^\gamma(\varphi_0)$, with $\varphi_0 := \min\{\frac{\pi}{2}, -\frac{\beta}{\gamma}(\pi-\frac{\pi}{2}\alpha)+\frac{\pi}{\gamma}-\frac{\pi}{2}\}$.*

(ii) *If $0 \in \rho(A)$, then the γ-resolvent family generated by $-A^\beta$, $S_\gamma^\beta(t)$, can be represented by*

$$S_\gamma^\beta(t) = \frac{1}{2\pi i}\int_{\Gamma_\omega} E_\gamma(-\mu^\beta t^\gamma)(A-\mu)^{-1}d\mu, \quad t>0,$$

where Γ_ω is a smooth path in the resolvent set of A from $\infty e^{-i\omega}$ to $\infty e^{i\omega}$, avoiding the negative axis and zero, with $\omega \in (\pi - \alpha\frac{\pi}{2}, \frac{1}{\beta}(\pi - \gamma\frac{\pi}{2}))$.

The purpose of the following result is to characterize the domains $D(A^\gamma)$, $0<\gamma<1$, of fractional powers of sectorial operators. Recall that $-A$ generates a bounded analytic α-resolvent family for some $0<\alpha<2$ if and only if A is sectorial; see Theorem 1.23.

Theorem 2.27 ([14, Theorem 4.8]). *Let $-A$ be the generator of a bounded analytic α-resolvent family $\{S_\alpha(t)\}_{t\ge 0}$ for some $0<\alpha<1$. Let $0<\gamma<1$ and $x\in X$. The following assertions are equivalent:*

(i) *$x \in D(A^\gamma)$;*

(ii) *$\int_0^1 g_{\beta-\gamma}(t)S_\alpha(t^{1/\alpha})x\,dt \in D(A^\beta)$ for all (some) $\gamma<\beta<1$;*

(iii) *$\lim_{\epsilon\to 0}\int_\epsilon^\infty g_{\beta-\gamma}(t)A^\beta S_\alpha(t^{1/\alpha})x\,dt$ exists for all (some) $\gamma<\beta<1$.*

Moreover, for all $x \in D(A^\gamma)$ and $0<\gamma<\beta<1$:

$$A^\gamma x = s - \lim_{\epsilon\to 0}\frac{\Gamma(1-\alpha(\beta-\gamma))}{\Gamma(1-\beta+\gamma)}\int_\epsilon^\infty g_{\beta-\gamma}(t)A^\beta S_\alpha(t^{1/\alpha})x\,dt.$$

An analogous result holds in case $1 \le \alpha < 2$; see [14, Theorem 4.9]. Concerning integrated α-resolvent families, see [14, Theorem 4.15]. For a related result, see Zacher [79, Theorem 3.2, Remark 3.2(ii) and Theorem 3.3].

Concerning square root reductions we recall that if $-A$ generate a bounded 2-resolvent family, that is, a bounded cosine operator function $\{C(t)\}_{t\in\mathbb{R}}$, on a *UMD* space X, then $iA^{1/2}$ generates a bounded C_0-group $\{U(t)\}_{t\in\mathbb{R}}$ on X and

$$C(t) = \frac{U(t)+U(-t)}{2}, \quad t\in\mathbb{R}.$$

It is surprising that for a bounded analytic α-resolvent family $\{S_\alpha(t)\}_{t\ge 0}$ with some $\alpha \in (0,2)$, the square root reduction of $S_\alpha(t)$ always exists on a Banach space (not necessarily to be *UMD*).

Theorem 2.28 ([15, Proposition 5.6]). *Let $\alpha \in (0,2)$. The operator A generates a bounded analytic α-resolvent family $\{S_\alpha(t)\}_{t\ge 0}$ on X if and only if $\pm i[(-A)]^{1/2}$ generate* *bounded analytic $\alpha/2$-resolvent families* $\{S^{\pm}_{\alpha/2}(t)\}_{t\geq 0}$ *on* X. *Moreover,*

$$S_\alpha(t) = \frac{S^{+}_{\alpha/2}(t) + S^{-}_{\alpha/2}(t)}{2}, \quad t \geq 0.$$

A complementary result is the following: As we know, when A generates a C_0-group, then A^2 also generates a cosine operator function. The extension of this result to α-resolvent families read as follows.

Theorem 2.29 ([42, Proposition 2.8]). *Suppose that* A *and* $-A$ *both generate an* α-*resolvent family for some* $\alpha \in (0,1]$. *Then* A^2 *generates a* 2α-*resolvent family.*

2.9 Spatial regularity

In this section, we state some results on spatial regularity for α-resolvent families $\{S_\alpha(t)\}_{t\geq 0}$; see also [71, Theorem 2.2 p. 57].

Theorem 2.30 ([6, Proposition 2.15]). *Let* $\alpha \in (0,2)$ *and assume* $A \in \mathcal{A}^\alpha(\theta_0, \omega_0)$ *then for any* $x \in X$ *and* $t > 0$ *we have* $S_\alpha(t)x \in D(A)$ *and*

$$\|AS_\alpha(t)\| \leq Ce^{\omega t}(1 + t^{-\alpha}), \quad t > 0, \quad \omega > \omega_0.$$

Theorem 2.31 ([14, Lemma 4.1]). *Let* $-A$ *be the generator of a bounded analytic* α-*resolvent family* $\{S_\alpha(t)\}_{t\geq 0}$ *for some* $0 < \alpha < 2$. *Then:*

(i) *If* $1 \leq \alpha < 2$, *then* $S_\alpha(t)x \in D(A^\infty) := \bigcap_{k=1}^{\infty} D(A^k)$ *for all* $x \in X$, $t > 0$, *and*

$$\|A^\beta S_\alpha(t)\| \leq Ct^{-\alpha\beta},$$

for each $\beta > 0$. *Moreover,* $D(A^\infty)$ *is dense in* X.

(ii) *If* $0 < \alpha < \frac{1}{m}$ *for some* $m \in \mathbb{N}$, *then for each* $x \in D(A^{m-1})$ *we have* $S_\alpha(t)x \in D(A^m)$ *and*

$$\|A^m S_\alpha(t)x\| \leq Ct^{-m\alpha}\|x\| + \sum_{k=1}^{m} g_{1-k\alpha}(t)\|A^{m-k}x\|, \quad t > 0.$$

Theorem 2.32 ([76, Theorem 3.3]). *Let* $0 < \alpha \leq 1$ *and* $A \in \Theta^\gamma_\omega(X)$, $-1 < \gamma < 0 < \omega < \pi/2$. *Let* $0 < \beta < 1 - \gamma$. *For all* $x \in D(A)$ *and* $t > 0$,

$$\|AS_\alpha(t)x\| \leq Ct^{-\alpha(1+\gamma)}\|Ax\|,$$

where C *is a constant depending on* γ, α.

For the next results, recall that $\{R_\alpha(t)\}_{t>0}$ denotes an (α,α)-resolvent family.

Theorem 2.33 ([75, Theorem 2.3]). *Let $0 < \alpha < 1$ and $A \in \mathcal{A}^\alpha(\theta_0, \omega_0)$. Then there exists a constant $C > 0$ such that*

$$\|R_\alpha(t)\| \le Ce^{\omega t}(1 + t^{\alpha-1}), \quad t > 0. \tag{2.12}$$

Theorem 2.34 ([80, Section 2.2.2 Lemma 2.10 (iv)]). *Let $0 < \alpha < 1$. Let A be the generator of an exponentially stable semigroup $\{T(t)\}_{t\ge 0}$, that is, there exist constants $\delta > 0$ and $M > 0$ such that $\|T(t)\| \le Me^{-\delta t}$. Then*

$$\|R_\alpha(t)\| \le Mt^{\alpha-1}E_{\alpha,\alpha}(-\delta t^\alpha), \quad t > 0.$$

The following important result is due to Cuesta [18]. However, it should be noted that his notion of sectorial operator differs from ours.

Theorem 2.35 ([18, Theorem 1]). *Suppose that A is sectorial of negative type and angle $\theta \in [0, \pi(1 - \alpha/2))$ then there exists $C > 0$ depending solely on θ and α such that*

$$\|S_\alpha(t)\| \le \frac{CM}{1 + |\omega|t^\alpha}.$$

For additional results, see the end of Subsection 2.3.

3 The fractional abstract Cauchy problem

Recall that the Caputo fractional derivative of order $\alpha > 0$ is defined by

$${}^C D_t^\alpha f(t) := J_t^{m-\alpha} \frac{d^m}{dt^m} f(t)$$

where m is the smallest integer greater than or equal to α. We consider the solutions of fractional Cauchy problems. First, we give the definition of solutions to the inhomogeneous initial value problem

$$\begin{cases} {}^C D_t^\alpha u(t) = Au(t) + f(t), & t \in (0, \tau) \\ u^{(k)}(0) = x_k, & k = 0, 1, \dots, m-1 \end{cases} \tag{3.1}$$

where $\tau \in (0, \infty]$, $f \in L^1_{\mathrm{loc}}([0, \tau), X)$ and A is a closed densely defined operator in a Banach space X. Recall that the connection between ${}^C D_t^\alpha$ and the Riemann–Liouville fractional derivative ${}^{\mathrm{RL}} D_t^\alpha$ is given by

$${}^C D_t^\alpha u(t) = {}^{\mathrm{RL}} D_t^\alpha \left(u(t) - \sum_{k=0}^{m-1} g_{k+1}(t) x_k \right).$$

Definition 3.1. A function $u \in C([0, \tau), X)$ is called a strong (or classical) solution of (3.1) if $u(t)$ satisfies:

(a) $u \in C([0,\tau), D(A)) \cap C^{m-1}([0,\tau), X)$.
(b) $g_{m-\alpha} * (u - \sum_{k=0}^{m-1} g_{k+1}x_k) \in C^m([0,\tau), X)$.
(c) $u(t)$ satisfies (3.1).

Definition 3.2. A function $u \in C([0,\tau), X)$ is called a *mild solution* of (3.1) if $(g_\alpha * u)(t) \in D(A)$ and

$$u(t) = \sum_{k=0}^{m-1} g_{k+1}(t)x_k + A(g_\alpha * u)(t) + (g_\alpha * f)(t), \quad t \in [0,\tau).$$

If A is the generator of an α-resolvent family $\{S_\alpha(t)\}_{t\geq 0}$, then the mild solution of (3.1) on $\mathbb{R}_+$ can be represented by

$$u(t) = \sum_{k=0}^{m-1} (g_k * S_\alpha)(t)x_k + \frac{d}{dt}(g_\alpha * S_\alpha * f)(t), \quad t \geq 0, \tag{3.2}$$

whenever $x_k \in X$ for all $k = 0, \dots, m-1$.

3.1 The fractional-order homogeneous problem

We briefly analyze the homogeneous fractional Cauchy problem:

$$\begin{cases} {}^C D_t^\alpha u(t) = Au(t), & t \in (0,\tau) \\ u^{(k)}(0) = x_k, & k = 0,1,\dots,m-1. \end{cases} \tag{3.3}$$

In case $\tau = \infty$ and analogously to the cases $\alpha = 1$ and $\alpha = 2$, we have the following concept of well-posedness in the sense of Hadamard [71].

Definition 3.3. The problem (3.3) is called well-posed if for any $x_k \in D(A)$, $k = 0,1,\dots,m-1$, there is a unique strong solution $u_k(t)$ of (3.3), and $x_{k,n} \in D(A)$, $x_{k,n} \to 0$ as $n \to \infty$, imply $u_{k,n}(t) \to 0$ as $n \to \infty$ in X, uniformly on compact intervals.

By [71, Proposition 1.1] the problem (3.3) is well-posed if and only if A is the generator of an α-resolvent family $\{S_\alpha(t)\}_{t\geq 0}$, and the unique strong solution is given by

$$u(t) = \sum_{k=0}^{m-1} (g_k * S_\alpha)(t)x_k, \quad t \geq 0,$$

whenever $x_k \in D(A)$ for all $k = 0,1,\dots,m-1$. In particular,

$$u(t) = S_\alpha(t)x$$

is the unique strong solution of the problem

$$\begin{cases} {}^C D_t^\alpha u(t) = Au(t), & t > 0 \\ u(0) = x, \quad u^{(k)}(0) = 0, & k = 1,\dots,m-1, \end{cases} \tag{3.4}$$

for each $x \in D(A)$ (recall that g_0 is the Dirac delta); see, for example, [13, Proposition 3.3]. According to Theorem 1.15, if A is the generator of an α-resolvent family, then the study of well-posedness for the problem (3.4) should be restricted to the case $0 < \alpha \leq 2$. However, note that the concept of well-posedness may vary.

3.2 The fractional-order inhomogeneous problem

We now turn to the following problem:

$$\begin{cases} {}^C D_t^\alpha u(t) = Au(t) + f(t), & t \in (0, \tau) \\ u^{(k)}(0) = 0, & k = 0, 1, \ldots, m-1. \end{cases} \tag{3.5}$$

For the existence of strong solutions of (3.5), we have the following.

Theorem 3.4 ([41, Proposition 4.3]). *Let $\alpha \in (0, 2]$. Suppose that A is the generator of a strongly continuous α-resolvent family $\{S_\alpha(t)\}_{t\geq 0}$ and $f \in C([0, \tau), X)$. Then the following statements are equivalent:*
(a) (3.5) *has a strong solution on* $[0, \tau)$.
(b) $S_\alpha * f$ *is differentiable on* $[0, \tau)$.
(c) $\frac{d}{dt}(g_\alpha * S_\alpha * f)(t) \in D(A)$ *for* $t \in [0, \tau)$ *and* $A(\frac{d}{dt}(g_\alpha * S_\alpha * f)(t))$ *is continuous on* $[0, \tau)$.

In the case $\alpha \in [1, 2]$, the condition (c) *can be replaced by*
(c)′ $(g_{\alpha-1} * S_\alpha * f)(t) \in D(A)$ *for* $t \in [0, \tau)$ *and* $A(g_{\alpha-1} * S_\alpha * f)(t)$ *is continuous on* $[0, \tau)$.

As consequence, we have the following.

Corollary 3.5 ([41, Corollary 4.4], [43, Corollary 3.4]). *Let $\alpha \in (0, 2]$. Suppose that A is the generator of a strongly continuous α-resolvent family. Then* (3.5) *has a strong solution on* $[0, \tau)$ *if one of the following conditions is satisfied:*
(a) f *is continuously differentiable on* $[0, \tau)$.
(b) $\alpha \in [1, 2]$, $f(t) \in D(A)$ *for* $t \in [0, \tau)$ *and* $Af \in L^1_{\mathrm{loc}}([0, \tau), X)$.
(c) $\alpha \in (0, 1)$, $f(t) \in D(A)$ *for* $t \in [0, \tau)$ *and* $g_\alpha * f$ *is continuously differentiable on* $[0, \tau)$.
(d) $\alpha \in (1, 2)$, $(g_{\alpha-1} * f) \in L^1((0, \tau), D(A))$.

Maximal regularity results for (3.5) has been studied by a number of authors under different methods and in several classes of spaces of vector valued functions. We refer the interested reader to [10–12, 28, 53] and references therein. A characterization of existence of mild solutions for (3.5) with boundary condition $u(0) = u(\tau)$, in terms of α-resolvent families, has been proved in [29].

If $-A$ is an m-accretive operator defined on a Hilbert space H, and $f \in L^2([0, \tau]; H)$ maximal regularity conditions for (3.5) with Riemann–Liouville fractional derivative of order $\alpha \in (1,2]$, was studied by Bazhlekova [6]. She rewrites the problem as a system of equations of order $\alpha/2$ in $\mathcal{H} := D((-A)^{1/2}) \times H$. If $-A$ is a positive operator, the survey [17] reviews some regularity results for (3.5). See also [16] for more on regularity properties of solutions of fractional evolution equations.

The solvability and maximal L^p regularity of the linear nonautonomous problem, that is, $A = A(t)$ where $A(t)$ is a family of linear closed densely defined operators defined on a Banach space X, such that the domain of $A(t)$ does not depend on t, with the Riemann–Liouville fractional derivative, was considered by Bazhlekova [6, Section 6.1].

3.3 The fractional-order Cauchy problem: $0 < \alpha \le 2$

We consider the fractional-order linear Cauchy problem:

$$\begin{cases} {}^C D_t^\alpha u(t) = Au(t) + f(t), & t \ge 0 \\ u(0) = u_0, \quad \max\{0, \alpha - 1\}(u'(0) - u_1) = 0, & 0 < \alpha \le 2. \end{cases} \tag{3.6}$$

According to the previous subsections, if A is the generator of an α-resolvent family $\{S_\alpha(t)\}_{t\ge0}$ and $0 < \alpha \le 1$ then the mild solution of (3.6) can be represented by

$$u(t) = S_\alpha(t)u_0 + \frac{d}{dt}(g_\alpha * S_\alpha * f)(t), \quad t \ge 0. \tag{3.7}$$

In the case that A is the generator of an (α,α)-resolvent family $\{R_\alpha(t)\}_{t>0}$, we obtain that $S_\alpha(t) := (g_{1-\alpha} * R_\alpha)(t)$ is an α-resolvent family generated by A and the representation

$$u(t) = S_\alpha(t)u_0 + \int_0^t R_\alpha(t-s)f(s)ds, \quad t \ge 0. \tag{3.8}$$

Therefore, starting with A as the generator of an (α,α)-resolvent family is more natural and appropriate; see the remark at the end of Subsection 1.3. These are the more general hypotheses to be used in concrete applications.

It should be observed that if the problem (3.6) is considered with the Riemann–Liouville fractional derivative, then we have

$$u(t) = R_\alpha(t)u_0 + \int_0^t R_\alpha(t-s)f(s)ds, \quad t > 0. \tag{3.9}$$

See also the paper of Fan [24] for additional information and results. Using the subordination principle, one can give explicit descriptions of (α,β)-resolvent families in some cases. This fact has been widely used in the literature. For example, if A is the generator of a C_0-semigroup $\{S_1(t)\}_{t\ge0}$ and $0 < \alpha < 1$ then we have that $R_\alpha(t) = t^{\alpha-1}P_\alpha(t)$,

where

$$P_\alpha(t) = \alpha \int_0^\infty s\Phi_\alpha(s)S_1(st^\alpha)ds, \quad t > 0,$$

is the (α, α)-resolvent family generated by A, and

$$S_\alpha(t) = \int_0^\infty \Phi_\alpha(s)S_1(st^\alpha)ds, \quad t > 0,$$

is the α-resolvent family generated by A. For up to date information, applications and historical remarks concerning this formulation, see the monograph of Gal and Warma [25, Section 2].

In the case that $1 < \alpha \leq 2$, we have the representation

$$u(t) = S_\alpha(t)u_0 + (g_1 * S_\alpha)(t)u_1 + (g_{\alpha-1} * S_\alpha * f)(t), \quad t \geq 0. \tag{3.10}$$

Assuming that A is the generator of an α-resolvent family $\{S_\alpha(t)\}_{t\geq 0}$, we obtain that $R_\alpha(t) = (g_{\alpha-1} * S_\alpha)(t)$ is an (α, α)-resolvent family generated by A and

$$u(t) = S_\alpha(t)u_0 + \int_0^t S_\alpha(s)u_1 ds + \int_0^t R_\alpha(t-s)f(s)ds, \quad t \geq 0. \tag{3.11}$$

Using again the subordination principle, we can give an explicit description: If A is the generator of a strongly continuous cosine family $\{S_2(t)\}_{t\geq 0}$ and $1 < \alpha < 2$ then we have that

$$S_\alpha(t) = \int_0^\infty \Phi_{\alpha/2}(s)S_2(st^{\alpha/2})ds,$$

is the α-resolvent family generated by A.

To complete the picture, we observe that for the Riemann–Liouville fractional derivative, we have the following representation of the solutions:

$$u(t) = L_{\alpha,\alpha-1}(t)u_0 + \int_0^t L_{\alpha,\alpha-1}(s)u_1 ds + \int_0^t R_\alpha(t-s)f(s)ds \tag{3.12}$$

where $\{L_{\alpha,\alpha-1}(t)\}_{t>0}$ is an $(\alpha, \alpha - 1)$-resolvent family generated by A and $R_\alpha(t) := (g_1 * L_{\alpha,\alpha-1})(t)$ is an (α, α)-resolvent family. Note that if A is a bounded operator, then we have

$$L_{\alpha,\alpha-1}(t) = t^{\alpha-2}E_{\alpha,\alpha-1}(At^\alpha), \quad t > 0, \quad 1 < \alpha \leq 2.$$

The article [30] investigates the solvability of the fractional-order inhomogeneous Cauchy problem (3.6) using integrated α-resolvent families for $0 < \alpha \le 1$. When A is the generator of a β-times integrated semigroup on a Banach space X, with $\beta \ge 0$, explicit representations of mild and classical solutions of the above problem in terms of the integrated semigroup are derived. Then the existence, uniqueness, and regularity of the solutions of problem (3.6) are studied. The results are applied to the fractional diffusion equation with nonhomogeneous, Dirichlet, Neumann, and Robin boundary conditions and to the time fractional- order Schrödinger equation $\mathbb{D}_t^\alpha u(t,x) = e^{i\theta}\Delta_p u(t,x) + f(t,x)$, $t > 0$, $x \in \mathbb{R}^N$ where $\pi/2 \le \theta < (1-\alpha/2)\pi$ and Δ_p is a realization of the Laplace operator on $L^p(\mathbb{R}^N)$, $1 \le p < \infty$. In the case of $1 < \alpha \le 2$, analogous results are proved in the paper [31]. The solvability of the fractional-order Cauchy problem (3.6) with the Riemann–Liouville fractional derivative has been studied in the paper [64] by Mophou and N'Guérékata.

3.4 Comments and open problems

A theory of resolvent families in order to consider fractional abstract equations in the form

$$\begin{cases} {}^C D^\alpha u(t) + c_1 \, {}^C D^{\beta_1} u(t) + \cdots + c_d \, {}^C D^{\beta_d} u(t) = Au(t), & t > 0 \\ u^{(k)}(0) = x_j, & k = 0,1,\ldots,m-1 \end{cases} \tag{3.13}$$

where $\alpha > \beta_1 > \cdots > \beta_d > 0$, c_j are constants and m is the smallest integer greater that or equal to α, has been introduced by G. G. Li, Kostic, M. Li, and Piskarev [38]. They derive generation theorems, algebraic equations, and approximation theorems for such resolvent families.

Since the domain of the generator of (α,α)-resolvent families is dense in the case $0 < \alpha = \beta < 1$, it would be interesting to know if the same property remains true in case $0 < \alpha < 1$ and $0 < \beta < 1$.

It should be noted that the theory of (a,k)-regularized families introduced in [49] covers most of the results about (α,β)-resolvents. A more general notion of (local) (a,k)-regularized C-resolvent family was introduced by Kostić in [36]. If $a(t) = g_\alpha(t)$ and $k(t) = 1$, we obtain the class of α-times C-regularized resolvent families introduced in 2010 by Chen and Li [13]. A connection of this general theory with abstract time-fractional equations appears in [35].

It is well known that both strongly continuous semigroups and strongly continuous cosine functions are necessarily exponentially bounded. However, whether an (α,β)-resolvent family is exponentially bounded is unknown in general.

When $1 < \alpha \le 2$, a careful study of $(\alpha,\alpha-1)$-resolvent families seems to be missing in the literature, except for the general properties stated in this chapter. They are necessary in order to study the existence, uniqueness, and qualitative behavior of solutions for the fractional abstract Cauchy problem with Riemann–Lioville fractional-order derivative for both, the linear and nonlinear cases. For instance, an explicit representation in terms of an strongly continuous cosine family generated by A and some special density function should be very useful.

Recall that a family of bounded and linear operators $\{S(t)\}_{t\geq 0}$ is called exponentially stable if there are constants $M > 0$ and $\omega > 0$ such that

$$\|S(t)\| \leq Me^{-\omega t}, \quad t \geq 0. \tag{3.14}$$

The next result shows that when $0 < \alpha < 1$, an α-resolvent family is never exponentially stable.

Theorem 3.6 ([42, Proposition 2.7]). *Suppose that $\alpha \in (0,1)$ and $\{S_\alpha(t)\}_{t\geq 0}$ is a strongly continuous α-resolvent family generated by A, then $\{S_\alpha(t)\}_{t\geq 0}$ does not decay exponentially.*

The same happens in case of $\alpha = 2$ because of the D'Alembert functional equation [74].

A criteria for stability of α-resolvent families is unknown, except in the finite dimensional case, where the most well-known criteria is Matignon's stability theorem [61]. An extension of Matignon's result to the case $1 < \alpha < 2$ is given in [72]. The general abstract case, when A is generator of an α-resolvent family, remains widely open and seems to be a very difficult task. The study of spectral mapping theorems for (α,β)-resolvent families, which are also missing in the literature, could be a remarkable advance to such objective.

There is a large amount of literature dealing with the subject of this chapter; see, for instance, [32–34] and references therein. However, the list of references do not escape the usual rule of being incomplete. In general, I have listed those papers which are more close to the topics discussed here. But, even for those papers, the list is far from being exhaustive and I apologize for omissions.

Bibliography

[1] L. Abadias, C. Lizama, and P. J. Miana, Sharp extensions and algebraic properties for solution families of vector-valued differential equations, *Banach J. Math. Anal.*, **10**(1) (2016), 169–208.

[2] L. Abadias and P. J. Miana, A subordination principle on Wright functions and regularized resolvent families, *J. Funct. Spaces*, (2015), Article ID 158145, 9 pages.

[3] O. V. Antonyuk, A. N. Kochubei, and S. I. Piskarev, On the compactness and the uniform continuity of a resolvent family for a fractional differential equation, *Dopov. Nats. Akad. Nauk Ukr. Mat. Prirodozn. Tekh. Nauki*, **6** (2014), 7–12.

[4] D. Araya and C. Lizama, Almost automorphic mild solutions to fractional differential equations, *Nonlinear Anal., Theory Methods Appl.*, **69** (2008), 3692–3705.

[5] W. Arendt and H. Kellermann, Integrated solutions of Volterra integrodifferential equations and applications., in *Volterra Integrodifferential Equations in Banach Spaces and Applications (Trento, 1987)*. Pitman Res. Notes Math. Ser., vol. 190, pp. 21–51, Longman Sci. Tech., Harlow, 1989.

[6] E. Bajlekova, Fractional evolution equations in Banach spaces, Ph.D. thesis, Department of Mathematics, Einhoven University of Technology, 2001.

[7] E. Bazhlekova, The abstract Cauchy problem for the fractional evolution equation, *Fract. Calc. Appl. Anal.*, **1**(3) (1998), 255–.

[8] E. Bazhlekova, Perturbation properties for abstract evolution equations of fractional order, *Fract. Calc. Appl. Anal.*, **2** (1999), 359–366.

[9] E. Bazhlekova, Subordination principle for fractional evolution equations, *Fract. Calc. Appl. Anal.*, **3** (2000), 213–230.

[10] S. Bu, Mild well-posedness of equations with fractional derivative, *Math. Nachr.*, **285**(2–3) (2012), 202–209.

[11] S. Bu, Well-posedness of equations with fractional derivative via the method of sum, *Acta Math. Sin. Engl. Ser.*, **28**(1) (2012), 37–44.

[12] S. Bu and G. Cai, Well-posedness of degenerate differential equations with fractional derivative in vector-valued functional spaces, *Math. Nachr.*, **290**(5–6) (2017), 726–737.

[13] C. Chen and M. Li, On fractional resolvent operator functions, *Semigroup Forum*, **80** (2010), 121–142.

[14] C. Chen and M. Li, Characterizations of domains of fractional powers for non-negative operators, *J. Math. Anal. Appl.*, **435** (2016), 179–209.

[15] C. Chen, M. Li, and F. Li, On boundary values of fractional resolvent families, *J. Math. Anal. Appl.*, **384**(2) (2011), 453–467.

[16] Ph. Clément, G. Gripenberg, and S.-O. Londen, Regularity properties of solutions of fractional evolution equations, in G. Lumer and L. Weiss (eds.) *Evolution Equations and Their Applications in Physical and Life Sciences*, Marcel Dekker Inc., Basel, 2000.

[17] Ph. Clément and S.-O. Londen, Regularity aspects of fractional evolution equations, *Rend. Ist. Mat. Univ. Trieste*, **31** (2000), 19–30.

[18] E. Cuesta, Asymptotic behaviour of the solutions of fractional integro-differential equations and some time discretizations, *Discrete Contin. Dyn. Syst.* (Supplement) (2007), 277–285.

[19] C. R. Day, Spectral mapping theorem for integrated semigroups, *Semigroup Forum*, **47**(3) (1993), 359–372.

[20] J. C. De-Souza and C. Lizama, The complex inversion formula in UMD spaces for families of bounded operators, *Appl. Anal.*, **91**(15) (2012), 937–946.

[21] M. M. El-Borai, Some probability densities and fundamental solutions of fractional evolution equations, *Chaos Solitons Fractals*, **14** (2002), 433–440.

[22] K. J. Engel and R. Nagel, *One-parameter Semigroups for Linear Evolution Equations*, Graduate Texts in Mathematics, vol. 194, Springer-Verlag, New York, 2000.

[23] Z. Fan, Characterization of compactness for resolvents and its applications, *Appl. Math. Comput.*, **232** (2014), 60–67.

[24] Z. Fan, Existence and regularity of solutions for evolution equations with Riemann–Liouville fractional derivatives, *Indag. Math.*, **25**(3) (2014), 516–524.

[25] C. Gal and M. Warma, Fractional-in-time semilinear parabolic equations and applications. HAL Id: hal-01578788, https://hal.archives-ouvertes.fr/hal-01578788.

[26] H. Gou, and B. Li, Existence of mild solutions for fractional nonautonomous evolution equations of Sobolev type with delay, *J. Inequal. Appl.*, **19** (2017), Paper No. 252, 20 pages.

[27] M. Haase, *The Functional Calculus for Sectorial Operators*, Oper. Theory Adv. Appl., vol. 169, Birkhäuser-Verlag, Basel, 2006.

[28] V. Keyantuo and C. Lizama, A characterization of periodic solutions for time-fractional differential equations in *UMD* spaces and applications, *Math. Nachr.*, **284**(4) (2011), 494–506.

[29] V. Keyantuo, C. Lizama, and M. Warma, Spectral criteria for solvability of boundary value problems and positivity of solutions of time-fractional differential equations, *Abstr. Appl. Anal.*, **2013** (2013), Article ID 614328, 11 pages, DOI:10.1155/2013/614328.

[30] V. Keyantuo, C. Lizama, and M. Warma, Existence, regularity and representation of solutions of time fractional diffusion equations, *Adv. Differ. Equ.*, **21**(9–10) (2016), 837–886.

[31] V. Keyantuo, C. Lizama, and M. Warma, Existence, regularity and representation of solutions of time fractional wave equations, *Electron. J. Differ. Equ.*, **222** (2017), 1–42.

[32] P. A. Kirichuk, Correctness of the abstract Cauchy problem and an operator-valued function of the Mittag-Leffler type, *Dokl. Akad. Nauk Ukr. SSR, Ser. A. Fiz.-Mat. Teh. Nauki*, **12** (1989), 10–12.

[33] A. N. Kochubei, A Cauchy problem for evolution equations of fractional order, *Differ. Equ.*, **25**(8) (1989), 967–974.

[34] A. N. Kochubei, Fractional differential equations: α-entire solutions, regular and irregular singularities, *Fract. Calc. Appl. Anal.*, **12** (2009), 135–158.

[35] M. Kostić, Abstract time-fractional equations: existence and growth of solutions, *Fract. Calc. Appl. Anal.*, **14**(2) (2011), 301–315.

[36] M. Kostić, Abstract Volterra equations in locally convex spaces, *Sci. China Math.*, **55**(9) (2012), 1797–1825.

[37] V. A. Kostin, The Cauchy problem for an abstract differential equation with fractional derivatives, *Russ. Acad. Sci., Dokl. Math.*, **46**(2) (1993), 316–319.

[38] G. G. Li, M. Kostic, M. Li, and S. Piskarev, On a class of time-fractional differential equations, *Fract. Calc. Appl. Anal.*, **15**(4) (2012), 639–667.

[39] K. Li and J. Peng, Fractional resolvents and fractional evolution equations, *Appl. Math. Lett.*, **25**(5) (2012), 808–812.

[40] K. Li, J. Peng, and J. Jia, Cauchy problems for fractional differential equations with Riemann–Liouville fractional derivatives, *J. Funct. Anal.*, **263** (2012), 476–510.

[41] M. Li, C. Chen, and F. Li, On fractional powers of generators of fractional resolvent families, *J. Funct. Anal.*, **259** (2010), 2702–2726.

[42] M. Li and Q. Zheng, On spectral inclusions and approximations of α-times resolvent families, *Semigroup Forum*, **69** (2004), 356–368.

[43] Y. Li, Regularity of mild solutions for fractional abstract Cauchy problem with order $\alpha \in (1, 2)$, *Z. Angew. Math. Phys.*, **66** (2015), 3283–3298.

[44] Y. N. Li and H. R. Sun, Integrated fractional resolvent operator function and fractional abstract Cauchy problem, *Abstr. Appl. Anal.*, **2014** (2014), Article ID 430418, 9 pages.

[45] Y. N. Li and H. R. Sun, Regularity of mild solutions to fractional Cauchy problems with Riemann–Liouville fractional derivative, *Electron. J. Differ. Equ.*, **184** (2014), 1–13.

[46] C. Lizama, Uniform continuity and compactness for resolvent families of operators, *Acta Appl. Math.*, **38**(2) (1995), 131–138.

[47] C. Lizama, A representation formula for strongly continuous resolvent families, *J. Integral Equ. Appl.*, **9**(4) (1997), 321–327.

[48] C. Lizama, A characterization of uniform continuity for Volterra equations in Hilbert spaces, *Proc. Am. Math. Soc.*, **126**(12) (1998), 3581–3587.

[49] C. Lizama, Regularized solutions for abstract Volterra equations, *J. Math. Anal. Appl.*, **243**(2) (2000), 278–292.

[50] C. Lizama, On approximation and representation of k-regularized resolvent families, *Integral Equ. Oper. Theory*, **41**(2) (2001), 223–229.

[51] C. Lizama and P. J. Miana, A Kallman-Rota inequality for generators of families of bounded operators, *J. Math. Anal. Appl.*, **371** (2010), 614–623.

[52] C. Lizama and F. Poblete, On a functional equation associated with (a, k)-regularized resolvent families, *Abstr. Appl. Anal.*, **2012** (2012), Article ID 495487, 23 pages, DOI:10.1155/2012/495487.

[53] C. Lizama and V. Poblete, Periodic solutions of fractional differential equations with delay, *J. Evol. Equ.*, **11**(1) (2011), 57–70.

[54] C. Lizama, R. Ponce, and A. Pereira, On the compactness of fractional resolvent operator functions, *Semigroup Forum*, **93**(2) (2016), 363–374.

[55] C. Lizama, R. Ponce, and A. Pereira, Norm continuity for strongly continuous families of operators, *Int. J. Evol. Equ.*, **10**(2) (2017), 161–178.

[56] C. Lizama and H. Prado, Rates of approximation and ergodic limits of regularized operator families, *J. Approx. Theory*, **122** (2003), 42–61.

[57] C. Lizama and H. Prado, On duality and spectral properties of (a, k)-regularized resolvents, *Proc. R. Soc. Edinb., Sect. A, Math.*, **139**(3) (2009), 505–517.

[58] C. Lizama and J. Sánchez, On perturbation of k-regularized resolvent families, *Taiwan. J. Math.*, **7**(2) (2003), 217–227.

[59] D. Lutz, On bounded time dependent perturbations of operator cosine functions, *Aequ. Math.*, **23** (1981), 197–203.

[60] C. Martinez and M. Sanz, *The Theory of Fractional Powers of Operators*, North-Holland Math. Stud., vol. 187, North-Holland, Amsterdam, 2001.

[61] D. Matignon, Stability properties for generalized fractional differential systems, in *Systemes Differentiels Fractionnaires—Modeles, Methodes et Applications*, ESAIM: Proc., vol. 5, 1998.

[62] Z. D. Mei, J. G. Peng, and J. H. Gao, General fractional differential equations of order $\alpha \in (1, 2)$ and type $\beta \in [0, 1]$ in Banach spaces, *Semigroup Forum*, **94**(3) (2017), 712–737.

[63] Z. D. Mei, J. G. Peng, and Y. Zhang, A characteristic of fractional resolvents, *Fract. Calc. Appl. Anal.*, **16**(4) (2013), 777–790.

[64] G. M. Mophou and G. M. N'Guérékata, On a class of fractional differential equations in a Sobolev space, *Appl. Anal.*, **91**(1) (2012), 15–34.

[65] Y.-Y. Mu and M. Li, Generation theorems for fractional resolvent families, *Contemp. Anal. Appl. Math.*, **3**(1) (2015).

[66] B. Nagy, On cosine operator functions in Banach spaces, *Acta Sci. Math.*, **36** (1974), 281–290.

[67] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Applied Math. Sc., vol. 44, Springer, New York, 1983.

[68] J. Peng and K. Li, A note on property of the Mittag-Leffler function, *J. Math. Anal. Appl.*, **370** (2010), 635–638.

[69] J. Peng and K. Li, A novel characteristic of solution operator for the fractional abstract Cauchy problem, *J. Math. Anal. Appl.*, **385**(2) (2012), 786–796.

[70] R. Ponce, Existence of mild solutions to nonlocal fractional Cauchy problems via compactness, *Abstr. Appl. Anal.*, **2016** (2016), Article ID 4567092, 15 pages.

[71] J. Prüss, *Evolutionary Integral Equations and Applications*, Monographs Math., vol. 87, Birkhäuser Verlag, 1993.

[72] J. Sabatier, M. Moze, and C. Farges, LMI stability conditions for fractional order systems, *Comput. Math. Appl.*, **59** (2010), 1594–1609.

[73] S.-Y. Shaw, Convergence rates of ergodic limits and approximate solutions, *J. Approx. Theory*, **75** (1993), 157–166.

[74] S.-Y. Shaw, Growth order and stability of semigroups and cosine operator functions, *J. Math. Anal. Appl.*, **357** (2009), 340–348.
[75] X. B. Shu, Y. Lai, and Y. Chen, The existence of mild solutions for impulsive fractional partial differential equations, *Nonlinear Anal., Theory Methods Appl.*, **74**(5) (2011), 2003–2011.
[76] R. N. Wang, D. H. Chen, and T. J. Xiao, Abstract fractional Cauchy problems with almost sectorial operators, *J. Differ. Equ.*, **252** (2012), 202–235.
[77] K. Yosida, Fractional powers of infinitesimal generators and the analyticity of the semigroups generated by them, *Proc. Jpn. Acad.*, **36** (1960), 86–89.
[78] P. You, Characteristic conditions for a C_0-semigroup with continuity in the uniform operator topology for $t > 0$ in Hilbert space, *Proc. Am. Math. Soc.*, **116** (1992), 991–997.
[79] R. Zacher, Maximal regularity of type L_p for abstract parabolic Volterra equations, *J. Evol. Equ.*, **5** (2005), 79–103.
[80] Y. Zhou, *Fractional Evolution Equations and Inclusions: Analysis and Control*, Elsevier/Academic Press, London, 2016.

Carlos Lizama

Abstract nonlinear fractional evolution equations

Abstract: In this paper, we review some of the main results in the field of abstract nonlinear fractional evolution equations. We study solutions of the semilinear Cauchy problem in the subdiffusive and superdiffusive cases, mainly with the Caputo and Riemann–Liouville fractional-order derivative, in the setting of the real semiaxis and real axis, and under various assumptions on the main data of the given equations. We consider in our analysis several kinds of perturbed systems, for example, delay, control, and stochastic properties, even with nonlocal conditions and impulses. We provide a complete description of the representation of mild solutions in terms of associated solution families of operators.

Keywords: Semilinear fractional Cauchy problem, Caputo fractional derivative, Riemann–Liouville fractional derivative, mild solutions

MSC 2010: 35R11, 26A33, 37L05

For about two decades, semilinear abstract Cauchy problems of the form

$$\begin{cases} {}^{C}D_t^{\alpha}u(t) = Au(t) + f(t, u(t)), \quad t \geq 0, \quad 0 < \alpha \leq 2; \\ u(0) = u_0; \quad \max\{0, \alpha - 1\}(u'(0) - u_1) = 0, \end{cases} \tag{0.1}$$

in a Banach space X have been studied in many different settings. Here, A denotes the generator of a strongly continuous family of bounded and linear operators in X and f a nonlinear function.

This abstract and general approach enables that the results can be applied to a broad variety of problems, including both ordinary fractional equations and partial fractional equations problems.

The basic idea for this type of approach is to consider the integral equation

$$u(t) = S_\alpha(t)x_0 + \int_0^t R_\alpha(t-s)f\big(s, u(s)\big)ds, \quad t \geq 0, \quad 0 < \alpha \leq 1, \tag{0.2}$$

where $\{R_\alpha(t)\}_{t>0}$ is an (α, α)-resolvent family generated by A and $S_\alpha(t) := (g_{1-\alpha} * R_\alpha)(t)$ is an α-resolvent family, also generated by the same operator A. The reason is that under appropriate conditions on the initial data x_0 and the forcing term f, a solution of the equation (0.2) corresponds to a strong (or classical) solution of the equation (0.1); see

Carlos Lizama, Universidad de Santiago de Chile, Departamento de Matemática y Ciencia de la Computación, Las Sophoras 173, Estación Central, Santiago, Chile, e-mail: carlos.lizama@usach.cl

https://doi.org/10.1515/9783110571660-022

the chapter "Abstract linear fractional evolution equations" for more information on this subject.

As a consequence of the definition of fractional derivative of order $\alpha > 0$, in the situation that $1 < \alpha \le 2$, one should consider the integral equation

$$u(t) = S_\alpha(t)u_0 + (g_1 * S_\alpha)(t)u_1 + \int_0^t R_\alpha(t-s)f(s,u(s))ds \tag{0.3}$$

where $\{S_\alpha(t)\}_{t\ge 0}$ is an α-resolvent family generated by A and $R_\alpha(t) := (g_{\alpha-1} * S_\alpha)(t)$ is an (α,α)-resolvent family, generated by A. Note the inversion of roles between the families $S_\alpha(t)$ and $R_\alpha(t)$ in the cases $0 < \alpha \le 1$ and $1 < \alpha \le 2$, respectively.

Solutions of the vector-valued integral equation (0.2) (resp., (0.3)) are called *mild* solutions of (0.1). It should be noted that the definition of mild solution for abstract fractional differential equations has been misunderstood for some researchers, contrasting with those known in the literature on the subject [11, Section 4]. This has been observed in some papers since some time ago [29, 47, 61].

If A is the generator of a C_0-semigroup $\{S_1(t)\}_{t\ge 0}$ and $0 < \alpha < 1$ then, using the subordination principle, we can show an explicit description of the families of bounded and linear operators in (0.2) as follows: The (α,α)-resolvent family generated by A is given by

$$R_\alpha(t) = t^{\alpha-1}P_\alpha(t) \quad \text{where } P_\alpha(t) := \alpha\int_0^\infty s\Phi_\alpha(s)S_1(st^\alpha)ds, \quad t > 0 \tag{0.4}$$

and the α-resolvent family generated by A is represented by

$$S_\alpha(t) = \int_0^\infty \Phi_\alpha(s)S_1(st^\alpha)ds, \quad t > 0; \tag{0.5}$$

see Subsection 2.2 in the chapter "Abstract linear fractional evolution equations" for details. Here, Φ_α are the functions of Wright-type defined by

$$\Phi_\alpha(s) := \sum_{n=0}^\infty \frac{(-s)^n}{n!\alpha(-\alpha n + 1 - \alpha)} = \frac{1}{\pi\alpha}\sum_{n=1}^\infty (-s)^{n-1}\frac{\Gamma(n\alpha+1)}{n!}\sin(n\pi\alpha),$$

valid for $0 < \alpha < 1$ and $s \ge 0$. In the formulation (0.2) (resp., (0.3)), the unbounded operator A does only appear in terms of the family $R_\alpha(t)$ (resp., $S_\alpha(t)$), which makes, for example, the application of fixed-point theorems for the solution of the integral equation possible.

A typical argument uses the contraction mapping principle. In such case, we suppose that the nonlinearity f is continuous and globally Lipschitz. This argument has been refined in many directions, for example, localizing the Lipschitz condition, or replacing it by compactness assumptions involving measures of noncompactness and applying fixed-point theorems for set contractions, or by allowing A to depend on t. This way also a qualitative theory for mild solutions of (0.1) can be developed.

There is a already a number of papers available where these ideas have been carried out in various settings, most of them in the autonomous one. In the nonautonomous case, that is, when A is time dependent, much work remains to be done.

In what follows, for transparency reasons, we separately discuss the cases

$$0 < \alpha \leq 1 \quad \text{and} \quad 1 < \alpha \leq 2.$$

1 The semilinear Cauchy problem: $0 < \alpha \leq 1$

1.1 Caputo fractional derivative

As standard model, we consider the semilinear Cauchy problem

$$\begin{cases} {}^C D_t^\alpha u(t) = Au(t) + f(t, u(t)), & t \geq 0, \quad 0 < \alpha \leq 1, \\ u(0) = u_0. \end{cases} \tag{1.1}$$

This is by far the most studied fractional abstract model in the existing literature. The analysis cover the existence and uniqueness of mild solutions for the fractional evolution equation (1.1) under different conditions on the operator A, the nonlinear term f and the initial condition u_0. The study of fractional controllability, fractional inclusions, and fractional stochastic evolution equations and inclusions are the most common subjects of research.

Recall that a mild solution of (1.1) is a solution of the integral equation

$$u(t) = S_\alpha(t)x_0 + \int_0^t R_\alpha(t-s)f(s, u(s))ds, \quad t \geq 0, \tag{1.2}$$

where $S_\alpha(t) = (g_{1-\alpha} * R_\alpha)(t)$, being $\{R_\alpha(t)\}_{t>0}$ an (α, α)-resolvent family generated by A. Existence and uniqueness of local and global mild solutions for (1.1) were investigated by Chen, Li, Chen, and Feng [15] when A is the generator of an uniformly bounded and immediately norm continuous C_0-semigroup. As methods, they have mainly used Sadovskii's fixed-point theorem and the technique of the measure of noncompactness. See also the paper [36] for a related result using the Kuratowski measure of noncompactness. However, until now, the most complete study of the semilinear problem (1.1) is contained in the monograph [21, Section 3] by Gal and Warma, which also includes several interesting applications, examples, and historical remarks.

If A is the generator of a positive, compact, and uniformly bounded C_0-semigroup, the existence of minimal and maximal mild solutions for the problem (1.1) with periodic boundary conditions has been studied by Mu and Li [54]. They used the method of upper and lower solutions coupled with a monotone iterative technique and the properties of positive C_0-semigroups.

Under the hypothesis that A is a sectorial operator on a Banach space X (i. e., A is the generator of an analytic semigroup), the authors, Shu, Lai, and Chen [61], proved the existence of unique mild solutions of (1.1) with impulses. Under the same hypothesis of sectoriality of A, Guswanto and Suzuki [26] studied the existence and uniqueness of a local mild solution for the problem (1.1) with nonlinear term in the form $f(u(t))$. They put some conditions on f and the initial data u_0 in terms of the fractional powers of A. By applying Banach's fixed- point theorem, they obtain a unique local mild solution with smoothing effects, estimates, and a behavior at t close to 0. In the same line of ideas, and under some local Lipschitz conditions on f, De Andrade, Carvalho, Carvahlo-Neto, and Marin-Rubio [18] proved an existence and uniqueness theorem for local mild solutions to (1.1), as well as continuation, noncontinuation (due to blow-up) and global existence results. They also investigate critical cases by proving the existence of the so-called ϵ-regular mild solutions using a technique of fractional power spaces associated to the operator A. For some other results under the hypothesis of sectoriality on A, see the recent book of Zhou [74, Section 2.1.4].

Assuming that A is the generator of a compact semigroup, Chauhan and Dabas [14] proved existence of mild solutions for (1.1) with nonlocal conditions

$$u(0) + g(u) = u_0, \tag{1.3}$$

and impulses. Moreover, in [14], the nonlinear term admits the general form $f(t, u(t), u(a_1(t)), \ldots, u(a_m(t)))$ where a_i are scalar functions defined on a finite interval. In the range $1/2 < \alpha < 1$, Ponce [58, Theorem 23] proved existence of mild solutions for (1.1) with the nonlocal conditions (1.3) but assuming that A is the generator of an (α, α)-resolvent family $R_\alpha(t)$ and $(\lambda - A)^{-1}$ is compact for some $\lambda \in \rho(A)$.

We remark that the nonlocal condition (1.3) has a better effect on the solution of (1.1) and is more precise for physical measurements than the classical condition alone.

If the semigroup generated by A is noncompact, Gou and Li [23] investigated local and global existence of mild solution for (1.1) with impulses and an additional nonlinear term in the form

$$\int_0^t q(t-s)g(s, u(s))ds.$$

Existence results under general and weak assumptions on f by utilizing Schaefer and O'Regan fixed-point theorems have been proved by Wang, Zhou, and Feckan [71]. If A generates a bounded analytic semigroup, existence of mild solutions for fractional-order equations with infinite delay and an integral nonlinearity in the form $\int_0^t a(t,s)f(s,u(s),u_s)ds$ has been analyzed by Aissani and Benchohra [7], by means of the application of Mönch's fixed-point theorem combined with the Kuratowski measure of noncompactness.

An interesting way to deal with many kinds of nonlinearities at once, is the use of the notion of causal operator due to Tonelli [65]. This approach has been recently pursued by Agarwal, Asma, Lupulescu, and O'Regan [3]. The main idea is consider the model

$$\begin{cases} {}^C D_t^\alpha u(t) = Au(t) + (Qu)(t), & \text{a.e. } t \in [0,T], \\ u|_{[-\sigma,0]} = \varphi \in C([-\sigma,T],X), & \sigma \geq 0, \end{cases} \tag{1.4}$$

where $Q : C([-\sigma,T],X) \to L^p([0,T],X)$ is a causal operator, that is, for each $\tau \in [0,T)$ and for all $u, v \in L^p([-\sigma,T],X)$ with $u(t) = v(t)$ for every $t \in [0,\tau]$, we have $Qu(t) = Qv(t)$ for a.e. $t \in [0,\tau]$. Using this approach, the authors in [3] proved existence of mild solution of (1.4) assuming that A is the generator of an immediately norm continuous semigroup.

In [59], Sakthivel, Ren, and Mahmudov established sufficient conditions for the approximate controllability of the problem

$${}^C D_t^\alpha u(t) = Au(t) + Bx(t) + f\big(t,u(t)\big), \quad t \in [0,T], \tag{1.5}$$

with initial condition $u(0) = u_0$. Here, the state variable $u(\cdot)$ takes values in the Hilbert space H; A is the generator of a C_0-semigroup; the control function $x(\cdot)$ is given in $L^2([0,T],U)$, U is a Hilbert space; and B is a bounded linear operator from U into H. The results are established under the assumption that the associated linear system is approximately controllable. Further, the authors extend their results to study the approximate controllability of fractional systems with nonlocal conditions as in (1.3); see also the papers [32, 68] for further results on this problem.

We point out that if the operator B is compact, or the C_0-semigroup generated by A is compact, then the controllability operator is also compact and hence the inverse of it does not exist if the state space is infinite dimensional; see [28] and [55]. Thus, the concept of exact controllability for fractional differential equations is too strong in infinite dimensional spaces and the notion of approximate controllability is more appropriate.

In [22] Ge, Zhou, and Kou studied the approximate controllability of the semilinear fractional evolution equation (1.5) with nonlocal and impulsive conditions. The impulsive functions in that paper are supposed to be continuous and the nonlocal item is divided into two cases: Lipschitz continuous and only continuous, which generalizes previous contributions.

In the article [19], Debbouche and Torres studied the approximate controllability of (1.5), where the control function depends on multidelay arguments and where the nonlocal condition is fractional. In [38], Liu and Fu studied controllability for (1.1) with a nonlinear term in the form $f(t,u(t)) + g(t)x(t)$ and a mixed nonconvex constraint on the control $x(\cdot)$. We also note the work of Mophou [52] where she studied the approximate controllability of a fractional semilinear differential equation (1.5) but involving the right fractional Caputo derivative.

Sufficient conditions for the approximate controllability of (1.1) with bounded delay, namely, the class

$$\begin{cases} {}^{C}D_t^{\alpha}u(t) = Au(t) + Bx(t) + f(t, u(t-h)), & t \in (0, \tau], \quad \frac{1}{2} < \alpha \le 1, \\ u(t) = \varphi(t), & t \in [-h, 0], \end{cases}$$

have been considered by Kumar and Sakavanam [34].

Fractional semilinear differential inclusions in Banach spaces has been studied by Wang and Zhou [70]. By using the Bohnenblust–Karlin's fixed-point theorem, an existence result of mild solutions for the multivalued version of (1.1) was obtained in [70] under the assumption that A generates a compact semigroup. In such paper, also controllability results are discussed. The paper of Liu and Liu [39] studied also the same topic but relaxing the conditions in the nonlinearity, admiting f nonconvex; see also [40, 69, 74] for additional research on this topic.

Under the general assumption that A is the generator of an (α, α)-resolvent family, relative controllability for a class of semilinear stochastic fractional differential equation with nonlocal conditions of the form

$$\begin{cases} {}^{C}D_t^{\alpha}u(t) = Au(t) + Bx(t) + f(t, u(t)) + \sigma(t, u(t))\frac{dw(t)}{dt}, & t \in [0, T], \\ u(0) + g(u) = u_0, \end{cases}$$

in Hilbert spaces, were studied by Guendouzi and Hamada [25]; see also the article of Saktivel, Revathi, and Ren [60], where impulsive fractional stochastic differential equations with infinite delay under the same hypothesis has been studied. In [66], Toufik studied existence and controllability results for fractional stochastic semilinear differential inclusions of the above form. The results are obtained by using the Bohnenblust–Karlin fixed-point theorem. More recently, Ahmed [6] proved the existence of mild solutions for a very general class of semilinear neutral fractional stochastic integrodifferential equations with nonlocal conditions. Ahmed, derived sufficient conditions with the help of the Sadovskii fixed-point theorem.

When $A = A(t)$ are bounded operators and $0 < \alpha < 1$, the problem of existence and uniqueness of solutions on an interval $[0, T]$ was studied by Balachandran and Park [10], as well as the existence and uniqueness of solutions with nonlocal condition of the form (1.3) where g is a given function satisfying certain Lipschitz-type conditions. In the paper [9], the authors deal with fractional impulsive evolution equations. Under classical assumptions and by using the Banach contraction principle, the authors proved the existence and uniqueness of solutions.

1.2 Riemann–Liouville fractional derivative

We consider the model

$$ {}^{\mathrm{RL}}D_t^{\alpha}u(t) = Au(t) + f\big(t, u(t)\big), \quad t \ge 0, \quad 0 < \alpha \le 1, \tag{1.6}$$

with initial condition $(g_{1-\alpha} * u)(0) = u_0$.

We observe that the definition of Riemann–Liouville fractional initial condition $(g_{1-\alpha} * u)(0) = u_0$ is difficult to interpret, but play an important role in some practical problems. Heymans and Podlubny [30] have demonstrated that it is possible to attribute physical meaning to initial conditions expressed in terms of Riemann–Liouville fractional derivatives on the field of the viscoelasticity.

Assuming that A is the generator of an (α,α)-resolvent family $\{R_\alpha(t)\}_{t>0}$, the appropriate and more general notion of mild solution for (1.6) is a locally integrable function u such that $g_{1-\alpha} * u$ is absolutely continuous and satisfy the integral equation

$$u(t) = R_\alpha(t)u_0 + \int_0^t R_\alpha(t-s)f(s,u(s))ds, \quad t > 0, \tag{1.7}$$

see [56, Lemma 4] and the chapter "Abstract linear fractional evolution equations." In the paper [58, Theorem 26], Ponce proved existence of mild solutions for the problem (1.6) with nonlocal initial conditions. He assumes that A is the generator of an immediately norm continuous (α,α)-resolvent family and that the resolvent operator $(\lambda - A)^{-1}$ is compact for some $\lambda \in \rho(A)$. In the article [56], Pan, Li, and Zhao investigated the solvability and optimal controllability for the following semilinear control system

$${}^{\mathrm{RL}}D_t^\alpha u(t) = Au(t) + f(t,u(t)) + Bx(t), \quad t \in J := (0,T], \tag{1.8}$$

with initial condition $(g_{1-\alpha} * u)(0) = u_0$. Here, the control function $x(\cdot)$ is given in a suitable admissible control set, and B is a linear operator from a separable reflexive Banach space Y into X. If A is the generator of a compact C_0-semigroup, then sufficient conditions for the existence of mild solutions of (1.8) are proved. Further, optimal control results corresponding to the admissible control sets are shown. More recently, approximate controllability was studied in [31] when A is the generator of an (α,α)-resolvent family; see also the paper of Liu and Li [43] where existence of mild solutions of (1.8) in the space $C_{1-\alpha}(J;X) := \{u : t^{1-\alpha}u(t) \in C(J;X)\}$ as well as approximate controllability is studied. The article [41] consider also impulses.

If A is almost sectorial, that is, $A \in \Theta_\omega^p(X)$ where $-1 < p < 0 < \omega < \pi/2$, and the associated C_0-semigroup generated by A is compact, then Zhou [74, Section 2.1.3] proved that, under suitable conditions on f, the problem (1.6) has at least one mild solution in $B_r^{(\alpha)}(J)$, for every $x_0 \in D(A^\beta)$ with $\beta > 1 + p$. Here $B_r^{(\alpha)}(J)$ is the ball of radius r of the Banach space $X^{(\alpha)}(J) := \{u \in C(J;X) : \lim_{t\to 0^+} t^{1+\alpha p}u(t)$ exists and is finite$\}$ provided with the norm $\|u\|_\alpha := \sup_{t\in J} t^{1+\alpha p}\|u(t)\|$.

Fractional evolution inclusions for (1.6) with nonconvex right-hand side has been only recently studied by Liu, Bin, and Liu [42]. Assuming that A is the generator of an (α,α)-resolvent family, they proved existence of the extreme solution and the relationship of the solution sets between the original problem and the convexified problem.

It should be noted that in recent years a practical and useful way to treat the Caputo and Riemann–Liouville fractional abstract Cauchy problem, simultaneously, has been investigated by means of the Hilfer fractional derivative. The notion of mild solution in such case is a solution of

$$u(t) = (g_{\gamma(1-\alpha)} * R_\alpha)(t)u_0 + \int_0^t R_\alpha(t-s)f(s,u(s))ds, \quad t > 0, \tag{1.9}$$

where $0 \le \gamma \le 1$ and $0 < \alpha \le 1$. Here, $\{R_\alpha(t)\}_{t>0}$ is an (α,α)-resolvent family generated by A. Note that $(g_{\gamma(1-\alpha)} * R_\alpha)(t)$ is an $(\alpha, \alpha+\gamma(1-\alpha))$-resolvent family with the same generator. When $\gamma = 0$, the Hilfer fractional derivative corresponds to the classical Riemann–Liouville fractional derivative and (1.9) is the same that (1.7) ($g_0 \equiv$ Dirac delta). When $\gamma = 1$, the Hilfer fractional derivative corresponds to the classical Caputo fractional derivative and (1.9) reduces to (1.2) in Section 1.1. For more details on this type of approach for the study of existence of mild solutions for nonlinear fractional nonautonomous evolution equations of Sobolev type with delay, see the recent paper of Gou and Li [23].

2 The semilinear Cauchy problem: $1 < \alpha \le 2$

2.1 Caputo fractional derivative

In this section, we deal with the problem

$$^{C}D_t^\alpha u(t) = Au(t) + f(t,u(t)), \quad t \ge 0, \quad 1 < \alpha \le 2, \tag{2.1}$$

with initial conditions $u(0) = u_0$ and $u'(0) = u_1$. Recall that a mild solution of (2.1) is understood as a solution of the integral equation

$$u(t) = S_\alpha(t)u_0 + (g_1 * S_\alpha)(t)u_1 + \int_0^t R_\alpha(t-s)f(s,u(s))ds \tag{2.2}$$

where $\{S_\alpha(t)\}_{t\ge 0}$ is an α-resolvent family generated by A and $R_\alpha(t) := (g_{\alpha-1} * S_\alpha)(t)$ is an (α,α)-resolvent family with the same generator.

If A is the generator of a strongly continuous cosine family $\{S_2(t)\}_{t\ge 0}$ and $1 < \alpha < 2$ then, using the subordination principle, we have the following explicit representation:

$$S_\alpha(t) = \int_0^\infty \Phi_{\alpha/2}(s)S_2(st^{\alpha/2})ds.$$

Existence and uniqueness of mild solutions of (2.1) with nonlocal initial conditions have been proved by Ponce [58, Theorems 20 and 21] under the hypothesis that A is the generator of an α-resolvent family and $(\lambda - A)^{-1}$ is compact for some $\lambda \in \rho(A)$.

Assuming that A is the generator of an α-resolvent family, Li [35] proved the existence of mild solutions of (2.1) with a nonlinear term in the form

$$\int_0^t h(t,s,u(s))ds + g(t).$$

The paper by Mophou and N'Guérékata [53] is concerned with the semilinear differential system of fractional order with infinite delay:

$$ {}^C D_t^\alpha u(t) = Au(t) + Bx(t) + f(t,u_t), \quad t \in (0,T], \tag{2.3}$$

$u(t) = \phi(t), t \in (-\infty, 0]$. The authors proved that the system is controllable when A generates an α-resolvent family $(S_\alpha(t))_{t\geq 0}$ on a complex Banach space X and the control $x \in L^2([0,T];X)$. This problem has been also recently studied by Shukla, Sukanavam, and Pandey [63].

Assuming that A is the generator of a compact (α,α)-resolvent family, Shukla, Sukavanam, and Pandey proved in [62] that the problem

$$ {}^C D_t^\alpha u(t) = Au(t) + Bx(t) + f(t,u(t)), \quad t \in [0,T],$$

with initial conditions $u(0) = u_0$ and $u'(0) = u_1$ is approximately controllable. They used Schauder's fixed-point theorem in order to achieve their results. We remark that necessary and sufficient conditions for the compactness of $\{R_\alpha(t)\}_{t\geq 0}$ have been recently studied by Lizama, Pereira, and Ponce [48, 58]. Under essentially the same hypothesis on A, Guendouzi and Farahi considered in [24] the approximate controllability for a class of fractional semilinear stochastic dynamic systems with nonlocal conditions in Hilbert spaces of the form

$$\begin{aligned} {}^C D_t^\alpha u(t) &= Au(t) + Bx(t) + f(t,u(t),u(b_1(t),\dots,u(b_m(t))) \\ &\quad + \sigma(t,u(t),u(a_1(t),\dots,a_n(t))\frac{dw(t)}{dt}, \quad t \in [0,T]. \end{aligned}$$

Existence of solutions of (2.1) in the nonautonomous case, that is, when $A = A(t)$ are bounded linear operators, and considering impulses and antiperiodic boundary value conditions in the equation, have been proved by Agarwal and Ahmad [2]. The contraction mapping principle and Krasnoselskii's fixed-point theorem are applied to prove the main results. An extension of such result to the case of mixed boundary values has been studied by Zhang, Wang, and Song [73].

Dos Santos et al. [20] studied the existence of mild solutions for abstract fractional neutral equations of the type (2.1), but that includes an additional integral term and a state-dependent delay, by using the Leray–Schauder alternative fixed-point theorem. If the integral term is not present, the resulting mild solutions coincides with those given by (2.2). Very recently, Tamilalagan and Balasubramaniam [64] consider a class of fractional stochastic differential inclusions, that includes problem

(2.1), driven by fractional Brownian motion in Hilbert space with Hurst parameter $H \in (1/2, 1)$. Sufficient conditions for the existence and asymptotic stability of mild solutions are derived in mean square moment by employing (α, α)-resolvent families and Bohnenblust–Karlin's fixed- point theorem. For other contributions in this direction, see the references in [64].

2.2 Riemann–Liouville fractional derivative

We consider the model problem

$$ {}^{\mathrm{RL}}D_t^{\alpha} u(t) = Au(t) + f\big(t, u(t)\big), \quad t \geq 0, \quad 1 < \alpha \leq 2, \tag{2.4} $$

with initial conditions $(g_{2-\alpha} * u)(0) = u_0$ and $(g_{2-\alpha} * u)'(0) = u_1$. We note that in this case, controllability, stability analysis, and other qualitative and quantitative properties have not received much attention from researchers, and hence many problems are still open.

For the problem (2.4), a mild solution should be understood as a solution of the following integral equation:

$$ u(t) = L_\alpha(t)u_0 + R_\alpha(t)u_1 + \int_0^t R_\alpha(t-s)f\big(s, u(s)\big)ds, \quad t \geq 0, $$

where $\{L_\alpha(t)\}_{t\geq 0}$ is an $(\alpha, \alpha-1)$-resolvent family generated by A and $R_\alpha(t) := (g_1 * L_\alpha)(t)$ is an (α, α)-resolvent family, with the same generator A. However, except for the information provided by the general theory of (a, k)-regularized families, there exists little material in the literature concerning the family of operators $\{L_\alpha(t)\}_{t\geq 0}$. See the chapter "Abstract linear fractional evolution equations" for details.

When A is the generator of an $(\alpha, \alpha-1)$-resolvent family and $(\lambda - A)^{-1}$ is compact for some $\lambda \in \rho(A)$, Ponce [58, Theorems 24 and 25] proved the existence of at least one mild solution for (2.4) with nonlocal initial conditions. He assumed that f satisfies a Carathéodory-type condition and then uses the Krasnoselskii theorem.

When $u_0 = 0$, a unified approach to (2.4) with Riemann–Liouville and Caputo fractional-order derivative of order $1 < \alpha \leq 2$ has been developed by Mei, Peng, and Gao [51]. They used the Hilfer fractional derivative and obtain a representation of the homogeneous problem (2.4) with $u_0 = 0$ by means of $(\alpha, \alpha + \gamma(2-\alpha))$-resolvent families, where $0 \leq \gamma \leq 1$. This method can be used to develop a more complete theory for the linear and nonlinear problem, at least in this case. In a very general form, the linear problem (2.4) is included in the paper [33] by Kostic, where strong and mild solutions are considered.

3 The semilinear Cauchy problem on the line: $0 < \alpha \le 1$

We consider the model problem

$$ {}_{-\infty}D_t^{\alpha} u(t) = Au(t) + f\big(t, u(t)\big), \quad t \in \mathbb{R}, \quad 0 < \alpha \le 1, \tag{3.1} $$

where ${}_{-\infty}D_t^{\alpha}$ is the Liouville–Weyl fractional derivative. Suppose that A is the generator of an (α,α)-resolvent family $\{R_\alpha(t)\}_{t\ge 0}$. In such general case, a mild solution of (3.1) is a solution of the equation

$$ u(t) = \int_{-\infty}^{t} R_\alpha(t-s) f\big(s, u(s)\big)\,ds, \quad t \in \mathbb{R}, \tag{3.2} $$

whenever the above integral exists. The existence can be proved, for example, when the nonlinear term is bounded and A is the generator of an exponentially stable C_0-semigroup $\{S_1(t)\}_{t\ge 0}$ since in such case one can appeal to the representation

$$ R_\alpha(t) = t^{\alpha-1} P_\alpha(t) \quad \text{where } P_\alpha(t) := \alpha \int_0^{\infty} s\Phi_\alpha(s) S_1(st^\alpha)\,ds, \quad t > 0, $$

and use [74, Property 1.10(ii) and (iii)] and [74, Property 1.11(v)]. Using this representation, and under the hypothesis that A is the generator of an exponentially stable C_0-semigroup, which is in addition positive or compact, Zhou [74, Section 2.2.3] established some sufficient conditions for the existence and uniqueness of periodic solutions, S-asymptotically periodic solutions, and other types of bounded solutions when $f : \mathbb{R}\times X \to X$ satisfies some ordering hypothesis on X or Lipschitz conditions in f. The main methods are the monotone iterative technique and Banach contraction principle.

Bounded mild solutions to (3.1) when in the nonlinear term we add a perturbation in the form

$$ \int_{-\infty}^{t} a(t-s) A u(s)\,ds $$

have been studied by Ponce [57]. However, until now the study of the model (3.1) is still undeveloped and much work remains to be done.

4 The semilinear Cauchy problem on the line: $1 < \alpha \leq 2$

We consider the problem

$$ {}_{-\infty}D_t^{\alpha} u(t) = Au(t) + f\big(t, u(t)\big), \quad t \in \mathbb{R}, \quad 1 < \alpha \leq 2. \tag{4.1} $$

A mild solution of (4.1) is a fixed point of the equation

$$ u(t) = \int_{-\infty}^{t} R_\alpha(t-s) f\big(s, u(s)\big) ds, \quad t \in \mathbb{R}, $$

whenever the above integral exists. Here, $\{R_\alpha(t)\}_{t\geq 0}$ denotes an (α,α)-resolvent family generated by A. This extension of the notion of mild solution from the border cases $\alpha = 1$ and $\alpha = 2$ to the intermediary case $1 < \alpha < 2$ was first noted by Araya and Lizama [8]. Assuming that A is the generator of an integrable (α,α)-resolvent family $\{R_\alpha(t)\}_{t\geq 0}$, that is, such that

$$ \|R_\alpha(t)\| \leq \varphi_\alpha(t), \quad t > 0, \quad \varphi_\alpha \in L^1(\mathbb{R}_+), $$

and that f satisfies a global Lipschitz condition, it is proved in [8] the existence and uniqueness of an almost automorphic mild solution of the semilinear equation (4.1). See also the recent paper of Liu, Cheng and Zhang [37] that establish the existence of antiperiodic mild solutions. After the paper [8], Cuevas and Lizama [17] studied almost automorphic mild solutions of the equation (4.1) with forcing term $f(t,u(t)) := D_t^{\alpha-1} g(t,u(t))$. In such case, a mild solution of (4.1) is a fixed point of the equation

$$ u(t) = \int_{-\infty}^{t} S_\alpha(t-s) g\big(s, u(s)\big) ds, \quad t \in \mathbb{R}, $$

where $\{S_\alpha(t)\}_{t\geq 0}$ is an α-resolvent family generated by A. It should be observed that according to Cuesta [16], a such family $\{S_\alpha(t)\}_{t\geq 0}$ exists and is integrable whenever A is a sectorial operator of negative type.

Following this approach, in [4] Agarwal, Cuevas, and Soto proved sufficient conditions for the existence and uniqueness of a pseudo-almost periodic solutions of the equation (4.1) with forcing term $f(t,u(t)) := D_t^{\alpha-1} g(t,u(t))$. With the same forcing term, Cao, Yang, and Huang [12], proved existence of antiperiodic mild solutions, Chang, Zhang, and N'Guérékata [13] proved the existence of weighted pseudo almost automorphic mild solutions and He, Cao, and Yang [27] established sufficient criteria for the existence and uniqueness of a weighted Stepanov-like pseudo-almost automorphic mild solution. For a review of regularity results in several classes of vector-valued subspaces of the space of continuous and bounded functions, see the paper

[49] by Lizama and Poblete. Wang and Xia [67] proved the existence and uniqueness of (μ, ν)-pseudo almost automorphic mild solution. See also the paper [72] by Xia, Fan, and Agarwal where the same property is investigated but with nonlinearity in the form $D_t^{\alpha-1} f(t, Bu(t))$, being B a bounded linear operator. Also, in the reference [52], Mophou studied the existence and uniqueness of weighted pseudo almost automorphic mild solution to the semilinear fractional equation (4.1) with $f(t, u(t)) := g(t, u(t), Bu(t))$. This extends a previous paper of Agarwal, de Andrade, and Cuevas [5] where the case $B = 0$ was considered. The results obtained are utilized to study the existence and uniqueness of a weighted pseudo-almost automorphic solution to fractional diffusion wave equation with Dirichlet conditions.

Although existence and uniqueness of solutions of this equation has been studied in several subspaces of the vector-valued space of bounded functions, still some development in other lines of research could be interesting to pursue, as for instance discrete settings. In this line, recently some papers have appeared [1, 44–46, 50].

Bibliography

[1] L. Abadías and C. Lizama, Almost automorphic mild solutions to fractional partial difference-differential equations, *Appl. Anal.*, **95**(6) (2016), 1347–1369.

[2] R. P. Agarwal and B. Ahmad, Existence of solutions for impulsive anti-periodic boundary value problems of fractional semilinear evolution equations, *Dyn. Contin. Discrete Impuls. Syst., Ser. A Math. Anal.*, **18**(4) (2011), 457–470.

[3] R. P. Agarwal, Asma, V. Lupulescu, and D. O'Regan, Fractional semilinear equations with causal operators, *Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat.*, **111**(1) (2017), 257–269.

[4] R. P. Agarwal, C. Cuevas, and H. Soto, Pseudo-almost periodic solutions of a class of semilinear fractional differential equations, *J. Appl. Math. Comput.*, **37**(1–2) (2011), 625–634.

[5] R. P. Agarwal, B. de Andrade, and C. Cuevas, Weighted pseudo-almost periodic solutions of a class of semilinear fractional differential equations, *Nonlinear Anal., Real World Appl.*, **11** (2010), 3532–3554.

[6] H. Ahmed, Semilinear neutral fractional stochastic integro-differential equations with nonlocal conditions, *J. Theor. Probab.*, **28**(2) (2015), 667–680.

[7] K. Aissani and M. Benchohra, Semilinear fractional order integro-differential equations with infinite delay in Banach spaces, *Arch. Math.*, **49**(2) (2013), 105–117.

[8] D. Araya and C. Lizama, Almost automorphic mild solutions to fractional differential equations, *Nonlinear Anal., Theory Methods Appl.*, **69** (2008), 3692–3705.

[9] K. Balachandran and S. Kiruthika, Existence of solutions of abstract fractional impulsive semilinear evolution equations, *Electron. J. Qual. Theory Differ. Equ.*, **2010** (2010), No. 4, 12 pages.

[10] K. Balachandran and J. Y. Park, Nonlocal Cauchy problem for abstract fractional semilinear evolution equations, *Nonlinear Anal., Theory Methods Appl.*, **71**(10) (2009), 4471–4475.

[11] E. Bazhlekova, The abstract Cauchy problem for the fractional evolution equation, *Fract. Calc. Appl. Anal.*, **1**(3) (1998), 255–270.

[12] J. Cao, Q. Yang, and Z. Huang, Existence of anti-periodic mild solutions for a class of semilinear fractional differential equations, *Commun. Nonlinear Sci. Numer. Simul.*, **17**(1) (2012), 277–283.

[13] Y. K. Chang, R. Zhang, and G. M. N'Guérékata, Weighted pseudo almost automorphic mild solutions to semilinear fractional differential equations, *Comput. Math. Appl.*, **64**(10) (2012), 3160–3170.

[14] A. Chauhan and J. Dabas, Existence of mild solutions for impulsive fractional-order semilinear evolution equations with nonlocal conditions, *Electron. J. Differ. Equ.*, **2011** (2011), 107, 10 pages.

[15] P. Chen, Y. Li, Q. Chen, and B. Feng, On the initial value problem of fractional evolution equations with noncompact semigroup, *Comput. Math. Appl.*, **67**(5) (2014), 1108–1115.

[16] E. Cuesta, Asymptotic behaviour of the solutions of fractional integro-differential equations and some time discretizations, *Discrete Contin. Dyn. Syst., Ser. A* (Supplement) (2007), 277–285.

[17] C. Cuevas and C. Lizama, Almost automorphic solutions to a class of semilinear fractional differential equations, *Appl. Math. Lett.*, **21** (2008), 1315–1319.

[18] B. de Andrade, A. N. Carvalho, P. M. Carvalho-Neto, and P. Marin-Rubio, Semilinear fractional differential equations: global solutions, critical nonlinearities and comparison results, *Topol. Methods Nonlinear Anal.*, **45**(2) (2015), 439–467.

[19] A. Debbouche and D. F. M. Torres, Approximate controllability of fractional nonlocal delay semilinear systems in Hilbert spaces, *Int. J. Control*, **86**(9) (2013), 1577–1585.

[20] J. P. C. dos Santos, M. M. Arjunan, and C. Cuevas, Existence results for fractional neutral integro-differential equations with state-dependent delay, *Comput. Math. Appl.*, **62** (2011), 1275–1283.

[21] C. Gal and M. Warma, Fractional-in-time semilinear parabolic equations and applications, HAL Id: hal-01578788, https://hal.archives-ouvertes.fr/hal-01578788.

[22] F. D. Ge, H. G. Zhou, and C. H. Kou, Approximate controllability of semilinear evolution equations of fractional order with nonlocal and impulsive conditions via an approximating technique, *Appl. Math. Comput.*, **275** (2016), 107–120.

[23] H. Gou and B. Li, Local and global existence of mild solution to impulsive fractional semilinear integro-differential equation with noncompact semigroup, *Commun. Nonlinear Sci. Numer. Simul.*, **42** (2017), 204–214.

[24] T. Guendouzi and S. Farahi, Approximate controllability of semilinear fractional stochastic dynamic systems with nonlocal conditions in Hilbert spaces, *Mediterr. J. Math.*, **13**(2) (2016), 637–656.

[25] T. Guendouzi and I. Hamada, Relative controllability of semilinear fractional stochastic control systems in Hilbert spaces, *Filomat*, **27**(8) (2013), 1485–1495.

[26] B. H. Guswanto and T. Suzuki, Existence and uniqueness of mild solutions for fractional semilinear differential equations, *Electron. J. Differ. Equ.*, **2015** (2015), 168, 16 pages.

[27] B. He, J. Cao, and B. Yang, Weighted Stepanov-like pseudo-almost automorphic mild solutions for semilinear fractional differential equations, *Adv. Differ. Equ.*, **2015** (2015), 74, 36 pages.

[28] E. Hernández and D. O'Regan, Controllability of Volterra–Fredholm type systems in Banach spaces, *J. Franklin Inst.*, **346** (2009), 95–101.

[29] E. Hernández, D. O'Regan, and K. Balachandran, On recent developments in the theory of abstract differential equations with fractional derivatives, *Nonlinear Anal., Theory Methods Appl.*, **73**(10) (2010), 3462–3471.

[30] N. Heymans and I. Podlubny, Physical interpretation of initial conditions for fractional differential equations with Riemann–Liouville fractional derivatives, *Rheol. Acta*, **45**(5) (2006), 765–771.

[31] B. H. E. Ibrahim, Z. Fan, and G. Li, Approximate controllability for functional equations with Riemann–Liouville derivative by iterative and approximate method, *J. Funct. Spaces*, **2017** (2017), Art. ID 2508165, 7 pages.

[32] S. Ji, Approximate controllability of semilinear nonlocal fractional differential systems via an approximating method, *Appl. Math. Comput.*, **236** (2014), 43–53.

[33] M. Kostić, Abstract degenerate multi-term fractional differential equations with Riemann–Liouville derivatives, *Bull. Cl. Sci. Math. Nat. Sci. Math.*, **41** (2016), 1–20.

[34] S. Kumar and N. Sukavanam, Approximate controllability of fractional order semilinear systems with bounded delay, *J. Differ. Equ.*, **252**(11) (2012), 6163–6174.

[35] K. Li, Fractional order semilinear Volterra integrodifferential equations in Banach spaces, *Topol. Methods Nonlinear Anal.*, **47**(2) (2016), 439–455.

[36] K. Li, J. Peng, and J. Gao, Existence results for semilinear fractional differential equations via Kuratowski measure of noncompactness, *Fract. Calc. Appl. Anal.*, **15**(4) (2012), 591–610.

[37] J. Liu, S. Cheng, and L. Zhang, Anti-periodic mild solutions to semilinear fractional differential equations, *J. Appl. Math. Comput.*, **48**(1–2) (2015), 381–393.

[38] X. Liu and X. Fu, Relaxation in control systems of fractional semilinear evolution equations, *Electron. J. Differ. Equ.*, **2014** (2014), No. 26, 16 pages.

[39] X. Liu and Z. Liu, Existence results for fractional semilinear differential inclusions in Banach spaces, *J. Appl. Math. Comput.*, **42**(1–2) (2013), 171–182.

[40] X. Liu and Z. Liu, On the 'bang-bang' principle for a class of fractional semilinear evolution inclusions, *Proc. R. Soc. Edinb., Sect. A, Math.*, **144**(2) (2014), 333–349.

[41] Z. Liu and M. Bin, Approximate controllability of impulsive Riemann-Liouville fractional equations in Banach spaces, *J. Integral Equ. Appl.*, **26**(4) (2014), 527–551.

[42] Z. Liu, M. Bin, and X. Liu, On the "bang-bang" principle for a class of Riemann–Liouville fractional semilinear evolution inclusions, *Math. Slovaca*, **66**(6) (2016), 1329–1344.

[43] Z. Liu and X. Li, Approximate controllability of fractional evolution systems with Riemann–Liouville fractional derivatives, *SIAM J. Control Optim.*, **53**(4) (2015), 1920–1933.

[44] C. Lizama, l_p-maximal regularity for fractional difference equations on UMD spaces, *Math. Nachr.*, **288**(17/18) (2015), 2079–2092.

[45] C. Lizama, The Poisson distribution, abstract fractional difference equations, and stability, *Proc. Am. Math. Soc.*, **145**(9) (2017), 3809–3827.

[46] C. Lizama and M. Murillo-Arcila, Maximal regularity in l_p spaces for discrete time fractional shifted equations, *J. Differ. Equ.*, **263**(6) (2017), 3175–3196.

[47] C. Lizama and G. M. N'Guérékata, Mild solutions for abstract fractional differential equations, *Appl. Anal.*, **92**(8) (2013), 1731–1754.

[48] C. Lizama, A. Pereira, and R. Ponce, On the compactness of fractional resolvent operator functions, *Semigroup Forum*, **93**(2) (2016), 363–374.

[49] C. Lizama and F. Poblete, Regularity of mild solutions for a class of fractional order differential equations, *Appl. Math. Comput.*, **224** (2013), 803–816.

[50] C. Lizama and M. P. Velasco, Weighted bounded solutions for a class of nonlinear fractional equations, *Fract. Calc. Appl. Anal.*, **19**(4) (2016), 1010–1030.

[51] Z. D. Mei, J. G. Peng, and J. H. Gao, General fractional differential equations of order $\alpha \in (1,2)$ and type $\beta \in [0,1]$ in Banach spaces, *Semigroup Forum*, **94**(3) (2017), 712–737.

[52] G. M. Mophou, Weighted pseudo almost automorphic mild solutions to semilinear fractional differential equations, *Appl. Math. Comput.*, **217**(19) (2011), 7579–7587.

[53] G. M. Mophou and G. M. N'Guerekata, Controllability of semilinear neutral fractional functional evolution equations with infinite delay, *Nonlinear Stud.*, **18**(2) (2011), 195–209.

[54] J. Mu and Y. Li, Periodic boundary value problems for semilinear fractional differential equations, *Math. Probl. Eng.*, **2012** (2012), Article ID 746872, 16 pages. DOI:10.1155/2012/746872.

[55] V. Obukhovski and P. Zecca, Controllability for systems governed by semilinear differential inclusions in a Banach space with a noncompact semigroup, *Nonlinear Anal., Theory Methods Appl.*, **70** (2009), 3424–3436.

[56] X. Pan, X. Li, and J. Zhao, Solvability and optimal controls of semilinear Riemann–Liouville fractional differential equations, *Abstr. Appl. Anal.*, **2014** (2014), Art. ID 216919, 11 pages.

[57] R. Ponce, Bounded mild solutions to fractional integro-differential equations in Banach spaces, *Semigroup Forum*, **87**(2) (2013), 377–392.

[58] R. Ponce, Existence of mild solutions to nonlocal fractional Cauchy problems via compactness, *Abstr. Appl. Anal.*, **2016** (2016), Art. ID 4567092, 15 pages.

[59] R. Sakthivel, Y. Ren, and N. I. Mahmudov, On the approximate controllability of semilinear fractional differential systems, *Comput. Math. Appl.*, **62**(3) (2011), 1451–1459.

[60] R. Saktivel, P. Revathi, and Y. Ren, Existence of solutions for nonlinear fractional stochastic differential equations, *Nonlinear Anal., Theory Methods Appl.*, **81** (2013), 70–86.

[61] X. B. Shu, Y. Lai, and Y. Chen, The existence of mild solutions for impulsive fractional partial differential equations, *Nonlinear Anal., Theory Methods Appl.*, **74**(5) (2011), 2003–2011.

[62] A. Shukla, N. Sukavanam, and D. N. Pandey, Approximate controllability of fractional semilinear control system of order $\alpha \in (1, 2]$ in Hilbert spaces, *Nonlinear Stud.*, **22**(1) (2015), 131–138.

[63] A. Shukla, N. Sukavanam, and D. N. Pandey, Approximate controllability of semilinear fractional control systems of order $\alpha \in (1, 2]$ with infinite delay, *Mediterr. J. Math.*, **13**(5) (2016), 2539–2550.

[64] P. Tamilalagan and P. Balasubramaniam, Moment stability via resolvent operators of fractional stochastic differential inclusions driven by fractional Brownian motion, *Appl. Math. Comput.*, **305** (2017), 299–307.

[65] L. Tonelli, Sulle equazioni funzionali di Volterra, *Bull. Calcutta Math. Soc.*, **20** (1930), 31–48.

[66] G. Toufik, Existence and controllability results for fractional stochastic semilinear differential inclusions, *Differ. Equ. Dyn. Syst.*, **23**(3) (2015), 225–240.

[67] D. Wang and Z. Xia, Pseudo almost automorphic solution of semilinear fractional differential equations with the Caputo derivatives, *Fract. Calc. Appl. Anal.*, **18**(4) (2015), 951–971.

[68] J. Wang, Z. Fan, and Y. Zhou, Nonlocal controllability of semilinear dynamic systems with fractional derivative in Banach spaces, *J. Optim. Theory Appl.*, **154**(1) (2012), 292–302.

[69] J. Wang, A. G. Ibrahim, and M. Feckan, Nonlocal Cauchy problems for semilinear differential inclusions with fractional order in Banach spaces, *Commun. Nonlinear Sci. Numer. Simul.*, **27**(1–3) (2015), 281–293.

[70] J. Wang and Y. Zhou, Existence and controllability results for fractional semilinear differential inclusions, *Nonlinear Anal., Real World Appl.*, **12**(6) (2011), 3642–3653.

[71] J. Wang, Y. Zhou, and M. Feckan, On the nonlocal Cauchy problem for semilinear fractional order evolution equations, *Cent. Eur. J. Math.*, **12**(6) (2014), 911–922.

[72] Z. Xia, M. Fan, and R. P. Agarwal, Pseudo almost automorphy of semilinear fractional differential equations in Banach spaces, *Fract. Calc. Appl. Anal.*, **19**(3) (2016), 741–764.

[73] L. Zhang, G. Wang, and G. Song, On mixed boundary value problem of impulsive semilinear evolution equations of fractional order, *Bound. Value Probl.*, **2012** (2012), 17, 8 pages.

[74] Y. Zhou, *Fractional Evolution Equations and Inclusions: Analysis and Control*, Elsevier/Academic Press, London, 2016.