Alexandria University
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Theory & Applications Of Fractional Calculus

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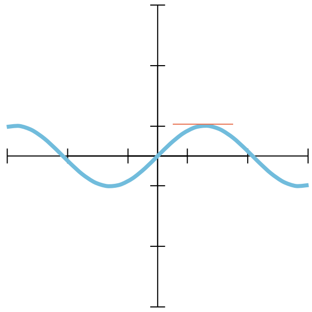
1 Introduction

When learning calculus, you are probably accustomed to the idea of higher order derivatives

The first derivative $\frac{\mathrm{d}y}{\mathrm{d}x}$ indicates the slope of a graph.



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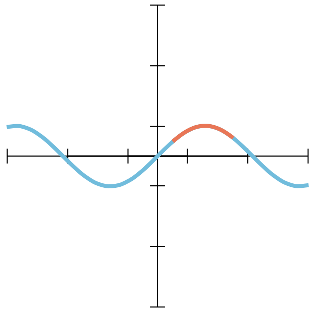
And the second derivative $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ indicates concavity



And so on calculating the n^{th} derivative of a function $\frac{\mathrm{d}^n f(x)}{\mathrm{d}x^n}$ is taking the derivative of it n times

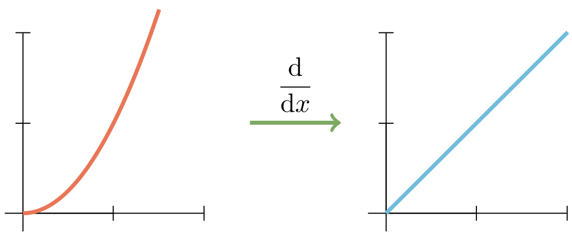
$$\underbrace{\left(\frac{\mathrm{d}}{\mathrm{d}x}\cdots\frac{\mathrm{d}}{\mathrm{d}x}\right)}_{\text{n times}} f(x)$$

And this made sense but what does it mean to take a fractional derivative

$$\frac{\mathrm{d}^{\frac{1}{2}} f(x)}{\mathrm{d}x^{\frac{1}{2}}} = ???$$

We're going to be exploring another branch of calculus **FRACTIONAL CALCULUS**.

The expression $\frac{\mathrm{d}^n f(x)}{\mathrm{d}x^n}$ can have multiple meanings first it can be thought of as a repeated differentiation so if we take the n^{th} derivative of a function it means taking the derivative n times however this only makes sense for positive integers if we are to extend this to other numbers we must think of this expression as a transformation something that takes in a function as an input and gives a function as an output



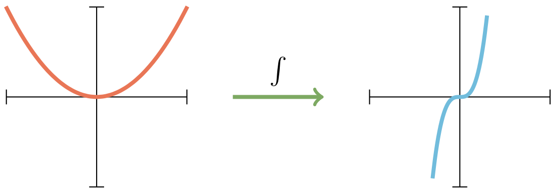
And not as repeated differentiation
We will look at $\frac{\mathrm{d}^n f(x)}{\mathrm{d}x^n}$ as an operator that transforms f into it's n^{th} derivative

2 Fractional Integral

We will start with the Fractional Integral although integrals are often harder to compute, but they often relate more nicely to each other and are less picky about what functions you throw into them. For example any continuous function can be integrated but NOT every continuous function can be differentiated. Like the Weierstrass function which is a continuous function that has no derivative anywhere along it due to its fractal jaggedness.
Let's establish a bit of notation $If(x)$ mean the indefinite integral of the function from 0 to x

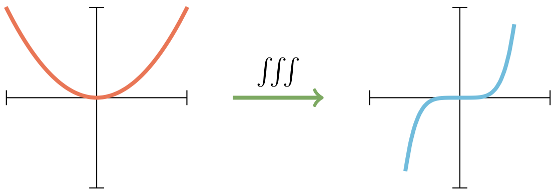
$$If(x) = \int_0^x f(t)\ dt$$

As we talked about earlier we can think of this as a transformation something that takes in a function and outputs a function



Similar to differentiation we put an exponent like thing to denote the n^{th} integral of $f(x)$
So for example the third integral would look something like this

$$I^3 f(x) = \int_0^x \left[\int_0^t \left[\int_0^s f(u)\ du\right] ds\right] dt$$



2.1 Cauchy's Repeated Integration Formula

As we increase the number the expression gets more and more complicated But Cauchy found a way to look at repeated integrals like this and put it in the form of a single integral of convolution type which is Cauchy's Formula for Repeated Integration.

First we take the integral of a function $f(t)$

$$If(t) = \int_0^t f(s)\ ds$$

Repeating this process gives

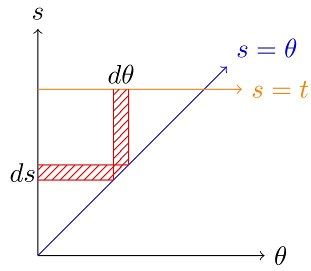
$$I^2 f(t) = \int_0^t If(s)\ ds = \int_0^t \int_0^s f(\theta)\ d\theta\ ds$$

Augustin Louis Cauchy

Baron Augustin-Louis Cauchy (1789-1857) was a French mathematician and physicist he's well known the most for his significant contributions in analysis, calculus, and number theory although he didn't make direct contributions to fractional calculus in the way we understand it today, his integral formula provided a crucial tool for mathematicians who later explored this area. Essentially, Cauchy's formula helped establish the foundation for defining fractional differentiation and integration.

We can interchanging the order of integration using Fubini's theorem

$$I^2 f(t) = \int_0^t \int_\theta^t ds\ f(\theta)\ d\theta = \int_0^t (t-\theta) f(\theta)\ d\theta$$



Similarly we can get $I^3 f(t)$

$$\begin{aligned} I^3 f(t) &= \int_0^t \int_0^s (s-\theta) f(\theta)\ d\theta\ ds \\ &= \int_0^t \frac{(t-\theta)^2}{2} f(\theta)\ d\theta = \int_0^t \frac{(t-\theta)^2}{2!} f(\theta)\ d\theta \end{aligned}$$

And so on we can get

$$I^n f(t) = \int_0^t \frac{(t-\theta)^{n-1}}{(n-1)!} f(\theta)\ d\theta \quad , \quad n = 1, 2, 3, \dots \tag{2.1}$$

2.2 Riemann-Liouville Integral

Now the real question is how do we define this formula for any positive number ?
The answer lies within the gamma function $\Gamma(n)$.

Gamma Function

The Gamma Function is defined as follows

$$\Gamma(n) = \int_0^\infty e^{-t} t^{n-1}\ dt \quad , \quad \forall n > 0$$

And has the following properties

- $\Gamma(n+1) = n\Gamma(n)$
- $\Gamma(n+1) = n!$
- $\Gamma(1) = 1 \quad ; \quad \Gamma(\frac{1}{2}) = \sqrt{\pi}$

The goal of the gamma function was to define a smooth curve that would go through factorial points
It gives us a way to extend the domain of factorials from positive integers to the positive real numbers and even the complex numbers for $Re(n) \notin \mathbb{Z}^- \cup \{0\}$

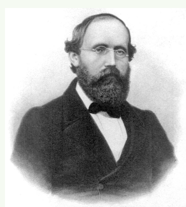
Since the main thing restricting the domain of a formula for repeated integration (2.1) is the factorial we can replace this with the gamma function now we can plug in any positive number for n and get a value for this integral

$$I^n f(t) = \frac{1}{\Gamma(n)} \int_0^t (t-s)^{n-1} f(s)\ ds \quad , \quad \forall n \in \mathbb{C}, Re(n) > 0 \tag{2.2}$$

This is a valid operator this particular operator is called the Riemann Liouville integral or RL integral for short although there are many other ways of going about fractional integration the RL integral is probably the easiest to understand

Bernhard Riemann

Though he is best known for his contributions to geometry and analysis, he also made huge steps in the development of fractional calculus. While not published until after his death, Riemann's work explored a definition for fractional integration laid the groundwork for what is now known as the Riemann-Liouville fractional integral, a cornerstone of the field.



Definition 2.1 — Riemann-Liouville Fractional Integral. The fractional integral of the function $f \in L_1[0,b]$ is the integral of (arbitrary) order $\alpha > 0$ is defined by

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\ ds \tag{2.3}$$

And we define $I^0 f(t) = f(t)$

Lemma 2.1 The definition (2.3) of fractional integral operator is satisfied in any point for the continuous functions and in almost every point for the absolutely integrable functions. i.e

1. If $f(t)$ continuous $\Longrightarrow I^\alpha f(t)$ exist $\forall t \in [0,b]$
2. If $f(t)$ absolutely integrable ($\in L_1[0,b]$) $\Longrightarrow I^\alpha f(t)$ exist a.e (almost everywhere)

In case $\alpha > 1$ the integral $I^\alpha f(t)$ exist $\forall t \in [0,b]$ since the integrand is a product of an integrable function $f(t)$ and continuous function $(t-s)^{\alpha-1}$

In case $0 < \alpha < 1$ we can rewrite $I^\alpha f(t)$ as following

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{\infty} \psi(t-s) f(s)\ ds = \frac{1}{\Gamma(\alpha)} \psi(t) * f(t)$$

Where $\psi(u) = \begin{cases} u^{\alpha-1} & u \in [0,b] \\ 0 & \text{else} \end{cases}$

Now from lemma 2.1 and Young's convolution inequality we get the following results

Result 1 If $f, \psi \in L_1 \Longrightarrow \psi * f \in L_1$ exist a.e

Result 2 If $f \in C \quad , \quad \psi \in L_1 \Longrightarrow \psi * f \in L_\infty$ exist $\forall$points

Young's Convolution Inequality

Suppose that f, g are two functions such that $f \in L_p[\mathbb{R}^d]$ and $g \in L_q[\mathbb{R}^d]$ Then

$$||f * g||_r \leq ||f||_p\ ||g||_q$$

i.e $f * g \in L_r[\mathbb{R}^d]$ where

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1 \quad ; \quad p, q, r \in [1, \infty] \tag{2.4}$$

In particular
1. If $p\ ,\ q = 1$ we get that $r = 1$ (Result 1)
2. If $p = \infty\ ,\ q = 1$ we get that $r = \infty$ (Result 2)

Now in case $0 < \alpha < 1$ we get that $\displaystyle\int_0^b |\psi(s)|\ ds < \infty \Longrightarrow \psi \in L_1[0,b]$

Joseph Liouville

Joseph Liouville deserves credit for sparking the entire field of fractional calculus. Even earlier than Riemann, Liouville, in 1832, first proposed the idea of generalizing derivatives and integrals to non-integer orders. While he didn't establish a complete framework, Liouville's pioneering thought and experiment opened the door for mathematicians like Riemann to develop the specific definitions and tools used in fractional calculus today.

■ **Example 2.2.1** Evaluate $I^{\alpha}(f(t))$ where $f(t) = c$ ((Constant function))

Sol.

$$\begin{aligned} I^{\alpha}(c) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\ c\ ds \\ &= \frac{c}{\Gamma(\alpha)} \left[\frac{(t-s)^{\alpha}}{\alpha} \right]_0^t \\ &= \frac{c\ t^{\alpha}}{\Gamma(\alpha+1)} \end{aligned}$$

■

■ **Example 2.2.2** Evaluate $I^{\alpha}(f(t))$ where $f(t) = t^n \quad , \quad n > -1$ (to make sure Γ is defined)

Sol.

$$I^{\alpha}(t^n) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\ s^n\ ds$$

Substitute $\begin{cases} s = t\theta \\ ds = t\ d\theta \\ 0 \to 1 \end{cases}$

$$\begin{aligned} &= \frac{1}{\Gamma(\alpha)} \int_0^1 (t-t\theta)^{\alpha-1}(t\theta)^n\ t\ d\theta \\ &= \frac{1}{\Gamma(\alpha)} t^{n+\alpha} \int_0^1 (1-\theta)^{\alpha-1}(\theta)^n\ d\theta \\ &= \frac{1}{\Gamma(\alpha)} t^{n+\alpha} \beta(\alpha, n+1) \\ &= \frac{1}{\Gamma(\alpha)} t^{n+\alpha} \frac{\Gamma(\alpha)\Gamma(n+1)}{\Gamma(n+\alpha+1)} \\ &= t^{n+\alpha} \frac{\Gamma(n+1)}{\Gamma(n+\alpha+1)} \end{aligned}$$

■

Beta Function

The Beta Function is defined as follows

$$\beta(\alpha, \gamma) = \int_0^1 (1-t)^{\gamma-1} t^{\alpha-1}\ dt$$

And has the following property

- $\beta(\alpha, \gamma) = \dfrac{\Gamma(\gamma)\Gamma(\alpha)}{\Gamma(\gamma+\alpha)}$

We can do a test to show that this formula is sensible applying a half-integral twice should have the same effect as a regular single integral.

■ **Example 2.2.3** Evaluate $I^{\frac{1}{2}}I^{\frac{1}{2}}(f(t))$ where $f(t) = t^n \quad , \quad n > -1$

Sol.

$$\begin{aligned}
\because I^{\alpha}(t^n) &= t^{n+\alpha}\frac{\Gamma(n+1)}{\Gamma(n+\alpha+1)} \\
I^{\frac{1}{2}}I^{\frac{1}{2}}(t^n) &= I^{\frac{1}{2}}\left[t^{n+\frac{1}{2}}\frac{\Gamma(n+1)}{\Gamma(n+\frac{3}{2})}\right] \\
&= \frac{\Gamma(n+1)}{\Gamma(n+\frac{3}{2})}I^{\frac{1}{2}}\left[t^{n+\frac{1}{2}}\right] \\
&= \frac{\Gamma(n+1)}{\Gamma(n+\frac{3}{2})}t^{n+\frac{1}{2}+\frac{1}{2}}\frac{\Gamma(n+\frac{1}{2}+1)}{\Gamma(n+\frac{1}{2}+\frac{1}{2}+1)} \\
&= \frac{\Gamma(n+1)}{\Gamma(n+2)}t^{n+1} = \frac{t^{n+1}}{n+1}
\end{aligned}$$

■

We're not limited to just half-integrals, of course. Using the same trick, you can similarly derive a formula for a one-third integral and show that applying it 3 successive times results in a single integral.

$$I^{\frac{1}{3}}I^{\frac{1}{3}}I^{\frac{1}{3}}f(t) = If(t)$$

■ **Example 2.2.4** Evaluate $I_a^{\alpha}(f(t))$ where $f(t) = (t-a)^n$

Sol.

$$I_a^{\alpha}(t-a)^n = \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}(s-a)^n \; ds$$

$$\text{Substitute}\begin{cases}\dfrac{s-a}{t-a} = \theta \\[2ex] ds = (t-a)\; d\theta \\[2ex] 0 \to 1\end{cases}$$

$$\begin{aligned}
&= \frac{1}{\Gamma(\alpha)}\int_0^1 (t-a-(t-a)\theta)^{\alpha-1}(t-a)^n\theta^n \; (t-a)\; d\theta \\
&= \frac{1}{\Gamma(\alpha)}(t-a)^{\alpha-1+n+1}\int_0^1 (1-\theta)^{\alpha-1}\theta^n \; d\theta \\
&= \frac{1}{\Gamma(\alpha)}(t-a)^{\alpha+n}\beta(\alpha, n+1) \\
&= \frac{\Gamma(n+1)}{\Gamma(n+\alpha+1)}(t-a)^{\alpha+n}
\end{aligned}$$

■

The notation I_a^{α} is specifying the lower terminal of the integral and we also can write ${}_aI_t^{\alpha}$ to specify the lower terminal and the upper terminal

$${}_0I_t^{\alpha} = I^{\alpha}$$

2.3 Properties Of RL Integral

Let $\alpha, \beta > 0$ for $f, g \in L_1$ the following properties of the operator I^α holds

Property 1 Semi-Group Property

We can generalize the idea of example (2.2.3) by the following property

$$I^\alpha I^\beta f(t) = I^\beta I^\alpha f(t) = I^{\alpha+\beta} f(t)$$

Proof.

$$\begin{aligned} I^\alpha I^\beta f(t) &= I^\alpha \left[\frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s)\ ds\right] \\ &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^t (t-s)^{\alpha-1} \int_0^s (s-\theta)^{\beta-1} f(\theta)\ d\theta\ ds \\ &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^t \underbrace{\int_\theta^t (t-s)^{\alpha-1}(s-\theta)^{\beta-1}\ ds}_{J}\ f(\theta)\ d\theta \end{aligned} \tag{2.5}$$

Let's handle the inner integral first

$$J = \int_\theta^t (t-s)^{\alpha-1}(s-\theta)^{\beta-1}\ ds$$

$$\text{Substitute} \begin{cases} s-\theta = \eta \\ ds = d\eta \\ 0 \to t-\theta \end{cases}$$

$$\begin{aligned} &= \int_0^{t-\theta} (t-\theta-\eta)^{\alpha-1}(\eta)^{\beta-1}\ d\eta \\ &= (t-\theta)^{\alpha-1} \int_0^{t-\theta} (1-\frac{\eta}{t-\theta})^{\alpha-1}(\eta)^{\beta-1}\ d\eta \end{aligned}$$

$$\text{Substitute} \begin{cases} \eta = (t-\theta)\xi \\ d\eta = (t-\theta)\ d\xi \\ 0 \to 1 \end{cases}$$

$$\begin{aligned} &= (t-\theta)^{\alpha-1} \int_0^1 (1-\xi)^{\alpha-1}(t-\theta)^{\beta-1}\ \xi^{\beta-1}\ (t-\theta)\ d\xi \\ &= (t-\theta)^{\alpha+\beta-1} \int_0^1 (1-\xi)^{\alpha-1}\ \xi^{\beta-1}\ d\xi \\ &= (t-\theta)^{\alpha+\beta-1}\beta(\alpha,\beta) = (t-\theta)^{\alpha+\beta-1}\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \end{aligned}$$

Substitute in (2.5) we get that

$$\begin{aligned} I^\alpha I^\beta f(t) &= \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\alpha+\beta)} \int_0^t (t-\theta)^{\alpha+\beta-1}\ f(\theta)\ d\theta \\ &= \frac{1}{\Gamma(\alpha+\beta)} \int_0^t (t-\theta)^{\alpha+\beta-1}\ f(\theta)\ d\theta = I^{\alpha+\beta} f(t) \end{aligned}$$

∎

Theorem 2.2 — Fundamental Theorem of Calculus for Lebesgue Integrable functions. I^1 (ordinary integral) maps $L_1[a,b]$ to $AC[a,b]$. Moreover, for each $f(t) \in L_1[a,b]$

$$\frac{\mathrm{d}}{\mathrm{d}t} I^1 f(t) = f(t) \quad \text{for a.e } t \in [a,b]$$

And for $f(t) \in C[a,b]$

$$\frac{\mathrm{d}}{\mathrm{d}t} I^1 f(t) = f(t) \quad \text{for all } t \in [a,b]$$

Property 2 Fractional Version Of The Fundamental Theorem

Now to Proof that

$$\frac{\mathrm{d}}{\mathrm{d}t} I^{\alpha+1} f(t) = I^{\alpha} f(t)$$

We can use the semi group property to get

$$\frac{\mathrm{d}}{\mathrm{d}t} I^{\alpha+1} f(t) = \frac{\mathrm{d}}{\mathrm{d}t} I^1 I^{\alpha} f(t)$$

Now if $I^{\alpha} f(t) \in L_1[a,b]$ (i.e $f(t) \in L_1[a,b]$ and $t^{\alpha-1} \in L_1[a,b]$) we get

$$= I^{\alpha} f(t) \quad \text{for a.e } t \in [a,b]$$

And if $I^{\alpha} f(t) \in C[a,b]$ (i.e $f(t) \in C[a,b]$) we get

$$= I^{\alpha} f(t) \quad \text{for each } t \in [a,b]$$

Property 3 Continuity With Respect To The Order

Let $f(t) \in L_1[a,b]$

$$\lim_{\alpha \to n} I^{\alpha} f(t) = I^n f(t) \quad , \quad n \in \mathbb{N}^+$$

Proof.

$$\lim_{\alpha \to n} I^{\alpha} f(t) = \lim_{\alpha \to n} \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \ f(s) \ ds$$

Using Lebesgue's Dominated Convergence Theorem the limit can surpass the integral

$$\begin{aligned} &= \frac{1}{\Gamma(n)} \int_0^t \lim_{\alpha \to n} (t-s)^{\alpha-1} \ f(s) \ ds \\ &= \frac{1}{\Gamma(n)} \int_0^t (t-s)^{n-1} \ f(s) \ ds \\ &= I^n f(t) \end{aligned}$$

■

Lebesgue's Dominated Convergence Theorem

Let $\{f_n(t)\}$ be a sequence of functions converges Pointwise to $f(t)$ on A, and suppose that

$$|f_n(t)| < \phi(t) \quad , \quad n = 0, 1, 2, \ldots$$

Where ϕ is an integrable function on A. Then $f(t)$ is integrable on A and

$$\lim_{n \to \infty} \int_A f_n(t) dt = \int_A \lim_{n \to \infty} f_n(t) dt = \int_A f(t) dt$$

Property 4 Linearity

Let a, b be constants

$$I^\alpha[a\ f(t) + b\ g(t)] = a\ I^\alpha f(t) + b\ I^\alpha g(t)$$

Property 5 Effect On Zero Functions

$$I^\alpha f(t) = 0 \Longleftrightarrow f(t) = 0 \quad a.e$$

Proof. It's clear if $f(t) = 0$

Now let $I^\alpha f(t) = 0$

In case of $\alpha > 1$ using the (Semi-group) property it can be treated as $I^\alpha = I^n I^\beta$ where n is the integer part of α

$$I^n I^\beta f(t) = 0$$

And because I^n is the normal integral thus

$$I^\beta f(t) = 0 \quad , \quad 0 < \beta < 1$$

Now we can treat it as the next case

In case of $0 < \alpha < 1$

$$\begin{aligned} I^\alpha f(t) &= 0 \\ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\ f(s)\ ds &= 0 \\ \int_0^t (t-s)^{\alpha-1}\ f(s)\ ds &= 0 \\ (t-s)^{\alpha-1} f(s) &= 0 \end{aligned}$$

Because $0 < \alpha < 1$ the power of $(t-s)$ is negative therefore $(t-s)^{\alpha-1} = 0$ only if $(t-s) = \pm\infty$ and that leave us to the other case that is

$$f(s) = 0$$

■

Property 6 One-To-One

$$I^\alpha f(t) = I^\alpha g(t) \Longleftrightarrow f(t) = g(t) \qquad a.e$$

Proof. It's clear if $f(t) = g(t)$

Now let $I^\alpha f(t) = I^\alpha g(t)$

$$\begin{aligned} I^\alpha f(t) &= I^\alpha g(t) \\ I^\alpha f(t) - I^\alpha g(t) &= 0 \end{aligned}$$

From (Linearity) property

$$I^\alpha (f(t) - g(t)) = 0$$

And from (Effect on zero functions) property we get that

$$f(t) - g(t) = 0$$

Thus

$$f(t) = g(t)$$

■

Property 7 Limit At Zero

If f is bounded measurable function such that $\lim_{t\to 0} f(t)$ exists then

$$\lim_{t\to 0} t^{-\alpha} I^{\alpha} f(t) = \frac{1}{\Gamma(1+\alpha)} \lim_{t\to 0} f(t)$$

Proof.

$$\begin{aligned}
t^{-\alpha} I^{\alpha} f(t) &= \frac{1}{\Gamma(\alpha)} t^{-\alpha} \int_0^t (t-s)^{\alpha-1} f(s) ds \\
&= \frac{1}{\Gamma(\alpha)} \int_0^t \frac{(t-s)^{\alpha-1}}{t^{\alpha}} f(s) ds \\
&= \frac{1}{\Gamma(\alpha)} \int_0^t \left(\frac{t-s}{t}\right)^{\alpha} \frac{f(s)}{(t-s)} ds
\end{aligned}$$

Substitute $\begin{cases} \dfrac{t-s}{t} = u \\ ds = -t du \\ 1 \to 0 \end{cases}$

$$\begin{aligned}
&= \frac{1}{\Gamma(\alpha)} \int_1^0 u^{\alpha} \frac{f(t(1-u))}{tu} (-t) du \\
&= \frac{1}{\Gamma(\alpha)} \int_0^1 u^{\alpha-1} f(t(1-u)) du
\end{aligned}$$

take the limit for both sides as $t \to 0$

$$\lim_{t\to 0} t^{-\alpha} I^{\alpha} f(t) = \lim_{t\to 0} \frac{1}{\Gamma(\alpha)} \int_0^1 u^{\alpha-1} f(t(1-u)) du$$

Using the boundedness and measurability of f along with the Dominated Convergence Theorem, we interchange the limit and the integral

$$= \frac{1}{\Gamma(\alpha)} \int_0^1 u^{\alpha-1} \lim_{t\to 0} f(t(1-u)) du$$

We can say that $\lim_{t\to 0} f(t(1-u)) = \lim_{t\to 0} f(t)$

$$\begin{aligned}
&= \frac{1}{\Gamma(\alpha)} \int_0^1 u^{\alpha-1} \lim_{t\to 0} f(t) du \\
&= \frac{1}{\Gamma(\alpha)} \lim_{t\to 0} f(t) \int_0^1 u^{\alpha-1} du \\
&= \frac{1}{\Gamma(\alpha)} \lim_{t\to 0} f(t) \left[\frac{u^{\alpha}}{\alpha}\right]_0^1 \\
&= \frac{1}{\Gamma(\alpha)} \lim_{t\to 0} f(t) \frac{1}{\alpha} \\
&= \frac{1}{\Gamma(\alpha+1)} \lim_{t\to 0} f(t)
\end{aligned}$$

■

Theorem 2.3 For $\alpha > 0$

① $I^\alpha : L_p[0,b] \Longrightarrow L_p[0,b]$ is bounded linear (continuous) operator $\forall p \in [1,\infty]$ i.e the fractional integration maps $L_p[0,b]$ continuously into itself
In particular $0 < \alpha < 1$ then $I^\alpha := L_1[0,b] \Longrightarrow L_{\frac{1}{1-\alpha}-\epsilon}[0,b]$

② $I^\alpha : C[0,b] \Longrightarrow C[0,b]$ is bounded linear (continuous) operator $\forall p \in [1,\infty]$ i.e the fractional integration maps $C[0,b]$ continuously into itself

Proof. Define $g(t) := \dfrac{t^{(\alpha-1)}}{\Gamma(\alpha)} \Longrightarrow L_1[0,b]$ and it's norm

$$||g(t)||_{L_1} = \int_0^b \frac{t^{(\alpha-1)}}{\Gamma(\alpha)}\ dt = \left[\frac{t^\alpha}{\alpha\Gamma(\alpha)}\right]_0^b = \frac{b^\alpha}{\Gamma(\alpha+1)}$$

① If $f \in L_p[0,b]$ then by Young's convolution inequality (2.4)

$$\begin{aligned} \frac{1}{p} + \frac{1}{q} &= \frac{1}{r} + 1 \\ \frac{1}{p} + 1 &= \frac{1}{r} + 1 \\ p &= r \end{aligned}$$

Then $I^\alpha f = f * g \in L_p[0,b]$
Thus I^α is Linear

Also

$$||I^\alpha f||_{L_p} = ||f * g|| \leq ||f||_{L_p}\ ||g||_{L_1} \leq ||f||_{L_p} \frac{b^\alpha}{\Gamma(\alpha+1)}$$

Thus I^α is bounded
And from functional analysis An operator between two normed spaces is a bounded linear operator if and only if it is continuous operator.

Thus I^α is continuous

Now if $0 < \alpha < 1$ then

$$\begin{aligned} \int_0^b |g(t)|^q\ dt &= \int_0^b \left|\frac{t^{(\alpha-1)}}{\Gamma(\alpha)}\right|^q\ dt \\ &= \int_0^b \frac{t^{q(\alpha-1)}}{\Gamma(\alpha)}\ dt \\ &= \left[\frac{t^{q(\alpha-1)+1}}{\Gamma(\alpha)(q(\alpha-1)+1)}\right]_0^b < \infty \quad , \quad \forall q(\alpha-1)+1 > 0 \end{aligned}$$

$$\begin{aligned} q(\alpha-1)+1 &> 0 \\ q &< \frac{1}{1-\alpha} \\ q &= \frac{1}{1-\alpha} - \epsilon \quad , \quad \epsilon > 0 \\ q &\in \left[1, \frac{1}{1-\alpha} - \epsilon\right] \\ \Longrightarrow g &\in L_{\frac{1}{1-\alpha}-\epsilon}[0,b] \quad , \quad \epsilon > 0 \end{aligned}$$

Now by Young's convolution inequality (2.4)
If $f \in L_1$

$$\begin{aligned}\frac{1}{p}+\frac{1}{q}&=\frac{1}{r}+1\\ 1+\frac{1}{q}&=\frac{1}{r}+1\\ r=q&=\frac{1}{1-\alpha}-\epsilon\end{aligned}$$

$$\therefore I^{\alpha}f = f*g \in L_{\frac{1}{1-\alpha}-\epsilon}[0,b]$$

2 Let $f \in C[0,b] \subset L_{\infty}[0,b]$ and $g \in L_1[0,b]$ then by Young's convolution inequality (2.4)

$$\frac{1}{\infty}+1=\frac{1}{r}+1 \quad \Longrightarrow \quad r=\infty$$

Thus $I^{\alpha}f = f*g \in C[0,b]$

Also

$$\begin{aligned}|I^{\alpha}f| &= \left|\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\ f(s)\ ds\right|\\ &\leq \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|f(s)|\ ds\\ &\leq \underset{t\in[0,b]}{max}|f(t)|\ \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\ ds\\ &\leq ||f(t)||_C\ \frac{1}{\Gamma(\alpha)}\left[\frac{(t-s)^{\alpha}}{\alpha}\right]_0^t\\ &\leq ||f(t)||_C\ \frac{b^{\alpha}}{\Gamma(\alpha+1)}\end{aligned}$$

Thus I^{α} is bounded and since it's Linear then it is continuous operator

$$\therefore I^{\alpha} : C[0,b] \Longrightarrow C[0,b]$$

■

Lemma 2.4 $I^{\alpha}f(t)$ vanishes at $t=0$ i.e $\lim_{t\to 0} I^{\alpha}f(t) = I^{\alpha}f(0) = 0$

Proof.

$$\begin{aligned}\lim_{t\to 0} I^{\alpha}f(t) &= \lim_{t\to 0} t^{\alpha}t^{-\alpha}I^{\alpha}f(t)\\ &= \lim_{t\to 0} t^{\alpha} \times \lim_{t\to 0} t^{-\alpha}I^{\alpha}f(t)\\ &= \underbrace{\lim_{t\to 0} t^{\alpha}}_{\to 0}\ \frac{1}{\Gamma(\alpha+1)}\ \underbrace{\lim_{t\to 0} f(t)}_{\text{Exist}} = 0\end{aligned}$$

■

Theorem 2.5 The fractional integral operator maps non-negative a.e non-decreasing functions continuously into a functions of the same type (non-negative a.e non-decreasing).

Proof. Let $\alpha > 0$ and f be a non-negative and a.e non-decreasing function on $[0, b]$.
And $f \in L_1[0, b]$ thus $I^\alpha f(t)$ exist a.e on $[0, b]$

Now let $t_1, t_2 \in [0, b]$ such that $t_1 \le t_2$ then $0 \le f(t_1) \le f(t_2)$

$$I^\alpha f(t_1) = \int_0^{t_1} \frac{(t_1 - s)^{\alpha-1}}{\Gamma(\alpha)} f(s)\ ds$$

Substitute $\begin{cases} t_1 - s = u \\ ds = -du \\ t_1 \to 0 \end{cases}$

$$\begin{aligned} &= \int_{t_1}^{0} \frac{(u)^{\alpha-1}}{\Gamma(\alpha)} f(t_1 - u)(-du) \\ &= \frac{1}{\Gamma(\alpha)} \int_0^{t_1} u^{\alpha-1} f(t_1 - u)\ du \\ &\le \frac{1}{\Gamma(\alpha)} \int_0^{t_1} u^{\alpha-1} f(t_2 - u)\ du \end{aligned}$$

Substitute $\begin{cases} t_2 - u = \theta \\ du = -d\theta \\ t_2 \to 0 \end{cases}$

$$\le \frac{1}{\Gamma(\alpha)} \int_0^{t_2} (t_2 - \theta)^{\alpha-1} f(\theta)\ d\theta = I^\alpha f(t_2)$$

Thus $I^\alpha f(t_1) \le I^\alpha f(t_2)$

The images of non-negative and a.e non-decreasing functions are also non-negative and a.e non-decreasing function ■

Mittag-Leffler Function

Mittag-Leffler Function $E_a(z)$ is direct generalization of the exponential series defined as follows

$$E_a(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(ak + 1)}$$

The series is uniformly convergent. Hence $E_a(z)$ is continuous
also there is Mittag-Leffler Function of two parameters $\forall a, b > 0$

$$E_{a,b}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(ak + b)}$$

Special Values

$$E_{a,1}(z) = E_a(z) \qquad E_{a,b}(0) = 1 \qquad E_{1,1}(z) = e^z$$

$$E_{0,1}(z) = \frac{1}{1-z} \qquad E_{2,1}(z) = cosh(\sqrt{z}) \qquad E_{1,2}(z) = \frac{e^z - 1}{z}$$

$$E_{2,2}(z) = \frac{sinh(\sqrt{z})}{\sqrt{z}} \qquad E_{1,3}(z) = \frac{e^z - z - 1}{z^2} \qquad E_{2,1}(-z^2) = cos(z)$$

$$E_{\frac{1}{2},1}(\sqrt{z}) = \frac{2}{\sqrt{\pi}} e^{-z} \mathrm{erfc}(-\sqrt{z}) \qquad E_{1,b}(z) = \frac{1}{z^{z-1}} \left[e^z - \sum_{k=0}^{b-2} \frac{z^k}{\Gamma(z+1)} \right]$$

■ **Example 2.3.1** Evaluate $I^\alpha(E_{a,b}(\lambda t))$

Sol.

$$I^\alpha(E_{a,b}(\lambda t)) = I^\alpha\left(\sum_{k=0}^{\infty}\frac{(\lambda t)^k}{\Gamma(ak+b)}\right) \tag{2.6}$$

Now because the series in (2.6) is uniformly convergent we can interchange the I^α operator with the summation

$$I^\alpha(E_{a,b}(\lambda t)) = \sum_{k=0}^{\infty}\frac{I^\alpha(\lambda t)^k}{\Gamma(ak+b)} = \sum_{k=0}^{\infty}\frac{\lambda^k I^\alpha(t^k)}{\Gamma(ak+b)}$$

And we know that

$$I^\alpha(t^n) = t^{n+\alpha}\frac{\Gamma(n+1)}{\Gamma(n+\alpha+1)}$$

Thus

$$I^\alpha(E_{a,b}(\lambda t)) = \sum_{k=0}^{\infty}\frac{\Gamma(k+1)}{\Gamma(k+\alpha+1)}\frac{(\lambda^k t^{k+\alpha})}{\Gamma(ak+b)} = t^\alpha\sum_{k=0}^{\infty}\frac{\Gamma(k+1)}{\Gamma(k+\alpha+1)}\frac{(\lambda t)^k}{\Gamma(ak+b)}$$

When $a, b = 1$

$$\begin{aligned} I^\alpha(E_{1,1}(\lambda t)) = I^\alpha(e^{\lambda t}) &= t^\alpha\sum_{k=0}^{\infty}\frac{\Gamma(k+1)}{\Gamma(k+\alpha+1)}\frac{(\lambda t)^k}{\Gamma(k+1)} \\ &= t^\alpha\sum_{k=0}^{\infty}\frac{(\lambda t)^k}{\Gamma(k+1+\alpha)} = t^\alpha E_{1,1+\alpha}(\lambda t) \end{aligned}$$

■

■ **Example 2.3.2** Evaluate $I^\alpha_{-\infty}(e^{\lambda t})$

Sol.

$$I^\alpha_{-\infty}(e^{\lambda t}) = \frac{1}{\Gamma(\alpha)}\int_{-\infty}^{t}(t-s)^{\alpha-1}e^{\lambda s}\,ds$$

Substitute $\begin{cases} \xi = \lambda(t-s) \\ d\xi = -\lambda\ ds \\ \infty \to 0 \end{cases}$

$$\begin{aligned} &= \frac{1}{\Gamma(\alpha)}\int_{\infty}^{0}\left(\frac{\xi}{\lambda}\right)^{\alpha-1}\left(e^{\lambda\left(t-\frac{\xi}{\lambda}\right)}\right)\left(-\frac{1}{\lambda}\ d\xi\right) \\ &= \frac{1}{\lambda^\alpha\Gamma(\alpha)}\int_{0}^{\infty}\xi^{\alpha-1}e^{\lambda t-\xi}\ d\xi \\ &= \frac{e^{\lambda t}}{\lambda^\alpha\Gamma(\alpha)}\int_{0}^{\infty}\xi^{\alpha-1}e^{-\xi}\ d\xi \\ &= \frac{e^{\lambda t}}{\lambda^\alpha\Gamma(\alpha)}\Gamma(\alpha) = \frac{e^{\lambda t}}{\lambda^\alpha} \end{aligned}$$

Which is similar to the normal Integral of the exponential function ■

Definition 2.2 — Compact Operator. An operator is said to be Compact if it is continuous and maps bounded sets into relatively compact sets

Definition 2.3 — Relatively Compact Set. A set A is said to be relatively compact set or precompact if it's closure $\bar{A}$ is Compact

Theorem 2.6 — Arzelà-Ascoli Theorem. A subset or subspace $A \subset C[a,b]$ is relatively compact if and only if

1. $\forall x \in [a,b]$, $\sup\limits_{f \in A} |f(x)| < \infty$ (i.e uniformly bounded)
2. A is equicontinuous i.e. $\forall \epsilon > 0$, $\exists \delta > 0$ such that

$$|x-y| < \delta \implies |g(x)-g(y)| \leq \epsilon \quad , \quad g \in A$$

So for the normed space $(C[a,b], ||\cdot||)$ we have

$$\text{Compact Sets} = \text{Closed} + \text{Bounded} + \text{Equicontinuous}$$

The Generalized Hölder inequality

Let $\Omega \subset \mathbb{R}$ and $p,q,r \in [1,\infty]$ such that $\frac{1}{p}+\frac{1}{q}=\frac{1}{r}$ now if $f \in L_{p[\Omega]}, g \in L_{q[\Omega]}$ then $fg \in L_{r[\Omega]}$ (this is normal multiplication not convolution) and

$$||fg||_{L_{r[\Omega]}} \leq ||f||_{L_{p[\Omega]}} ||g||_{L_{q[\Omega]}}$$
$$\left(\int_\Omega |f(t)g(t)|^r \, dt\right)^{\frac{1}{r}} \leq \left(\int_\Omega |f(t)|^r \, dt\right)^{\frac{1}{p}} \left(\int_\Omega |g(t)|^q \, dt\right)^{\frac{1}{q}}$$

Most well known case when $r=1$

$$\int_\Omega f(t)g(t) \, dt \leq \left(\int_\Omega |f(t)|^r \, dt\right)^{\frac{1}{p}} \left(\int_\Omega |g(t)|^q \, dt\right)^{\frac{1}{q}}$$

Where $\frac{1}{p}+\frac{1}{q}=1$

Definition 2.4 — Hölderian Function. A function f is Called Hölderian of order λ if

$$|f(\alpha)-f(\beta)| \leq A|\alpha-\beta|^\lambda \qquad \forall \alpha,\beta \in [a,b]$$
$$f(t) \in \mathcal{H}^\lambda$$

Where A is real constants ≥ 0

Theorem 2.7 Let $\alpha > 0$ if $p > \max\left\{1, \frac{1}{\alpha}\right\}$ then

$$I^\alpha : L_p[0,b] \Longrightarrow C[0,b]$$

Is Compact operator

In particular, if $\alpha \in (0,1]$, then

$$I^\alpha : L_p[0,b] \Longrightarrow \mathcal{H}^{\alpha-\frac{1}{p}}[0,b]$$

Where $\mathcal{H}^{\alpha-\frac{1}{p}}[0,b]$ is the Hölder space of order $\alpha - \frac{1}{p}$

Proof.
Take $f \in L_p[0,b]$ and Let $q \in [1,\infty]$ such that $\frac{1}{p}+\frac{1}{q}=1$ so that we can use Hölder inequality

$$\begin{aligned}
|I^\alpha f(t)| &\leq \frac{1}{\Gamma(\alpha)} \left(\int_0^t (t-s)^{q(\alpha-1)}\ ds\right)^{\frac{1}{q}} \left(\int_0^t |f(s)|^p\ ds\right)^{\frac{1}{p}} \\
&\leq \frac{1}{\Gamma(\alpha)} \left(\frac{t^{q(\alpha-1)+1}}{q(\alpha-1)+1}\right)^{\frac{1}{q}} \left(\int_0^b |f(s)|^p\ ds\right)^{\frac{1}{p}} \\
&\leq \frac{1}{\Gamma(\alpha)} \left(\frac{t^{\alpha-1+\frac{1}{q}}}{(q(\alpha-1)+1)^{\frac{1}{q}}}\right) ||f(s)||_{L_p[0,b]} = \frac{1}{\Gamma(\alpha)} \left(\frac{t^{\alpha-\frac{1}{p}}}{(q(\alpha-1)+1)^{\frac{1}{q}}}\right) ||f(s)||_{L_p[0,b]}
\end{aligned}$$

For $I^\alpha f(t)$ to be bounded the following must happen

$$\begin{aligned}
(q(\alpha-1)+1)^{\frac{1}{q}} &> 0 \\
q(\alpha-1)+1 &> 0 \\
q(\alpha-1) &> -1 \\
\alpha &> 1-\frac{1}{q} \\
\alpha &> \frac{1}{p} \\
p &> \frac{1}{\alpha}
\end{aligned}$$

And the upper limit of $I^\alpha f(t)$ is given by

$$||I^\alpha f(t)|| = \max_{t\in[0,b]} |I^\alpha f(t)| \leq \frac{1}{\Gamma(\alpha)} \left(\frac{b^{\alpha-\frac{1}{p}}}{(q(\alpha-1)+1)^{\frac{1}{q}}}\right) ||f(s)||_{L_p[0,b]}$$

Therefore we get the boundedness (hence the continuity) of the linear operator I^α

Now let $0 \leq t_1 \leq t_2 \leq b$

$$\begin{aligned}
\Gamma(\alpha)|I^\alpha f(t_2) - I^\alpha f(t_1)| &= \left|\int_0^{t_2} (t_2-s)^{\alpha-1} f(s)\ ds - \int_0^{t_1} (t_1-s)^{\alpha-1} f(s)\ ds\right| \\
&= \left|\int_0^{t_1} (t_2-s)^{\alpha-1} f(s)\ ds + \int_{t_1}^{t_2} (t_2-s)^{\alpha-1} f(s)\ ds - \int_0^{t_1} (t_1-s)^{\alpha-1} f(s)\ ds\right| \\
&= \left|\int_0^{t_1} \left((t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}\right) f(s)\ ds + \int_{t_1}^{t_2} (t_2-s)^{\alpha-1} f(s)\ ds\right| \\
&\leq \int_0^{t_1} \left|(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}\right| |f(s)|\ ds + \int_{t_1}^{t_2} \left|(t_2-s)^{\alpha-1}\right| |f(s)|\ ds
\end{aligned}$$

Using Hölder inequality

$$\begin{aligned}
&\leq \left(\int_0^{t_1} \left|(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}\right|^q\ ds\right)^{\frac{1}{q}} \left(\int_0^{t_1} |f(s)|^p\ ds\right)^{\frac{1}{p}} \\
&+ \left(\int_{t_1}^{t_2} \left|(t_2-s)^{\alpha-1}\right|^q\ ds\right)^{\frac{1}{q}} \left(\int_0^{t_1} |f(s)|^p\ ds\right)^{\frac{1}{p}} \\
&\leq ||f||_{L_p[0,b]} \left[\left(\int_0^{t_1} \left|(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}\right|^q\ ds\right)^{\frac{1}{q}} + \left(\int_{t_1}^{t_2} \left|(t_2-s)^{\alpha-1}\right|^q\ ds\right)^{\frac{1}{q}}\right] \\
&\leq ||f||_{L_p[0,b]} \left[\left(\int_0^{t_1} \left|(t_2-s)^{q(\alpha-1)} - (t_1-s)^{q(\alpha-1)}\right|\ ds\right)^{\frac{1}{q}} + \left(\int_{t_1}^{t_2} (t_2-s)^{q(\alpha-1)}\ ds\right)^{\frac{1}{q}}\right]
\end{aligned}$$

Now because the change in the power of the terms inside the integrals will change the value of the absolute value and the value of the integrals we will consider the following:

1 In case of $\alpha = 1$

$$\begin{aligned}\Gamma(\alpha)|I^\alpha f(t_2) - I^\alpha f(t_1)| &\leq \int_0^{t_1} \left|(t_2 - s)^0 - (t_1 - s)^0\right| \, ds + \left(\int_{t_1}^{t_2} \left|(t_2 - s)^0\right|^q \, ds\right)^{\frac{1}{q}} \\ &\leq \int_0^{t_1} 0 \, ds + \left(\int_{t_1}^{t_2} 1 \, ds\right)^{\frac{1}{q}} = (t_2 - t_1)^{\frac{1}{q}}\end{aligned}$$

2 In case of $\alpha \neq 1$

$$\Gamma(\alpha)|I^\alpha f(t_2) - I^\alpha f(t_1)| \leq ||f||_{L_p[0,b]} \left[\left(\int_0^{t_1} \left|(t_2 - s)^{q(\alpha-1)} - (t_1 - s)^{q(\alpha-1)}\right| \, ds\right)^{\frac{1}{q}} + \left(\frac{(t_2 - t_1)^{q(\alpha-1)+1}}{q(\alpha-1)+1}\right)^{\frac{1}{q}}\right]$$

2.1 In case of $\alpha > 1$

Because $t_2 > t_1$ therefore $(t_2 - s)^{q(\alpha-1)} > (t_1 - s)^{q(\alpha-1)}$

Thus

$$\begin{aligned}\int_0^{t_1} \left|(t_2 - s)^{q(\alpha-1)} - (t_1 - s)^{q(\alpha-1)}\right| \, ds &= \int_0^{t_1} (t_2 - s)^{q(\alpha-1)} - (t_1 - s)^{q(\alpha-1)} \, ds \\ &= \left[\frac{-(t_2 - s)^{q(\alpha-1)+1} + (t_1 - s)^{q(\alpha-1)+1}}{q(\alpha-1)+1}\right]_0^{t_1} \\ &= \frac{-(t_2 - t_1)^{q(\alpha-1)+1} + (t_2)^{q(\alpha-1)+1} - (t_1)^{q(\alpha-1)+1}}{q(\alpha-1)+1}\end{aligned}$$

Because $t_2 > t_1$ and $q(\alpha-1)+1 > 0$ therefore $\dfrac{-(t_2 - t_1)^{q(\alpha-1)+1}}{q(\alpha-1)+1}$ is negative if we remove it the value will get bigger

$$\leq \frac{(t_2)^{q(\alpha-1)+1} - (t_1)^{q(\alpha-1)+1}}{q(\alpha-1)+1}$$

2.2 In case of $\alpha < 1$

Because $t_2 > t_1$ therefore $(t_2 - s)^{q(\alpha-1)} < (t_1 - s)^{q(\alpha-1)}$

Thus

$$\begin{aligned}\int_0^{t_1} \left|(t_2 - s)^{q(\alpha-1)} - (t_1 - s)^{q(\alpha-1)}\right| \, ds &= \int_0^{t_1} \left|(t_1 - s)^{q(\alpha-1)} - (t_2 - s)^{q(\alpha-1)}\right| \, ds \\ &= \int_0^{t_1} (t_1 - s)^{q(\alpha-1)} - (t_2 - s)^{q(\alpha-1)} \, ds \\ &= \left[\frac{-(t_1 - s)^{q(\alpha-1)+1} + (t_2 - s)^{q(\alpha-1)+1}}{q(\alpha-1)+1}\right]_0^{t_1} \\ &= \frac{(t_2 - t_1)^{q(\alpha-1)+1} + (t_1)^{q(\alpha-1)+1} - (t_2)^{q(\alpha-1)+1}}{q(\alpha-1)+1}\end{aligned}$$

Because $t_2 > t_1$ and $q(\alpha-1)+1 > 0$ therefore $\dfrac{(t_1)^{q(\alpha-1)+1} - (t_2)^{q(\alpha-1)+1}}{q(\alpha-1)+1}$ is negative if we remove it the value will get bigger

$$\leq \frac{(t_2 - t_1)^{q(\alpha-1)+1}}{q(\alpha-1)+1}$$

Therefore we get

$$|I^\alpha f(t_2) - I^\alpha f(t_1)| \leq \frac{||f||_{L_p[0,b]}}{\Gamma(\alpha)} \begin{cases} \left(\frac{(t_2)^{q(\alpha-1)+1} - (t_1)^{q(\alpha-1)+1}}{q(\alpha-1)+1}\right)^{\frac{1}{q}} + \left(\frac{(t_2-t_1)^{q(\alpha-1)+1}}{q(\alpha-1)+1}\right)^{\frac{1}{q}} & \text{if } \alpha > 1 \\ (t_2-t_1)^{\frac{1}{q}} & \text{if } \alpha = 1 \\ \left(\frac{(t_2-t_1)^{q(\alpha-1)+1}}{q(\alpha-1)+1}\right)^{\frac{1}{q}} + \left(\frac{(t_2-t_1)^{q(\alpha-1)+1}}{q(\alpha-1)+1}\right)^{\frac{1}{q}} & \text{if } \alpha < 1 \end{cases}$$

Simplifying it we get

$$|I^\alpha f(t_2) - I^\alpha f(t_1)| \leq \frac{||f||_{L_p[0,b]}}{\Gamma(\alpha)} \begin{cases} \frac{\left((t_2)^{q(\alpha-1)+1} - (t_1)^{q(\alpha-1)+1}\right)^{\frac{1}{q}} + (t_2-t_1)^{\alpha-\frac{1}{p}}}{(q(\alpha-1)+1)^{\frac{1}{q}}} & \text{if } \alpha > 1 \\ (t_2-t_1)^{\frac{1}{q}} & \text{if } \alpha = 1 \\ 2\frac{(t_2-t_1)^{\alpha-\frac{1}{p}}}{(q(\alpha-1)+1)^{\frac{1}{q}}} & \text{if } \alpha < 1 \end{cases}$$

In the case $\alpha > 1$ we may use the Mean Value Theorem

The Mean Value Theorem

If f is a continuous function on the closed interval $[a, b]$ and differentiable on the open interval (a, b), then there exists a point c in (a, b) such that the tangent at c is parallel to the secant line through the endpoints $(a, f(a))$ and $(b, f(b))$, that is

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Let $h(s) = s^{q(\alpha-1)+1}$ and $a = t_1, b = t_2$ then applying the Mean Value Theorem we get

$$h'(c) = \frac{h(t_2) - h(t_1)}{t_2 - t_1}$$
$$(q(\alpha-1)+1)c^{q(\alpha-1)}(t_2 - t_1) = (t_2)^{q(\alpha-1)+1} - (t_1)^{q(\alpha-1)+1}$$

BUT $c \in [t_1, t_2] \subset [0, b]$ therefore

$$(t_2)^{q(\alpha-1)+1} - (t_1)^{q(\alpha-1)+1} \leq (q(\alpha-1)+1)b^{q(\alpha-1)}(t_2 - t_1)$$

Thus we can say

$$|I^\alpha f(t_2) - I^\alpha f(t_1)| \leq \frac{||f||_{L_p[0,b]}}{\Gamma(\alpha)} \begin{cases} \frac{\left((q(\alpha-1)+1)b^{q(\alpha-1)}(t_2-t_1)\right)^{\frac{1}{q}} + (t_2-t_1)^{\alpha-\frac{1}{p}}}{(q(\alpha-1)+1)^{\frac{1}{q}}} & \text{if } \alpha > 1 \\ (t_2-t_1)^{\frac{1}{q}} & \text{if } \alpha = 1 \\ 2\frac{(t_2-t_1)^{\alpha-\frac{1}{p}}}{(q(\alpha-1)+1)^{\frac{1}{q}}} & \text{if } \alpha < 1 \end{cases}$$

In all cases the expression on the right hand side of converges to 0 as $t_1 \to t_2$ which confirms that $I^\alpha f \in C[0, b]$

Now because $I^\alpha f$ is equicontinuity and uniformly bounded From Arzela Ascoli theorem

$$I^\alpha : L_p[0,b] \Longrightarrow C[0,b]$$

Is Compact operator

In particular, in case $\alpha \in (0,1]$

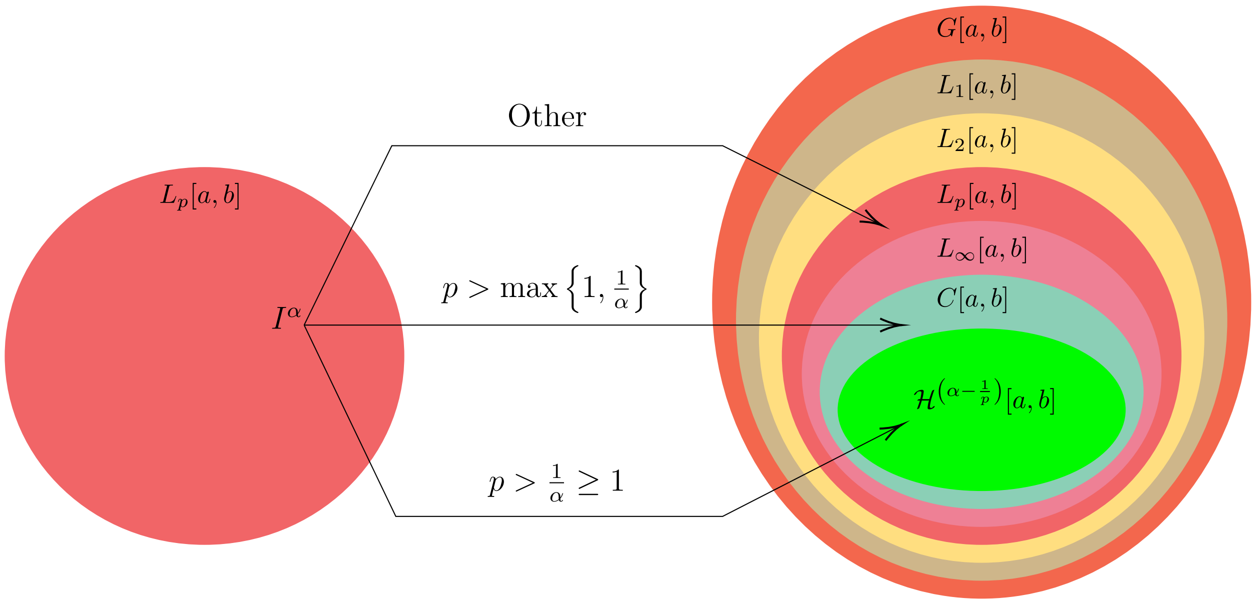
$$|I^\alpha f(t_2) - I^\alpha f(t_1)| \leq \frac{||f||_{L_p[0,b]}}{\Gamma(\alpha)} \begin{cases} (t_2 - t_1)^{\frac{1}{q}} & \text{if } \alpha = 1 \\ \\ 2\dfrac{(t_2 - t_1)^{\alpha - \frac{1}{p}}}{(q(\alpha - 1) + 1)^{\frac{1}{q}}} & \text{if } \alpha < 1 \end{cases}$$

$$\leq \frac{||f||_{L_p[0,b]}}{\Gamma(\alpha)} \begin{cases} A_1(t_2 - t_1)^{1 - \frac{1}{p}} & \text{if } \alpha = 1 \\ \\ A_2(t_2 - t_1)^{\alpha - \frac{1}{p}} & \text{if } \alpha < 1 \end{cases}$$

In both cases $I^\alpha f(t)$ is Holderian of order $\alpha - \frac{1}{p}$

$$I^\alpha : L_p[0,b] \Longrightarrow \mathcal{H}^{\alpha - \frac{1}{p}}[0,b]$$

$\blacksquare$



The figure above Showing the mapping of I^α of the space L_p

3 Fractional Differentiation

One might assume that fractional differentiation can be accomplished by assuming that $\frac{\mathrm{d}^n}{\mathrm{d}x^n} = I^{-n}$ but this doesn't work because the gamma doesn't actually extend to ALL the real numbers. Gamma is actually undefined for non-positive integer inputs, so we wouldn't be able to plug in $n = -1$ into gamma to compute a derivative using the Fractional Integral formula (2.3).
But even for fractional negative orders like $\alpha = -\frac{1}{2}$, it turns out the integral expression becomes divergent, since an integral of the form $t^{(\alpha-1)}$ is divergent near $t = 0$ for non-positive values of α

3.1 Riemann-Liouville Fractional Derivative

We know that from classical calculus integer order derivatives and integrals they're supposed to be inverses of each other, and should cancel each other out.

i.e $\forall n > 0, n \in \mathbb{Z}^+$ the n^{th} derivative of the n^{th} integral is the function itself (The First Fundamental Theorem of Calculus)

$$\frac{\mathrm{d}^n}{\mathrm{d}t^n} I^n f(t) = f(t)$$

So we can use this formula indirectly to compute a half-derivative by say first compute a half-integral and then taking the ordinary whole derivative of that

Basically using the fractional integral to get us some kind of fractional order then using ordinary derivatives to sort of "lower" that order to where we actually want it.

This technique is called the Riemann-Liouville fractional derivative

Definition 3.1 — Riemann-Liouville Fractional Derivative. Let $\alpha > 0$ for a positive integer m such that $\alpha \in [m-1, m]$ we define the Riemann-Liouville fractional derivative of order α by

$$D^\alpha f(t) = \frac{\mathrm{d}^m}{\mathrm{d}t^m} I^{m-\alpha} f(t) \tag{3.1}$$

And by using the Riemann-Liouville fractional integration formula (2.3) we can rewrite a formula for the fractional derivative as follows

$$D^\alpha f(t) = \frac{1}{\Gamma(m-\alpha)} \frac{\mathrm{d}^m}{\mathrm{d}t^m} \int_0^t (t-s)^{m-\alpha-1}\ f(s)\ ds \tag{3.2}$$

When α is equal to 0.5 we call this a semi-derivative

■ **Example 3.1.1** Evaluate $D^\alpha(f(t))$ where $f(t) = t^n$, $n > -1$

Sol.

$$D^\alpha(t^n) = \frac{\mathrm{d}^m}{\mathrm{d}t^m} \left(I^{m-\alpha}(t^n)\right)$$

And we know that

$$I^{m-\alpha}(t^n) = \left[t^{n+\alpha} \frac{\Gamma(n+1)}{\Gamma(n+\alpha+1)}\right]_{\alpha=m-\alpha}$$

$$\begin{aligned}
\therefore \frac{\mathrm{d}^m}{\mathrm{d}t^m} \left(t^{n+m-\alpha} \frac{\Gamma(n+1)}{\Gamma(n+1+m-\alpha)}\right) &= \frac{\Gamma(n+1)}{\Gamma(n+1+m-\alpha)} \frac{\mathrm{d}^m}{\mathrm{d}t^m} t^{n+m-\alpha} \\
&= \frac{\Gamma(n+1)}{\Gamma(n+1+m-\alpha)} \frac{\Gamma(n+1+m-\alpha)}{\Gamma(n+1+m-\alpha-m)} t^{n-\alpha} \\
&= t^{n-\alpha} \frac{\Gamma(n+1)}{\Gamma(n-\alpha+1)}
\end{aligned}$$

■

■ **Example 3.1.2** Evaluate $D^\alpha(t^n)$ when $n = \alpha$ and when $n = \alpha - k$ for $k = 1, 2, 3, \ldots$

Sol.

$$D^\alpha(t^n) = t^{n-\alpha}\frac{\Gamma(n+1)}{\Gamma(n-\alpha+1)}$$

At $n = \alpha$

$$D^\alpha(t^\alpha) = t^{\alpha-\alpha}\frac{\Gamma(\alpha+1)}{\Gamma(\alpha-\alpha+1)} = \Gamma(\alpha+1)$$

At $n = \alpha - k$

$$D^\alpha(t^{\alpha-k}) = t^{\alpha-k-\alpha}\frac{\Gamma(\alpha-k+1)}{\Gamma(\alpha-k-\alpha+1)} = \frac{\Gamma(\alpha-k+1)}{t^k\Gamma(1-k)}$$

We know that $\Gamma(x)$ maps non-positive integer numbers to $\{\infty, -\infty\}$

Thus for $k = 1, 2, 3, \ldots, m-1$

$$D^\alpha(t^{\alpha-k}) = 0 \tag{3.3}$$

That for $k < m$ to make sure that $\Gamma(\alpha - k + 1)$ is not $\{\infty, -\infty\}$ ■

3.2 Caputo Fractional Derivative

Someone might say instead of taking the fractional integral of order $m - \alpha$ for a function then differentiate it $m - times$ we can differentiate it $m - times$ first then taking the fractional integral of order $m - \alpha$ for it

This what Prof. Michele Caputo did in 1967 when he introduced the Caputo fractional derivative definition but this alternative definition doesn't always give the same results as the original

Definition 3.2 — Caputo Fractional Derivative. Let $\alpha > 0$, $m - 1 < \alpha < m$, $m \in \mathbb{N}^+$ Caputo Fractional Derivative is defined by

$$^C D^\alpha f(t) = I^{m-\alpha}\frac{\mathrm{d}^m}{\mathrm{d}t^m}f(t) = \frac{1}{\Gamma(m-\alpha)}\int_0^t (t-s)^{m-\alpha-1}\frac{\mathrm{d}^m f(s)}{\mathrm{d}s^m}\,ds \tag{3.4}$$

■ **Example 3.2.1** Evaluate $D^\alpha(f(t))$ where $f(t) = t^n$ and $n > m - 1$, $n \in \mathbb{R}$

Sol.

$$\begin{aligned}
^C D^\alpha(t^n) &= I^{m-\alpha}\left[\frac{\mathrm{d}^m}{\mathrm{d}t^m}t^n\right] = I^{m-\alpha}\left[t^{n-m}\frac{\Gamma(n+1)}{\Gamma(n-m+1)}\right] \\
&= \frac{\Gamma(n+1)}{\Gamma(n-m+1)}\left[\frac{\Gamma(n-m+1)}{\Gamma(n+1-m+m-\alpha)}t^{n-m+m-\alpha}\right] \\
&= \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)}t^{n-\alpha}
\end{aligned}$$

If $n < m - 1$, $n \in \mathbb{N}$

$$^C D^\alpha(t^n) = I^{m-\alpha}\left[\frac{\mathrm{d}^m}{\mathrm{d}t^m}t^n\right] = I^{m-\alpha}[0] = 0 \tag{3.5}$$

■

Michele Caputo

Professor Michele Caputo is a distinguished mathematician known for his contributions to the field of fractional calculus, particularly his creation of the Caputo derivative. His work has significantly advanced our understanding of fractional calculus and its applications. With a career marked by innovation and scholarly excellence, Caputo has established himself as a leading authority in the mathematical community

3.3 The Differences And Properties Of Riemann-Liouville And Caputo Derivative

1. **Restrictions On The Order**

In Riemann-Liouville For $\alpha \in [m-1, m]$, m can be any positive natural number we are not restricted to make it small as possible

$$D^{\frac{1}{2}} f(t) = \frac{\mathrm{d}}{\mathrm{d}t} I^{\frac{1}{2}} f(t) = \frac{\mathrm{d}^5}{\mathrm{d}t^5} I^{4.5} f(t) = D^{4.5} f(t) = D^{8.5} f(t) = \dots$$

That's due to the (semi-group) property of the fractional integral

$$\frac{\mathrm{d}^5}{\mathrm{d}t^5} I^{4.5} f(t) = \frac{\mathrm{d}^5}{\mathrm{d}t^5} I^4 I^{\frac{1}{2}} f(t) = \frac{\mathrm{d}}{\mathrm{d}t} I^{\frac{1}{2}} f(t) = D^{\frac{1}{2}} f(t)$$

In Caputo This is not true with Caputo

$${}^C D^{\frac{1}{2}} f(t) = I^{\frac{1}{2}} \frac{\mathrm{d}}{\mathrm{d}t} f(t) \neq I^{4.5} \frac{\mathrm{d}^5}{\mathrm{d}t^5} f(t) \neq I^{8.5} \frac{\mathrm{d}^9}{\mathrm{d}t^9} f(t)$$

To show that let $f(t) = t^2$

$${}^C D^{\frac{1}{2}} t^2 = I^{\frac{1}{2}} \frac{\mathrm{d}}{\mathrm{d}t} t^2 = I^{\frac{1}{2}} 2t$$

$${}^C D^{\frac{1}{2}} t^2 = I^{\frac{5}{2}} \frac{\mathrm{d}^3}{\mathrm{d}t^3} t^2 = I^{\frac{5}{2}} 0 = 0$$

2. **Restrictions On The Function**

In Riemann-Liouville $f \in L_1[0, b]$ ((may exist even for some discontinuous functions))

In Caputo f is differentiable and $f' \in L_1[0, b]$ ((at least $f \in AC[0, b]$))

Absolute Continuous Functions

A function f is said to be Absolute Continuous i.e $f \in AC^n[0, b]$ if it's defined on $[0, b]$ and have continuous derivatives up to order $(n-1)$ on $[0, b]$ and $f^{(n-1)}$ is absolutely continuous on $[0, b]$

3. **Linearity**

In Riemann-Liouville Let a, b be constants

$$D^\alpha [a\ f(t) + b\ g(t)] = a\ D^\alpha f(t) + b\ D^\alpha g(t)$$

In Caputo Let a, b be constants

$${}^C D^\alpha [a\ f(t) + b\ g(t)] = a\ {}^C D^\alpha f(t) + b\ {}^C D^\alpha g(t)$$

4. **Effect On Constant**

In Riemann-Liouville

$$D^\alpha c = \frac{t^{-\alpha}}{\Gamma(1-\alpha)}$$

In Caputo

$${}^C D^\alpha c = I^{m-\alpha} \left[\frac{\mathrm{d}^m}{\mathrm{d}t^m} c \right] = I^{m-\alpha} [0] = 0$$

Which is most important difference between RL and Caputo and it consider an advantage for Caputo over RL and show us that

$${}^C D^\alpha f(t) \neq D^\alpha f(t)$$

5. Continuity With Respect To The Order Of Derivation

We naturally expect that $D_a^\alpha f(t)$ and ${}^C D_a^\alpha f(t)$ to be a continuous functions. It is clear that complications may occur only at points which represent the integer-order derivatives.

Let $m-1<\alpha<m$ and $f(t)$ has a sufficient number of continuous derivatives.

In Riemann-Liouville

$$\lim_{\alpha\to m} D_a^\alpha f(t)=\lim_{\alpha\to m}\left\{\frac{1}{\Gamma(m-\alpha)}\frac{\mathrm{d}^m}{\mathrm{d}t^m}\int_a^t (t-s)^{m-\alpha-1}\ f(s)\ ds\right\}$$

Integrate by part $(m+1)-times$

$$\begin{aligned}
&=\lim_{\alpha\to m}\frac{\mathrm{d}^m}{\mathrm{d}t^m}\left[\sum_{k=0}^{m}\frac{(t-a)^{m+k-\alpha}f^{(k)}(a)}{\Gamma(m+k+1-\alpha)}+\int_a^t\frac{(t-s)^{2m-\alpha}}{\Gamma(2m-\alpha+1)}\ f^{(m+1)}(s)\ ds\right]\\
&=\lim_{\alpha\to m}\left[\sum_{k=0}^{m}\frac{(t-a)^{k-\alpha}f^{(k)}(a)}{\Gamma(k+1-\alpha)}+\int_a^t\frac{(t-s)^{m-\alpha}}{\Gamma(m-\alpha+1)}\ f^{(m+1)}(s)\ ds\right]
\end{aligned}$$

All the summation terms except the last one will be zero due to the Gamma in the denominator which will be $\{-\infty,\infty\}$ when α is replaced by m

$$\begin{aligned}
&=f^{(m)}(a)+\int_a^t f^{(m+1)}(s)\ ds\\
&=f^{(m)}(a)+f^{(m)}(t)-f^{(m)}(a)=f^{(m)}(t)
\end{aligned}$$

This does not hold for points of the discontinuity of some derivative involved.

In Caputo

The calculation of the left limit is similar and even more easier

$$\lim_{\alpha\to m^-}{}^C D_a^\alpha f(t)=\lim_{\alpha\to m^-}\left\{\frac{1}{\Gamma(m-\alpha)}\int_a^t (t-s)^{m-\alpha-1}\ f^{(m)}(s)\ ds\right\}$$

Integrate by part

$$\begin{aligned}
&=\lim_{\alpha\to m^-}\left[\frac{(t-a)^{m-\alpha}f^{(m)}(a)}{\Gamma(m+1-\alpha)}+\int_a^t\frac{(t-s)^{m-\alpha}}{\Gamma(m-\alpha+1)}\ f^{(m+1)}(s)\ ds\right]\\
&=f^{(m)}(a)+\int_a^t f^{(m+1)}(s)\ ds=f^{(m)}(t)
\end{aligned}$$

The right limit for the Caputo derivative is as follows

$$\lim_{\alpha\to(m-1)^+}{}^C D_a^\alpha f(t)=\int_a^t f^{(m)}(s)\ ds=f^{(m-1)}(t)-f^{(m-1)}(a)$$

This result destroys our hope in the continuity of the Caputo derivative with respect to α. The function $f(t)$ would have to fulfill $f^{(m-1)}(a)=0$ and it seems like a very strong restriction.
But, most functions used in fractional calculus satisfy this condition so it is not such a big complication.
In addition we could expect this result because it coincides with one of the requirement for the Caputo derivative the zero value of all derivatives of a constant function.

6. Equivalence Of The Approaches

We impose $f(t)$ to be $(m-1)-times$ continuously differentiable and the m^{th} derivative of $f(t)$ to be integrable.
We suppose as usual $\alpha > 0$, but $\alpha \neq m-1$

$$D_a^\alpha f(t) = \frac{1}{\Gamma(m-\alpha)} \frac{\mathrm{d}^m}{\mathrm{d}t^m} \int_a^t (t-s)^{m-\alpha-1} \ f(s) \ ds$$

Integrate by part $m - times$

$$\begin{aligned}
&= \frac{\mathrm{d}^m}{\mathrm{d}t^m} \left[\sum_{k=0}^{m-1} \frac{(t-a)^{m+k-\alpha} f^{(k)}(a)}{\Gamma(m+k+1-\alpha)} + \int_a^t \frac{(t-s)^{2m-\alpha-1}}{\Gamma(2m-\alpha)} \ f^{(m)}(s) \ ds \right] \\
&= \sum_{k=0}^{m-1} \frac{(t-a)^{k-\alpha} f^{(k)}(a)}{\Gamma(k+1-\alpha)} + \int_a^t \frac{(t-s)^{m-\alpha-1}}{\Gamma(m-\alpha)} \ f^{(m)}(s) \ ds \\
&= \sum_{k=0}^{m-1} \frac{(t-a)^{k-\alpha} f^{(k)}(a)}{\Gamma(k+1-\alpha)} + {}^C D_a^\alpha f(t)
\end{aligned}$$

A more interesting situation occurs when $a \to -\infty$ because due to $k - \alpha < 0$ for all k, the power functions are zero for all values α and we obtain

$$D_{-\infty}^\alpha f(t) = {}^C D_{-\infty}^\alpha f(t)$$

7. Equivalence Of Functions Derivatives

In Riemann-Liouville

$$D^\alpha f(t) = D^\alpha g(t) \Longleftrightarrow f(t) = g(t) + \sum_{k=1}^{m} C_k t^{\alpha-k}$$

Proof. Let

$$f(t) = g(t) + \sum_{k=1}^{m} C_k t^{\alpha-k}$$

Take D^α for both sides

$$\begin{aligned}
D^\alpha f(t) &= D^\alpha g(t) + D^\alpha \sum_{k=1}^{m} C_k t^{\alpha-k} \\
&= D^\alpha g(t) + \sum_{k=1}^{m} C_k D^\alpha t^{\alpha-k}
\end{aligned}$$

From equation (3.3) we got that $D^\alpha t^{\alpha-k} = 0$ Thus

$$D^\alpha f(t) = D^\alpha g(t)$$

Conversely let

$$D^\alpha f(t) = D^\alpha g(t)$$

Take I^α for both sides

$$\begin{aligned}
f(t) &= g(t) + H(t) \\
f(t) &= g(t) + \sum_{k=1}^{m} C_k t^{\alpha-k}
\end{aligned}$$

■

In Caputo

$$^{C}D^{\alpha}f(t) = {}^{C}D^{\alpha}g(t) \iff f(t) = g(t) + \sum_{k=1}^{m} C_k t^{m-k}$$

Proof. Let

$$f(t) = g(t) + \sum_{k=1}^{m} C_k t^{m-k}$$

Take ${}^{C}D^{\alpha}$ for both sides

$$\begin{aligned} {}^{C}D^{\alpha}f(t) &= {}^{C}D^{\alpha}g(t) + {}^{C}D^{\alpha}\sum_{k=1}^{m} C_k t^{\alpha-k} \\ &= {}^{C}D^{\alpha}g(t) + \sum_{k=1}^{m} C_k \, {}^{C}D^{\alpha}t^{m-k} \end{aligned}$$

From equation (3.5) we got that ${}^{C}D^{\alpha}t^{m-k} = 0$ Thus

$${}^{C}D^{\alpha}f(t) = {}^{C}D^{\alpha}g(t)$$

Conversely let

$${}^{C}D^{\alpha}f(t) = {}^{C}D^{\alpha}g(t)$$

Take I^{α} for both sides

$$\begin{aligned} f(t) &= g(t) + H(t) \\ f(t) &= g(t) + \sum_{k=1}^{m} C_k t^{m-k} \end{aligned}$$

$\blacksquare$

8. **Effect Of Integer Derivative**

For $k = 0, 1, 2, \ldots$

In Riemann-Liouville

$$\begin{aligned} \frac{\mathrm{d}^k}{\mathrm{d}t^k} D^{\alpha} f(t) &= \frac{\mathrm{d}^k}{\mathrm{d}t^k} \frac{\mathrm{d}^m}{\mathrm{d}t^m} I^{m-\alpha} f(t) \\ &= \frac{\mathrm{d}^{m+k}}{\mathrm{d}t^{m+k}} I^{m-\alpha} f(t) \\ &= \frac{\mathrm{d}^{m+k}}{\mathrm{d}t^{m+k}} I^{m+k-k-\alpha} f(t) = \frac{\mathrm{d}^{m+k}}{\mathrm{d}t^{m+k}} I^{(m+k)-(\alpha+k)} f(t) \\ &= D^{\alpha+k} f(t) \end{aligned}$$

In Caputo

$$\begin{aligned} {}^{C}D^{\alpha} \frac{\mathrm{d}^k}{\mathrm{d}t^k} f(t) &= I^{m-\alpha} \frac{\mathrm{d}^m}{\mathrm{d}t^m} \frac{\mathrm{d}^k}{\mathrm{d}t^k} f(t) \\ &= I^{m-\alpha} \frac{\mathrm{d}^{m+k}}{\mathrm{d}t^{m+k}} f(t) \\ &= I^{m+k-k-\alpha} \frac{\mathrm{d}^{m+k}}{\mathrm{d}t^{m+k}} f(t) = I^{(m+k)-(\alpha+k)} \frac{\mathrm{d}^{m+k}}{\mathrm{d}t^{m+k}} f(t) \\ &= {}^{C}D^{\alpha+k} f(t) \end{aligned}$$

9. The Integral Of Derivative

In Caputo

$$
\begin{aligned}
I^{\alpha}\,{}^{C}D^{\alpha}f(t) &= I^{\alpha}\left[\frac{1}{\Gamma(m-\alpha)}\int_0^t (t-s)^{m-\alpha-1}\frac{\mathrm{d}^m}{\mathrm{d}s^m}f(s)\ ds\right] \\
&= \frac{1}{\Gamma(\alpha)\Gamma(m-\alpha)}\int_0^t (t-s)^{\alpha-1}\int_0^s (s-\theta)^{m-\alpha-1}\frac{\mathrm{d}^m}{\mathrm{d}\theta^m}f(\theta)\ d\theta\ ds \\
&= \frac{1}{\Gamma(\alpha)\Gamma(m-\alpha)}\int_0^t \underbrace{\int_\theta^t (t-s)^{\alpha-1}(s-\theta)^{m-\alpha-1}ds}_{J}\ \frac{\mathrm{d}^m}{\mathrm{d}\theta^m}f(\theta)\ d\theta \qquad (3.6)
\end{aligned}
$$

Let's handle the inner integral first

$$
J = \int_\theta^t (t-s)^{\alpha-1}(s-\theta)^{m-\alpha-1}\ ds
$$

$$
\text{Substitute}\begin{cases} s-\theta=\eta \\ ds = d\eta \\ 0 \to t-\theta \end{cases}
$$

$$
\begin{aligned}
&= \int_0^{t-\theta} (t-\theta-\eta)^{\alpha-1}(\eta)^{m-\alpha-1}\ d\eta \\
&= (t-\theta)^{\alpha-1}\int_0^{t-\theta} (1-\frac{\eta}{t-\theta})^{\alpha-1}(\eta)^{m-\alpha-1}\ d\eta
\end{aligned}
$$

$$
\text{Substitute}\begin{cases} \eta=(t-\theta)\xi \\ d\eta = (t-\theta)\ d\xi \\ 0 \to 1 \end{cases}
$$

$$
\begin{aligned}
&= (t-\theta)^{\alpha-1}\int_0^1 (1-\xi)^{\alpha-1}(t-\theta)^{m-\alpha-1}\ \xi^{m-\alpha-1}\ (t-\theta)\ d\xi \\
&= (t-\theta)^{\alpha+m-\alpha-1}\int_0^1 (1-\xi)^{\alpha-1}\ \xi^{m-\alpha-1}\ d\xi \\
&= (t-\theta)^{m-1}\beta(\alpha, m-\alpha)
\end{aligned}
$$

Substitute in (3.6) we get that

$$
\begin{aligned}
I^{\alpha}\,{}^{C}D^{\alpha}f(t) &= \frac{\beta(\alpha,m-\alpha)}{\Gamma(\alpha)\Gamma(m-\alpha)}\int_0^t (t-\theta)^{m-1}\frac{\mathrm{d}^m}{\mathrm{d}\theta^m}f(\theta)\ d\theta \\
&= \frac{1}{\Gamma(m)}\int_0^t (t-\theta)^{m-1}\frac{\mathrm{d}^m}{\mathrm{d}\theta^m}f(\theta)\ d\theta
\end{aligned}
$$

Integrate by part $(m-1)-times$

$$
= f(t) - \sum_{k=0}^{m-1}\frac{t^{m-k-1}f^{(m-k-1)}(0)}{\Gamma(m-k)}
$$

If we put $k = m-k-1$ the summation value will not change only it's order

$$
= f(t) - \sum_{k=0}^{m-1}\frac{t^k f^{(k)}(0)}{\Gamma(k+1)} = f(t) - \sum_{k=0}^{m-1}\frac{t^k}{k!}f^{(k)}(0)
$$

In Riemann-Liouville
Using the Equivalence Of The Approaches relation

$$\begin{aligned}
I^{\alpha}D^{\alpha}f(t) &= I^{\alpha}\left[\sum_{k=0}^{m-1}\frac{t^{k-\alpha}f^{(k)}(0)}{\Gamma(k+1-\alpha)} + {}^{C}D^{\alpha}f(t)\right] \\
&= \sum_{k=0}^{m-1}\frac{f^{(k)}(0)}{\Gamma(k+1-\alpha)}I^{\alpha}t^{k-\alpha} + I^{\alpha}\,{}^{C}D^{\alpha}f(t) \\
&= \sum_{k=0}^{m-1}\frac{f^{(k)}(0)}{\Gamma(k+1-\alpha)}t^{k}\frac{\Gamma(k+1-\alpha)}{\Gamma(k+1)} + f(t) - \sum_{k=0}^{m-1}\frac{t^{k}}{k!}f^{(k)}(0) \\
&= \sum_{k=0}^{m-1}\frac{t^{k}}{k!}f^{(k)}(0) + f(t) - \sum_{k=0}^{m-1}\frac{t^{k}}{k!}f^{(k)}(0) = f(t)
\end{aligned}$$

10. The Interchange Of The Differentiation Operators

In Riemann-Liouville

$$D_a^k\left(D_a^{\alpha}f(t)\right) = D_a^{\alpha}\left(D_a^k f(t)\right) = D_a^{\alpha+k}f(t)$$

Is allowed under the conditions

$$f^{(i)}(a) = 0 \quad , \quad i = 0, 1, 2, \ldots, k \quad , \quad k = 0, 1, 2, \ldots \quad , \quad m-1 < \alpha < m$$

In Caputo

$${}^{C}D_a^k\left({}^{C}D_a^{\alpha}f(t)\right) = {}^{C}D_a^{\alpha}\left({}^{C}D_a^k f(t)\right) = {}^{C}D_a^{\alpha+k}f(t)$$

Is allowed under the conditions

$$f^{(i)}(a) = 0 \quad , \quad i = m, m+1, \ldots, k \quad , \quad k = 0, 1, 2, \ldots \quad , \quad m-1 < \alpha < m$$

Contrary to the RL approach in the case of the Caputo derivative there are no restrictions on the values $f^{(i)}(a)$ for $i = 0, 1, 2, \ldots, m-1$

11. **Non-Locality**

Normally when we take first or second derivatives the output of the derivative only depends on the input we give it this is called locality i.e. $f^{(n)}(t)$ only depends on t. now if we go back to the definition of a fractional derivative we have constant a at the bottom of the integral

$$D_a^\alpha f(t) = \frac{1}{\Gamma(m-\alpha)} \frac{\mathrm{d}^m}{\mathrm{d}t^m} \int_{\boxed{a}}^t (t-s)^{m-\alpha-1}\ f(s)\ ds$$

Thus the fractional derivative has Non-locality which means that it's value at a point depends on the function values over an interval, not just at that point.
The next two examples will show the effect of this property

■ **Example 3.3.1** Evaluate $D_0^\alpha(E_{a,b}(\lambda t))$

Sol.

$$D_0^\alpha(E_{a,b}(\lambda t)) = D_0^\alpha\left(\sum_{k=0}^{\infty} \frac{(\lambda t)^k}{\Gamma(ak+b)}\right) \tag{3.7}$$

Because the series in (3.7) is uniformly convergent we can interchange D_0^α with the summation

$$D_0^\alpha(E_{a,b}(\lambda t)) = \sum_{k=0}^{\infty} \frac{D_0^\alpha((\lambda t)^k)}{\Gamma(ak+b)} = \sum_{k=0}^{\infty} \frac{\lambda^k D_0^\alpha(t^k)}{\Gamma(ak+b)}$$

And we know that

$$D_0^\alpha(t^n) = t^{n-\alpha}\frac{\Gamma(n+1)}{\Gamma(n-\alpha+1)}$$

Thus

$$D_0^\alpha(E_{a,b}(\lambda t)) = \sum_{k=0}^{\infty} \frac{\Gamma(k+1)}{\Gamma(k-\alpha+1)} \frac{(\lambda^k t^{k-\alpha})}{\Gamma(ak+b)} = t^{-\alpha}\sum_{k=0}^{\infty} \frac{\Gamma(k+1)}{\Gamma(k-\alpha+1)} \frac{(\lambda t)^k}{\Gamma(ak+b)}$$

At $a, b = 1$

$$\begin{aligned} D_0^\alpha(E_{1,1}(\lambda t)) = D_0^\alpha(e^{\lambda t}) &= t^{-\alpha}\sum_{k=0}^{\infty} \frac{\Gamma(k+1)}{\Gamma(k-\alpha+1)} \frac{(\lambda t)^k}{\Gamma(k+1)} \\ &= t^{-\alpha}\sum_{k=0}^{\infty} \frac{((\lambda t)^k)}{\Gamma(k+1-\alpha)} = t^{-\alpha}(E_{1,1-\alpha}(\lambda t)) \end{aligned}$$

■

■ **Example 3.3.2** Evaluate $D_{-\infty}^\alpha(e^{\lambda t})$

Sol.

$$D_{-\infty}^\alpha(e^{\lambda t}) = \frac{1}{\Gamma(m-\alpha)} \frac{\mathrm{d}^m}{\mathrm{d}t^m} \int_{-\infty}^t (t-s)^{m-\alpha-1}(e^{\lambda s})\ ds$$

Substitute $\begin{cases} \dfrac{\xi}{\lambda} = (t-s) \\ d\xi = -\lambda\ ds \\ \infty \to 0 \end{cases}$

$$\begin{aligned} &= \frac{1}{\Gamma(m-\alpha)} \frac{\mathrm{d}^m}{\mathrm{d}t^m} \int_{\infty}^0 \left(\frac{\xi}{\lambda}\right)^{m-\alpha-1} (e^{\lambda t-\xi}) \left(-\frac{1}{\lambda}\right)\ d\xi \\ &= \frac{1}{\Gamma(m-\alpha)} \frac{\mathrm{d}^m}{\mathrm{d}t^m} \left(\frac{1}{\lambda}\right)^{m-\alpha} e^{\lambda t} \int_0^\infty \xi^{m-\alpha-1}(e^{-\xi})\ d\xi \\ &= \frac{\lambda^{\alpha-m}}{\Gamma(m-\alpha)} \frac{\mathrm{d}^m}{\mathrm{d}t^m} e^{\lambda t}\ \Gamma(m-\alpha) \\ &= \lambda^{\alpha-m}\lambda^m e^{\lambda t} = \lambda^\alpha e^{\lambda t} \end{aligned}$$

■

Advantages Of Nonlocality:

1. Modeling Complex Phenomena : Fractional derivatives are useful Analyzing functions that not only depend on time for example some phenomenon in the real world have something called a memory effect which means that the current state not only depends on time but also in previous States Many physical systems exhibit such behaviors, including viscoelastic materials, anomalous diffusion, and certain types of signal processing. traditional differential equations have a hard time modeling phenomenon like this but fractional derivatives can make the task easier
2. Anomalous Diffusion : They are also useful for describing anomalous diffusion phenomena, where the mean squared displacement of particles does not follow a linear relationship with time. Such phenomena are prevalent in complex systems like porous media, biological tissues, and turbulent flows.

Disadvantages Of Nonlocality:

1. Computational Challenges : Numerical approximation and computation of fractional derivatives can be computationally intensive and require specialized algorithms. This can pose challenges, especially when dealing with large-scale systems or real-time applications.
2. Interpretation Difficulty : Non-locality introduced by fractional derivatives can sometimes make it difficult to interpret the physical meaning of the derived equations. Understanding the behavior of systems described by fractional calculus may require a deeper conceptual grasp compared to classical systems.

This raises the question of how exactly we can interpret fractional derivatives. since they're non-local and are influenced by the function's behavior far away from a given input point, they must represent something different than the slope of a tangent line, which is all about measuring a function's local behavior near a point.

Despite ordinary derivatives and integrals having pretty straightforward geometric and physical meanings, it seems no one has come up with a truly satisfying, general interpretation of the fractional operators, or at least not one that's widely accepted.

Trying too hard to interpret fractional calculus in terms of ordinary calculus is probably like insisting on interpreting the equation $e^{i\pi} = -1$ in terms of repeated multiplication. $\underbrace{ee\ldots ee}_{i\pi\ times} = -1$

Note that some familiar properties Don't Work Anymore

12. **The Chain Rule**

$$Df(g(t)) = f'(g(t))g'(t)$$

Proof. counter-example let $g(t) = t^2$ and $f(g) = g^2$
If we use the chain rule

$$D^\alpha f(g(t)) = D^\alpha g(t)^2 = g(t)^{2-\alpha}\frac{\Gamma(3)}{\Gamma(3-\alpha)}D^\alpha g(t)$$

$$D^\alpha g(t) = D^\alpha t^2 = t^{2-\alpha}\frac{\Gamma(3)}{\Gamma(3-\alpha)}$$

Thus

$$D^\alpha f(g(t)) = t^{2(2-\alpha)}\frac{\Gamma(3)}{\Gamma(3-\alpha)}t^{2-\alpha}\frac{\Gamma(3)}{\Gamma(3-\alpha)} = t^{3(2-\alpha)}\left(\frac{\Gamma(3)}{\Gamma(3-\alpha)}\right)^2$$

On the other hand if we apply the fractional derivative on the equivalent function $f(g(t)) = t^4$

$$D^\alpha t^4 = t^{4-\alpha}\frac{\Gamma(5)}{\Gamma(5-\alpha)}$$

■

13. **The Product Rule**

$$Df(t)g(t) = f'(t)g(t) + g'(t)f(t)$$

Proof. Counter example let $f(t) = t^2$ and $g(t) = t^2$
If we use the Product rule

$$D^\alpha f(t) = D^\alpha t^2 = t^{2-\alpha}\frac{\Gamma(3)}{\Gamma(3-\alpha)}$$

$$D^\alpha g(t) = D^\alpha t^2 = t^{2-\alpha}\frac{\Gamma(3)}{\Gamma(3-\alpha)}$$

Thus

$$D^\alpha f(t)g(t) = t^{2-\alpha}\frac{\Gamma(3)}{\Gamma(3-\alpha)}t^2 + t^2 t^{2-\alpha}\frac{\Gamma(3)}{\Gamma(3-\alpha)} = 2t^{4-\alpha}\frac{\Gamma(3)}{\Gamma(3-\alpha)}$$

On the other hand if we apply the fractional derivative on the equivalent function $f(t)g(t) = t^4$

$$D^\alpha t^4 = t^{4-\alpha}\frac{\Gamma(5)}{\Gamma(5-\alpha)}$$

■

14. **Semi-Group Property**

$$D^m D^n f(t) = D^n D^m f(t) = D^{m+n} f(t)$$

Proof. Counter example let $f(t) = t^\alpha$ and $m = \alpha$, $n = \alpha + 1$

$$\begin{aligned} D^\alpha D^{\alpha+1} f(t) &= D^\alpha D^{\alpha+1} t^\alpha \\ &= D^\alpha D^{\alpha+1} t^{(\alpha+1)-1} \\ &= D^\alpha 0 = 0 \end{aligned}$$

Now if we change it's sequence

$$\begin{aligned} D^{\alpha+1} D^\alpha f(t) &= D^{\alpha+1} D^\alpha t^\alpha \\ &= D^{\alpha+1}\Gamma(1+\alpha) \\ &= \Gamma(1+\alpha) D^{\alpha+1} t^0 \\ &= \Gamma(1+\alpha) t^{-\alpha-1}\frac{\Gamma(1)}{\Gamma(-\alpha)} \\ &= t^{-\alpha-1}\frac{\Gamma(1+\alpha)}{\Gamma(-\alpha)} \end{aligned}$$

And if we apply $D^{\alpha+\alpha+1} = D^{2\alpha+1}$

$$\begin{aligned} D^{2\alpha+1} f(t) &= D^{2\alpha+1} t^\alpha \\ &= t^{-1-\alpha}\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha-2\alpha-1)} = t^{-1-\alpha}\frac{\Gamma(1+\alpha)}{\Gamma(-\alpha)} \end{aligned}$$

■

3.4 The Motivation For Caputo Derivative

The fractional differentiation of the Riemann Liouville type played an important role in the development of the theory of fractional derivatives and integrals and for it's applications in pure mathematics (solution of integer-order differential equations, definitions of new function classes, summation of series, ...)

However, the demands of modern technology require a certain revision of the well-established pure mathematical approach. There have appeared a number of works, especially in the theory of viscoelasticity and in hereditary solid mechanics, where fractional derivatives are used for a better description of material properties. mathematical modeling based on enhanced rheological models naturally leads to differential equations of fractional order and to the necessity of the formulation of initial conditions to such equations.

Applied problems require definitions of fractional derivatives allowing the utilization of physically interpret-able initial conditions, which contain $f(a), f'(a), \ldots$

Unfortunately. the Riemann-Liouville approach leads to initial conditions containing the limit values of the Riemann-Liouville fractional derivatives at the lower terminal $t = a$, for example

$$\begin{gathered}\lim_{t\to a} D^{\alpha-1} f(t) = \beta_1 \\ \lim_{t\to a} D^{\alpha-2} f(t) = \beta_2 \\ \vdots \\ \lim_{t\to a} D^{\alpha-m} f(t) = \beta_m\end{gathered}$$

Where β_k for $k = 1, 2, \ldots, m$ are given constants. In spite of the fact that initial value problems with such initial conditions can be successfully solved mathematically their solutions are practically useless, because there is no known physical interpretation for such types of initial conditions.

Here we observe a conflict between the well-established and polished mathematical theory and practical needs. A certain solution to this conflict was proposed by M. Caputo first in his paper and two years later in his book , and recently (in Banach spaces) by A.M.A.El-Sayed.

The main advantage of Caputo's approach is that the initial conditions for fractional differential equations with Caputo derivatives take on the same form as for differential equations

$$f^{(k)}(a) = \beta_k \quad , \quad k = 0, 1, 2, \ldots, m-1$$

Thus the Caputo derivative allows utilization of initial values of classical integer-order derivatives with known physical interpretations (position,velocity,acceleration,...)

It is very important to understand which type of definition of fractional derivative (in other words, which type of initial conditions) must be used.

■ **Example 3.4.1** Evaluate $D^{\alpha}_{-\infty}(sin(at))$

Sol.

$$\begin{aligned}
D^{\alpha}_{-\infty}(sin(at)) &= D^{\alpha}_{-\infty}\left(\frac{e^{iat}-e^{-iat}}{2i}\right) \\
&= \frac{D^{\alpha}_{-\infty}(e^{iat}) - D^{\alpha}_{-\infty}(e^{-iat})}{2i} \\
&= \frac{(ia)^{\alpha}e^{iat} - (-ia)^{\alpha}e^{-iat}}{2i} \\
&= a^{\alpha}\frac{e^{i\left(at+\frac{\alpha\pi}{2}\right)} - e^{-i\left(at+\frac{\alpha\pi}{2}\right)}}{2i} = a^{\alpha}sin\left(at+\frac{\alpha\pi}{2}\right)
\end{aligned}$$

In the same manner

$$D^{\alpha}_{-\infty}(cos(at)) = a^{\alpha}cos\left(at+\frac{\alpha\pi}{2}\right)$$

■

■ **Example 3.4.2** Evaluate ${}^{C}D^{\alpha}(E_{a,b}(\lambda t))$

Sol.

$${}^{C}D^{\alpha}(E_{a,b}(\lambda t)) = {}^{C}D^{\alpha}\left(\sum_{k=0}^{\infty}\frac{(\lambda t)^{k}}{\Gamma(ak+b)}\right)$$

Now because the series is uniformly convergent we can interchange the D operator with the summation

$${}^{C}D^{\alpha}(E_{a,b}(\lambda t)) = \sum_{k=0}^{\infty}\frac{{}^{C}D^{\alpha}((\lambda t)^{k})}{\Gamma(ak+b)} = \sum_{k=0}^{\infty}\frac{\lambda^{k}\,{}^{C}D^{\alpha}(t^{k})}{\Gamma(ak+b)}$$

And we know that

$${}^{C}D^{\alpha}(t^{n})\begin{cases}0 & k<m \\ t^{n-\alpha}\dfrac{\Gamma(n+1)}{\Gamma(n-\alpha+1)} & k\geq m\end{cases}$$

Thus

$${}^{C}D^{\alpha}(E_{a,b}(\lambda t)) = \sum_{k=m}^{\infty}\frac{\Gamma(k+1)}{\Gamma(k-\alpha+1)}\frac{(\lambda^{k}t^{k-\alpha})}{\Gamma(ak+b)}$$

Substitute $k=m+k$

$$\begin{aligned}
&= \sum_{k=0}^{\infty}\frac{\Gamma(m+k+1)}{\Gamma(m+k-\alpha+1)}\frac{(\lambda^{m+k}t^{m+k-\alpha})}{\Gamma(a(m+k)+b)} \\
&= \lambda^{m}t^{m-\alpha}\sum_{k=0}^{\infty}\frac{\Gamma(m+k+1)}{\Gamma(m+k-\alpha+1)}\frac{(\lambda t)^{k}}{\Gamma(a(m+k)+b)}
\end{aligned}$$

When $a,b=1$

$$\begin{aligned}
{}^{C}D^{\alpha}(E_{1,1}(\lambda t)) = {}^{C}D^{\alpha}(e^{\lambda t}) &= \lambda^{m}t^{m-\alpha}\sum_{k=0}^{\infty}\frac{\Gamma(m+k+1)}{\Gamma(m+k-\alpha+1)}\frac{(\lambda t)^{k}}{\Gamma(m+k+1)} \\
&= \lambda^{m}t^{m-\alpha}\sum_{k=0}^{\infty}\frac{(\lambda t)^{k}}{\Gamma(m+k-\alpha+1)} = \lambda^{m}t^{m-\alpha}(E_{1,1-\alpha+m}(\lambda t))
\end{aligned}$$

■

4 Integral Transformations

Integral transformations are mathematical operations that map functions from one space to another, typically defined on the real line or in higher dimensions. They represent a fundamental tool for solving problems and simplifying complex mathematical operations we will take a look at Laplace,Fourier, Mellin transformations for fractional integral and fractional derivatives that are essential to solve a lot of differential equation

4.1 Laplace Transform

The Laplace Transform of the function $f(t)$ defined by

$$\mathcal{L}\left[f(t)\right] = \int_0^\infty e^{-st} f(t) dt = F(s) \tag{4.1}$$

The function $F(s)$ is of complex variable s for the existence of this integral $f(t)$ must be exponential of order α this means that $\exists M, T \in \mathbb{R}^+$ Such that

$$e^{-\alpha t} f(t) \leq M \quad , \quad \forall t > T$$

i.e $f(t)$ must not grow faster than a certain exponential function when $t \to \infty$

And to restore the original function $f(t)$ from $F(s)$ we use the inverse Laplace Transform

$$f(t) = \mathcal{L}^{-1}\left[F(s)\right] = \int_{\gamma - i\infty}^{\gamma + i\infty} e^{st} F(s)\ ds$$

Where the integration is done along the vertical line $Re(s) = \gamma$ in the complex plane such that γ is greater than the real part of all singularities of $F(s)$ and $F(s)$ is bounded on the line

For example if the contour path is in the region of convergence. If all singularities are in the left half-plane, or $F(s)$ is an entire function, then γ can be set to zero.

Now some of the useful properties of the Laplace Transform that we will use later

Property 1 The Laplace Transform Of The Convolution

$$f(t) * g(t) = \int_0^t f(t-s) g(t)\ dt = \int_0^t f(t) g(t-s)\ dt$$

$$\mathcal{L}\left[f(t) * g(t)\right] = \mathcal{L}\left[f(t)\right] \mathcal{L}\left[g(t)\right] = F(s) G(s) \tag{4.2}$$

Property 2 The Laplace Transform Of The Derivative Of An Integer Order n

$$\mathcal{L}\left[f^{(n)}(t)\right] = s^n F(s) - \sum_{k=0}^{n-1} s^{n-k-1} f^{(k)}(0) \tag{4.3}$$

Pierre Simon Laplace

Laplace (1749-1827) a prominent figure in mathematics and physics, made significant contributions to a wide range of fields during the 18th and 19th centuries. Laplace's mark on fractional calculus was significant, though not all-encompassing. He provided a crucial stepping stone in 1812 by defining a fractional derivative through an integral equation. This definition, along with subsequent work by mathematicians like Riemann and Liouville, paved the way for fractional calculus to become a robust field studying derivatives and integrals with "in-between" orders.

OR

$$\mathcal{L}\left[f^{(n)}(t)\right] = s^n F(s) - \sum_{k=0}^{n-1} s^k f^{(n-k-1)}(0) \tag{4.4}$$

Which can be obtained from (4.1) by integrating by parts

4.1.1 Laplace Transform Of Riemann-Liouville Fractional Integral

We will start with Laplace Transform of Riemann-Liouville fractional Integral

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\, ds$$

We know that we can rewrite it as a convolution

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} g(t) * f(t)$$

Where $g(t) = t^{\alpha-1}$

Now take the Laplace Transform for both sides

$$\mathcal{L}\left[I^\alpha f(t)\right] = \mathcal{L}\left[\frac{1}{\Gamma(\alpha)} g(t) * f(t)\right] = \frac{1}{\Gamma(\alpha)} \mathcal{L}\left[g(t)\right] \mathcal{L}\left[f(t)\right]$$

And we know that

$$\mathcal{L}\left[t^{\alpha-1}\right] = \Gamma(\alpha) s^{-\alpha}$$

Therefore we obtain Laplace Transform of Riemann-Liouville fractional Integral

$$\mathcal{L}\left[I^\alpha f(t)\right] = \frac{1}{\Gamma(\alpha)} \Gamma(\alpha) s^{-\alpha} F(s) = s^{-\alpha} F(s) \tag{4.5}$$

4.1.2 Laplace Transform Of Riemann-Liouville Fractional Derivative

Now let us try to get Laplace Transform for Riemann-Liouville fractional derivative

$$D^\alpha f(t) = \frac{\mathrm{d}^m}{\mathrm{d}t^m} I^{m-\alpha} f(t)$$

Now take the Laplace Transform of it

$$\mathcal{L}\left[D^\alpha f(t)\right] = \mathcal{L}\left[\frac{\mathrm{d}^m}{\mathrm{d}t^m} I^{m-\alpha} f(t)\right] = \mathcal{L}\left[\frac{\mathrm{d}^m}{\mathrm{d}t^m} g(t)\right]$$

Using the formula for The Laplace Transform of the derivative of an integer order (4.4)

$$\mathcal{L}\left[\frac{\mathrm{d}^m}{\mathrm{d}t^m} g(t)\right] = s^m \mathcal{L}\left[g(t)\right] - \sum_{k=0}^{m-1} s^k \left[\frac{\mathrm{d}^{m-k-1}}{\mathrm{d}t^{m-k-1}} g(t)\right]_{t=0}$$

Now Substitute for $g(t) = I^{m-\alpha} f(t)$ we get

$$\begin{aligned}
\mathcal{L}\left[D^\alpha f(t)\right] &= s^m s^{\alpha-m} F(s) - \sum_{k=0}^{m-1} s^k \left[\frac{\mathrm{d}^{m-k-1}}{\mathrm{d}t^{m-k-1}} I^{m-\alpha} f(t)\right]_{t=0} \\
&= s^\alpha F(s) - \sum_{k=0}^{m-1} s^k \left[\frac{\mathrm{d}^{m-k-1}}{\mathrm{d}t^{m-k-1}} I^{m-k-1-\alpha+k+1} f(t)\right]_{t=0} \\
&= s^\alpha F(s) - \sum_{k=0}^{m-1} s^k \left[D^{\alpha-k-1} f(t)\right]_{t=0}
\end{aligned}$$

This is the Laplace transform of the Riemann Liouville fractional derivative However, it's practical applicability is limited by the absence of the physical interpretation of the limit values of fractional derivatives at the lower terminal $t = 0$. such an interpretation is not known.

4.1.3 Laplace Transform Of Caputo Derivative

The Caputo derivative is defined by

$$ {}^{C}D^{\alpha}f(t) = I^{m-\alpha}\frac{\mathrm{d}^m}{\mathrm{d}t^m}f(t) $$

Now take the Laplace Transform of both sides

$$ \mathcal{L}\left[{}^{C}D^{\alpha}f(t)\right] = \mathcal{L}\left[I^{m-\alpha}\frac{\mathrm{d}^m}{\mathrm{d}t^m}f(t)\right] = s^{\alpha-m}\mathcal{L}\left[\frac{\mathrm{d}^m}{\mathrm{d}t^m}f(t)\right] $$

Using the formula for The Laplace Transform of the derivative of an integer order (4.3)

$$ \mathcal{L}\left[{}^{C}D^{\alpha}f(t)\right] = s^{\alpha-m}\left[s^m F(s) - \sum_{k=0}^{m-1} s^{m-k-1}f^{(k)}(0)\right] = s^{\alpha}F(s) - \sum_{k=0}^{m-1} s^{\alpha-k-1}f^{(k)}(0) $$

Since this formula for the Laplace transform of the Caputo derivative involves the values of the function $f(t)$ and it's derivatives at the lower terminal $t = 0$ for which a certain physical interpretation exists (for example, $f(0)$ is the initial position, $f^{(1)}(0)$ is the initial velocity, $f^{(2)}(0)$ is the initial acceleration), we can expect that it can be useful for solving applied problems leading to linear fractional differential equations with constant coefficients with accompanying initial conditions in traditional form.

4.2 Fourier Transform

The Fourier Transform of a continuous function $f(t)$ absolutely integrable on $(-\infty, \infty)$ is defined by

$$\mathcal{F}\left[f(t)\right] = \int_{-\infty}^{\infty} e^{-i\omega t} f(t)\ dt = F(\omega)$$

And the original $f(t)$ can be restored from it's Fourier Transform $F(\omega)$ by the inverse Fourier transform

$$\mathcal{F}^{-1}\left[F(\omega)\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t} F(\omega)\ d\omega = f(t)$$

Fourier Transform have the following properties

Property 1 The Fourier Transform Of The Convolution

Let $f(t)$ and $g(t)$ be two functions defined on $(-\infty, \infty)$,

$$f(t) * g(t) = \int_{-\infty}^{\infty} f(t-s)g(t)\ dt = \int_{-\infty}^{\infty} f(t)g(t-s)\ dt$$

The Fourier Transform of their convolution is equal to the product of their Fourier transforms

$$\mathcal{F}\left[f(t) * g(t)\right] = \mathcal{F}\left[f(t)\right]\mathcal{F}\left[g(t)\right] = F(\omega)G(\omega) \tag{4.7}$$

Property 2 The Fourier Transform Of Derivatives

Let $f(t)$ and it's derivatives vanish for $t \to \pm\infty$ then the Fourier transform of the n^{th} derivative of $f(t)$ is

$$\mathcal{F}\left[\frac{\mathrm{d}^n}{\mathrm{d}t^n} f(t)\right] = (-i\omega)^n F(\omega) \tag{4.8}$$

4.2.1 Fourier Transform Of Riemann-Liouville Integral

As we did in Laplace Transform here as well we will evaluate the Fourier transform of the Riemann-Liouville fractional integral but first let's change it's form with lower terminal $-\infty$ and assume that $0 < \alpha < 1$ we get

$$I^{\alpha}_{-\infty} f(t) = \frac{1}{\Gamma(\alpha)}\int_{-\infty}^{t} (t-s)^{\alpha-1} f(s)\ ds$$

Let us define $g(t)\begin{cases} \dfrac{t^{\alpha-1}}{\Gamma(\alpha)} & , \quad t > 0 \\ 0 & , \quad t \leq 0 \end{cases}$

Now to get it's Fourier Transform

$$G(\omega) = \mathcal{F}\left[g(t)\right] = \frac{1}{\Gamma(\alpha)}\int_{0}^{\infty} t^{\alpha-1} e^{-i\omega t}\ dt = (-i\omega)^{-\alpha}$$

Jean-Baptiste Joseph Fourier

Joseph Fourier (1768–1830) was a brilliant French mathematician and physicist although he laid the groundwork for many areas of mathematics, he didn't directly contribute to fractional calculus, a field that emerged later. However, the concept of Fourier transforms plays a role in modern fractional calculus. Fractional Fourier transforms are mathematical tools used to analyze and solve equations involving non-integer order derivatives.

Now we can find the Fourier transform of the Riemann-Liouville fractional integral , which can be written as a convolution of the functions $g(t)$ and $f(t)$

$$\mathcal{F}\left[I^{\alpha}_{-\infty}f(t)\right] = [g(t) * f(t)] = G(\omega)F(\omega) = (-i\omega)^{-\alpha}F(\omega)$$

4.2.2 Fourier Transform Of Riemann-Liouville Derivative

Now for the Fourier Transform of Riemann-Liouville fractional derivative

$$D^{\alpha}_{-\infty}f(t) = \frac{\mathrm{d}^m}{\mathrm{d}t^m} I^{m-\alpha}_{-\infty} f(t)$$

Now take the Fourier Transform of it

$$\mathcal{F}\left[D^{\alpha}_{-\infty}f(t)\right] = \mathcal{F}\left[\frac{\mathrm{d}^m}{\mathrm{d}t^m} I^{m-\alpha}_{-\infty} f(t)\right] = \mathcal{F}\left[\frac{\mathrm{d}^m}{\mathrm{d}t^m} g(t)\right]$$

Using the formula for The Fourier Transform of derivatives (4.8)

$$\mathcal{F}\left[\frac{\mathrm{d}^m}{\mathrm{d}t^m} g(t)\right] = (-i\omega)^m \mathcal{F}\left[g(t)\right]$$

Now Substitute for $g(t) = I^{m-\alpha}_{-\infty} f(t)$ we get

$$\mathcal{F}\left[D^{\alpha}_{-\infty}f(t)\right] = (-i\omega)^m(-i\omega)^{\alpha-m}F(\omega) = (-i\omega)^{\alpha}F(\omega)$$

4.2.3 Fourier Transform Of Caputo Derivative

The Caputo derivative is defined as follows

$${}^{C}D^{\alpha}_{-\infty}f(t) = I^{m-\alpha}_{-\infty}\frac{\mathrm{d}^m}{\mathrm{d}t^m} f(t)$$

Now take the Fourier Transform of it

$$\mathcal{F}\left[{}^{C}D^{\alpha}_{-\infty}f(t)\right] = \mathcal{F}\left[I^{m-\alpha}_{-\infty}\frac{\mathrm{d}^m}{\mathrm{d}t^m} f(t)\right] = (-i\omega)^{\alpha-m}\mathcal{F}\left[\frac{\mathrm{d}^m}{\mathrm{d}t^m} f(t)\right]$$

Using the formula for The Fourier Transform of derivatives (4.8)

$$\begin{aligned}\mathcal{F}\left[{}^{C}D^{\alpha}_{-\infty}f(t)\right] &= (-i\omega)^{\alpha-m}\mathcal{F}\left[\frac{\mathrm{d}^m}{\mathrm{d}t^m} f(t)\right] \\ &= (-i\omega)^{\alpha-m}(-i\omega)^m F(\omega) = (-i\omega)^{\alpha}F(\omega)\end{aligned}$$

Which is the same as Riemann-Liouville fractional derivative Since both start from $-\infty$

4.3 Mellin Transforms

The Mellin integral Transform $F(s)$ of a function $f(t)$. which is defined in the interval $(0, \infty)$ is

$$F(s) = \mathcal{M}\left[f(t)\right] = \int_0^\infty f(t)t^{s-1}\,dt \tag{4.9}$$

Where s is complex variable , such as

$$\gamma_1 < Re(s) < \gamma_2$$

The Mellin transform exists if the function $f(t)$ is piecewise continuous in every closed interval $[a, b] \subset (0, \infty)$ and

$$\int_0^1 |f(t)|t^{\gamma_1 - 1}\,dt < \infty \quad , \quad \int_1^\infty |f(t)|t^{\gamma_2 - 1}\,dt < \infty$$

If the function $f(t)$ also satisfies the Dirichlet conditions in every closed interval $[a, b] \subset (0, \infty)$, then the function $f(t)$ can be restored using the inverse Mellin transform formula

$$f(t) = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} F(s)t^{-s}\,ds \quad , \quad (0 < t < \infty)$$

Properties of Mellin transform

Property 1 Shift Property
It follows from the definition (4.9) that

$$\mathcal{M}\left[t^\alpha f(t)\right] = \mathcal{M}\left[f(t)\right]_{(s=s+\alpha)} = F(s+\alpha) \tag{4.10}$$

Property 2 The Mellin Transform Of The Mellin Convolution

$$f(t) * g(t) = \int_0^\infty f(ts)g(s)\,ds$$

Is given by the formula

$$\mathcal{M}\left[\int_0^\infty f(ts)g(s)\,ds\right] = F(s)G(1-s) \tag{4.11}$$

Property 3 The Shift Of Convolution

$$\begin{aligned}
\mathcal{M}\left[t^\lambda \int_0^\infty f(ts)g(s)\,ds\right] &= \left[\mathcal{M}\left[\int_0^\infty f(ts)g(s)\,ds\right]\right]_{(s=s+\lambda)} \\
&= F(s)\left[G(1-s)\right]_{(s=s+\lambda)} \\
&= F(s+\lambda)G(1-s-\lambda)
\end{aligned}$$

Robert Hjalmar Mellin

Hjalmar Mellin (1854 - 1933) was a Finnish mathematician His work in the early 1900s laid the foundation for applying fractional derivatives and integrals to real-world problems. Mellin's contributions involved defining the Mellin transform This transformation allowed mathematicians to express fractional-order derivatives and integrals in terms of more familiar operations. Mellin's work opened doors for further development of fractional calculus, making it a valuable tool in various scientific fields.

Property 4 The Mellin Transform Of An Integer Order Derivative

$$\mathcal{M}\left[f^{(n)}(t)\right] = \int_0^\infty f^{(n)}(t)t^{s-1}dt$$

Using integration by parts

$$\begin{aligned}
&= \left[f^{(n-1)}(t)t^{s-1}\right]_0^\infty - (s-1)\int_0^\infty f^{(n-1)}(t)t^{s-2}dt \\
&= \left[f^{(n-1)}(t)t^{s-1}\right]_0^\infty - (s-1)\mathcal{M}\left[f^{(n-1)}(t)\right]_{(s=s-1)} \\
&\vdots \\
&= \sum_{k=0}^{n-1}(-1)^k\frac{\Gamma(s)}{\Gamma(s-k)}\left[f^{(n-k-1)}(t)t^{s-k-1}\right]_0^\infty + (-1)^n\frac{\Gamma(s)}{\Gamma(s-n)}F(s-n)
\end{aligned}$$

But we know that

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{sin(\pi x)}$$

Put $x = s - k$

$$\begin{aligned}
\Gamma(s-k)\Gamma(1-s+k) &= \frac{\pi}{sin(\pi(s-k))} \\
&= \frac{\pi}{sin(\pi s)cos(\pi k) - cos(\pi s)sin(\pi k)}
\end{aligned}$$

$sin(\pi k) = 0 \quad , \quad cos(\pi k) = (-1)^k$

$$= \frac{\pi}{sin(\pi s)}(-1)^k = \Gamma(s)\Gamma(1-s)(-1)^k$$

Thus

$$\frac{\Gamma(1-s+k)}{\Gamma(1-s)} = (-1)^k\frac{\Gamma(s)}{\Gamma(s-k)}$$

Therefore

$$\mathcal{M}\left[f^{(n)}(t)\right] = \sum_{k=0}^{n-1}\frac{\Gamma(1-s+k)}{\Gamma(1-s)}\left[f^{(n-k-1)}(t)t^{s-k-1}\right]_0^\infty + \frac{\Gamma(1-s+n)}{\Gamma(1-s)}F(s-n) \tag{4.12}$$

If $f(t)$ and $Re(s)$ are such that all substitutions of the limits $t = 0$ and $t = \infty$ give zero, then the formula takes it's simplest form

$$\mathcal{M}\left[f^{(n)}(t)\right] = \frac{\Gamma(1-s+n)}{\Gamma(1-s)}F(s-n) \tag{4.13}$$

4.3.1 Mellin Transforms Of Riemann-Liouville Integral

Let us evaluate the mellin transform of the Riemann Liouville fractional integral.

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s)\ ds$$

Substitute $\begin{cases} s = t\theta \\ ds = t\ d\theta \\ 0 \to 1 \end{cases}$

$$= \frac{t^\alpha}{\Gamma(\alpha)}\int_0^1 (1-\theta)^{\alpha-1}f(t\theta)\ d\theta$$

Let us define $g(t) \begin{cases} (1-\theta)^{\alpha-1} & , \quad 0 \le t < 1 \\ 0 & , \quad t \ge 1 \end{cases}$

$$= \frac{t^\alpha}{\Gamma(\alpha)} \int_0^\infty g(\theta) f(t\theta)\ d\theta$$

Take the Mellin transform for it and using the Shift of Convolution property (3)

$$\mathcal{M}\left[I^\alpha f(t)\right] = \frac{1}{\Gamma(\alpha)} F(s+\alpha) G(1-s-\alpha)$$

The Mellin transform of the function $g(t)$ gives simply the beta function.

$$\mathcal{M}\left[g(t)\right] = G(s) = \beta(\alpha, s) = \frac{\Gamma(\alpha)\Gamma(s)}{\Gamma(\alpha+s)}$$

Thus

$$\mathcal{M}\left[I^\alpha f(t)\right] = \frac{1}{\Gamma(\alpha)} F(s+\alpha) \beta(\alpha, 1-s-\alpha) = \frac{\Gamma(1-s-\alpha)}{\Gamma(1-s)} F(s+\alpha)$$

4.3.2 Mellin Transforms Of Riemann-Liouville Derivative

Now let us get the Mellin transform of the RL fractional derivative

$$D^\alpha f(t) = \frac{\mathrm{d}^m}{\mathrm{d}t^m} I^{m-\alpha} f(t)$$

Take the Mellin transform of it

$$\mathcal{M}\left[D^\alpha f(t)\right] = \mathcal{M}\left[\frac{\mathrm{d}^m}{\mathrm{d}t^m} I^{m-\alpha} f(t)\right] = \mathcal{M}\left[\frac{\mathrm{d}^m}{\mathrm{d}t^m} g(t)\right]$$

Using The Mellin transform of an integer order derivative property

$$= \sum_{k=0}^{m-1} \frac{\Gamma(1-s+k)}{\Gamma(1-s)} \left[\frac{\mathrm{d}^{m-k-1}}{\mathrm{d}t^{m-k-1}} g(t) t^{s-k-1}\right]_0^\infty + \frac{\Gamma(1-s+m)}{\Gamma(1-s)} G(s-m)$$

Now Substitute for $g(t)$

$$= \sum_{k=0}^{m-1} \frac{\Gamma(1-s+k)}{\Gamma(1-s)} \left[\frac{\mathrm{d}^{m-k-1}}{\mathrm{d}t^{m-k-1}} I^{m-\alpha} f(t) t^{s-k-1}\right]_0^\infty + \frac{\Gamma(1-s+m)}{\Gamma(1-s)} \mathcal{M}\left[I^{m-\alpha} f(t)\right]_{s=s-m}$$

$$= \sum_{k=0}^{m-1} \frac{\Gamma(1-s+k)}{\Gamma(1-s)} \left[D^{\alpha-k-1} f(t) t^{s-k-1}\right]_0^\infty$$

$$+ \frac{\Gamma(1-s+m)}{\Gamma(1-s)} \left[\frac{\Gamma(1-s-(m-\alpha))}{\Gamma(1-s)} F(s+(m-\alpha))\right]_{s=s-m}$$

$$= \sum_{k=0}^{m-1} \frac{\Gamma(1-s+k)}{\Gamma(1-s)} \left[D^{\alpha-k-1} f(t) t^{s-k-1}\right]_0^\infty$$

$$+ \frac{\Gamma(1-s+m)}{\Gamma(1-s)} \frac{\Gamma(1-(s-m)-(m-\alpha))}{\Gamma(1-(s-m))} F((s-m)+(m-\alpha))$$

$$= \sum_{k=0}^{m-1} \frac{\Gamma(1-s+k)}{\Gamma(1-s)} \left[D^{\alpha-k-1} f(t) t^{s-k-1}\right]_0^\infty + \frac{\Gamma(1-s+\alpha)}{\Gamma(1-s)} F(s-\alpha) \tag{4.14}$$

If $0 < \alpha < 1$ then

$$\mathcal{M}\left[D^\alpha f(t)\right] = \left[I^{1-\alpha} f(t) t^{s-1}\right]_0^\infty + \frac{\Gamma(1-s+\alpha)}{\Gamma(1-s)} F(s-\alpha) \tag{4.15}$$

If $f(t)$ and $Re(s)$ are such that all substitutions of the limits $t = 0$ and $t = \infty$ give zero, then the formula takes it's simplest form

$$\mathcal{M}\left[D^\alpha f(t)\right] = \frac{\Gamma(1-s+\alpha)}{\Gamma(1-s)} F(s-\alpha) \tag{4.16}$$

4.3.3 Mellin Transforms Of Caputo Derivative

Let $m-1 \leq \alpha < m$ and we know that

$$^{C}D^{\alpha}f(t) = I^{m-\alpha}\frac{\mathrm{d}^m}{\mathrm{d}t^m}f(t)$$

Now take the Mellin Transform of it

$$\mathcal{M}\left[^{C}D^{\alpha}f(t)\right] = \mathcal{M}\left[I^{m-\alpha}\frac{\mathrm{d}^m}{\mathrm{d}t^m}f(t)\right] = \frac{\Gamma(1-s-m+\alpha)}{\Gamma(1-s)}\mathcal{M}\left[\frac{\mathrm{d}^m}{\mathrm{d}t^m}f(t)\right]_{s=s+m-\alpha}$$

Using the formula for The Mellin Transform of derivatives (4.13)

$$\begin{aligned}
&= \frac{\Gamma(1-s-m+\alpha)}{\Gamma(1-s)}\mathcal{M}\left[\frac{\mathrm{d}^m}{\mathrm{d}t^m}f(t)\right]_{s=s+m-\alpha} \\
&= \frac{\Gamma(1-s-m+\alpha)}{\Gamma(1-s)}\left\{\sum_{k=0}^{m-1}\frac{\Gamma(1-s+k)}{\Gamma(1-s)}\left[f^{(m-k-1)}(t)t^{s-k-1}\right]_0^\infty \right. \\
&\left. +\frac{\Gamma(1-s+m)}{\Gamma(1-s)}F(s-m)\right\}_{s=s+m-\alpha} \\
&= \frac{\Gamma(1-s-m+\alpha)}{\Gamma(1-s)}\left\{\sum_{k=0}^{m-1}\frac{\Gamma(1-(s+m-\alpha)+k)}{\Gamma(1-(s+m-\alpha))}\left[f^{(m-k-1)}(t)t^{(s+m-\alpha)-k-1}\right]_0^\infty \right. \\
&\left. +\frac{\Gamma(1-(s+m-\alpha)+m)}{\Gamma(1-(s+m-\alpha))}F((s+m-\alpha)-m)\right\} \\
&= \sum_{k=0}^{m-1}\frac{\Gamma(1-(s+m-\alpha)+k)}{\Gamma(1-s)}\left[f^{(m-k-1)}(t)t^{(s+m-\alpha)-k-1}\right]_0^\infty \\
&+ \frac{\Gamma(1-s+\alpha)}{\Gamma(1-s)}F(s-\alpha)
\end{aligned}$$

If we put $k=m-k-1$ the summation value will not change only it's order

$$= \sum_{k=0}^{m-1}\frac{\Gamma(\alpha-s-k)}{\Gamma(1-s)}\left[f^{(k)}(t)t^{s+k-\alpha}\right]_0^\infty + \frac{\Gamma(1-s+\alpha)}{\Gamma(1-s)}F(s-\alpha) \tag{4.17}$$

If $0<\alpha<1$ then

$$\mathcal{M}\left[^{C}D^{\alpha}f(t)\right] = \frac{\Gamma(\alpha-s)}{\Gamma(1-s)}\left[f(t)t^{s-\alpha}\right]_0^\infty + \frac{\Gamma(1-s+\alpha)}{\Gamma(1-s)}F(s-\alpha) \tag{4.18}$$

If $f(t)$ and $Re(s)$ are such that all substitutions of the limits $t=0$ and $t=\infty$ give zero, then the formula takes it's simplest form

$$\mathcal{M}\{D^{\alpha}f(t)\} = \frac{\Gamma(1-s+\alpha)}{\Gamma(1-s)}F(s-\alpha) \tag{4.19}$$

5 Other Fractional Integral And Derivative Definitions

In the previous sections we only talked about RL and Caputo Definition of Fractional Integral And Derivative but there is a lot of other Definitions That we will show some of them now

5.1 Riemann Liouville Integral Definitions

Until now, we considered the fractional integral and derivatives with fixed lower terminal a and moving upper terminal t. Moreover we supposed that $a < t$. However, it is also possible to consider fractional integral and derivatives with moving lower terminal t and fixed upper terminal b. Let us suppose that the function $f(t)$ is defined in the interval $[a, b]$

1. The Left Riemann Liouville Fractional Integral

$$I_{a+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s)\ ds \tag{5.1}$$

2. The Right Riemann Liouville Fractional Integral

$$I_{b-}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (s-t)^{\alpha-1} f(s)\ ds \tag{5.2}$$

3. The Liouville Fractional Integral We define the fractional integral according to Liouville by setting $a = -\infty$, $b = +\infty$

$${}^{L}I_{+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^t (t-s)^{\alpha-1} f(s)\ ds \tag{5.3}$$

$${}^{L}I_{-}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_t^{+\infty} (s-t)^{\alpha-1} f(s)\ ds \tag{5.4}$$

4. The Riemann Fractional Integral We define the fractional integral according to Riemann by setting $a, b = 0$

$${}^{R}I_{+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\ ds \tag{5.5}$$

$${}^{R}I_{-}^{\alpha} = \frac{1}{\Gamma(\alpha)} \int_t^0 (s-t)^{\alpha-1} f(s)\ ds \tag{5.6}$$

This some examples to show the difference Between The Liouville sense and Riemann sense

$$\begin{aligned}
&{}^{L}I_{+}^{\alpha}(e^{kt}) = k^{-\alpha} e^{kt} && k, t > 0 \\
&{}^{L}I_{-}^{\alpha}(e^{kt}) = (-k)^{-\alpha} e^{kt} && k < 0 \\
&{}^{R}I_{+}^{\alpha}(e^{kt}) = (k)^{-\alpha} e^{kt} \left(1 - \frac{\gamma(\alpha, kt)}{\Gamma(\alpha)}\right) && t > 0 \\
&{}^{R}I_{-}^{\alpha}(e^{kt}) = (-k)^{-\alpha} e^{kt} \left(1 - \frac{\gamma(\alpha, kt)}{\Gamma(\alpha)}\right) && t < 0
\end{aligned}$$

Where $\gamma(\alpha, kt)$ is the incomplete gamma function

Incomplete Gamma Function

The upper and lower incomplete gamma functions are special functions defined by
The upper incomplete gamma function

$$\Gamma(s, x) = \int_x^{\infty} t^{s-1} e^{-t} dt$$

The lower incomplete gamma function

$$\gamma(s, x) = \int_0^x t^{s-1} e^{-t} dt$$

The notions of left and right fractional derivatives can be considered from the physical and the mathematical viewpoints. Sometimes the following physical interpretation of the left and right derivative can be helpful. Let us suppose that t is time and the function $f(t)$ describes a certain dynamical process developing in time. If we take $s < t$, where t is the present moment, then the state $f(s)$ of the process f belongs to the past of this process and if we take $s > t$, then $f(s)$ belongs to the future of the process f.

5.2 Riemann Liouville Derivative Definitions

For the simple case $0 < \alpha < 1$ we obtain

1. The Liouville Fractional Derivatives

$$ {}^{L}D_{+}^{\alpha} f(t) = \frac{\mathrm{d}}{\mathrm{d}t} \, {}^{L}I_{+}^{1-\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}t} \int_{-\infty}^{t} (t-s)^{-\alpha} f(s) \, ds \tag{5.7} $$

$$ {}^{L}D_{-}^{\alpha} f(t) = \frac{\mathrm{d}}{\mathrm{d}t} \, {}^{L}I_{-}^{1-\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}t} \int_{t}^{+\infty} (s-t)^{-\alpha} f(s) \, ds \tag{5.8} $$

2. The Riemann Fractional Derivatives

$$ {}^{R}D_{+}^{\alpha} f(t) = \frac{\mathrm{d}}{\mathrm{d}t} \, {}^{R}I_{+}^{1-\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}t} \int_{0}^{t} (t-s)^{-\alpha} f(s) \, ds \tag{5.9} $$

$$ {}^{R}D_{-}^{\alpha} f(t) = \frac{\mathrm{d}}{\mathrm{d}t} \, {}^{R}I_{-}^{1-\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}t} \int_{t}^{0} (s-t)^{-\alpha} f(s) \, ds \tag{5.10} $$

3. The Liouville-Caputo Fractional Derivatives

$$ {}^{LC}D_{+}^{\alpha} f(t) = {}^{L}I_{+}^{1-\alpha} \frac{\mathrm{d}}{\mathrm{d}t} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_{-\infty}^{t} (t-s)^{-\alpha} \frac{\mathrm{d}f(s)}{\mathrm{d}s} \, ds \tag{5.11} $$

$$ {}^{LC}D_{-}^{\alpha} f(t) = {}^{L}I_{-}^{1-\alpha} \frac{\mathrm{d}}{\mathrm{d}t} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_{t}^{+\infty} (s-t)^{-\alpha} \frac{\mathrm{d}f(s)}{\mathrm{d}s} \, ds \tag{5.12} $$

4. The Riemann-Caputo Fractional Derivative (The Caputo Fractional Derivative)

$$ {}^{C}D_{+}^{\alpha} f(t) = {}^{R}I_{+}^{1-\alpha} \frac{\mathrm{d}}{\mathrm{d}t} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_{0}^{t} (t-s)^{-\alpha} \frac{\mathrm{d}f(s)}{\mathrm{d}s} \, ds \tag{5.13} $$

$$ {}^{C}D_{-}^{\alpha} f(t) = {}^{R}I_{-}^{1-\alpha} \frac{\mathrm{d}}{\mathrm{d}t} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_{t}^{0} (s-t)^{-\alpha} \frac{\mathrm{d}f(s)}{\mathrm{d}s} \, ds \tag{5.14} $$

5.3 Differ-Integral Operator

If we combine fractional differentiation and fractional integration we get a unified derivative-integral operator that works basically as a piecewise combination of the two where plugging in a positive order uses the fractional derivative formula, and plugging in a negative order uses the fractional integral formula. This combined operator is known as a "Differ-Integral" and it's defined by

$$ {}^{RL}_{\;a}D_{t}^{\alpha} f(t) = \begin{cases} D^{\alpha} f(t) = \dfrac{1}{\Gamma(m-\alpha)} \dfrac{\mathrm{d}^{m}}{\mathrm{d}t^{m}} \displaystyle\int_{a}^{t} (t-s)^{m-\alpha-1} f(s) \, ds & \text{if } \alpha > 0 \\ f(t) & \text{if } \alpha = 0 \\ I^{\alpha} f(t) = \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_{a}^{t} (t-s)^{\alpha-1} f(s) \, ds & \text{if } \alpha < 0 \end{cases} $$

5.4 Fractional Derivative According To Fourier

The Fractional Derivative represented using Fourier series is

$$ f(x) = a_0 + \sum_{k=1}^{\infty} a_k cos(kx) + b_k sin(kx) $$

$$ {}^{\mathcal{F}}D^{\alpha} f(x) = \sum_{k=1}^{\infty} a_k k^{\alpha} cos(kx + \frac{\pi}{2}\alpha) + b_k k^{\alpha} sin(kx + \frac{\pi}{2}\alpha) $$

5.5 The Grünwald-Letnikov Fractional Derivative

Another definition of a fractional derivative in terms of a limit of finite differences. Starting with the definition of the first derivative as

$$\frac{\mathrm{d}}{\mathrm{d}t}f(t) = \lim_{h\to 0}\frac{\Delta f(x)}{h} = \lim_{h\to 0}\frac{f(x)-f(x-h)}{h}$$

And for higher derivatives

$$\begin{aligned}
\frac{\mathrm{d}^2}{\mathrm{d}t^2}f(t) &= \lim_{h\to 0}\frac{\Delta^2 f(x)}{h^2} = \lim_{h\to 0}\frac{\Delta(f(x)-f(x-h))}{h^2}\\
&= \lim_{h\to 0}\frac{f(x)-2f(x-h)+f(x-2h)}{h^2}\\
\frac{\mathrm{d}^3}{\mathrm{d}t^3}f(t) &= \lim_{h\to 0}\frac{\Delta^3 f(x)}{h^3} = \lim_{h\to 0}\frac{\Delta^2(f(x)-f(x-h))}{h^3}\\
&= \lim_{h\to 0}\frac{\Delta(f(x)-2f(x-h)+f(x-2h))}{h^3}\\
&= \lim_{h\to 0}\frac{f(x)-3f(x-h)+3f(x-2h)-f(x-3h)}{h^3}
\end{aligned}$$

And so on we can deduce the formula for the n^{th} derivative

$$\frac{\mathrm{d}^n}{\mathrm{d}t^n}f(t) = \lim_{h\to 0}\frac{\Delta^n f(x)}{h^n} = \lim_{h\to 0}\left[\frac{1}{h^n}\sum_{k=0}^{n}(-1)^k\binom{n}{k}f(t-kh)\right]$$

Where $\binom{n}{k} = \dfrac{n!}{k!\,(n-k)!}$ if we change the factorial to the gamma function we reach The Grünwald-Letnikov Fractional Derivative

$$\begin{aligned}
{}^{GL}D^{\alpha}f(t) &= \lim_{h\to 0}\frac{\Delta^{\alpha} f(x)}{h^{\alpha}}\\
&= \lim_{h\to 0}\left[\frac{1}{h^{\alpha}}\sum_{k=0}^{\infty}(-1)^k\binom{\alpha}{k}f(t-kh)\right]\\
&= \lim_{h\to 0}\left[\frac{1}{h^{\alpha}}\sum_{k=0}^{\infty}(-1)^k\frac{\Gamma(\alpha)}{k!\,\Gamma(\alpha-k)}f(t-kh)\right]
\end{aligned}$$

5.6 The Canavati Fractional Derivative

There is another definition of fractional derivatives that is useful in deriving inequalities. This is the Canavati fractional derivative. It is ”between” the Riemann-Liouville derivative and the Caputo derivative.

Let $m-1<\alpha<m$. Then, the Canavati derivative of order α is defined as

$${}^{Can}_{\;\;a}D^{\alpha}_{t}f(t) = \frac{1}{\Gamma(m-\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}\int_a^t (t-s)^{n-\alpha-1}\frac{\mathrm{d}^{n-1}}{\mathrm{d}s^{n-1}}f(s)ds$$

For $f(t)\in C^{\alpha}[a,b]$

$$C^{\alpha}[a,b] := \left\{f\in C^{m-1}[a,b] \quad | \quad {}_aI^{n-1}_t f(t)\in C^1[a,b]\right\}$$

5.7 The Marchaud Fractional Derivative

The Left Marchaud fractional derivative of the order $0<\alpha<1$ for $f(t)\in\mathcal{H}^{\lambda}[a,b]$, $\lambda>\alpha$ defined by

$${}^{M}_{a}D^{\alpha}_{t}f(t) = \frac{f(t)}{\Gamma(1-\alpha)(t-a)^{\alpha}} + \frac{\alpha}{\Gamma(1-\alpha)}\int_a^t \frac{f(t)-f(s)}{(t-s)^{1+\alpha}}\,ds$$

The Right Marchaud fractional derivative is defined as

$${}^{M}_{t}D^{\alpha}_{b}f(t) = \frac{f(t)}{\Gamma(1-\alpha)(b-t)^{\alpha}} + \frac{\alpha}{\Gamma(1-\alpha)}\int_t^b \frac{f(t)-f(s)}{(s-t)^{1+\alpha}}\,ds$$

5.8 The Riesz Fractional Integral And Derivative

The most prominent approach is the Riesz fractional integral and fractional derivative

It is a linear combination of both left and right fractional Liouville integrals (5.3) and (5.4)

$$
\begin{aligned}
{}^{RZ}I^{\alpha} f(t) &= \frac{{}^{L}I_{+}^{\alpha} + {}^{L}I_{-}^{\alpha}}{2\cos\left(\frac{\pi}{2}\alpha\right)} f(t) \\
&= \frac{1}{2\Gamma(\alpha)\cos\left(\frac{\pi}{2}\alpha\right)} \int_{-\infty}^{+\infty} |t-s|^{\alpha-1} f(s)\ ds
\end{aligned}
$$

This is fractional Riesz integral

In order to derive the explicit form of the Riesz fractional derivative we first present the left and right Liouville derivative (5.7) and (5.8) in an alternative form

$$
\begin{aligned}
{}^{L}D_{+}^{\alpha} f(t) &= \frac{\alpha}{\Gamma(1-\alpha)} \int_{0}^{\infty} \frac{f(t)-f(t-s)}{s^{\alpha+1}} ds \\
{}^{L}D_{-}^{\alpha} f(t) &= \frac{\alpha}{\Gamma(1-\alpha)} \int_{0}^{\infty} \frac{f(t)-f(t+s)}{s^{\alpha+1}} ds
\end{aligned}
$$

Which follows from (5.7)

$$
\begin{aligned}
{}^{L}D_{+}^{\alpha} f(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\, {}^{L}I_{+}^{1-\alpha} f(t) \\
&= \frac{1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}t} \int_{-\infty}^{t} (t-s)^{-\alpha} f(s)\ ds
\end{aligned}
$$

Substitute $\begin{cases} (t-s) = \xi \\ d\xi = -ds \\ \infty \to 0 \end{cases}$

$$
\begin{aligned}
&= \frac{1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}t} \int_{\infty}^{0} \xi^{-\alpha} f(t-\xi)\ (-d\xi) \\
&= \frac{1}{\Gamma(1-\alpha)} \int_{0}^{\infty} \xi^{-\alpha} \frac{\partial}{\partial t} f(t-\xi)\ d\xi \\
&= \frac{1}{\Gamma(1-\alpha)} \int_{0}^{\infty} \xi^{-\alpha} \left(-\frac{\partial}{\partial \xi} f(t-\xi)\right)\ d\xi \\
&= \frac{\alpha}{\Gamma(1-\alpha)} \left(\int_{0}^{\infty} \frac{f(t)}{\xi^{\alpha+1}}\ d\xi - \int_{0}^{\infty} \frac{f(t-\xi)}{\xi^{\alpha+1}}\ d\xi\right) \\
&= \frac{\alpha}{\Gamma(1-\alpha)} \int_{0}^{\infty} \frac{f(t)-f(t-\xi)}{\xi^{\alpha+1}}\ d\xi
\end{aligned}
$$

And similarly for (5.8)

Now applying the formula

$$
\Gamma(\alpha)\Gamma(1-\alpha) = \frac{\pi}{\sin(\pi\alpha)}
$$

We get

$$
\frac{\alpha}{\Gamma(1-\alpha)} = \Gamma(1+\alpha)\frac{\sin(\pi\alpha)}{\pi}
$$

With the definition of the fractional derivative according to Riesz

$$
{}^{RZ}D^{\alpha} f(t) = -\frac{{}^{L}D_{-}^{\alpha} + {}^{L}D_{+}^{\alpha}}{2\cos\left(\frac{\pi}{2}\alpha\right)} f(t)
$$

We explicitly obtain

$$
\begin{aligned}
{}^{RZ}D^{\alpha} f(t) &= \Gamma(1+\alpha)\frac{\sin(\pi\alpha)}{2\pi\cos\left(\frac{\pi}{2}\alpha\right)}\int_0^{\infty}\frac{f(t+s)-2f(t)+f(t-s)}{s^{\alpha+1}}\,ds \\
&= \Gamma(1+\alpha)\frac{\sin\left(\frac{\pi}{2}\alpha\right)}{\pi}\int_0^{\infty}\frac{f(t+s)-2f(t)+f(t-s)}{s^{\alpha+1}}\,ds
\end{aligned}
$$

5.9 The Feller Fractional Integral And Derivative

A possible generalization for the Riesz fractional derivative was proposed by Feller (1952). He suggested a general superposition of both fractional Liouville integrals (5.3) and (5.4)

$$
{}^{F}I^{\alpha}_{\theta} = c_{-}(\theta,\alpha)\,{}^{L}I^{\alpha}_{+} + c_{+}(\theta,\alpha)\,{}^{L}I^{\alpha}_{-}
$$

Introducing a free parameter $0 < \theta < 1$ which is a measure for the influence of both components

$$
\begin{aligned}
c_{-}(\theta,\alpha) &= \frac{\sin\left(\frac{\pi}{2}(\alpha-\theta)\right)}{\sin(\pi\theta)} \\
c_{+}(\theta,\alpha) &= \frac{\sin\left(\frac{\pi}{2}(\alpha+\theta)\right)}{\sin(\pi\theta)}
\end{aligned}
$$

The fractional Feller derivative is then given as

$$
{}^{F}D^{\alpha}_{\theta} = -\left[c_{+}(\theta,\alpha)\,{}_{L}D^{\alpha}_{+} + c_{+}(\theta,\alpha)\,{}_{L}D^{\alpha}_{-}\right]
$$

For the special case $\theta = 0$ we obtain

$$
c_{+}(0,\alpha) = c_{-}(0,\alpha) = \frac{1}{2\cos\left(\alpha\frac{\pi}{2}\right)}
$$

Which exactly corresponds to the definition of the Riesz derivative.

5.10 Hadamard Fractional Integral And Derivative

The Hadamard fractional Integral of order α of a function $f(t) \in L_p[a,b]$ and $a \neq 0$ is defined by

$$
{}^{H}I^{\alpha}_{a} f(t) = \frac{1}{\Gamma(\alpha)}\int_a^t \left(\ln\frac{t}{s}\right)^{\alpha-1}\frac{f(s)}{s}\,ds
$$

Let $m-1 < \alpha < m$ the Hadamard fractional derivative of order α of a function $f(t) \in AC^{m}[a,b]$ is defined by

$$
{}^{H}D^{\alpha}_{a} f(t) = \frac{1}{\Gamma(m-\alpha)}\frac{\mathrm{d}^m}{\mathrm{d}t^m}\int_a^t \left(\ln\frac{t}{s}\right)^{n-\alpha-1}\frac{f(s)}{s}\,ds
$$

And the Hadamard-Caputo fractional derivative of order α is defined by

$$
{}^{HC}D^{\alpha}_{a} f(t) = \frac{1}{\Gamma(m-\alpha)}\int_a^t \left(\ln\frac{t}{s}\right)^{n-\alpha-1}\frac{f^{(m)}(s)}{s}\,ds
$$

5.11 Generalized Fractional Integration And Differentiation

When we used the repeated integration formula we used Riemann definition of the integration but if we use Riemann-Stieltjes Integral

$$\mathcal{J}f(t) = \int_0^t f(s)\ dg(s)$$

We get

$$\begin{aligned}
\mathcal{J}f(t) &= \int_0^t f(s)\ dg(s) \\
\mathcal{J}^2 f(t) &= \int_0^t \int_0^s f(\theta)\ dg(\theta)\ dg(s)
\end{aligned}$$

We can interchanging the order of integration using Fubini's theorem

$$\begin{aligned}
&= \int_0^t \int_\theta^t\ dg(s)\ f(\theta)\ dg(\theta) \\
&= \int_0^t (g(t) - g(\theta))f(\theta)\ dg(\theta) = \int_0^t (g(t) - g(s))f(s)\ dg(s) \\
\mathcal{J}^3 f(t) &= \int_0^t \int_0^s (g(s) - g(\theta))f(\theta)\ dg(\theta)\ dg(s) \\
&= \int_0^t \int_\theta^s (g(s) - g(\theta))\ dg(s)\ f(\theta)\ dg(\theta) \\
&= \int_0^t \frac{(g(t) - g(\theta))^2}{2} f(\theta)\ dg(\theta)
\end{aligned}$$

And so on we get that

$$\mathcal{J}^n f(t) = \int_0^t \frac{(g(t) - g(s))^{n-1}}{(n-1)!} f(s)\ dg(s)$$

Now change the factorial to the Gamma function and the order n to an arbitrary order α we get the Generalized Fractional integral

$$\mathcal{J}^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (g(t) - g(s))^{\alpha-1} f(s)\ dg(s)$$

Or

$$\mathcal{J}^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (g(t) - g(s))^{\alpha-1} g'(s) f(s)\ ds$$

In case of RL Fractional integral $g(t)$ is chosen to be t and in hadamard is chosen to be $ln(t)$ and using this definition we can deduce the Generalized Fractional Derivative of Riemann sense

$$\mathfrak{D}^\alpha f(t) = \frac{\mathrm{d}^m}{\mathrm{d}t^m} \mathcal{J}^{m-\alpha} f(t) = \frac{1}{\Gamma(m-\alpha)} \frac{\mathrm{d}^m}{\mathrm{d}t^m} \int_0^t (g(t) - g(s))^{m-\alpha-1} f(s)\ dg(s)$$

And Caputo sense

$${}^{C}\mathfrak{D}^\alpha f(t) = \mathcal{J}^{m-\alpha} \frac{\mathrm{d}^m}{\mathrm{d}t^m} f(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (g(t) - g(s))^{m-\alpha-1} \frac{\mathrm{d}^m f(s)}{\mathrm{d}s^m}\ dg(s)$$

5.12 Kolwankar And Gangal (1994)

Local fractional calculus is a new branch of mathematics (is also called Fractal calculus) was first introduced by Kolwankar and Gangal. It deals with derivatives and integrals of the functions defined on fractal sets. And it is explain the behavior of continuous but nowhere differentiable function.

They proposed the following definition for the local fractional derivative of a function defined on fractal sets

$$ {}^{KG}_{x_0}D^{\alpha}_{x} f(x) = \lim_{x \to x_0} \frac{\mathrm{d}^{\alpha}(f(x) - f(y))}{\mathrm{d}(x-y)^{\alpha}} \quad , \quad 0 < \alpha < 1 $$

And the local fractional integrals of a function defined on fractal sets is defined as

$$ {}^{KG}_{a}I^{\alpha}_{b} f(x) = \lim_{N \to \infty} \sum_{i=0}^{N-1} f(x_i^*) \frac{d^{-\alpha} 1_{dx_i(x)}}{d(x_{i+1} - x_i)} $$

Where $1_{dx_i(x)}$ is the unit function defined upon $[x_i, x_{i+1}]$ and x_i^* is a point in the interval $[x_i, x_{i+1}]$ and the intervals $[x_i, x_{i+1}]$, $i = 0, 1, 2, \dots, N-1$ are partitions of the interval $[a, b]$

5.13 The Gohar Fractional Integral And Derivative

And last but not least The Gohar definition a new local fractional derivative introduced by Abdelrahman Gohar ,Mayada Younes and Salah B.Doma in 2023 which generalizes the classical limit definition of the derivative.

Given a function $f : [0, \infty) \to \mathbb{R}$, the GFD of f of order α, denoted by G_α is defined by

$$ G_\alpha f(t) = \lim_{h \to 0} \frac{1}{h} \left[f\left(t \left\{ 1 + \ln\left(1 + \frac{h\Gamma(\eta)}{\Gamma(\eta - \alpha + 1)} t^{-\alpha} \right) \right\} \right) - f(t) \right] $$

For $t > 0$, $\alpha \in (0, 1)$, $\eta \in \mathbb{R}^+$

And for $t \geq 0$ if f is a function defined on $(0, t]$, then the GFI of f, of order α, is defined by

$$ \mathfrak{T}^{\alpha} f(t) = \frac{\Gamma(\eta - \alpha + 1)}{\Gamma(\eta)} \int_0^t \frac{f(s)}{s^{1-\alpha}} ds $$

And there is a lot of others different formulations for fractional derivatives like

- Chen Derivative
- Cossar Derivative
- Davidson-Essex Derivative
- Hilfer Derivative
- Jumarie Derivative
- Osler Derivative
- Weyl Derivative
- Yang Derivative

And there's no definitive version because each one of them was trying to preserve some of the ordinary derivative properties but there is no such Definition for the fractional derivative that preserve all the ordinary derivative properties

6 Fractional Differential Equations (FDEs)

Differential equations involving fractional differential operators have recently proved to be valuable tools in the modeling of many physical phenomena.

We will focus our analysis in this section on FDEs of the Caputo type of the form

$$ {}^{C}D^{\alpha} y(t) = f(t, y(t)) \tag{6.1} $$

And some parts will deal with a slightly more general class of problems then we will look into some of the methods to solve linear and nonlinear FDEs

The reason why we will use the Caputo derivative that RL fractional derivative D^{α} in order to obtain a particular solution to the straightforward form of a FDE

$$ D^{\alpha} y(t) = f(t, y(t)) $$

We need to specify m initial conditions corresponding to it and it must be of the form

$$ \left[D^{\alpha-k} y(t)\right]_{t=0} = \beta_k \quad , \quad k = 1, 2, \ldots, m $$

With given values β_k. Thus we are forced to specify some fractional derivatives of the function y. In practical applications, these values are frequently not available, and it may not even be clear what their physical meaning is

Therefore Caputo has suggested that one should incorporate the classical derivatives (of integer order) of the function y, as they are commonly used in initial value problems with integer order equations, into the fractional order equation, using the relation between RL and Caputo derivatives

$$ {}^{C}D^{\alpha} y(t) = D^{\alpha}(y - T_{m-1}[y]) $$

Where $T_{m-1}[y]$ is the Taylor polynomial of order $m-1$ for y, centered at 0. Then, one can specify the initial conditions in the classical form

$$ y^{(k)}(0) = \beta_k \quad , \quad k = 0, 1, 2, \ldots, m-1 $$

In the classical theory of integer order ordinary differential equations, it is well known that unique solutions can only be expected if the differential equation is accompanied by certain additional conditions. The same observation is true in the fractional case. The question is then where on the $t - axis$ such condition(s) should be imposed.

By choosing the differential operator starting point, that is, the point 0 we have answered this question. Interpreting the free variable t as a time variable, this amounts to providing information at the beginning of the process that the differential equation describes and to seeking the process behavior for times that are in the future of this instant.

6.1 Initial Value Problems For Single Term Equations

We will study IVPs in the form (6.1) Since exactly one differential operator occurs in this equation, this type of equations is known as a single term FDE.

Thus the IVP will be in the form

$$ \begin{cases} {}^{C}D^{\alpha} y(t) = f(t, y(t)) \\ I.C \Longrightarrow y^{(k)}(0) = \beta_k \quad , \quad k = 0, 1, 2, \ldots, m-1 \end{cases} \tag{6.2} $$

We can Consider the classical theory as a special case. Many (but not all) classical results (and their proofs) can be generalized to this fractional setting

6.1.1 Existence And Uniqueness Of Solutions

The most important results in the classical theory, Peano's existence theorem and the Picard-Lindelöf uniqueness theorem, remain valid in the fractional setting too but with some more conditions. Their proofs are based on an equivalence between the IVP and a Volterra integral equation.

Consider the IVP (6.2) let f be a Continuous function if we apply I^α for both sides we get

$$^C D^\alpha y(t) = f(t, y(t))$$
$$y(t) - \sum_{k=0}^{m-1} \frac{t^k}{k!} y^{(k)}(0) = I^\alpha f(t, y(t))$$

Now we will state and proof the Theorem that makes function $y(t) \in C[0,T]$ a solution of this nonlinear Volterra integral equation of the second kind

$$y(t) = \sum_{k=0}^{m-1} \frac{t^k}{k!} y^{(k)}(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y(s))\, ds \tag{6.3}$$

And then we will state the conditions that makes the equation (6.3) equivalent to the IVP (6.2)

BUT first let's build some definitions that we will use in our proof

Definition 6.1 — Complete Space. The metric space X is said to be Complete if every Cauchy sequence in X converges to a limit point in X

Definition 6.2 — Banach Space. A Banach Space is a complete normed space
Examples : $\mathbb{R}^n, \mathbb{C}^n, C[a,b]$ are Banach

Lemma 6.1 A closed subspace of a Banach space is a Banach space

Proof. Let X be a Banach Space then it is Complete

$\therefore$ Every Cauchy sequence in X converges to a limit point in X

$\therefore$ X is Closed and bounded

Now let E be Closed subspace of X

$\therefore$ It is bounded with the same boundaries of X

$\therefore$ E is closed and bounded $\implies$ it contains all it's limit point

$\therefore$ Every Cauchy sequence in E converges to a limit point in E then E is Complete normed space then it is Banach

■

Definition 6.3 — Continuous Operator. An operator T is said to be continuous at a point x_0 if

$$\forall \epsilon > 0, \exists \delta > 0 \text{ such that } ||x - x_0|| \implies ||Tx - Tx_0|| < \epsilon$$

T is continuous on X if T is continuous at every point, $x \in X$

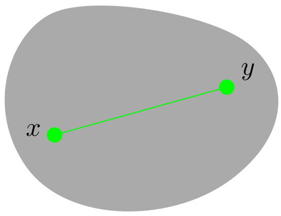
Let $T : X \to Y$ be a linear operator then

1. T is continuous if and only if , T is bounded
2. If T is continuous at a point then it's continuous

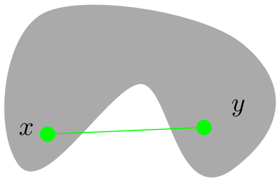
Definition 6.4 — Convex Set. A subset A of X is said to be Convex if for $x, y \in A$ we have

$$M := \{z \in X : z = \alpha x + (1-\alpha)y \quad , \quad 0 \leq \alpha \leq 1\} \subset A$$

M is called a closed segment with boundary points x, y



Convex Set



Not Convex Set

Lemma 6.2 A ball in a normed Space is Convex

Proof. Let x, y be two points in a ball $B(a, r) \implies ||x-a|| \leq r, ||y-a|| < r$
Now let

$$\begin{aligned} z &= \alpha x + (1-\alpha)y \quad , \quad 0 \leq \alpha \leq 1 \\ z - a &= \alpha x + (1-\alpha)y - a \\ &= \alpha x - \alpha a + (1-\alpha)y - a + \alpha a \\ &= \alpha(x-a) + (1-\alpha)y - (1-\alpha)a \\ &= \alpha(x-a) + (1-\alpha)(y-a) \\ &\leq \alpha r + (1-\alpha)r \\ &\leq r \end{aligned}$$

Thus $z \in B(a, r)$

$$\therefore \{z \in X : z = \alpha x + (1-\alpha)y \quad , \quad 0 \leq \alpha \leq 1\} \subset B(a,r) \implies B(a,r) \text{ is Convex}$$

■

Definition 6.5 — Fixed Point. The fixed point of a mapping of a set X into itself $T : X \to X$ is an element $x \in X$, which is mapped by T onto itself, that is $Tx = x$.

Examples let $T : \mathbb{R} \to \mathbb{R}$

- Consider the mapping $Tx = x^2$ it's fixed points are $\{0, 1\}$
- Consider the mapping $Tx = x$ it's fixed points are the whole $\mathbb{R}$
- Consider the mapping $Tx = x + 1$ it has no fixed point

Theorem 6.3 — Schauder Fixed Point Theorem.
Let Q be a Convex and Closed subset of a Banach space then a continuous and Compact Operator

$$T : Q \to Q$$

Has at least one fixed point

Now we have all the properties and definitions that we need now consider the following existence Theorem

Theorem 6.4 — **Peano's Existence Theorem.**
Let $0 \leq m-1 < \alpha < m$ and let $y(0), y^{(1)}(0), \ldots, y^{(m-1)}(0) \in \mathbb{R}^d$, $\eta > 0, K > 0$
Consider the domain $D := \left\{ (t, y) : t \in [0, \eta] \quad , \quad \left\| y - \sum_{k=0}^{m-1} \frac{t^k}{k!} y^{(k)}(0) \right\| \leq K \right\}$
And that $f := D \to \mathbb{R}^d$ is continuous and bounded, with $M := \sup_{(t,y) \in D} |f(t, y)|$. Moreover , let

$$\Omega := \begin{cases} \eta & \textit{if M=0} \\ \min \left\{ \eta, \left(\frac{K\Gamma(\alpha+1)}{M} \right)^{\frac{1}{\alpha}} \right\} & \text{else} \end{cases} \tag{6.4}$$

Then there exists a function $y(t) \in C[0, \Omega]$ solves the nonlinear Volterra integral equation (6.3).

Proof. Let

$$\begin{aligned} y(t) &= \sum_{k=0}^{m-1} \frac{t^k}{k!} y^{(k)}(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y(s)) \, ds \\ &= g(t) + I^\alpha f(t, y(t)) \end{aligned} \tag{6.5}$$

And let the set

$$U := \{ y \in C[0, \eta] \quad , \quad ||y - g|| \leq K \}$$

U is a closed since the less than or "equal" condition and convex since it is a ball centered at g subset of the Banach space of all continuous functions on $[0, \eta]$ Hence, U is a Banach space too Since the polynomial $g(t)$ is an element of U, we also see that U is not empty.

On this set U we define the operator T by

$$Ty = g(t) + I^\alpha f(t, y(t))$$

Using this operator, the equation (6.5) whose solvability we need to prove can be rewritten as

$$Ty = y$$

And thus, in order to prove our desired existence result, we have to show that T has a fixed point.

We therefore proceed by investigating the properties of the operator T more closely.

Our first goal in this context is to show that $Ty \in U$ for $y \in U$.
It's clear that Ty is Continuous

$$\begin{aligned} Ty &= g(t) + I^\alpha f(t, y(t)) \\ &= polynomial + I^\alpha(Continuous) \\ &= polynomial + Continuous = Continuous \end{aligned}$$

And for Ty to be element in U

$$\begin{aligned} |Ty - g| &= \frac{1}{\Gamma(\alpha)} \left| \int_0^t (t-s)^{\alpha-1} f(s, y(s)) \, ds \right| \\ &\leq \frac{1}{\Gamma(\alpha)} \int_0^t \left| (t-s)^{\alpha-1} \right| |f(s, y(s))| \, ds \\ &\leq \frac{M}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \, ds \\ &\leq \frac{M}{\Gamma(\alpha)} \frac{t^\alpha}{\alpha} = \frac{M t^\alpha}{\Gamma(\alpha+1)} \end{aligned}$$

We want this to be less than K therefore

$$\frac{Mt^\alpha}{\Gamma(\alpha+1)} \leq K$$

$$t \leq \left(\frac{K\Gamma(\alpha+1)}{M}\right)^{\frac{1}{\alpha}}$$

Therefore t must not grow more than $\left(\frac{K\Gamma(\alpha+1)}{M}\right)^{\frac{1}{\alpha}}$ and in the same time t is limited by η therefore we can put the condition

$$t \in [0, \Omega] \quad , \quad \Omega = \min\left\{\eta, \left(\frac{K\Gamma(\alpha+1)}{M}\right)^{\frac{1}{\alpha}}\right\}$$

Thus, we have shown that $Ty \in U$ if $y \in U[0, \Omega]$ i.e T maps the set U to itself.

Since we want to apply Schauder's Fixed Point Theorem, all that remains now is to show that $T(U) := \{Ty : y \in U\}$ is a relatively compact set. This can be done by means of the Arzela Ascoli Theorem . For $y \in U$ we find that, for all $t \in [0, \Omega]$

$$\begin{aligned}
|Ty| &= \left|g(t) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s, y(s))\ ds\right| \\
&\leq ||g(t)||_\infty + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|f(s, y(s))|\ ds \\
&\leq ||g(t)||_\infty + \frac{M}{\Gamma(\alpha)}\frac{t^\alpha}{\alpha} \\
&\leq ||g(t)||_\infty + \frac{M\Omega^\alpha}{\Gamma(\alpha+1)} \\
&\leq ||g(t)||_\infty + K
\end{aligned}$$

Chebyshev Norm

the supremum norm, the Chebyshev norm, the infinity norm on a set S is defined as

$$||f(t)||_\infty = \sup_{t\in S}|f(t)|$$

Which is the required boundedness property.

Moreover, the equicontinuity property let $0 \leq t_1 \leq t_2 \leq \Omega$

$$\begin{aligned}
|Ty(t_2) - Ty(t_1)| &= |g(t_2) - g(t_1) + I^\alpha f(t_2, y(t_2)) - I^\alpha f(t_1, y(t_1))| \\
&\leq |g(t_2) - g(t_1)| + I^\alpha\ |f(t_2, y(t_2)) - f(t_1, y(t_1))|
\end{aligned}$$

Thus if $|t_2 - t_1| < \delta$

$$\begin{aligned}
&\leq \epsilon_1 + I^\alpha \epsilon_2 \\
&\leq \epsilon_1 + \epsilon_2 \frac{t^\alpha}{\Gamma(\alpha+1)} \\
&\leq \epsilon_1 + \epsilon_2 \frac{\Omega^\alpha}{\Gamma(\alpha+1)} \\
&\leq \epsilon_1 + \epsilon_2 \frac{K}{M} < \epsilon
\end{aligned}$$

Therefore the set $T(U)$ is equicontinuous.
Thus Arzela Ascoli Theorem yields that $T(U)$ is relatively compact, and hence Schauder's Fixed Point Theorem asserts that T has a fixed point. ■

We reached to that there exists a Continuous solution $y(t)$ that solves the Volterra integral equation (6.3) to make it solve the IVP (6.2) we must prove the equivalence between them

$$y(t) = g(t) + I^\alpha f(t, y(t))$$

Applying Caputo Derivative for both sides we get

$$\begin{aligned}
{}^C D^\alpha y(t) &= {}^C D^\alpha g(t) + {}^C D^\alpha I^\alpha f(t, y(t)) \\
&= {}^C D^\alpha I^\alpha f(t, y(t)) \\
&= I^{m-\alpha} \frac{\mathrm{d}^m}{\mathrm{d}t^m} I^\alpha f(t, y(t)) \\
&= I^{m-\alpha} \frac{\mathrm{d}^m}{\mathrm{d}t^m} I^{\alpha+m-m} f(t, y(t)) \\
&= I^{m-\alpha} D^{m-\alpha} f(t, y(t)) = f(t, y(t))
\end{aligned}$$

A condition must be taking into account that the right hand side must be Caputo differentiable thus we must put the condition that $I^\alpha f(t, y(t)) \in AC^m[0, \Omega]$ and this yields that

$$y(t) = g(t) + I^\alpha f(t, y(t)) = polynomial + AC^m \in AC^m$$

This condition remove the counter Example that have been used in [10] that is by taking

$$f(t, y(t)) = D^\alpha \mathcal{W}(t)$$

Where $\mathcal{W}(t)$ is Weierstrass function because $D^\alpha \mathcal{W}(t)$ is continuous we can say that

$$\begin{aligned}
y(t) &= g(t) + I^\alpha f(t, y(t)) \\
&= g(t) + I^\alpha D^\alpha \mathcal{W}(t) \\
&= g(t) + \mathcal{W}(t)
\end{aligned}$$

If we apply Caputo Derivative the Derivative of $\mathcal{W}(t)$ the expression will be "meaningless"

Thus $f(t, y(t))$ being continuous is not enough to get the equivalence of the IVP and Volterra IE

Schauder fixed point Theorem only guarantee the existence to get the uniqueness consider the following

The classical Picard-Lindelöf theorem can be generalized to the fractional setting in the same way: If the given function f is continuous and bounded and satisfies a Lipschitz condition with respect to the second variable, then uniqueness of the continuous solution to the integral equation (6.3) can be guaranteed.

Theorem 6.5 — **Picard-Lindelöf Uniqueness Theorem.** Assume the hypotheses of Theorem (6.4). Moreover, let f fulfill a Lipschitz condition with respect to the second variable, that is,

$$||f(t, y_1) - f(t, y_2)|| \leq L\, ||y_1 - y_2||$$

Where L is Lipschitz constant Then there exists a uniquely defined function $y(t) \in C[0, \Omega]$ solves the integral equation (6.3)

To Proof the uniqueness we are going to use the successive approximation method

$$y_n(t) = g(t) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y_{n-1}(s))\, ds$$

In the case $n = 0$ it is obvious.

$$y_0(t) = g(t)$$

If $n = 1$, then we have:

$$\begin{aligned}
|y_1(t) - y_0(t)| &= \left| \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y_0(s)) \; ds \right| \\
&\leq \frac{1}{\Gamma(\alpha)} \int_0^t \left|(t-s)^{\alpha-1}\right| |f(s, y_0(s))| \; ds \\
&\leq \frac{M}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \; ds \\
&\leq \frac{M}{\Gamma(\alpha)} \frac{t^\alpha}{\alpha} \leq \frac{M\Omega^\alpha}{\Gamma(\alpha+1)} < K \\
|y_2(t) - y_1(t)| &= \left| \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left[f(s, y_1(s)) - f(s, y_0(s))\right] \; ds \right| \\
&\leq \frac{L}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |y_1 - y_0| \; ds \\
&\leq \frac{LK}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \; ds \\
&< \frac{LK^2}{M} \\
|y_3(t) - y_2(t)| &= \left| \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left[f(s, y_2(s)) - f(s, y_1(s))\right] \; ds \right| \\
&\leq \frac{L}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |y_2 - y_1| \; ds \\
&\leq \frac{L^2K^2}{M\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \; ds \\
&< \frac{L^2K^3}{M^2}
\end{aligned}$$

And so on we get

$$|y_{n+1}(t) - y_n(t)| \leq K \left(\frac{LK}{M}\right)^n$$

Now summing over n to get the solutions

$$y_0(t) + \sum_{k=0}^{n} y_{k+1}(t) - y_k(t) = y_0(t) + y_{n+1}(t) - y_n(t) + y_n(t) - \cdots - y_0(t) = y_{n+1}(t)$$

Take the limit as $n \to \infty$ we get

$$\begin{aligned}
\lim_{n\to\infty} y_{n+1}(t) &= y_0(t) + \lim_{n\to\infty} \sum_{k=0}^{n} y_{k+1}(t) - y_k(t) \\
y(t) &= y_0(t) + \sum_{k=0}^{\infty} y_{k+1}(t) - y_k(t)
\end{aligned}$$

Using Weierstrass Test

$$\sum_{k=0}^{\infty} |y_{k+1}(t) - y_k(t)| \leq \eta \sum_{k=0}^{\infty} \left(\frac{LK}{M}\right)^k$$

The series is convergent if $\dfrac{LK}{M} < 1$

Thus the sequence $\{y_n(t)\}$ is uniform convergent to a function $y(t)$ for $t \in [0, \Omega]$.

The Weierstrass Test

Suppose that $\{f_n(t)\}$ is a sequence of real functions defined on a set A, and there is a sequence of positive numbers $\{R_n\}$ satisfying:

$$\forall n > 0, \forall t \in A \quad , \quad |f_n(t)| < R_n \quad , \quad \sum_{n=0}^{\infty} R_n < \infty$$

Then the series

$$\sum_{n=0}^{\infty} f_n(t)$$

Is convergent

Proof Theorem (6.5). Let there is two solutions y_1, y_2 satisfy the equation

$$y(t) = g(t) + I^{\alpha} f(t, y(t))$$

Therefore

$$\begin{aligned}
y_1(t) - y_2(t) &= g(t) - g(t) + I^{\alpha} f(t, y_1(t)) - I^{\alpha} f(t, y_2(t)) \\
||y_1(t) - y_2(t)|| &\leq I^{\alpha} ||f(t, y_1(t)) - f(t, y_2(t))|| \\
&\leq L I^{\alpha} ||y_1(t) - y_2(t)|| \\
&\leq ||y_1(t) - y_2(t)|| L \frac{t^{\alpha}}{\Gamma(\alpha+1)} \\
&\leq ||y_1(t) - y_2(t)|| \frac{LK}{M}
\end{aligned}$$

$$||y_1(t) - y_2(t)||(1 - \frac{LK}{M}) \leq 0$$

Because $\frac{LK}{M} < 1$

$$\begin{aligned}
||y_1(t) - y_2(t)|| &\leq 0 \\
||y_1(t) - y_2(t)|| &= 0 \\
y_1(t) &= y_2(t)
\end{aligned}$$

■

6.1.2 Stability

In many important situations, the solutions to the differential equation exist on the unbounded interval $[0, \infty)$. In such a case, it is often required to investigate the behavior of solutions as $t \to \infty$

The Solutions of a problem are called Stable if they depends on the given data in a continuous way

In integer differential equation the given data was only referring to the initial conditions

Consider

$$\frac{dy(t)}{dt} = f(t, y(t)) \qquad t > 0$$

Let $y(t), y^*(t)$ be solutions to the equation with different initial values $y(0) = a, y^*(0) = b$

We say that the solutions of equation are **stable** if and only if

$$\forall \epsilon > 0 \ , \ \exists \delta > 0 \text{ s.t } |a - b| < \delta \implies |y(t) - y^*(t)| \leq \epsilon$$

Moreover we say that the solutions of equation are **asymptotically stable** if and only if they satisfies the previous Conditions and $\lim\limits_{t \to \infty} (y(t) - y^*(t)) = 0$

Furthermore we say that the solutions of equation are **unstable** if there exists a unique initial value such that the solution of the differential equation subject to that initial condition is bounded and the solutions to the differential equation subject to other initial values are unbounded

One important difference between the fractional and the classical setting is the meaning of the expression "the given data".

In the classical theory, we usually assumes that the initial values and the function f to be given and then the behavior of the solution under perturbations of these expressions is discussed.

In the fractional setting, however, it is additionally possible to perturb the order α of the differential equation, and so this new feature must be taken into account as well.

Well-Posedness

A problem is called well-posed if it has the following three properties:

1. A solution exists
2. The solution is unique
3. The solution depends on the given data in a continuous way (stable)

6.1.3 Separation Of Solutions

Another result from the theory of first order differential equations states that the graphs of two solutions of the same differential equation that satisfy different initial conditions can never meet or cross each other if the given function f satisfies a Lipschitz condition. This statement is indeed valid only for first-order problems, that is, for problems with exactly one initial condition. Thus a similar result for FDE can only be shown for IVPs with exactly one initial condition, that is, for problems with an order $\alpha \in (0, 1)$

It can be explained by the visualization indicated in Figure 1.

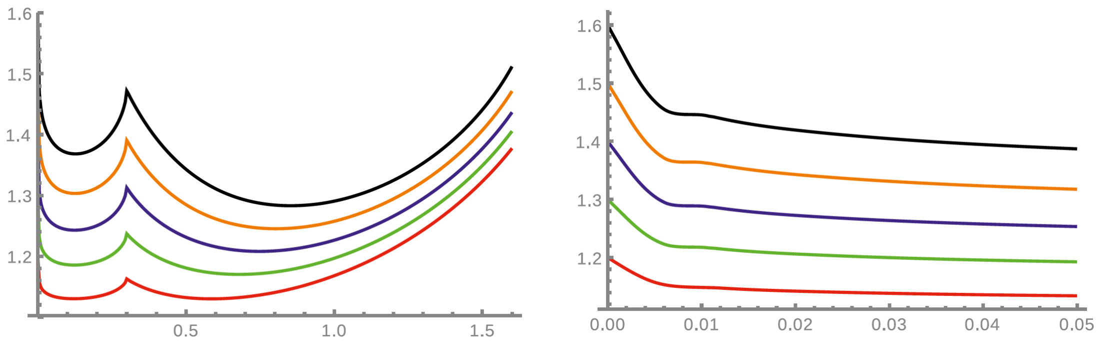


Figure 1: Graphs of solutions to the differential equation ${}^{C}D^{0.28}y(t) = \sqrt{|0.3-t|}\,\sin(3y(t)) + 0.3t^3$ with initial conditions $y(0) = 1.2$ (red), $y(0) = 1.3$ (green), $y(0) = 1.4$ (blue), $y(0) = 1.5$ (orange) and $y(0) = 1.6$ (black), plotted over the interval $[0, 1.6]$ (left) and zoom of this picture to the interval $[0, 0.05]$ (right). It can be observed that the graphs never meet or cross each other.

6.2 Linear Fractional Differential Equations (LFDE)

A linear FDE is an equation of form

$$\left({}^{C}D^{\alpha_m} + a_{m-1}(t)\,{}^{C}D^{\alpha_{m-1}} + \cdots + a_1(t)\,{}^{C}D^{\alpha_1} + a_1(t)\right) y(t) = f(t)$$

With the conditions:

$$y^{(k)}(0) = \beta_k \quad , \quad k = 0, 1, 2, \ldots, m-1$$

Theorem 6.6 — Existence And Uniqueness Of LFDE. If $f(t)$ is bounded on $(0, T)$ and $a_k(t)$, $k = 0, 1, \ldots, m-1$ are continuous functions on $[0, T]$, the equation has a unique solution.

In the particular case where the given differential equation is linear, it is often possible to write up the solutions in closed form.

■ **Example 6.2.1** Show that

$$y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\; ds$$

Solves the IVP

$$\begin{cases} {}^{C}D^{\alpha}y(t) = f(t) & , \quad m-1 < \alpha < m \\ I.C \Longrightarrow y^{(k)}(0) = 0 & , \quad k = 0, 1, 2, \ldots, m-1 \end{cases}$$

Sol. We apply the Laplace transform

$$\begin{aligned} \mathcal{L}\left[{}^{C}D^{\alpha}y(t)\right] &= \mathcal{L}[f(t)] \\ s^{\alpha}Y(s) &= F(s) \\ Y(s) &= s^{-\alpha}F(s) \\ \mathcal{L}[y(t)] &= \mathcal{L}\left[I^{\alpha}f(t)\right] \\ y(t) &= I^{\alpha}f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\; ds \end{aligned}$$

■

Laplace of Mittag Leffler Function

$$\mathcal{L}\left[t^{\beta-1}E_{\alpha,\beta}(\lambda t^{\alpha})\right] = \frac{s^{\alpha-\beta}}{s^{\alpha} - \lambda}$$

■ **Example 6.2.2** Show that

$$y(t) = \sum_{k=0}^{m-1} \beta_k t^k E_{\alpha,k+1}(\lambda t^\alpha)$$

Solves the IVP

$$\begin{cases} {}^C D^\alpha y(t) = \lambda y(t) & , \quad m-1 < \alpha < m \\ I.C \Longrightarrow y^{(k)}(0) = \beta_k & , \quad k = 0, 1, 2, \ldots, m-1 \end{cases}$$

Sol. We apply the Laplace transform

$$\mathcal{L}\left[{}^C D^\alpha y(t)\right] = \lambda \mathcal{L}[y(t)]$$

$$s^\alpha Y(s) - \sum_{k=0}^{m-1} s^{\alpha-k-1} y^{(k)}(0) = \lambda Y(s)$$

$$Y(s) = \sum_{k=0}^{m-1} \frac{s^{\alpha-k-1}}{s^\alpha - \lambda} \beta_k$$

$$\mathcal{L}[y(t)] = \sum_{k=0}^{m-1} \mathcal{L}\left[\beta_k t^k E_{\alpha,k+1}(\lambda t^\alpha)\right]$$

$$\mathcal{L}[y(t)] = \mathcal{L}\left[\sum_{k=0}^{m-1} \beta_k t^k E_{\alpha,k+1}(\lambda t^\alpha)\right]$$

$$y(t) = \sum_{k=0}^{m-1} \beta_k t^k E_{\alpha,k+1}(\lambda t^\alpha)$$

■

In example (6.1.4) if we take special case when $m - 1 = 0$ and for $\lambda = -1$.

$$y(t) = \beta_0 E_{\alpha,1}(-t^\alpha)$$

The cases $\alpha = 1$ and $\alpha = 2$ reduce to the well known statements that are the exponential function $E_{1,1}(-t) = e^{-t}$ and the cosine $E_{2,1}(-t^2) = cos(t)$ solve the given first and second order IVPs.

This observation indicates that the solutions to the general problem of order α decay in a monotonic way for $\alpha = 1$ and exhibit persistent oscillations for $\alpha = 2$ if λ is a negative real number. And the behavior of the solution in the cases $1 < \alpha < 2$ and $0 < \alpha < 1$. The associated results are illustrated in Figure 2

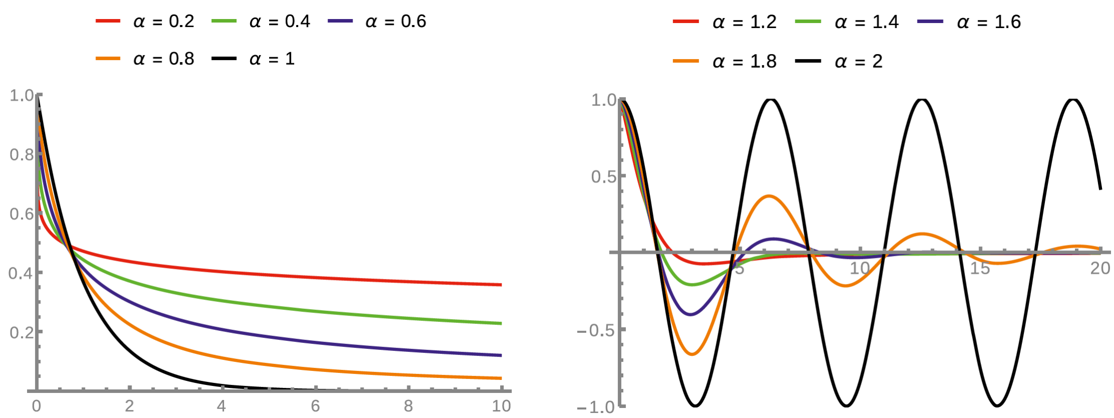


Figure 2: Plots of $y(t) = E_\alpha(-t^\alpha)$ for various $\alpha \in (0, 1]$ (left) and $\alpha \in (1, 2]$ (right).

■ **Example 6.2.3** Show that

$$y(t) = \beta + t^{\alpha}\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{\Gamma(n+\alpha+1)}t^{n}$$

Solves the IVP

$$\begin{cases} {}^{C}D^{\alpha}y(t) = f(t) \quad , \quad 0<\alpha<1 \\ I.C \Longrightarrow y(0) = \beta \end{cases}$$

Sol. We expand $f(t)$ using Taylor expansion and apply the Laplace transform

$$\begin{aligned}
\mathcal{L}[{}^{C}D^{\alpha}y(t)] &= \mathcal{L}[f(t)] \\
s^{\alpha}Y(s) - \beta s^{\alpha-1} &= \mathcal{L}\left[\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{\Gamma(n+1)}t^{n}\right] \\
Y(s) &= \frac{\beta}{s} + \sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{\Gamma(n+1)}\frac{1}{s^{\alpha}}\mathcal{L}[t^{n}] \\
\mathcal{L}[y(t)] &= \mathcal{L}[\beta] + \sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{\Gamma(n+1)}\frac{1}{s^{\alpha}}\frac{\Gamma(n+1)}{s^{n+1}} \\
\mathcal{L}[y(t)] &= \mathcal{L}[\beta] + \sum_{n=0}^{\infty}f^{(n)}(0)\frac{1}{s^{n+\alpha+1}} \\
\mathcal{L}[y(t)] &= \mathcal{L}[\beta] + \sum_{n=0}^{\infty}f^{(n)}(0)\mathcal{L}\left[\frac{t^{n+\alpha}}{\Gamma(n+\alpha+1)}\right] \\
y(t) &= \beta + t^{\alpha}\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{\Gamma(n+\alpha+1)}t^{n}
\end{aligned}$$

■

■ **Example 6.2.4** Solve the in-homogeneous IVP

$$\begin{cases} {}^{C}D^{\alpha}y(t) = \lambda y(t) + Q(t) \quad , \quad m-1<\alpha<m \\ I.C \Longrightarrow y^{(k)}(0) = \beta_k \quad , \quad k = 0,1,2,\dots,m-1 \end{cases}$$

Where $Q \in C[0,h]$ is a given function

Sol. As we did previously we expand $Q(t)$ using Taylor expansion and apply the Laplace transform

$$\mathcal{L}\left[{}^{C}D^{\alpha}y(t)\right] = \lambda\mathcal{L}[y(t)] + \mathcal{L}\left[\sum_{n=0}^{\infty}\frac{Q^{(n)}(0)}{\Gamma(n+1)}t^{n}\right]$$

$$s^{\alpha}Y(s) - \sum_{k=0}^{m-1}s^{\alpha-k-1}y^{(k)}(0) = \lambda Y(s) + \sum_{n=0}^{\infty}Q^{(n)}(0)s^{-n-1}$$

$$\begin{aligned}
Y(s) &= \sum_{k=0}^{m-1}\frac{s^{\alpha-k-1}}{s^{\alpha}-\lambda}\beta_k + \sum_{n=0}^{\infty}Q^{(n)}(0)\frac{s^{-n-1}}{s^{\alpha}-\lambda} \\
\mathcal{L}[y(t)] &= \sum_{k=0}^{m-1}\mathcal{L}\left[\beta_k t^{k}E_{\alpha,k+1}(\lambda t^{\alpha})\right] + \sum_{n=0}^{\infty}Q^{(n)}(0)\frac{s^{\alpha-\alpha-n-1}}{s^{\alpha}-\lambda} \\
\mathcal{L}[y(t)] &= \mathcal{L}\left[\sum_{k=0}^{m-1}\beta_k t^{k}E_{\alpha,k+1}(\lambda t^{\alpha})\right] + \sum_{n=0}^{\infty}\mathcal{L}\left[Q^{(n)}(0)t^{\alpha+n}E_{\alpha,\alpha+n+1}(\lambda t^{\alpha})\right] \\
\mathcal{L}[y(t)] &= \mathcal{L}\left[\sum_{k=0}^{m-1}\beta_k t^{k}E_{\alpha,k+1}(\lambda t^{\alpha})\right] + \mathcal{L}\left[\sum_{n=0}^{\infty}Q^{(n)}(0)t^{\alpha+n}E_{\alpha,\alpha+n+1}(\lambda t^{\alpha})\right]
\end{aligned}$$

$$y(t)=\sum_{k=0}^{m-1}\beta_k t^k E_{\alpha,k+1}(\lambda t^\alpha)+\sum_{n=0}^{\infty}Q^{(n)}(0)t^{\alpha+n}E_{\alpha,\alpha+n+1}(\lambda t^\alpha) \tag{6.7}$$

Using Mittag Leffler function recursion relation $E_{\alpha,\alpha+\beta}(z)=\frac{1}{z}\left[E_{\alpha,\beta}(z)-\frac{1}{\Gamma(\beta)}\right]$

$$\begin{aligned}
&=\sum_{k=0}^{m-1}\beta_k t^k E_{\alpha,k+1}(\lambda t^\alpha)+\sum_{n=0}^{\infty}Q^{(n)}(0)t^{\alpha+n}\frac{1}{\lambda t^\alpha}\left[E_{\alpha,n+1}(z)-\frac{1}{\Gamma(n+1)}\right]\\
&=\sum_{k=0}^{m-1}\beta_k t^k E_{\alpha,k+1}(\lambda t^\alpha)+\frac{1}{\lambda}\sum_{n=0}^{\infty}Q^{(n)}(0)t^{n}E_{\alpha,n+1}(z)-\frac{Q(t)}{\lambda}
\end{aligned}$$

Another way using the superposition principle

The Superposition Principle

The solution of the in-homogeneous equation can be written as the sum of the solution of the homogeneous equation, and a particular solution of the in-homogeneous equation.

We can break the problem into the following two problems

$$\begin{cases}{}^{C}D^{\alpha}y_h(t)=\lambda y_h(t) & , \quad m-1<\alpha<m\\ I.C \Longrightarrow y_h^{(k)}(0)=\beta_k & , \quad k=0,1,2,\dots,m-1\end{cases}$$

$$\begin{cases}{}^{C}D^{\alpha}y_p(t)=\lambda y_p(t)+Q(t) & , \quad m-1<\alpha<m\\ I.C \Longrightarrow y_p^{(k)}(0)=0 & , \quad k=0,1,2,\dots,m-1\end{cases}$$

Where $y=y_h+y_p$ solves the original problem. From example (6.3.2) we got that

$$y_h(t)=\sum_{k=0}^{m-1}\beta_k t^k E_{\alpha,k+1}(\lambda t^\alpha)$$

Now for the second problem take Laplace Transform for it

$$\begin{aligned}
\mathcal{L}\left[{}^{C}D^{\alpha}y_p(t)\right]&=\lambda\mathcal{L}[y_p(t)]+\mathcal{L}\left[\sum_{n=0}^{\infty}\frac{Q^{(n)}(0)}{\Gamma(n+1)}t^n\right]\\
s^\alpha Y_p(s)&=\lambda Y_p(s)+\sum_{n=0}^{\infty}Q^{(n)}(0)s^{-n-1}\\
Y_p(s)&=\sum_{n=0}^{\infty}Q^{(n)}(0)\frac{s^{-n-1}}{s^\alpha-\lambda}=\sum_{n=0}^{\infty}Q^{(n)}(0)\frac{s^{\alpha-\alpha-n-1}}{s^\alpha-\lambda}\\
\mathcal{L}[y_p(t)]&=\sum_{n=0}^{\infty}\mathcal{L}\left[Q^{(n)}(0)t^{\alpha+n}E_{\alpha,\alpha+n+1}(\lambda t^\alpha)\right]\\
\mathcal{L}[y_p(t)]&=\mathcal{L}\left[\sum_{n=0}^{\infty}Q^{(n)}(0)t^{\alpha+n}E_{\alpha,\alpha+n+1}(\lambda t^\alpha)\right]\\
y_p(t)&=\sum_{n=0}^{\infty}Q^{(n)}(0)t^{\alpha+n}E_{\alpha,\alpha+n+1}(\lambda t^\alpha)
\end{aligned}$$

Adding y_h+y_p we get the result (6.7) ∎

Duhamel's Principle

If one can solve an IVP for a homogeneous linear differential equation then an in-homogeneous linear differential equation can be solved as well.

6.3 Nonlinear Equations

Now we will discuss One of the methods to solve Nonlinear FDE which is The Adomian Decomposition Method

The Adomian method applied to the ordinary and partial differential equations of integer order was extended also to the case of FDE.

6.3.1 The Adomian Decomposition Method (ADM)

Before we talk about how to solve FDE using ADM let us first see how it work

Consider the problem

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} + L(u(x,t)) + N(u(x,y)) = g(x,t) \\ I.C \Longrightarrow u(x,0) = \phi(x) \end{cases} \tag{6.8}$$

This is nonlinear partial differential equation of integer order where $L(u(x,t))$ is the linear part and $N(u(x,t))$ is the non-linear part We will discuss it's solution by ADM

Now, Integrate (6.8) from $0 \to t$

$$u(x,t) = \phi(x) - \int_0^t L(u(x,s))\ ds - \int_0^t N(u(x,s))\ ds + \int_0^t g(x,s)\ ds \tag{6.9}$$

Set

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,y) \tag{6.10}$$

$$N(u(x,t)) = \sum_{n=0}^{\infty} A_n(x,y) \tag{6.11}$$

Where $A_0, A_1, A_2, \dots$ are **Adomian Polynomials** defined as:

$$A_n(x,y) = \frac{1}{n!}\frac{\mathrm{d}^n}{\mathrm{d}\lambda^n}\left(N\left(\sum_{j=0}^{n} \lambda^j u_j\right)\right)$$

Substitute equations (6.10),(6.11) into (6.9) we get

$$u(x,y) = \sum_{n=0}^{\infty} u_n(x,t) = \phi(x) + \int_0^t g(x,s)\ ds - \int_0^t L\left(\sum_{n=0}^{\infty} u_n(z,t)\right) - \int_0^t \sum_{n=0}^{\infty} A_n(x,s)\ ds$$

Now,

$$\begin{aligned} u_0 &= \phi(x) + \int_0^t g(x,s)\ ds \\ u_1 &= -\int_0^t L(u_0)\ ds - \int_0^t A_0\ ds \\ u_2 &= -\int_0^t L(u_1)\ ds - \int_0^t A_1\ ds \\ &\ \vdots \\ u_n &= -\int_0^t L(u_{n-1})\ ds - \int_0^t A_{n-1}\ ds \end{aligned}$$

$$\therefore u(x,t) = \sum_{n=0}^{\infty} u_n(x,y) = u_0 + u_1 + u_2 + \dots$$

This will get the solution for the problem (6.8)

Note that The Adomian Decomposition Method (ADM) is a numerical technique used to approximate solutions of differential equations. Whether ADM converges or diverges depends on the specific problem and how it is applied. The convergence and divergence of ADM can be influenced by several factors, including the complexity of the problem, the choice of the decomposition functions, and the behavior of the nonlinear terms in the differential equation.

■ **Example 6.3.1** Solve the nonlinear differential equation using ADM

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = x^2 - \dfrac{1}{4}\left(\dfrac{\partial u(x,t)}{\partial x}\right)^2 \\ I.C \Longrightarrow u(x,0) = 0 \end{cases} \tag{6.12}$$

Sol. Set

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t)$$
$$N(u) = \sum_{n=0}^{\infty} A_n(x,t)$$

Where $N(u)$ represents the nonlinear form of u

In our case in equation (6.12) $N(u) = \left(\dfrac{\partial u}{\partial x}\right)^2$

$$A_n(x,t) = \left[\frac{1}{n!}\frac{d^n}{d\lambda^n} N\left(\sum_{i=0}^{n} \lambda^i u_i(x,t)\right)\right]_{\lambda=0}$$
$$A_n(x,t) = \left[\frac{1}{n!}\frac{d^n}{d\lambda^n}\left(\sum_{i=0}^{n} \lambda^i \frac{\partial u_i(x,t)}{\partial x}\right)^2\right]_{\lambda=0}$$

Integrating equation (6.12) from $0 \to t$

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) = x^2 t - \frac{1}{4}\int_0^t \sum_{n=0}^{\infty} A_n(x,\theta)\, d\theta$$

Now we get A_0,A_1,$A_2,\ldots$

$$A_0(x,\theta) = \left[\sum_{i=0}^{0} \lambda^i \frac{\partial u_i(x,\theta)}{\partial x}^2\right]_{\lambda=0} = \left(\frac{\partial u_0(x,\theta)}{\partial x}\right)^2$$

$$A_1(x,\theta) = \left[\frac{d}{d\lambda}\left(\sum_{i=0}^{1} \lambda^i \frac{\partial u_i(x,\theta)}{\partial x}\right)^2\right]_{\lambda=0}$$
$$= \left[\frac{d}{d\lambda}\left(\frac{\partial u_0(x,\theta)}{\partial x} + \lambda \frac{\partial u_1(x,\theta)}{\partial x}\right)^2\right]_{\lambda=0}$$
$$= 2\left[\left(\frac{\partial u_0(x,\theta)}{\partial x} + \lambda \frac{\partial u_1(x,\theta)}{\partial x}\right)\frac{\partial u_1(x,\theta)}{\partial x}\right]_{\lambda=0} = 2\frac{\partial u_0(x,\theta)}{\partial x}\frac{\partial u_1(x,\theta)}{\partial x}$$

$$A_2(x,\theta) = \left[\frac{1}{2!}\frac{d^2}{d\lambda^2}\left(\sum_{i=0}^{2} \lambda^i \frac{\partial u_i(x,\theta)}{\partial x}\right)^2\right]_{\lambda=0}$$
$$= \left[\frac{1}{2}\frac{d^2}{d\lambda^2}\left(\frac{\partial u_0(x,\theta)}{\partial x} + \lambda\frac{\partial u_1(x,\theta)}{\partial x} + \lambda^2\frac{\partial u_2(x,\theta)}{\partial x}\right)^2\right]_{\lambda=0}$$

$$
\begin{aligned}
&= \left(\frac{\partial u_1(x,\theta)}{\partial x}\right)^2 + 2\left(\frac{\partial u_0(x,\theta)}{\partial x}\frac{\partial u_2(x,\theta)}{\partial x}\right) \\
A_3(x,\theta) &= 2\frac{\partial u_1(x,\theta)}{\partial x}\frac{\partial u_2(x,\theta)}{\partial x} + 2\frac{\partial u_0(x,\theta)}{\partial x}\frac{\partial u_3(x,\theta)}{\partial x}
\end{aligned}
$$

Now because

$$u_0 + u_1 + u_2 + \cdots = u(x,t) = x^2 t - \frac{1}{4}\int_0^t [A_0 + A_1 + A_2 + \ldots]\, d\theta$$

Then

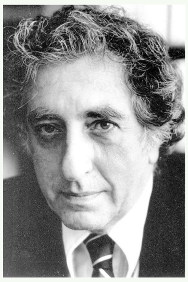
$$
\begin{aligned}
u_0 &= x^2 t \\
u_1 &= -\frac{1}{4}\int_0^t A_0 d\theta = -\frac{1}{4}\int_0^t \left(\frac{\partial u_0(x,\theta)}{\partial x}\right)^2 = -\int_0^t x^2\theta^2 d\theta = \frac{-1}{3}x^2 t^3 \\
u_2 &= \frac{2}{15}x^2 t^5 \\
u_3 &= \frac{-17}{315}x^2 t^7 \\
&\vdots
\end{aligned}
$$

$$u(x,t) = x^2\left[t - \frac{1}{3}t^3 + \frac{2}{15}t^5 - \frac{17}{315}t^7 \ldots\right] = x^2\tanh(t)$$

■

George Adomian

George Adomian (March 21, 1922 – June 17, 1996) was an American mathematician of Armenian descent who developed the Adomian decomposition method (ADM) for solving nonlinear differential equations, both ordinary and partial. The method is explained among other places in his book ***"Solving Frontier Problems in Physics: The Decomposition Method"***. He was a faculty member at the University of Georgia (UGA) from 1966 through 1989. While at UGA, he started the Center for Applied Mathematics. Adomian was also an aerospace engineer.



6.3.2 Decomposition Of Nonlinear FDE

We consider the following nonlinear FDE

$$\begin{cases} {}^{C}D^{\alpha}y(t) + L(y(t)) + N(y(t)) = f(t) \\ I.C \Longrightarrow y^{(k)}(0) = \beta_k \quad , \quad k = 0, 1, 2, \ldots, m-1 \end{cases} \tag{6.13}$$

We apply the Laplace Transform to the equation

$$\mathcal{L}[{}^{C}D^{\alpha}y(t)] = s^{\alpha}Y(s) - \sum_{k=0}^{m-1} s^{\alpha-k-1}y^{(k)}(0)$$

We use the following decomposition of $y(t)$

$$y(t) = \sum_{n=0}^{\infty} y_n(t)$$

And

$$N(y(t)) = \sum_{n=0}^{\infty} A_n$$

Where A_n are Adomian polynomials

The equation (6.13) become

$$\mathcal{L}\left[\sum_{n=0}^{\infty} y_n(t)\right] = \sum_{k=0}^{m-1} \frac{\beta_k}{s^{k+1}} - \frac{1}{s^{\alpha}}\mathcal{L}\left[L\left(\sum_{n=0}^{\infty} y_n(t)\right)\right] - \frac{1}{s^{\alpha}}\mathcal{L}\left[\sum_{n=0}^{\infty} A_n\right] + \frac{1}{s^{\alpha}}\mathcal{L}\left[f(t)\right]$$

Now we can get

$$\begin{aligned} Y_0 &= \mathcal{L}[y_0] = \sum_{k=0}^{m-1} \frac{\beta_k}{s^{k+1}} + \frac{1}{s^{\alpha}}\mathcal{L}[f(t)] \\ Y_1 &= \mathcal{L}[y_1] = -\frac{1}{s^{\alpha}}\mathcal{L}[L(y_0(t))] - \frac{1}{s^{\alpha}}\mathcal{L}[A_0] \\ Y_2 &= \mathcal{L}[y_2] = -\frac{1}{s^{\alpha}}\mathcal{L}[L(y_1(t))] - \frac{1}{s^{\alpha}}\mathcal{L}[A_1] \\ &\vdots \\ Y_n &= \mathcal{L}[y_n] = -\frac{1}{s^{\alpha}}\mathcal{L}[L(y_{n-1}(t))] - \frac{1}{s^{\alpha}}\mathcal{L}[A_{n-1}] \end{aligned}$$

■ **Example 6.3.2** Solve the nonlinear FDE using the Adomian decomposition method

$$\begin{cases} {}^{C}D^{\alpha}y(t) = t + y^2 \quad , \quad 1 < \alpha \leq 2 \\ I.C \Longrightarrow y(0) = 0 \quad , \quad y^{(1)}(0) = 1 \end{cases}$$

Sol. In order to solve the equation we apply the Laplace transform

$$\begin{aligned} \mathcal{L}\left[{}^{C}D^{\alpha}y(t)\right] &= \mathcal{L}\left[t + y^2\right] \\ s^{\alpha}Y(s) - s^{\alpha-1}y(0) - s^{\alpha-2}y^{(1)}(0) &= \mathcal{L}\left[t + y^2\right] \\ s^{\alpha}Y(s) - s^{\alpha-2} &= \mathcal{L}\left[t + y^2\right] \end{aligned}$$

So that, for the decomposition

$$y(t) = \sum_{n=0}^{\infty} y_n(t)$$

We obtain

$$Y(s) = \sum_{n=0}^{\infty} Y_n(s) \quad , \quad t + y^2 = \sum_{n=0}^{\infty} A_n$$

The Adomian polynomials A_n are

$$A_0 = t + y_0^2 \quad , \quad A_1 = 2y_0y_1 \quad , \quad A_2 = y_1^2 + 2y_0y_2 \quad , \quad A_2 = 2y_1y_2 + 2y_0y_3 \quad , \ldots$$

$$Y_0 = \frac{1}{s^2} \quad \Longrightarrow \quad y_0 = t \quad \Longrightarrow \quad A_0 = t + t^2$$

$$Y_1 = \frac{1}{s^\alpha}\mathcal{L}\left[A_0\right] \quad \Longrightarrow \quad y_1 = \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + 2\frac{t^{\alpha+2}}{\Gamma(\alpha+3)} \quad \Longrightarrow \quad A_1 = 2y_0y_1$$

And so on

$$y(t) = y_0 + y_1 + y_2 + \ldots$$

■

6.4 Fractional Systems of Differential Equations

Solving System of FDE is not very different from solving single FDE there are Linear Systems which can be solved using Laplace Transform and we will show example of them and there are Nonlinear Systems which can be solved by Method of Successive Approximations and The Adomian Decomposition Method

■ **Example 6.4.1** Solve the system of FDE

$$\begin{cases} {}^CD^\alpha x(t) = {}^CD^\beta y(t) + 1 \quad , \quad 0 < \alpha < 1 \\ {}^CD^\beta y(t) = 2\,{}^CD^\alpha x(t) - 1 \quad , \quad 0 < \beta < 1 \\ I.C \Longrightarrow x(0) = 1 \quad , \quad y(0) = 1 \end{cases}$$

Sol. We apply the Laplace Transform method

$$\begin{cases} \mathcal{L}\left[{}^CD^\alpha x(t)\right] = \mathcal{L}\left[{}^CD^\beta y(t)\right] + \mathcal{L}\left[1\right] \\ \mathcal{L}\left[{}^CD^\beta y(t)\right] = 2\mathcal{L}\left[{}^CD^\alpha x(t)\right] - \mathcal{L}\left[1\right] \end{cases}$$

We obtain the system

$$\begin{cases} s^\alpha X(s) - s^{\alpha-1} = s^\beta Y(s) - s^{\beta-1} + \dfrac{1}{s} \\ s^\beta Y(s) - s^{\beta-1} = 2s^\alpha X(s) - 2s^{\alpha-1} - \dfrac{1}{s} \end{cases}$$

With the solution:

$$\begin{cases} X(s) = \dfrac{1}{s} \\ \\ Y(s) = \dfrac{1}{s} - \dfrac{1}{s^{\beta+1}} \end{cases}$$

Take the Laplace inverse we get the solution for the original system

$$\begin{cases} x(t) = 1 \\ \\ y(t) = 1 - \dfrac{t^\beta}{\Gamma(\beta+1)} \end{cases}$$

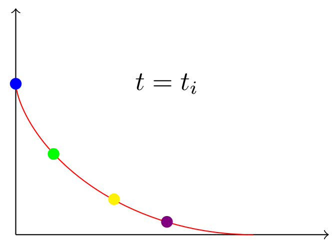
■

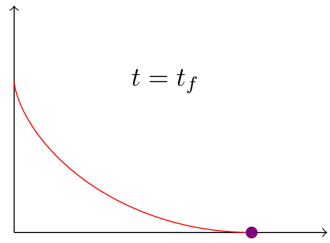
7 Applications Of Fractional Calculus

Fractional differential equations and integral equations appear in several physical systems In this section we will review some of their applications.

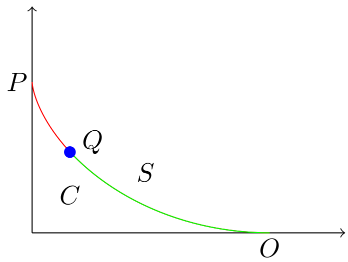
7.1 Abel's Fractional Integral Equation Of Tautochrone

The Abel's problem is to find a curve where the time that takes a ball to reach the bottom of the curve is same irrespective of the position of release of ball in friction less system.





Let S be the arc length measured along curve C from point O to an arbitrary point Q on C



The gain in Kinetic Energy while the ball descends is loss in the potential energy and is given by

$$\frac{1}{2}m\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2 = mg(y-\eta)$$
$$ds = -\sqrt{2g(y-\eta)}\ dt$$

The negative square root indicates that the distance (arc-length) decreases as the time increases. The equation thus to be solved is

$$dt = -\frac{1}{\sqrt{2g(y-\eta)}}\ ds$$

The time of decent from P to O is T , which is constant thus

$$T = -\frac{1}{\sqrt{2g}}\int_P^O \frac{1}{\sqrt{(y-\eta)}}\ ds$$

Now taking, the arc length as a function $s = h(\eta)$, where h depends on the shape of C.
We can write in differential form the curve as $ds = \frac{\mathrm{d}}{\mathrm{d}\eta}h(\eta)\ d\eta$, so this substitution gives

$$T = -\frac{1}{\sqrt{2g}}\int_y^0 \frac{1}{\sqrt{(y-\eta)}}\frac{\mathrm{d}}{\mathrm{d}\eta}h(\eta)\ d\eta$$
$$\sqrt{2g}T = \int_0^y (y-\eta)^{-\frac{1}{2}}\frac{\mathrm{d}}{\mathrm{d}\eta}h(\eta)\ d\eta$$

Meaning that the integral of right hand side of above, when a constant will be solution to get constant time of decent (T). In the above expression rewriting the integral of right hand side as

$$\sqrt{2g}T = \int_0^y (y-t)^{-\frac{1}{2}} f(t)\ dt$$

This is Abel's integral equation where $f(t) = \dfrac{\mathrm{d}}{\mathrm{d}t}h(t)$.
Abel solved this problem using convolutions and Laplace transforms but using fractional calculus we can see a much quicker solution if we divide both sides by $\Gamma(\frac{1}{2})$ which is equal to $\sqrt{\pi}$ we get this

$$\frac{\sqrt{2g}}{\sqrt{\pi}}T = \frac{1}{\Gamma(\frac{1}{2})}\int_0^y (y-t)^{-\frac{1}{2}} f(t)\ dt$$

We notice that the integral in the right hand side is the semi integral of $f(t)$
In terms of fractional order Integral, this equation can be written as

$$\sqrt{\frac{2g}{\pi}}T = I^{\frac{1}{2}} f(t)$$

Now by taking half derivative of RL type for both sides we get

$$D^{\frac{1}{2}}\sqrt{\frac{2g}{\pi}}T = f(t)$$

Thus

$$f(t) = \sqrt{\frac{2g}{\pi}}\frac{T}{\sqrt{t}}$$

Doing algebraic manipulation the following is obtained.

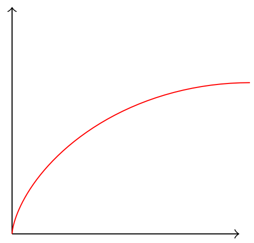
$$f(y) = \frac{\mathrm{d}}{\mathrm{d}y}h(y) = \frac{\mathrm{d}s}{\mathrm{d}y} = \frac{\sqrt{dx^2+dy^2}}{dy} = \sqrt{1+\left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2}$$

Thus

$$\frac{\mathrm{d}x}{\mathrm{d}y} = \sqrt{f(y)^2-1}$$

$$x = \int_0^y \sqrt{\frac{2gT^2}{\pi^2\eta} - 1}\ d\eta + C$$

If we make the curve start from (0,0)



We can get that $C = 0$

$$\text{Now Substitute}\begin{cases} a = \dfrac{2gT^2}{\pi^2} \\ \eta = 2a\ sin^2(\xi) \\ d\eta = 4a\ sin(\xi)\ cos(\xi)\ d\xi \\ 0 \to \beta \\ \beta = sin^{-1}\left(\sqrt{\dfrac{y}{2a}}\right) \end{cases}$$

We get

$$x = 4a\int_0^\beta \sqrt{\csc^2(\xi)-1}\ sin(\xi)\ cos(\xi)\ d\xi$$

$$x = 4a \int_0^\beta \cot(\xi)\ sin(\xi)\ cos(\xi)\ d\xi$$
$$x = 4a \int_0^\beta cos^2(\xi)\ d\xi$$

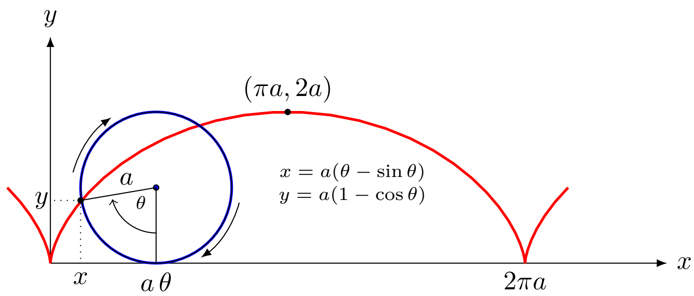
Thus we obtain

$$x = 2a\left(\beta - \frac{1}{2}\sin(2\beta)\right)$$
$$y = 2a\sin^2(\beta)$$

Put $\theta = 2\beta$ and we know that $a = \dfrac{gT^2}{\pi^2}$ we obtain

$$x = a\,(\theta - \sin(\theta))$$
$$y = a(1 - \cos(\theta))$$

The parametric equation of the cycloid
The cycloid is a curve you get when you roll a circle and focus on a point



Cycloid is the tautochrone.
A point be mentioned about the cycloid curve shape is that this is too a curve for brachistochrone problem solved by Bernoulli. That is determination of shape of the curve giving minimum time of descent. Therefore, the constant time T is also minimum time of descent in a cycloid.

7.2 Viscoelasticity

Viscoelasticity is a the field that interested in studying the property of materials that exhibit both viscous and elastic characteristics when undergoing deformation and it seems to be the field of the most extensive applications of fractional differential and integral operators.

There are two main quantities in viscoelasticity, a stress $\sigma(t)$ and a strain $\epsilon(t)$.
The relationships between stress and strain for solids(Hooke's law)

$$\sigma(t) = E\epsilon(t)$$

And for Newtonian fluids

$$\sigma(t) = \eta\frac{\mathrm{d}\epsilon(t)}{\mathrm{d}t}$$

Where E and η are constants. These functions satisfy the fractional differential equations

$$D^\alpha\sigma(t) = \frac{\Gamma(1-\alpha)t^{-\alpha}}{\Gamma(1-2\alpha)}\sigma(t)$$

$$D^\alpha\epsilon(t) = \Gamma(1+\alpha)t^{-\alpha}\epsilon(t)$$

7.3 Fractional Damped Motion

Consider a ball falling freely under gravity in viscous fluid having constituent equation as

$$D^1 v(t) + D^\alpha v(t) + v(t) = 1$$

With initial condition $v(0) = 0$
Using Laplace Transformation

$$\begin{aligned} sV(s) + s^\alpha V(s) + V(s) &= \frac{1}{s} \\ (s + s^\alpha + 1)V(s) &= \frac{1}{s} \\ V(s) &= \frac{1}{s(s + s^\alpha + 1)} \\ &= \frac{\left[1 - (-s^{-1} - s^{\alpha-1})\right]^{-1}}{s^2} \end{aligned}$$

Expanding numerator as binomial series, where $(s^{-1} + s^{\alpha-1}) < 1$ and $\alpha < 1$, for large s
The expansion is

$$V(s) = \sum_{n=0}^{\infty} (-1)^n \sum_{r=0}^{\infty} \binom{n}{r} \frac{1}{s^{n+2-r\alpha}}$$

Using Laplace Inverse

$$v(t) = \sum_{n=0}^{\infty} (-1)^n \sum_{r=0}^{\infty} \binom{n}{r} \frac{t^{n+1-r\alpha}}{\Gamma(n+2-r\alpha)}$$

Consider another fractional damping system where the inertia plays negligible role given by

$$D^{\frac{1}{3}} x(t) + x(t) = f(t) \quad , \quad x(0) = 0$$

And $f(t) = H(t)$ Heaviside's step function
The Laplace transforms gives

$$X(s) = \frac{1}{s(1 + s^{\frac{1}{3}})} = \frac{\left[1 - (-s^{-\frac{1}{3}})\right]^{-1}}{s^{\frac{4}{3}}}$$

With $|s| << 1$, the above is expanded as the following series

$$X(s) = \sum_{n=4}^{\infty} \frac{(-1)^n}{s^{\frac{n}{3}}}$$

And Using Laplace Inverse the temporal expression for above fractional dynamics equation's solution is

$$x(t) = \sum_{n=4}^{\infty} (-1)^n \frac{t^{\frac{n}{3}-1}}{\Gamma(\frac{n}{3})}$$

7.4 Fractional Diffusion Equations

Fractional differential equations are applied to models in relaxation and diffusion problems. Fractional calculus is used to formulate and to solve different physical models allowing a continuous transition from relaxation to oscillation phenomena. An application to an anomalous diffusion process demonstrates that the method used is also useful for more than one independent variable

7.5 Fractional Order Multipoles In Electromagnetism

It is well known that the axial multipole expansion of the electrostatic potential of electric charge distribution in three dimensions is

$$\Phi_n(r) = \frac{q}{4\pi\epsilon} \sum_{n=0}^{\infty} \frac{1}{r^{n+1}} P_n(\cos(\theta))$$

Where

- q is the called electric monopole moment
- ϵ is constant permittivity of the homogeneous isotropic medium
- $r = \sqrt{x^2 + y^2 + z^2}$
- $P_n(\cos(\theta))$ is the Legendre function of n degree.

The electrostatic potential functions for monopole (2^0), dipole (2^1), and quadrupole (2^2) are respectively, given by

$$\begin{aligned} \Phi_0(r) &= \frac{q}{4\pi\epsilon} \frac{1}{r} \\ \Phi_1(r) &= \frac{q}{4\pi\epsilon} \frac{\cos(\theta)}{r^2} \\ \Phi_2(r) &= \frac{q}{4\pi\epsilon} \frac{1}{r^3} P_n(\cos(\theta)) \end{aligned} \tag{7.1}$$

The scientist Nader Engheta generalized the idea of the integer order multipoles related to powers of 2 to the fractional order multipoles that are called 2^α-poles

He obtained the potential function for 2^α-poles ($0 < \alpha < 1$) along the $z - axis$, in terms of the Riemann-Liouville fractional derivatives in the form

$$\Phi_{2^\alpha}(r) = \frac{ql^\alpha}{4\pi\epsilon} {}_{-\infty}D_z^\alpha \frac{1}{r} \tag{7.2}$$

Where l is a constant with dimension of length so that the usual dimension of the resulting volume charge density is Coulomb/m^3

Evaluating the fractional derivative (7.2) yields the following result for the electrostatic potential

$$\Phi_{2^\alpha}(r) = \frac{ql^\alpha \Gamma(\alpha+1)}{4\pi\epsilon r^{\frac{(\alpha+1)}{2}}} P_\alpha\left(-\frac{z}{r}\right) \tag{7.3}$$

Where $P_\alpha(x)$ is the Legendre function of the first kind and of fractional degree α.
When $\alpha = 0$, $\alpha = 1$, and $\alpha = 2$, the potentials (7.3) reduce to those given by (7.1).

Conclusion

It's pretty clear why there hasn't been any significant applications of fractional Calculus in the past 300 years computing these without a computer is pretty tedious and difficult

Although fractional calculus doesn't have that many application it demo an extremely important lesson in mathematics and that's to try to break the rules and see what happens this is what led to the discovery of so many things like complex numbers when people tried to take the square root of negative numbers

And although it's hard to see fractional derivatives geometrically it's quite a fascinating topic

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1695

Leibniz

Can the meaning of derivative with integer order be generalized to derivative with non-integer order?!

L'Hopital

What if the order will be $\frac{1}{2}$?!

September 30, 1695

Leibniz

It will lead to a paradox, from which one day useful consequences will be drawn