THEORY OF FRACTIONAL CALCULUS

AND THEIR APPLICATIONS

By

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Preface

"There is a universe of mathematics laying in between the complete differentiations and integrations"

"Each generation has found something of interest to say about the fractional calculus. Perhaps the next generation will also"

This book is intended for a two semesters introduction to fractional calculus "*integral and derivatives of fractional order*". A search of the library using the keywords *fractional calculus* or *integral and derivatives of fractional order* will produce a much longer list of books. These books are listed in approximate order of increasing difficulty. Some of these books will be readable by the beginner, some will be quite advanced and will be difficult to understand without extensive background. A search on the keywords *fractional differential equations* and *fractional integral equations* will also produce a number of more specialized manuscripts on the subject matter of this course. If you wish to see what is going on at the frontier of the subject, you might take a look at some recent issues of the journals of nonlinear integral and differential equations which you may find in our library.

Instead of spending a lot of time going over background material, we go directly into the primary subject matter. We discuss proof methods and necessary background as the need arises. Nevertheless, you should at least skim the Preliminaries section where some of this material can be found so that you will know where to look if you need some fact or technique.

Although fractional calculus has many applications in engineering, computer science and physics, the thought processes one learns in this course may be more valuable than specific subject matter. In this course, one learns, perhaps for the first time, how mathematics is organized in a rigorous manner. A great deal of importance is placed on *understanding*. Every detail should be understood. Students should not expect to obtain this understanding without considerable effort. My advice is to learn each definition as soon as it is covered in class (if not earlier) and to make a real effort to understanding each point in the book. Many problems require the construction of a proof. Even if you are not able to find a particular proof, the effort spent trying to do so will help to increase your understanding of the proof when you see it. With sufficient effort, your ability to successfully prove statements on your own will increase.

We assume that students have some familiarity with basic functional analysis, measure theory, real analysis and ordinary differential equations. But little of this nature will be needed. Hopefully, the student's level of mathematical maturity will increase as the course progresses
The aim of our book is many-fold. In the one hand, we gather together the

main concepts and the well-known results of the fractional calculus. Some emphasizes examples, definitions, theorems and proofs are demonstrated. In the other hand, beside the successive approximations method used to construct explicit solution to the fractional differential equations, the Laplace transform also used to obtain the exact solution of some linear fractional integral and differential equations. Furthermore, based on the linear functional over a Banach space and on the definition of fractional integrals of real-valued functions, we define the fractional Pettis-integrals of vector-valued functions and the corresponding fractional derivatives. Also, we show that a well-known properties of fractional calculus over the domain of Lebesgue integrable functions also hold in the Pettis space. To encompass the full scope of this research, we investigate the existence of solutions to some initial and boundary value problems of the fractional type.

Finally, some applications of fractional calculus are given.

Contents

Introduction

In a letter to L'Hospital in 1695 Leibniz raised the following question: **Can the meaning of derivative with integer order be generalized to derivative with non-integer order?!** L'Hospital was somewhat curious about that question and replied by another question to Leibniz: **What if the order will be $\frac{1}{2}$?!**

Leibnitz in a letter dated September $30, 1695$ the exact birthday of the fractional calculus! replied: **It will lead to a paradox, from which one day useful consequences will be drawn.**

The *Fractional Calculus* may be considered an old and yet novel topic. It is an old topic since, starting from some speculations of G. W. Leibniz (1695-1763) and L. Euler (1730), it has been developed up to nowadays. However, it may be considered a novel topic as well, since only from a little more than twenty years it has been object of specialized conferences and treatises. The year (1974) saw the first international specialized conferences on the fractional calculus, held at the university of new Hayen. In (1984) the second international conferences was sponsored by the university of Strathclyde. The third international conferences was held at Nihon university in Tokyo in (1989).
The *Fractional Calculus* has its origin in the question of extension of meaning. In generalized integration and differentiation the question of extension of meaning is:
Can the meaning of the derivatives of integral order $\frac{d^n y}{dx^n}$ be extended to have meaning where n is any number irrational, fractional or complex?!
In 1819 the first mention of a derivative of arbitrary order appears in a text. The French mathematician, S. F. Lacroix, published a 700 page text on differential and integration calculus in which he devoted less than two pages to this topic. It has taken 279 year since L'Hospital first raised the question for a text to appears solely devoted to this topic. Euler and Fourier made mention of a derivative of arbitrary order but they gave no applications or examples. So the honor of making the first application belongs to Henrik Apple. He applied the *Fractional Calculus* in the solution of the integral equation which arises in the formulation of the **Tautochrone Problem.** In recent years considerable interest in *Fractional Calculus* has been stimulated by the applications that calculus finds in numerical analysis and different areas of physics and engineering, possible including fractal phenomena. However, in the last few decades many authors pointed out that derivatives and integrals of non-integer order are very suitable for the description of properties of various real materials, e.g. polymers. It has been shown that new fractional-order models

are more adequate than previously used integer-order models.
The development of *Fractional Calculus* within the framework of classical functions is now well-known and no purpose would be served here by a detailed exposition, there are many different starting points for a discussion of classical fractional calculus.It should be mentioned that from the view point of applications in physics, chemistry and engineering it was undoubtedly the book written by K. B. Oldham and J. Spanier which played an outstanding role in the development of the subject which can be called applied *Fractional Calculus*. Moreover, it was the first book which was entirely devoted to a systematic presentation of the ideas, methods and applications of the *Fractional Calculus*. Later there appeared eight fundamental works on various aspects of the *Fractional Calculus*, including the encyclopedic monograph by S. Samko, A. Kilbas, and O. Marichev [169], books by R. Gorenflo and S. Vessella [90], A. C. McBride [123], K. S. Miller and B. Ross [126], B. Rubin [146], lecture notes by F. Mainardi and R. Gorenflo about the book [44], and an extensive survey by Yu. Rossikhin and M. Shitikova [145].

The fractional calculus *"Integral and derivatives of fractional order"* is one of the singular integral and integro-differential operators. In this notes, the basic definitions of the fractional-order integration and differentiation are mentioned, which are used in the other chapters and play the most important role in the theory of differential and integral equations of fractional-order. Properties of the considered fractional derivatives and integrals are introduced, including composition rules, and some applications on fractional calculus.

1. Brief overview of results

In the following, we will quote a short history note for different existence theorems that were proved for some of the preceding integral equations and fractional order differential equations:

Such equations, for example, some methods use the classical Banach or Schauder fixed point principles that require strong hypotheses and do not give sufficiently general results (see e.g [6]). Other methods are developed to fit a specific class as in the case of integral equations of convolution type. Several authors tried to develop a technique that depends on the Darbo fixed point theorem with the Hausdorff or Kuratowski measure of noncompactness. However, these measures do not have convenient formulas in applications, hence strong conditions like Lipschitzian conditions on the functions involved in the considered integral equation have to be assumed to establish an existence theorem [6, 16]. Later, the measure of weak noncompactness appeared with a convenient formula in the space $L_1(I)$ of Lebesgue integrable functions on the interval I [58]. This measure coincides with the Hausdorff measure of noncompactness on subsets of $L_1(I)$ that are compact in measure. This feature allows the study of several types of integral equations such as: Fredholm, Volterra, Hammerstein, Urysohn, mixed type, Fredholm Stieltjes, Volterra Stieltjes, Hammerstein Stieltjes and Urysohn Stieltjes for different classes of functions [17]. In particular, Banas presented an approach that depends on some monotonicity conditions on the nonlinear term, the measure of noncompactness and the Darbo fixed point theorem. This approach is applied to prove the existence of monotonic solutions of integral equations of various types in the space of Lebesgue integrable functions [16, 15, 18, 19, 20, 27, 28, 29]. A different approaches that dispenses the monotonicity assumptions the nonlinear term, gives general results for the existence of the solution of functional integral equations in $L_1(I)$ was introduced by Emmanuele [73, 74, 75]. Emmanuele in his work assumes a rather strong condition on the kernel of the integral operator (Carathéodory conditions). Nevertheless, this restriction is naturally satisfied in the discussion of some initial value problems of fractional orders. The main interest in these works is to prove the existence of solutions of nonlinear differential equations and integral equations of various types in the space of continuous functions or Lebesgue integrable functions. Several recent papers give a treatment of the class of functional integral equations in a more general setting [140, 141]. For example, O'Regan in [141] presents a discussion of the Volterra-Hammerstein integral equation in a reflexive Banach space. In his work the main tools were the Schauder Tychonoff fixed point theorem in a locally convex topological space and the fact that in a reflexive Banach space the weakly compact subsets coincide with those subsets that are weakly closed and norm bounded. We also remark, that in nonreflexive Banach space, the existence of solutions of the Hammerstein integral equations have been considered, for the first time, by Cichon and Kubiaczyk [48].

1. In (2008), H. Salem [157], investigated the existence of solutions of the operator equations $p + \lambda G f x = x$ in the Banach space $C[I, E]$. It is assumed the vector-valued function f is nonlinear Pettis-integrable. Some additional assumptions imposed on f are expressed in terms of a weak measure of noncompactness. In this investigation the space E assumed to be arbitrary Banach space endowed with its weak topology.
2. In (2001) M. Cichoń, et al. [47], prove the existence of integrable solutions to integral equation of fractional type
$$x(t) = h(t) + I^{\alpha} f(t, x(\varphi(t))),\ t \in [0,1],\ 0 < \alpha < 1,$$
where $\varphi : [0,1] \to [0,1]$ is monotonic function. It is assumed that f is a real valued Carathéodory function satisfies a sufficiently small linear growth condition. The inclusion problem of the type
$$x(t) \in h(t) + I^{\alpha} F(t, x(\varphi(t))),\ t \in [0,1],\ 0 < \alpha < 1,$$
has been considered. Here $F : [0,1] \times \mathbb{R} \to 2^{\mathbb{R}}$ assumed to have a nonempty closed values, $F(t, \cdot) : \mathbb{R} \to \mathbb{R}$ is lower semicontinuous and $F(\cdot, \cdot)$ is measurable.
3. In (2009), H. Salem [161], prove the existence of weak solutions to some quadratic integral equations of fractional type
$$x(t) = h(t) + g(t, x(t)) I^{\alpha} f(t, x(t)),\ t \in [0,1],\ \alpha > 0,$$
in a reflexive Banach space relative to the weak topology under the weak-weak continuity assumption imposed on f. A special case was considered.
4. In (2011) H. Salem [162] give a sufficient conditions for the existence of positive continuous solutions to the *possibly singular* quadratic integral equation of fractional type
$$x(t) = H(t, x(t)) + \ x(t) I^{\alpha} \varphi(s)(f(x(t)) + g(x(t))),\ t \in [0,1],\ \alpha \in (0,1).$$
It is assumed that $f : [0,\infty) \to [0,\infty)$ and $g : (0,\infty) \to [0,\infty)$ are nonlinear continuous functions such that f is nondecreasing while g is nonincreasing and possibly singular, that is, the possibility of $g(0)$ being undefined is allowed. Meanwhile, the existence of maximal and minimal solutions of the above problem was proved. The method used in the considerations depends on both Schauder and Schauder-Tychonoff fixed point principles. Unlike all previous contributions of the same type, no

assumptions in terms of the measure of noncompactness were imposed on the nonlinearity of the given functions.

Now, we give a brief overview of the results of existence theorems for differential equations of fractional order on a finite interval of the real axis. We remark first that, most of the investigations in this field involve the existence of solutions to the fractional order differential equations with Riemann-Liouville fractional derivative. The Cauchy type problems of fractional orders were studied by many authors (see e.g. [32, 47, 51, 52, 76, 77, 78, 79, 96]). But the above investigations were not complete, however. Most researchers have obtained results not for the initial value problem, but for the corresponding Volterra integral equations. Some authors consider only particular cases. Moreover, some of the results obtained contained errors in the proof of the equivalence of the initial value problems and the Volterra integral equations and in the proof of the uniqueness theorem. In this regard, see the survey paper by Kilbas and Trujillo [104].

Barret [31] in (1954) first consider the Cauchy type problem for the linear differential equation

$$D^{\alpha}x(t) - \lambda x(t) = f(t),\ \alpha \in [n-1, n),\ n > 1.$$

with the initial conditions

$$D^{\alpha-j}x(0) = b_j,\ j = 1, 2, \cdots, n.$$

Barret proved that, if $f \in L_1(0,1)$, then such a problem the unique solution x in some subspace of $L_1(0,1)$.

Al-Bassam [4] first consider the Cauchy type problem

$$\begin{cases} D^{\alpha}x(t) = f(t, x(t)),\ t \in [0,1],\ \alpha \in (0,1], \\ I^{1-\alpha}x(0) = b, \end{cases} \tag{1}$$

in the space $C[0,1]$, provided that f is a real-valued continuous function and that it satisfies the Lipschitz condition.

Delbosco and Rodino [62] consider the nonlinear Cauchy problem

$$\begin{cases} D^{\alpha}x(t) = f(t, x(t)),\ t \in [0,1],\ \alpha > 0 \\ x^{(j)}(0) = b_j,\ j = 0, 1, 2, \cdots, [\alpha], \end{cases} \tag{2}$$

with f a continuous function on $[0,1] \times \mathbb{R}$. They used Schauder's fixed point theorem to prove that the problem (2) has at least a continuous solution x defined on $[0, \delta]$ for a suitable $\delta \in (0,1]$, provided $t^{\sigma} f(t, x)$ is continuous for some $\sigma \in [0, \alpha)$.

Hayek et al. [97] investigate the following Cauchy type problem for a system of linear differential equations

$$\begin{cases} D^{\alpha}x(t) = f(t, x(t)), \ t \in [0,1], \ \alpha \in (0,1], \\ x(0) = b, \ b \in \mathbb{R}^n, \end{cases} \tag{3}$$

with a real-valued vector function x, provided that f is continuous and Lipschitzain with respect to x.

Using the approach of Al-Bassam [4] and Dzhrbashyan and Nersesyan [69], I. Podlubny ([135], Section 3.2) consider the problem of the existence and uniqueness for the nonlinear Cauchy type problem of the form

$$\begin{cases} D^{\sigma}x(t) = f(t, x(t)), \ t \in [0,1], \ \alpha \in (0,1], \\ D^{\sigma_j}x(t)\mid_{t=0} = b_j, \end{cases} \tag{4}$$

with the sequential fractional derivatives D^{σ} given by

$$D^{\sigma}x(t) := D^{\sigma_k}x(t) + \sum_{k=0}^{n-1} a_k(t) D^{\sigma_{n-k-1}}x(t) + a_n(t)x(t),$$

where $D^{\sigma_k} = D^{\sigma_{k-1}}D^{\sigma_{k-2}} \cdots D^{\sigma_0}$.

Diethelm ([60], Theorem 4.1) proved the uniqueness and the existence of a local solutions in the space $C(0,1]$ to the Cauchy type problem

$$\begin{cases} D^{\alpha}x(t) = f(t, x(t)), \ t \in [0,1], \ \alpha \in (n-1,n], \\ D^{\alpha-j}x(0) = b_j, \ j = 1, 2, \cdots, n, \end{cases} \tag{5}$$

Provided f is continuous and Lipschitzian function.

The more general form of a Cauchy type problem, namely the form

$$\begin{cases} D^{\alpha}x(t) = f(t, x(t), D^{\alpha_1}x(t), \cdots, D^{\alpha_m}x(t)), \\ D^{\alpha-j}x(0) = b_j, \ j = 1, 2, \cdots, n, \end{cases} \tag{6}$$

$\alpha_1 < \alpha_2 < \cdots < \alpha \in (n-1, n]$, was studied by Kilbas and Marzan [105]. They assume that the function $f \in L^1[0,1]$ satisfies the condition

$$|f(t,V) - f(t,U)| \leq A_m \left[\sum_{j=0}^{m} |x_j - X_j| \right], \ t \in [0,1],$$

where $V = (x, x_1, \cdots, x_n)$, and $U = (X, X_1, \cdots, X_n)$.
Lakshmikantham and Vatsala [115, 116] study systematically the basic theory of fractional differential equations involving RiemannLiouville differential operators instead of deducing the basic existence and uniqueness results from the fixed point theory, they follow the classical proofs of differential equations so that one can compare and contrast the differences as well as understand

the intricacies that might result in the investigation. They start with the fundamental theory of inequalities, which provide necessary comparison results that are useful in further study of qualitative and quantitative properties of solutions of fractional differential equations. Then, they prove Peanos local existence result and consider the existence of extremal solutions. As an application of the comparison result developed, they discuss a global existence result in the end.

The above investigations were devoted to fractional differential equations with the Riemann-Liouville fractional derivative $D^\alpha x$ on a finite interval $[a, b]$ of the real axis $\mathbb{R}$. Such equations with the Caputo fractional derivative $D_c^\alpha x$ are not studied extensively. Gorenflo and Mainardi [91] applied the Laplace transformation to solve fractional differential equation

$$D_c^\alpha x(t) - \lambda x(t) = f(t)$$

with the Caputo fractional derivative of order $\alpha > 0$ and the initial conditions

$$x^{(j)}x(0) = b_j,\ j = 0, 1, \cdots, n-1,\ \alpha \in (n-1, n].$$

They discussed the key role of the Mittag-Leffler function for the cases $1 < \alpha < 2$ and $2 < \alpha < 3$. In this regard, see also the papers by Gorenflo and Mainardi [92] and Gorenflo et al. [93].

Seredynska and Hanyga [173] consider the equation

$$x''(t) + kD_c^\alpha x(t) = f(t),$$

with constant k, and prove the existence and the uniqueness of a solution $x \in C^{(2)}[0, 1]$ for the initial conditions

$$x(0) = x_0,\ x'(0) = y_0.$$

Diethelm and Ford [63] and Kilbas and Marazan [106] investigate the nonlinear Cauchy problem

$$\begin{cases} D_c^\alpha x(t) = f(t, x(t)),\ t \in [0, 1],\ \alpha > 0 \\ x^{(j)}(0) = b_j,\ j = 0, 1, 2, \cdots, n-1, \end{cases} \tag{7}$$

in the space $C^{n-1}[0, 1]$.

Using the A domain Decomposition method, Daftardar-Gejji and Jafari [52] derived analytical solution of a more general system of differential equations with Caputo derivatives and illustrated the results obtained.

1. In (2002) H. Salem et all [149], prove the existence of Continuous solutions of some the nonlinear fractional order differential equation

$$\begin{aligned} D^{\alpha}x(t) = f(x(t)) + g(x(t)),\ t \in [0,1], \\ x(0) = 0,\ \alpha \in (0,1), \end{aligned} \tag{8}$$

   Provided, the functions f, g satisfy
   (a) $f : [0,\infty) \to [0,\infty)$ is continuous nondecreasing
   (b) $g : (0,\infty) \to [0,\infty)$ is continuous and nonincreasing.

2. In (2004) H. Salem and M. Väth [150] prove an abstract generalization of a Gronwall lemma which gives a priori estimates for various (functional) differential and integral equations, of Volterra type under a linear growth condition on the nonlinearity. Further, Salem and Väth apply a simple special case of this abstract result to obtain the existence of global solutions of the functional differential equation of fractional type

$$\begin{aligned} D^{\alpha}x(t) \;\; = \;\; & f(t, x(t-c_1), \cdots, x(t-c_n), \\ & D^{\alpha_1}x(t-a_1), \cdots, D^{\alpha_m}x(t-a_m), \\ & I^{\beta_1}x(t-b_1), \cdots, I^{\beta_k}x(t-b_k)), \end{aligned}$$

   under a linear growth condition on f. Here $\alpha > \alpha_k > 0$ denote the, not necessarily integer, order of the corresponding (either Riemann-Liouville or Caputo) differential operators while $\beta_j > 0$ denote the, not necessarily integer, order of the (Abel) integral operators. The inclusion problem also treated.

3. In (2005), based on the linear functional over a Banach space E and on the definition of fractional integrals of real-valued functions, H. Salem et al. [151, 152] define the fractional Pettis-integrals of $E-$valued functions and the corresponding fractional derivatives. Also, they show that a well-known properties of fractional calculus over to domains of the Lebesgue integrable also hold in the Pettis space. To encompass the full scope of the papers, Salem et al. apply this abstract result to investigate the existence of weak-solutions to the Cauchy-type problem of fractional order in the Banach space $C[I, E]$ under weak continuity assumption imposed on the nonlinear term.

4. In (2007) H. Salem [153] present an existence of monotonic solutions for a nonlinear multi-term non-autonomous fractional differential equation

$$\begin{cases} \left(D^{\alpha_n} - \sum_{i=1}^{n-1} a_i D^{\alpha_i}\right) x(t) = f(t, x\varphi(t)),\ \textit{a.e.}\ \text{ on } (0,1), \\ \left(\sum_{i=1}^{n} a_i I^{1-\alpha_i} x\right)(0) = 0,\ \alpha_i \in (0,1), \end{cases} \tag{9}$$

in the Banach space of summable functions. The concept of measure of noncompactness and a fixed point theorem due to G. Emmanuele is the main tool in carrying out the proof.
5. In (2008) H. Salem [155] prove existence of continuous solutions for the set-valued differential equation of fractional type

$$\begin{cases} \left(D^{\alpha_n} - \sum_{i=1}^{n-1} a_i D^{\alpha_i}\right) x(t) \in F(t, x\varphi(t)), \quad \text{a.e. on } (0,1), \\ I^{1-\alpha_n} x(0) = c,\ \alpha_i \in (0,1),\ i = 1, 2, \cdots, n \end{cases} \tag{10}$$

where $F(t,\cdot)$ is lower semicontinuous from $\mathbb{R}$ into $\mathbb{R}$ and $F(\cdot,\cdot)$ is measurable.
6. In (2008) H. Salem [154], establish the existence of continuous solutions to some nonlinear fractional integral and differential equations. Quadratic system of fractional integral equations cases has been considered.The analysis rely on Krasnoselskii's fixed point theorem on a cone.
7. In (2008) H. Salem [156] analyze the nonlinear functional differential equations of the fractional type

$$\begin{cases} D^{\alpha} x(t) &= f(t, x(t), D^{\alpha_1} x(t-r), \cdots, D^{\alpha_n} x(t-nr)),\ \text{a.e.on } (0,1), \\ D^{j} x(t) &= 0,\ t \le 0,\ j = 0, 1, 2, \cdots, n. \end{cases} \tag{11}$$

Here r is positive constant, $\alpha \in (n, n+1]$, $\alpha_k \in (k-1, k]$, $k = 1, 2, \cdots, n$ $\alpha_0 = 0$ and D^{α_k} denotes the Caputo differential operator of order α_k. It assumed that f is monotonic and Carathéodory function satisfies *a not necessarily small* linear growth condition. The main tools used in [156] are the Darbo fixed point theorem and the measure of noncompactness.

We remark that A.M. El-Sayed [78] was the first author to consider the above problem.
8. In (2009) H. Salem [158] deal with the existence of weak solutions for the multi-term differential equation of the fractional type

$$\begin{cases} \left(D^{\alpha_n} - \sum_{i=1}^{n-1} a_i D^{\alpha_i}\right) x(t) = f(t, x(t)),\ t \in [0,1] \\ x(0) = 0. \end{cases} \tag{12}$$

Here, x takes values in a reflexive Banach space E endowed with the weak topology, $a_1, a_2, \cdots, a_{n-1}$, are constants, $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_n < 1$ and D^{α_j} denotes the weakly-(Riemann-Liouville) fractional differential operator of order α_j. Salem assumed that the vector-valued function f is weakly-weakly continuous. The case when E is not necessary reflexive was considered by Ravi P. Agarwal et. all in (2015)(see [1]).

9. Recently, in (2014) K. Furati [84], investigates an existence results to the Cauchy-type nonlinear problem for a class of fractional differential equations with sequential derivatives

$$\begin{cases} D^{\alpha}\left(t^{\gamma}D^{\beta}\right)x(t) = f(t, x(t)),\ t > 0,\ \alpha \in (0,1),\ \beta \in [0,1],\ \gamma < \alpha, \\ \lim\limits_{t\to 0} I^{1-\alpha}\left[t^{\gamma}D^{\beta}x\right](t) = c_1,\ \lim\limits_{t\to 0} I^{1-\beta}x(t) = c_0,\ c_0, c_1 \in \mathbb{R}. \end{cases} \tag{13}$$

in the space of weighted continuous functions.

10. In (2014), W. Yang [195] investigates the coupled integral boundary value problem for systems of nonlinear fractional q-difference equations. The nonlinear terms are continuous and semipositone. Firstly the corresponding Greens function for the boundary value problem and some of its properties are given. Moreover, by applying the nonlinear alternative of Leray Schauder type and Krasnoselskii's fixed point theorems, the author derive an interval of image such that any image lying in this interval, the semipositone boundary value problem has one or multiple positive solutions.

11. In (2015), R. Agarwal et all, [2] [3] and establish an existence result for the fractional differential equation

$$\begin{cases} D^{\alpha}x(t) = f(t, x(t)),\ t \in [0,1] \\ x(0) = x_0. \end{cases}$$

where $D^{\alpha}x(\cdot)$ is a fractional pseudo-derivative of a weakly absolutely continuous and pseudo-differentiable function $x(\cdot) : \Delta \to E$, the function $f(t, \cdot) : \Delta \times E \to E$ is weaklyweakly sequentially continuous for every $t \in \Delta$ and $f(\cdot, x(\cdot))$ is Pettis integrable for every weakly absolutely continuous function $x(\cdot) : \Delta \to E$, Δ is a bounded interval of real numbers and E is not necessary reflexive Banach space.

On the other hand, devoted by the applications of the boundary-value problems in various sciences such as physics, mechanics, chemistry, engineering, ect., the existence and multiplicity of positive solutions for boundary-value problems of second order have been received a great deal of attention [46, 80, 94, 118, 192]. However, to the authors knowledge, few papers can be found in the literature for multi-point boundary value problems of fractional order in Banach spaces. In scalar spaces, the fractional boundary value problems has provoked some interest in [13].

In the paper by Benchohraa et. all [34] (see also [198]), the authors claim that the boundary value problem of the fractional type

$$\begin{cases} D_c^\alpha y(t) = f(t, y(t)),\ t \in [0,1], \\ y(0) = g(y),\ y(1) = y_1, \end{cases}$$

is equivalent to the problem

$$y(t) = I^\alpha f(t, y(t)) - tI^\alpha f(1, y(1)) - (t-1)g(y) + ty_1$$

which is (in general) false:

We remark that the simple necessarily condition for the existence of a solution y to the boundary value problem fractional (Caputo sense) is that $y^{''} \in L_1$.

By looking at the main results of [34] and [198], we see that the authors proved that the integral equation has a continuous, solution y. Of course the continuity of y can not implies the existence of the fractional derivative (Caputo sense) of y. So, if the integral has a continuous solution, this solution does not (in general) satisfy the corresponding boundary value problem of fractional order.

Indeed, in the Caputo-definition of fractional differential operator, the equivalence between the boundary value problem and the corresponding integral equation holds if and only if the integrand satisfies the assumptions of the fundamental theorem of calculus. That is, if the integral equation have only continuous, but not absolutely continuous, solution we can not show that this solution satisfies the corresponding fractional BVP "Caputo sense".

1. In (2008), H. Salem [157], investigated the existence of pseudo-solutions to the following boundary value problem of fractional type

$$\begin{cases} D^\alpha x(t) + \lambda f(t, x(t)) = 0,\ \alpha \in (1,2],\ a.e. \text{ on } [0,1], \\ x(0) = x(1) = 0, \end{cases}$$

   where, x takes values in a reflexive Banach space E and D^α denotes the Pseudo-differential operator of fractional order. We note that no compactness condition assumed on the nonlinearity of f, this is will be due to the fact that a subset of a reflexive Banach space is weakly compact if and only if it is weakly closed and norm bounded.
2. In (2009) H. Salem [159], investigate the existence of pseudo-solutions for the nonlinear $m-$point boundary value problem of fractional type

$$D^\alpha x(t) + q(t) f(t, x(t)) = 0,\ a.e. \text{ on } [0,1],\ \alpha \in (n-1, n],\ n \geq 2,$$

$$x(0) = x^{'}(0) = x^{''}(0) = \cdots = x^{(n-2)}(0) = 0,\ x(1) = \sum_{i=1}^{m-2} \zeta_i x(\eta_i),$$

where $0 < \eta_1 < \cdots < \eta_{m-2} < 1$, $\zeta_i > 0$ with $\sum_{i=1}^{m-2} \zeta_i \eta_i^{\alpha-1} < 1$. It is assumed that q is real-valued continuous function and f is nonlinear Pettis integrable function. A particular cases of the above problem in case real-valued functions, were consider by many authors (see e.g. [191] and the references therein.)

3. In (2010) H. Salem [160], investigate the existence of Pseudo-solutions to the following fractional-order boundary value problem

$$\begin{cases} D^\alpha x(t) + a(t) f(t, x(t)) = 0, \ t \in [0,1], \ \alpha \in (n-1, n], \ n \geq 2, \\ x(1) + \displaystyle\int_0^1 u(\tau) x(\tau) d\tau = l, \ x^{(k)}(0) = 0, \ k = 0, 1, \cdots, n-2, \end{cases} \tag{14}$$

in the Banach space $C[I, E]$ under Pettis integrability assumptions imposed on f. This assumption is not sufficient for the existence of solutions in nonreflexive Banach spaces. So, we impose a weak noncompactness type condition expressed in terms of measure of weak noncompactness. Our results extend all previous results of the same type in the Bochner integrability setting and in the Pettis integrability one. Here, $u \in L_p$, $a \in L_q$ and $l \in E$.

The case when the nonlinear term of the problem (14) depends on the fractional derivative of an unknown function investigated in (2014) by H. Salem (see [163]).

2. **Preliminaries**

The basic idea behind fractional calculus is intimately related to a classical standard results from (classical) calculus and functional analysis. For this, we exhibit here certain ideas and propositions in mathematical and functional analysis necessary for our purposes. Also various notations and statements known in analysis courses are given.

Furthermore, during the course of the text we will occasionally state theorems from classical analysis for purposes of comparison with their fractional counterparts or in order to illustrate the ideas behind the generalizations. These classical theorems are usually well known and no proofs will be provided for them.

2.1. **Definitions and Notations.**

In the following pages, we will generally follow the following notations

1. By $\Gamma(\cdot)$ we denote the gamma function which defined by

$$\Gamma(x) := \int_0^\infty t^{x-1}e^{-t}dt, \; x > 0.$$

Its also possible to extend this function to negative values by inverting the functional equation (which becomes a definition identity for $x \in (-1, 0)$)

$$\Gamma(x) = \frac{\Gamma(1+x)}{x},$$

and for example $\Gamma(-0.5) = -2\Gamma(0.5)$. For the negative integer values of x, we have $1/\Gamma(1 + x) = 0$. In addition, the formula

$$\Gamma(\frac{1}{2} - k) = \frac{(-4)^k k!}{(2k)!}\sqrt{\pi}, \; k = 0, 1, 2, \cdots \tag{15}$$

is also useful for computing the values of the gamma function for some negative real values of x. However, we can calculate the value of the gamma function for all real values of x.

2. $B(\cdot, \cdot)$ denotes the well-known beta function. We remark here that the name beta function was introduced for the first time by Jacques Binet (1786-1856) in 1839 and he made various contributions on the subject. The beta function is symmetric and may be computed by mean of the gamma function thanks to the relation

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x + y)}, \; x, y \notin \{0, -1, -2, \cdots\},$$

3. $C[0, b]$ denotes the Banach space of continuous functions on the interval $[0, b]$,

4. For $m \in \mathbb{N} := \{1, 2, 3, \cdots\}$, define the space

$$C^m[0,b] := \{f : [0,b] \to \mathbb{R},\ f \ \text{ has a continuous } m^{th} \text{ derivative}\}.$$

5. By $AC^m[0,b], m \in \mathbb{N} := \{1, 2, 3, \cdots\}$, we denotes the space of functions f defined on $[0,b]$ which have continuous derivatives up to order $m-1$ on $[0,b]$ with $f^{(n-1)}$ is absolutely continuous $[0,b]$.
6. According to the custom $L_p[0,b],\ 1 \leq p \leq \infty$ will denote the Banach space of real-valued measurable functions f defined over $[0,b]$ and having $|f|^p$ be a Lebesgue integrable function on the interval $[0,b]$, and $L_\infty[0,b]$ denote the Banach space of real-valued essentially bounded and measurable functions defined over $[0,b]$. We say that the pairs $p, q \in [1,\infty]$ are of "conjugate exponents" if p, q connected by the relation $1/p + 1/q = 1$ for $1 < p < \infty$ with the convention that $1/\infty = 0$,
7. We say that the function f satisfies the Hölder condition of order $\lambda \in \mathbb{R}$ on the interval $[0,b]$, $0 < b \leq \infty$ or $f \in \mathcal{H}^\lambda[0,b]$ if

$$|f(t+h) - f(t)| \leq A|h|^\lambda,\ t, t+h \in [0,b], \tag{16}$$

where A is constant independent on h. The space $\mathcal{H}^\lambda[0,b]$ is called the Hölder space of a fixed order λ and the condition (16) is called Hölder condition on $[0,b]$. The Hölder space is an example of a locally convex topological vector space (see e.g. [67]). We define $\mathcal{H}^0[0,b] := C[0,b]$.

If $f \in \mathcal{H}^1[0,b]$ then f is called Lipschitzian. Clearly, the Hölderain functions are continuous while the continuously differentiable functions satisfy the Hölder condition. The fact that the continuously differentiable functions on bounded closed interval are, by mean value theorem, Lipschitzian. This yields $C^1[0,b] \subset \mathcal{H}^1[0,b]$. However, it is not hard to show, for $0 < \lambda_1 < \lambda_2 < 1$, that

$$C^1 \subset \mathcal{H}^1 \subset \mathcal{H}^{\lambda_2} \subset \mathcal{H}^{\lambda_1} \subset\ C \text{ and } \mathcal{H}^1 \subset AC[0,b] := AC^1[0,b].$$

It is simple to see that under such a definition only the case $0 < \lambda \leq 1$ is of interest, since if $\lambda > 1$, then the space $\mathcal{H}^\lambda$ contains only the constant functions. Evidently, for $\lambda > 1$ and any $t \in [0,b]$, it follows from (16), that

$$\lim_{t\to 0} \left| \frac{f(t+h) - f(t)}{h} - 0 \right| \leq A \lim_{h\to 0} |h|^{\lambda-1} = 0.$$

That is $f'(t) = 0$ for every $t \in [0,b]$. so f is constant.

For convenience, we introduce the following well-known inequalities (see e.g. [136])

Lemma 2.1. Jensen's inequality
Let $\zeta_i \geq 0$ such that $\sum_i \zeta_i = 1$. For the convex function $\psi : \mathbb{R} \to \mathbb{R}$ (e.g. $|x|^p,\ p \geq 1$ or $e^{ax},\ a \in \mathbb{R}$ or $x \log x,\ x \in \mathbb{R}^+$), we have

$$\psi\left(\sum_i \zeta_i x_i\right) \leq \sum_i \zeta_i \psi(x_i). \tag{17}$$

For concave ψ (e.g. $x^\alpha,\ \alpha \in [0,1], x \in \mathbb{R}^+$ or $\log x,\ x \in \mathbb{R}^+$), the reverse holds.

For $\lambda \geq 1$, we have the following is a special case of Jensen's inequality

$$\left(\sum_{i=1}^m |a_i|\right)^\lambda \leq m^{\lambda-1}\left[\sum_{i=1}^m |a_i|^\lambda\right],\ a_i \in \mathbb{R},\ i = 1, 2, \cdots,\ m \in \mathbb{N}. \tag{18}$$

Also, for $\lambda \in (0,1]$ and $t, s \in [0,b]$ with $s < t$. we have

$$|t^\lambda - s^\lambda| \leq (t-s)^\lambda,\ t^\lambda \leq \lambda t + (1-\lambda) \tag{19}$$

Further we shall prove the following inequality

$$|t-s|^\lambda \leq |t^\lambda - s^\lambda|,\ \lambda \geq 1,\ t, s \geq 0. \tag{20}$$

Clearly the result is true when $\lambda = 1$. To see that the result holds when $\lambda > 1$, we define $f(x) := (t-x)^\lambda - (s-x)^\lambda$. Without loss of generality, assume that $t \geq s$. Then f is continuous on $[0,s]$, and

$$f'(x) = -\lambda\left[(t-x)^{\lambda-1} - (s-x)^{\lambda-1}\right]$$

is negative on $(0,s)$, because $\lambda > 1$. By classical standard results from (classical) calculus, it follows that f is strictly decreasing on $[0,s]$, in particular $f(s) < f(0)$. Thus

$$(t-s)^\lambda \leq (t^\lambda - s^\lambda),\ \lambda \geq 1,\ t \leq s.$$

This is the claimed inequality.

It is worth to mentation that the Hölder space and nowhere differentiable functions are related. An immediate example of this is the Weierstrass function

$$\mathcal{W}_\alpha(t) := \sum_{n=0}^\infty \frac{e^{ib^n t}}{b^{n\alpha}},\ b > 1,\ \alpha > 0. \tag{21}$$

It is well-known (see [109] and [170]) that the Weierstrass function is continuous (even Hölderain function) when $0 < \alpha \leq 1$ but nowhere differentiable.

8. Let $m \in \mathbb{N} \cup \{0\}$, $\sigma \in (0,1]$. We say that $f \in \mathcal{H}^{m+\sigma,k}[0,b]$, $k \in \mathbb{R}^+$ if $f \in C^m[0,b]$ and
$$\left|f^{(m)}(t+h) - f^{(m)}(t)\right| \leq A|h|^\sigma \left(\ln \frac{1}{|h|}\right)^k, \ t, t+h \in [0,b], \ |h| < \frac{1}{2}.$$
9. Let E and F be Banach spaces. The operator $T : E \to F$ is called compact if the image of bounded sets under T is relatively compact,
10. We define the most important special functions used in fractional calculus, namely the Mittag-Leffler function which defined for $\alpha \geq 0$, $z \in \mathbb{C}$ by
$$E_\alpha(z) = \sum_{n=o}^{\infty} \frac{z^n}{\Gamma(n\alpha+1)}, \ z \in \mathbb{C}. \tag{22}$$
The series (22) is convergent in the whole complex plane (that is, it is uniformly convergent). Hence E_α is continuous. It is clear that
$$E_\alpha(0) = 1, \ E_0 = \frac{1}{1-z}, \ E_1(z) = e^z, \text{ and } E_2(z) = \cosh(\sqrt{z}).$$
In addition, we have $E_\alpha(\lambda t^\alpha) = e^{\lambda t^\alpha}$ and
$$\frac{d}{dt}E_\alpha(\lambda t^\alpha) = \frac{d}{dt}\left(1 + \sum_{n=1}^{\infty} \frac{\lambda^n t^{n\alpha}}{\Gamma(n\alpha+1)}\right) = \sum_{n=1}^{\infty} \frac{\lambda^n t^{n\alpha-1}}{\Gamma(n\alpha)}, \ \alpha \geq 0, \ \lambda \in \mathbb{R}. \tag{23}$$
Generally, for $\alpha, \beta > 0$ and $z \in \mathbb{C}$, we define
$$E_{\alpha,\beta}(z) := \sum_{n=o}^{\infty} \frac{z^n}{\Gamma(n\alpha+\beta)}. \tag{24}$$
Obviously $E_{\alpha,1} \equiv E_\alpha$. We also have the particular cases:
$$E_{1,2}(z) = \frac{e^z - 1}{z}, \ E_{2,2}(z) = \frac{\sinh(\sqrt{z})}{\sqrt{z}} \text{ and } E_{1,3}(z) = \frac{e^z - z - 1}{z^2}.$$
In general, for $m \in \mathbb{N}$, we have
$$E_{1,m}(z) = \frac{1}{z^{z-1}}\left[e^z - \sum_{k=0}^{m-2} \frac{z^k}{\Gamma(z+1)}\right].$$
11. Beside the definitions (22) and (24) of the Mittag-Leffler function, we define for $\alpha, \beta, \gamma > 0$, the generalized Mittag-Leffler functions $E_{\alpha,\beta,\gamma}$ by
$$E_{\alpha,\beta,\gamma}(z) := \sum_{k=0}^{\infty} c_k z^k,$$

where

$$c_0 = 1,\ c_k = \prod_{r=1}^{k} \frac{\Gamma[\alpha(j\beta+\gamma)+1]}{\Gamma[\alpha(j\beta+\gamma+1)+1]},\ (k \in \mathbb{N}).$$

For the remainder of this section, we gather together some results which well be used throughout this notes.

Definition 2.1. Rearrangements and unconditional convergence
Let E be a real Banach space and E^* be its topological dual. We remind that a series $\sum_{n=0}^{\infty} a_n$ in E converges absolutely if $\sum_{n=0}^{\infty} \|a_n\| < \infty$ and it is called unconditionally convergent if each of its rearrangements $\sum_{n=0}^{\infty} a_{\pi(n)}$ converges in the norm of E. That is, a series is said to be unconditionally convergent if all rearrangements of the series are convergent to the same value.

To illustrate the importance of the unconditionally convergent, we consider the alternative series $\sum_n \frac{(-1)^n}{n}$. This series converges, but does not absolutely converges. To show that this series is conditionally converges, we note that each of sub-series

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\cdots \text{ and } \frac{-1}{3}+\frac{-1}{5}+\frac{-1}{7}+\cdots$$

is diverges. Hence there must exists $m > 0$ such that

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\cdots+\frac{1}{m} > 1.$$

Then, there must exists $N > m$ such that

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\cdots+\frac{1}{m}-\frac{1}{3}+\frac{1}{m+1}+\cdots+\frac{1}{N} > 2.$$

Continuing in this way, we see that

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\cdots+\frac{1}{m}-\frac{1}{3}+\frac{1}{m+1}+\cdots+\frac{1}{N}-\frac{1}{5}+\cdots$$

is a rearrangement of $\sum_n \frac{(-1)^n}{n}$ which diverges to ∞.

An Equivalent formulation of unconditionally converges (see Theorem 1 in [113]): A series is unconditionally convergent if for every sequence $\{\xi_n\}$ with $\xi_n \in \{-1, 1\}$, the series $\sum_{n=0}^{\infty} \xi_n a_n$ converges. For example, by putting $\xi_n \in (-1)^n$, we infer that the alternative series $\sum_n \frac{(-1)^n}{n}$ is conditionally convergent.

It is not difficult to see that the series$\sum_{n=0}^{\infty} a_n$ converges unconditionally in E if and only if only if $\sum_{n=0}^{\infty} \varphi a_n < \infty$ for all $\varphi \in E^*$ and

$$\lim_n \sup_{\|\varphi\|\le 1} \sum_{k\ge n} \varphi a_k = 0.$$

In $\mathbb{R}$, every unconditionally convergent series is necessary absolutely convergent. However, the converse in not true in the infinite-dimensional Banach spaces. Dvoretzky-Rogers [68] asserts that every infinite-dimensional Banach space admits an unconditionally convergent series that is not absolutely convergent.

Example 2.1. Let $E = \ell_2$ and $a_n = (0, 0, \cdots, \frac{1}{n}, 0, \cdots)$, where the non-zero coordinate is in the $n-$th place. Then $\sum_n a_n$ converges to the element $(1, \frac{1}{2}, \frac{1}{3}, \cdots)$ of ℓ_2 and for any choice of $\xi_n \in \{-1, 1\}$, the series $\sum_{n=1}^{\infty} \xi_n a_n$ clearly converges, that is, the series is unconditionally convergent. However, since $\|a_n\|_{\ell_2}^2 = \frac{1}{n^2}$, for any n, it follows

$$\sum_{n=1}^{\infty} \|a_n\|_{\ell_2} = \sum_{n=1}^{\infty} \frac{1}{n} = \infty.$$

and so $\sum_{n=0} a_n$ does not converges absolutely. Thus, the series $\sum_{n=1}^{\infty} a_n$ converges unconditionally but not absolutely.

Lemma 2.2. The generalized Minkowski inequality
Let $p \in [1, \infty]$ and $f : I \times J \to \mathbb{R}$, then

$$\left(\int_I \left(\int_J |f(t,s)| \, ds \right)^p dt \right)^{\frac{1}{p}} \leq \int_J \left[\int_I |f(t,s)|^p dt \right]^{\frac{1}{p}} ds.$$

In other words,

$$\left\| \int_J f(\cdot, s) \, ds \right\|_p \leq \int_J \|f(\cdot, s)\|_p \, ds.$$

Lemma 2.3. The generalized Hölder inequality
Let $\Omega \subset \mathbb{R}$ and $p, q, r \in [1, \infty]$ such that $1/p + 1/q = 1/r$. If $f \in L_p[\Omega]$ and $g \in L_q[\Omega]$, then $fg \in L_r[\Omega]$ and

$$\|fg\|_{L_r[\Omega]} = \left(\int_\Omega |f(t)g(t)|^r dt \right)^{\frac{1}{r}} \leq \|f\|_{L_p[\Omega]} \|g\|_{L_q[\Omega]} .$$

For the case when p and q are in the open interval $(1, \infty)$ with $1/p + 1/q = 1$, we have

$$\sum_{k=1}^{\infty} |x_k \, y_k| \leq \left(\sum_{k=1}^{\infty} |x_k|^p \right)^{\frac{1}{p}} \left(\sum_{k=1}^{\infty} |y_k|^q \right)^{\frac{1}{q}} \quad \text{for} \quad \{x_n\} \in \ell_p, \; \{y_n\} \in \ell_q. \tag{25}$$

We would like to remark here that when the triple $(\mathbb{R}, \Omega, \mu)$ is a finite measure space, then for any $1 \leq p \leq q \leq \infty$, we have $L_q(\mathbb{R}, \Omega, \mu) \subset L_p(\mathbb{R}, \Omega, \mu)$. This is not true in infinite measure spaces. The counterexample: $f(x) = x^{-1} \in L_2[1, \infty]$, but clearly it does not belong to $\in L_1[1, \infty]$.

Further, if $f \in L_p[\Omega]$, an application of Hölder inequality with $r = 1$ to $f = 1 \cdot f$ shows that $f \in L_q[\Omega]$ and

$$\|f\|_{L_q[\Omega]} \leq |\Omega|^{1/q-1/p} \|f\|_{L_p[\Omega]}\,,\ 1 \leq q \leq p.$$

Repeated application of this inequality gives the following generalization: Let $p, p_i \in [1, \infty], i = 1, 2, \cdots, n$ such that $\sum_{i=1}^{n} 1/p_i = 1/p$. If $f_i \in L_{p_i}[\Omega]$ for $1 \leq i \leq n$, then

$$f := \prod_{i=1}^{n} f_i \in L_p[\Omega] \text{ and } \|f\|_p \leq \prod_{i=1}^{n} \|f_i\|_{p_i}\,.$$

However, we have

Lemma 2.4. Interpolation Theorem
Let $\Omega \subseteq \mathbb{R}$, $1 \leq p \leq r \leq q \leq \infty$ and $\theta \in [0, 1]$ be such that

$$1/r = \theta/p + (1 - \theta)/q.$$

If $f \in L_p[\Omega] \bigcap L_q[\Omega]$, then $f \in L_r[\Omega]$ and

$$\|f\|_{L_r[\Omega]} \leq \|f\|^{\theta}_{L_p[\Omega]} \|f\|^{1-\theta}_{L_q[\Omega]}\,.$$

To prove this, we use $r\theta/p + r(1 - \theta)/q = 1$ and then we apply the Hölder inequality in the following way

$$\int_{\Omega} |f(t)|^r dt = \int_{\Omega} |f(t)|^{\theta r} |f(t)|^{(1-\theta)r} dt \leq \|f\|^{\theta}_{L_p[\Omega]} \|f\|^{1-\theta}_{L_q[\Omega]}\,.$$

Lemma 2.5. Hölder's inequality for negative exponents *(*[98] *page 191)*
Let $0 < p < 1$ and $q \in \mathbb{R}$ be such that $1/p + 1/q = 1$ (hence $q < 0$)). If f and g are measurable functions, then

$$\|fg\|_{L_1[\Omega]} = \left(\int_{\Omega} |f(t)g(t)| dt\right) \leq \|f\|_{L_p[\Omega]} \|g\|_{L_q[\Omega]}\,.$$

unless $\|g\|_{L_q[\Omega]} = 0$.

Next proposition gives a sufficient condition to guarantee that the convolution product of f and g defined by

$$f * g(t) := \int_{\Omega} f(t - s)g(s)\, ds = g * f(t),\ \Omega \in \mathbb{R},$$

is well-defined a.e. in Ω.

Proposition 2.1. (Yong Inequality)
Let $\Omega \subset \mathbb{R}$ and $p, q, r \in [1, \infty]$ such that $1/p+1/q = 1/r+1$. If $f \in L_p[\Omega]$ and $g \in L_q[\Omega]$, then for a.e. $t \in \Omega$, the function $s \longmapsto f(t - s)g(s)$ is integrable and

$$\|f * g\|_{L_r[\Omega]} \leq \|f\|_{L_p[\Omega]} \|g\|_{L_q[\Omega]}\,.$$

In particular, if $r = p$ then $q = 1$ and consequently $f * g \in L_p[\Omega]$ and

$$\|f * g\|_{L_p[\Omega]} \leq \|f\|_{L_p[\Omega]} \|g\|_{L_1[\Omega]} .$$

Moreover, if $r = \infty$ (equivalently if $1/p + 1/q = 1$) then $f * g$ is continuous function on Ω.

Proposition 2.2. (Clarkson Inequality) *(see see Adams, Sobolev Spaces.)* Let $f, g \in L_p[\Omega]$. Then we have the following inequalities:

1. If $p \in [1, 2)$, then

$$\left\|\frac{f+g}{2}\right\|_{L_p}^{q} + \left\|\frac{f-g}{2}\right\|_{L_p}^{q} \leq \left(\frac{\|f\|_{L_p}^{p} + \|g\|_{L_p}^{p}}{2}\right)^{q-1},$$

2. If $p \in [2, \infty)$, then

$$\left\|\frac{f+g}{2}\right\|_{L_p}^{p} + \left\|\frac{f-g}{2}\right\|_{L_p}^{p} \leq \left(\frac{\|f\|_{L_p}^{q} + \|g\|_{L_p}^{q}}{2}\right)^{p-1}.$$

Definition 2.2. A family $M := \{x_i,\ i \in J,\ J \text{ some index set}\} \subset C[0, b]$, is said to be equicontinuous if given $\epsilon > 0$, there exists $\delta > 0$ such that, for $t, s \in [0, b]$, if $|t - s| < \delta$, then

$$|x_i(t) - x_i(s)| < \epsilon \text{ for all } i \in J.$$

Definition 2.3. Let (E, d) be a metric space and $M \subset E$. The set M is called relatively compact in E if the closure of M is a compact subset of E.

A helpful classical result from analysis in connection with such sets is as follows. The proof can be found in many standard textbooks, e.g. [65].

Theorem 2.1. (Arzelà-Ascoli) *A set $M \subset C[0, b]$ is relatively compact if, and only if the functions $x \in M$ are uniformly bounded and equicontinuous.*

According to Tychonoff's theorem in topological products and Arzelà-Ascoli theorem the compactness criteria for the Banach space of $n-$times products of $C[0, 1]$

$$C := C[0, 1] \times C[0, 1] \times \cdots \times C[0, 1], \tag{26}$$

equipped by the norm

$$|||\bar{x}||| := \max_i \|x_i\|,\ i = 1, 2, \cdots, n$$

is readily available (see, for example [67]).

Theorem 2.2. (Kolmogorov Compactness Criterion) [67] *The bounded subset Ω of $L_p(0, 1)$ is relatively compact if, and only if*

$$\lim_{h \to 0} \|T_h x - x\|_{L_p} = 0,$$

uniformly on Ω, that is

$$\lim_{h\to 0}\left\{\sup_{x\in\Omega}\|T_h x - x\|_{L_p}\right\} = 0, \text{ where } T_h x(t) \ = \ \frac{1}{h}\int_t^{t+h} x(s)\ ds.$$

The following results are well-know and standard

Theorem 2.3. (Beppo-Levi Theorem)
Let $\{f_n\}$ be a sequence of Lebesgue functions on the interval $[0,b]$ and suppose that

$$\sum_{n=1}^{\infty}\int_0^b |f_n(s)|\,ds < \infty.$$

Then $\sum f_n(t)$ converges to a finite limit for almost all $t \in [0,b]$, the sum is integrable, and

$$\int_0^b \sum_{n=1}^{\infty} f_n(s)\,ds = \sum_{n=1}^{\infty}\int_0^b f_n(s)\,ds.$$

Theorem 2.4. (Generalized Dominated Convergent Theorem) *(Royden real Analysis Book)*
If $\{f_n\} \in L_1(E)$ for each n, $f_n \to f$ a.e. on E and $f \in L_1(E)$, then

$$\int_E |f_n - f| \to 0, \ \textit{iff} \ \lim_{n\to\infty}\int_E |f_n| = \int_E |f|$$

We note, if $\int_E |f_n - f| \to 0$, then $\left|\int_E (f_n - f)\right| \to 0$ and so $\lim_{n\to\infty}\int_E f_n = \int_E f$. That is

$$\lim_{n\to\infty}\int_E |f_n| = \int_E |f| \Longrightarrow \lim_{n\to\infty}\int_E f_n = \int_E f.$$

Based on Lebesgue dominated theorem, Bukhalov proves the following fact: The linear integral operator K maps L_p into L_p ($p,q \geq 1$ are of conjugate exponents), if and only if for every sequence $\{f_n\}$ which is dominated by an L_p- function there holds the implication

$$y_n \to 0 \text{ in measure } \ \Longrightarrow \ Ky_n \to 0 \ \text{ almost everywhere } (*).$$

The case $\Longrightarrow$ comes from Lebesgue's dominated convergence theorem in the variant mentioned above). The case when (*) holds implies that K is actually an integral operator is of course much harder to show (this implication is the result for which Bukhalov is famous for).

Theorem 2.5. (Vitali Convergent Theorem) *(Royden real Analysis Book)*
Let $\{f_n\} \in L_1(E)$ be a sequence so that $f_n \to f$ a.e. on E. If the sequence

$\{f_n\}$ satisfies the Vitali equi-integrability condition (That is, for every $\epsilon > 0$ there is a $\delta > 0$ so that if B is an open set with $\mu(B) < \delta$ then

$$\int_E |f_n(t)|\chi_B(t)dt < \epsilon,$$

for all $n \in N$), then $f \in L_1(E)$ and

$$\lim_{n\to\infty} \int_E f_n = \int_E f$$

Note that if $\{f_n\}$ dominated by a function $g \in L_1(E)$, then the Vitali equi-integrability condition automatically satisfied since

$$\int_E |f_n(t)|\chi_B(t)dt \leq \int_E g(t)\chi_B(t)dt < \epsilon.$$

Thus, if $\{f_n\} \in L_1(E)$ be a sequence of nonnegative functions so that $f_n \to 0$ a.e. on E , then

$$\lim_{n\to\infty} \int_E f_n = 0,$$

if and only if, $\{f_n\}$ satisfies the Vitali equi-integrability condition on E (*cf.* e.g. Exercise 242 in [188]).

Recall that a linear operator T whose domain D is contained in a normed vector space E is a closed operator if $\lim x_n = x$ for a sequence $\{x_n\}$ in D, and $\lim T(x_n) = y$, then x is in D and $T(x) = y$. We state the following well-known theorem

Theorem 2.6. (Closed Graph Theorem)
A linear map defined on a Banach space E is bounded if, and only if it is closed.

The following results are folklore

Theorem 2.7. (Egorov's Theorem)
Let $\{x_n\}$ be a sequence of measurable, almost everywhere finite functions on the interval I. If x_n converge to x in measure, then there exists a subsequence x_{n_k} such that $x_{n_k} \to x$ almost everywhere.

Lemma 2.6. (Mazur's lemma)
Let $(E, \|\cdot\|)$ be a Banach space and let $\{a_n\}$ be a sequence in E that converges weakly to some $a \in E$. Then there exists a function $N : \mathbb{N} \to \mathbb{N}$ and a sequence of sets of positive real numbers $\{\xi(n)_i : i = n, \cdots, N(n)\}$ such that $\sum_{i=n}^{N(n)} \xi(n)_i = 1$ such that the sequence $\{b_n\}$ defined by the convex combination

$$b_n := \sum_{i=n}^{N(n)} \xi(n)_i a_i$$

converges strongly in E to a

The proof of Mazur's lemma is a direct consequence of the fact that closed convex sets are weakly closed. The following proposition is a direct consequence of Baire's category theorem

Proposition 2.3. *Let $X \subseteq [a,b]$ be non-empty closed set. If $\{X_n\}$ is an increasing sequence of closed sets with $\cup_{n\in\mathbb{N}} X_n = X$, then there exists $N \in \mathbb{N}$ and a sub-interval $[u,v]$ of $[a,b]$ such that $X_N \cap [u,v]$ is non-empty and*

$$X \cap [u,v] = X_N \cap [u,v].$$

The following result asserts that a measurable function f is nearly continuous in the sense that for all $\varepsilon > 0$ there is a set Ω of measure less than ε such that f is continuous on the complement of Ω.

Theorem 2.8. (Lusin's Theorem)
For an interval $[a,b]$, let $f : [a,b] \to \mathbb{R}$ be a measurable function. Then, for every $\varepsilon > 0$, there exists a compact $\Omega \subset [a,b]$ such that f restricted to Ω is continuous and

$$\mu(\Omega^c) < \varepsilon.$$

Theorem 2.9. (Hahn-Banach Theorem)
If φ is bounded linear functional on a subspace of a normed space, then φ extended to the whole space with preservation of norm.

The Theorem of Hahn-Banach claims that, there exists an extended linear functional ψ of φ on a normed space X, which preserves the norm of the functional ψ on some linear subspace Y of X. But it does no claim that, every $\varphi \in Y^*$ the dual of Y, can be extended to a linear functional in ψ on Y^*: Really, recall that Y subspace of X if elements of Y forms a vector subspace of X and Y endowed with the same norm of X. For example $(L_\infty, \|\cdot\|_\infty)$ not subspace of $(L_1, \|\cdot\|_1)$ but $(L_\infty, \|\cdot\|_1)$ is a subspace of $(L_1, \|\cdot\|_1)$. In the following example, we let $Y = (L_\infty, \|\cdot\|_\infty)$ and $X = (L_\infty, \|\cdot\|_1)$, we will show that there exits $\varphi \in Y^*$ which can not be extended to be an of the elements of X^*:
Indeed, if y belongs to $X = L_1$ but not to $Y = L_\infty$, then the functional $\varphi(x) := \int y(t)x(t)dt$ would define an element from Y^* which is an unbounded in the L_1 norm (that is the linear functional is unbounded from L_1 into $\mathbb{R}$) and thus cannot be in X^*. However, if $(Y, \|\cdot\|_Y)$ is a subspace of the Banach space $(X, \|\cdot\|_X)$, then any $\varphi \in Y^*$ has an extension to an element from X^* if and only if φ is bounded with respect to $\|\cdot\|_X$. Indeed, the necessity is trivial, and sufficiency follows from the Hahn-Banach extension theorem 2.9.

Denote by $S = S(0,1)$ the set of all Lebesgue measurable function acting from (0,1) into $\mathbb{R}$. Let us furnish this set with the metric

$$\rho_s(x,y) := \inf[a + meas\{s : |x(s) - y(s)| \geq a\} : a > 0],$$

where the symbol $measD$ stands for the Lebesgue measure of the set D. Then $S(0,1)$ becomes a complete metric space if we identify functions which are equal a.e. on (0,1) [67]. The complete description of compactness in measure (i.e. the compactness in the space $S(0,1)$ was given by Fréchet [67]. We will not quote this criterion because it has rather complicated form. Based on this criterion we have the following result [16].

Theorem 2.10. *Let X be a bounded subset of $L_1[0,1]$ consisting of functions which are a.e. nondecreasing on the interval $[0,1]$. Then X is compact in measure.*

Definition 2.4. Let E be a measurable set and let c be a real number. The density of E at c is defined by

$$d_c E := \lim_{h \to 0^+} \frac{\mu[E \cap (c-h, c+h)]}{2h}$$

provided the limit exists.

It is clear that $0 \leq d_c E \leq 1$, when it exists. The point c is a density point of E if $d_c E = 1$ and a point of dispersion of E if $d_c E = 0$.

A density point of E is certainly a limit (accumulation) point of E, but not conversely. For example: Let $A = \{\frac{1}{n},\ n \in \mathbb{N}\}$. Then A has no points of density but has 0 as a limit point. In general, a set of zero measure has no points of density. Every real number is a density point of the set of irrational numbers.

The Laplace transform

The Laplace transform is a powerful tool in applied mathematics and engineering. Evidently it is indispensable in certain areas of control theory. Virtually every beginning subsection in differential equations at the undergraduate level introduces this technique for solving linear differential equations.

Indeed, the Laplace transform method is an extremely useful tool for the analysis of linear (fractional or classical) initial value problems. In particular, it allows us to replace a differential equation by an algebraic equation.

It is interesting to remark here that the the Laplace transform is considered, recently, in terms of Henstok-Kurzweil integral (see [171])

Definition 2.5. The Laplace transform

Given a function f defined for $0 < t < \infty$, the Laplace transform F is defined as

$$F(s) := \pounds[f(t)] = \int_0^\infty f(t)e^{-st}dt, \tag{27}$$

at least for those s for which the integral converges.

Properties of Laplace transform

For $\pounds[f(t)] = F(s)$ and $\pounds[g(t)] = G(s)$ we have the following useful properties of Laplace transform which are applied in this research

1.
$$\pounds[f(t) + g(t)] = F(s) + G(s),$$

2.
$$\pounds[t^\beta] = \frac{\Gamma(\beta+1)}{s^{\beta+1}}, \qquad \beta > -1, \tag{28}$$

3.
$$\pounds[t^{n-\frac{1}{2}}] = \frac{1.3.5.\cdots.(2n-1)\sqrt{\pi}}{2^n s^{n+\frac{1}{2}}} = \frac{(2n)!\sqrt{\pi}}{2^{2n}(n!)s^{n+\frac{1}{2}}},\ n = 1, 2, 3, \cdots \tag{29}$$

4.
$$\pounds[D^n f(t)] = s^n F(s) - \sum_{k=0}^{n-1} s^{n-(k+1)} D^k f(t), \tag{30}$$

5.
$$\pounds[t^n f(t)] = (-1)^n F^{(n)}(s),$$

6.
$$\pounds\left[\int_0^t f(x)dx\right] = \frac{F(s)}{s},$$

7.
$$\pounds\left[\int_0^t f(t-x)g(x)dx\right] = F(s)G(s).$$

8.
$$\pounds\left[\frac{e^{-\zeta^2/4t}}{\sqrt{\pi t}}\right] = \frac{e^{-\zeta\sqrt{s}}}{\sqrt{s}},\ \zeta > 0,$$

Definition 2.6. The inverse Laplace transform

The function f in (27) is called the inverse Laplace transform of F and will be denoted by $f(t) = \pounds^{-1}[F(s)]$ in the research. In practice when one uses the Laplace transform to, for example, solve a differential equation, one has to at some point invert the Laplace transform by finding the function f

which corresponds to some specified F. The Inverse Laplace Transform of F is defined as:

$$f(t) = \pounds^{-1}[F(s)] := \frac{1}{2\pi i} \lim_{\tau\to\infty} \int_{\sigma-i\tau}^{\sigma+i\tau} e^{st}F(s)ds, \tag{31}$$

Where σ is large enough that F is defined for the real part of $s \geq \sigma$.

Surprisingly, formula (31) is not really useful in applications.

The following lemma is folklore

Lemma 2.7. 1. *If f is continuous $[0, b]$, then*

$$D^m I^m f(t) = f(t), \ \text{for all } t \in [0, b].$$

In addition $D^m I^m f(t) = f(t)$, a.e. $t \in [0, b]$ if $f \in L_1[0, b]$.

2. *If the derivative of order m to the function f is absolutely integrable on $[0, b]$, then*

$$I^m D^m f(t) = f(t) - \sum_{k=0}^{m-1} f^{(k)}(0)\frac{t^k}{k!}, \tag{32}$$

3. *There exists a $\varphi \in L_1[0, b]$ such that*

$$f \in AC^m[0, b] \Leftrightarrow f(t) = I^m\varphi(t) + \sum_{k=0}^{m-1} c_k t^k \tag{33}$$

Here, it is obvious that $\varphi = D^m f$ a.e. and $c_k = f^{(k)}(0)/k!$.

2.2. **Fixed point theorems.** Fixed point theory is an important area in the fast growing fields of nonlinear analysis and nonlinear operators. It is relatively young and vigorously developing area for research.

The theory of fixed point is concerned with the conditions which guarantee that $T : X \to X$ of a topological space X onto itself admit one or more fixed points, that is, points $x \in X$ for which $x = T(x)$. This theory began in (1912) with the work of Brouwer, who proved that any continuous self mapping of a closed unit ball of $n-$dimensional Euclidean space has at least one fixed point. The other fundamental result of Brouwer's fixed point theorems was given by Banach in (1922) which is popularly known as a Banach Contraction Principle or contraction mapping theorem of Banach.

In what follow, we introduce some well-known fixed Point theorems. Firstly, we introduce the classical Banach's fixed point theorem

Theorem 2.11. (Banach fixed point theorem) [59, 65] *Let* $(U, \|\cdot\|)$ *be a nonempty complete normed space. Let* $T : U \to U$ *be a map such that for every* $u, v \in U$, *the relation*

$$\|Tu - Tv\| \leq \lambda \|u - v\|, \ 0 \leq \lambda < 1,$$

holds. Then the operator T *has a unique fixed point* $u \in U$.

Secondly, we introduce Schauder's fixed point theorem [59]

Theorem 2.12. (Schauder fixed point theorem)
Let U *be a convex subset of a Banach space* F, *and* $T : U \longrightarrow U$ *is compact, continuous map (that is, completely continuous map). Then* T *has at least one fixed point in* U.
In other words:
Let U *be compact convex subset of a Banach space* F, *and* $T : U \longrightarrow U$ *is continuous map. Then* T *has at least one fixed point in* U.

The following theorem is an extension of Schauder's fixed point theorem

Theorem 2.13. (Tychonov Fixed Point Theorem) *Let* U *be a closed convex subset of a locally convex space* F, *and* $T : U \longrightarrow U$ *is continuous and* $T(U)$ *is relatively compact in* F. *Then* T *has at least one fixed point in* U.

Theorem 2.14. (Nonlinear alternative of Laray-Schauder type) [59] *Let* U *be an open subset of a closed convex set* D *in a Banach space* E. *Assume* $0 \in U$ *and* $T : \bar{U} \to D$ *is compact. Then either (= at least one of the following holds)*
(A_1) *:* T *has a fixed point in* $\bar{U}$, *or*
(A_2) *: There exists* $\gamma \in (0, 1)$ *and* $x \in \partial U$ *such that* $x = \gamma T x$.

Based on the Leray-Schauder principle, with $D = E$, we have

Theorem 2.15. (Rothe Fixed Point Theorem) [59] *Let* U *be an open and bounded subset of a Banach space* E, *let* $T : \bar{U} \to E$ *be compact. Then* T *has a fixed point if the following condition holds*

$$T(\partial U) \subseteq \bar{U}. \tag{34}$$

We conclude this section by stating the following fixed point theorem which is a trivial application of the fixed point index of a compact operator $T : Q \to K$ in the "Absolute neighborhood retract" of a closed convex cone K on Q where $Q = B_{r_0} \cap K$ (for details, *cf.* e.g. [59, 190, 189]).

Theorem 2.16. *Let* K *be the closed convex cone. Suppose that* $T : Q \to K$ *is compact, continuous map. Then either (= at least one of the following holds)*
(A_1) T *has a fixed point in* $\overline{Q}$, *or*

(A_2) *There is $\gamma \in (0,1)$ and $x \in K \cap \partial B_{r_0}$ (i.e. in the relative boundary of Q with respect to K) such that $x = \gamma T x$.*

Proof. If (1) and (2) are both false, the homotopy $H(t,x) := tTx$ has no fixed point on the relative boundary of Q with respect to K, and so

$$\mathrm{ind}_K(T,Q) = \mathrm{ind}_K(H(0,\cdot),Q) = \mathrm{ind}_K(H(1,\cdot),Q) = \mathrm{ind}_K(I,Q) = 1,$$

i.e. T has a fixed point in Q that contradicting the assumption. □

Theorem 2.17. ***(Emmanuele's Fixed Point Theorem)*** [72]
Let Q be a nonempty, closed, bounded, compact in measure and convex subset of a real Banach space $L_1(0,1)$ and $T : Q \longrightarrow Q$ be $\beta -$ condensing i.e., it is continuous and have the property that there is a constant $\xi \in [0,1)$ such that $\beta(TX) \leq \xi\beta(X)$ for any subset X of Q, where $\beta(X)$ is the De Blasi measure of weak noncompactness (see [58]*) given by the formula* [7],

$$\beta(X) := \lim_{\epsilon \to 0} \sup_{x \in X} \left(\sup_{D} \left[\int_D |x(t)|dt : D \subset (0,1), measD \leq \epsilon \right] \right). \tag{35}$$

Then T has at least one fixed point in Q.

Remark 2.1. To apply the Emmanuele's Fixed Point theorem, we usually make use of Theorem 2.10.

Definition 2.7. A Banach space C endowed with a closed cone Q is an ordered Banach space(C,Q) with partial order " $\leq$ " in C as follows $x \leq y$, if $y - x \in Q$.

Theorem 2.18. *(Krasnoselskii's fixed point Theorem.) Let (C,Q) be an order Banach space. Let Q_1 and Q_2 be open subsets of C with $\theta \in Q_1$ and $\bar{Q}_1 \subset Q_2$ and let $T : Q \cap (\bar{Q}_2/Q_1) \to Q$ be completely continuous. Further suppose either*

1- $\|Tx\| \leq \|x\|$ *for* $x \in Q \cap \partial Q_1$ *and* $\|Tx\| \geq \|x\|$ *for* $x \in Q \cap \partial Q_2$,
2- $\|Tx\| \geq \|x\|$ *for* $x \in Q \cap \partial Q_1$ *and* $\|Tx\| \leq \|x\|$ *for* $x \in Q \cap \partial Q_2$.

Then T has a fixed point in $Q \cap (\bar{Q}_2 \cap /Q_1)$.

This seems to be a good place to recall some important fact about the space $BV[a,b]$ of the functions of bounded variation on $[a,b]$, which, strictly speaking, is not related to the above results: The pair $(BV[a,b], \|\cdot\|_{BV})$, where $\|\varphi\|_{BV} := Var[\varphi,[a,b]]$ is not complete: In fact $\|\cdot\|_{BV}$ is seminorm but it is not a norm because variation of any constant function equals 0.
However, if we define $\|\varphi\|_{BV} := |\varphi(a)| + Var[\varphi,[a,b]]$, we obtain a norm on $BV[a,b]$:

(I) If $\varphi = 0$, then $\|\varphi\|_{BV} = 0$ (seminorm),

(II) If $\|\varphi\|_{BV} = 0$, then $\varphi(a) = 0$ and $Var[\varphi, [a, b]] = 0$. This occurs only when φ is the constant zero function.

Thus the pair $\left(BV_0[a, b], \|\cdot\|_{BV_0}\right)$, where $\|\varphi\|_{BV_0} := Var[\varphi, [a, b]]$ is complete.

2.3. **Gauge integrals.** *Also known as the generalized Riemann integral, the Henstock integral, the kurzweil integral, the Henstock-kurzweil integral, the HK-integral, the Denjoy-Perron integral, etc.*

The theory of the Riemann integral was not fully satisfactory. Many important functions do not have a Riemann integral – even after we extend the class of integrable functions slightly by allowing "improper" Riemann integrals. Moreover, even for integrable functions, it is difficult to prove good convergence theorems using only the tools ordinarily associated with Riemann integrals. A pointwise, bounded limit of Riemann integrable functions is not necessarily Riemann integrable. (For instance, since the rationals are enumerable, the characteristic function of the rationals can be represented as the pointwise limit of a sequence of characteristic functions of finite sets.)

In 1902, Henri Lebesgue devised a new approach to integration, overcoming many of the defects of the Riemann integral. Lebesgue's definition is appreciably more complicated, but Lebesgue's techniques yield better convergence theorems and, for the most part, more integrable functions. The Lebesgue integral has become the "standard" integral in our graduate courses in analysis.

The Lebesgue integral is strictly more general than the proper Riemann integral – i.e., it can integrate a wider class of functions. However, in comparing the improper Riemann integral with the Lebesgue integral, we find that neither is strictly more general than the other.

Neither the improper Riemann integral nor the Lebesgue integral yielded a fully satisfactory construction of antiderivatives. Slightly more satisfactory answers – i.e., more general notions of integral – were given by Arnaud Denjoy (1912) and Oskar Perron (1914). Denjoy's and Perron's definitions turned out to be equivalent; both were very complicated.

Decades later, independently, Ralph Henstock (1955) and Jaroslav Kurzweil (1957) found a much simpler formulation of the Denjoy-Perron integral. In fact, the Henstock-Kurzweil formulation – the Gauge integral – is considerably simpler than the Lebesgue idea, and its definition is only slightly different from the definition of the Riemann integral. Consequently, interest in this integral has been rising over the last few decades, and some mathematicians have even advocated that we should teach the Gauge integral either alongside or in place of either the Riemann integral or the Lebesgue integral.

Defects of Riemann/ Lebesgue Integrals:

1. The Riemann integrable function must be bounded. Thus, the function $f : [0,1] \to \mathbb{R}$ defined by $f(t) := \frac{1}{\sqrt{t}}$ is not Riemann integrable on [0,1],
2. Lebesgue integral must be absolutely convergent, that is $\int \mid f(t) \mid dt < \infty$. So, the integral
$$\int_0^1 \frac{\sin(1/t)}{t^\eta} dt, \;\; \eta \in [1,2) \tag{36}$$
can not be Lebesgue (hence not Riemann) integrable on [0,1].
3. **Fundamental Theorem of Calculus:** The following is false

   If $F : [a,b] \to \mathbb{R}$, is continuous and F' exists on (a,b). Then F' is Reimann (or Lebesgue) integrable and
$$\int_a^t F'(s)\, ds = F(t) - F(a), \quad \text{for all } t \in [a,b].$$

Example: Define the continuous function $F : [0,1] \to \mathbb{R}$ by
$$F(t) := \begin{cases} t^2 \sin\left(\frac{1}{t^2}\right), & t \in (0,1] \\ 0, & x = 0 \end{cases}$$

$$F'(t) = \begin{cases} 2t\left[\sin\left(\frac{1}{t^2}\right) - \left(\frac{1}{t^2}\right)\cos\left(\frac{1}{t^2}\right)\right], & t \in (0,1] \\ 0, & t = 0 \end{cases} \tag{37}$$

F' is neither Lebesgue nor Riemann because
$$\int_0^1 |F'(s)|\, ds = \infty.$$

Before embarking on the next section, we gather together the main properties of Gauge integral (see e.g. [30] and [197] for the Definition and the properties)

Basic Properties of Gauge integral

1. Gauge integrable functions are measurable
2. Any Lebesgue integrable function is Gauge integrable, and the two integrals coincide within this class of functions.

3. The small gap between $G[a,b]$ and $L_1[a,b]$ contains the highly oscillatory functions whose indefinite integral is not of bounded variations. Recall that the function $f \in G[a,b]$ is Lebesgue on $[a,b]$ if, and only if $F := \int_a^t f \in BV[a,b]$. In this case
$$\int_a^b |f| = Var(F[a,b]) < \infty,$$
where $Var(F[a,b])$ is the variation of F on $[a,b]$
4. The normed space $(G[a,b], \|\cdot\|_G)$, is not complete, where
$$\|f\|_G := \sup_{J\subset[a,b]} \left|\int_J f(t)dt\right| \geq \left|\int_a^b f(t)dt\right|.$$
5. $f, |f| \in G[a,b]$ if, and only if $f \in L_1[a,b]$. Therefore, the non negative Gauge integrable functions are also Lebesgue integrable.
6. The well-known property that, if $f \in L_1[a,b]$ then $f\chi_E \in L_1[a,b]$ for each measurable $E \subset [a,b]$, need not to hold for Gauge integrable functions. However, if $f \in G[a,b]$, then $f \in G[E]$ for any measurable $E \subset [a,b]$ but, $f\chi_E$ is not necessary Gauge integrable on $[a,b]$.
7. A Gauge integrable functions on $[a,b]$ is Lebesgue integrable on some sub-interval $[u,v]$ of $[a,b]$.
8. The measurable function f is Gauge integrable on $[a,b]$ if, and only if, there exists $\alpha, \beta \in G[a,b]$ such that
$$\alpha(t) \leq f(t) \leq \beta(t), \ a.e \text{ on } [a,b].$$
For example, the function $f : [0,1]\times\mathbb{R} \to \mathbb{R}$ given by $f(t,x) := t^2 \sin x + t^{-\eta}\sin(1/t)$, $\eta \in [1,2)$ is Gauge (but neither Lebesgue nor Riemann) integrable on $[0,1]$ for every $x \in \mathbb{R}$. To see this, let $\alpha(t) = -t^2 + t^{-\eta}\sin(1/t)$ and $\beta(t) = t^2 + t^{-\eta}\sin(1/t)$.

9. (Test of Gauge integrability): If f and g are Gauge integrable on $[a,b]$ such that $|f(t)| \leq g(t)$ for all $t \in [a,b]$, then $f \in L_1[a,b]$. Consequently the non-negative Gauge integrable functions on $[a,b]$ are "in fact" Lebesgue integrable on $[a,b]$ (note, in this case, that $|f(t)| = f(t)$ for all $t \in [a,b]$).
10. Even for the continuous function $g \in C[a,b]$, the product fg, where $f \in G[a,b]$, need not to be Gauge integrable on $[a,b]$. However, if $g \in BV[a,b]$ then $fg \in G[a,b]$. In particular, if $g \in BV_0[a,b] := \{\varphi \in BV[a,b] : \varphi(a) = 0\}$, then $fg \in G[a,b]$. However, if $F := \int_a^t f$ we have
$$\int_a^b fg = \int_a^b g dF = F(b)g(b) - \int_a^b F dg, \tag{38}$$

and

$$\left|\int_a^b fg\right| \le \inf_{t\in[a,b]} |g(t)| \left|\int_a^b f\right| + \|f\|_{G[a,b]} Var(g,[a,b]), \tag{39}$$

where $Var(g,[a,b])$ is the variation of g on $[a,b]$.

11. $G^*[a,b]$ " the dual of $G[a,b]$" isometrically isomorphic to the space all multiplies of the functions on $G[a,b]$. In particular, the space $BV[a,b]$ is a proper subset of the conjugate space of $G^*[a,b]$.
12. There is no improper Gauge integral (Hake's Theorem).
13. The Fundamental Theorem (I): If $f:[a,b]\to\mathbb{R}$ is a.e. derivative of continuous function $F\in C[a,b]$ (where the null set of non differentiability is at most countable) then $f\in G[a,b]$ and
$$\int_a^t f(s)\,ds = F(t)-F(a),\ t\in[a,b].$$
14. The Fundamental Theorem (II): If $f:[a,b]\to\mathbb{R}$ is Gauge integrable, then any indefinite integral F is continuous and
$$F' = f,\ a.e. \text{ on } [a,b].$$

Example: Given a sequence $\{c_n\}$. Define the function $f:[0,1]\to\mathbb{R}$, by

$$f(t) := \begin{cases} c_n & t\in[\frac{1}{n+1},\frac{1}{n}), \\ 0 & \text{else}. \end{cases}$$

Let $a_n = c_n\left(\frac{1}{n+1}-\frac{1}{n}\right)$ so that

$$\int_\epsilon^1 f(s)ds = \sum_{k\le\epsilon^{-1}}^{\infty} a_k + c_{[1/\epsilon]}\left(\frac{1}{[1/\epsilon]}-\epsilon\right),\ \epsilon\in(0,1). \tag{40}$$

By Hake's Theorem, it follows that $f\in G[0,1]$ if, and only if, $\sum_k^\infty a_k$ is convergent (f is even in $L_1[0,1]$ if, and only if, $\sum_k^\infty |a_k|<\infty$) (see [30] Example 2.8).

Example: ([183]) Let $\{a_n\}\in\ell_\infty$. Define the continuous function f : $[0,\infty]\to\mathbb{R}$ by

$$f(t) := \sum_{k=1}^{\infty} a_k \chi_{[k-1,k)}. \tag{41}$$

Then $f\in G[0,\infty]$ if, and only if $\sum a_k$ is convergent. In particular, if $\sum a_k$ is absolutely convergent, then $f\in L_1[0,\infty]$.

The following results play an important rule in our investigation of the fractional calculus in Gauge space

Proposition 2.4. *(cf.* [197] *Theorem 2.4.8)*
If $f \in G[a,b]$, then given $\epsilon > 0$ there exists $\delta > 0$ such that

$$\left| \int_{[u,v]} f \right| < \epsilon, \text{ whenever } [u,v] \subset [a,b] \text{ with } \mu([u,v]) < \delta.$$

Theorem 2.19. *(cf.* [88] *or Theorem 4.2.9 and Lemma 4.4.3 in* [197]*)*
Let $f \in G[a,b]$, then

1. *There exists an increasing sequence of $\{X_n\}$ of closed subsets of $[a,b]$ such that $\bigcup_{n=1}^{\infty} X_n = [a,b]$, the sequence $\{f\chi_{X_n}\}$ is in $L_1[a,b]$ and*
$$\lim_{n\to 0} \|f - f\chi_{X_n}\|_{G[a,b]} = 0.$$
In particular
$$(G)\int_a^t f(s)\,ds = \lim_{n\to\infty} (L)\int_a^t f\chi_{X_n}(s)\,ds = \lim_{n\to\infty} (L)\int_{X_n\cap[a,t]} f(s)\,ds,$$
uniformly on $[a,b]$.
2. *There exists a sequence $\{(a_i,b_i)\}$ of pairwise disjoint open sub-intervals of (a,b) such that $f\chi_{[a,b]/\cup_i(a_i,b_i)} \in L_1[a,b]$ and the series $\sum_{i=1}^{\infty} \|f\|_{L_1[a_i,b_i]}$ converges.*

Theorem 2.20. *(Dominated convergence theorem for Gauge integrals)*
Let $\{f_n\}$ be a sequence in $G[a,b]$ such that $f = \lim_{n\to\infty} f_n$ a.e. on $[a,b]$. Suppose that there exists Gauge integrable functions ζ and η such that

$$\zeta(t) \le f_n(t) \le \eta(t) \text{ a.e. on } [a,b] \text{ and for all } n \in \mathbb{N}.$$

Then $f \in G[a,b]$ and

$$\lim_{n\to\infty} \int_a^b f_n = \int_a^b f.$$

As a simpler substitute for a Fubini-type theorem for the Gauge integral, we have the following integration-by-parts formula from [197, Lemma 6.1.2]

Theorem 2.21. *If $f \in G[a,b]$ and $g \in L_1[a,b]$, then $f(\cdot)\int_a^{(\cdot)} g(s)\,ds \in G[a,b]$ and*

$$(G)\int_a^b \left(f(t) \int_a^t g(s)\,ds \right) dt = \int_a^b \left((G)\int_s^b f(t)dt \right) g(s)\,ds.$$

3. Fractional calculus of real-valued functions

3.1. **Integrals of fractional order.** The term fractional is a misnomer, as fractional calculus refers to any non-integer order integro-differentiation, whether that be rational, irrational or complex. This section concentrates on the generalization of the integral discovered by Riemann and Liouville (hence the name). It is the simplest example of fractional calculus (others do exist, see the Appendix section) and also probably, because of its applications, the most important in one dimension.

The notation of fractional integral in a natural consequence of the well-known formula, that reduces the calculation of the n-fold integral of a function f to a single integral of convolution type. In fact, for the function f defined on $(0, \infty)$, let's define

$$If(t) := \int_0^t f(s)\ ds.$$

Repeating process gives

$$\begin{aligned} I^2 f(t) &= \int_0^t If(s)\, ds = \int_0^t \left(\int_0^s f(\theta) d\theta \right) ds \\ &= \int_0^t \int_0^s f(\theta) d\theta\, ds. \end{aligned}$$

Interchanging the order of integration implies

$$I^2 f(t) = \int_0^t \int_\theta^t f(\theta)\, ds d\theta = \int_0^t (t-\theta) f(\theta) d\theta$$

Analogously, we have

$$\begin{aligned} I^3 f(t) &= \int_0^t \left(\int_0^s (s-\theta) f(\theta) d\theta \right) ds \\ &= \int_0^t \frac{(t-\theta)^2}{2} f(\theta)\ d\theta. \end{aligned}$$

Thus

$$I^3 f(t) = \int_0^t \frac{(t-\theta)^2}{2!} f(\theta)\ d\theta$$

This can be extended arbitrarily to obtain

$$I^n f(t) = \int_0^t \frac{(t-\theta)^{n-1}}{(n-1)!} f(\theta) d\theta,\ n = 1, 2, \cdots$$

which leads to a straightforward way using the Gamma function to remove the discrete nature of the fractional function. Indeed noting that $(n-1)! =$

$\Gamma(n)$, and introduce the arbitrary positive real number α, one can define the fractional integral of order α by

Definition 3.1. Definition of fractional integrals

The fractional (arbitrary) order integral of the function $f \in L_1[0,b]$ of order $\alpha > 0$ is defined by

$$I^\alpha f(t) := \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)\ ds.$$

For complementation, we define $I^0 f := f$.

It is nice to remark that, since the Gamma function is defined for all $\mathbb{C}/\{0,-1,\cdots\}$, we can define fractional calculus of complex order, however, in this book, we will only consider real orders.

As a direct consequence of a classical result on Lebesgue integration ([194], Theorem 4.2d) we arrive at

Lemma 3.1. *The definition of fractional integral operator is satisfied in any point for the continuous functions and in almost every point for the absolutely integrable functions.*

Evidently, Lemma 3.1 is also a direct consequence of the Young's inequality (*cf.* Proposition 2.1):

If we define $\psi : [0,b] \to \mathbb{R}$ by

$$\psi(s) := \begin{cases} (t-s)^{\alpha-1}, \ s \in [0,t]\ ,\ t > 0 \\ 0, \ \text{otherwise.} \end{cases}$$

it follows that $\psi \in L_\infty[0,b]$ when $\alpha \geq 1$. Also when $0 < \alpha < 1$, we have $\int_0^b |\psi(u)| du < \infty$, meaning that $\psi \in L_1[0,b]$. The result now is direct consequence of the long-known inequality due to Young (*cf.* Proposition 2.1).

Before we come to a detailed study of the mathematical properties of fractional integral operator, let us take a brief look at some simple examples

Example 3.1. Let $f(t) = c$ (a constant function). According to the definition of fractional integral, we have

$$\begin{aligned} I^\alpha c &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\ c\ ds = \frac{c}{\alpha\Gamma(\alpha)} \left[\frac{(t-s)^\alpha}{-1}\right]_o^t \\ &= \frac{c\, t^\alpha}{\Gamma(1+\alpha)} \end{aligned}$$

Therefore

$$I^{\frac{1}{2}}5 = \frac{5t^{\frac{1}{2}}}{\Gamma(1+\frac{1}{2})} = \frac{5\sqrt{t}}{\frac{1}{2}\Gamma(\frac{1}{2})} = \frac{10\sqrt{t}}{\sqrt{\pi}}.$$

Example 3.2. If $f(t) = t$, then

$$I^{\alpha}t = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\, s\, ds$$

using integration by part with $u = s,\ dv = (t-s)^{\alpha-1}\ ds$ implies

$$\begin{aligned} I^{\alpha}t &= \frac{1}{\Gamma(\alpha)}\left[\left(\frac{s(t-s)^{\alpha}}{-\alpha}\right)_0^t - \int_0^t \frac{(t-s)^{\alpha}}{-\alpha} ds\right] \\ &= \frac{-1}{\Gamma(1+\alpha)}\left[\left(\frac{(t-s)^{\alpha+1}}{(\alpha+1)}\right)_0^t\right] \\ &= \frac{1}{\Gamma(1+\alpha)}\left[\frac{t^{\alpha+1}}{(\alpha+1)}\right] = \frac{t^{1+\alpha}}{\Gamma(2+\alpha)}. \end{aligned}$$

Therefore

$$I^{\frac{1}{2}}t = \frac{t^{\frac{3}{2}}}{\Gamma(2+\frac{1}{2})} = \frac{t^{\frac{3}{2}}}{\frac{3}{2}\Gamma(\frac{3}{2})} = \frac{t^{\frac{3}{2}}}{\frac{3}{2}\frac{\sqrt{\pi}}{2}} = \frac{4t^{\frac{3}{2}}}{3\sqrt{\pi}}.$$

In the view of the above examples, one expect a natural generalization of the well-know properties of the integral of the power functions. However, the following result is precisely what one would expect from a sensible generalization of the integral operator.

Lemma 3.2. If $\alpha > 0, \gamma > -1$, then

$$I^{\alpha}t^{\gamma} = \frac{\Gamma(1+\gamma)}{\Gamma(1+\gamma+\alpha)}\, t^{\alpha+\gamma}.$$

Proof. Firstly, we remark that our assumption that $\gamma > -1$ result in $t \to t^{\gamma} \in L_1$, and so $I^{\alpha}t^{\gamma}$ makes sense (*cf.* Lemma 3.1).
Now, for the arbitrary value of $\alpha > 0$, we have by the definition of fractional integrals

$$I^{\alpha}t^{\gamma} = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\, s^{\gamma}\, ds = \frac{1}{\Gamma(\alpha)}\int_0^t t^{\alpha-1}\left(1-\frac{s}{t}\right)^{\alpha-1} s^{\gamma}\, ds$$

Putting $s = ut$ implies

$$I^{\alpha}t^{\gamma} = \frac{t^{\alpha-1}}{\Gamma(\alpha)}\int_0^1 (1-u)^{\alpha-1}\, (ut)^{\gamma} t du = \frac{t^{\alpha+\gamma}}{\Gamma(\alpha)}\int_0^1 (1-u)^{\alpha-1}\, u^{\gamma} du.$$

By the definition and the properties of the Beta-function we obtain

$$I^{\alpha}t^{\gamma} = \frac{t^{\alpha+\gamma}}{\Gamma(\alpha)}\beta(\alpha,\gamma+1) = \frac{\Gamma(1+\gamma)}{\Gamma(1+\gamma+\alpha)}\, t^{\alpha+\gamma}$$

□

As an immediate consequence of Lemma 3.2 is the following

Example 3.3.

$$I^{\frac{1}{2}}t^{2} = \frac{16}{15}\sqrt{\frac{t^5}{\pi}}, \text{ and } It^{\gamma} = \frac{t^{1+\gamma}}{1+\gamma}, \ \gamma > -1$$

Example 3.4. Let $\alpha \in (1,2)$. Based on Lemma 3.2, an explicit calculation shows that the functions

$$x(t) := 1 \pm \frac{\sqrt{\Gamma(1-\alpha/2)}}{\sqrt{\Gamma(1+\alpha/2)}} t^{\frac{\alpha}{2}}, \ t \in [0,1] \tag{42}$$

are solutions to the integral equation

$$x(t) = 1 + I^{\alpha}\left(\frac{1}{x(t)-1}\right), \ \alpha \in (1,2), \ t \in [0,1]. \tag{43}$$

Thus we observe that the integral equation formulation has more than one continuous solution. However we also see that neither of these solutions is differentiable at $t = 0$.

Now we introduce an example where the continuity condition imposed on the function is not fulfilled

Example 3.5. Let us consider the following function $f : [0,2] \to \mathbb{R}$ with one point of discontinuity

$$f(t) := \begin{cases} t, & t \in [0,1), \\ 1-t, & t \geq 1. \end{cases} \tag{44}$$

Of course, $I^{\alpha}f$ makes sense because $f \in L_1[0,2]$.
Now, there is no problem to find $I^{\alpha}f$ in case $t < 1$ (we use Lemma 3.2), but we can easily see the influence of the discontinuity during calculation of the function f for $t \geq 1$:
Indeed, for $\alpha > 0$, we have

$$I^{\alpha}f(t) = \begin{cases} \frac{t^{1+\alpha}}{\Gamma(2+\alpha)}, & t \in [0,1), \\ \frac{1}{\Gamma(\alpha)}\left[\int_0^1 (t-s)^{\alpha-1}s \ ds + \int_1^t (t-s)^{\alpha-1}(1-s) \ ds\right], & t \geq 1. \end{cases}$$

$$
= \begin{cases} \frac{t^{1+\alpha}}{\Gamma(2+\alpha)}, & t \in [0,1), \\ \dfrac{1}{\Gamma(\alpha)} \left[\displaystyle\int_{t-1}^{t} \zeta^{\alpha-1}(t-\zeta)\ d\zeta + \int_{1}^{t} (t-s)^{\alpha-1}(1-s)\ ds \right], & t \geq 1, \end{cases}
$$

where we use the substitution $s = t - \zeta$. Now putting $s - 1 = u(t-1)$ implies

$$
I^\alpha f(t) = \begin{cases} \frac{t^{1+\alpha}}{\Gamma(2+\alpha)}, \ t \in [0,1), \\ \dfrac{1}{\Gamma(\alpha)} \left[\displaystyle\int_{t-1}^{t} \zeta^{\alpha-1}(t-\zeta)\ d\zeta - \int_{0}^{1} (1-u)^{\alpha-1} u(t-1)^{\alpha+1}\ du \right], t \geq 1. \end{cases}
$$

By the definition and the properties of the Beta-function we obtain

$$
I^\alpha f(t) = \begin{cases} \frac{t^{1+\alpha}}{\Gamma(2+\alpha)}, & t \in [0,1), \\ \dfrac{1}{\Gamma(\alpha)} \left[\left(t\dfrac{\zeta^\alpha}{\alpha} - \dfrac{\zeta^{\alpha+1}}{\alpha+1} \right)_{t-1}^{t} - (t-1)^{\alpha+1} B(\alpha, 2) \right], & t \geq 1. \end{cases}
$$

Therefore

$$
I^\alpha f(t) = \begin{cases} \frac{t^{1+\alpha}}{\Gamma(2+\alpha)}, & t \in [0,1), \\ \left(t\dfrac{\zeta^\alpha}{\Gamma(1+\alpha)} - \dfrac{\zeta^{\alpha+1}}{\Gamma(\alpha)(1+\alpha)} \right)_{t-1}^{t} - \dfrac{(t-1)^{\alpha+1}}{\Gamma(\alpha+2)}, & t \geq 1. \end{cases}
$$

Putting in mind that

$$
\begin{aligned}
\left(t\frac{\zeta^\alpha}{\Gamma(1+\alpha)} - \frac{\zeta^{\alpha+1}}{\Gamma(\alpha)(1+\alpha)} \right)_{t-1}^{t} &= \left(\frac{t^{\alpha+1}}{\Gamma(1+\alpha)} - \frac{t^{\alpha+1}}{\Gamma(\alpha)(1+\alpha)} \right) \\
&\quad - \left(\frac{t(t-1)^\alpha}{\Gamma(1+\alpha)} - \frac{(t-1)^{\alpha+1}}{\Gamma(\alpha)(1+\alpha)} \right) \\
&= t^{\alpha+1} \frac{(1+\alpha) - \alpha}{\Gamma(2+\alpha)} - \frac{(t-1)^\alpha}{\Gamma(\alpha)} \left[\frac{t(1+\alpha) - \alpha(t-1)}{\alpha(\alpha+1)} \right] \\
&= \frac{t^{\alpha+1}}{\Gamma(2+\alpha)} - \frac{(t-1)^\alpha}{\Gamma(2+\alpha)} [\alpha + t],
\end{aligned}
$$

implies

$$
I^\alpha f(t) = \begin{cases} \frac{t^{1+\alpha}}{\Gamma(2+\alpha)}, & t \in [0,1), \\ \dfrac{t^{1+\alpha}}{\Gamma(2+\alpha)} - \dfrac{(t-1)^\alpha(\alpha + 2t - 1)}{\Gamma(2+\alpha)}, & t \geq 1. \end{cases} \tag{45}
$$

We remark that although f is not continuous at $t = 1$, the map $I^\alpha f$ is continuous at this point. Indeed $I^\alpha f \in AC[0,2]$. This gives a reason to believe that $I^\alpha f$ is better than $f \notin C[0,1]$

Example 3.6. Let us consider the function $f(t) = (1-t)^{-1/2},\ t \in (0,1)$. We have

$$I^{\frac{1}{2}} \frac{1}{\sqrt{1-t}} = \frac{1}{\sqrt{\pi}} \int_0^t \frac{1}{\sqrt{(t-s)(1-s)}}\, ds,$$

A little experimentation with the substitution

$$t - s = \zeta^2(1-s) \Leftrightarrow s = (t-\zeta^2)/(1-\zeta^2)$$

yields

$$I^{\frac{1}{2}} \frac{1}{\sqrt{1-t}} = \frac{2}{\sqrt{\pi}} \int_0^{\sqrt{t}} \frac{1}{(1-\zeta^2)} d\zeta = \frac{2}{\sqrt{\pi}} \tanh^{-1} \sqrt{t}.$$

That is

$$I^{\frac{1}{2}} \frac{1}{\sqrt{1-t}} = \frac{1}{\sqrt{\pi}} \ln \frac{1+\sqrt{t}}{1-\sqrt{t}}. \tag{46}$$

Example 3.7. Let $\alpha, \beta > 0$ and $f(t) = t^{\beta-1} \ln t$. Then

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} s^{\beta-1} \ln s \ \ ds.$$

The change $s = ut$ of variables implies

$$\begin{aligned} I^\alpha f(t) &= \int_0^1 \frac{(t-ut)^{\alpha-1}(ut)^{\beta-1}\ln(ut)}{\Gamma(\alpha)} t du \\ &= \int_0^1 \frac{(1-u)^{\alpha-1} t^{\alpha+\beta-1} u^{\beta-1} \ln(ut)}{\Gamma(\alpha)} du \\ &= \frac{t^{\alpha+\beta-1}}{\Gamma(\alpha)} \int_0^1 \left[(1-u)^{\alpha-1} u^{\beta-1} (\ln u + \ln t)\right] du \\ &= \frac{t^{\alpha+\beta-1}}{\Gamma(\alpha)} (c_2 + c_1 \ln t), \end{aligned}$$

where

$$c_1 = \int_0^1 (1-u)^{\alpha-1} u^{\beta-1} du = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)},\ c_2 = \int_0^1 (1-u)^{\alpha-1} u^{\beta-1} \ln u \ du$$

Here c_2 may evaluated by differentiating c_1 with respect to β. Thus

$$\begin{aligned} c_2 &= \frac{\Gamma(\alpha+\beta)\Gamma(\alpha)\Gamma'(\beta) - \Gamma'(\alpha+\beta)\Gamma(\alpha)\Gamma(\beta)}{(\Gamma(\alpha+\beta))^2} \\ &= \frac{\Gamma(\alpha)\Gamma'(\beta)}{\Gamma(\alpha+\beta)} - \frac{\Gamma'(\alpha+\beta) c_1}{\Gamma(\alpha+\beta)} \\ &= c_1(\Psi(\beta) - \Psi(\alpha+\beta)) \end{aligned}$$

where Ψ is the Euler psi-function defined by $\Psi(u) := \dfrac{\Gamma'(u)}{\Gamma(u)}$. Therefore

$$\begin{aligned} I^{\alpha} t^{\beta-1} \ln t &= \frac{c_1 t^{\alpha+\beta-1}}{\Gamma(\alpha)}((\Psi(\beta) - \Psi(\alpha+\beta)) + \ln t) \\ &= \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)} t^{\alpha+\beta-1} \left[\ln t + \Psi(\beta) - \Psi(\alpha+\beta)\right]. \end{aligned}$$

In particular if $\beta = 1$, we have

$$I^{\alpha} \ln t = \frac{t^{\alpha}}{\Gamma(1+\alpha)} \left[\ln t + \Psi(1) - \Psi(1+\alpha)\right], \tag{47}$$

further, if $\beta = 2$, we obtain

$$I^{\alpha}(t \ln t) = \frac{t^{\alpha+2}}{\Gamma(1+\alpha)} \left[\ln t + \Psi(2) - \Psi(2+\alpha)\right]. \tag{48}$$

We ask the reader to check the following properties of the operator I^{α}

Let $\alpha, \beta > 0$ $\lambda \in \mathbb{R}$ and $f, g \in L_1$, then we have the following

1. Interpolation (continuity):
$$\lim_{\alpha \to n} I^{\alpha} f(t) = I^{n} f(t), \; n \in \mathbb{N},$$
2. Linearity:
$$I^{\alpha}\left[a\, f(t) + b\, g(t)\right] = a\, I^{\alpha} f(t) + b\, I^{\alpha} g(t), \; a, b \in \mathbb{R},$$
3. Effect on a.e. zero functions: Let $f \in L_1[0, b]$. Then we have
$$I^{\alpha} f = 0 \Longleftrightarrow f \equiv 0 \; a.e., \tag{49}$$
4. Limit at zero (see [67] page 353): If f is bounded measurable function such that $\lim_{t\to 0} f(t)$ exists, then
$$\lim_{t \to 0} t^{-\alpha} I^{\alpha} f(t) = \frac{1}{\Gamma(1+\alpha)} \lim_{t \to 0} f(t). \tag{50}$$

In the following Lemma, we indicate some important properties of the fractional integral operators

Lemma 3.3. *Let $\alpha > 0$. Then the following holds*

1. $I^{\alpha} : L_p[0, b] \to L_p[0, b]$ *is bounded linear (hence continuous) operator for every $p \in [1, \infty]$. That is, the fractional integral operator maps $L_p[0, b]$ continuously into itself. In particular, if $0 < \alpha < 1$, then $I^{\alpha} : L_1[0, b] \to L_{1/(1-\alpha)-\epsilon}[0, b]$ however small $\epsilon > 0$ is.*

2. $I^\alpha : C[0,b] \to C[0,b]$ *is bounded linear (hence continuous) operator. That is, the fractional integral operator maps* $C[0,b]$ *continuously into itself. In addition,* $I^\alpha f(0) := \lim_{t\to 0} I^\alpha f(t) = 0$.

Proof. At the beginning, we remark that in all case, the operator I^α makes sense (cf. Lemma 3.1). Now, define $g(t) := \frac{t^{\alpha-1}}{\Gamma(\alpha)}$, then $g \in L_1[0,b]$ and $\|g\| = b^\alpha/\Gamma(1+\alpha)$. we consider the following cases:

1. If $f \in L_p[0,b]$, it follows by Young inequality with $r = p,\ q = 1$ (*cf.* Proposition 2.1) that $I^\alpha f = f * g \in L_p[0,b]$ for every $p \geq 1$. Also

$$\|I^\alpha f\|_p \leq \frac{b^\alpha}{\Gamma(1+\alpha)} \|f\|_p . \tag{51}$$

Thus the boundedness (hence the continuity) of the linear operator I^α is now obvious.

Now, if $0 < \alpha < 1$, it can be easily seen that $\int_0^b (g(t))^q dt < \infty$ for every $q \in [1, 1/(1-\alpha))$. This means that $g \in L_{1/(1-\alpha)-\epsilon}[0,b],\ \epsilon > 0$. So by Young inequality with $r = q = 1/(1-\alpha) - \epsilon,\ p = 1$ it follows that $I^\alpha f = f * g \in L_{1/(1-\alpha)-\epsilon}[0,b]$ however small $\epsilon > 0$ is.

2. Let $f \in C[0,b]$. Since $C[0,b] \subset L_\infty[0,b]$ and $g \in L_1[0,b]$, it follows by Young inequality with $q = 1, p = \infty$ that $I^\alpha f = f * g \in C[0,b]$. Moreover, we have

$$|I^\alpha f(t)| \leq \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \max_{s\in[0,b]} |f(s)|\, ds \leq \frac{t^\alpha \|f\|_{C[0,b]}}{\Gamma(1+\alpha)}. \tag{52}$$

Thus

$$\|I^\alpha f\|_{C[0,b]} \leq c \|f\|_{C[0,b]} , \text{ where } c := \frac{b^\alpha}{\Gamma(1+\alpha)}.$$

Thus the mapping $I^\alpha : C[0,b] \to C[0,b]$ is bounded (hence continuous) linear operator.

Now, to prove that $I^\alpha f$ vanishes at $t = 0$ for any $f \in C[0,b]$, we apply the formula (50) as follows

$$\lim_{t\to 0} I^\alpha f(t) = \lim_{t\to 0} t^\alpha t^{-\alpha} I^\alpha f(t) = 0.$$

This fact also is an immediate consequence of inequality (52).

□

As a direct consequence of part 2 of Lemma 3.3, we have

Corollary 3.1. *If $\alpha > 0$ and $f \in C[0,b]$, then*

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \begin{cases} \displaystyle\int_0^t (t-s)^{\alpha-1} f(s)\, ds & t \in (0,b]. \\ 0 & t = 0. \end{cases}$$

Lemma 3.4. *The fractional integral operator maps, nonnegative a.e. nondecreasing functions continuously into a functions of the same type.*

Proof. Let $\alpha > 0$ and suppose f is nonnegative, a.e. nondecreasing function in $[0,b]$. Then $f \in L_1[0,b]$ and consequently, $I^\alpha f$ exists a.e. on $[0,b]$. Let $t_1,\ t_2 \in [0,b]$. Without loss of generality we may assume that $t_1 \le t_2$. Then $0 \le f(t_1) \le f(t_2)$. With a bit work using the substitution $u = t_1 - s$ we have

$$\begin{aligned} I^\alpha f(t_1) &= \int_0^{t_1} \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)\, ds = \int_{t_1}^0 \frac{u^{\alpha-1}}{\Gamma(\alpha)} f(t_1-u)(-du) \\ &= \int_0^{t_1} \frac{u^{\alpha-1}}{\Gamma(\alpha)} f(t_1-u) du \le \int_0^{t_2} \frac{u^{\alpha-1}}{\Gamma(\alpha)} f(t_2-u) du \\ &= \int_0^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)\, ds = I^\alpha f(t_2). \end{aligned}$$

From which, we deduce that $0 \le I^\alpha f(t_1) \le I^\alpha f(t_2)$. Thus, the function $I^\alpha f$ is nonnegative, nondecreasing function in $[0,b]$.
The boundedness (hence the continuity) of I^α follows immediately by Lemma 3.3, since the set of nonnegative, nondecreasing functions in $[0,b]$ is a subset of $L_1[0,b]$. This completes the proof. □

Example 3.8. Let $\alpha > 0$ and $\beta > 1$. Consider the continuous nonnegative, nondecreasing function $f : [0,1] \to \mathbb{R}^+$ defined by $f(t) = t^{\beta-1} \ln t,\ t \in (0,1]$, $f(0) = 0$. It follows by Example 3.7 that the function $I^\alpha f$ is continuous nonnegative, nondecreasing function in $[0,1]$ (this is precisely one would expect in view of Lemma 3.4).

One important property of integer-order integral operators is preserved by our generalization: Indeed, the so called " Index Law, Commutative property or Semi-group property" of fractional integration operator is given by the following

Lemma 3.5. *(**Semi-group property**) Let $\alpha, \beta > 0$. If $f \in L_1[0,b]$, then $I^\alpha I^\beta f(t) = I^\beta I^\alpha f(t) = I^{\alpha+\beta} f(t)$ is satisfied at almost every point $t \in [0,b]$. This property is true in any point if $f \in C[0,b]$.*
In general, we have

$$(I^\alpha)^n f(t) := I^\alpha I^\alpha \cdots I^\alpha f(t) = I^{n\alpha} f(t),\ \text{a.e. in } [0,b],\ n = 1,2,\cdots.$$

Proof. Let $f \in L_1[0,b]$. We first note that the term " a.e." follows from Lemma 3.1.In the view of Lemma 3.3, we have

$$\begin{aligned} I^\alpha I^\beta f(t) &= \int_0^t \frac{(t-\theta)^{\alpha-1}}{\Gamma(\alpha)} \int_0^\theta \frac{(\theta-s)^{\beta-1}}{\Gamma(\beta)} f(s)\, dsd\theta \\ &= \int_0^t \int_0^\theta \frac{(t-\theta)^{\alpha-1}(\theta-s)^{\beta-1}}{\Gamma(\alpha)\Gamma(\beta)} f(s)\, dsd\theta \end{aligned}$$

whence applying Fubini's theorem may interchange the order of integration

$$I^\alpha I^\beta f(t) = \int_0^t \int_s^t \frac{(t-\theta)^{\alpha-1}(\theta-s)^{\beta-1}}{\Gamma(\alpha)\Gamma(\beta)} f(s) d\theta\, ds.$$

The substitution $\theta = s + u(t-s)$ with a little experimentation yields

$$I^\alpha I^\beta f(t) = \int_0^t f(s)\, ds \int_0^1 \frac{u^{\beta-1}(t-s)^{\beta+\alpha-1}}{\Gamma(\alpha)\Gamma(\beta)} (1-u)^{\alpha-1}\, du.$$

By the properties of Gamma function, we have

$$\int_0^1 u^{\beta-1}(1-u)^{\alpha-1} du = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}.$$

Therefore

$$I^\alpha I^\beta f(t) = \int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\beta+\alpha)} f(s)\, ds = I^{\alpha+\beta} f(t),$$

almost everywhere on $[0,b]$. Similarly $I^\beta I^\alpha f(t) = I^{\alpha+\beta} f(t)$ almost everywhere on $[0,b]$ as required. □

The following lemma allows us to recognize about the functions which for $t > 0$ admit the same fractional integrals.

Lemma 3.6. *For any $\alpha > 0$, we have*

$$I^\alpha f(t) = I^\alpha g(t) \Leftrightarrow f(t) = g(t) \ a.e..$$

Evidently, if we define $z := f - g$, then $I^\alpha z = 0$, (The integrals of lebesgue integrable functions over null sets vanishes). Consequently,

$$0 = I^{1-\alpha} I^\alpha z = \int_0^t z(s)\, ds.$$

The result now follows by the main properties of the Lebesgue integrals.

Definition 3.2. A point $t_0 \in [0,b]$ is said to be a Lebesgue point of a function $f \in L_1[0,b]$ if

$$\lim_{u \to 0} \frac{1}{u} \int_0^u \left[f(t_0 - s) - f(t_0) \right] \ ds = 0$$

It is well-known from the classical books of functional analysis that almost all points $t_0 \in [0,b]$ are Lebesgue points of $f \in L_1[0,b]$. Indeed we have the following lemma (see e.g. [181])

Lemma 3.7. *Let $f \in L_1[0,b]$. Then for almost all points $t \in [0,b]$, we have*

$$\lim_{u\to 0} \frac{1}{u} \int_0^u [f(t-s) - f(t)] \;\; ds = 0.$$

We are in the position to state and prove the following

Theorem 3.1. *Let $f \in L_1[0,b]$. Then*

$$\lim_{\alpha\to 0} I^\alpha f(t) = f(t),$$

for any Lebesgue point of a function f and consequently almost everywhere on $[0,b]$.

Proof. Let $t_0 \in [0,b]$ be a Lebesgue point of f. Define the function $\psi : [0,b] \to \mathbb{R}$ by

$$\Psi(u) := \int_{t_0-u}^{t_0} f(s) \;\; ds = \int_0^u f(t_0 - s) \;\; ds.$$

Thus, $\Psi'(u) = f(t_0 - u)$, and $d\Psi(u) = f(t_0 - u)du$. Also, we have

$$\frac{\Psi(u)}{u} - f(t_0) = \frac{1}{u}\int_0^u [f(t_0 - s) - f(t_0)] \; ds \to 0 \text{ as soon as } u \to 0.$$

Therefore $\Psi(u) = u[f(t_0)+\theta(u)]$, where θ is a continuous function such that $|\theta(u)| < \epsilon$ as soon as $u \in (0, \delta(\epsilon))$. Consequently

$$\begin{aligned}
I^\alpha f(t_0) &= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)\, ds = \int_0^{t_0} \frac{s^{\alpha-1}}{\Gamma(\alpha)} f(t_0 - s)\, ds = \frac{1}{\Gamma(\alpha)} \int_0^{t_0} s^{\alpha-1} d\Psi \\
&= \frac{\Psi(t_0)}{\Gamma(\alpha) t_0^{1-\alpha}} - \left[\frac{\Psi(s)}{\Gamma(\alpha)s^{1-\alpha}}\right]_{s=0} + \frac{1-\alpha}{\Gamma(\alpha)} \int_0^{t_0} \frac{\Psi(s)}{s^{2-\alpha}} \;\; ds \\
&= \frac{\Psi(t_0)}{\Gamma(\alpha) t_0^{1-\alpha}} - \left[\frac{s^\alpha(f(t_0) + \theta(s))}{\Gamma(\alpha)}\right]_{s=0} + \frac{1-\alpha}{\Gamma(\alpha)} \int_0^{t_0} \frac{s(f(t_0) + \theta(s))}{s^{2-\alpha}} \;\; ds \\
&= \frac{1-\alpha}{\Gamma(\alpha)} \int_0^{t_0} s^{\alpha-1}\theta(s)\, ds + \frac{\Psi(t_0)}{\Gamma(\alpha) t_0^{1-\alpha}} + \frac{1-\alpha}{\Gamma(\alpha)} f(t_0) \int_0^{t_0} s^{\alpha-1}\, ds.
\end{aligned}$$

Hence

$$\begin{aligned}
I^\alpha f(t_0) - f(t_0) &= \frac{1-\alpha}{\Gamma(\alpha)} \int_0^{t_0} s^{\alpha-1}\theta(s)\, ds + \frac{\Psi(t_0)}{\Gamma(\alpha) t_0^{1-\alpha}} \\
&+ \left[\frac{(1-\alpha)t_0^\alpha}{\alpha\Gamma(\alpha)} - 1\right] f(t_0).
\end{aligned}$$

Putting in mind that $\lim_{z\to 0}(\Gamma(z))^{-1} = 0$, one may pass to the limit under the last integral sign. So

$$\lim_{\alpha\to 0} |I^\alpha f(t_0) - f(t_0)| \le \lim_{\alpha\to 0} \left[\frac{(1-\alpha)t_0^\alpha}{\Gamma(1+\alpha)} - 1\right] |f(t_0)| = 0.$$

Hence the required result. □

Next we discuss the interchange of the infinite sum operation and fractional integration.

Lemma 3.8. ***(Generalized linearity of the fractional integrals)***

Let $\alpha > 0$ and suppose $f_n \in C[0,b],\ n \in \mathbb{N}$. If the series $\sum_{n=1}^{\infty} f_n(t)$ is uniformly convergent on $[0,b]$. Then term-wise fractional integration is possible, that is

$$I^\alpha\left(\sum_{n=1}^{\infty} f_n(t)\right) = \left(\sum_{n=1}^{\infty} I^\alpha f_n(t)\right).$$

Proof. Define $f := \sum_{n=1}^{\infty} f_n$ and note that the uniformly convergence of the series $\sum_{n=1}^{\infty} f_n$ yields $f \in C[0,b]$. So, for a fixed integer $m \ge 1$ we have

$$\begin{aligned}
\Delta &:= \left|I^\alpha f - \sum_{n=1}^{m} I^\alpha f_n\right| \\
&= \frac{1}{\Gamma(\alpha)}\left|\int_0^t (t-s)^{\alpha-1}\sum_{n=1}^{\infty} f_n(s)\,ds - \sum_{n=1}^{m}\int_0^t (t-s)^{\alpha-1} f_n(s)\,ds\right| \\
&\le \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left|\sum_{n=m+1}^{\infty} f_n(s)\right| ds.
\end{aligned}$$

Since $\left|\sum_{n=m+1}^{\infty} f_n(s)\right| < \epsilon$, it follow $\Delta < \frac{b^\alpha}{\Gamma(1+\alpha)}\epsilon$ which proves the lemma. □

As an immediate consequence of Lemma 3.8 is the following

Example 3.9. Let $\alpha, \lambda \in \mathbb{R}$ and $f(t) = \exp(\lambda t)$. In the case $\alpha \in \mathbb{N},\ \lambda > 0$ we obviously have $I^\alpha f(t) = \lambda^{-\alpha}\exp(\lambda t)$. However, this result does not generalize in a straightforward way to the case $\alpha \notin \mathbb{N}$. Rather, in view of Lemma 3.8, we have

$$I^\alpha e^{\lambda t} = I^\alpha \sum_{n=0}^{\infty} \frac{(\lambda t)^n}{\Gamma(1+n)} = \sum_{n=0}^{\infty} \frac{\lambda^n t^{n+\alpha}}{\Gamma(1+n+\alpha)} = t^\alpha E_{1,1+\alpha}(\lambda t). \tag{53}$$

It is worth to remark here that the series on the right-hand side is not $\exp(\lambda t)$. That is, the fractional integral operators do not reproduce exponential functions in the same way as integrals of integer order do. Incidentally the same problem arises when we compute fractional integrals of other non-polynomial functions that have very simple integer-order integrals such as the logarithmic function. We encourage the reader to work out the details as an exercise.

Example 3.10. Let α and $\lambda \in \mathbb{R}$. Based on the formula (53), we deduce

$$I^{\alpha}\sinh \lambda t = \frac{t^{\alpha}}{2}\left(E_{1,1+\alpha}(\lambda t) - E_{1,1+\alpha}(-\lambda t)\right).$$

and

$$I^{\alpha}\cosh \lambda t = \frac{t^{\alpha}}{2}\left(E_{1,1+\alpha}(\lambda t) + E_{1,1+\alpha}(-\lambda t)\right).$$

Example 3.11. Let us consider Weierstrass function (see the Formula (21))

$$\mathcal{W}_1(t) := \sum_{n=0}^{\infty} f_n(t), \text{ where } f_n(t) = \frac{e^{ib^n t}}{b^n}, \; b > 1. \tag{54}$$

Since $b > 1$, it follows that the series $\sum\limits_{n=1}^{\infty} f_n$ is uniformly convergence, then $\mathcal{W}_1 \in C[0,1]$. By Lemma 3.8 and Formula (53), we get

$$I^{\alpha}\mathcal{W}_1(t) = \sum_{n=0}^{\infty} I^{\alpha}\frac{e^{ib^n t}}{b^n} = \sum_{n=0}^{\infty}\frac{t^{\alpha}}{b^n}E_{1,1+\alpha}(ib^n t).$$

Example 3.12. Let $\alpha,\ \beta > 0$, $\lambda \neq 0$ and $\gamma \geq 0$. We will compute the fractional integrals of Mittag-Leffler type functions multiplied by a suitable power function. For the computation we can use the `generalized` linearity of the fractional integrals given by Lemma 3.8 because Mittag-Leffler functions fulfill all needed conditions. Hence the problem is reduced to the calculation of the fractional integral of the power function which obtained by Lemma 3.2. We have

$$\begin{aligned} I^{\alpha}\left[t^{\beta-1}\ E_{\gamma,\beta}(\lambda t^{\gamma})\right] &= I^{\alpha}\left(\sum_{k=0}^{\infty}\frac{\lambda^k t^{\gamma k+\beta-1}}{\Gamma(\gamma k+\beta)}\right) = \sum_{k=0}^{\infty}\frac{\lambda^k}{\Gamma(\gamma k+\beta)}I^{\alpha}\left(t^{\gamma k+\beta-1}\right) \\ &= \sum_{k=0}^{\infty}\frac{\Gamma(\gamma k+\beta)}{\Gamma(\gamma k+\beta+\alpha)}\frac{\lambda^k t^{\gamma k+\beta-1+\alpha}}{\Gamma(\gamma k+\beta)} = \sum_{k=0}^{\infty}\frac{\lambda^k t^{\gamma k+\beta-1+\alpha}}{\Gamma(\gamma k+\beta+\alpha)} \\ &= t^{\alpha+\beta-1}\sum_{k=0}^{\infty}\frac{\lambda^k t^{\gamma k}}{\Gamma(\gamma k+\beta+\alpha)} = t^{\alpha+\beta-1}\ E_{\gamma,\alpha+\beta}(\lambda t^{\gamma}). \end{aligned} \tag{55}$$

In the following lemma, we will compute the fractional integrals of a sufficiently smooth functions multiplied by a suitable power function

Lemma 3.9. *Let $\alpha > 0$ and p is a nonnegative integer. Then we have*

$$I^\alpha(t^p f(t)) = \sum_{j=0}^{p} \frac{(-1)^j \Gamma(\alpha+j)}{j!\Gamma(\alpha)} \left[D^j t^p\right] I^{\alpha+j} f(t). \tag{56}$$

In particular

$$I^\alpha(tf(t)) = tI^\alpha f(t) - \alpha I^{\alpha+1} f(t). \tag{57}$$

Proof. Firstly, we note for any $j, p \in \mathbb{N}$ with $j \leq n$, that

$$\frac{d^j t^p}{dt^j} = p(p-1)\cdots(p-j+1)t^{p-j} = \frac{p!}{(p-j)!} t^{p-j}.$$

Thus putting in mind that

$$s^p = (t-(t-s))^p = \sum_{j=0}^{p} (-1)^j \frac{(-1)^j p!}{J!(p-j)!} (t-s)^j t^{p-j} = \sum_{j=0}^{p} \frac{(-1)^j (t-s)^j}{J!} D^j t^p,$$

we obtain directly by the definition of fractional integrals that

$$\begin{aligned}
I^\alpha(t^p f(t)) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} s^p f(s)\ ds \\
&= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left\{ \sum_{j=0}^{p} (-1)^j \frac{(t-s)^j}{J!} D^j t^p \right\} f(s)\ ds \\
&= \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{p} \int_0^t \frac{(-1)^j}{J!} \left[D^j t^p\right] (t-s)^{\alpha+j-1} f(s)\ ds \\
&= \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{p} \left[D^j t^p\right] \frac{(-1)^j \Gamma(\alpha+j)}{J!} I^{\alpha+j} f(t) \\
&= \sum_{j=0}^{p} \frac{(-1)^j \Gamma(\alpha+j)}{j!\Gamma(\alpha)} \left[D^j t^p\right] I^{\alpha+j} f(t).
\end{aligned}$$

□

This leads us to make the following

Example 3.13. Let $\alpha > 0$ and $\lambda \in \mathbb{R}$. We will compute the fractional integrals of the exponential type function multiplied by t. For the computation we can use a particular case of Lemma 56. Indeed, by (53) and formula (57) with $f(t) = e^{\lambda t}$, we deduce

$$I^\alpha \left\{te^{\lambda t}\right\} = t^{\alpha+1} E_{1,1+\alpha}(\lambda t) - \alpha t^{1+\alpha} E_{1,2+\alpha}(\lambda t). \tag{58}$$

Example 3.14. Let $\alpha > 0$. We will compute the fractional integrals of the logarithmic type function multiplied by t. For the computation we can use a particular case of Lemma 56. Indeed, from (47) and formula (57) with $f(t) = \ln t$, we have in the view of $\psi(1+x) = \psi(x) + x^{-1}$ that

$$\begin{aligned}
I^\alpha(t\ln t) &= \frac{t^{\alpha+1}}{\Gamma(1+\alpha)}\left[\ln t + \Psi(1) - \Psi(1+\alpha)\right] \\
&- \alpha\left[\frac{t^{\alpha+1}}{\Gamma(2+\alpha)}\left[\ln t + \Psi(1) - \Psi(2+\alpha)\right]\right] \\
&= \frac{t^{\alpha+1}}{\Gamma(1+\alpha)}\left[\ln t + \Psi(1) - \Psi(2+\alpha) + \frac{1}{1+\alpha}\right] \\
&- \alpha\left[\frac{t^{\alpha+1}}{\Gamma(2+\alpha)}\left[\ln t + \Psi(1) - \Psi(2+\alpha)\right]\right] \\
&= \frac{t^{\alpha+1}}{\Gamma(1+\alpha)}\left[\ln t + \Psi(1) - \Psi(2+\alpha)\right]\left(1 - \frac{\alpha}{(1+\alpha)}\right) + \frac{t^{\alpha+1}}{\Gamma(2+\alpha)} \\
&= \frac{t^{\alpha+1}}{\Gamma(2+\alpha)}\left[\ln t + \Psi(2) - 1 - \Psi(2+\alpha)\right] + \frac{t^{\alpha+1}}{\Gamma(2+\alpha)} \\
&= \frac{t^{\alpha+1}}{\Gamma(2+\alpha)}\left[\ln t + \Psi(2) - \Psi(2+\alpha)\right].
\end{aligned}$$

Which is the same formula given by (48).

Example 3.15. Let $\alpha = 1/2$. By (46) and formula (57) with

$$f(t) = (1-t)^{-1/2} \in L_1[0,1],$$

we deduce in the view of Lemma 3.5 that

$$\begin{aligned}
I^{1/2}\left(\frac{t}{\sqrt{1-t}}\right) &= tI^{1/2}\left(\frac{1}{\sqrt{1-t}}\right) - \frac{1}{2}II^{1/2}\left(\frac{1}{\sqrt{1-t}}\right) \\
&= \frac{2t}{\sqrt{\pi}}\tanh^{-1}\sqrt{t} - \frac{1}{\sqrt{\pi}}\int_0^t \tanh^{-1}\sqrt{s}\; ds.
\end{aligned}$$

Now, using integration by parts and then the substitution $u = \sqrt{s}$ result in

$$\begin{aligned}
\int_0^t \tanh^{-1}\sqrt{s}\; ds &= \left[s\tanh^{-1}\sqrt{s}\right]_0^t - \int_0^t \frac{s}{2\sqrt{s}(1-s)}\; ds \\
&= t\tanh^{-1}\sqrt{t} - \left[-\sqrt{s} + \tanh^{-1}\sqrt{s}\right]_0^t \\
&= t\tanh^{-1}\sqrt{t} - \left(-\sqrt{t} + \tanh^{-1}\sqrt{t}\right) \\
&= (t-1)\tanh^{-1}\sqrt{t} + \sqrt{t}.
\end{aligned}$$

Therefore

$$\begin{aligned} I^{1/2}\left(\frac{t}{\sqrt{1-t}}\right) &= \frac{2t}{\sqrt{\pi}}\tanh^{-1}\sqrt{t} - \frac{1}{\sqrt{\pi}}\left[(t-1)\tanh^{-1}\sqrt{t}+\sqrt{t}\right] \\ &= \frac{t+1}{\sqrt{\pi}}\tanh^{-1}\sqrt{t} - \sqrt{\frac{t}{\pi}} \\ &= \frac{1}{\sqrt{\pi}}\left[\frac{t+1}{2}\ln\frac{1+\sqrt{t}}{1-\sqrt{t}} - \sqrt{t}\right]. \end{aligned}$$

Lemma 3.10. *If the function f is absolutely continuous on $[0,b]$, then the function*

$$f_{1-\alpha}(t) := I^{1-\alpha}f(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}f(s)\ ds,\ 0<\alpha<1, \tag{59}$$

is also absolutely continuous on $[0,b]$. And the converse is not (in general) true.

Proof. We note first that the absolute continuity of f is equivalent to the fundamental theorem of calculus, i.e., equivalent to $IDf(t) = f(t) - f(0)$. Then, we may substitute the function $f(t) = f(0) + \displaystyle\int_0^t f'(\theta)\ d\theta$ into (59) so that

$$f_{1-\alpha} = I^{1-\alpha}\left[f(0) + I^1 f'(t)\right] = \frac{t^{1-\alpha}f(0)}{\Gamma(2-\alpha)} + I^{1-\alpha}I^1 f'(t).$$

Since $t^{1-\alpha} = (1-\alpha)\displaystyle\int_0^t s^{-\alpha}\ ds$, then the first term of (60) is absolutely continuous. Moreover, by the semi-group property of the fractional integral operators, it follows $I^{1-\alpha}I^1 f'(t) = I^1 I^{1-\alpha} f'(t)$. That is

$$I^{1-\alpha}I^1 f'(t) = \int_0^t I^{1-\alpha}f'(s)\,ds,\ I^{1-\alpha}f' \in L_1[0,b].$$

Then the second term in (60) is also absolutely continuous and we are finished. Now, to show that the converse is not (in general) true, it is sufficient to consider the function f with one point of discontinuity given by Example 3.5. This function is not continuous while $I^{1-\alpha}f$ is absolutely continuous. □

Lemma 3.11. *Let $\lambda_1, \lambda_2, \cdots, \lambda_m \in \mathbb{R}^+$ such that $\lambda_1 < \lambda_2 < \cdots < \lambda_m$. Then:*

$$I^{\lambda_k}|f(t)| \le \frac{\Gamma(\lambda_1)\Gamma(\lambda_2)\cdots\Gamma(\lambda_{k-1})}{\left[\min_{i\in[1,m]}\{\Gamma(\lambda_i)\}\right]^{k-1}} I^{\lambda_1}|f(t)|,\ f\in L_1[0,b].$$

Proof. We have $\lambda_k < \lambda_{k+1},\ k = 1, 2, \cdots, n$. Then

$$(t-s)^{\lambda_k - 1} > (t-s)^{\lambda_{k+1}-1},\ t, s \in [0,b],\ s < t.$$

Define $\Gamma(\lambda) := \min_{i\in[1,m]} \{\Gamma(\lambda_i)\}$ and note that

$$\begin{aligned} I^{\lambda_{k+1}}|f(t)| &\leq \frac{1}{\Gamma(\lambda_{k+1})}\int_0^t (t-s)^{\lambda_{k+1}-1}|f(s)| \ ds \\ &\leq \frac{1}{\Gamma(\lambda)}\int_0^t (t-s)^{\lambda_k-1}|f(s)| \ ds = \frac{\Gamma(\lambda_k)}{\Gamma(\lambda)}I^{\lambda_k}|f(t)|. \end{aligned}$$

Thus

$$\begin{aligned} I^{\lambda_k}|f(t)| &\leq \frac{\Gamma(\lambda_{k-1})}{\Gamma(\lambda)}I^{\lambda_{k-1}}|f(t)| \\ &\leq \frac{\Gamma(\lambda_{k-1})\Gamma(\lambda_{k-2})}{[\Gamma(\lambda)]^2}I^{\lambda_{k-2}}|f(t)| \\ &\leq \frac{\Gamma(\lambda_{k-1})\Gamma(\lambda_{k-2})\Gamma(\lambda_{k-3})}{[\Gamma(\lambda)]^3}I^{\lambda_{k-3}}|f(t)| \\ &\leq \cdots \leq \frac{\Gamma(\lambda_{k-1})\Gamma(\lambda_{k-2})\cdots\Gamma(\lambda_1)}{[\Gamma(\lambda)]^{k-1}}I^{\lambda_1}|f(t)|. \end{aligned}$$

□

The statements revealing haw much the fractional integral $I^\alpha f$ is *better* than the function $f \in L_p,\ p \geq 1$ are more important. Indeed, we have

Lemma 3.12. *Let $\alpha > 0$. If $p > \max\{(1/\alpha), 1\}$, the map $I^\alpha : L_p[0,b] \to C[0,b]$ is compact operator (That is, it is continuous and maps bounded sets into relatively compact sets). In particular, if $\alpha \in (0,1]$, then $I^\alpha : L_p[0,b] \to \mathcal{H}^{\alpha-\frac{1}{p}}[0,b]$ is compact operator, where $\mathcal{H}^{\alpha-\frac{1}{p}}[0,b]$ is the Hölder space of order $\alpha - \frac{1}{p}$ (if we define $I^\alpha f(0) := 0$).*

Proof. Take $f \in L_p[0,b]$ and let $p > \max\{(1/\alpha), 1\}$. If $q \in [1,\infty]$ such that $1/p+1/q = 1$, then $q(\alpha-1) > -1$. Whence the function $g : t \to \frac{t^{\alpha-1}}{\Gamma(\alpha)} \in L_q[0,b]$. Consequently, by Young inequality it follows $I^\alpha f = f * g \in C[0,b]$. Also by Hölder inequality we obtain

$$\begin{aligned} \Gamma(\alpha)\,|I^\alpha f(t)| &\leq \left(\int_0^t (t-s)^{q(\alpha-1)}\,ds\right)^{1/q}\left(\int_0^t |f(s)|^p\,ds\right)^{1/p} \\ &\leq \left(\frac{t^{(\alpha-1)q+1}}{(\alpha-1)q+1}\right)^{1/q}\left(\int_0^b |f(s)|^p\,ds\right)^{1/p} \\ &= \frac{t^{\alpha-1+1/q}}{[q(\alpha-1)+1]^{1/q}}\,\|f\|_{L_p[0,b]} = \frac{t^{\alpha-1/p}\,\|f\|_{L_p[0,b]}}{[q(\alpha-1)+1]^{1/q}}. \end{aligned}$$

Thus, owing to $\alpha > 1/p$ we get

$$\|I^\alpha f\|_{C[0,b]} = \max_{t\in[0,b]} |I^\alpha f(t)| \le \frac{b^{\alpha-1/p}}{\Gamma(\alpha)[q(\alpha-1)+1]^{1/q}} \|f\|_{L_p[0,b]} \,. \tag{60}$$

,

$$\lim_{t\to 0} I^\alpha f(t) = 0,$$

This yields the boundedness (hence the continuity) of the linear operator I^α as a mapping from $L_p[0,b]$ into $C[0,b]$.

However, Hölder inequality and Formula (20) yield for each $0 \le \tau \le t \le b$ that

$$\begin{aligned}
|I^\alpha f(t) - I^\alpha f(\tau)|\Gamma(\alpha) &= \left| \int_0^t (t-s)^{\alpha-1} f(s)\,ds - \int_0^\tau (\tau-s)^{\alpha-1} f(s)\,ds \right| \\
\le & \left(\int_0^\tau |(t-s)^{\alpha-1} - (\tau-s)^{\alpha-1}||f(s)|\,ds + \int_\tau^t (t-s)^{\alpha-1}|f(s)|\,ds \right) \\
= & \left[\left(\int_0^\tau |(t-s)^{\alpha-1} - (\tau-s)^{\alpha-1}|^q\,ds \right)^{1/q} + \left(\int_\tau^t (t-s)^{(\alpha-1)q}\,ds \right)^{1/q} \right] \|f\|_{L_p[0,b]} \\
\le & \left[\left(\int_0^\tau |(t-s)^{q(\alpha-1)} - (\tau-s)^{q(\alpha-1)}|\,ds \right)^{1/q} + \left(\frac{(t-\tau)^{(\alpha-1)q+1}}{(\alpha-1)q+1} \right)^{1/q} \right] \|f\|_{L_p[0,b]} \\
= & \left[\left(\int_0^\tau |(t-s)^{q(\alpha-1)} - (\tau-s)^{q(\alpha-1)}|\,ds \right)^{1/q} + \frac{(t-\tau)^{\alpha-1/p}}{[q(\alpha-1)+1]^{1/q}} \right] \|f\|_{L_p[0,b]} \,.
\end{aligned}$$

Now by noting that, $(\alpha-1)q > 0$ when $\alpha > 1$ and that $(\alpha-1)q \in (-1,0)$ when $\alpha \in (0,1)$, it can be easily seen that

$$\int_0^\tau |(t-s)^{q(\alpha-1)} - (\tau-s)^{q(\alpha-1)}|\,ds \le \frac{1}{q(\alpha-1)+1} \begin{cases} (t-\tau)^{q(\alpha-1)+1} & \text{if } \alpha < 1, \\ 0 & \text{if } \alpha = 1 \\ t^{q(\alpha-1)+1} - \tau^{q(\alpha-1)+1} & \text{if } \alpha > 1. \end{cases}$$

Consequently

$$\left(\int_0^\tau |(t-s)^{q(\alpha-1)} - (\tau-s)^{q(\alpha-1)}|\,ds \right)^{1/q} \le \begin{cases} \frac{(t-\tau)^{\alpha-1/p}}{[q(\alpha-1)+1]^{1/q}} & \text{if } \alpha < 1, \\ 0 & \text{if } \alpha = 1 \\ \frac{\left[t^{q(\alpha-1)+1} - \tau^{q(\alpha-1)+1}\right]^{1/q}}{[q(\alpha-1)+1]^{1/q}} & \text{if } \alpha > 1. \end{cases}$$

A combination of these results yields

$$|I^\alpha f(t)-I^\alpha f(\tau)| \leq \frac{\|f\|_p}{\Gamma(\alpha)} \begin{cases} \dfrac{2(t-\tau)^{\alpha-1/p}}{[q(\alpha-1)+1]^{1/q}} & \text{if } \alpha<1, \\ (t-\tau)^{1-1/p} & \text{if } \alpha=1 \\ \dfrac{(t-\tau)^{\alpha-1/p}+\left[t^{q(\alpha-1)+1}-\tau^{q(\alpha-1)+1}\right]^{1/q}}{[q(\alpha-1)+1]^{1/q}} & \text{if } \alpha>1. \end{cases} \tag{61}$$

In all cases the expression on the right-hand side of (61) converges to 0 as $\tau \to t$ which confirms that $I^\alpha f \in C[0,b]$. In particular, in case $0<\alpha<1$, it follows by (61) that I^α maps $L_p[0,b]$ into the Hölder space $\mathcal{H}^{\alpha-\frac{1}{p}}[0,b]$.

Now, the equicontinuity of the image of bounded subsets of $L_p[0,b]$ under I^α is an immediate consequence of (61). Moreover, the uniform boundedness of this image follows from (60). That is I^α takes bounded subsets of $L_p[0,b]$ into uniformly bounded equicontinuous subsets of $C[0,b]$. Owing to Arzelà-Ascoli theorem (Theorem 2.1), it follows for $p>\max\{(1/\alpha),1\}$ that the map $I^\alpha : L_p[0,b] \to C[0,b]$ is compact. $\square$

A particular case of Lemma 3.12 is the following

Corollary 3.2. *Define* $g(t) := \int_0^t f(s)\,ds$. *If* $f \in L_p[0,b]$ *for all* $p>1$, *then* g *is Hölderain of order* $(1-\epsilon)$ *however small* $\epsilon>0$ *is.*

The proof of Corollary 3.2 is an immediate consequence of Lemma 3.12 with $\alpha=1$.

Corollary 3.3. *For any* $f \in L_p[0,b]$, $0 \leq 1/p < \alpha < 1$, *the function* $I^\alpha f$ *can be identified to a continuous "more precisely to a Hölderian continuous function with exponent* $(\alpha-1/p)$*" vanishing at* $t=0$.

The proof of this corollary is a direct consequence of Lemma 3.12 and Corollary 3.1.

Somewhat more challenging is to prove the following

Theorem 3.2. *The fractional integral operators sends* $L_q[0,b]$ *continuously into* $L_p[0,b]$ *if* $p \in [1,\infty]$ *satisfy* $q > 1/(\alpha+(1/p))$ *(a deep result from interpolation theory implies that even* $q=1/(\alpha+(1/p))$ *is allowed if* $1<p<\infty$ *). In particular,* $I^\alpha : L_p[0,b] \to L_p[0,b]$ *is compact for each* $p \in [1,\infty]$.

The proof of the above theorem is an immediate consequence of the Young inequality and the Kolmogorov Compactness Criterion (Theorem 2.2). For the proof see e.g. [90].

We now consider additional mapping properties of the operator I^α. Roughly speaking, we shall see that fractional integration improves the smoothness properties of functions. To be a bit more precise, we can say that $I^\alpha f$ is the sum of two expressions one of which (denoted by Ψ in the theorem below) is usually better behaved than f itself.

Theorem 3.3. *Let $\alpha \in (0,1)$ and $\lambda \in [0,1]$. If $f \in \mathcal{H}^\lambda[0,b]$, then the fractional integral $I^\alpha f$ has the form*

$$I^\alpha f(t) = \frac{f(0)t^\alpha}{\Gamma(1+\alpha)} + \Psi(t),\ \Psi(t) \in \begin{cases} \mathcal{H}^{\lambda+\alpha}[0,b] & \text{if } \lambda+\alpha \neq 1, \\ \mathcal{H}^{\lambda+\alpha,1}[0,b] & \text{if } \lambda+\alpha = 1. \end{cases} \tag{62}$$

A stronger statement in the case $\lambda + \alpha > 1$ is

$$\Psi \in C^1[0,b] \text{ and } \Psi' \in \mathcal{H}^{\lambda+\alpha-1}[0,b].$$

The proof of this theorem makes use of deeper theorems of functional analysis and is beyond the scope of this Book. For the proof see Theorem 3.1 in [169].

Now, we state and prove some Gronwall-type lemmas. It is well-known that the Gronwall-type lemma can be interpreted as a result which gives *a priori* bounds for the norm of a solution of an implicit inequality under the assumption of a linear growth estimate.

Lemma 3.13. ***(Gronwall-type lemma)*** *Let $\alpha > 0,\ \lambda \in \mathbb{R}$ and $a \in L_1[0,b]$. If $f \in L_1[0,b]$ satisfies the integral equation*

$$f(t) = a(t) + \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\ ds,\ t \in [0,b],$$

then

$$\|f\| \le \|a\|\, E_\alpha(|\lambda| b^\alpha),$$

where $E_\alpha(|\lambda|b^\alpha)$ is the Mittag-Leffler function of order α evaluated at $|\lambda|b^\alpha$.

Proof. For $t \in [0,b]$ we have $|f(t)| \le |a(t)| + |\lambda| I^\alpha |f(t)|$. By the semi-group property, it follows

$$|f(t)| \le |a(t)| + |\lambda| I^\alpha \left(|a(t)| + |\lambda| I^\alpha |f(t)|\right) = |a(t)| + |\lambda| I^\alpha |a(t)| + |\lambda|^2 I^{2\alpha} |f(t)|.$$

Repeating application of this process gives

$$|f(t)| \le \sum_{k=0}^{m-1} |\lambda|^k I^{\alpha k} |a(t)| + |\lambda|^m I^{m\alpha} |f(t)|,\ m \in \mathbb{N}.$$

Thus

$$\|f\|_{L_1[0,b]} \leq \sum_{k=0}^{m-1} |\lambda|^k \left\| I^{\alpha k}|a| \right\|_{L^1[0,b]} + |\lambda|^m \left\| I^{m\alpha}|f| \right\|_{L^1[0,b]}.$$

By (51), it follows

$$\|f\|_{L_1[0,b]} \leq \sum_{k=0}^{m-1} \frac{|\lambda|^k b^{\alpha k} \|a\|_{L_1[0,b]}}{\Gamma(1+k\alpha)} + \frac{|\lambda|^m b^{\alpha m} \|f\|_{L_1[0,b]}}{\Gamma(1+m\alpha)}.$$

Taking the limit as $m \to \infty$ and putting in mind that

$$\frac{|\lambda|^m b^{\alpha m} \|f\|_{L_1[0,b]}}{\Gamma(1+m\alpha)} \to 0 \text{ as } m \to \infty,$$

we have

$$\|f\|_{L_1[0,b]} \leq \sum_{k=0}^{\infty} \frac{|\lambda|^k b^{\alpha k} \|a\|_{L_1[0,b]}}{\Gamma(1+k\alpha)} = \|a\|_{L_1[0,b]} E_\alpha(|\lambda| b^\alpha).$$

□

The following Gronwall-type lemma was introduced by Ye *et. all* [196].

Lemma 3.14. *Let $\alpha > 0$, a is a nonnegative locally integrable on $[0, b]$ and g is nonnegative, nondecreasing continuous on $[0, b]$. If f is a nonnegative locally integrable on $[0, b]$ such that*

$$f(t) \leq a(t) + \frac{g(t)}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) \ ds, \ t \in [0, b],$$

then

$$f(t) \leq a(t) + \int_0^t \left[\sum_{n=1}^{\infty} \frac{(g(t))^n}{\Gamma(n\alpha)} (t-s)^{n\alpha-1} a(s) \right] \ ds, \ t \in [0, b].$$

Moreover if, in addition, a is nondecreasing, then

$$f(t) \leq a(t) E_\alpha(g(t) t^\alpha).$$

Now consider the non-linear operator

$$Tx(t) = g(t) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s)) \ ds, \ t \in [0, b], \ \alpha > 0, \tag{63}$$

where $g \in L_p[0, b]$, $p \in [1, \infty]$ and $f : [0, b] \times \mathbb{R} \to \mathbb{R}$ is a function satisfying Carathéodory conditions (that is $f(t, \cdot)$ is continuous for $t \in [0, b]$ and $f(\cdot, x)$ measurable for $x \in \mathbb{R}$). Moreover, let f satisfies the linear growth condition

$$|f(t, x)| \leq a(t) + k|x|,$$

with $k \in [0, \infty)$ and a, necessary nonnegative measurable function a with the property that $I^\alpha a \in L_p[0, b]$. Then we have the following lemma (see Theorem 4.1 in [150])

Lemma 3.15. *Let $1 \leq p \leq \infty$. There is a constant $M < \infty$ depending only on the above data such that each $x \in L_p[0, b]$ which satisfies*

$$|x(t)| \leq |Tx(t)|, \text{ for almost every } t \in [0, b],$$

must satisfy the norm estimate $\|x\|_p < M$.

3.2. **Derivatives of fractional order.** After the notation of fractional integral, that of fractional derivative becomes a natural requirement. At the beginning, we note that the integer-order derivatives and integrals are uniquely determined in the classical analysis. It is also the same for the fractional integral, since in all the standard literatures an unique definition is used (this is already known from the previous section). Unlike the fractional integral case, the situation for the fractional derivative is more complicated. There are different definitions, which do not coincide in general. Possibly it is due to the fact, that the different authors try to preserve different properties of the classical integer-order derivatives. As a consequence, we commence with a preliminary definition.

Definition 3.3. Riemann-Liouville fractional derivative:
Let $\alpha > 0$. For the positive integer m such that $\alpha \in (m-1, m)$, we define the Riemann-Liouville fractional derivative of order α by

$$D^{\alpha} f(t) := D^{m} I^{m-\alpha} f(t), \qquad m-1 < \alpha < m.$$

In particular, if $\alpha \in (0,1)$, we have $D^{\alpha} f(t) = \dfrac{d}{dt} I^{1-\alpha} f(t)$.
Defining for complementation $D^{0} f := f$.

Remark 3.1. Obviously, the restriction that the positive integer m is as small as possible is not serious requirement. Indeed arbitrary natural number k allowed as long as the inequality $k > \alpha$ is sufficient. This is evident by looking at our assumption and the semigroup property of fractional integration which imply

$$D^{k} I^{k-\alpha} f = D^{m} D^{k-m} I^{k-m} I^{m-\alpha} f = D^{m} I^{m-\alpha} f.$$

e.g:

$$D^{\frac{1}{2}} f = D I^{\frac{1}{2}} f = D^{5} I^{4.5} f = D^{81} I^{80.5} f.$$

The following is a natural generalization of this well-known when the order of the derivative is positive integer

Lemma 3.16. *If $\alpha > 0, \gamma > -1$, we have*

$$D^{\alpha} t^{\gamma} = \frac{\Gamma(1+\gamma)}{\Gamma(1-\alpha+\gamma)} t^{\gamma-\alpha}, \qquad t > 0,\ m-1 < \alpha < m.$$

Proof. Firstly, we note for any $m, n \in \mathbb{N}$ with $m \le n$, that

$$\frac{d^{m} t^{n}}{dt^{m}} = n(n-1)\cdots(n-m+1) t^{n-m} = \frac{n!}{(n-m)!} t^{n-m}.$$

Also, for any real number $c \notin \{-1, -2, \cdots\}$ we conclude that

$$\frac{d^{m} t^{c}}{dt^{m}} = \frac{\Gamma(1+c)}{\Gamma(1+c-m)!} t^{c-m}.$$

By the definition of the Riemann-liouville fractional derivative, a bit work using Lemma 3.2 yields

$$\begin{aligned} D^\alpha t^\gamma &= \frac{d^m}{dt^m} I^{m-\alpha} t^\gamma = \frac{d^m}{dt^m}\left[\frac{\Gamma(1+\gamma)}{\Gamma(1+m-\alpha+\gamma)} t^{m-\alpha+\gamma}\right] \\ &= \frac{\Gamma(1+\gamma)}{\Gamma(1+m-\alpha+\gamma)}\left[\frac{\Gamma(1+m-\alpha+\gamma)}{\Gamma(1+m-\alpha+\gamma-m)} t^{-\alpha+\gamma}\right] \\ &= \frac{\Gamma(1+\gamma)}{\Gamma(1-\alpha+\gamma)} t^{\gamma-\alpha}. \end{aligned}$$

This completes the proof. □

As a consequence of Lemma 3.16, we give the following

Example 3.16. 1. $D^{\frac{1}{2}} t = \dfrac{\Gamma(2)}{\Gamma(1-1/2+1)} t^{1-1/2} = \dfrac{t^{1/2}}{\Gamma(3/2)} = \dfrac{2\sqrt{t}}{\sqrt{\pi}}$

2. $D^{\frac{1}{2}} \sqrt{t} = \dfrac{\sqrt{\pi}}{2}$, $D^{\frac{1}{4}} \sqrt{t} = 0.97774 t^{1/4}$.

3. $D^{\frac{1}{4}} t^{-\frac{1}{4}} = 0.6913 t^{-\frac{1}{2}}$, $D^{\frac{1}{2}} t^{-\frac{1}{4}} = 0.3380 t^{-\frac{3}{4}}$.

Using the old-known fact that "$\dfrac{1}{\Gamma(1+x)} \equiv 0$ for any negative integer x", it can be easily prove the following particular cases of Lemma (3.16)

Proposition 3.1. *Let* $\alpha \in (m-1, m),\ m \in \mathbb{N}$*, then*

$$D^\alpha t^\gamma = \begin{cases} \Gamma(1+\alpha), & \text{if } \gamma = \alpha, \\ 0, & \text{if } \gamma = \alpha - k, \quad k = 1, 2, \cdots, m. \end{cases} \tag{64}$$

As a consequence of Proposition 3.1, we have the following

Example 3.17. If $\alpha \in (m-1, m),\ m \in \mathbb{N}$, then the general solution of the differential equation

$$D^\alpha x(t) = 0,\ t \in [0, b],$$

given by

$$x(t) = \sum_{k=1}^{m} c_k t^{\alpha-k},\ t \in [0, b]$$

where $c_k,\ k = 1, 2, \cdots, m$ are arbitrary are constants.

The following example allows us to recognize about the functions which for $t > 0$ admit the same Reimann-Liouville fractional derivatives.

Example 3.18. For $\alpha > 0$, with $m - 1 < \alpha \leq m, m \in \mathbb{N}$, we have "in the view of the formula (64)" that

$$D^\alpha f(t) = D^\alpha g(t) \Leftrightarrow f(t) = g(t) + \sum_{k=1}^{m} c_k t^{\alpha-k}.$$

Here the coefficients c_k are arbitrary constants.

In the classical calculus the properties of the operators of differentiation with respect to law of commutation and additivity of their (integer) exponents are well known. The trivial law

$$D^m D^n = D^n D^m = D^{m+n}$$

where $m, n \in \mathbb{N}$ can be referred as the *law of exponents* for the standard operators of differentiation. In the fractional calculus the *law of exponents* is known to be generally true for operators of fractional integration thanks to their semigroup property.

Unlike the classical operators of differentiation and fractional integration, the fractional differential operators does not satisfy either the commutative property or the semigroup property. To show how the *law of exponents* does not necessarily hold for the Riemann-Liouville fractional derivative, we provide an example (with power function) for which

$$D^\alpha D^\beta f(t) \neq D^{\alpha+\beta} f(t), \text{ and } D^\alpha D^\beta f(t) \neq D^\beta D^\alpha f(t), \qquad (\#)$$

e.g. Let $\alpha > 0$, $\beta = 1 + \alpha$. Then by the Formula (64) it follows

$$D^\alpha D^\beta t^\alpha = D^\alpha D^{\alpha+1} t^\alpha = D^\alpha 0 = 0,$$

$$D^\beta D^\alpha t^\alpha = D^\beta \left(\Gamma(1+\alpha)\right) = \frac{\Gamma(1+\alpha) t^{-1-\alpha}}{\Gamma(-\alpha)},$$

while

$$D^{\alpha+\beta} t^\alpha = D^{1+2\alpha} t^\alpha = \frac{\Gamma(1+\alpha)}{\Gamma(-\alpha)} t^{-1-\alpha},$$

Thus the assertion (#) holds with power function $f(t) = t^\alpha$. Another example for which (#) holds is to consider $\alpha = \beta,\ f(t) = t^{-\alpha}$.

Another major difference between the usual differential operator and the Riemann-Liouville fractional derivative is that

$$D^\alpha g(t) f(t) \neq g(t) D^\alpha f(t) + f(t) D^\alpha g(t).$$

An instructive example to show this: $\alpha = 1/2,\ f(t) = g(t) := \sqrt{t}$.

Example 3.19. We use Lemma 3.16 for deriving the Riemann-liouville fractional derivative to the discontinuous function

$$f(t) := \begin{cases} t, & t \in [0,1), \\ 1-t, & t \geq 1. \end{cases}$$

Indeed, for $\alpha \in (0,1)$ we have in the view of the formula (45) that

$$I^{1-\alpha}f(t) = \begin{cases} \frac{t^{2-\alpha}}{\Gamma(3-\alpha)}, & t \in [0,1), \\ \dfrac{t^{2-\alpha}}{\Gamma(3-\alpha)} - \dfrac{(t-1)^{1-\alpha}(-\alpha+2t)}{\Gamma(3-\alpha)}, & t \geq 1. \end{cases}$$

Therefore, by noting that $-\alpha + 2t = 2(t-1) + (2-\alpha)$ and the usual differentiations, it can be seen that

$$D^{\alpha}f(t) = DI^{1-\alpha}f(t) = \begin{cases} \frac{t^{1-\alpha}}{\Gamma(2-\alpha)}, \ t \in [0,1), \\ \dfrac{t^{1-\alpha}}{\Gamma(2-\alpha)} - \dfrac{(t-1)^{-\alpha}}{\Gamma(1-\alpha)} - 2\dfrac{(t-1)^{1-\alpha}}{\Gamma(2-\alpha)}, t \geq 1. \end{cases} \tag{65}$$

We note here that f and $D^{\alpha}f$ are not continuous at $t = 1$.

Let's consider the following miscellaneous examples

Example 3.20. Consider the following fractional order differential equation

$$D^{\frac{1}{4}}x(t) = \frac{\sqrt{\pi}}{\sqrt[4]{t}\ \Gamma(\frac{1}{4})}x(t), \ t > 0.$$

It can directly shown, by Lemma 3.16, that the exact non-zero solution of this equation given by $x(t) = 1/\sqrt{t} \in L_1[0,b]$.

Example 3.21. Consider the following differential equation of fractional order $\alpha = 1/4$

$$D^{\frac{1}{4}}x(t) = t^{\frac{1}{3}}[x(t)]^2, \ t > 0.$$

It can directly shown, by Lemma 3.16, that the exact non-zero solution of this equation given by

$$x(t) = \frac{\Gamma(\frac{5}{12})}{\Gamma(\frac{1}{6})}t^{-\frac{7}{12}},$$

and this solution belongs to the space $L_1[0,b]$.

Example 3.22. Consider the following differential equation of fractional order $\alpha = 1/2$:

$$D^{\frac{1}{2}}x(t) = t^{\frac{1}{4}}\sqrt{x(t)}, \ t > 0.$$

It can directly shown, by Lemma 3.16, that the exact non-zero solution of this equation given by

$$x(t) = \left[\frac{4}{3\sqrt{\pi}}\right]^2 t^{\frac{3}{2}},$$

and this solution belongs to the space $C[0,b]$.

Generally speaking, we have

Example 3.23. Let $\alpha, \beta > 0$ and $m \in \mathbb{N},\ m \neq 1$. If $\dfrac{\alpha+\beta}{1-m} > -1$, then the differential equation

$$D^\alpha x(t) = t^\beta [x(t)]^m,\ t > 0.$$

has the explicit solution

$$x(t) = \left[\frac{\Gamma\left(\frac{\alpha+\beta}{1-m}+1\right)}{\Gamma\left(\frac{m\alpha+\beta}{1-m}+1\right)} \right]^{\frac{1}{m-1}} t^{\frac{\alpha+\beta}{1-m}},$$

and this solution belongs to the space $C[0,b]$ for $m < 1$.

Example 3.24. Here, we will study a more complicated example, namely, we compute the Riemann-Liouville fractional derivative of the logarithmic function. For the computation, we will use the formula (47). Let $\alpha \in (m-1, m), m \in \mathbb{N}$. By the formula (47) we have for $t > 0$

$$\begin{aligned} D^\alpha \ln t &= \frac{d^m}{dt^m} I^{m-\alpha} \ln t = D^{m-1} \frac{d}{dt} \left(I^{m-\alpha} \ln t \right) \\ &= D^{m-1} \frac{d}{dt} \left\{ \frac{t^{m-\alpha}}{\Gamma(1+m-\alpha)} [\ln t + \Psi(1) - \Psi(1+m-\alpha)] \right\} \\ &= D^{m-1} \left\{ \frac{t^{m-\alpha-1}}{\Gamma(1+m-\alpha)} + \frac{t^{m-\alpha}}{\Gamma(m-\alpha)} [\ln t + \Psi(1) - \Psi(1+m-\alpha)] \right\} \end{aligned}$$

Because $\psi(x+1) = \psi(x) + x^{-1}$, we can rewrite this as

$$\begin{aligned} \frac{d^m}{dt^m} I^{m-\alpha} \ln t &= D^{m-1} \left\{ \frac{t^{m-\alpha-1}}{\Gamma(m-\alpha)} \left[\ln t + \Psi(1) - \Psi(m-\alpha) - \frac{1}{m-\alpha} \right] \right. \\ &+ \left. \frac{t^{m-\alpha-1}}{\Gamma(1+m-\alpha)} \right\} \\ &= D^{m-1} \left\{ \frac{t^{m-\alpha-1}}{\Gamma(m-\alpha)} [\ln t + \Psi(1) - \Psi(m-\alpha)] \right\} \\ &= D^{m-2} \frac{d}{dt} \left\{ \frac{t^{m-\alpha-1}}{\Gamma(m-\alpha)} [\ln t + \Psi(1) - \Psi(m-\alpha)] \right\} \\ &= D^{m-2} \left\{ \frac{t^{m-\alpha-2}}{\Gamma(m-\alpha-1)} [\ln t + \Psi(1) - \Psi(m-\alpha-1)] \right\} \end{aligned}$$

By repeating this $m-1$ of times we conclude

$$D^\alpha \ln t = \frac{t^{-\alpha}}{\Gamma(1-\alpha)} [\ln t + \Psi(1) - \Psi(1-\alpha)],\ \alpha \in (m-1, m),\ t > 0. \quad (66)$$

Example 3.25. Here, we compute the Riemann-Liouville fractional derivative the logarithmic type function multiplied by t. For convenience, we consider only the interesting case when $\alpha \in (0,1)$ the more general case is straightforward and we left it to the readers. For the computation we let $t > 0$ and putting in mind the formula (48) which implies the following chain of inequalities

$$\begin{aligned} D^{\alpha}(t\ln t) &= \frac{d}{dt} I^{1-\alpha}(t\ln t) = \frac{d}{dt}\left\{\frac{t^{2-\alpha}}{\Gamma(3-\alpha)}\left[\ln t + \Psi(2) - \Psi(3-\alpha)\right]\right\} \\ &= \frac{(2-\alpha)t^{1-\alpha}}{\Gamma(3-\alpha)}\left[\ln t + \Psi(2) - \Psi(3-\alpha)\right] + \frac{t^{1-\alpha}}{\Gamma(3-\alpha)} \\ &= \frac{(2-\alpha)t^{1-\alpha}}{\Gamma(3-\alpha)}\left[\ln t + \Psi(2) - \Psi(2-\alpha) - \frac{1}{2-\alpha}\right] + \frac{t^{1-\alpha}}{\Gamma(3-\alpha)} \\ &= \frac{t^{1-\alpha}}{\Gamma(2-\alpha)}\left[\ln t + \Psi(2) - \Psi(2-\alpha)\right]. \end{aligned} \tag{67}$$

It turns out that the Riemann-liouville derivatives have certain disadvantages when trying to model real-world phenomena with fractional differential equations. We shall therefore now discuss a modified concept of a fractional derivative. As we will see below when comparing the two ideas, this second one seems to be better suited to such tasks. The following definition was introduced by Italian mathematician Caputo in (1967) [42].

Definition 3.4. Caputo fractional derivative:
Let $\alpha > 0$. For the positive integer m such that $\alpha \in (m-1, m)$, we define the Caputo fractional derivative of order α by

$$\frac{d^{\alpha} f(t)}{dt^{\alpha}} := I^{m-\alpha} D^m f(t), \qquad m-1 < \alpha < m$$

In particular, if $\alpha \in (0,1)$, we have $\dfrac{d^{\alpha} f(t)}{dt^{\alpha}} = I^{1-\alpha}\dfrac{d}{dt} f(t)$.

Defining for complementation $\dfrac{d^0 f(t)}{dt^0} f := f$.

Remark 3.2. We are able to find the Caputo fractional derivative of arbitrary order $\alpha \in (m-1, m),\ m \in \mathbb{N}$ of f by finding the Caputo derivative of order $\beta := \alpha - (m-1)$ of the $(m-1)^{th}$ derivative of the function. Note that $\alpha - (m-1)$ is real number between 0 and 1. In fact, we can define the Caputo fractional derivative of order α by

$$\frac{d^{\alpha} f(t)}{dt^{\alpha}} := I^{\alpha-(m-1)} D^m f(t) = I^{\alpha-(m-1)} D[D^{m-1} f(t)] = \frac{d^{\beta}}{dt^{\beta}} D^{m-1} f(t).$$

Lemma 3.17. For $\alpha \in (m-1, m),\ m \in \mathbb{N}$, the Caputo fractional derivative of power function satisfies

$$\frac{d^\alpha}{dt^\alpha} t^\gamma = \begin{cases} \dfrac{\Gamma(1+\gamma)}{\Gamma(1-\alpha+\gamma)} t^{\gamma-\alpha} = D^\alpha t^\gamma, & \gamma > m-1,\ \gamma \in \mathbb{R}, \\ 0, & \gamma \leq m-1,\ \gamma \in \mathbb{N}, \\ \text{no formula determind yet}, & \gamma \leq m-1,\ \gamma \in \mathbb{R}. \end{cases}$$

Proof. The proof of the second case ($\frac{d^\alpha}{dt^\alpha} t^\gamma = 0,\ \gamma \leq m-1,\ \gamma \in \mathbb{N}$) follows immediately by the definition of Caputo fractional derivative. The more interesting case is the first one:
Let $\gamma > m-1,\ \gamma \in \mathbb{R}$. By the definition of the Caputo fractional derivative, a bit work yields

$$\begin{aligned} \frac{d^\alpha}{dt^\alpha} t^\gamma &= I^{m-\alpha} \frac{d^m}{dt^m} t^\gamma = I^{m-\alpha} \left[\frac{\Gamma(1+\gamma)}{\Gamma(1+\gamma-m)} t^{\gamma-m} \right] \\ &= \frac{\Gamma(1+\gamma)}{\Gamma(1+\gamma-m)} \frac{\Gamma(1+\gamma-m)}{\Gamma(1+\gamma-m+m-\alpha)} t^{\gamma-\alpha} \\ &= \frac{\Gamma(1+\gamma)}{\Gamma(1-\alpha+\gamma)} t^{\gamma-\alpha}. \end{aligned}$$

This completes the proof. □

Remark 3.3. As in the Reimann-Liouville definition of fractional derivatives, we assume in the Caputo definition that $\alpha \in (m-1, m)$. However, in the Reimann-Liouville definition we had seen, in Remark 3.1, that this restriction actually is not necessary. One may use any $k \in \mathbb{N}$ with $k > m$ in the Riemann.Liouville case.

For the newly introduced definition due to Caputo, the situation is different: Here we may not replace $m \in \mathbb{N}$ by some $k \in \mathbb{N}$ with $k > m$. This is evident by looking at the simple example function $f(t) = t^2$:

$$\frac{d^{\frac{1}{2}}}{dt^{\frac{1}{2}}} t^2 = I^{\frac{1}{2}} D t^2 = 2 I^{\frac{1}{2}} t = \frac{2\Gamma(2)}{\Gamma(2.5)} t^{1.5}, \text{ but } \frac{d^{\frac{1}{2}}}{dt^{\frac{1}{2}}} t^2 = I^{2.5} D^3 t^2 = 0,$$

The following fact is a direct consequence of the definition of Caputo fractional derivative

Proposition 3.2. *If* $\alpha \in (m-1, m),\ m \in \mathbb{N}$, *then the general solution of the differential equation*

$$\frac{d^\alpha x(t)}{dt^\alpha} = 0,\ t \in [0, b],$$

given by

$$x(t) = \sum_{k=1}^{m} c_k t^{m-k}.$$

where $c_k,\ k = 1, 2, \cdots, m$ *are arbitrary are constants.*

The following lemma allows us to recognize about the functions which for $t > 0$ admit the same Caputo fractional derivatives. Indeed, by Proposition 3.2 we have

Lemma 3.18. *For* $\alpha > 0$, *with* $m - 1 < \alpha \leq m, m \in \mathbb{N}$, *we have*

$$\frac{d^\alpha f(t)}{dt^\alpha} = \frac{d^\alpha g(t)}{dt^\alpha} \Leftrightarrow f(t) = g(t) + \sum_{k=1}^{m} c_k t^{m-k}.$$

Here, the coefficients c_k *are arbitrary constants.*

Now we will see that the trivial law $D^m D^n = D^n D^m = D^{m+n}$ where $m, n \in \mathbb{N}$ does not necessarily hold for the the Caputo fractional derivative: To see this we provide the following example

Example 3.26. Let $f(t) = t, \alpha = \frac{7}{10}, \beta = \frac{7}{10}$, then

$$\frac{d^{\alpha+\beta} t}{dt^{\alpha+\beta}} = \frac{d^{\frac{7}{5}} t}{dt^{\frac{7}{5}}} = I^{\frac{3}{5}} D^2 t = 0,$$

while

$$\frac{d^\alpha t}{dt^\alpha} = \frac{d^{\frac{7}{10}} t}{dt^{\frac{7}{10}}} = \frac{t^{\frac{3}{10}}}{\Gamma(\frac{3}{10})},$$

Therefore

$$\frac{d^\alpha}{dt^\alpha}\left(\frac{d^\beta t}{dt^\beta}\right) = \frac{d^{\frac{7}{10}}}{dt^{\frac{7}{10}}}\left(\frac{d^{\frac{7}{10}} t}{dt^{\frac{7}{10}}}\right) = \frac{t^{\frac{-2}{5}}}{\Gamma(\frac{3}{5})}.$$

Thus, in general we have

$$\frac{d^\alpha}{dt^\alpha}\frac{d^\beta}{dt^\beta} \neq \frac{d^{\alpha+\beta}}{dt^{\alpha+\beta}}.$$

It is also simple to show (in general) that

$$\frac{d^\alpha f(t)g(t)}{dt^\alpha} \neq g(t)\frac{d^\alpha f(t)}{dt^\alpha} + f(t)\frac{d^\alpha g(t)}{d^\alpha}$$

What is quite obvious is the following

Example 3.27. If $f(t) = c$ be a constant function, then

$$\frac{d^\alpha f(t)}{dt^\alpha} = 0,$$

holds for any $\alpha > 0$.

Example 3.28. If $f(t) = t^2$, we have $\dfrac{d^{1/2} t^2}{dt^{1/2}} = \dfrac{2t^{1.5}}{\Gamma(2.5)} = \dfrac{8\sqrt{t^3}}{3\sqrt{\pi}}$.

Example 3.29. If $f(t) = t$, we have $\dfrac{d^{1/2}t}{dt^{1/2}} = \dfrac{t^{0.5}}{\Gamma(1.5)} = \dfrac{2\sqrt{t}}{\sqrt{\pi}}$.

The following example, exhibit a very different types of the behavior between the Reimann-Liouville and the Caputo differential operators.

Example 3.30. Let $f(t) = c$ be a constant function and assume that $\alpha \in (0, 1)$. Then we have for $t > 0$ that

$$\begin{aligned} D^\alpha f(t) &= \frac{d}{dt} I^{1-\alpha} c = \frac{d}{dt}\left[\frac{ct^{1-\alpha}}{\Gamma(2-\alpha)}\right] \\ &= \frac{c(1-\alpha)t^{-\alpha}}{(1-\alpha)\Gamma(1-\alpha)} = \frac{ct^{-\alpha}}{\Gamma(1-\alpha)}, \end{aligned}$$

while

$$\frac{d^\alpha f(t)}{dt^\alpha} = I^{1-\alpha}\frac{dc}{dt} = 0.$$

What is quite obvious is, in general, that

$$D^\alpha f \neq \frac{d^\alpha f}{dt^\alpha}$$

Example 3.31. Consider the following differential equation of fractional order $\alpha = 1/2$:

$$\frac{d^{\frac{1}{2}}}{dt^{\frac{1}{2}}} x(t) = t^{\frac{1}{4}}[x(t)]^2, \ t > 0.$$

It can directly shown, by Lemma 3.17, that the exact non-zero solution of this equation given by

$$x(t) = \frac{12\Gamma(\frac{3}{4})}{5\Gamma(\frac{1}{4})} t^{-\frac{3}{4}},$$

and this solution belongs to the space $L_1[0, b]$.

Example 3.32. Consider the following initial value problem in the case of the inhomogeneous Bagley-Torvik equation

$$\frac{d^2x}{dt^2} + \frac{d^{\frac{3}{2}}x}{dt^{\frac{3}{2}}} + x(t) = 1 + t \tag{68}$$

$$x(0) = x'(0) = 1$$

It can directly shown, by Lemma 3.17, that the exact solution of this equation given by

$$x(t) = 1 + t,$$

and this solution belongs to the space $C[0, b]$.

Example 3.33. Consider the following fractional differential equation:

$$\frac{d^\alpha}{dt^\alpha}x(t) = \lambda x(t),\ \alpha \in (0,1],\ t > 0.$$

It can directly shown, by Lemma 3.8, that the exact non-zero solution of this equation given by $x(t) = E_\alpha(\lambda t^\alpha)$.

In fact, by the formula (23) and Lemma 3.8, we have

$$\begin{aligned} I^{1-\alpha}\left[\frac{d}{dt}E_\alpha(\lambda t^\alpha)\right] &= \sum_{n=1}^{\infty} I^{1-\alpha}\frac{\lambda^n t^{n\alpha-1}}{\Gamma(n\alpha)} = \sum_{n=1}^{\infty}\frac{\lambda^n t^{\alpha(n-1)}}{\Gamma(\alpha(n-1)+1)} \\ &= \lambda\sum_{n=0}^{\infty}\frac{\lambda^n t^{n\alpha}}{\Gamma(n\alpha+1)}. \end{aligned}$$

Thus

$$\frac{d^\alpha}{dt^\alpha}E_\alpha(\lambda t^\alpha) = \lambda E_\alpha(\lambda t^\alpha).$$

In the next example, we will show that in the presence of singularities we may lose the equivalence between the initial value problems and the corresponding Volterra integral equations

Example 3.34. Consider the integral equation

$$x(t) = 1 + I^\alpha\left(\frac{1}{x(t)-1}\right),\ \alpha \in (1,2),\ t \in [0,1]. \tag{69}$$

As we remark in Example 3.4, the absolutely continuous function

$$x(t) := 1 \pm \frac{\sqrt{\Gamma(1-\alpha/2)}}{\sqrt{\Gamma(1+\alpha/2)}}t^{\frac{\alpha}{2}},\ t \in [0,1] \tag{70}$$

is a solution to (69). However, x fails to satisfies the initial value problem

$$\begin{cases} \dfrac{d^\alpha}{dt^\alpha}x(t) = \dfrac{1}{x(t)-1},\ \alpha \in (1,2),\ t > 0, \\ x(0) = x'(0) = 1, \end{cases} \tag{71}$$

Indeed, by the aid of Lemma 3.2 we get

$$\begin{aligned} I^{1-\alpha}\frac{d}{dt}\left[\frac{\sqrt{\Gamma(1-\alpha/2)}}{\sqrt{\Gamma(1+\alpha/2)}}t^{\frac{\alpha}{2}}\right] &= I^{1-\alpha}\left[\frac{\sqrt{\Gamma(1-\alpha/2)}}{\sqrt{\Gamma(1+\alpha/2)}}(\frac{\alpha}{2})t^{\frac{\alpha}{2}-1}\right] \\ &= \frac{\sqrt{\Gamma(1-\alpha/2)}}{\sqrt{\Gamma(1+\alpha/2)}}\frac{\frac{\alpha}{2}\Gamma(\frac{\alpha}{2})}{\Gamma(1-\frac{\alpha}{2})}t^{-\frac{\alpha}{2}}, \end{aligned}$$

This estimates yield

$$\frac{d^\alpha x(t)}{dt^\alpha} = \frac{d^\alpha}{dt^\alpha}\left[\frac{\sqrt{\Gamma(1-\alpha/2)}}{\sqrt{\Gamma(1+\alpha/2)}}t^{\frac{\alpha}{2}}\right] = \frac{\sqrt{\Gamma(1+\alpha/2)}}{\sqrt{\Gamma(1-\alpha/2)}}t^{-\frac{\alpha}{2}}.$$

That is x satisfies the fractional differential equation

$$\frac{d^\alpha}{dt^\alpha}x(t) = \frac{1}{x(t)-1}, \ \alpha \in (1,2], \ t > 0.$$

Since x fails to be differentiable at $t = 0$, that is, $x'(0)$ undefined then x fails to be a solution to the problem (71). So the the initial value problem (71) is not equivalent to the corresponding integral equations.

In the next example, we discuss the same problem which has been discussed in [100] by the so-called modified homotoy perturbation method. We will solve by the separation of variables method

Example 3.35. Consider the following unidimensional diffusion equation of the fractional order $\alpha \in (0,1] \subset \mathbb{R}$

$$\frac{\partial^\alpha u}{\partial t^\alpha} = -\frac{\partial u}{\partial x}, \text{ and } u(x,0) = e^{-x}. \tag{72}$$

Here $u = u(x,t)$, $x \in (-\infty,\infty)$, $t > 0$. We suppose a solution with the form $u(x,t) = X(x)T(t)$.

It can directly shown by substitution in (72) that

$$\frac{\frac{d^\alpha T}{dt^\alpha}}{T} = -\frac{\frac{dX}{dx}}{X} = \lambda,$$

where $\lambda \in \mathbb{R}$. We first consider the fractional differential equation

$$\frac{d^\alpha T}{dt^\alpha} = \lambda T.$$

So, by the result of Example 3.33, we deduce $T(t) = E_\alpha(\lambda t^\alpha)$, where E_α is the one-parameter Mittag-Leffler function. Using the initial condition we have $X(x)T(0) = e^x$. Since $T(0) = 1$, then $X(x) = e^x$.
Substituting this result in differential equation involving $X(x)$ we obtain

$$-X(x) = \lambda X(x), \ i.e. \ \lambda = -1.$$

So the solution of the fractional diffusion equation problem given by

$$u(x,t) = e^x E_\alpha(-t^\alpha).$$

Next we discuss the interchange of the infinite sum operation and fractional differential operators.

Lemma 3.19. ***(Generalized linearity of D^α)***
For $\alpha > 0$, let $D^\alpha f_n$ exists for each $n \in \mathbb{N}$, and suppose that for every subinterval of $[0, b]$ that the series $\sum_{n=0}^{\infty} f_n$ and $\sum_{n=0}^{\infty} D^\alpha f_n$ are uniformly convergent. Then

$$D^\alpha \left(\sum_{n=0}^{\infty} f_n(t) \right) = \sum_{n=0}^{\infty} \left(D^\alpha f_n(t) \right)$$

Proof. Define $f := \sum_{n=0}^{\infty} f_n(t)$. We start with the case $\alpha \in (0, 1)$. By Lemma 3.8 we obtain

$$D^\alpha f = D I^{1-\alpha} f = D I^{1-\alpha} \left(\sum_{n=0}^{\infty} f_n(t) \right) = D \left(\sum_{n=0}^{\infty} I^{1-\alpha} f_n(t) \right). \tag{73}$$

Now because $\sum_{n=0}^{\infty} d/dt \left(I^{1-\alpha} f_n(t) \right) = \sum_{n=0}^{\infty} D^\alpha f_n(t)$ is uniformly convergent on every compact subinterval of $[0, b]$, we can differentiate (73) term-by-term and obtain the desired result. As for $\alpha \geq 1$ we need to find $m \in \mathbb{N}$ such that $m \leq \alpha < m + 1$ and then repeat the argument on $D^\alpha f = D^{m+1} I^{m+1-\alpha} f$. $\square$

The following Lemma is direct consequence of Lemma 3.8

Lemma 3.20. ***(Generalized linearity of the $\frac{d^\alpha}{dt^\alpha}$)***
Let $\alpha > 0$ and suppose $f_n \in C[0, b]$ such that f_n' exists for each $n \in \mathbb{N}$. If $\sum_{n=1}^{\infty} f_n(t)$ and $\sum_{n=1}^{\infty} f_n'(t)$ are uniformly convergent on $[0, b]$. Then term-wise Caputo fractional derivative is possible, that is

$$\frac{d^\alpha}{dt^\alpha} \left(\sum_{n=1}^{\infty} f_n(t) \right) = \left(\sum_{n=1}^{\infty} \frac{d^\alpha f_n(t)}{dt^\alpha} \right).$$

Example 3.36. In the view of Lemma 3.19 and Lemma 3.16, we have

$$D^\alpha e^{\lambda t} = \sum_{n=0}^{\infty} D^\alpha \frac{(\lambda t)^n}{\Gamma(1+n)} = \sum_{n=0}^{\infty} \frac{\lambda^n}{\Gamma(1-\alpha+n)} t^{n-\alpha} = t^{-\alpha} E_{1,1-\alpha}(\lambda t).$$

Example 3.37. Let $\alpha \in (m-1, m)$. In the view of Lemma 3.20 and Lemma 3.17, we have

$$\frac{d^\alpha}{dt^\alpha} e^{\lambda t} = \sum_{n=m}^{\infty} D^\alpha \frac{(\lambda t)^n}{\Gamma(1+n)} = \sum_{n=m}^{\infty} \frac{\lambda^n}{\Gamma(1-\alpha+n)} t^{n-\alpha} = \sum_{n=0}^{\infty} \frac{\lambda^{n+m}}{\Gamma(1-\alpha+n+m)} t^{n+m-\alpha}.$$

Thus

$$\frac{d^\alpha}{dt^\alpha} e^{\lambda t} = \lambda^m t^{m-\alpha} E_{1,1-\alpha+m}(\lambda t)$$

In particular, if $\alpha \in (0,1)$ it follows

$$\frac{d^\alpha}{dt^\alpha} e^{\lambda t} = \lambda t^{1-\alpha} E_{1,2-\alpha}(\lambda t)$$

Now, we introduce the following comparison:

The main advantages and disadvantages of the two definitions of the fractional derivatives are that:

1. The Caputo definition is more restrictive than the Riemann-Liouville definition: On the one hand, we observe that, the Caputo derivative of a function f exists provided that f has integrable derivative (that is, f differentiable with $f' \in L_1[0,b]$) (At least $f \in AC[0,b]$). However, the following example points out that there is differentiable functions for which the Caputo derivative is no longer necessarily exists.
   Consider the following continuous $f : [0,1] \to \mathbb{R}$

$$f(t) := \begin{cases} t^2 \sin\left(\frac{1}{t^2}\right), & t \in (0,1] \\ 0, & x = 0 \end{cases} \Rightarrow$$

$$f'(t) = \begin{cases} 2t\left[\sin\left(\frac{1}{t^2}\right) - \left(\frac{1}{t^2}\right)\cos\left(\frac{1}{t^2}\right)\right], & t \in (0,1] \\ 0, & t = 0 \end{cases}$$

   It is clear that $f' \notin L_1[0,1]$. So the Caputo derivative of f does not exist.
   On the other hand, Riemann-Liouville derivative may exit even for some discontinuous functions (see e.g Examples 3.5 and 3.19.)
2. For the two definitions of fractional derivative, we also note a difference with respect to the *formal* limit as $\alpha \to (m-1)$. From the two definitions we obtain

$$\lim_{\alpha \to (m-1)} D^\alpha f(t) = D^m I f(t) = D^{m-1} f(t) = f^{(m-1)}(t),$$

$$\lim_{\alpha \to (m-1)} \frac{d^\alpha f(t)}{dt^\alpha} = I D^m f(t) = f^{(m-1)}(t) - f^{(m-1)}(0).$$

   The corresponding rule for the *formal* limit as $\alpha \to 1$ will be

$$\lim_{\alpha \to 1} D^\alpha f(t) = f'(t), \text{ and } \lim_{\alpha \to 1} \frac{d^\alpha f(t)}{dt^\alpha} = f'(t) - f'(0).$$

3. There is also another difference between the Riemann-Liouville and the Caputo approaches, which we would like to mention here and which

seems to be important for applications. Namely for the Caputo derivative we have for $\alpha \in (m-1, m),\ m \in \mathbb{N}$

$$\frac{d^\alpha D^k f(t)}{dt^\alpha} = \frac{d^{\alpha+k} f(t)}{dt^{\alpha+k}},\ k = 0, 1, 2, \cdots$$

while for the Riemann-Liouville derivative

$$D^k D^\alpha f(t) = D^{\alpha+k} f(t),\ k = 0, 1, 2, \cdots .$$

To present the next property of sequential fractional derivatives, we need the following

Definition 3.5. Let $I^\alpha(L_p[0, b]), \alpha > 0$ denote the space of function f represented by the fractional integral of order α of an integrable function:

$$f = I^\alpha \varphi,\ \varphi \in L_p[0, b], 1 \le p < \infty.$$

That is

$$I^\alpha(L_p[0, b]) := \{f :\ f = I^\alpha \varphi,\ \varphi \in L_p[0, b]\}$$

Obviously $I^\alpha(L_1[0, b]) \supset I^\alpha(L_2[0, b]) \supset \cdots$

The following theorem is a useful characterization of the space $I^\alpha(L_1[0, b])$:

Theorem 3.4. *In order that $f \in I^\alpha(L_1[0, b]), \alpha > 0$, it is necessary and sufficient that*

$$I^{m-\alpha} f \in AC^m[0, b],\ \alpha \in (m-1, m) \tag{74}$$

and that

$$\frac{d^k}{dt^k} I^{m-\alpha} f(0) = 0,\ k = 0, 1, 2, \cdots, m-1 \tag{75}$$

Proof. Necessity. Let $f = I^\alpha \varphi,\ \varphi \in L_1[o, b]$. This estimate justifies the use of the semigroup property to get, $I^{m-\alpha} f = I^m \varphi$. The conditions (74) and (75) now follow from lemma 2.7.

Sufficiency. By (33), putting in mind (75), a function $\varphi \in L_1[o, b]$ exists such that $I^{m-\alpha} f = I^m \varphi$. Consequently, $I^{m-\alpha} f = I^{m-\alpha} I^\alpha \varphi$ owing to the semigroup property of the fractional integral operators. Hence $I^{m-\alpha}(f - I^\alpha \varphi) = 0$. Since $m - \alpha > 0$, we have $f - I^\alpha \varphi = 0$ by (49) which completes the proof. □

Lemma 3.21. *Let $\alpha > 0$. If $f \in I^\alpha(L_1[0, b])$, then f have the Riemann-Liouville fractional derivatives of order $\alpha > 0$ almost everywhere on $[0, b]$.*

Proof. Let $\alpha \in (m-1, m),\ m \in \mathbb{N}$. By our assumption imposed on f, there exists $\varphi \in L_1[0, b]$ such that $f = I^\alpha \varphi$. The definition of the Riemann-Liouville differential operator and the semigroup property of the integral operators allows us to write

$$D^\alpha f = D^m I^{m-\alpha} I^\alpha \varphi = D^m I^m \varphi = \varphi \ a.e..$$

This completes the proof □

Lemma 3.22. *Let $\alpha, \beta > 0$. If $f \in I^{\alpha+\beta}(L_1[0,b])$, then*

$$D^{\alpha}D^{\beta}f = D^{\alpha+\beta}f \ a.e..$$

Proof. Let $\alpha \in (m_1-1, m_1)$ and $\beta \in (m_2-1, m_2)$ for some $m_1, m_2 \in \mathbb{N}$. By our assumption on f, there exists $\varphi \in L_1[0,b]$ such that $f = I^{\alpha+\beta}\varphi$. The definition of the Riemann-Liouville differential operator and the semigroup property of the integral operators allows us to write

$$D^{\alpha}D^{\beta}f = D^{\alpha}D^{m_2}I^{m_2-\beta}I^{\alpha+\beta}\varphi = D^{\alpha}D^{m_2}I^{m_2}I^{\alpha}\varphi.$$

In the view of the fundamental theorem of classical calculus (*cf.* Lemma 2.7) and the fact that the orders of the integral and differential operators involved on the last equation are natural numbers, we find that this is equivalent to

$$D^{\alpha}D^{\beta}f = D^{\alpha}I^{\alpha}\varphi = D^{m_1}I^{m_1-\alpha}I^{\alpha}\varphi = D^{m_1}I^{m_1}\varphi.$$

Once again, by the fundamental theorem of classical calculus, we deduce

$$D^{\alpha}D^{\beta}f = \varphi \ a.e..$$

The proof that $D^{\alpha+\beta}f = \varphi$ *a.e.* goes along similar lines:

$$D^{\alpha+\beta}f = D^{m_1+m_2}I^{(m_1+m_2)-(\alpha+\beta)}I^{\alpha+\beta}\varphi = D^{m_1+m_2}I^{m_1+m_2}\varphi = \varphi \ a.e..$$

□

It seems to be a good place to remark that a Riemann-Liouville fractional derivative of any order $\alpha \in (0,1]$ of an absolutely continues function f exists a.e. on $[0,b]$. However, the absolutely continuity assumption is sufficient for the existence of $D^{\alpha}f$ for almost every $t \in [0,b]$ but it is not necessary condition as the following lemma asserts

Lemma 3.23. *Let f be Hölderian of order $\lambda \in (0,1]$ on $[0,b]$. Then f has the Riemann-Liouville fractional derivative of all orders $\alpha \in (0,\lambda)$ on $(0,b]$. In addition*

$$D^{\alpha}f(t) = \frac{f(0)(t)^{-\alpha}}{\Gamma(1-\alpha)} + \Psi(t), \ t \in (0,b], \tag{76}$$

where Ψ is Hölderian of order $(\lambda-\alpha)$ (hence continuous) on $[0,b]$.

Proof. By Theorem 3.3, we have

$$I^{1-\alpha}f(t) = \frac{f(0)t^{1-\alpha}}{\Gamma(2-\alpha)} + \Phi(t), \tag{77}$$

Since $\lambda - \alpha > 0$ "that is $\lambda + 1 - \alpha > 1$", then $\Phi \in C^1[0,b]$. Therefore $D^{\alpha}f$ exists a.e. and for any $t > 0$,

$$\frac{d}{dt}I^{1-\alpha}f(t) = \frac{f(0)t^{-\alpha}}{\Gamma(1-\alpha)} + \frac{d\Phi(t)}{dt}.$$

Hence the formula (76) with $\Psi = \Phi' \in \mathcal{H}^{\lambda-\alpha}[0,b]$ is obtained. □

It is worth now to remark, in the view of the formula (76), that

$$D^\alpha f(t) = \Psi(t) \in \mathcal{H}^{\lambda-\alpha}[0,b],\ t \in [0,b],\ \text{for } f(0) = 0. \tag{78}$$

Indeed we have the following statement

Corollary 3.4. *Any function Hölderian of order $\lambda \in (0,1]$ on $[0,b]$ and vanishing at $t = 0$ has continuous Riemann-Liouville fractional derivatives of any order $\alpha < \lambda$ for all $t \in [0,b]$.*

Remark 3.4. Earlier (*cf.* Lemma 3.10 and Lemma 3.21), it was established that the absolutely continuous functions and the Hölderian functions have the Riemann- Liouville fractional derivatives of order $\alpha \in (0,1)$ almost everywhere. A stronger result of this statement is given by Lemma 3.23. We remark also that, when $p > 1/\alpha$, $\alpha \in (0,1]$, any function $f \in I^\alpha(L_p[0,b])$ takes the form $f = I^\alpha\varphi,\ \varphi \in L_p[0,b]$. Hence, in the view of Lemma 3.12

$$f \in \mathcal{H}^{\alpha-\frac{1}{p}}[0,b].$$

This implies the following interesting inclusion relation

$$I^\alpha(L_p[0,b]) \subset \mathcal{H}^{\alpha-\frac{1}{p}}[0,b].$$

A deep result is (see [169] §18.4)

$$\mathcal{H}^{\alpha,p}[0,b] = I^\alpha(L_p[0,b]),\ 1 < p < 1/\alpha.$$

Now, based on the formula (67) and Lemma 3.23, we are able introduce some miscellaneous examples of a continuous functions that have no first order derivative at a countable set of points but but have Riemann-Liouville fractional derivatives of all orders less than one for all points. Indeed, we have

Example 3.38. In this example we construct an example of continuous function f that has Riemann-Liouville fractional derivatives of all orders $\alpha \in (0,1)$ at $t = 0$, but has no derivative of order one at $t = 0$. To do this, we define the continuous function $f : [0,1] \to \mathbb{R}$ by

$$f(t) := \begin{cases} t\ln t,\ t \in (0,1] \\ 0,\ t = 0. \end{cases} \tag{79}$$

It is easy to see that the function f is continuous on $[0,1]$ (and even Hölderian of order 1). Evidently, an explicit calculation using L'Hospital's rule reveals that $\lim\limits_{t\to 0} f(t) = 0$. Further, by noting that f is continuous on [0,1] and differentiable on $(0,1)$, we obtain in the view of the (classical) mean value theorem that

$$|f(t+h) - f(t)| \le A|h|,\ t, t+h \in [0,b].$$

Here $A \geq f'(\zeta)$ for some $\zeta \in (0,1)$. On the one hand, f has no derivative at the point $t=0$. This is evidently follows by noting that the limit

$$\lim_{h\to 0^-} \frac{f(h)-f(0)}{h} = \lim_{h\to 0^-} \ln h,$$

does not exist. On the other hand, by Lemma 3.23, the function f has the Riemann-Liouville fractional derivatives of orders $\alpha \in (0,1)$ on $[0,1]$. This also follows directly from the formula (67) where

$$D^\alpha f(t) = \begin{cases} \dfrac{t^{1-\alpha}}{\Gamma(2-\alpha)} \left[\ln t + \Psi(2) - \Psi(2-\alpha)\right], \ t \in (0,1], \\ 0, \ t=0. \end{cases} \tag{80}$$

Once again by L'Hospital's rule it follows $\lim_{t\to 0} D^\alpha f(t) = 0$. That is, f has the Riemann-Liouville fractional derivatives of orders $\alpha \in (0,1)$, all of them being continuous everywhere including the point $t=0$.

In what follows, we construct two examples of a functions that have no finite first order derivative at a countable set of points but have Riemann-Liouville fractional derivatives of all orders less than one. To do this we define

$$C(t) := \sum_{n=1}^{\infty} \frac{\cos nt}{n^2}, \ t \geq 0, \tag{81}$$

$$S(t) := \sum_{n=1}^{\infty} \frac{\sin nt}{n^2}, \ t \geq 0. \tag{82}$$

Both of this series are uniformly converges on $[0,b]$:

$$\left|\frac{\cos nt}{n^2}\right| \leq \frac{1}{n^2}, \text{ for every } n \in \mathbb{N},$$

$$\left|\frac{\sin nt}{n^2}\right| \leq \frac{1}{n^2}, \text{ for every } n \in \mathbb{N}.$$

Since the series $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n^2}$ converges, then by $M-$Weierstrass test the series $C, \ S$ are uniformly converges on $[0,b]$ where $b>0$.
Looking now to the function $(t \to C(t))$. This function is as an elementary function:

$$\sum_{n=1}^{\infty} \frac{\cos nt}{n^2} = \frac{t^2}{4} - \frac{\pi|t|}{2} + \frac{\pi^2}{6}, \ t \in (-2\pi, 2\pi),$$

which may be obtained by direct expansion of the right-hand side into Fourier series.

As regards the function $(t \to S(t))$, it has the following integral representation:

$$\sum_{n=1}^{\infty} \frac{\sin nt}{n^2} = \int_0^t \log\left(\frac{1}{2|\sin(s/2)|}\right) ds, \ 0 \le t \le 2\pi,$$

which follows from [199] page 5 (see also [144] page 151).Thus

$$C(t) = \frac{t^2}{4} - \frac{\pi|t|}{2} + \frac{\pi^2}{6}, \ -2\pi \le t \le 2\pi, \tag{83}$$

$$S(t) = \int_0^t \log\left(\frac{1}{2|\sin(s/2)|}\right) ds, \ 0 \le t \le 2\pi. \tag{84}$$

Now, we are in the position to discuss the following

Example 3.39. Here, we will show that the function S defined by the formula (82) has the Riemann-Liouville fractional derivatives, which are continuous for all $t \ge 0$. Also, we will show that the function S has the first order continuous derivative for all $t \ge 0$ except the points $t = 2m\pi, m \in \mathbb{N}$. Evidently, since S is periodic of period 2π, it follows directly from (84) that

$$\frac{dS(t+2m\pi)}{dt} = \frac{dS(t)}{dt} = \log\left(\frac{1}{2|\sin(t/2)|}\right), \ m \in \mathbb{N}, \ t \in [0, 2\pi],$$

which implies the existence of the first order derivative of S for all $t \ge 0$ except the points $t = 2m\pi, m \in \mathbb{N}$. To consider the fractional derivatives we note by (84) and Corollary 3.2 that the function S is Hölderian of order $(1-\epsilon), \ \epsilon > 0$. Since $S(0) = 0$, Corollary 3.4 guarantees the existence of the Riemann-Liouville fractional derivatives $D^\alpha S$ of all orders $\alpha \in (0, 1)$ which are continuous for all $t \ge 0$. Now we calculate the fractional derivative to S: From (84) we have

$$D^\alpha S = D^\alpha I^1 f = \frac{d}{dt} I^{1-\alpha} I^1 f, \ \text{where } f(t) := -\log|2\sin(t/2)|.$$

The Semi-group property (Lemma 3.5) yields

$$D^\alpha S(t) = \frac{d}{dt} I^1 I^{1-\alpha} f(t) = \frac{-1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \log|2\sin(s/2)| ds, \ t > 0.$$

Similarly, for the function C defined by the formula (81), we have

Example 3.40. Here, we will show that the function C has the Riemann-Liouville fractional derivatives, which are continuous for all $t > 0$. Since C is periodic of period 2π, it follow directly from (83) that the function C has the first order continuous derivative for all $t > 0$ except the points $t = 2m\pi, m \in \mathbb{N}$. We remark also that C is a continuous periodic function. So, by (83) it is a continuous piece-wise differentiable function and, therefore, Lipschitzian.

Then, by Corollary 3.4 its fractional derivative $D^\alpha C$ is continuous (and even Hölderian of order $(1-\alpha)$ beyond the point $t=0$.

To consider the fractional derivatives we note by (83) that

$$D^\alpha C(t) = \frac{1}{4} D^\alpha t^2 - \frac{\pi}{2} D^\alpha t + \frac{\pi^2}{6} D^\alpha (1)$$

which yields

$$D^\alpha C(t) = \frac{3t^{2-\alpha} - 3\pi(2-\alpha)t^{1-\alpha} + \pi^2(2-\alpha)(1-\alpha)t^{-\alpha}}{6\Gamma(3-\alpha)}, \ t > 0.$$

Now look at the constant function, $g(t) = c$. This function is Hölderian of any order $\lambda \geq 0$ on $[0,1]$ and have a derivatives of any integer order on $[0,1]$. But g does not have a Riemann-Liouville fractional derivative of any order $\alpha > 0$ at $t = 0$ unless $c = 0$.

However, the following interesting question may raised in our mind:

`Is there exists a continuous functions that have no first order derivative might have a Riemann-Liouville fractional derivative of all orders less than one?!`

Obviously, by recalling that the Hölder space contains the class of continuous, nowhere differentiable functions, it follows in view of Corollary 3.4, that the answer is positive. Evidently, we will we consider the following

Example 3.41. At the beginning, we remark that the Riemann-Liouville fractional derivatives even of a very good function f is infinite at $t = 0$, if $f(0) \neq 0$. By this reason, we define the continuous function $\mathcal{W}$ by

$$\mathcal{W}(t) := \mathcal{W}_1(t) - \mathcal{W}_1(0), \ t \in [0,\infty), \tag{85}$$

where, $\mathcal{W}_1(t) = \sum_{n=0}^{\infty} b^{-n} e^{ib^n t}$, $b > 1$ denotes the Weierstrass function (see formula (21)). Since $\mathcal{W}(0) = 0$, then $D^\alpha \mathcal{W}$ (if exists) is finite at $t = 0$. However, the function $\mathcal{W}$ has continuous and bounded Riemann-Liouville fractional derivatives of any order less than one, but nowhere has the first order derivative.

In fact, for $\alpha \in (0,1)$, we have in the view of Example 3.11

$$I^{1-\alpha}\mathcal{W}(t) = \sum_{n=0}^{\infty} \frac{1}{b^n} A_n(t), \ A_n(t) = t^{1-\alpha}\left[E_{1,2-\alpha}(ib^n t) - \frac{1}{\Gamma(2-\alpha)}\right].$$

By the properties of $E_{\alpha,\beta}$, we have for any for each $n \in \mathbb{N}$

$$\begin{aligned} A_n(t) &= t^{1-\alpha}\left[\sum_{n=1}^{\infty} \frac{(ib^n t)^n}{\Gamma(n+2-\alpha)}\right] = t^{1-\alpha}\left[\sum_{n=0}^{\infty} \frac{(ib^n t)^{n+1}}{\Gamma(n+3-\alpha)}\right] \\ &= t^{1-\alpha}\left[(ib^n t) E_{1,3-\alpha}(ib^n t)\right] = ib^n t^{2-\alpha} E_{1,3-\alpha}(ib^n t). \end{aligned} \tag{86}$$

Obviously A_n is continuous on $[0,\infty)$ for any for each $n \in \mathbb{N}$. The uniform convergency of the series $\sum_{n=0}^{\infty} \frac{1}{b^n} A'_n(t)$ and easy calculations yields (*cf.* Theorem 2 in [144])

$$D^{\alpha}\mathcal{W}(t) = \sum_{n=0}^{\infty} it^{1-\alpha} E_{1,2-\alpha}(ib^n t).$$

For further details, see Example 3.42 below.

Example 3.42. Let $\alpha \in (0,1)$ and consider the function $\mathcal{W}$ given by the formula (85). Obviously

$$\mathcal{W}(t) = \sum_{n=0}^{\infty} f_n(t),\ t \in [0,\infty), \quad \text{Where } f_n(t) = b^{-n}(e^{ib^n t} - 1).$$

By Example 3.36 and the properties of $E_{\alpha,\beta}$, we have for any for each $n \in \mathbb{N}$

$$\begin{aligned} D^{\alpha} f_n(t) &= \frac{t^{-\alpha}}{b^n}\left(E_{1,1-\alpha}(ib^n t) - \frac{1}{\Gamma(1-\alpha)}\right) \\ &= \frac{t^{-\alpha}}{b^n}\left(\frac{1}{\Gamma(1-\alpha)} + \sum_{n=1}^{\infty} \frac{(ib^n t)^n}{\Gamma(n+1-\alpha)} - \frac{1}{\Gamma(1-\alpha)}\right) \\ &= \frac{t^{-\alpha}}{b^n} \sum_{n=0}^{\infty} \frac{(ib^n t)^{n+1}}{\Gamma(n+2-\alpha)} = it^{1-\alpha} E_{1,2-\alpha}(ib^n t). \end{aligned} \tag{87}$$

Thus $D^{\alpha} f_n$ exists and continuous. Since the series $\sum_{n=0}^{\infty} f_n$ and $\sum_{n=0}^{\infty} D^{\alpha} f_n$ are uniformly convergent. Therefore, by Lemma 3.19 we infer ([144])

$$D^{\alpha}\mathcal{W}(t) = \sum_{n=0}^{\infty} D^{\alpha} f_n(t) = \sum_{n=0}^{\infty} it^{1-\alpha} E_{1,2-\alpha}(ib^n t) \in C[0,1].$$

Remark: The result similar to stated in Examples 3.41 and 3.36 for the function $\mathcal{W}$ given by (85) is valid if we replace $\mathcal{W}_1$ by its real valued version $\mathcal{W}_1 = \sum_{n=0}^{\infty} b^{-n} \cos(b^n t),\ b > 1$. (Remark 4 in [144]).

Relations between fractional differential and integral operators

Having established a theory of fractional differential and integral operators separately, we now investigate how they interact. A very important first result in this context is that the fractional differential operator is left (but not right) inverse of the fractional integral operator. Indeed, we have the following

Lemma 3.24. *Let $\alpha, \beta > 0$. Then the equations*

$$D^0 f = f \text{ and } D^{\beta} I^{\alpha} f = \begin{cases} D^{\beta-\alpha} f, & 0 < \alpha \le \beta, \\ I^{\alpha-\beta} f, & \alpha \ge \beta > 0, \end{cases} \tag{88}$$

are satisfied a.e. on $[0,b]$ *for* $f \in L_1[0,b]$ *and hold for all* $t \in [0,b]$ *for* $f \in C[0,b]$. *In particular, when* $\alpha = \beta$, *(88) means that* D^α *is the left (but not necessary right) inverse of* I^α. *In addition, if* $\beta = k \in \mathbb{N}$, *we have*

$$D^k I^\alpha f = I^{\alpha-k} f, \text{ if } \alpha \geq k \text{ and } D^k I^\alpha f = D^{k-\alpha} f \text{ if } \alpha \leq k. \tag{89}$$

Proof. The first claim, i.e. $DIf = f$, follows from the fundamental theorem of ordinary calculus. Let $\beta \in (m-1, m), m \in \mathbb{N}$. Then we have by the definition of Riemann-Liouville fractional differential operators, semi-group property (*cf.* Lemma 3.5) and in the view of $DIf = f$ that

$$D^\beta I^\alpha f = D^m I^{m-\beta} I^\alpha f = \begin{cases} D^m I^{m-(\beta-\alpha)} f = D^{\beta-\alpha} f, \ 0 < \alpha \leq \beta, \\ D^m I^{m+\alpha-\beta} f = D^m I^m I^{\alpha-\beta} f = I^{\alpha-\beta} f, \ \alpha \geq \beta > 0, \end{cases}$$

Hence the result holds. Now, an instructive example to show that D^α is not necessary right inverse of I^α is

$$I^\alpha D^\alpha t^{\alpha-1} = I^\alpha\{0\} = 0 \neq t^{\alpha-1}.$$

□

Example 3.43. Let $f \in L_1[0,b]$, $\alpha \in (m-1, m)$, $m \in \mathbb{N}$. By the Formula (64) and Lemma 88, the general solution of the fractional differential equation

$$D^\alpha x(t) = f(t), \ t \in [0,b], \tag{90}$$

given by

$$x(t) = I^\alpha f(t) + \sum_{k=1}^{m} c_k t^{\alpha-k}, \ t \in [0,b] \tag{91}$$

where $c_k, \ k = 1, 2, \cdots, m$ are arbitrary are constants. Here the additional $c_k t^{\alpha-k}$ terms are is taken into account because $D^\alpha t^{\alpha-k} = 0, \ k = 1, 2, \cdots, m$. However, if x given by (91), then x satisfies the fractional differential equation (90) but (since x undefined at $t = 0$) it fails to be a solution to the problem

$$\begin{cases} D^\alpha x(t) = f(t), \ t \in [0,1], \ \alpha \in (0,1), \\ x(0) = x_0, \end{cases} \tag{92}$$

Generally, the equivalence between the fractional order initial value problem and the corresponding integral equation fails.

Example 3.44. Let α, β be positive number with $\alpha + \beta = 1$. For any $f \in L_1[0,b]$, the implication

$$D^\alpha D^\beta u(t) = f(t) \Rightarrow u(t) = \int_0^t f(s)\, ds + a + bt^{\beta-1}$$

holds a.e. on $[0, b]$, where a, b are constants.
To see this, we suppose, for a given function f that $D^\alpha D^\beta u(t) = f(t)$. Recalling that $D^\alpha I^\alpha f(t) = f(t)$ a.e. on $[0, b]$, we obtain

$$D^\beta u(t) = I^\alpha f(t) + ct^{\alpha-1}, \text{ where } c \text{ is constant,}$$

Thus, we conclude that

$$u(t) = I^\beta \left(I^\alpha f(t) + ct^{\alpha-1}\right) + c_0 t^{\beta-1}, \text{ where } c_0 \text{ is constant.}$$

Putting in mind that $\alpha + \beta = 1$ and that $I^{1-\alpha} t^{\alpha-1} = \Gamma(\alpha)$, we get

$$u(t) = If(t) + c\Gamma(\alpha) + c_0 t^{\beta-1}.$$

This completes the proof.

Further examples

Example 3.45. Let $\alpha,\ \beta > 0$, $\lambda \neq 0$ and $\gamma \geq 0$. We will compute the Riemann-Liouville fractional derivative to the Mittag-Leffler type functions multiplied by a suitable power function. However, we have

$$\begin{aligned} D^\alpha \left[t^{\beta-1}\ E_{\gamma,\beta}(\lambda t^\gamma)\right] &= D^\alpha \left(\sum_{k=0}^{\infty} \frac{\lambda^k t^{\gamma k+\beta-1}}{\Gamma(\gamma k+\beta)}\right) = \sum_{k=0}^{\infty} \frac{\lambda^k}{\Gamma(\gamma k+\beta)} D^\alpha t^{\gamma k+\beta-1} \\ &= \sum_{k=0}^{\infty} \frac{\lambda^k t^{\gamma k+\beta-1-\alpha}}{\Gamma(\gamma k+\beta-\alpha)} \\ &= t^{\beta-\alpha-1}\ E_{\gamma,\beta-\alpha}(\lambda t^\gamma). \end{aligned}$$

In particular, if $\beta = 1$ and $\lambda = \alpha \neq 1, 2, \cdots, m-1$, we have

$$\begin{aligned} D^\alpha \left[\ E_{\alpha,1}(\lambda t^\alpha)\right] &= t^{-\alpha}\ E_{\alpha,1-\alpha}(\lambda t^\alpha) = t^{-\alpha} \sum_{k=0}^{\infty} \frac{(\lambda t^\alpha)^k}{\Gamma(\alpha k+1-\alpha)} \\ &= t^{-\alpha} \lambda \frac{1}{\Gamma(1-\alpha)} + \lambda \sum_{k=1}^{\infty} \frac{(\lambda t^\alpha)^{k-1}}{\Gamma(\alpha(k-1)+1)} \\ &= \frac{\lambda t^{-\alpha}}{\Gamma(1-\alpha)} + \lambda E_{\alpha,1}(\lambda t^\alpha). \end{aligned}$$

Furthermore, if $\alpha = \gamma$ and $\beta - \alpha \in \{0, -1, -2, \cdots\}$, then Formula (94) implies

$$\begin{aligned} D^\alpha \left[t^{\beta-1}\ E_{\gamma,\beta}(\lambda t^\alpha)\right] &= t^{\beta-\alpha-1} \sum_{k=0}^{\infty} \frac{\lambda^k t^{k\alpha}}{\Gamma(k\alpha+\beta-\alpha)} \\ &= t^{\beta-\alpha-1} \left[\frac{1}{\Gamma(\beta-\alpha)} + \sum_{k=1}^{\infty} \frac{\lambda^k t^{k\alpha}}{\Gamma(k\alpha+\beta-\alpha)}\right]. \end{aligned}$$

Putting in mind that $[\Gamma(\beta-\alpha)]^{-1}=0$, we arrive at

$$\begin{aligned} D^\alpha\left[t^{\beta-1}\ E_{\alpha,\beta}(\lambda t^\alpha)\right] &= \lambda t^{\beta-1}\sum_{k=1}^{\infty}\frac{\lambda^{k-1}t^{(k-1)\alpha}}{\Gamma(\alpha(k-1)+\beta)} = \lambda t^{\beta-1}\sum_{k=0}^{\infty}\frac{\lambda^k t^{k\alpha}}{\Gamma(k\alpha+\beta)} \\ &= \lambda t^{\beta-1}E_{\alpha,\beta}(\lambda t^\alpha). \end{aligned}$$

Consequently, the function $x(t)=t^{\beta-1}\ E_{\alpha,\beta}(\lambda t^\alpha)$ is a solution to the fractional differential equation

$$D^\alpha x(t)-\lambda x(t)=0,\ t>0.$$

Example 3.46. For any $\alpha,\ \beta>0,\ \lambda\neq 0$ and $\gamma\geq 0$, we have (see Examples 3.12 and 3.45)

$$I^\alpha\left[t^{\beta-1}\ E_{\gamma,\beta}(\lambda t^\gamma)\right]=t^{\alpha+\beta-1}\ E_{\gamma,\alpha+\beta}(\lambda t^\gamma), \tag{93}$$

$$D^\alpha\left[t^{\beta-1}\ E_{\gamma,\beta}(\lambda t^\gamma)\right]=t^{\beta-\alpha-1}\ E_{\gamma,\beta-\alpha}(\lambda t^\gamma). \tag{94}$$

Now, if $f(t)=t^{\beta-1}\ E_{\gamma,\beta}(\lambda t^\gamma)$, we deduce in the view of (93) and (94), that

$$D^\alpha I^\alpha f(t)=D^\alpha\left[t^{\alpha+\beta-1}\ E_{\gamma,\alpha+\beta}(\lambda t^\gamma)\right]=t^{\alpha+\beta-\alpha-1}\ E_{\gamma,\alpha+\beta-\alpha}(\lambda t^\gamma)=f(t).$$

Also, we can show that $I^\alpha D^\alpha f=f$.

Example 3.47. By the formulas (53) and Example 3.36, we deduce

$$D^\alpha e^{\lambda t}=t^{-\alpha}E_{1,1-\alpha}(\lambda t)\text{ and }I^\alpha e^{\lambda t}=t^\alpha E_{1,1+\alpha}(\lambda t). \tag{95}$$

Thus $D^\alpha I^\alpha e^{\lambda t}=I^\alpha D^\alpha e^{\lambda t}=e^{\lambda t}$.

Relations between Caputo and Riemann-Liouville derivatives

Assuming that the passage of $m-$derivative under the integral is legitimate, one recognizes that the link between the Riemann-Liouville and Caputo fractional derivatives is given by the following statement

$$\frac{d^\alpha f(t)}{dt^\alpha}=D^\alpha f(t)-\sum_{k=0}^{m-1}\frac{t^{k-\alpha}}{\Gamma(k+1-\alpha)}f^{(k)}(0),\ m-1<\alpha<m. \tag{96}$$

Indeed we have

Lemma 3.25. *Let $\alpha\in(m-1,m),\ m\in\mathbb{N}$. If $f\in AC^{(m-1)}[0,b]$, then (96) holds almost everywhere on [0,b].*

Proof. Observe the formula (32) and the assumption imposed on f. Then, in the view of Lemma 3.24, we have

$$\begin{aligned}\frac{d^\alpha f(t)}{dt^\alpha} &= I^{m-\alpha}D^m f(t) = D^\alpha I^\alpha[I^{m-\alpha}D^m f(t)]\\ &= D^\alpha I^m D^m f(t) = D^\alpha\left[f(t) - \sum_{k=0}^{m-1} f^{(k)}(0)\frac{t^k}{k!}\right]\\ &= D^\alpha f(t) - \sum_{k=0}^{m-1}\frac{t^{k-\alpha}}{\Gamma(k+1-\alpha)}f^{(k)}(0).\end{aligned}$$

Hence the result holds. □

Example 3.48. By (96) and formula (95), we obtain, for any $\alpha \in (m-1, m)$, $m \in \mathbb{N}$ that

$$\begin{aligned}\frac{d^\alpha e^{\lambda t}}{dt^\alpha} &= t^{-\alpha}E_{1,1-\alpha}(\lambda t) - \sum_{k=0}^{m-1}\frac{\lambda^k t^{k-\alpha}}{\Gamma(k+1-\alpha)}\\ &= \sum_{k=0}^{\infty}\frac{\lambda^k t^{k-\alpha}}{\Gamma(k+1-\alpha)} - \sum_{k=0}^{m-1}\frac{\lambda^k t^{k-\alpha}}{\Gamma(k+1-\alpha)} = \sum_{k=m}^{\infty}\frac{\lambda^k t^{k-\alpha}}{\Gamma(k+1-\alpha)}\\ &= \sum_{k=0}^{\infty}\frac{\lambda^{k+m} t^{k+m-\alpha}}{\Gamma(k+m+1-\alpha)} = \lambda^m t^{m-\alpha}E_{1,1+m-\alpha}(\lambda t).\end{aligned}$$

Remark 3.5. 1. In Lemma 3.25, if we assume that the first $m-1$ derivatives of the function f is absolutely continuous and that f along with its first $m-1$ derivatives vanishes at $t=0$, it follows that

$$\frac{d^\alpha}{dt^\alpha}f(t) = D^\alpha f(t).$$

2. As we remark before, Caputo fractional derivative of the function f has a disadvantage that, it completely lose its meaning if f fails to be (almost everywhere) differentiable. For this reason, we are able to use Lemma 3.25 to define the Caputo fractional derivative in general, that is, we put

$$\frac{d^\alpha f(t)}{dt^\alpha} := D^\alpha f(t) - \sum_{k=0}^{m-1}\frac{t^{k-\alpha}}{\Gamma(k+1-\alpha)}f^{(k)}(0),\ m-1<\alpha<m. \tag{97}$$

Thus, if we assume that the first $m-1$ derivatives of the function f is absolutely continuous, the Definition 97 coincides with the usual definition of the Caputo fractional derivative.

Lemma 3.26. *Let $\alpha \in (m-1, m),\ m \in \mathbb{N}$. For any $f \in AC^{m-1}[0,b]$ the operator $\frac{d^\alpha}{dt^\alpha}$ is left (but not necessary right) inverse of I^α. However, if $f \in AC^{m-1}[0,b]$ then*

$$\frac{d^\alpha}{dt^\alpha} I^\alpha f(t) = f(t)$$

Proof. For the sake of simplicity we restrict ourselves to the case when $\alpha \in (0,1)$. Since $f \in AC[0,b]$, then $I^\alpha f(0) = 0$.

By the the Definition 97 of the Caputo fractional derivative, with f replaced by $I^\alpha f \in AC[0,b]$, we get

$$\frac{d^\alpha}{dt^\alpha} I^\alpha f(t) = D^\alpha I^\alpha f(t) - \frac{t^{-\alpha}}{\Gamma(1-\alpha)} I^\alpha f(0) = f(t).$$

Next, to show that he fractional differential operator $\frac{d^\alpha}{dt^\alpha}$ is not necessary right inverse of I^α, it is sufficient to consider the simple counterexample: $\alpha = 3.5$, $f(t) = t,\ t > 0$. In this case we observe

$$I^\alpha \frac{d^\alpha}{dt^\alpha} f(t) = 0 \neq f.$$

□

Example 3.49. Let $f \in AC^{m-1}[0,b]$, $\alpha \in (m-1, m),\ m \in \mathbb{N}$. By the Proposition 3.2 and Lemma 3.26, the general solution of the fractional differential equation

$$\frac{d^\alpha x(t)}{dt^\alpha} = f(t),\ t \in [0,b],$$

given by

$$x(t) = I^\alpha f(t) + \sum_{k=1}^{m} c_k t^{m-k},\ t \in [0,b]$$

where $c_k,\ k = 1, 2, \cdots, m$ are arbitrary are constants.

Arguing similarly as in the Example 3.44, we can show, by the aid of Example 3.49 that

Example 3.50. Let $\alpha,\ \beta$ be positive number with $\alpha + \beta = 1$. For any $f \in AC[0,b]$, the implication

$$\frac{d^\alpha}{dt^\alpha}\frac{d^\beta}{dt^\beta} u(t) = f(t) \Rightarrow u(t) = \int_0^t f(s)\,ds + at^\beta + b$$

holds on $[0,b]$, where a, b are constants.

Remark 3.6. The assertion of Lemma 3.26 fails if we enlarge the class $AC[0,b]$ up to $C[0,b]$: To see this, consider the *continuous* function $f := D^\alpha \mathcal{W},\ \alpha \in (0,1)$, where $\mathcal{W}(t) = \mathcal{W}_1(t) - \mathcal{W}_1(0)$, $\mathcal{W}_1$ denotes the Weierstrass function. As we remarked above, the function $\mathcal{W}$ has continuous Riemann-Liouville fractional derivatives of any order less than one, but nowhere has the first order derivative. In view of Formula (55) and the generalized linearity of the fractional integrals, it can be easily seen that

$$I^\alpha D^\alpha \mathcal{W}(t) = \sum_{n=0}^{\infty} it\ E_{1,2}(ib^n t) = \sum_{n=0}^{\infty} it \frac{e^{ib^n t} - 1}{ib^n t} = \mathcal{W}(t).$$

That is $I^\alpha f = \mathcal{W}$.

Therefore, the function $I^\alpha f$, is nowhere differentiable on $[0, b]$, meaning that the Caputo fractional derivatives of $I^\alpha f$ does not exists on a set of positive measure.

In the next example, we will show that we may lose the equivalence between the initial/boundary value problems and the corresponding integral equations whenever the nonlinear term of the unknown functions fails to absolutely continuous.

Example 3.51. Consider the boundary value problem

$$\begin{cases} \dfrac{d^\alpha x(t)}{dt^\alpha} = f(t, x(t)),\ t \in [0,1],\ \alpha \in (1,2), \\ x(0) = x(1) = 0, \end{cases} \tag{98}$$

where $f(\cdot, x(\cdot)) \in C[0,1]/AC[0,1]$ for every $x \in C[0,1]$. To obtain *formally* the integral equation modeled off the problem (98), we keep Example 3.49 in mind and put

$$x(t) = I^\alpha f(t, x(t)) + c_1 t + c_2. \tag{99}$$

By the assumptions $x(0) = x(1) = 0$, we infer that $c_2 = 0$ and $c_1 = -I^\alpha f(1, x(1))$. Thus, if x is a continuous (but not absolutely continuous) solution to (99), then x may fails to satisfies the problem (98): To show this, let $\alpha = 1 + \beta$, $\beta \in (0,1)$, and $f(t, x(t)) = D^\beta \mathcal{W} \in C[0,1]$. Then the integral equation (99) takes the form

$$x(t) = I^1(I^\beta D^\beta \mathcal{W}(t)) + c_1 t \tag{100}$$

Consequently, proceed as in the Remark 3.6, it follows for almost every $t \in [0,1]$ that

$$\frac{dx}{dt} = I^\beta D^\beta \mathcal{W}(t)) = \mathcal{W}(t) + c_1.$$

Since $\mathcal{W}$ is continuous nowhere differentiable, it follows that x' fails to be differentiable on a subset of positive measure contains in [0,1]. That is, there exists no subset of positive measure on which the Caputo fractional derivatives of order $\alpha \in (1,2)$ of x exists. Therefore, the boundary value problem (98) is not equivalent to the corresponding integral equation. Indeed, the problem (98) completely lost it meaning when $f(t, x(t)) = D^\beta \mathcal{W}(t)$.

In the case where f is of a general form and not as simple as in Lemma 3.26, using formula (32) we are able to prove the following similar, albeit slightly weaker, observation.

Lemma 3.27. *Let $\alpha \in (m-1, m),\ m \in \mathbb{N}$. If the function f has absolute integrable derivative of order m on $[0, b]$ (that is $f^{(m)} \in L_1[0, b]$), then*

$$I^\alpha \frac{d^\alpha}{dt^\alpha} f(t) = f(t) - \sum_{k=0}^{m-1} f^{(k)}(0) \frac{t^k}{k!}.$$

In particular, if $0 < \alpha < 1$ and $f \in AC[0, b]$, then

$$I^\alpha \frac{d^\alpha}{dt^\alpha} f(t) = f(t) - f(0).$$

Corollary 3.5. *Let $\alpha \in (m-1, m),\ m \in \mathbb{N}$. If the function f has absolute integrable derivative of order m on $[0, b]$, then*

$$I^\alpha \frac{d^\alpha}{dt^\alpha} f(t) = f(t) + c_1 + c_2 t + \cdots + c_m t^{m-1},$$

with some $c_k \in \mathbb{R},\ k = 1, 2, \cdots, m$. In particular, if $0 < \alpha < 1$ and $f \in AC[0, b]$, then

$$I^\alpha \frac{d^\alpha}{dt^\alpha} f(t) = f(t) + c,\ c \in \mathbb{R}.$$

The following property is also easy to be proved

Lemma 3.28. *Let $\beta \in (m-1, m),\ m \in \mathbb{N}$ and $\alpha \geq \beta$. If f is a function such that $I^{m-\beta} f \in AC^m[0, b]$, then*

$$I^\alpha D^\beta f = I^{\alpha-\beta} f - \sum_{k=0}^{m-1} \frac{t^{\alpha-k-1}}{\Gamma(\alpha-k)} \left[D^{m-k-1} I^{m-\beta} f(t)\right]_{t=0} \tag{101}$$

In particular,

$$I^\alpha D^\alpha f = f - \sum_{k=0}^{m-1} \frac{t^{\alpha-k-1}}{\Gamma(\alpha-k)} \left[D^{m-k-1} I^{m-\alpha} f(t)\right]_{t=0}. \tag{102}$$

If $\alpha \in (0, 1)$ it follows

$$I^\alpha D^\alpha f = f - \frac{t^{\alpha-1}}{\Gamma(\alpha)} I^{1-\alpha} f(0) (= f,\ \text{if } f \in C[0, b]).$$

Proof. The condition $I^{m-\beta}f \in AC^m[0,b]$ implies that the fractional derivative $D^\beta f = D^m I^{m-\beta} f \in L_1[0,b]$. Thus, by (33), $\varphi \in L_1[o,b]$ exists such that

$$\begin{aligned} I^{m-\beta}f(t) &= I^m\varphi(t) + \sum_{k=0}^{m-1} \frac{t^k}{\Gamma(1+k)} \left[D^k I^{m-\beta} f(t)\right]_{t=0} \\ &= I^m\varphi(t) + \sum_{k=0}^{m-1} \frac{t^{m-1-k}}{\Gamma(m-k)} \left[D^{m-1-k} I^{m-\beta} f(t)\right]_{t=0}. \quad (103) \end{aligned}$$

Consequently, $D^\beta f(t) = D^m I^{m-\beta} f(t) = \varphi(t)$. Owing to Lemma 3.24 and the semigroup property of the fractional integral operators we deduce

$$I^\alpha D^\beta f(t) = I^\alpha \varphi(t) = I^{\alpha-\beta} I^\beta \varphi(t). \quad (104)$$

Moreover, by (103) and the semigroup property, it follows

$$I^{m-\beta}\left(f - I^\beta\varphi\right)(t) = \sum_{k=0}^{m-1} \frac{t^{m-1-k}}{\Gamma(m-k)} \left[D^{m-1-k} I^{m-\beta} f(t)\right]_{t=0}.$$

Since $m-\beta > 0$, Putting in mind Lemma 3.2, we get

$$f(t) - I^\beta\varphi(t) = \sum_{k=0}^{m-1} \frac{t^{\beta-1-k}}{\Gamma(\beta-k)} \left[D^{m-1-k} I^{m-\beta} f(t)\right]_{t=0}.$$

That is

$$I^\beta\varphi(t) = f(t) - \sum_{k=0}^{m-1} \frac{t^{\beta-1-k}}{\Gamma(\beta-k)} \left[D^{m-1-k} I^{m-\beta} f(t)\right]_{t=0}.$$

Therefore, by Equation (104) we get

$$\begin{aligned} I^\alpha D^\beta f(t) &= I^{\alpha-\beta}\left(f(t) - \sum_{k=0}^{m-1} \frac{t^{\beta-1-k}}{\Gamma(\beta-k)} \left[D^{m-1-k} I^{m-\beta} f(t)\right]_{t=0}\right) \\ &= I^{\alpha-\beta} f(t) - \sum_{k=0}^{m-1} \frac{\Gamma(\beta-k) t^{\alpha-1-k}}{\Gamma(\beta-k)\Gamma(\alpha-k)} \left[D^{m-1-k} I^{m-\beta} f(t)\right]_{t=0}, \end{aligned}$$

which completes the proof. □

Corollary 3.6. *Let $\beta \in (m-1,m)$, $m \in \mathbb{N}$ and $\alpha \geq \beta$. If f is a function such that $I^{m-\beta}f \in AC^m[0,b]$, then*

$$I^\alpha D^\beta f = I^{\alpha-\beta} f + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_m t^{\alpha-m-2},$$

with some $c_k \in \mathbb{R}$, $k = 1, 2, \cdots, m$. In particular, if $\alpha = \beta \in (0,1)$

$$I^\alpha D^\alpha f = f + ct^{\alpha-1}, \ c \in \mathbb{R}.$$

Now, we are in the position to state and prove an important property of the composition of the fractional integral operators with the fractional differential operators

Lemma 3.29. *Let $\beta \in (m-1, m)$, $m \in \mathbb{N}$ and $\alpha \geq \beta$. If $f \in I^{\beta}(L_1[0,b])$, then*

$$I^{\alpha} D^{\beta} f = I^{\alpha-\beta} f \text{ a.e. on } [0,b] \tag{105}$$

In particular, we have

$$I^{\alpha} D^{\alpha} f = f \text{ a.e. on } [0,b]. \tag{106}$$

Proof. For any $f \in I^{\beta}(L_1[0,b])$, there exists $\varphi \in L_1[o,b]$ such that $f = I^{\beta}\varphi$. Thus

$$\begin{aligned} I^{\alpha} D^{\beta} f &= I^{\alpha} D^{m} I^{m-\beta} I^{\beta} \varphi = I^{\alpha} D^{m} I^{m} \varphi \\ &= I^{\alpha} \varphi = I^{\alpha-\beta} I^{\beta} \varphi = I^{\alpha-\beta} f, \end{aligned}$$

owing to the semigroup property of the fractional integral operators. This completes the proof. □

Earlier (*cf.* Lemma 3.10), it was established that the absolutely continuous functions f have the Riemann-Liouville fractional derivatives of order $\alpha \in (0,1)$ almost everywhere. Here we shall extend these assertions to a wider space of functions:

$$f(t) = \frac{\widetilde{f}(t)}{t^{\mu}(b-t)^{\nu}}, \ \mu, \nu \in [0, 1-\alpha), \ \alpha \in (0,1). \tag{107}$$

Lemma 3.30. *Let $\alpha \in (0,1)$, $\mu, \nu \in [0, 1-\alpha)$ and assume that $\widehat{f} \in AC[0,b]$ (i.e. $\widehat{f}$ is absolutely continuous on $[o,b]$). If $f(t) = \widetilde{f}(t) t^{-\mu}(b-t)^{-\nu}, t \in [0,b]$, then $f \in I^{\alpha}(L_1[0,b])$. Consequently, the Riemann-Liouville fractional derivatives to the function f exist almost everywhere on $[o,b]$.*

Proof. For the sake of simplicity we restrict ourselves to the case $\nu = 0$, the general case being reducible to this. Since $\widehat{f} \in AC[0,b]$, then there exists $\psi \in L_1[0,b]$ such that $\widehat{f}(t) = \widehat{f}(0) + \int_0^t \psi(s)\ ds$. Consequently

$$f(t) = \widehat{f}(0) t^{-\mu} + t^{-\mu} \int_0^t \psi(s)\ ds$$

Since $\mu + \alpha < 1$, we obtain in the view of Lemma 3.2

$$\widehat{f}(0) t^{-\mu} = \widehat{f}(0) \frac{\Gamma(1-\mu)}{\Gamma(1-\mu-\alpha)} I^{\alpha} t^{-\mu-\alpha}.$$

Thus $\widehat{f}(0)t^{-\mu} \in I^\alpha(L_1[0,b])$. It remains for us to show that the function $t^{-\mu}\int_0^t \psi(s)\ ds$ is representable as required in our lemma. That is, we well show that

$$t^{-\mu}\int_0^t \psi(s)\ ds \in I^\alpha(L_1[0,b]).$$

To do this, we define the function $\varphi : [0,b] \to \mathbb{R}$ by

$$\varphi(t) := \int_0^t \psi(s)u(t,s)\ ds, \text{ where } u(t,s) := \frac{\partial}{\partial t}\int_s^t \zeta^{-\mu}(t-\zeta)^{-\alpha}\ d\zeta. \tag{108}$$

By method of integration by parts we have

$$\begin{aligned} u(t,s) &= \frac{\partial}{\partial t}\left[\left(\frac{-\zeta^{-\mu}(t-\zeta)^{1-\alpha}}{1-\alpha}\right)_s^t - \mu\int_s^t \frac{\zeta^{-\mu-1}(t-\zeta)^{1-\alpha}}{1-\alpha}\ d\zeta\right] \\ &= \frac{\partial}{\partial t}\left[\frac{s^{-\mu}(t-s)^{1-\alpha}}{1-\alpha} - \mu\int_s^t \frac{\zeta^{-\mu-1}(t-\zeta)^{1-\alpha}}{1-\alpha}\ d\zeta\right] \\ &= s^{-\mu}(t-s)^{-\alpha} - \mu\frac{\partial}{\partial t}\int_s^t \frac{\zeta^{-\mu-1}(t-\zeta)^{1-\alpha}}{1-\alpha}\ d\zeta \\ &= s^{-\mu}(t-s)^{-\alpha} - \mu\int_s^t \frac{\partial}{\partial t}\frac{\zeta^{-\mu-1}(t-\zeta)^{1-\alpha}}{1-\alpha}\ d\zeta \\ &= s^{-\mu}(t-s)^{-\alpha} - \mu\int_s^t \frac{1}{\zeta^{\mu+1}(t-\zeta)^\alpha}\ d\zeta. \end{aligned}$$

Consequently

$$\begin{aligned} \frac{\partial u(t,s)}{\partial s} &= \alpha s^{-\mu}(t-s)^{-\alpha-1} - \mu s^{-1-\mu}(t-s)^{-\alpha} - \mu\frac{\partial}{\partial s}\int_s^t \frac{1}{\zeta^{\mu+1}(t-\zeta)^\alpha}\ d\zeta \\ &= \frac{\alpha}{s^\mu(t-s)^{\alpha+1}} - \frac{\mu}{s^{1+\mu}(t-s)^\alpha} + \frac{\mu}{s^{\mu+1}(t-s)^\alpha} \\ &= \frac{\alpha}{s^\mu(t-s)^{\alpha+1}} > 0. \end{aligned} \tag{109}$$

Therefore, the function $u(t,\cdot)$ is monotonic increasing functions. Moreover, the substitution $\zeta = t\tau$ yields

$$\begin{aligned} u(t,0) &= \frac{\partial}{\partial t}\int_0^t \zeta^{-\mu}(t-\zeta)^{-\alpha}\ d\zeta = \frac{\partial}{\partial t}\int_0^1 \tau^{-\mu}(1-\tau)^{-\alpha}t^{1-\mu-\alpha}\ d\tau \\ &= \frac{\partial}{\partial t}\left[t^{1-\mu-\alpha}B(1-\mu,1-\alpha)\right] \\ &= (1-\mu-\alpha)t^{-\mu-\alpha}B(1-\mu,1-\alpha) > 0. \end{aligned} \tag{110}$$

Here $B(\cdot,\cdot)$ is the well-known beta function. Thus the monotonicity of $u(t,\cdot)$ and (110) implies that $u(t,s) > 0$ for all $s > 0$. So

$$
\begin{aligned}
\int_s^b |u(t,s)|dt &= \int_s^b u(t,s)dt = \int_s^b \left[s^{-\mu}(t-s)^{-\alpha} - \mu \int_s^t \frac{1}{\zeta^{\mu+1}(t-\zeta)^{\alpha}}\, d\zeta \right] dt \\
&= \frac{s^{-\mu}(b-s)^{1-\alpha}}{1-\alpha} - \mu \int_s^b \left[\int_s^t \frac{d\zeta}{\zeta^{\mu+1}(t-\zeta)^{\alpha}} \right] dt \\
&= \frac{s^{-\mu}(b-s)^{1-\alpha}}{1-\alpha} - \mu \int_s^b \left[\int_\zeta^b \frac{dt}{\zeta^{\mu+1}(t-\zeta)^{\alpha}} \right] d\zeta \\
&= \frac{s^{-\mu}(b-s)^{1-\alpha}}{1-\alpha} - \mu \int_s^b \left[\frac{(b-\zeta)^{1-\alpha}}{(1-\alpha)\zeta^{\mu+1}} \right] d\zeta.
\end{aligned}
$$

Putting in mind that $\mu \notin \mathbb{N}$. A little experimentation using the substitution $\tau = \dfrac{b-\zeta}{b-s}$ will lead the reader to

$$
\begin{aligned}
\int_s^b |u(t,s)|dt &= \frac{s^{-\mu}(b-s)^{1-\alpha}}{1-\alpha} - \frac{\mu}{1-\alpha} \int_0^1 \tau^{1-\alpha}(b-s)^{2-\alpha}[b(1-\tau)+\tau s]^{-\mu-1} d\tau \\
&\leq \frac{s^{-\mu}(b-s)^{1-\alpha}}{1-\alpha} - \frac{\mu}{(1-\alpha)} \int_0^1 b^{-\mu-1}\tau^{1-\alpha}(b-s)^{2-\alpha}(1-\tau)^{-\mu-1} d\tau \\
&= \frac{s^{-\mu}(b-s)^{1-\alpha}}{1-\alpha} - \frac{\mu(b-s)^{2-\alpha}}{b^{\mu+1}(1-\alpha)} B(-\mu, 2-\alpha) := \omega > 0.
\end{aligned}
$$

Now, we make sure that $\varphi \in L_1[o,b]$. Indeed

$$
\begin{aligned}
\|\varphi\|_{L_1} &= \int_0^b \left[\int_0^t |\psi(s)u(t,s)| \;\, ds \right] dt \\
&= \int_0^b |\psi(s)| \left[\int_s^b u(t,s)\; dt \right] \; ds \\
&= \omega \, \|\psi\| < \infty.
\end{aligned}
$$

To verify our aim that $t^{-\mu} \int_0^t \psi(s)\, ds \in I^{\alpha}(L_1[0,b])$, we note that

$$
\begin{aligned}
I^{\alpha}\varphi(t) &= \int_0^t (t-s)^{\alpha-1} \left[\int_0^s \psi(\theta)u(s,\theta)d\theta \right] \, ds \\
&= \int_0^t \psi(\theta) \left[\int_\theta^t (t-s)^{\alpha-1} u(s,\theta)\, ds \right] \; d\theta
\end{aligned}
$$

The inner integral is easily evaluated after the change of variable $s = \theta + \tau(t-\theta)$ as follows

$$\begin{aligned}
\int_\theta^t (t-s)^{\alpha-1} u(s,\theta)\, ds &= \int_\theta^t (t-s)^{\alpha-1}\left[\theta^{-\mu}(s-\theta)^{-\alpha} - \int_\theta^s \frac{\mu d\zeta}{\zeta^{\mu+1}(s-\zeta)^\alpha}\right] ds \\
&= \int_0^1 (t-\theta)^{1-\alpha}\theta^{-\mu}\tau^{-\alpha}[t-\theta-\tau(t-\theta)]^{\alpha-1} d\tau \\
&- \int_\theta^t (t-s)^{\alpha-1}\left[\int_\theta^s \frac{\mu d\zeta}{\zeta^{\mu+1}(s-\zeta)^\alpha}\right] ds \\
&= \int_0^1 (t-\theta)^{1-\alpha}\theta^{-\mu}\tau^{-\alpha}[1-\tau]^{\alpha-1} d\tau \\
&- \mu \int_\theta^t \zeta^{\mu+1}\left[\int_\zeta^t \frac{(t-s)^{\alpha-1}\, ds}{(s-\zeta)^\alpha}\right] d\zeta = \theta^{-\mu} B(\alpha, 1-\alpha) \\
&- \mu \int_\theta^t \zeta^{\mu+1}\left[\int_\zeta^t (t-s)^{\alpha-1}(s-\zeta)^{-\alpha}\ \ ds\right] d\zeta.
\end{aligned}$$

Change of variable $s = \zeta + \eta(t-\zeta)$ implies

$$\begin{aligned}
\int_\theta^t (t-s)^{\alpha-1} u(s,\theta)\ \ ds &= \theta^{-\mu} B(\alpha, 1-\alpha) \\
&- \mu \int_\theta^t \zeta^{\mu+1}\left[\int_0^1 (1-\eta)^{\alpha-1}\eta^{-\alpha} d\eta\right] d\zeta \\
&= \theta^{-\mu} B(\alpha, 1-\alpha) - \mu \int_\theta^t \left[\zeta^{-1-\mu} B(\alpha, 1-\alpha)\right] d\zeta \\
&= \theta^{-\mu} B(\alpha, 1-\alpha) - \left[-\zeta^{-\mu}\right]_\theta^t B(\alpha, 1-\alpha) \\
&= \theta^{-\mu} B(\alpha, 1-\alpha) - \left[\theta^{-\mu} - t^{-\mu}\right] B(\alpha, 1-\alpha) \\
&= t^{-\mu} B(\alpha, 1-\alpha).
\end{aligned}$$

Therefore,

$$I^\alpha \varphi(t) = \int_0^t \psi(\theta)\left[t^{-\mu} B(\alpha, 1-\alpha)\right]\ d\theta,$$

which gives

$$t^{-\mu}\int_0^t \psi(\theta) d\theta = \frac{1}{B(\alpha, 1-\alpha)} I^\alpha \varphi(t),$$

and thereby proves the representability of the function of the function $t^{-\mu}\widetilde{f}(t)$ by a fractional integral. Thus, there always exists $\varphi \in L_1[o,b]$ such that $t^{-\mu}\widetilde{f}(t) = I^\alpha\varphi(t)$. Consequently $f \in I^\alpha(L_1[0,b])$ an so the function $I^{1-\alpha}f$ is differentiable a.e. on $[o,b]$ and this is, of course, implies the existence of the

Riemann-Liouville fractional derivative $D^{\alpha}\left(t^{-\mu}\widetilde{f}(t)\right)$. This completes the proof. $\square$

Further properties

If $f(t) = t^{\lambda}\ln t\ \zeta(t)$, where $\lambda > -1$ and $\zeta(t) = \sum_{n=0}^{\infty} a_n t^n$ having a positive radius ξ of convergence, then for $0 \leq t < \xi$, the following three formulas are valid

$$\begin{cases} \mu \geq 0 \text{ and } 0 \leq \nu \leq \mu & \Rightarrow D^{\nu} I^{\mu} f(t) = I^{\mu-\nu} f(t), \\ \mu \geq 0 \text{ and } \nu > \mu & \Rightarrow D^{\nu} I^{\mu} f(t) = D^{\mu-\nu} f(t), \\ 0 \leq \mu < \lambda + 1 \text{ and } \nu \geq 0 & \Rightarrow D^{\nu} D^{\mu} f(t) = D^{\mu+\nu} f(t), \end{cases}$$

At least in the case of f without the factor $\ln t$, the proof of this formulas is straightforward. In this proof, we use the definitions of fractional integration, Riemann-Liouville differentiation and apply the semi-group property of fractional integration to the infinite series you meet in the calculations. Of course, the condition that the function ζ be analytic can be considerable relaxed, it only need to be "sufficiently smooth."

4. A First Steps towards the Fractional calculus in Gauge space

Earlier, the fractional integrals and derivatives were considered for the functions on the Lebesgue space. Here we shall extend these assertions to much wider class of functions, namely the Gauge space. It is well-known that the Gauge integral (also known as the generalized Riemann integral, the Henstock integral, the kurzweil integral, the Henstock-kurzweil integral, the HK-integral, the Denjoy-Perron integral, etc.) generalizes the integrals of Riemann and Lebesgue as well as the Riemann and Lebesgue improper integrals. So, it is worth to define here the fractional integral and differential operators using Gauge integral. We start with the following

Definition 4.1. Let $f \in G[0,b]$. The fractional (arbitrary order) Gauge-integral "shortly FGI " of f of order $\alpha > 0$ is defined by

$$I_G^\alpha f(t) := (G)\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)\,ds,\ t \in [0,b]. \tag{111}$$

In the above definition the sign "$(G)\displaystyle\int$" denotes the Gauge integral.

Remark: It what follows we consider the most interesting case when $\alpha \in (0,1)$

Obviously, the Definition 4.1 makes sense for any $f \in L_1[0,b]$. It is also exists for the following Gauge (but not Lebesgue) integrable functions $f, g : [0,1] \to \mathbb{R}$ defined by

$$f(t) := \begin{cases} \dfrac{\sin(1/t)}{t^\gamma}, & t \in (0,1],\ \gamma \in [1,2) \\ 0, & t = 0. \end{cases},\ g(t) := \frac{d}{dt}\begin{cases} t^2\sin(1/t^2), & t \in (0,1], \\ 0, & t = 0. \end{cases}$$

Indeed, for $t > 0$, we can simply separate the integral into an integral over $[0,\delta]$ and over $[\delta,t]$ (with $0 < \delta < t$). The former causes no difficulty, because $(t-s)^{\alpha-1}$ is of bounded variation for $s \in [0,\delta]$. The latter causes has no difficulty too. This because f and g are in $L_1[\delta,t]$ (even in $L_\infty[\delta,t]$ if it is bounded and measurable on $[\delta,t]$). Thus, $I_G^\alpha, \alpha > 0$ is defined for f and g. However, devoted by Hake's Theorem we define the fractional integral operator I_G^α of order $\alpha > 0$ for the Gauge integrable function $f \in G[0,b]$ by

$$I_G^\alpha f(t) := \lim_{c_n \to 0} (G)\int_0^{t-c_n} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)\,ds,\ t \in [0,b]. \tag{112}$$

where $\{c_n\}$ is a decreasing sequence of positive real numbers which converges to 0.

We say that $I_G^\alpha f$ exists at the point $t = t_0 \in [0,b]$ if the limit on the right

hand side of (112) exists at t_0 and $I_G^\alpha f$ exists a.e. on $[0,b]$ if the limit on the right hand side of (112) exists for almost all points on $[0,b]$.
However, if $f:[0,b]\to\mathbb{R}$ is Gauge integrable function on $[0,b]$, such that the the limit on the rite had side of (112) exists at $t_0\in[0,b]$ it follows by Hake' Theorem that

$$I_G^\alpha f(t) := (G)\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)\,ds,$$

exists at the point $t=t_0\in[0,b]$. It is worth to remark here, that the function $s\to(t-s)^{\alpha-1}$ is monotonic (hence of bounded variation) on $[0,t-c_n]$ for any $t>0$. So, by the Multiplier Theorem it follows that the integral on the right hand side of (112) makes sense for any $f\in G[0,b]$.
Moreover, if $f\in L_1[0,b]$, the definition (112) coincide with the usual definition of the Riemann-Liouville fractional integral: To see this, let $\{c_n\}$ be any decreasing sequence of positive real numbers which converges to 0. If $f\in L_1[0,b]$, it follows by the Lebesgue Dominated (or Vitali) Convergence Theorem that

$$\begin{aligned}
\lim_{c_n\to 0}(G)\int_0^{t-c_n}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f(s)\,ds &= \lim_{c_n\to 0}\left[(L)\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f(s)\,ds\right.\\
&- \left.(L)\int_{t-c_n}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f(s)\,ds\right]\\
&= I_L^\alpha f(t) - \lim_{c_n\to 0}(L)\int_0^1 \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f\chi_{[t-c_n,t]}(s)\,ds\\
&= I_L^\alpha f(t) - (L)\int_0^1 \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\lim_{c_n\to 0} f\chi_{[t-c_n,t]}(s)\,ds\\
&= I_L^\alpha f(t),
\end{aligned}$$

where I_L^α denotes the standard Riemann-Liouville fractional integral operator. By Young inequality, it follows that $I_L^\alpha f$ exists a.e. on $[0,b]$. Consequently, by Hake's Theorem $I_G^\alpha f$ exists a.e. on $[0,b]$.

Proposition 4.1. *Let $f\in G[0,b]$, then for any $t>0$, we have*

$$\lim_{c_n\to 0}(G)\int_0^{t-c_n}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f(s)\,ds = \lim_{c_n\to 0}(G)\int_{c_n}^{t-c_n}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f(s)\,ds, \quad (113)$$

either both sides of (113) are defined or none.

Proof. Let $f\in G[0,b]$, then

$$\begin{aligned}
\left|(G)\int_0^{t-c_n}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f(s)\,ds \right.&- \left.(G)\int_{c_n}^{t-c_n}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f(s)\,ds\right|\\
&= \left|(G)\int_0^{c_n}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f(s)\,ds\right|.
\end{aligned}$$

If $s \to (t-s)^{\alpha-1} f(s) \in G(\cup_n^\infty [0, t-c_n])$, it follows by Proposition 2.4 that

$$\left|(G)\int_0^{c_n} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)\,ds\right| \to 0 \text{ as } n \to \infty.$$

□

Proposition 4.2. *The limit on the right hand side of the Definition (112) (if exists) independent of the choice of the sequence $\{c_n\}$.*

Proof. Fix $t > 0$ and suppose that the limit on the write had side of (112) exists at the point t. Let $\{c_n\}$ and $\{d_n\}$ be two different decreasing sequences which converges to 0. Let $f \in G[0,b]$ and note that

$$\begin{aligned}\left|(G)\int_0^{t-c_n} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)\,ds \right.&-\left.(G)\int_0^{t-d_n} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)\,ds\right| \\ &= \left|(G)\int_{t-d_n}^{t-c_n} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)\,ds\right|,\ c_n \le d_n \\ &= \left|(G)\int_{[t-d_n, t-c_n]} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)\,ds\right|.\end{aligned}$$

If $s \to (t-s)^{\alpha-1} f(s) \in G(\cup_n^\infty [t-d_n, t-c_n]) = G([0,t])$, it follows by Proposition 2.4

$$\lim_{n\to\infty}\left|(G)\int_{[t-d_n, t-c_n]} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)\,ds\right| = 0.$$

□

==

Devoted by Theorem 2.19, we define the fractional Gauge-integral "shortly FGI " I_G^α of order $\alpha > 0$ for the function $f \in G[0,b]$ as follows:
Let $f \in G[0,b]$ and $\alpha > 0$. We say that f has FGI if there exists an increasing sequence $\{X_n\}$ of closed subsets such that $f \in L_1[X_n]$ for each n ($= f\chi_{X_n} \in L_1[0,b]$) and for almost every $t \in [0,b]$, the limit

$$\lim_{n\to\infty}(L)\int_{[0,t]\cap X_n} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)\,ds$$

exists. In this case we define

$$I_G^\alpha f(t) := \lim_{n\to\infty}(L)\int_{[0,t]\cap X_n} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)\,ds,\ t \in [0,b]. \tag{114}$$

==

In fact, this is the whole difficulty: If the function f is too much oscillatory on $(t-\epsilon, t)$ (though still Gauge integrable) it can happen that multiplication with $g(s) := (t-s)^{1-\alpha}$ lets the product on $(t-\epsilon, t)$ oscillate even more so much that this product can fail to be Gauge integrable (and so limit in (112) may fail to exist, Also, the limit in (114) might still exist for some sequence $\{X_n\}$, but perhaps not for all, and for those sequences for which it does exists, it may have different values).

==

The integral in the right hand side of the Definition (114) makes sense: This is a direct consequence of Young inequality since $f \in L_1([0,t] \cap X_n)$ ($=f\chi_{X_n} \in L_1[0,b]$), for any $t \in [0,b]$.

Moreover, if the limit in the right hand side exists for all $\{X_n\}$ which satisfies the assumption of Theorem 2.19, we write

$$I_G^\alpha f(t) = \frac{1}{\Gamma(\alpha)}(G)\int_0^t (t-s)^{\alpha-1} f(s)\, ds,\ t \in [0,b].$$

Obviously, when $f \in L_1[0,b]$, then $f\chi_{X_n} \in L_1[0,b]$ for each n and so, by

Lebesgue dominated theorem it follows

$$\lim_{n\to\infty}(L)\int_{[0,t]\cap X_n} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)\, ds = \lim_{n\to\infty}(L)\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)\chi_{X_n}(s)\, ds = I_L^\alpha f(t),$$

meaning that the Definition (114) coincides with the classic definition of the Riemann-Liouville fractional integral operator whenever the integrand is Lebesgue integrable on $[0,b]$. Moreover, if the function f so that the sequence $\{f\chi_{X_n}\} \subset L_1[0,b]$ satisfies the Vitali equi-integrability condition, it follows in the view of Vitali Convergent Theorem (Theorem 2.5) that $I_G^\alpha f = I_L^\alpha f$.

Lemma 4.1. *The two definition (112) and (114) of the* FGI *are equivalent.*

For any subset X and Y of $[0,b]$, we write $X\Delta Y = (X/Y) \cup (Y/X)$. We are now ready to state and prove the following lemma

Lemma 4.2. *The limit on the right hand side of the Definition (114) (if exists) is unique.*

The proof is a simple consequence of the Generalized Dominated Convergent Theorem (Theorem 2.4). Indeed if $\{Y_n\}$ another increasing sequence of closed subsets whose union is $[0,b]$ such that $f \in L_1[Y_n]$ for each n. Since the sequences $\{X_n\}$ and $\{Y_m\}$ are increasing, then there exists m,n such that (for a fixed $t \in [0,b]$) that (If m,n are the smallest integers such that $X_n \bigcap Y_m \neq \phi$, it follows for any $n > N$, where $N = \max\{m, n\}$)

$$
\begin{aligned}
&\left|(L)\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f\chi_{X_n}(s)\,ds - (L)\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f\chi_{Y_n}(s)\,ds\right| \\
\leq\ & (L)\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |f\chi_{X_n}(s) - f\chi_{Y_n}(s)|\,ds \\
\leq\ & (L)\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |f\chi_{X_n\Delta Y_n}(s)|\,ds| \\
=\ & (L)\int_{[0,t]\cap(X_n\Delta Y_n)} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |f(s)|\,ds \to 0 \text{ as } n\to\infty.
\end{aligned}
$$

===

The proof of Lemma 4.1 is incorrect: One can assert that the last limit in the proof is zero only if f is Lebesgue integrable on the *union* (over all n) of the sets $X_n\Delta Y_n$ (which probably essentially means that f is Lebesgue integrable on $[0, t]$):

If $f \in L_1[0, b]$, then $\int_E f \to 0$ when $E \subset [0, b]$ with $\mu(E) \to 0$

===

Example 4.1. Define the highly oscillatory function $f : [0, 1] \to \mathbb{R}$ by

$$
f(t) := \begin{cases} (-1)^k k, & t \in [\frac{1}{k+1}, \frac{1}{k}],\ k = 1, 2, \cdots \\ 0, & \text{otherwise.} \end{cases}
$$

Since the series $\sum_k \frac{(-1)^k}{k+1}$ is converges but not absolutely converges, it can be easily seen that f is Gauge (but it is not Lebesgue) integrable on $[0, 1]$ (see e.g. Example 7.2 in [30]) and

$$
(D)\int_0^1 f(s)ds = \sum_k^{\infty} \frac{(-1)^k}{k+1}.
$$

Define $X_n = [\frac{1}{n}, 1]$. Obviously, $\{X_n\}$ is an increasing sequence of closed subsets whose union is $[0, 1]$ and $f \in L_1[X_n]$ for each n. Fix $t > 0$ and assume that $t \in [\frac{1}{n_0+1}, \frac{1}{n_0}]$ for some $n_0 \in \mathbb{N}$ where $n_0 = [1/t]$. Thus, we have

$$\begin{aligned}(L)\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)\chi_{X_n}(s)\,ds &= \sum_{k=n_0}^{n} (L)\int_{\frac{1}{k+1}}^{\frac{1}{k}} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}(-1)^k k\,ds \\ &+ (L)\int_{\frac{1}{n_0}}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}(-1)^{n_0}\,ds \\ &= \sum_{k=n_0}^{n} \frac{(-1)^k}{\alpha} k\left[\left(t-\frac{1}{k+1}\right)^{\alpha} - \left(t-\frac{1}{k}\right)^{\alpha}\right] \\ &+ \frac{(-1)^{n_0}}{\alpha} n_0 \left(t-\frac{1}{n_0}\right)^{\alpha}\end{aligned}$$

Now, we will show that the sequence $\{(-1)^k a_k\}$, where $a_k = k\left[\left(t-\frac{1}{k+1}\right)^{\alpha} - \left(t-\frac{1}{k}\right)^{\alpha}\right]$ converges for a fixed $t > \frac{1}{k}$. At the beginning note that

$$\begin{aligned}\lim_{k\to\infty} a_k &= \lim_{k\to\infty} \frac{\left(t-\frac{1}{k+1}\right)^{\alpha} - \left(t-\frac{1}{k}\right)^{\alpha}}{\frac{1}{k}} \\ &= \lim_{k\to\infty} \alpha \frac{\left(t-\frac{1}{k+1}\right)^{\alpha-1}(k+1)^{-2} - \left(t-\frac{1}{k}\right)^{\alpha-1}(k)^{-2}}{\frac{-1}{k^2}} \\ &= \alpha \lim_{k\to\infty}\left\{\left(t-\frac{1}{k}\right)^{\alpha-1} - \left(t-\frac{1}{k+1}\right)^{\alpha-1}\left(\frac{k}{k+1}\right)^2\right\} = 0. \quad (115)\end{aligned}$$

On the other hand, for any $k \geq n_0$ we have

$$\begin{aligned}a_{k+1} - a_k &= (k+1)\left(t-\frac{1}{k+2}\right)^{\alpha} + k\left(t-\frac{1}{k}\right)^{\alpha} - (2k+1)\left(t-\frac{1}{k+1}\right)^{\alpha} \\ &= (2k+1)\left[\frac{k+1}{2k+1}\left(t-\frac{1}{k+2}\right)^{\alpha} + \frac{k}{2k+1}\left(t-\frac{1}{k}\right)^{\alpha} - \left(t-\frac{1}{k+1}\right)^{\alpha}\right].\end{aligned}$$

Applying Jensen's inequality (*cf.* Lemma 2.1) with $\psi(x) = x^{\alpha},\ x \in \mathbb{R}_+$ yields

$$\begin{aligned}\frac{k+1}{2k+1}\left(t-\frac{1}{k+2}\right)^{\alpha} &+ \frac{k}{2k+1}\left(t-\frac{1}{k}\right)^{\alpha} \\ &\leq \left[\frac{k+1}{2k+1}\left(t-\frac{1}{k+2}\right) + \frac{k}{2k+1}\left(t-\frac{1}{k}\right)\right]^{\alpha} \\ &= \left(t-\frac{k+1}{(2k+1)(k+2)} - \frac{1}{2k+1}\right)^{\alpha} \\ &= \left(t-\frac{2k+3}{(2k+1)(k+2)}\right)^{\alpha} < \left(t-\frac{1}{k+1}\right)^{\alpha}.\end{aligned}$$

This implies that $a_{k+1} < a_k$ for each $k \geq n_0$, which in the view of Leibniz's criterion together with (115) yields the convergency of the alternative series $\sum_k (-1)^k a_k$. So the fractional integral of the function f exists a.e. on [0.1] and

$$\begin{aligned} I_G^\alpha f(t) &= \lim_{n\to\infty} (L)\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)\chi_{X_n}(s)\,ds \\ &= \sum_{k=[1/t]}^{\infty} \frac{(-1)^k}{\Gamma(1+\alpha)} k\left[\left(t-\frac{1}{k+1}\right)^\alpha - \left(t-\frac{1}{k}\right)^\alpha\right] \\ &+ \frac{(-1)^{[1/t]}}{\Gamma(1+\alpha)}[1/t]\left(t-\frac{1}{[1/t]}\right)^\alpha . \end{aligned}$$

Example 4.2. Define the highly oscillatory function $f : [0,1] \to \mathbb{R}$ by

$$f(t) := \begin{cases} (-1)^k 2^k a_k, & t \in [1-\frac{1}{2^{k-1}}, 1-\frac{1}{2^k}],\ k = 1,2,\cdots \\ 0, & \text{otherwise.} \end{cases}$$

where $\{a_n\}$ is a decreasing sequence so that the series $\sum_n (-1)^n a_n$ is conditionally convergent (that is, it is not absolutely convergent). It is well-know that f is Gauge (but it is not Lebesgue) integrable on $[0,1]$ and

$$(D)\int_0^1 f(s)ds = \sum_n^\infty (-1)^n a_n.$$

Define $X_n = [0, c_n] = [0, 1-\frac{1}{2^n}]$. Obviously, $\{X_n\}$ is an increasing sequence of closed subsets whose union is $[0,1]$ and $f \in L_1[X_n]$ for each n. Fix $t > 0$ and assume that $t \in (1-\frac{1}{2^{n_0-1}}, 1-\frac{1}{2^{n_0}}]$ for some $n_0 \in \mathbb{N}$. Thus, we have

$$\begin{aligned} (L)\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)\chi_{X_n}(s)\,ds &= \begin{cases} \sum_{k=1}^n (L)\int_{[c_{k-1},c_k]} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}(-1)^k 2^k a_k\,ds, & n < n_0 \\ (L)\int_{c_{n-1}}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} 2^n a_n\,ds, & n = n_0 \\ (L)\int_{c_{n_0-1}}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} 2^{n_0} a_{n_0}\,ds, & n_0 < n. \end{cases} \\ &= \begin{cases} \sum_{k=1}^n 2^k a_k \frac{(t-c_k)^\alpha - (t-c_{k-1})^\alpha}{\Gamma(\alpha+1)} & n < n_0 \\ 2^n a_n \frac{(t-c_{n-1})^\alpha}{\Gamma(\alpha+1)}, & n = n_0 \\ 2^{n_0} a_{n_0} \frac{(t-c_{n_0-1})^\alpha}{\Gamma(\alpha+1)}, & n_0 < n. \end{cases} \end{aligned}$$

By the mean value theorem, there exists $\zeta_k \in (t-c_{k-1}, t-c_k)$ such that

$$
\begin{aligned}
(L)\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)\chi_{X_n}(s)\,ds &= \begin{cases} \displaystyle\sum_{k=1}^{n} 2^k a_k (c_k - c_{k-1}) \frac{\zeta_n^{\alpha-1}}{\Gamma(\alpha)} & n < n_0 \\ 2^n a_n \frac{(t-c_{n-1})^{\alpha}}{\Gamma(\alpha+1)}, & n = n_0 \\ 2^{n_0} a_{n_0} \frac{(t-c_{n_0-1})^{\alpha}}{\Gamma(\alpha+1)}, & n_0 < n. \end{cases} \\
&= \begin{cases} \displaystyle\sum_{k=1}^{n} a_k \frac{\zeta_k^{\alpha-1}}{\Gamma(\alpha)} & n < n_0 \\ 2^n a_n \frac{(t-c_{n-1})^{\alpha}}{\Gamma(\alpha+1)}, & n = n_0 \\ 2^{n_0} a_{n_0} \frac{(t-c_{n_0-1})^{\alpha}}{\Gamma(\alpha+1)}, & n_0 < n. \end{cases}
\end{aligned}
$$

How we chose a_n such that the limit exists????

Example 4.3. Consider the function $f : [0,1] \to \mathbb{R}$ defined by

$$
f(t) := \begin{cases} \dfrac{\sin(1/t)}{t^{\delta}}, & t \in (0,1],\ \delta \in [1,2) \\ 0, & t = 0. \end{cases},
$$

This function is Gauge (but not Lebesgue) integrable on $[0,1]$. It is clear that $f \in L_1[\frac{1}{n}, 1]$ for each n. Thus if we consider the increasing sequence $\{X_n\}$, where $X_n = [\frac{1}{n}, 1]$, then by the Definition (114), we have

$$
I_G^{\alpha} f(t) = \lim_{n\to\infty} \frac{1}{\Gamma(\alpha)} (L) \int_{\frac{1}{n}}^{t} (t-s)^{\alpha-1} s^{-\delta} \sin\left(\frac{1}{s}\right)\,ds,\ t > 0.
$$

5. Fractional calculus in weighted spaces

In this section we take a brief look at the definitions and the properties of some different weighted spaces before turning our attention to the analytical study of fractional integral and differential operators in this spaces .

5.1. **Fractional calculus in weighted space of continuous functions.** In this section, we introduce the definition of the weighted space of continuous functions and define the sequential derivative. Also, we develop some properties and composition identities.

Definition 5.1. For $\gamma \in \mathbb{R}$, we define the weighted space $C_\gamma[0, b]$ of functions defined on $(0, b]$ by

$$C_\gamma[0, b] := \{f \in C(0, b] : t^\gamma f(t) \in C[0, b]\}.$$

In the case f is not defined at $t = 0$ or $\gamma < 0$ we define

$$t^\gamma f(t)|_{t=0} := \lim_{t \to 0^+} t^\gamma f(t).$$

The space $C_\gamma[0, b]$ is nonempty: Let $0 < \beta \leq \gamma$ and define the function $g : (0, 1] \to \mathbb{R}$ by $g(t) = t^{-\beta}$, then $g \in C_\gamma[0, b]$.

Remark 5.1. The space $C_\gamma[0, b]$ satisfy the following properties:

(1) If we define a norm in the space $C_\gamma[0, b]$ by

$$\|f\|_{C_\gamma} := \|t^\gamma f(t)\|_{C[0,b]} = \max_{t \in [0,b]} |t^\gamma f(t)|,$$

then it can be easily seen that the space $\left(C_\gamma[0, b], \|\cdot\|_{C_\gamma}\right)$ is a Banach space.

(2) Let $f \in C_\gamma[0, b]$. For any $\gamma \in (-\infty, 1)$, we have

$$\int_0^b |f(t)|dt = \int_0^b |t^\gamma f(t)|t^{-\gamma} dt \leq \frac{\max_{[0,b]} |t^\gamma f(t)|}{1 - \gamma} b^{1-\gamma},$$

which implies that $C_\gamma[0, b] \subset L_1[0, b]$. However, if $\gamma \in [0, 1)$, then $C[0, b] \subset C_\gamma[0, b] \subset L_1[0, b]$. Moreover, it is not hard to see for $\gamma < 0$ that $C_\gamma[0, b] \subset C[0, b] \subset L_1[0, b]$.

(3) The set of spaces $C_\gamma[0, b]$ is ordered by inclusion according to

$$p \leq q \Leftrightarrow C_p[0, b] \subset C_q[0, b],$$

(4) For any $\gamma \in \mathbb{R}$ we have $t^\gamma \in C(0, b]$. Thus, if $f \in C(0, b]$, then $(t \to t^\gamma f(t)) \in C(0, b]$. Therefore, if in addition, $\lim_{t\to 0} t^\gamma f(t)$ exists and is finite, then $(t \to t^\gamma f(t)) \in C[0, b]$. An immediate consequence is
$f \in C_\gamma[0, b]$ if, and only if, $f \in C(0, b]$ and $\lim\limits_{t\to 0^+} t^\gamma f(t)$ exists and is finite,

(5) For $\alpha > 0,\ \gamma \in \mathbb{R}$ and $f \in C_\gamma[0,b]$ we have, in account of the formula (50), that

$$\lim_{t\to 0} t^{-\alpha} I^\alpha t^\gamma f(t) = \frac{1}{\Gamma(1+\alpha)} \lim_{t\to 0} t^\gamma f(t). \tag{116}$$

Lemma 5.1. *Let* $0<\alpha<1$ *and* $f \in L_1[0,b]$. *If there exists a.e. the limit*

$$\lim_{t\to 0^+} t^{1-\alpha} f(t) = \kappa \in \mathbb{R},$$

Then there also exits a.e. the limit

$$I^{1-\alpha} f(0) := \lim_{t\to 0^+} I^{1-\alpha} f(t) = \kappa\Gamma(\alpha). \tag{117}$$

In particular, if $f \in C_{1-\alpha}[0,b],\ \alpha\in(0,1)$, *then the limit (117) exists for all points on* $[0,b]$.

Proof. Choose an arbitrary $\epsilon>0$. Then, there exists $\delta=\delta(\epsilon)$ such that

$$|t^{1-\alpha} f(t) - \kappa| < \frac{\epsilon\Gamma(1-\alpha)}{\Gamma(\alpha)}, \text{ for almost every } t \in (0,\delta).$$

According to the equality $I^{1-\alpha}t^{\alpha-1} = \Gamma(\alpha)$, we have

$$\begin{aligned}
|I^{1-\alpha} f(t) \ &-\ \kappa\Gamma(\alpha)| = \left|I^{1-\alpha} f(t) - \kappa I^{1-\alpha} t^{\alpha-1}\right| = \left|I^{1-\alpha}[f(t) - \kappa t^{\alpha-1}]\right| \\
&\le \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \left|f(s) - \kappa s^{\alpha-1}\right| ds \\
&= \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} s^{\alpha-1} \left|s^{1-\alpha} f(s) - \kappa\right| ds \\
&= \frac{\epsilon\Gamma(1-\alpha)}{\Gamma(\alpha)\Gamma(1-\alpha)} I^{1-\alpha} t^{\alpha-1} = \epsilon.
\end{aligned}$$

□

somewhat more challenging is to prove the following

Lemma 5.2. *(cf.* [84]*) Let* $\alpha>0$. *For any* $\gamma<1$, *we have*

$$I^\alpha : C_\gamma[0,b] \to C_{\gamma-\alpha}[0,b] \text{ is bounded linear operator.}$$

Proof. In the view of Remark 5.1, we have $C_\gamma[0,b] \subset L_1[0,b]$. Thus the operator I^α makes sense. Now, we will show, for any $f \in C_\gamma[0,b]$, that $I^\alpha f \in C_{\gamma-\alpha}[0,b]$. This is evident by showing for $f \in C_\gamma[0,b]$, that the limit $\lim\limits_{t\to 0^+} t^{\gamma-\alpha} I^\alpha f(t)$ exists and finite. To see this, we let $t\in(0,b]$ and define the *finite* number $\kappa := \lim_{t\to 0^+} t^\gamma f(t)$. By Lemma 3.2, we have

$$t^{\gamma-\alpha} I^\alpha t^{-\gamma} = \frac{\Gamma(1-\gamma)}{\Gamma(1-\gamma+\alpha)}.$$

Thus

$$\begin{aligned}\left|t^{\gamma-\alpha}I^{\alpha}f(t)-\kappa\frac{\Gamma(1-\gamma)}{\Gamma(1-\gamma+\alpha)}\right| &= t^{\gamma-\alpha}\left|I^{\alpha}f(t)-\kappa I^{\alpha}t^{-\gamma}\right|\\ &= t^{\gamma-\alpha}\left|I^{\alpha}[f(t)-\kappa t^{-\gamma}\}]\right|\\ &= t^{\gamma-\alpha}\left|I^{\alpha}[t^{-\gamma}\{t^{\gamma}f(t)-\kappa\}]\right|\end{aligned}$$

Since $t\to t^{\gamma}f(t)$ is continuous at $t=0$, then given $\epsilon>0$ there exists $\delta>0$ such that

$$|t^{\gamma}f(t)-\kappa|<\epsilon\frac{\Gamma(1-\gamma+\alpha)}{\Gamma(1-\gamma)},\ \text{for } 0<t<\delta.$$

So,

$$\begin{aligned}|t^{\gamma-\alpha}I^{\alpha}f(t) &- \kappa t^{\gamma-\alpha}I^{\alpha}t^{-\gamma}|\leq\frac{\Gamma(1-\gamma+\alpha)}{\Gamma(1-\gamma)}t^{\gamma-\alpha}\left|I^{\alpha}[t^{-\gamma}\{\epsilon\}]\right|\\ &\leq t^{\gamma-\alpha}t^{\alpha-\gamma}\epsilon\frac{\Gamma(1-\gamma+\alpha)}{\Gamma(1-\gamma)}\frac{\Gamma(1-\gamma)}{\Gamma(1-\gamma+\alpha)}=\epsilon.\end{aligned}$$

This yields the limit

$$\lim_{t\to0^{+}}t^{\gamma-\alpha}I^{\alpha}f(t)=\kappa\frac{\Gamma(1-\gamma)}{\Gamma(1-\gamma+\alpha)}. \tag{118}$$

Hence the continuity at the point $t=0$. Therefore I^{α} maps $C_{\gamma}[0,b]$ into $C_{\gamma-\alpha}[0,b]$.

Next, we will show that I^{α} is bounded on $C_{\gamma}[0,b]$. Indeed, we have

$$\begin{aligned}\|I^{\alpha}f\|_{C_{\gamma-\alpha}} &= \max_{t\in[0,b]}|t^{\gamma-\alpha}I^{\alpha}f(t)|\\ &= \max_{t\in[0,b]}t^{\gamma-\alpha}\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}s^{-\gamma}|s^{\gamma}f(s)|\ ds\\ &\leq \max_{t\in[0,b]}t^{\gamma-\alpha}\|f\|_{C_{\gamma}}I^{\alpha}t^{-\gamma}=\max_{t\in[0,b]}t^{\gamma-\alpha}\|f\|_{C_{\gamma}}\frac{\Gamma(1-\gamma)}{\Gamma(1-\gamma+\alpha)}t^{\alpha-\gamma}\\ &\leq \frac{\Gamma(1-\gamma)}{\Gamma(1-\gamma+\alpha)}\|f\|_{C_{\gamma}}.\end{aligned}$$

This completes the proof. □

Remark 5.2. Let $\alpha>0$, $\gamma<1$. If $f\in C_{\gamma}[0,b]$, we have in the view of Lemma 5.2 that $I^{\alpha}f\in C_{\gamma-\alpha}[0,b]$. Thus by (50) we deduce that

$$\lim_{t\to0}t^{-\alpha}I^{\alpha}\left[t^{\gamma-\alpha}I^{\alpha}f(t)\right]=\frac{1}{\Gamma(1+\alpha)}\lim_{t\to0}t^{\gamma-\alpha}f(t). \tag{119}$$

Definition 5.2. We denote by $CL_1(0,b]$ the space of all functions $f:(0,b]\to\mathbb{R}$ such that $f\in C(0,b]\cap L_1[0,b]$.

Remark 5.3. By Remark 5.1, the inclusion

$$C_\gamma[0,b] \subset CL_1(0,b], \tag{120}$$

holds for any $\gamma < 1$.

Now put the semi-group property and Lemma 3.24 in mind. Using (120), it can be easily prove the following

Lemma 5.3. *Let $\alpha, \beta > 0$ and $\gamma < 1$. If $f \in C_\gamma[0,b]$, then for any $t \in (0,b]$, we have*

1. $I^\alpha I^\beta f(t) = I^{\alpha+\beta} f(t)$,
2. $D^0 f(t) = f(t)$ *and*

$$D^\beta I^\alpha f(t) = \begin{cases} D^{\beta-\alpha} f(t), & 0 < \alpha \le \beta, \\ I^{\alpha-\beta} f(t), & \alpha \ge \beta > 0, \end{cases} \tag{121}$$

In what follows, we indicate an important property of the fractional integral operator in the space $CL_1(0,b]$. Indeed, we have

Lemma 5.4. *If $\alpha > 0$, then I^α maps $CL_1(0,b]$ into itself.*

Proof. Let $f \in CL_1(0,b]$. Clearly if $\alpha \ge 1$, then $I^\alpha f = II^{\alpha-1} f \in AC[0,b]$. For $0 < \alpha < 1$, it is follows by Lemma 3.3 that $I^\alpha f \in L_1[0,b]$. It remains to show that $I^\alpha f$ is continuous at every $a \in (0,b]$. For any $t \in (a,b]$ we have

$$\begin{aligned} |I^\alpha f(t) \; - \; I^\alpha f(a)| &= \left| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)\,ds - \int_0^a \frac{(a-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)\,ds \right| \\ &\le \frac{1}{\Gamma(\alpha)} \left(\int_0^a |(t-s)^{\alpha-1} - (a-s)^{\alpha-1}||f(s)|\,ds + \int_a^t (t-s)^{\alpha-1}|f(s)|\,ds \right) \\ &\le \frac{1}{\Gamma(\alpha)} \int_0^a |(t-s)^{\alpha-1} - (a-s)^{\alpha-1}||f(s)|\,ds + \frac{\max\limits_{s\in[a,t]} |f(s)|}{\Gamma(1+\alpha)} (t-a)^\alpha. \end{aligned}$$

As $t \to a$ we have $(t-s)^{\alpha-1} \to (a-s)^{\alpha-1}$. Then by Lebesgue dominated theorem, the limit of the right-hand side vanishes and the proof is complete. □

Remark 5.4. A particular case of Lemma 5.4 is the following implication

$$I^\alpha : C_\gamma[0,b] \to CL_1(0,b].$$

Now we present a version of the fundamental theorem of fractional calculus.

Lemma 5.5. *Let $\alpha \in (m-1,m)$, $m \in \mathbb{N}$. If $f \in C(0,b]$ such that $D^\alpha f \in CL_1(0,b]$ then $f \in CL_1(0,b]$ and*

$$I^\alpha D^\alpha f(t) = f(t) - \sum_{j=1}^m \frac{D^{m-j} I^{m-\alpha} f(0)}{\Gamma(1+\alpha-j)} t^{\alpha-j}, \text{ for all t} \in (0,\text{b}].$$

In particular when $\alpha \in (0,1)$, we get

$$I^\alpha D^\alpha f(t) = f(t) - \frac{I^{1-\alpha} f(0)}{\Gamma(\alpha)} t^{\alpha-1}, \text{ for all t} \in (0, \mathrm{b}].$$

Proof. Since $D^\alpha f \in CL_1(0,b]$, we obtain in the view of Lemma 3.24, the relation $D^\alpha I^\alpha D^\alpha f(t) = D^\alpha f(t)$, for all $t \in (0,b]$ which we can write as $D^\alpha[I^\alpha D^\alpha f(t) - f(t)] = 0$. This by the formula (64) implies for some constants $c_i,\ i = 1, 2, \cdots, m$, that

$$f(t) - I^\alpha D^\alpha f(t) = \sum_{j=1}^{m} c_j t^{\alpha-j}. \tag{122}$$

Since Lemma 5.4 implies that $I^\alpha D^\alpha f \in CL_1[0,b]$, we also have by (122) that $f \in CL_1[0,b]$. Also, if we apply $I^{m-\alpha}$ to both sides of (122) we obtain

$$I^{m-\alpha} f(t) - I^m D^\alpha f(t) = \sum_{j=1}^{m} c_j t^{m-j} \frac{\Gamma(1+\alpha-j)}{\Gamma(1+m-j)}.$$

Therefore, by differentiation $(m-k)$-times, $k = 1, 2, \cdots, m$ we obtain

$$D^{m-k} I^{m-\alpha} f(t) - I^k D^\alpha f(t) = \sum_{j=1}^{k} c_j t^{k-j} \frac{\Gamma(1+\alpha-j)}{\Gamma(1+k-j)},\ k = 1, 2, \cdots, m.$$

Taking the limit as $t \to 0$ and noting that $I^k D^\alpha f = I I^{k-1} D^\alpha f$ yields

$$D^{m-k} I^{m-\alpha} f(0) = c_k \Gamma(1+\alpha-k).$$

Hence, the result follows immediately from (122) by replacing $c_k,\ k = 1, 2, \cdots, m$ with

$$\frac{D^{m-k} I^{m-\alpha} f(0)}{\Gamma(1+\alpha-k)}$$

Which completes the proof. □

Definition 5.3. For $\alpha > 0$ and $\xi \in \mathbb{R}$, we define the operators $I^{\alpha,\xi}$ and $D^{\alpha,\xi}$ respectively by

$$I^{\alpha,\xi} f(t) := t^{-\xi} I^\alpha f(t),\ t \in [0,b],$$

and

$$D^{\alpha,\xi} f(t) := t^{\xi} D^\alpha(t),\ t \in [0,b].$$

Example 5.1. Let $\alpha > 0$ and $\xi \in \mathbb{R}$. If $\gamma > -1$, then

1. By Lemma 3.2 we deduce

$$I^{\alpha,\xi} t^\gamma = t^{-\xi} I^\alpha t^\gamma = t^{-\xi} \frac{\Gamma(1+\gamma)}{\Gamma(1+\gamma+\alpha)}\, t^{\alpha+\gamma} = \frac{\Gamma(1+\gamma)}{\Gamma(1+\gamma+\alpha)}\, t^{\alpha+\gamma-\xi},$$

2. By Lemma 3.16 we deduce

$$D^{\alpha,\xi}t^{\gamma}=t^{\xi}D^{\alpha}t^{\gamma}=t^{\xi}\frac{\Gamma(1+\gamma)}{\Gamma(1+\gamma-\alpha)}\,t^{\gamma-\alpha}=\frac{\Gamma(1+\gamma)}{\Gamma(1+\gamma-\alpha)}\,t^{\gamma-\alpha+\xi}.$$

In the view of the above example, it is not hard to prove the following

Lemma 5.6. *If $\alpha\in(m-1,m), m\in\mathbb{N}$, the well-known property of the gamma function for the negative integer values of x (i.e. $[\Gamma(1+x)]^{-1}=0, x=-1,-2,\cdots$) implies*

$$D^{\alpha,\xi}t^{\alpha-j}=0,\ \ j=1,2,\cdots,m.$$

In the view of the formula (50), we have the following

Lemma 5.7. *If f is bounded measurable function such that $\lim_{t\to0}f(t)$ exists, then*

$$\lim_{t\to0}I^{\alpha,\alpha}f(t)=\frac{1}{\Gamma(1+\alpha)}\lim_{t\to0}f(t),\ \ \alpha>0. \tag{123}$$

Also, from Lemma 5.2 we have the following property

Lemma 5.8. *Let $\alpha>0$, $\gamma<1$ and $\xi\in\mathbb{R}$, then*

$$I^{\alpha,\xi}:C_{\gamma}[0,b]\to C_{\gamma+\xi-\alpha}[0,b]\ \textit{is bounded linear operator.}$$

In particular $I^{\alpha,\alpha}:C_{\gamma}[0,b]\to C_{\gamma}[0,b]$ is bounded linear operator.

Proof. For any $f\in C_{\gamma}[0,b]$, we have in the view of Lemma 5.2, that $I^{\alpha}f\in C_{\gamma-\alpha}[0,b]$. Thus the function $(t\to t^{\gamma-\alpha}I^{\alpha}f(t))\in C[0,b]$. The assertion follows immediately since

$$t^{\gamma+\xi-\alpha}I^{\alpha,\xi}f(t)=t^{\gamma+\xi-\alpha}\left[t^{-\xi}I^{\alpha}f(t))\right]=t^{\gamma-\alpha}I^{\alpha}f(t)\in C[0,b].$$

Next, we have

$$\begin{aligned}
\left\|I^{\alpha,\xi}f\right\|_{C_{\gamma+\xi-\alpha}} &= \max_{t\in[0,b]}|t^{\gamma+\xi-\alpha}I^{\alpha,\xi}f(t)|=\max_{t\in[0,b]}|t^{\gamma+\xi-\alpha}t^{-\xi}I^{\alpha}f(t)|\\
&= \max_{t\in[0,b]}t^{\gamma-\alpha}\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}s^{-\gamma}|s^{\gamma}f(s)|\ ds\\
&\le \max_{t\in[0,b]}t^{\gamma-\alpha}\left\|f\right\|_{C_{\gamma}}I^{\alpha}t^{-\gamma}=\max_{t\in[0,b]}t^{\gamma-\alpha}\left\|f\right\|_{C_{\gamma}}\frac{\Gamma(1-\gamma)}{\Gamma(1-\gamma+\alpha)}t^{\alpha-\gamma}\\
&\le \frac{\Gamma(1-\gamma)}{\Gamma(1-\gamma+\alpha)}\left\|f\right\|_{C_{\gamma}}.
\end{aligned} \tag{124}$$

Hence $I^{\alpha,\xi}$ is bounded on $C_{\gamma}[0,b]$. □

Lemma 5.9. *Let $\alpha>0$, $\gamma<1$ and $\xi\in\mathbb{R}$. If $\gamma+\xi-\alpha<1$, then $I^{\alpha,\xi}:C_{\gamma}[0,b]\to L_1[0,b]$ is bounded linear operator.*

Proof. Let $f \in C_\gamma[0,b]$. For any $t \in (0,b]$, we have

$$\begin{aligned}
|I^{\alpha,\xi}f(t)| &\leq \left|t^{-\xi}I^\alpha f(t)\right| \leq \frac{t^{-\xi}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}s^{-\gamma}|s^\gamma f(s)|\ ds \\
&\leq t^{-\xi}\left\|f\right\|_{C_\gamma} I^\alpha t^{-\gamma} = t^{-\xi}\left\|f\right\|_{C_\gamma}\frac{\Gamma(1-\gamma)}{\Gamma(1-\gamma+\alpha)}t^{\alpha-\gamma} \\
&= \left\|f\right\|_{C_\gamma}\frac{\Gamma(1-\gamma)}{\Gamma(1-\gamma+\alpha)}t^{-(\gamma+\xi-\alpha)}.
\end{aligned}$$

Thus, if $\gamma+\xi-\alpha<1$, we get

$$\left\|I^{\alpha,\xi}f\right\|_{L_1[0,b]} = \int_0^b |t^{-\xi}I^\alpha f(t)|dt \leq \frac{\Gamma(1-\gamma)}{(1-(\gamma+\xi-\alpha))\Gamma(1-\gamma+\alpha)}b^{1-(\gamma+\xi-\alpha)}\left\|f\right\|_{C_\gamma}.$$

This completes the proof. □

Lemma 5.10. *Let $\alpha>0$ and $\xi\in\mathbb{R}$. If $\xi<\alpha$, then $I^{\alpha,\xi}$ maps $CL_1(0,b]$ into itself.*

Proof. Let $f\in CL_1(0,b]$. When $\xi\leq 0$ then $t\to t^{-\xi}\in C[0,b]$. It follows, by Lemma 5.4, that $I^\alpha f\in CL_1(0,b]$ and thus $t\to t^{-\xi}I^\alpha f(t)\in CL_1(0,b]$. When $\xi>0$ then $t\to t^{-\xi}\in C(0,b]$. It follows, by Lemma 5.4, that $I^\alpha f\in CL_1(0,b]$ and thus $t\to t^{-\xi}I^\alpha f(t)\in C(0,b]$.

Thus, we only need to show that $t\to t^{-\xi}I^\alpha f(t)\in L_1[0,b]$. To do this, we put in mind that $\xi>0$ and observe the following inequality

$$\begin{aligned}
\left\|I^{\alpha,\xi}f\right\|_{L_1[0,b]} &= \int_0^b \left|t^{-\xi}I^\alpha f(t)\right|dt \leq \int_0^b t^{-\xi}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|f(s)|\,dsdt \\
&\leq \int_0^b t^{-\xi}\int_0^t \frac{(t-s)^{\alpha-1}(t-s)^{-\xi}(t-s)^{\xi}}{\Gamma(\alpha)}|f(s)|\,dsdt \\
&\leq \int_0^b t^{-\xi}t^{\xi}\int_0^t \frac{(t-s)^{\alpha-\xi-1}}{\Gamma(\alpha)}|f(s)|\,dsdt \\
&\leq \int_0^b \frac{\Gamma(\alpha-\xi)}{\Gamma(\alpha)}I^{\alpha-\xi}f(t)dt.
\end{aligned}$$

By Lemma 3.3, $I^{\alpha-\xi}f\in L_1[0,b]$ and thus $\left\|I^{\alpha,\xi}f\right\|_{L_1[0,b]}<\infty$. This completes the proof. □

Definition 5.4. For $\alpha,\beta>0$ and $\xi\in\mathbb{R}$, we define the operators $I_\beta^{\alpha,\xi}$ and $D_\beta^{\alpha,\xi}$ respectively by

$$I_\beta^{\alpha,\xi}f(t) := I^\beta I^{\alpha,\xi}f(t) = I^\beta t^{-\xi}I^\alpha f(t),\ t\in[0,b].$$

and

$$D_\beta^{\alpha,\xi}f(t) := D^\beta D^{\alpha,\xi}f(t) = D^\beta t^{\xi}D^\alpha f(t),\ t\in[0,b].$$

By Example 5.1, we have

Lemma 5.11. *Let $\alpha > 0$ and $\xi \in \mathbb{R}$. If $\gamma > -1$ and $\alpha \geq \xi$, then*

$$I_\beta^{\alpha,\xi} t^\gamma = I^\beta \frac{\Gamma(1+\gamma)}{\Gamma(1+\gamma+\alpha)}\, t^{\alpha+\gamma-\xi} = \frac{\Gamma(1+\gamma)}{\Gamma(1+\gamma+\alpha)} \frac{\Gamma(1+\gamma+\alpha-\xi)}{\Gamma(1+\gamma+\alpha-\xi+\beta)} t^{\gamma+\alpha-\xi+\beta}.$$

Furthermore, by Lemma 3.24, we have

Lemma 5.12. *Let $\alpha, \beta > 0$ and $\xi \in \mathbb{R}$. If $f \in CL_1(0, b]$ such that $t \to t^{-\xi} I^\alpha f(t) \in CL_1(0, b]$. Then*

$$D_\beta^{\alpha,\xi} I_\alpha^{\beta,\xi} f(t) = f(t),\ t \in (0, b].$$

Proof. This follows directly:

$$D_\beta^{\alpha,\xi} I_\alpha^{\beta,\xi} f(t) = D^\beta t^\xi D^\alpha I^\alpha t^{-\xi} I^\beta f(t) = D^\beta t^\xi t^{-\xi} I^\beta f(t) = f(t).$$

□

From Lemma 3.16 we have the following formula for the power function.

Lemma 5.13. *Let $\alpha, \beta > 0$ and $\xi \in \mathbb{R}$. If $\rho > \max\{-1, \beta - \xi - 1\}$, then for any $t \in (0, b]$ we have*

$$D_\beta^{\alpha,\xi} t^\rho = D^\beta t^\xi D^\alpha t^\rho = \frac{\Gamma(1+\rho)\Gamma(1+\rho+\xi-\beta)}{\Gamma(1+\rho-\beta)\Gamma(1+\rho+\xi-\beta-\alpha)} t^{\rho+\xi-\alpha-\beta}.$$

Moreover, by Lemma 3.2 and (64) we directly have

Lemma 5.14. (1) *Let $\alpha > 0$, $\beta \in (0, 1)$ and $\xi \in \mathbb{R}$. Then for any $t \in (0, b]$ we have*

$$D_\beta^{\alpha,\xi} t^{\beta-1} = 0.$$

(2) *Let $\beta > 0$, $\alpha \in (0, 1)$ and $\xi \in \mathbb{R}$ such that $\xi < \alpha + \beta$. Then for any $t \in (0, b]$ we have*

$$D_\beta^{\alpha,\xi} t^{\alpha+\beta-\xi-1} = 0.$$

Lemma 5.15. *Let $\alpha, \beta > 0$ and $\xi < 1 + \alpha$. If $\gamma < \min\{1, 1 + \alpha - \xi\}$, then*

$$I_\beta^{\alpha,\xi} : C_\gamma[0, b] \to C_{\gamma+\xi-\alpha-\beta}[0, b] \textit{ is bounded linear operator.}$$

Proof. Let $f \in C_\gamma[0, b]$, then by Lemma 5.8 we have $I^{\alpha,\xi} f \in C_{\gamma+\xi-\alpha}[0, b]$. Thus, by Lemma 5.2 we have $I_\beta^{\alpha,\xi} f = I^\beta I^{\alpha,\xi} f \in C_{\gamma+\xi-\alpha-\beta}[0, b]$. Also, we have in account of (124)

$$\begin{aligned}
\left\|I_\beta^{\alpha,\xi}\right\|_{C_{\gamma+\xi-\alpha-\beta}} &= \max_{t\in[0,b]}|t^{\gamma+\xi-\alpha-\beta}I^\beta I^{\alpha,\xi}f(t)| = \max_{t\in[0,b]}|t^{\gamma+\xi-\alpha-\beta}I^\beta g(t)|,\ g(t)=I^{\alpha,\xi}f(t)\\
&\leq \max_{t\in[0,b]}t^{\gamma+\xi-\alpha-\beta}\frac{1}{\Gamma(\beta)}\int_0^t(t-s)^{\beta-1}s^{-(\gamma+\xi-\alpha)}|s^{(\gamma+\xi-\alpha)}g(s)|\ ds\\
&\leq \max_{t\in[0,b]}t^{\gamma+\xi-\alpha-\beta}\left\|g\right\|_{C_{\gamma+\xi-\alpha}} I^\beta t^{-(\gamma+\xi-\alpha)}\\
&= \max_{t\in[0,b]}t^{\gamma+\xi-\alpha-\beta}\left\|g\right\|_{C_{\gamma+\xi-\alpha}}\frac{\Gamma(1-(\gamma+\xi-\alpha))}{\Gamma(1-(\gamma+\xi-\alpha)+\beta)}t^{\beta-(\gamma+\xi-\alpha))}\\
&\leq \left\|g\right\|_{C_{\gamma+\xi-\alpha}}\frac{\Gamma(1-\gamma-\xi+\alpha)}{\Gamma(1-\gamma-\xi+\alpha+\beta)}\\
&\leq \frac{\Gamma(1-\gamma-\xi+\alpha)}{\Gamma(1-\gamma-\xi+\alpha+\beta)}\frac{\Gamma(1-\gamma)}{\Gamma(1-\gamma+\alpha)}\left\|f\right\|_{C_\gamma}.
\end{aligned}$$

Hence $I_\beta^{\alpha,\xi}$ is bounded on $C_\gamma[0,b]$. □

5.2. **Fractional calculus in weighted space of integrable functions.** In this section, we introduce the definition of the weighted space of integrable functions and develop some of the properties of the fractional integral operators in this space.

Definition 5.5. For $\gamma\in\mathbb{R}$ and $p\geq 1$, we define the weighted space $L_p^\gamma[0,b]$ of measurable functions on $[0,b]$ by

$$L_p^\gamma[0,b]:=\{f: f \text{ measurable on } [0,b],\ t^\gamma f(t)\in L_p[0,b]\}.$$

Remark 5.5. For the given p, we introduce the conjugate exponent q and consider $\gamma\in(-\infty,1/q)$. The space $L_p^\gamma[0,b]$ satisfies the following properties:

(1) Let $f\in L_p^\gamma[0,b]$. For any $\gamma\in(-\infty,1/q)$, the function $t\to t^\gamma f(t)$ is measurable on $(0,b)$: This is evidently proved by noting that the function is a composition of the measurable function $t\to(t^\gamma,f(t))$ with the continuous function $F(u,v):=uv$. Another way is to defined $F(t,u):=t^q u$. This function is Carathéodory "even continuous" function on $(0,1)\times\mathbb{R}$. Now we use that for every Carathéodory function $F(t,u)$ the corresponding superposition operator sends measurable functions into measurable functions, that is, $t\to F(t,f(t))$ is measurable for every measurable f (this is clear if f is a simple function, and the general case follows by approximation with simple functions and using the continuity of F with respect to u). As a consequence, we are able to define a norm in the space $L_p^\gamma[0,b]$ as

$$\|f\|_{L_p^\gamma}:=\left(\int_0^b|t^\gamma f(t)|^p dt\right)^{\frac{1}{p}}.$$

The space $\left(L_p^\gamma[0,b], \|\cdot\|_{L_p^\gamma}\right)$ is a Banach space in the view of the isometry

$$\|f\|_{L_p^\gamma} = \|t^\gamma f\|_{L_p}. \tag{125}$$

Owing to (125), the Hölder inequality for the spaces $L_p^\gamma[0,b]$ given by

$$\int_0^b f(t)g(t)dt \leq \|f\|_{L_p^\gamma} \|g\|_{L_q^{-\gamma}}, \ 1 < p < \infty.$$

This follows from

$$\int_0^b f(t)g(t)dt = \int_0^b \left(t^\gamma f(t)\right)\left(t^{-\gamma} g(t)\right) dt \leq \|t^\gamma f\|_{L_p} \left\|t^{-\gamma} g\right\|_{L_q}.$$

(2) For $\gamma \in (-\infty, 1/q)$, we have by Hölder inequality, that

$$\begin{aligned}\int_0^b |f(t)|dt &= \int_0^b t^{-\gamma}|t^\gamma f(t)|dt \leq \left(\int_0^b t^{-q\gamma}dt\right)^{\frac{1}{q}} \left(\int_0^b |t^\gamma f(t)|^p dt\right)^{\frac{1}{p}} \\ &\leq \left(\frac{b^{1-q\gamma}}{1-q\gamma}\right)^{\frac{1}{q}} \|f\|_{L_p^\gamma} = \frac{b^{1/q-\gamma}}{(1-q\gamma)^{\frac{1}{q}}} \|f\|_{L_p^\gamma},\end{aligned}$$

which yields $L_p^\gamma[0,b] \subset L_1[0,b]$. Thus if $f \in L_p^\gamma[0,b]$ we obtain, in the view of Lemma 3.3, that $I^\alpha f \in L_1[0,b]$.

(3) The set of spaces $L_p^\gamma[0,b]$ is ordered by inclusion according to

$$\gamma \leq \delta \Leftrightarrow L_p^\gamma[0,b] \subset L_p^\delta[0,b].$$

Moreover $L_p^0[0,b] \equiv L_p[0,b]$.

For Example, let $f(t) = t^{-1/8}$ and $p = 4$, hence $q = 4/3$. Then $f \in L_4[0,1]$ and therefore

$$t \to t^{1/2} f(t) = t^{3/8} \in L_\infty[0,1] \subset L_4[0,1].$$

$t \to t^{-1/2} f(t) = t^{-5/8} \in L_\beta[0,1]$, $\beta \in [1, 8/5)$. We note that $L_4[0,1] \subset L_\beta[0,1]$. We now consider some mapping properties of the operator I^α in the spaces $L_p^\gamma[0,b]$. Roughly speaking, we shall see that fractional integration improves the smoothness properties of functions from the spaces $L_p^\gamma[0,b]$.

Lemma 5.16. *Let $\alpha > 0$. Consider for a given $p > \max\{1, 1/\alpha\}$ the conjugate exponent q. For any $\gamma < (1/q)$ and $\zeta \geq 1/p - 1/q$, the map*

$$I^\alpha : L_p^\gamma[0,b] \to L_q^{\gamma-\alpha+\zeta}[0,b] \text{ is compact operator.}$$

In particular the map

$$I^\alpha : L_p^\gamma[0,b] \to L_q^\gamma[0,b] \text{ is compact operator.}$$

Proof. By Remark 5.5, we have $L_p^\gamma[0,b] \subset L_1[0,b]$. Thus the operator I^α makes sense. To show, for any $f \in L_p^\gamma[0,b]$, that $I^\alpha f \in L_q^{\gamma-\alpha+\zeta}[0,b]$, we firstly note, in the view of Hölder inequality with $1/p + 1/q = 1$ that

$$\begin{aligned} I^\alpha f(t) &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s)\,ds = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} s^{-\gamma} s^\gamma f(s)\,ds \\ &\le \frac{1}{\Gamma(\alpha)}\left(\int_0^t (t-s)^{q\alpha-q} s^{-q\gamma}\,ds\right)^{\frac{1}{q}}\left(\int_0^t |s^\gamma f(s)|^p\,ds\right)^{\frac{1}{p}} \\ &\le \frac{1}{\Gamma(\alpha)}\left(\Gamma(q\alpha-q+1) I^{q\alpha-q+1} t^{-q\gamma}\right)^{\frac{1}{q}} \|f\|_{L_p^\gamma} \\ &= \frac{1}{\Gamma(\alpha)}\left(\Gamma(q\alpha-q+1)\frac{\Gamma(1-q\gamma)}{\Gamma(2-q\gamma+q\alpha-q)}\right)^{\frac{1}{q}} t^{1/q-\gamma+\alpha-1}\|f\|_{L_p^\gamma} \\ &= \frac{1}{\Gamma(\alpha)}\left(\Gamma(q\alpha-q+1)\frac{\Gamma(1-q\gamma)}{\Gamma(2-q\gamma+q\alpha-q)}\right)^{\frac{1}{q}} t^{\alpha-\gamma-1/p}\|f\|_{L_p^\gamma} \qquad (126) \end{aligned}$$

Therefore

$$t^{\gamma-\alpha+\zeta} I^\alpha f(t) \le \frac{1}{\Gamma(\alpha)}\left(\frac{\Gamma(q\alpha-q+1)\Gamma(1-q\gamma)}{\Gamma(2-q\gamma+q\alpha-q)}\right)^{\frac{1}{q}} t^{\zeta-1/p}\|f\|_{L_p^\gamma},$$

which yields in the view of $q(\zeta-1) > -2$

$$\begin{aligned} \left(\int_0^b |t^{\gamma-\alpha+\zeta} I^\alpha f(t)|^q dt\right)^{1/q} &\le \frac{\|f\|_{L_p^\gamma}}{\Gamma(\alpha)}\left(\frac{\Gamma(q\alpha-q+1)\Gamma(1-q\gamma)}{\Gamma(2-q\gamma+q\alpha-q)}\right)^{\frac{1}{q}}\left[\frac{\left(t^{q\zeta-q/p+1}\right)_0^b}{q\zeta-q/p+1}\right]^{1/q} \\ &= \frac{\|f\|_{L_p^\gamma}}{\Gamma(\alpha)}\left(\frac{\Gamma(q\alpha-q+1)\Gamma(1-q\gamma)}{\Gamma(2-q\gamma+q\alpha-q)}\right)^{\frac{1}{q}}\left[\frac{\left(t^{q\zeta-(q-1)+1}\right)_0^b}{q(\zeta-1/p)}\right]^{1/q} \\ &= \frac{\|f\|_{L_p^\gamma}}{\Gamma(\alpha)}\left(\frac{\Gamma(q\alpha-q+1)\Gamma(1-q\gamma)}{\Gamma(2-q\gamma+q\alpha-q)}\right)^{\frac{1}{q}}\left[\frac{b^{q(\zeta-1)+2}}{q(\zeta-1)+2}\right]^{1/q} \\ &\le \frac{\|f\|_{L_p^\gamma}}{\Gamma(\alpha)}\left(\frac{\Gamma(q\alpha-q+1)\Gamma(1-q\gamma)}{\Gamma(2-q\gamma+q\alpha-q)}\right)^{\frac{1}{q}}\frac{b^{\zeta+1/q-1/p}}{[q(\zeta-1)+2]^{1/q}}. \end{aligned}$$

That is

$$\|I^\alpha f\|_{L_p^{\gamma-\alpha+\zeta}} \le \frac{\|f\|_{L_p^\gamma}}{\Gamma(\alpha)}\left(\frac{\Gamma(q\alpha-q+1)\Gamma(1-q\gamma)}{\Gamma(2-q\gamma+q\alpha-q)}\right)^{\frac{1}{q}}\frac{b^{\zeta+1/q-1/p}}{[q(\zeta-1)+2]^{1/q}}.$$

Thus the operator $I^\alpha : L_p^\gamma[0,b] \to L_q^{\gamma-\alpha+\zeta}[0,b]$ is bounded linear operator. Since $L_p^\gamma[0,b] \subset L_1[0,1]$ then the compactness of I^α is ready available from Theorem 3.2.

The particular case follows immediately by putting $\zeta = \alpha$. This completes the proof. $\square$

Lemma 5.17. *Let $\alpha > 0$. Consider for a given $p > \max\{1, 1/\alpha\}$ the conjugate exponent q. For any $\gamma \in (-\infty, \nu)$ where $\nu = \min\{\alpha - 1/p, 1/q\}$ the map $I^\alpha : L_p^\gamma[0,b] \to C[0,b]$ is compact.*

Remark 5.6. It is not hard to see that

$$\begin{cases} \alpha - 1/p < 1/q, & 0 < \alpha < 1 \\ \alpha - 1/p \geq 1/q, & \alpha \geq 1 \end{cases}$$

However, $q(\alpha - 1) > -1$ for any $\alpha > 0$.

Proof. of Lemma 5.17. Firstly, we observe that our assumption imposed on γ yields $\gamma < 1/q$. Therefore by Remark 5.5, the operator I^α makes sense. Now let $f \in L_p^\gamma[0,b]$. From inequality (126), we have

$$|I^\alpha f(t)| \leq \frac{1}{\Gamma(\alpha)} \left(\Gamma(q\alpha - q + 1) \frac{\Gamma(1 - q\gamma)}{\Gamma(2 - q\gamma + q\alpha - q)} \right)^{\frac{1}{q}} t^{\alpha - \gamma - 1/p} \|f\|_{L_p^\gamma},$$

which implies

$$I^\alpha f(t) \to 0 \text{ as } t \to 0.$$

Thus we are able to define $I^\alpha f(0) := \lim_{t\to 0} I^\alpha f(t) = 0$.

Since $\alpha - 1/p > 0$, by Hölder inequality with $1/p + 1/q = 1$, we have for each $\tau < t$ that

$$\begin{aligned}
|I^\alpha f(t) - I^\alpha f(\tau)|\Gamma(\alpha) &= \left| \int_0^t (t-s)^{\alpha-1} f(s)\, ds - \int_0^\tau (\tau - s)^{\alpha-1} f(s)\, ds \right| \\
&\leq \left(\int_0^\tau |(t-s)^{\alpha-1} - (\tau-s)^{\alpha-1}| s^{-\gamma} s^\gamma |f(s)|\, ds \right. \\
&+ \left. \int_\tau^t (t-s)^{\alpha-1} s^{-\gamma} s^\gamma |f(s)|\, ds \right) \\
&\leq \left[\left(\int_0^\tau |(t-s)^{\alpha-1} - (\tau-s)^{\alpha-1}|^q s^{-q\gamma}\, ds \right)^{1/q} \right. \\
&+ \left. \left(\int_\tau^t (t-s)^{(\alpha-1)q} s^{-q\gamma}\, ds \right)^{1/q} \right] \|f\|_{L_p^\gamma[0,b]}.
\end{aligned}$$

In order to evaluate the second integral we consider the two cases:

1. When $\gamma < 0$, we have in the view of $q\gamma < 0$ that

$$\begin{aligned}\int_\tau^t (t-s)^{(\alpha-1)q} s^{-q\gamma}\, ds &\le t^{-q\gamma}\int_\tau^t (t-s)^{(\alpha-1)q}\ \ ds\\ &= b^{-q\gamma}\left[-\frac{(t-s)^{(\alpha-1)q+1}}{(\alpha-1)q+1}\right]_\tau^t\\ &= b^{-q\gamma}\frac{(t-\tau)^{(\alpha-1)q+1}}{(\alpha-1)q+1},\end{aligned}$$

2. When $\gamma \in [0,\nu)$, putting in mind that $q\gamma \ge 0$ and proceed after the change of variable $s = \tau + \eta(t-\tau)$ as follows

$$\begin{aligned}\int_\tau^t (t-s)^{(\alpha-1)q} s^{-q\gamma}\, ds &\le \int_\tau^t (t-s)^{(\alpha-1)q}(s-\tau)^{-q\gamma}\, ds\\ &= \int_0^1 [(t-\tau)(1-\eta)]^{(\alpha-1)q}[\eta(t-\tau)]^{-q\gamma}(t-\tau)d\eta\\ &= \int_0^1 (1-\eta)^{(\alpha-1)q}\eta^{-q\gamma}(t-\tau)^{q(\alpha-1-\gamma)+1}d\eta\\ &= (t-\tau)^{q(\alpha-1-\gamma)+1}B(1-q\gamma, 1+q(\alpha-1-\gamma)),\end{aligned}$$

where B is the well-known beta function.

However for $\gamma \in (-\infty,\nu)$, we have

$$\left(\int_\tau^t (t-s)^{(\alpha-1)q}s^{-q\gamma}\, ds\right)^{(1/q)} \le |t-\tau|^\rho c_0(\alpha,\gamma,q),$$

where

$$\rho = \begin{cases}\alpha-\gamma-(1/p), & \gamma\in[0,\nu)\\ \alpha-(1/p), & \gamma<0.\end{cases}$$

Here, $c_0(\alpha,\gamma,q)$ is finite constant given by

$$c_0(\alpha,\gamma,q) = \begin{cases}[B(1-q\gamma,1+q(\alpha-1-\gamma))]^{(1/q)} & \text{, for } \gamma\in[0,\nu),\\ \dfrac{b^{-\gamma}}{[(\alpha-1)q+1]^{1/q}}, & \text{for } \gamma<0.\end{cases} \tag{127}$$

Therefore,

$$\begin{aligned}|I^\alpha f(t)-I^\alpha f(\tau)|\Gamma(\alpha) &\le \left(\int_0^\tau |(t-s)^{\alpha-1}-(\tau-s)^{\alpha-1}|^q s^{-q\gamma}\, ds\right)^{1/q}\|f\|_{L_p^\gamma[0,b]}\\ &+ \ c_0(\alpha,\gamma,q)|t-\tau|^\rho]\,\|f\|_{L_p^\gamma[0,b]}\,.\end{aligned} \tag{128}$$

Now, putting in mind that $\rho > 0$ and noting, by Lebesgue dominated theorem, that the first inner integral vanishes as $t\to\tau$, it follows by (128) that $I^\alpha f \in C[0,b]$.

Also, it can be seen in the view of $\alpha-\gamma-1/p>0$ that

$$\max_{t\in[0,b]}|I^{\alpha}f(t)| \leq \frac{1}{\Gamma(\alpha)}\left(\frac{\Gamma(q\alpha-q+1)\Gamma(1-q\gamma)}{\Gamma(2-q\gamma+q\alpha-q)}\right)^{\frac{1}{q}} b^{\alpha-\gamma-1/p}\,\|f\|_{L_p^{\gamma}}. \tag{129}$$

This yields the boundedness (hence the continuity) of the linear operator I^{α}. The equicontinuity of the image of bounded subsets of $L_p^{\gamma}[0,b]$ under I^{α} is an immediate consequence of (128). Moreover, the boundedness of this image follows from (129).

That is I^{α} takes bounded subsets of $L_p^{\gamma}[0,b]$ into uniformly bounded equicontinuous subsets of $C[0,b]$. By Arzelà-Ascoli theorem 2.1, it follows that the map $I^{\alpha}: L_p^{\gamma}[0,b]\to C[0,b]$ is compact. $\square$

Corollary 5.1. *Let $\alpha\in(0,1)$ and $p>\frac{1}{\alpha}$. For any $\gamma\in(-\infty,\alpha-\frac{1}{p})$, the map $I^{\alpha}: L_p^{\gamma}[0,b]\to\mathcal{H}^{\varrho}[0,b]$ is compact, where $\varrho=\min\{1-\alpha,\alpha-\dfrac{1}{p}-\gamma\}$.*

Remark 5.7. It is worth to remark that $\varrho=\alpha-\dfrac{1}{p}-\gamma$ in each of the following cases

$(A_1)\,:\,\alpha\in(0,\dfrac{1}{2}]$ and $\gamma\in[0,\alpha-\frac{1}{p}]$ Or,

$(A_2)\,:\,\alpha\in(\dfrac{1}{2},1),\ p>\dfrac{1}{2\alpha-1}$ and $\gamma\in[2\alpha-1-\frac{1}{p},\alpha-\frac{1}{p})$.

Otherwise $\varrho=1-\alpha$.

We observe that the assumptions of Lemma 5.17 with $\alpha\in(0,1)$ are satisfied. Moreover, in the case (A_2), putting in mind that $\frac{1}{2\alpha-1}>\frac{1}{\alpha}$ for $\alpha\in(\frac{1}{2},1)$ yields $p>\frac{1}{\alpha}$ which is necessary condition required to apply Lemma 5.17.

Remark 5.8. It is possible to show an even stronger statement in the case $\alpha\geq1$: Under this assumption, if $f\in L_p^{\gamma}[0,b],\ p>1/(1-\alpha)$ we have in the view of Lemma 5.17 that $I^{1-\alpha}f\in C[0,b]$ and consequently $I^{\alpha}f=I^{1}I^{1-\alpha}f\in AC[0,b]$.

Proof. of Corollary 5.1. Let $f\in L_p^{\gamma}[0,b]$. Because $\alpha\in(0,1)$, we have in the view of the formula (19) that

$$\left|\left(\frac{1}{t-s}\right)^{1-\alpha}-\left(\frac{1}{\tau-s}\right)^{1-\alpha}\right|\leq\left(\frac{1}{\tau-s}-\frac{1}{t-s}\right)^{1-\alpha},\ \tau<t.$$

Therefore we obtain by (128)

$$\begin{aligned}
|I^\alpha f(t)-I^\alpha f(\tau)|\frac{\Gamma(\alpha)}{\|f\|_{L_p^\gamma[0,b]}} &\leq \left(\int_0^\tau \left|\left(\frac{1}{t-s}\right)^{1-\alpha}-\left(\frac{1}{\tau-s}\right)^{1-\alpha}\right|^q s^{-q\gamma}\,ds\right)^{1/q} \\
&+ c_0(\alpha,\gamma,q)|t-\tau|^{(\alpha-1-\gamma)+1/q} \\
&\leq \left(\int_0^\tau \left(\frac{1}{\tau-s}-\frac{1}{t-s}\right)^{q(1-\alpha)} s^{-q\gamma}\,ds\right)^{1/q} \\
&+ c_0(\alpha,\gamma,q)|t-\tau|^{\alpha-1/p-\gamma} \\
&\leq \left(\int_0^\tau \left(\frac{t-\tau}{(\tau-s)(t-s)}\right)^{q(1-\alpha)} s^{-q\gamma}\,ds\right)^{1/q} \\
&+ c_0(\alpha,\gamma,q)|t-\tau|^{\alpha-1/p-\gamma} \\
&\leq |t-\tau|^{1-\alpha}\left(\int_0^\tau \left(\frac{1}{(\tau-s)^2}\right)^{q(1-\alpha)} s^{-q\gamma}\,ds\right)^{1/q} \\
&+ c_0(\alpha,\gamma,q)|t-\tau|^{\alpha-1/p-\gamma} \\
&= |t-\tau|^{1-\alpha}\left(\Gamma(2q(1-\alpha)+1)I^{2q(1-\alpha)+1}\tau^{-q\gamma}\right)^{1/q} \\
&+ c_0(\alpha,\gamma,q)|t-\tau|^{\alpha-1/p-\gamma}
\end{aligned}$$

Thus, we arrive at

$$|I^\alpha f(t)-I^\alpha f(\tau)|\frac{\Gamma(\alpha)}{\|f\|_{L_p^\gamma[0,b]}} \leq$$

$$\begin{cases} |t-\tau|^{(1-\alpha)}c_1(\alpha,\gamma,q)+c_0(\alpha,\gamma,q)|t-\tau|^{(\alpha-1/p-\gamma)}, & \gamma\in[0,\alpha-\frac{1}{p}), \\ |t-\tau|^{(1-\alpha)}c_1(\alpha,\gamma,q)+c_0(\alpha,\gamma,q)|t-\tau|^{(\alpha-1/p)}, & \gamma<0. \end{cases} \tag{130}$$

Where

$$c_1(\alpha,\gamma,q) := \left(\frac{\Gamma(2q(1-\alpha)+1)\Gamma(1-q\gamma)}{\Gamma(2q(1-\alpha)+2-q\gamma)}b^{2q(1-\alpha)+1-q\gamma}\right)^{1/q}$$

The inequality (130) takes the form

$$|I^\alpha f(t)-I^\alpha f(\tau)| \leq$$

$$\begin{cases} |t-\tau|^{\varrho}\left[c_1|t-\tau|^{(1-\alpha)-\varrho}+c_0|t-\tau|^{\alpha-1/p-\gamma-\varrho}\right]\dfrac{\|f\|_{L_p^\gamma[0,b]}}{\Gamma(\alpha)}, & \gamma\in[0,\alpha-\frac{1}{p}), \\ |t-\tau|^{1-\alpha}\left[c_1+c_0|t-\tau|^{2\alpha+\frac{1}{q}}\right]\dfrac{\|f\|_{L_p^\gamma[0,b]}}{\Gamma(\alpha)},, & \gamma<0. \end{cases}$$

where $\varrho=\min\{1-\alpha,\alpha-1/p-\gamma\}$. Now, putting in mind Lemma 5.17 finishes the proof. ∎

The following result follows immediately from Lemma 5.17

Lemma 5.18. *Let $\alpha > 0$. Consider for a given $p > \max\{1, 1/\alpha\}$ the conjugate exponent q. For any $\gamma < (-\infty, \nu)$ where $\nu = \min\{\alpha - 1/p, 1/q\}$, the map*

$$I^\alpha : L_p^\gamma[0,b] \to C_{\alpha-\gamma}[0,b] \text{ is bounded.}$$

Lemma 5.19. *Let $\alpha \in (0,1)$ and consider for a given $p > 1/\alpha$ that $0 \le \gamma < \alpha - 1/p$. Then the map*

$$I^\alpha : BL_p^\gamma[0,b] \to C_{-\varrho}[0,b] \text{ where } 0 < \varrho < \min\{\alpha, 1-\alpha, \alpha - \frac{1}{p} - \gamma\},$$

is bounded. Here $BL_p^\gamma[0,b] := \{f \in L_p^\gamma[0,b] : \lim_{t\to 0} f(t)$ exists and finite$\}$.

Proof. Firstly, we remark that (*cf.* [67] page 353) the limit $\lim\limits_{t\to 0^+} t^{-\alpha} I^\alpha f(t)$ exists and finite. Moreover,

$$t^{-\varrho} I^\alpha f(t)|_{t=0} := \lim_{t\to 0^+} t^{-\varrho} I^\alpha f(t) = \lim_{t\to 0^+} t^{\alpha-\varrho} t^{-\alpha} I^\alpha f(t) = 0.$$

Now, We will show, for any $f \in BL_p^\gamma[0,b]$, that $t^{-\varrho} I^\alpha f(t) \in C(0,b]$. To see this, we let $f \in L_p^\gamma[0,b]$. Since $\alpha - 1/p > 0$, by Hölder inequality with $1/p + 1/q = 1$, we have for each $0 < \tau < t$ that

$$\begin{aligned}
|t^{-\varrho} I^\alpha f(t) &- \tau^{-\varrho} I^\alpha f(\tau)|\Gamma(\alpha) = \left| t^{-\varrho} \int_0^t (t-s)^{\alpha-1} f(s)\, ds \right.\\
&- \left. \tau^{-\varrho} \int_0^\tau (\tau-s)^{\alpha-1} f(s)\, ds \right|\\
&\le \left| t^{-\varrho} \int_0^t (t-s)^{\alpha-1} f(s)\, ds - t^{-\varrho} \int_0^\tau (\tau-s)^{\alpha-1} f(s)\, ds \right|\\
&+ \left| t^{-\varrho} \int_0^\tau (\tau-s)^{\alpha-1} f(s)\, ds - \tau^{-\varrho} \int_0^\tau (\tau-s)^{\alpha-1} f(s)\, ds \right|\\
&+ \left| t^{-\varrho} - \tau^{-\varrho} \right| \int_0^\tau (\tau-s)^{\alpha-1} |f(s)|\, ds\\
&\le t^{-\varrho} \left(\int_0^\tau |(t-s)^{\alpha-1} - (\tau-s)^{\alpha-1}| s^{-\gamma} s^\gamma |f(s)|\, ds \right.\\
&+ \left. \int_\tau^t (t-s)^{\alpha-1} s^{-\gamma} s^\gamma |f(s)|\, ds \right)\\
&+ \left| t^{-\varrho} - \tau^{-\varrho} \right| \int_0^\tau (\tau-s)^{\alpha-1} s^{-\gamma} s^\gamma |f(s)|\, ds,
\end{aligned}$$

whence

$$\begin{aligned}
|t^{-\varrho}I^{\alpha}f(t) &- \tau^{-\varrho}I^{\alpha}f(\tau)|\Gamma(\alpha) \\
&\leq t^{-\varrho}\left[\left(\int_0^{\tau}|(t-s)^{\alpha-1}-(\tau-s)^{\alpha-1}|^q s^{-q\gamma}\,ds\right)^{1/q}\right. \\
&+ \left.\left(\int_{\tau}^{t}(t-s)^{(\alpha-1)q}s^{-q\gamma}\,ds\right)^{1/q}\right]\|f\|_{L_p^{\gamma}[0,b]} \\
&+ \left|t^{-\varrho}-\tau^{-\varrho}\right|\left[\left(\int_0^{\tau}(t-s)^{(\alpha-1)q}s^{-q\gamma}\,ds\right)^{1/q}\right]\|f\|_{L_p^{\gamma}[0,b]}\,.
\end{aligned}$$

By Lemma 3.2 we have

$$\begin{aligned}
\left(\int_0^{\tau}(t-s)^{(\alpha-1)q}s^{-q\gamma}\,ds\right)^{1/q} &= \left(\Gamma((\alpha-1)q+1)I^{(\alpha-1)q+1}\tau^{-q\gamma}\right)^{1/q} \\
&= \left(\Gamma(q\alpha-q+1)\frac{\Gamma(1-q\gamma)}{\Gamma(2-q\gamma+q\alpha-q)}\right)^{\frac{1}{q}}\tau^{\alpha-\gamma-1/p} \\
&\leq \left(\Gamma(q\alpha-q+1)\frac{\Gamma(1-q\gamma)}{\Gamma(2-q\gamma+q\alpha-q)}\right)^{\frac{1}{q}}b^{\alpha-\gamma-1/p}.
\end{aligned}$$

Putting in mind that $\alpha\in(0,1)$ and proceeding as in the proof of Corollary 5.1 yields

$$\begin{aligned}
|t^{-\varrho}I^{\alpha}f(t) &- \tau^{-\varrho}I^{\alpha}f(\tau)| \leq \left[t^{-\varrho}|t-\tau|^{\varrho}\left(c_1|t-\tau|^{(1-\alpha)-\varrho}+c_0|t-\tau|^{\alpha-1/p-\gamma-\varrho}\right)\right. \\
&+ \left.|t^{-\varrho}-\tau^{-\varrho}|\left(\frac{\Gamma(q\alpha-q+1)\Gamma(1-q\gamma)}{\Gamma(2-q\gamma+q\alpha-q)}\right)^{\frac{1}{q}}b^{\alpha-\gamma-1/p}\right]\frac{\|f\|_{L_p^{\gamma}[0,b]}}{\Gamma(\alpha)} \\
&\leq \left[c_1|t-\tau|^{(1-\alpha)-\varrho}+c_0|t-\tau|^{\alpha-1/p-\gamma-\varrho}\right]\frac{\|f\|_{L_p^{\gamma}[0,b]}}{\Gamma(\alpha)} \\
&+ |t^{-\varrho}-\tau^{-\varrho}|\left(\frac{\Gamma(q\alpha-q+1)\Gamma(1-q\gamma)}{\Gamma(2-q\gamma+q\alpha-q)}\right)^{\frac{1}{q}}b^{\alpha-\gamma-1/p}\frac{\|f\|_{L_p^{\gamma}[0,b]}}{\Gamma(\alpha)} \qquad (131)
\end{aligned}$$

with some finite constants $C_i=C_i(\alpha,\gamma,q), i=0,1$ depending only on $\alpha,\ \gamma$ and q. Therefore, the function $t\to t^{-\varrho}I^{\alpha}f(t)\in C(0,b]$. Finally, we will show that I^α is bounded from $BL_p^\gamma[0,b]$ to $C_{-\varrho}[0,b]$. Evidently, we have

$$\begin{aligned}
\|I^\alpha f\|_{C_{-\varrho}} &= \max_{t\in[0,b]} |t^{-\varrho} I^\alpha f(t)| \\
&\leq \max_{t\in[0,b]} t^{-\varrho}\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} s^{-\gamma}|s^\gamma f(s)| \; ds \\
&\leq \max_{t\in[0,b]} \frac{t^{-\varrho}}{\Gamma(\alpha)}\left(\int_0^t (t-s)^{q\alpha-q}s^{-q\gamma}\, ds\right)^{\frac{1}{q}}\left(\int_0^t |s^\gamma f(s)|^p\, ds\right)^{\frac{1}{p}} \\
&\leq \max_{t\in[0,b]} \frac{t^{-\varrho}}{\Gamma(\alpha)}\left(\frac{\Gamma(q\alpha-q+1)\Gamma(1-q\gamma)}{\Gamma(2-q\gamma+q\alpha-q)}\right)^{\frac{1}{q}} t^{\alpha-\gamma-1/p}\,\|f\|_{L_p^\gamma} \\
&\leq \frac{1}{\Gamma(\alpha)}\left(\frac{\Gamma(q\alpha-q+1)\Gamma(1-q\gamma)}{\Gamma(2-q\gamma+q\alpha-q)}\right)^{\frac{1}{q}} b^{\alpha-\gamma-1/p-\varrho}\,\|f\|_{L_p^\gamma}.
\end{aligned}$$

This completes the proof. □

6. Existence results of fractional differential and integral equations

In this part of the text we now discuss the classical questions concerning ordinary differential equations involving fractional derivatives, i.e. the questions of existence and uniqueness of solutions. We shall mainly be interested in initial value problems (Cauchy problems), and in particular in global results. However, the existence results in this Section can be extended (see section 9) to vector-valued functions (In particular, systems of fractional order differential equations).

To encompass the full scope of this section, we investigate the problem of the existence of continuous solutions to the quadratic integral equations of the fractional type.

6.1. **Existence results on the space of integrable functions.** In this subsection, we established conditions for the existence of summable solutions of Cauchy type problems for nonlinear differential equations on the basis of their equivalence to the corresponding Volterra integral equations.

It is well-known that there exist functions that have no first order derivative but have fractional derivative of all orders less than one (see e.g. Examples 3.38, 3.39 and 3.40, see also [144]), so our results is an essential generalization of existence theorems with usual derivatives. Now, we introduce the following set of examples

Example A: (Fractional Differential Equations in the space of integrable functions)
In this example we deal with the existence of integrable solutions for the multi-term differential equation of the fractional type

$$\begin{cases} L(D)x(t) = f(t, x(\varphi(t))), \text{ a.e. on } (0,1), \\ \left(\sum_{i=1}^{n} a_i I^{1-\alpha_i} x\right)(0) = c \in \mathbb{R}, \end{cases} \tag{132}$$

where $a_1, a_2, \cdots, a_{n-1}$ are nonnegative constants, $a_n = -1$, $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_n < 1$ and $L(D) := D^{\alpha_n} - a_{n-1}D^{\alpha_{n-1}} - \cdots - a_1 D^{\alpha_1}$ where D^{α_i} denotes the standard Riemann-Liouville fractional derivatives of order α_i. The symbol $I^{1-\alpha_i}x(0)$ means that the limit is taken at almost all points of the right-sided neighborhood $(0, \epsilon)$, $\epsilon > 0$ as follows:

$$I^{1-\alpha_i}x(0) := \lim_{t \to 0} I^{1-\alpha_i}x(t), \ (1 \le i \le n).$$

Our investigation based on reducing *formally* the problem (132) into the Volterra integral equation (*cf.* Lemma 3.28,)

$$x(t) = x_0 t^{\alpha_n - 1} + \sum_{i=1}^{n-1} a_i I^{\alpha_n - \alpha_i} x(t) + I^{\alpha_n} f(t, x(\varphi(t))), \quad \text{a.e. on } (0,1), \quad (133)$$

where $x_0 = \dfrac{-c}{\Gamma(\alpha_n)}$. To facilitate our discussion, let us first state the following assumptions:

1. $f : (0,1) \times \mathbb{R} \to \mathbb{R}$ be a function with the following properties:
   (a) For each $t \in (0,1)$, $f(t, \cdot)$ is continuous,
   (b) For each $x \in \mathbb{R}$, $f(\cdot, x)$ is measurable,
   (c) There exist two real functions $t \to a(t), t \to b(t)$ such that:

   $$|f(t,x)| \leq a(t) + b(t)|x| \quad \text{for each } t \in (0,1) \quad \text{and } x \in \mathbb{R},$$

   where $a(\cdot) \in L_1(0,1)$ and $b(\cdot)$ is measurable and bounded,
2. $\varphi : (0,1) \to (0,1)$ is nondecreasing, absolutely continuous and there is a constant $M > 0$ such that $\varphi' \geq M$ *a.e. on* $(0,1)$.

Thus, we are in a position to formulate and prove the following

Theorem 6.1. *Suppose that the assumptions (1-2) hold along with*

$$\sum_{i=1}^{n-1} \frac{|a_i|}{\Gamma(1 + \alpha_n - \alpha_i)} + \frac{\sup |b(t)|}{M\Gamma(\alpha_n + 1)} < 1 \quad (134)$$

Then equation (132) has at least one solution $x \in L_1(0,1)$.

Proof. Since f satisfies Carathéodory conditions (a), (b) and because φ satisfy assumption (2), then for any $x \in L_1(0,1)$, the function $f(\cdot, x(\varphi(\cdot)))$ is measurable and from (c) we have

$$\begin{aligned}
\int_0^1 |f(s, x(\varphi(s))| \; ds &\leq \int_0^1 \{|a(s)| + |b(s)| \; |x(\phi(s))|\} \; ds \\
&\leq \|a\| + \frac{\sup |b(t)|}{M} \int_0^1 |x(\varphi(s))||\varphi'(s)| \, ds \\
&\leq \|a\| + \frac{\sup |b(t)|}{M} \int_{\varphi(0)}^{\varphi(1)} |x(u)| du \\
&= \|a\| + \frac{\sup |b(t)| \, \|x\|}{M} \qquad (135)
\end{aligned}$$

This implies $f(\cdot, x(\varphi(\cdot))) \in L_1(0,1)$ for any $x \in L_1(0,1)$.

Next, let us define the operator T as

$$(Tx)(t) := x_0 t^{\alpha_n - 1} + \sum_{i=1}^{n-1} a_i I^{\alpha_n - \alpha_i} x(t) + I^{\alpha_n} f(t, x(\varphi(t))), \quad \text{a.e. on } (0,1). \tag{136}$$

We claim $T : L_1(0,1) \to L_1(0,1)$ is compact. To prove our claim, firstly we note that $f(\cdot, x(\varphi(\cdot))) \in L_1(0,1)$, for every $x \in L_1(0,1)$. That is, the operator T makes sense. Further, f is continuous in x (assumption 1) and I^α maps $L_1(0,1)$ continuously into itself (Theorem 3.2). That is $x \to I^\alpha f(t, x(\varphi(t)))$ is continuous in x. Since x is an arbitrary element in $L_1(0,1)$, then T well-defined. Also it is compact operator. Now, fix a constant r, where

$$r \leq \frac{\dfrac{|x_0|}{\alpha_n} + \dfrac{\|a\|}{\Gamma(1+\alpha_n)}}{1 - \left[\displaystyle\sum_{i=1}^{n-1} \frac{|a_i|}{\Gamma(1+\alpha_n-\alpha_i)} + \frac{\sup|b(s)|}{M\Gamma(1+\alpha_n)}\right]}.$$

We will show that $T : \bar{B}_r \to L_1(0,1)$, to do this, let x be an arbitrary element in the open set B_r. By our assumptions we have

$$\begin{aligned} \|Tx\| &\leq |x_0| \int_0^1 t^{\alpha_n - 1} dt \\ &+ \int_0^1 \int_0^t \left[\sum_{i=1}^{n-1} |a_i| \frac{(t-s)^{\alpha_n - \alpha_i - 1}}{\Gamma(\alpha_n - \alpha_i)} |x(s)| + \frac{(t-s)^{\alpha_n - 1}}{\Gamma(\alpha_n)} |f(s, x(\varphi(s)))|\right] ds\, dt. \end{aligned}$$

Interchanging the order of integration yields

$$\begin{aligned} \|Tx\| &\leq \frac{|x_0|}{\alpha_n} + \int_0^1 \int_s^1 \left[\sum_{i=1}^{n-1} |a_i| \frac{(t-s)^{\alpha_n - \alpha_i - 1}}{\Gamma(\alpha_n - \alpha_i)} |x(s)| \right. \\ &\quad \left. + \frac{(t-s)^{\alpha_n - 1}}{\Gamma(\alpha_n)} |f(s, x(\varphi(s)))|\right] dt\, ds \\ &\leq \frac{|x_0|}{\alpha_n} + \int_0^1 \left[\sum_{i=1}^{n-1} \frac{|a_i||x(s)|}{\Gamma(1+\alpha_n-\alpha_i)} + \frac{1}{\Gamma(1+\alpha_n)} \{|a(s)| + |b(s)||x(\varphi(s))|\}\right] ds \\ &\leq \frac{|x_0|}{\alpha_n} + \sum_{i=1}^{n-1} \frac{|a_i|\,\|x\|}{\Gamma(1+\alpha_n-\alpha_i)} + \frac{\|a\|}{\Gamma(1+\alpha_n)} + \int_0^1 \frac{b|x(\varphi(s))||\varphi'(s)|}{M\Gamma(1+\alpha_n)} ds, \end{aligned}$$

where $b := \sup|b(s)|$. The substitution $u = \varphi(s)$ yields

$$\begin{aligned}
\|Tx\| &\leq \frac{|x_0|}{\alpha_n} + \sum_{i=1}^{n-1} \frac{|a_i|\,\|x\|}{\Gamma(1+\alpha_n-\alpha_i)} + \frac{\|a\|}{\Gamma(1+\alpha_n)} + \frac{b}{M\Gamma(1+\alpha_n)} \int_{\varphi(0)}^{\varphi(1)} |x(u)|du \\
&\leq \frac{|x_0|}{\alpha_n} + \sum_{i=1}^{n-1} \frac{|a_i|\,\|x\|}{\Gamma(1+\alpha_n-\alpha_i)} + \frac{\|a\|}{\Gamma(1+\alpha_n)} + \frac{b}{M\Gamma(1+\alpha_n)} \int_0^1 |x(u)|du \\
&= \frac{|x_0|}{\alpha_n} + \sum_{i=1}^{n-1} \frac{|a_i|\,\|x\|}{\Gamma(1+\alpha_n-\alpha_i)} + \frac{\|a\|}{\Gamma(1+\alpha_n)} + \frac{b}{M\Gamma(1+\alpha_n)} \|x\|
\end{aligned}$$

Therefore

$$\|Tx\| \leq \frac{|x_0|}{\alpha_n} + \frac{\|a\|}{\Gamma(1+\alpha_n)} + \left[\sum_{i=1}^{n-1} \frac{|a_i|}{\Gamma(1+\alpha_n-\alpha_i)} + \frac{\sup|b(s)|}{M\Gamma(1+\alpha_n)}\right] \|x\| \quad (137)$$

Above inequality means, that the operator T maps B_r into $L_1(0,1)$. Moreover, if $x \in \partial B_r$ i.e. $\|x\| = r$. Then from inequality (137), we have $\|Tx\| \leq r$. i.e. the condition (34) of Theorem 2.15 is satisfied. Then Theorem 2.15 implies that T has a fixed point. Therefore, equation (133) has a solution $x \in L_1(0,1)$.

Now, let $x \in L_1(0,1)$ be a solution to the integral equation (133) a.e. on (0,1). Applying the operator D^{α_n} on both sides of (133) and use the properties of the fractional calculus in the space $L_1(0,1)$ (*cf.* Lemma 3.28) we have

$$D^{\alpha_n}x(t) = \sum_{i=1}^{n-1} a_i D^{\alpha_i}x(t) + f(t, x(\varphi(t))).$$

From here, we arrive at the equation (132). Now, we show that the initial condition of the problem (132) also hold. For this apply the operator $I^{1-\alpha_n}$ on both sides of (133) then

$$\begin{aligned}
I^{1-\alpha_n}x(t) &= x_0\Gamma(\alpha_n) + \sum_{i=1}^{n-1} a_i I^{1-\alpha_j}x(t) + \int_0^t f(s, x(\varphi(s)))\,ds \\
&= -c + \sum_{i=1}^{n-1} a_i I^{1-\alpha_j}x(t) + \int_0^t f(s, x(\varphi(s)))\,ds.
\end{aligned}$$

Thus

$$\sum_{i=1}^{n} a_i I^{1-\alpha_j}x(t) = c - \int_0^t f(s, x(\varphi(s)))\,ds.$$

Taking the limit as $t \to 0^+$, we obtain that the initial condition of the problem (132) makes sense. Thus the problem (132) has a solution $x \in L_1(0,1)$ which completes the proof. $\square$

Example B:(Monotonic solutions of fractional differential equations)

In this example, we present an existence of monotonic solutions for a nonlinear multi term non-autonomous fractional differential equation in the Banach space of integrable functions. The concept of measure of noncompactness and a fixed point theorem due to G. Emmanuele is the main tool in carrying out our proof. Indeed, we deal with the existence of nondecreasing solutions for the non-autonomous multi-term differential equation of the fractional type

$$\begin{cases} \left(D^{\alpha_n} - \sum_{i=1}^{n-1} a_i D^{\alpha_i}\right) x(t) = f(t, x(\varphi(t))), \quad \text{a.e. on } (0,1), \\ \left(\sum_{i=1}^{n} a_i I^{1-\alpha_i} x\right)(0) = 0. \end{cases} \tag{138}$$

where $a_1, a_2, \cdots, a_{n-1},$ are constans, $a_n = -1,\ 0 < \alpha_1 < \alpha_2 < \cdots < \alpha_n < 1$ and D^{α_i} denotes the standard Riemann-Liouville fractional derivative. Our investigation is based on reducing the problem (138) to the Volterra integral equation

$$x(t) = \sum_{i=1}^{n-1} a_i I^{\alpha_n - \alpha_i} x(t) + I^{\alpha_n} f(t, x(\varphi(t))), \quad \text{a.e. on } (0,1). \tag{139}$$

The notation $I^{1-\alpha} x(0)$ means that the limits taken at almost all points of the right-sided neighborhood $(0,\epsilon), \epsilon > 0$ of 0.
To facilitate our discussion, let us first state the following assumptions:

1. Suppose G be an open subset of $\mathbb{R}$ and let $f : (0,1) \times G \longrightarrow \mathbb{R}$ be a function with the following properties:
   (a) $f(t,\cdot)$ is continuous for each $t \in (0,1)$,
   (b) $f(\cdot, x)$ is measurable for each $x \in G$,
   (c) There exist two real functions $a(\cdot), b(\cdot)$ such that:
   $$f(t,x)| \leq a(t) + b(t)|x| \ \text{ for each } t \in (0,1), x \in G,$$
   where $a \in L_1(0,1)$ and b is measurable and bounded,
   (d) For each nondecreasing function x, the map $t \to f(t, x(\varphi(t)))$ is nondecreasing,
2. $\varphi : (0,1) \longrightarrow (0,1)$ is nondecreasing, absolutely continuous and there is a constant $M > 0$ such that $\varphi^{'} \geq M$ *a.e. on* $(0,1)$.

Theorem 6.2. *Let $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_n < 1$. Assume, in addition to the above assumptions that*

$$\sum_{i=1}^{n-1} \frac{|a_i|}{\Gamma(1+\alpha_n-\alpha_i)} + \frac{\sup|b(t)|}{M\Gamma(\alpha_n+1)} < 1. \tag{140}$$

Then equation (138) has at least one nonnegative monotonic solution.

Proof. Let us define the operator T as

$$(Tx)(t) := \sum_{i=1}^{n-1} a_i I^{\alpha_n-\alpha_i} x(t) + I^{\alpha_n} f(t, x(\varphi(t))), \quad \text{a.e. on } (0,1).$$

First of all observe, that for $x \in L_1(0,1)$, we have $x\varphi(\cdot) \in L_1(0,1)$ and $\|x\varphi\| \leq \|x\|$. Thus for every $x \in L_1(0,1)$, the function $f(\cdot, x(\varphi(\cdot)))$ is element of $L_1(I)$. That is, the operator T well-defined. Further, f is continuous in x (Assumption 1) and I^α maps $L_1(0,1)$ continuously into itself (Lemma 3.3). So $x \to I^\alpha f(t, x(\varphi(t)))$ is continuous in x. Since x is an arbitrary element in $L_1(0,1)$, then T maps $L_1(0,1)$ continuously into $L_1(0,1)$. Let x be an arbitrary element in the ball B_r (of radius r and center at θ). According to our Assumptions (1) and (2) we have

$$\begin{aligned}
\|Tx\| &= \int_0^1 |Tx(t)|dt \\
&\leq \int_0^1 \int_0^t \left[\sum_{i=1}^{n-1} |a_i| \frac{(t-s)^{\alpha_n-\alpha_i-1}}{\Gamma(\alpha_n-\alpha_i)} |x(s)| + \frac{(t-s)^{\alpha_n-1}}{\Gamma(\alpha_n)} |f(s, x(\varphi(s)))| \right] \, ds \, dt.
\end{aligned}$$

By interchanging the order of integration, we get

$$\begin{aligned}
\|Tx\| &\leq \int_0^1 \int_s^1 \left[\sum_{i=1}^{n-1} |a_i| \frac{(t-s)^{\alpha_n-\alpha_i-1}}{\Gamma(\alpha_n-\alpha_i)} |x(s)| + \frac{(t-s)^{\alpha_n-1}}{\Gamma(\alpha_n)} |f(s, x(\varphi(s)))| \right] dt \, ds \\
&= \int_0^1 \left[\sum_{i=1}^{n-1} \frac{|a_i|}{\Gamma(1+\alpha_n-\alpha_i)} |x(s)| + \frac{1}{\Gamma(1+\alpha_n)} |f(s, x(\varphi(s)))| \right] dt \, ds \\
&\leq \int_0^1 \left[\sum_{i=1}^{n-1} \frac{|a_i|}{\Gamma(1+\alpha_n-\alpha_i)} |x(s)| + \frac{1}{\Gamma(1+\alpha_n)} \{|a(s)| + |b(s)||x(\varphi(s))|\} \right] \, ds \\
&\leq \sum_{i=1}^{n-1} \frac{|a_i|}{\Gamma(1+\alpha_n-\alpha_i)} \|x\| + \frac{\|a\|}{\Gamma(1+\alpha_n)} + \frac{\sup|b(s)|}{M\Gamma(1+\alpha_n)} \int_0^1 |x(\varphi(s))||\varphi'(s)| \, ds
\end{aligned}$$

The substitution $u = \varphi(s)$ yields

$$\begin{aligned} \|Tx\| &\leq \sum_{i=1}^{n-1} \frac{|a_i|}{\Gamma(1+\alpha_n-\alpha_i)} \|x\| + \frac{\|a\|}{\Gamma(1+\alpha_n)} + \frac{\sup|b(s)|}{M\Gamma(1+\alpha_n)} \int_{\varphi(0)}^{\varphi(1)} |x(u)| \, du \\ &\leq \sum_{i=1}^{n-1} \frac{|a_i|}{\Gamma(1+\alpha_n-\alpha_i)} \|x\| + \frac{\|a\|}{\Gamma(1+\alpha_n)} + \frac{\sup|b(s)|}{M\Gamma(1+\alpha_n)} \int_0^1 |x(u)| \, du \end{aligned}$$

Therefore

$$\|Tx\| \leq \frac{\|a\|}{\Gamma(1+\alpha_n)} + \left[\sum_{i=1}^{n-1} \frac{|a_i|}{\Gamma(1+\alpha_n-\alpha_i)} + \frac{\sup|b(s)|}{M\Gamma(1+\alpha_n)}\right] \|x\|. \qquad (141)$$

Above inequality proves that the operator T maps B_r into itself, where

$$r := \frac{\dfrac{\|a\|}{\Gamma(1+\alpha_n)}}{1 - \left[\displaystyle\sum_{i=1}^{n-1} \frac{|a_i|}{\Gamma(1+\alpha_n-\alpha_i)} + \frac{\sup|b(s)|}{M\Gamma(1+\alpha_n)}\right]}.$$

Now, let Q_r denotes the subset of B_r consisting of all functions being a.e. nondecreasing on (0,1). Obviously Q_r is nonempty, closed, bounded and convex. Moreover, by Theorem 2.10, Q_r is compact in measure. We claim $T : Q_r \to Q_r$ is $\beta -$ condensing operator. Once our claim is established, the Emmanuele's fixed point theorem (Theorem 2.17) guarantees that T has a fixed point in Q_r, i.e. the problem (139) has a solution in Q_r. It remains to prove this claim. Firstly, for any $x \in Q_r$, Assumption (3) and Lemma 3.3 result that Tx is nonnegative and a.e. nondecreasing on (0,1). Thus the operator T maps Q_r into itself. Now, let X be a nonempty subset of Q_r. Fix $\epsilon > 0$, and take a measurable subset $D \subset (0,1)$ such that the measure of $D = measD \leq \epsilon$. For arbitrary $x \in X$ we deduce that

$$\|Tx\|_D \leq \frac{\|a\|_D}{\Gamma(1+\alpha_n)} + \left[\sum_{i=1}^{n-1} \frac{|a_i|}{\Gamma(1+\alpha_n-\alpha_i)} + \frac{\sup|b(s)|}{M\Gamma(1+\alpha_n)}\right] \|x\|_D .$$

Keeping in mind the formula expressing the De Blasi measure of weak noncompactness β defined by the formula (35) and taking into account that

$$\lim_{\epsilon\to 0} \sup_{x\in X} \left(\sup_D \left[\|a\|_D : D \subset (0,1), \ measD \leq \epsilon \ \right]\right) = 0,$$

we have ([16] and [72])

$$\beta(TX) \leq \xi\beta(X),$$

where

$$\xi = \left(\sum_{i=1}^{n-1} \frac{|a_i|}{\Gamma(1+\alpha_n-\alpha_i)} + \frac{\sup|b(t)|}{M\Gamma(\alpha_n+1)} \right) < 1.$$

Thus T has at least one fixed point in Q_r in the view of Theorem 2.17. Hence there exists at least one nondecreasing solution $x \in L_1(0,1)$ of (139).

Now, let $x \in L_1(0,1)$ be a nonnegative a.e. nondecreasing solution to the integral equation (139). Applying the operator D^{α_n} on both sides of (139) and use the properties of the fractional calculus in the space $L_1(0,1)$ we have

$$\begin{aligned} D^{\alpha_n}x(t) &= \sum_{i=1}^{n-1} a_i D I^{1-\alpha_n} I^{\alpha_n-\alpha_i} x(t) + f(t, x(\varphi(t))) \\ &= \sum_{i=1}^{n-1} a_i D^{\alpha_i} x(t) + f(t, x(\varphi(t))). \end{aligned}$$

From here, we arrive at the equation (138). It remains to show that the initial condition of the problem (138) also hold. For this, we apply the operator $I^{1-\alpha_n}$ on both sides of (139) then

$$\begin{aligned} I^{1-\alpha_n}x(t) &= \sum_{i=1}^{n-1} a_i I^{1-\alpha_j} x(t) + \int_0^t f(s, x(\varphi(s)))\ ds \\ &= \sum_{i=1}^{n-1} a_i I^{1-\alpha_j} x(t) + \int_0^t f(s, x(\varphi(s)))\ ds. \end{aligned}$$

Thus

$$\sum_{i=1}^{n} a_i I^{1-\alpha_j} x(t) = -\int_0^t f(s, x(\varphi(s)))\ ds.$$

Taking the limit as $t \to 0^+$, we obtain that the initial condition of the problem (138) makes sense. Thus the problem (138) has a nonnegative a.e. nondecreasing solution $x \in L_1(0,1)$ which completes the proof. $\square$

For a further illustration of this remark, we discuss a very simple example of a fractional differential equation with a right-hand side is continuous but does not fulfil a Lipschitz condition.

Example 6.1. Consider the following nonlinear fractional differential equation of order $\alpha \in (0,1)$:

$$\begin{cases} D^{\alpha}x(t) = \lambda t^{\beta}[x(t)]^m + Lt^{\gamma},\ a.e.\ on\ (0,1), \\ I^{1-\alpha}x(0) = 0. \end{cases} \tag{142}$$

where λ, L, β, m and γ are non-negative real numbers, $(\lambda \neq 0)$. Assume that $\beta > 0, m \in (0,1)$ and $\gamma = \dfrac{\beta + m\alpha}{1-m}$. We establish the conditions for the existence of monotonic solution of the equation (142). For this, we have to choose the domain G and check when the conditions of Theorem 6.2 with

$$f(t,x) = \lambda t^{\beta} x^{m} + L t^{\gamma},$$

are satisfied. In this case the, the function f is continuous but the Lipschitz condition is violated.

Let ω and $k > 0$ be constants such that $k < \dfrac{\Gamma(1+\alpha)}{|\lambda|}$, and $\omega \leq \dfrac{\beta}{1-m}$. We choose the following domain

$$D := \left\{ (t,x) \in \mathbb{R}^2 : \ t \in (0,1), \ 0 < x < k^{\frac{1}{m-1}} \ t^{\omega} \right\}.$$

This means that the domain G for x is unbounded for $\omega < 0$ and bounded for $\omega \geq 0$. To prove that f satisfies the condition $1(c)$, we note that, for $(t,x) \in D$

$$\begin{aligned} |f(t,x)| &\leq |\lambda| t^{\beta} |x|^{m} + |L| t^{\gamma} \\ &\leq |\lambda| t^{\beta} |x|^{m-1} |x| + |L| t^{\gamma} \\ &\leq |\lambda| t^{\beta} k t^{\omega(m-1)} |x| + |L| t^{\gamma} \\ &\leq |\lambda| k t^{\beta + \omega(m-1)} |x| + |L| t^{\gamma}. \end{aligned}$$

Since, $\beta + \omega(m-1) \geq 0$, the function f thus has the linear growth condition

$$|f(t,x)| \leq |L| t^{\gamma} + |\lambda| k |x|,$$

Therefore the Assumption (4) of Theorem 6.2 holds with

$$M = 1, a_1 = a_2 = \cdots = a_{n-1} = 0, \ \frac{|\lambda| k}{\Gamma(1+\alpha)} = \frac{\sup |b(t)|}{\Gamma(1+\alpha)} < 1.$$

According to Theorem 6.2, the differential equation (142) has at least a nondecreasing solution $x \in L_1(0,1)$. In fact, one can easily verified that the solution given by

$$x(t) = \mu t^{\left(\frac{\beta + \alpha}{1-m} \right)},$$

where μ is the positive solution of the transcendental equation (see [102])

$$\Gamma\left(\frac{\beta+\alpha}{1-m} + 1 - \alpha \right) [\lambda \mu^{m} + L] - \Gamma\left(\frac{\beta+\alpha}{1-m} + 1 \right) \mu = 0.$$

Example C: (Monotonic solutions of fractional differential equations)

In this example, based on the fixed point Theorem 2.16 we study the existence of a non-negative monotonic solution to the multi-term differential equations of the fractional type

$$\begin{cases} L(D)x(t) = f(t, x(\varphi(t))), \quad \text{a.e. on } (0,1), \\ \left(\sum_{i=1}^{n} a_i I^{1-\alpha_i} x\right)(0) = 0. \end{cases} \tag{143}$$

where $a_1, a_2, \cdots, a_{n-1}$ are nonnegative constants, $a_n = -1$, $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_n < 1$, $L(D) := D^{\alpha_n} - a_{n-1}D^{\alpha_{n-1}} - \cdots - a_1 D^{\alpha_1}$ and D^{α_i} denotes the standard Riemann-Liouville fractional derivatives of order α_i. The symbol $I^{1-\alpha_i}x(0)$ means that the limit is taken at almost all points of the right-sided neighborhood $(0, \epsilon)$, $\epsilon > 0$ as follows:

$$I^{1-\alpha_i}x(0) := \lim_{t \to 0} I^{1-\alpha_i}x(t), \ (1 \le i \le n).$$

Remark 6.1. In our investigation, we drop the requirement of the **Example B** that the growth condition imposed on the function f is sufficiently small.

In the view of Equation (64) and Lemma 3.28, it can be easily seen "formally" that the integral equation medelled off the problem (143) is given by the Volterra integral equation

$$x(t) = \sum_{i=1}^{n-1} a_i I^{\alpha_n - \alpha_i} x(t) + I^{\alpha_n} f(t, x(\varphi(t))), \quad \text{a.e. on } (0,1), \tag{144}$$

under the following set of assumptions

A. Let $I = [0,1]$ $G \subseteq \mathbb{R}$, $G^+ := G \cap \mathbb{R}^+$, $f : [0,1] \times G \longrightarrow \mathbb{R}$ be a function such that $f : [0,1] \times G^+ \longrightarrow \mathbb{R}^+$. Further, assume that $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_n < 1$ and that f satisfies the following properties:
1. $f(t, \cdot)$ is continuous for each $t \in [0,1]$,
2. $f(\cdot, x)$ is measurable for each $x \in G$,
3. f satisfies the linear growth condition
$$|f(t,x)| \le a(t) + b|x| \text{ for each } t \in (0,1) \text{ and } x \in G, \tag{145}$$
with $b \in [0, \infty)$ and a, necessarily nonnegative, measurable a with the property that $I^{\alpha_n} a \in L_p([0,1])$, $p \in [1, \infty)$,
4. f is nondecreasing with respect to each variable separately,

B. $\varphi : (0,1) \longrightarrow (0,1)$ is continuous nondecreasing function such that $\varphi(t) \le t$ *a.e.* in $(0,1)$.

Now, we are in the position to state and prove the following

Theorem 6.3. *Under the assumptions* $\boldsymbol{A}$ *and* $\boldsymbol{B}$*, the integral equation (144) has at least one on solution* $x \in L_p(0,1)$ *which is (nonnegative and a.e. nondecreasing) on (0,1).*

Proof. Fix for a moment a constant r_0 and define the closed convex cone K by

$$K := \{x \in L_p(I) : x \text{ nonnegative and nondecreasing } a.e. \text{ in } (0,1)\}$$

We will show that it is possible to choose the number r_0 in such a way to put $Q := B_{r_0} \cap K$ in Theorem 2.16 as a domain for T. An appropriate operator T to do this will be defined as

$$(Tx)(t) := \sum_{i=1}^{n-1} a_i I^{\alpha_n-\alpha_i} x(t) + I^{\alpha_n} f(t, x(\varphi(t))), \quad \text{a.e. on } (0,1). \tag{146}$$

First of all observe, that for $x \in Q$, $x\varphi(\cdot) \in Q$ and $\|x\varphi\|_p \le \|x\|_p$. By the result of Vainberg and Krasnoselskii, our Assumption $\boldsymbol{A}(3)$ is sufficient to ensure that $a \in L_1(I)$. Thus, by (135), for every $x \in Q$, the function $f(\cdot, x(\varphi(\cdot)))$ is element of $L_1(I)$. Moreover, since $|f(t,x)| \le a(t) + b|x|$, we have

$$\begin{aligned}
|I^{\alpha_n} f(t, x(\varphi(t)))| &\le I^{\alpha_n}|f(t, x(\varphi(t)))| \\
&\le I^{\alpha_n}[a(t) + b|x(\varphi(t))|] \\
&= I^{\alpha_n} a(t) + b I^{\alpha_n}|x(\varphi(t))|
\end{aligned}$$

By Minkowski's inequality, we deduce that

$$\begin{aligned}
\|I^{\alpha_n} f(\cdot, x(\varphi(\cdot)))\|_p &= \left(\int_0^1 |I^{\alpha_n} f(t, x(\varphi(t)))|^p dt\right)^{1/p} \\
&\le \left(\int_0^1 (I^{\alpha_n} a(t) + b I^{\alpha_n}|x(\varphi(t))|)^p dt\right)^{1/p} \\
&\le \|I^{\alpha_n} a\|_p + b \, \|I^{\alpha_n}|x(\varphi(\cdot))|\|_p .
\end{aligned}$$

That is $Tx \in L_p(I)$. By our assumptions and Lemma 3.3, T maps Q into K.

We claim $T : Q \to K$ is continuous and compact. Once our claim is established, we can ensure that T fulfills the assumptions of Theorem 2.16, further its fixed points are precisely the solutions of the integral equation (144). It remains to prove our claim. To do this, we note that the Volterra-Hammerstein

$$Ax(t) := \frac{1}{\Gamma(\alpha_n)} \int_0^t (t-s)^{\alpha_n - 1} f(s, x(\varphi(s)))\, ds$$

is a composition of the superposition operator $F_\varphi x(t) := f(t, x(\varphi(t)))$ and the fractional integral I^{α_n}. Clearly, the growth condition $\boldsymbol{A}(3)$ implies that

F_φ acts from $L_p(I)$ into $L_p(I)$ and is bounded. Since $p<\infty$ and f is a Carathéodory function, then $F_\varphi : L_p(I) \to L_p(I)$ is automatically continuous, see ([190], Theorem 5.2.1). Since I^{α_n} is compact in $L_p(I)$ by Theorem 3.2, $T : Q \to K$ is continuous and compact operator.

Now, we need also to claim that the second condition of Theorem 2.16 does not satisfied. From this alternative we deduce that T has a fixed point $\bar{Q}$. In fact, we have to verify that, for sufficiently large number $r_0>0$, the equation $x=\gamma Tx$ has no (nonnegative and a.e. nondecreasing) solution x with $\|x\|=r_0$ and $\gamma \in (0,1)$. To see this, let $\gamma \in (0,1)$ and assume that $x \in Q$ be a solution of $x=\gamma Tx$. Keeping Assumption ***B.*** in mind, we may write the following chain of inequalities

$$\begin{aligned}
x(t) &\leq \gamma \left[\sum_{i=1}^{n-1} |a_i| I^{\alpha_n-\alpha_i}|x(t)| + I^{\alpha_n}|f(t,x(\varphi(t)))|\right] \\
&< \left[\sum_{i=1}^{n-1} |a_i| I^{\alpha_n-\alpha_i}|x(t)| + I^{\alpha_n}|f(t,x(\varphi(t)))|\right] \\
&\leq \sum_{i=1}^{n-1} a_i I^{\alpha_n-\alpha_i}|x(t)| + I^{\alpha_n}\left(a(t)+b|x(\varphi(t)|\right) \\
&= I^{\alpha_n}a(t) + \left(\sum_{i=1}^{n-1} a_i I^{\alpha_n-\alpha_i}|x(t)| + bI^{\alpha_n}|x(\varphi(t))|\right) \\
&\leq A(t) + h\left(\sum_{i=1}^{n-1} I^{\alpha_n-\alpha_i}|x(t)| + I^{\alpha_n}|x(\varphi(t))|\right) \\
&\leq A(t) + h\left(\sum_{i=1}^{n-1} I^{\alpha_n-\alpha_i}|x(t)| + I^{\alpha_n}|x(t)|\right) \\
&\leq A(t) + h\left(\sum_{i=0}^{n-1} I^{\alpha_n-\alpha_i}|x(t)|\right),
\end{aligned}$$

where $\alpha_0=0$, $h=\max\{|a_1|,|a_2|,\cdots,|a_{n-1}|,b\}$ and $A=I^{\alpha_n}a$. If we define the operator $H : L_p(I) \to L_p(I)$ by

$$Hx(t) := h\sum_{i=0}^{n-1} I^{\alpha_n-\alpha_i}|x(t)|.$$

then $x(t)<A(t)+Hx(t)$. This yields $x(t)<A(t)+H\{A(t)+Hx(t)\}$ That is

$$x(t) < A(t) + HA(t) + H^2x(t)$$

Analogously

$$x(t) < A(t) + HA(t) + H^2A(t) + H^3x(t)$$

This can be extended arbitrarily to obtain

$$x(t) < \sum_{k=0}^{m-1} B^k A(t) + H^m x(t).$$

Thus,

$$\|x\|_p < \sum_{k=0}^{m-1} \left\|H^k A\right\|_p + \|H^m x\|_p .$$

Let us prove that $\|H^m x\|_p \to 0$ as $m \to \infty$. To see this, we notice, in the view of Lemma 3.11, that

$$Hx(t) \leq hCI^{\alpha_n-\alpha_{n-1}}|x(t)|,$$

with some finite constant C depending only on $\alpha_1, \alpha_2, \cdots, \alpha_{n-1}$. Note that the above estimate yields

$$H^m x(t) \leq (hC)^m I^{m(\alpha_n-\alpha_{n-1})}|x(t)|. \tag{147}$$

By induction and the properties of fractional integral operator, one can easily verify that the inequality (147) is true for every positive integer m. For sufficiently large value of m, we have $q(m(\alpha_n - \alpha_{n-1}) > 0$. Thus, by the formula (51), we deduce that

$$\|H^m x\|_p \leq \frac{(hC)^m \|x\|_p}{\Gamma(1 + m(\alpha_n - \alpha_{n-1}))}.$$

Therefore, $\|H^m x\| \to 0$ as $m \to \infty$.

Again, by the formula (51), it follows

$$\left\|I^\beta A\right\|_p \leq \frac{\|A\|_p}{\Gamma(1+\beta)}.$$

Therefore

$$\begin{aligned}
\|x\|_p &< \|A\|_p + \sum_{k=1}^{\infty} (hC)^k \left\| \int_0^t \frac{(t-s)^{k(\alpha_n-\alpha_{n-1})-1}}{\Gamma(k(\alpha_n - \alpha_{n-1}))} |A(s)|\, ds \right\|_p \\
&\leq \|A\|_p + \sum_{k=1}^{\infty} \frac{(hC)^k \|A\|_p}{\Gamma(1 + k(\alpha_n - \alpha_{n-1}))}.
\end{aligned}$$

This can be reduced to

$$\|x\|_p < \|A\|_p + \|A\|_p E_{\alpha_n-\alpha_{n-1}}(hC),$$

Where $E_{\alpha_n-\alpha_{n-1}}(hC)$ is the Mittag-Leffler function of order $\alpha_n - \alpha_{n-1}$ evaluated at hC. By choosing $r_0 \geq \|A\|_p + \|A\|_p \mathcal{E}_{\alpha_n-\alpha_{n-1}}(hC)$ we obtain that the equation $x = \gamma Tx$ has no nonnegative and a.e. nondecreasing solution x with $\|x\| = r_0$. This means that there is no solution $x \in K \cap \partial B_{r_0}$. Hence the claim is proved. □

Now, we restrict our attention to discuss the existence of monotonic solution to the problem (143).

Theorem 6.4. *If the Assumptions **A**(1)-**A**(4) and **B** hold, then the problem (143) has at least one (nonnegative and a.e. nondecreasing) solution $x \in L_p(0,1)$.*

Proof. Let $x \in L_p(0,1)$ satisfies the integral equation (144) a.e. on I. Applying the operator D^{α_n} on both sides of (144) and using Lemma 3.24 we obtain

$$\begin{aligned} D^{\alpha_n}x(t) &= \sum_{i=1}^{n-1} a_i D I^{1-\alpha_n} I^{\alpha_n-\alpha_i} x(t) + f(t, x(\varphi(t))) \\ &= \sum_{i=1}^{n-1} a_i D^{\alpha_i} x(t) + f(t, x(\varphi(t))). \end{aligned}$$

From here, we arrive at the equation (143). Now, we show that the initial condition of the problem (143) also hold. For this apply the operator $I^{1-\alpha_n}$ on both sides of (144). Then we obtain

$$I^{1-\alpha_n}x(t) = \sum_{i=1}^{n-1} a_i I^{1-\alpha_i} x(t) + \int_0^t f(s, x(\varphi(s)))\, ds,$$

that is

$$I^{1-\alpha_n}x(t) - \sum_{i=1}^{n-1} a_i I^{1-\alpha_i} x(t) = \int_0^t f(s, x(\varphi(s)))\, ds,$$

Taking the limit as $t \to 0^+$, we obtain the initial condition of the problem (143) as required. □

Arguing similarly as in the proof of Theorem 6.4, we are able to prove

Theorem 6.5. *If the Assumptions **A**(1), **A**(2), and **A**(3) are satisfied. Then the problem (143) (with $\varphi(t) = t$) has at least one nonnegative solution.*

Proof. The analysis is similar to that in Theorem 6.4, therefore, we omit the details. We define the closed convex cone K by

$$K := \{x \in L_p(I) : x(t) \geq 0,\ t \in (0,1)\}.$$

It is clear that the operator $T : Q := B_{r_0} \cap K \to K$ is compact, continuous operator. Now we are able to repeat our argumentation from Theorem 6.4 to prove that the operator T has a fixed point $x \in Q$. Once again we omit the detail since it is almost identical to that in the proof of Theorem 6.4. $\square$

Example 6.2. In order to illustrate the results proved in Theorem 6.4, let us consider the following problem:

$$\begin{cases} L(D)x(t) = -\ln(1-t) + \dfrac{\lambda t}{1+t} h(x(\sin(t))), \ \lambda \geq 0, \ t \in (0,1), \\ \left(\displaystyle\sum_{i=1}^{n} a_i I^{1-\alpha_i} x \right)(0) = 0. \end{cases} \tag{148}$$

where $h(x) := \max_{y \leq x}[g(y)]$ and g is defined by the formula

$$g(x) := \begin{cases} 0 \text{ for } x \leq 0, \\ x \sin x \text{ for } x > 0. \end{cases}$$

Denoting by $f(t,x)$ the right hand side of the equation (148), i.e.

$$f(t,x) = -\ln(1-t) + \lambda \frac{t}{1+t} h(x).$$

We can easily see that $f : (0,1) \times \mathbb{R}^+ \to \mathbb{R}^+$ and is nondecreasing on $(0,1) \times \mathbb{R}^+$ with respect to both variables. Apart of this f is continuous on $(0,1) \times \mathbb{R}$ and

$$f(t,x) \leq -\ln(1-t) + \frac{\lambda}{2}|x|, \text{ for } t \in (0,1) \text{ and } x \geq 0.$$

Thus we see that all assumptions of Theorem 6.4 are fulfilled, so the problem (148) possesses at least one solution being monotonic and integrable in the interval $(0,1)$.

Let us pay attention to the fact that the function f occurring in equation (148) does not satisfy the Lipschitz condition with respect to x. This means that the standard results of this type (see e.g. [107] Section 3) are not applicable to this equation. It is also worthwhile to mention that we only required that $\lambda \geq 0$ and need not to assume the value of λ is sufficiently small to ensure the existence of monotonic integrable solutions to the problem (148). Further, the existence of solution independent of the values of a_i's.

6.2. **Existence results on the space of continuous functions.** In this part of the text we now discuss the classical questions concerning ordinary differential equations involving fractional derivatives, i.e. the questions of existence of solutions. Our first result is an existence result that corresponds to the classical Peano's existence theorem for first order equations

Example D:

Consider the differential equations of the form

$$\frac{d^\alpha x(t)}{dt^\alpha} = f(t, x(t)),\ \alpha \in (0,1),\ t \in [0,1] \tag{149}$$

combined with appropriate boundary condition

$$x(0) + bx(1) = h \tag{150}$$

with certain constants $b,\ h \in \mathbb{R},\ b \neq -1$. To obtain the integral equation (modeled off the problem (149, 150), we keep the equation (149) in mind. In the view of Lemma 3.27, we have

$$x(t) = x(0) + I^\alpha f(t, x(t)). \tag{151}$$

with some (presently unknown) quantity $x(0)$. At $t = 1$, this reads

$$x(1) = x(0) + \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} f(s, x(s))\, ds.$$

We can plug this into the boundary condition $x(0) + bx(1) = h$ and derive

$$(1+b)x(0) + \frac{b}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} f(s, x(s))\, ds = h.$$

Thus

$$x(0) = \frac{h}{1+b} - \frac{b}{1+b} I^\alpha f(1, x(1)) \tag{152}$$

Now insert $x(0)$ into (151) which yields

$$x(t) = \frac{h}{1+b} + I^\alpha f(t, x(t)) - \frac{b}{1+b} I^\alpha f(1, x(1)). \tag{153}$$

Now, we are in the position to state and prove the following existence and uniqueness theorem

Theorem 6.6. *Assume that $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous and satisfies a Lipschitz condition with respect to the second variable with Lipschitz constant L. If*

$$L\left(1 + \frac{|b|}{1+|b|}\right) < \Gamma(1+\alpha) \tag{154}$$

then the boundary value problem (149 and 150), has a unique solution $x \in C[0,1]$.

Proof. Define the operator $T : C[0,1] \to C[0,1]$ by

$$Tx(t) = \frac{h}{1+b} + I^\alpha f(t, x(t)) - \frac{b}{1+b} I^\alpha f(1, x(1)). \tag{155}$$

We note that, for each $x \in C[0,1]$ we have $t \to x(t)$ is continuous mapping from $[0,1]$ to $\mathbb{R}$. Hence, $f(\cdot, x(\cdot))$ is compositions of this mapping with f. It follows that, for every $x \in C[0,1]$, the function $f(\cdot, x(\cdot))$ is continuous in $[0,1]$. In the view of our assumptions and Lemma 3.3, $Tx \in C[0,1]$ and consequently T well-defined. Moreover, for any $x, y \in C[0,1]$ we have

$$\begin{aligned}
|Tx(t) - Ty(t)| &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |f(s,x(s)) - f(s,y(s))| \, ds \\
&+ \frac{|b|}{\Gamma(1+\alpha)(1+|b|)} \int_0^1 (1-s)^{\alpha-1} |f(s,x(s)) - f(s,y(s))| \, ds \\
&\leq \frac{L}{\Gamma(\alpha)} \|x-y\| \left(\int_0^t (t-s)^{\alpha-1} \, ds + \frac{|b|}{|1+b|} \int_0^1 (1-s)^{\alpha-1} \, ds \right) \\
&\leq \frac{L}{\Gamma(1+\alpha)} \left(1 + \frac{|b|}{1+|b|} \right) \|x-y\| .
\end{aligned}$$

which implies, under our assumption, that T is a contraction. Thus, by Banachs fixed point theorem (Theorem 2.11), we obtain that T indeed has a unique fixed point.

To this end, we let $x \in C[0,1]$ be a solution to (153). Since $f(\cdot, x(\cdot))$ is continuous but it is not necessary absolutely continuous on $[0,1]$ and since the Caputo fractional derivative of the constant is zero, it follows from (153) that

$$\frac{d^\alpha x(t)}{dt^\alpha} = \frac{d^\alpha}{dt^\alpha} I^\alpha f(t, x(t)).$$

As far are we know, the right hand side of the last equation need not to equal $f(t, x(t))$ (it may even not exists (see Example 98). However, in this case we may use the Definition 97 of the Caputo fractional derivative and the formula (152) as follows

$$\begin{aligned}
\frac{d^\alpha x(t)}{dt^\alpha} &= D^\alpha x(t) - \frac{t^{-\alpha}}{\Gamma(1-\alpha)} x(0) \\
&= D^\alpha \left[\frac{h}{1+b} + I^\alpha f(t,x(t)) - \frac{b}{1+b} I^\alpha f(1,x(1)) \right] - \frac{t^{-\alpha}}{\Gamma(1-\alpha)} x(0) \\
&= \left[\frac{h}{1+b} - \frac{b}{1+b} I^\alpha f(1,x(1)) \right] \frac{t^{-\alpha}}{\Gamma(1-\alpha)} + D^\alpha I^\alpha f(t,x(t)) - \frac{t^{-\alpha}}{\Gamma(1-\alpha)} x(0) \\
&= x(0) \frac{t^{-\alpha}}{\Gamma(1-\alpha)} + D^\alpha I^\alpha f(t,x(t)) - \frac{t^{-\alpha}}{\Gamma(1-\alpha)} x(0) = f(t,x(t)).
\end{aligned}$$

Moreover, equation (153) implies

$$\begin{aligned} x(0)+bx(1) &= \frac{h}{1+b}-\frac{b}{1+b}I^\alpha f(1,x(1)) \\ &+ \frac{h}{1+b}+I^\alpha f(1,x(1))-\frac{1}{1+b}I^\alpha f(1,x(1))=h \end{aligned}$$

Therefore $x \in C[0,1]$ satisfies the problem (149 and 150).
Conversely, $x \in C[0,1]$ be the solution to the problem (149 and 150). Define $z(t) := f(t,x(t))$. By our assumption, z is a continuous function and $z(t) = \frac{d^\alpha}{dt^\alpha}x(t)$. Thus $x \in AC[0,1]$ is continuous too. By Lemma 3.27 we obtain $I^\alpha z(t) = x(t) - x(0)$. Therefore, we arrive at the integral equation (151). Consequently, in the view of the boundary condition, x is a solution of (153). □

We discuss a very simple example of a fractional differential equation with a right-hand side that does not fulfil a Lipschitz condition.

Example 6.3. Consider the differential equation

$$\frac{d^\alpha x(t)}{dt^\alpha} = [x(t)]^\gamma,\ x(0)=0,\ \alpha\in(0,1),\ t\in[0,1], \tag{156}$$

with $0<\gamma<1$. In this case the right-hand side of the equation is continuous but the Lipschitz condition is violated. We easily see that one solution is $x \equiv 0$. However, an explicit calculation reveals that the function x given by

$$x(t) = \sqrt[\gamma-1]{\frac{\Gamma(m+1)}{\Gamma(m+1-\alpha)}}t^m,\ m=\frac{\alpha}{1-\gamma}.$$

is also solution. Thus, we indeed see that, in general, the uniqueness of the solution cannot be expected without the Lipschitz condition.

Under slightly different conditions we can also derive an existence theorem

Theorem 6.7. *Let $\alpha \in (0,1)$, and $h \in \mathbb{R}$. Assume that $f : [0,1]\times\mathbb{R} \to \mathbb{R}$ is continuous. If additionally f is uniformly bounded by an absolute constant (meaning that, a constant $k > 0$ would exists such that $|f(t,x)| < k$ for every $x \in \mathbb{R}$), then the boundary value problem (149 and 150), has a solution $x \in C[0,1]$.*

Proof. Define the operator $T : C[0,1] \to C[0,1]$ by

$$Tx(t) = \frac{h}{1+b}+I^\alpha f(t,x(t))-\frac{b}{1+b}I^\alpha f(1,x(1)). \tag{157}$$

For each $x \in C[0,1]$ we have $t \to x(t)$ is continuous mapping from $[0,1]$ to $\mathbb{R}$. Hence, $f(\cdot,x(\cdot))$ is compositions of this mapping with f. It follows that, for every $x \in C[0,1]$, the function $f(\cdot,x(\cdot))$ is continuous in $[0,1]$. In the view of our assumptions and Lemma 3.3, $Tx \in C[0,1]$ and consequently T well-defined.

Our goal is now to show that it has at least one fixed point. To this end we want to invoke Tychonov's fixed point theorem (Theorem 10.2). Thus all we need to show is that $Q := \{Tx : x \in C[0,T]\}$ is a relatively compact set. A necessary and sufficient condition for this to hold is contained in the Arzelà-Ascoli Theorem (Theorem 2.1): We need to show that Q is uniformly bounded and equicontinuous. But the uniform boundedness of Q is a trivial consequence of the definition of T and the boundedness of f, and the equicontinuity can be shown just as in the proof of Lemma (3.3).
As in the proof of Theorem 6.6, $x \in C[0,1]$ satisfies the problem (149 and 150) and we are finished. □

Example E:(Continuous solutions of fractional differential equations)

In this example, we apply Krasnoselskii's fixed point theorem to obtain the existence of solutions for the fractional order differential equation of the following type

$$\begin{cases} D^\alpha x(t) = f(x(t)) + g(x(t)),\ \alpha \in (0,1),\ t \in [0,1], \\ x(0) = 0 \end{cases} \tag{158}$$

under the monotonicity condition of f, g. Here D^α denotes the, not necessarily integer, order Riemann-Liouville differential operator. For the physical point of view, positive solutions are interesting only. As usual we start with the following integral equation "which was modeled of the problem (158)"

$$x(t) = h(t) + \lambda I^\alpha[f(x(t)) + g(x(t))],\ t \in [0,1],\ \alpha \in (0,1). \tag{159}$$

To facilitate our discussion we let Q denotes the cone defined by

$$Q := \{x \in C[0,1] :\ x(t) \geq 0,\ t \in [0,1]\}.$$

Let us consider the following assumptions:

1- $h \in C[0,1]$ with $h(t) > 0,\ t \in [0,1]$,
2- $f : [0,\infty] \to [0,\infty]$ is continuous and nondecreasing,
3- $g : (0,\infty] \to [0,\infty]$ is continuous and nonincreasing,
4- Let $\lambda \geq 0$. Define $\mu := \min h(t)$ and suppose that there exist $\gamma > \mu$ such that

$$1 < \frac{\gamma}{\|h\| + \frac{\lambda}{\Gamma(1+\alpha)}\,[\ f(\gamma) +\ g(\mu)]}.$$

Now, we are in the position to state and prove the following

Theorem 6.8. *Under the above assumptions, the problem (159) has at least one positive continuous solution x such that $0 < \mu \leq \|x\| < \gamma$.*

Proof. Define $f^* : \mathbb{R} \to [f(\mu), f(\gamma)]$ and $g^* : \mathbb{R} \to [g(\gamma), g(\mu)]$ by

$$f^*(x) := \begin{cases} f(\gamma) & x \geq \gamma, \\ f(x) & \mu \leq x \leq \gamma \\ f(\mu) & x \leq \mu. \end{cases} \text{ and } g^*(x) := \begin{cases} g(\gamma) & x \geq \gamma, \\ g(x) & \mu \leq x \leq \gamma, \\ g(\mu) & x \leq \mu. \end{cases}$$

respectively. Furthermore, let the operator T defined by

$$Tx(t) = h(t) + \lambda I^{\alpha} \ [f^*(x(t)) + g^*(x(t))], \ t \in [0,1]. \tag{160}$$

We claim $T : C[0,1] \to C[0,1]$. To prove our claim, first note that, for each $x \in C[0,1]$ we have $t \to x(t)$ is continuous mapping from $[0,1]$ to $\mathbb{R}$. Hence, $f^*(x(\cdot))$ and $g^*(x(\cdot))$ are compositions of this mapping with f^* and g^* respectively. It follows that, for every $x \in C[0,1]$, the functions $f^*(x(\cdot))$ and $g^*(x(\cdot))$ are continuous in $[0,1]$. In the view of assumption (1) and Lemma 3.3, $Tx \in C[0,1]$ and consequently T well-defined. Arguing similarly as in the proof of Lemma 3.3 we can show for $x \in C[0,1]$ and $t_1, t_2 \in [0,1]$ that

$$\begin{aligned} |Tx(t_2) \ &- \ Tx(t_1)| \leq |h(t_2) - h(t_1)| \\ &+ \ \frac{\lambda}{\Gamma(1+\alpha)}[f(\gamma) + g(\mu)]\left\{2(t_2 - t_1)^{\alpha} + |t_2^{\alpha} - t_1^{\alpha}|\right\}. \end{aligned} \tag{161}$$

Now, we proceed to show that $T : Q \to Q$ is compact operator. Firstly, let $x_n \to x$ in Q, then $x_n(t) \to x(t)$ uniformly in $\mathbb{R}$ and

$$\begin{aligned} |Tx_n(t) - Tx(t)| \ &= \ \left|\lambda \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}(f^*(x_n(s)) + g^*(x_n(s)))\, ds \right. \\ &- \ \left. \lambda \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}(f^*(x(s)) + g^*(x(s))) \ ds \right| \\ &= \ \left|\lambda \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}(f^*(x_n(s)) - f^*(x(s)))\, ds \right. \\ &+ \ \left. \lambda \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}(g^*(x_n(s)) - g^*(x(s)))\, ds \right| \\ &\leq \ |\lambda| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|f^*(x_n(s)) - f^*(x(s))|\, ds \\ &+ \ |\lambda| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|g^*(x_n(s)) - g^*(x(s))|\, ds, \end{aligned}$$

That is

$$|Tx_n(t) - Tx(t)| \leq \frac{|\lambda|}{\Gamma(1+\alpha)} \left[\sup_{t\in[0,1]} |f^*(x_n(t)) - f^*(x(t))| + \sup_{t\in[0,1]} |g^*(x_n(t)) - g^*(x(t))| \right],$$

hence, by our assumptions $T : Q \to Q$ is continuous. Secondary, let $M \subset Q$ be bounded, for $x \in M$, we get

$$\|Tx\| \leq \|h\| + \frac{|\lambda|}{\Gamma(1+\alpha)}(f(\gamma) + g(\mu)). \tag{162}$$

According to assumption (4) we deduce $\|Tx\| < \gamma$, hence $T(M)$ is bounded. Finally, we use Arzelà-Ascoli theorem (Theorem 2.1, to show that $T : Q \to Q$ is compact. Immediately, we obtain from inequality (161) that for $x \in M$ we have $\|Tx(t_2) - Tx(t_1)\| \to 0$ as $t_1 \to t_2$. Thus, $T(M)$ is equicontinuous and in the view of Arzelà-Ascoli theorem (Theorem 2.1) $T(M)$ is compact which implying that the operator T is compact.

Now define

$$Q_1 := \{x \in C[0,1] : \|x\| < \mu\} \text{ and } Q_2 := \{x \in C[0,1] : \|x\| < \gamma\}$$

Note that for $x \in Q \cap \partial Q_1$ we have $0 \leq x \leq \mu, t \in [0,1]$. Then

$$Tx(t) \geq h(t) + \frac{\lambda}{\Gamma(1+\alpha)} \left[f^*(0) + g^*(\mu)\right] t^\alpha, \ \forall\ t \in [0,1]$$

that is

$$\|Tx\| \geq h(1) + \frac{\lambda}{\Gamma(1+\alpha)} \left[f^*(0) + g^*(\mu)\right].$$

Therefore, $\|Tx\| \geq \mu = \|x\|$. On the other hand, for $\bar{x} \in Q \cap \partial Q_2$, we have $0 \leq x(t) \leq \gamma,\ t \in [0,1]$. Then from equation (162) and assumption (4) we get $\|Tx\| < \gamma = \|x\|$. By Krasnoselskii's fixed point theorem (Theorem 2.18) the map

$$T : Q \cap (\bar{Q_2}/Q_1) \to Q$$

has a fixed point $x \in Q \cap (\bar{Q_2}/Q_1)$, that is

$$x(t) = h(t) + \frac{\lambda}{\Gamma(1+\alpha)} \int_0^t (t-s)^{\alpha-1} \left(f^*(x(s)) + g^*(x(s))\ \ ds,\ t \in [0,1]\right.$$

has a solution $x \in Q \cap (\bar{Q_2}/Q_1)$. Also, according to our assumptions however we see that $x(t) \geq \min h(t) = \mu,\ t \in [0,1]$. That is, $0 < \mu \leq x(t) \leq \gamma,\ t \in [0,1]$. Consequently, by the definition of f^* and g^*, we have $x \in Q \cap (\bar{Q_2}/Q_1)$ is a solution to the equation (159). This completes the proof. $\square$

Remark 6.2. Instead of applying the Krasnoselskii's fixed point theorem to investigate the existence of continuous solutions to the integral equation (159), we are able to proceed in different way making up the Schauder's fixed point theorem (Theorem 2.12) as follows

Proof. Let the operator T be given as in (160) and define the closed convex set Q by $Q := \{x \in C[0,1] : \mu \leq x(t) \leq \gamma,\ t \in [0,1]\}$. Therefore, one can apply Schauder's fixed point theorem to obtain the existence of at least one positive solution of the equation (159). We omit the details since it is almost identical to that in the proof in Theorem 6.8 with (small) necessary changes. □

Arguing similarly as in the proof of Theorem 6.8, we can show that

Theorem 6.9. *Suppose $h \in C[0,1]$ with $h(t) \geq 0,\ t \in [0,1]$ and $g : [0,\infty) \to [0,\infty)$. If the assumptions (2)-(4) of Theorem 6.8 hold along with $f + g : [0,\infty) \to (0,\infty)$. Then equation (159) has at least one continuous solution.*

Proof. The analysis is similar to that in Theorem 6.8 therefore, we omit the details. Let f^* and g^* be as defined in proof of Theorem 6.8 with $\mu = 0$ and let the operator T be given as in the equation (160). Defined the convex subset Q of $C[0,1]$ by $Q := \{c \in C[0,1] : \|x\| \leq \gamma\}$. One can show, with the aid of the proof of Theorem 6.8, that $T : Q \to Q$ is compact operator. The theorem, this time, follows from Schauder's fixed point theorem. Once again we omit the detail since it is almost identical to that in the proof of Theorem 6.8. □

Lemma 6.1. *If x is a continuous solution to the integral equation (159), then*

$$I^\alpha\left[f(x(\cdot)) + g(x(\cdot))\right] := \lim_{t\to 0} I^\alpha\left[f(x(t)) + g(x(t))\right] = 0,\ \alpha > 0.$$

Proof. The proof of this lemma follows immediately from the inequality

$$\begin{aligned}
\left|I^\alpha\left[f(x(t)) + g(x(t))\right]\right| &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left[f(x(s)) + g(x(s))\right] ds \\
&= \frac{(f(\gamma) + g(\mu)}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\, ds \\
&\leq \frac{f(\gamma) + g(\mu)}{\Gamma(1+\alpha)} t^\alpha.
\end{aligned}$$

Thus $I^\alpha\left[f(x(t)) + g(x(t))\right] \to 0$ as $t \to 0$. Since

$$I^\alpha\left[f(x(\cdot)) + g(x(\cdot))\right]$$

is continuous, and we complete the proof. □

Now we use the results of Theorem 6.9 to study the existence of continuous solutions for the fractional order differential equations (158).

Theorem 6.10. *Let $h(t) = 0$ and $\lambda = 1$. If the assumptions of Theorem 6.9 are satisfied, the problem (158) has at least one solution $x \in C[0, 1]$.*

Proof. Put $h(t) = 0$ and $\lambda = 1$. As in the proof of Theorem 6.8, one can show that, for every $x \in C[0, 1]$, the map the function $[f(x(\cdot)) + g(x(\cdot))]$ is continuous in [0,1]. Thus if $x \in C[0, 1]$ is a continuous solution to the problem (159), we obtain, in the view of Lemma 6.1 that $x(0) = 0$. Hence, by Lemma 3.24, $x \in C[0, 1]$ is a solution to the problem (158). □

Example 6.4. Let $\lambda = 1,\ h \equiv 0$. Define $f, g : [0, \infty] \to [0, \infty]$ by

$$f(x) := \cos\left(\frac{1}{1 + x^2}\right) \text{ and } g(x) := \frac{1}{e^x}.$$

If we put $\mu = 0$, it can be easily shown that the Assumptions of Theorem 6.10 with $\gamma > \frac{4}{\sqrt{\pi}}$, are satisfied and therefore the problem

$$\begin{cases} D^{\frac{1}{2}}x(t) & = \cos\left(\dfrac{1}{1 + x^2}\right) + \dfrac{1}{e^x} \\ x(0) & = 0, \end{cases}$$

has continuous solution x such that $0 \le x(t) \le \frac{4}{\sqrt{\pi}},\ t \in [0, 1]$

Example F (quadratic integral equations of fractional order)

The theory of quadratic integral equations with nonsingular kernels has received a lot of attention. Many authors studied the existence of solutions for several classes of nonlinear quadratic integral equations with nonsingular kernels. For example, Argyros [8], Banaś et al. [21, 25, 26], Caballero et al. [39, 40, 41], Darwish [53, 54], Leggett [117], Spiga et al. [178] and Stuart [179].

It is worth mentioning that, following the appearance of the paper [53], there has been a significant interest in the study of the existence of solutions for singular quadratic integral equations or fractional quadratic integral equations, (see e.g. [8, 55, 56, 57]). In most of the above investigations, the technique associated with the measures of noncompactness related to monotonicity is used to prove the existence of nondecreasing and continuous solutions to the quadratic integral equations. Here, we proceed in a different way by making up the technique associated with Schauder's fixed point theorem.

Motivated by the applications of the quadratic integral equations, we establish here a sufficient condition to ensure the existence of positive continuous solutions to the *possibly singular* quadratic integral equation of the fractional type

$$x(t) = H(t, x(t)) + x(t) I^{\alpha} \varphi(s)(f(x(t)) + g(x(t))),\ t \in [0,1],\ \alpha \in (0,1). \quad (163)$$

By the singularity, we mean that the possibility of $g(0)$ being undefined is permitted. To solve equation (163) it is necessary to find a fixed point of the operator $T : C[0,1] \to C[0,1]$ defined by

$$Tx(t) := H(t, x(t)) + \frac{x(t)}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \varphi(s)(f(x(s)) + g(x(s)))\, ds,\ t \in [0,1]. \quad (164)$$

To facilitate our discussion, let us first state the following assumptions

1. $H : [0,1] \times [0,\infty) \to [0,\infty)$ is continuous and satisfies the following conditions
   (a) $0 < \mu < \gamma$ exists such that
   $$H(t,x) \geq \mu \text{ holds for every } x \geq \mu \text{ and } \gamma \geq 2 \max_{t \in [0,1]} H(t,\mu),$$
   (b) There exists a function $b \in C[0,1]$ and a nondecreasing functions $c_i : [\mu, \infty) \to \mathbb{R}^+,\ i = 1,2$, such that
   $$|H(t,x) - H(s,y)| \leq c_1(r)|b(t) - b(s)| + c_2(r)|x - y|, \quad (165)$$
   for all $x, y \in [\mu, r]$ and $t, s \in [0,1]$.
2. $f : [0,\infty) \to [0,\infty)$ is continuous and nondecreasing,
3. $g : (0,\infty) \to [0,\infty)$ is continuous and nonincreasing,
4. $$0 \leq \varphi \in L_p(I), \text{ and } 2\left(c_2(\gamma) + \frac{[f(\gamma) + g(\mu)]\, \|\varphi\|_p}{\Gamma(\alpha)(q(\alpha-1)+1)^{1/q}}\right) \leq 1. \quad (166)$$

Theorem 6.11. *Let* $\alpha \in (0,1)$, $p > 1/\alpha$. *Assume that the assumptions* $(1)-(4)$ *be satisfied. Then equation (163) has at least one solution* $x \in C[0,1]$ *such that* $0 < \mu \leq x(t) \leq \gamma,\ t \in [0,1]$.

Proof. At the beginning, we observe, in the view of $p > 1/\alpha$, that $q(\alpha - 1) > -1$. Now, to solve equation (163) it is necessary to find a fixed point of the operator $T : C[0,1] \to C[0,1]$ defined by (164). We proceed by making up the technique associated with Schauder's fixed point theorem (Theorem 2.12). To do this, we define the subset $Q \subset C[0,1]$ (required by Schauder's fixed theorem) by:

$$\begin{aligned} Q \ := \ & \{x \in C[0,1] : \mu \leq x(t) \leq \gamma,\ \forall t \in [0,1] \text{ and } \forall t_1, t_2 \in [0,1], \\ & \text{we have } |x(t_1) - x(t_2)| \leq \frac{1}{1-K^*}\left[b_{1,2} + k|t_2 - t_1|^{\alpha - \frac{1}{p}}\right]\}, \end{aligned}$$

where

$$k := \frac{2(f(\gamma) + g(\mu))\gamma \left\|\varphi\right\|_p}{\Gamma(\alpha)(q(\alpha-1)+1)^{1/q}},$$

$$b_{1,2} := c_1(\gamma)|b(t_2) - b(t_1)|,$$

and

$$K^* := \left(c_2(\gamma) + \frac{(f(\gamma) + g(\mu)) \left\|\varphi\right\|_p}{\Gamma(\alpha)(q(\alpha-1)+1)^{1/q}}\right) < \frac{1}{2}.$$

Observe that Q is nonempty (since $\mu < \gamma$), closed, uniformly bounded, convex and equicontinuous (hence compact) subset of $C[0,1]$. We claim $T : Q \longrightarrow Q$. To prove our claim, firstly we note that, for each $x \in Q$, we have $t \to x(t)$ is continuous as a mapping from [0,1] into $[\mu, \gamma]$ then $f(x(\cdot))$ and $g(x(\cdot))$ are a compositions of this mapping with f and g respectively and thus, for each $x \in Q, f(x(\cdot)) : [0,1] \to [f(\mu), f(\gamma)]$ and $g(x(\cdot)) : [0,1] \to [g(\gamma), g(\mu)]$ are continuous. Therefore $\varphi(\cdot)[f(x(\cdot)) + g(x(\cdot))] \in L_p(0,1)$. Therefore, in the view of Lemma 3.1, the operator T makes sense.
Now, let $x \in Q$ and $t_1,\ t_2 \in [0,1]$ with $t_1 \leq t_2$. In the view of our assumptions

we obtain

$$
\begin{aligned}
|Tx(t_2) &- Tx(t_1)| \leq |H(t_2, x(t_2)) - H(t_1, x(t_1))| \\
&+ \left| x(t_2) \int_0^{t_2} \frac{(t_2 - s)^{\alpha-1}}{\Gamma(\alpha)} \varphi(s)(f(x(s)) + g(x(s)))\, ds \right. \\
&- \left. x(t_1) \int_0^{t_1} \frac{(t_2 - s)^{\alpha-1}}{\Gamma(\alpha)} \varphi(s)(f(x(s)) + g(x(s)))\, ds \right| \\
&\leq c_1(\|x\|)|b(t_2) - b(t_1)| + c_2(\|x\|)|x(t_2) - x(t_1)| \\
&+ \left| x(t_2) \int_0^{t_2} \frac{(t_2 - s)^{\alpha-1}}{\Gamma(\alpha)} (f(x(s)) + g(x(s)))\, ds \right. \\
&- \left. x(t_1) \int_0^{t_2} \frac{(t_2 - s)^{\alpha-1}}{\Gamma(\alpha)} \varphi(s)(f(x(s)) + g(x(s)))\, ds \right| \\
&+ \left| x(t_1) \int_0^{t_2} \frac{(t_2 - s)^{\alpha-1}}{\Gamma(\alpha)} \varphi(s)(f(x(s)) + g(x(s)))\, ds \right. \\
&- \left. x(t_1) \int_0^{t_1} \frac{(t_1 - s)^{\alpha-1}}{\Gamma(\alpha)} \varphi(s)(f(x(s)) + g(x(s)))\, ds \right| \\
&\leq c_1(\|x\|)|b(t_2) - b(t_1)| + c_2(\|x\|)|x(t_2) - x(t_1)| \\
&+ \left[\int_0^{t_2} \frac{(t_2 - s)^{\alpha-1}}{\Gamma(\alpha)} \varphi(s)(f(x(s)) + g(x(s)))\, ds \right] |x(t_2) - x(t_1)| \\
&+ \frac{|x(t_1)|}{\Gamma(\alpha)} \left| \int_0^{t_1} [(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}] \varphi(s)(f(x(s)) + g(x(s)))) \, ds \right. \\
&+ \frac{|x(t_1)|}{\Gamma(\alpha)} \left. \left| \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} \varphi(s)(f(x(s)) + g(x(s))) \ \ ds \right| \right. \\
&\leq c_1(\|x\|)|b(t_2) - b(t_1)| \\
&+ |x(t_2) - x(t_1)| \left[c_2(\|x\|) + \frac{f(\gamma) + g(\mu)}{\Gamma(\alpha)} \int_0^{t_2} (t_2 - s)^{\alpha-1} \varphi(s)\, ds \right] \\
&+ \frac{\|x\| \left(f(\gamma) + g(\mu)\right)}{\Gamma(\alpha)} \int_0^{t_1} \left| (t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1} \right| \varphi(s)\, ds \\
&+ \frac{\|x\| \left(f(\gamma) + g(\mu)\right)}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} \varphi(s)\, ds \\
&\leq c_1(\gamma)|b(t_2) - b(t_1)| + |x(t_2) - x(t_1)| \left[c_2(\gamma) + (f(\gamma) + g(\mu)) I^{\alpha} \varphi(t_2) \right] \\
&+ \left[\frac{\gamma(f(\gamma) + g(\mu))}{\Gamma(\alpha)} \left(\int_0^{t_1} \left| (t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1} \right| \varphi(s)\, ds \right. \right. \\
&+ \left. \left. \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} \varphi(s)\, ds \right) \right].
\end{aligned}
$$

Arguing similarly as in the proof of Lemma (3.12), we have

$$\begin{aligned}|Tx(t_2)-Tx(t_1)| &\le c_1(\gamma)|b(t_2)-b(t_1)|+K^*|x(t_2)-x(t_1)|\\ &+ \left[\frac{\gamma(f(\gamma)+g(\mu))}{\Gamma(\alpha)}\left(\frac{2\,\|\varphi\|_p}{(q(\alpha-1)+1)^{1/q}}\right)|t_2-t_1|^{\alpha-\frac{1}{p}}\right].\end{aligned}$$

Thus we have

$$|Tx(t_2)-Tx(t_1)|\le b_{1,2}+K^*\left[|x(t_2)-x(t_1)|+\frac{k}{K^*}|t_2-t_1|^{\alpha-\frac{1}{p}}\right] \tag{167}$$

The above inequality and our assumptions yield

$$|Tx(t_2)-Tx(t_1)|\to 0 \text{ as } t_2\to t_1,$$

then Tx is uniformly continuous in [0,1]. We claim that $T:Q\longrightarrow Q$ is continuous. Once our claim is established, according to Schauder's fixed point theorem, T has a fixed point in Q. It remain to prove our claim by showing that T maps Q into itself continuously. To see this, observe the equation (167) and the definition of Q. It can be easily seen that

$$\begin{aligned}|Tx(t_2)-Tx(t_1)| &\le b_{1,2}\\ &+ K^*\left[\frac{1}{1-K^*}\left[b_{1,2}+k|t_2-t_1|^{\alpha-\frac{1}{p}}\right]+\frac{k}{K^*}|t_2-t_1|^{\alpha-\frac{1}{p}}\right]\\ &= \frac{1}{1-K^*}\left[b_{1,2}+k|t_2-t_1|^{\alpha-\frac{1}{p}}\right]\end{aligned}$$

Moreover we have

$$Tx(t)\ge\min_{t\in[0,1]}H(t,x(t))+\ \mu(f(\mu)+g(\gamma))\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\varphi(s)\,ds\ge\mu.$$

Further, for every $t\in[0,1]$ and $x\in Q$, we have

$$\begin{aligned}\|Tx\| &\le \|H(\cdot,x(\cdot))\|+(f(\gamma)+g(\mu))\,\|x\|\left(\sup_{t\in[0,1]}\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\varphi(s)\,ds\right)\\ &= \|H(\cdot,x(\cdot))-H(\cdot,\mu)+H(\cdot,\mu)\|+\|x\|\,\frac{[f(\gamma)+g(\mu)]\,\|\varphi\|_p}{\Gamma(\alpha)(q(\alpha-1)+1)^{1/q}}\\ &\le \|H(\cdot,x(\cdot))-H(\cdot,\mu)\|+\|H(\cdot,\mu)\|+\|x\|\,\frac{[f(\gamma)+g(\mu)]\,\|\varphi\|_p}{\Gamma(\alpha)(q(\alpha-1)+1)^{1/q}}\\ &\le c_2(\|x\|)\,\|x\|+\max_{t\in[0,1]}H(t,\mu)+\|x\|\,\frac{[f(\gamma)+g(\mu)]\,\|\varphi\|_p}{\Gamma(\alpha)(q(\alpha-1)+1)^{1/q}}\\ &= \max_{t\in[0,1]}H(t,\mu)+\ \|x\|\left(c_2(\|x\|)+\frac{[f(\gamma)+g(\mu)]\,\|\varphi\|_p}{\Gamma(\alpha)(q(\alpha-1)+1)^{1/q}}\right).\end{aligned}$$

Thus

$$\|Tx\| \leq \max_{t\in[0,1]} H(t,\mu) + \gamma K^* \leq \frac{\gamma}{2} + \frac{\gamma}{2} \leq \gamma \tag{168}$$

Consequently $\mu \leq Tx(t) \leq \gamma,\ t \in [0,1]$. Hence $T : Q \to Q$ is well-defined. Next, let $x_n \to x$ in Q. For any $t \in [0,1]$ we have

$$\begin{aligned}
|Tx_n(t) &- Tx(t)| \leq |H(t,x_n(t)) - H(t,x(t))| \\
&+ \left| x_n(t) \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \varphi(s)(f(x_n(s)) + g(x_n(s)))\,ds \right. \\
&- \left. x(t) \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \varphi(s)(f(x_n(s)) + g(x_n(s)))\,ds \right| \\
&+ \left| x(t) \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \varphi(s)(f(x_n(s)) + g(x_n(s)))\,ds \right. \\
&- \left. x(t) \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \varphi(s)(f(x(s)) + g(x(s)))\,ds \right| \\
&\leq c_2(\gamma)\,\|x_n - x\| + \|x_n - x\|\,(f(\gamma) + g(\mu)) \left(\sup_{t\in[0,1]} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \varphi(s)\,ds \right) \\
&+ \gamma \left(\sup_{t\in[0,1]} |f(x_n(t)) - f(x(t))| + \sup_{t\in[0,1]} |g(x_n(t)) - g(x(t))| \right) \\
&\times \left(\sup_{t\in[0,1]} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \varphi(s)\,ds \right) \\
&\leq c_2(\gamma)\,\|x_n - x\| + \|x_n - x\| \frac{[f(\gamma) + g(\mu)]\,\|\varphi\|_p}{\Gamma(\alpha)(q(\alpha-1)+1)^{1/q}} \\
&+ \gamma \left(\sup_{t\in[0,1]} |f(x_n(t)) - f(x(t))| + \sup_{t\in[0,1]} |g(x_n(t)) - g(x(t))| \right) \\
&\times \left(\frac{\|\varphi\|_p}{\Gamma(\alpha)(q(\alpha-1)+1)^{1/q}} \right).
\end{aligned}$$

We have therefore shown that $T : Q \longrightarrow Q$ is continuous operator, hence by Schauder's fixed point (Theorem 2.12) theorem, $T : Q \longrightarrow Q$ has a fixed point. Consequently, the integral equation (163) has a positive continuous solution. □

Remark 6.3. We are able to relax the Assumption (4), if we assume that the function H independent of x, in this case we deduce that $c_2(\cdot) \equiv 0$ and then the Assumption (4) takes the form

$$2\left(\frac{[f(\gamma)+g(\mu)]\,\|\varphi\|_p}{\Gamma(\alpha)(q(\alpha-1)+1)^{1/q}}\right)\leq 1.$$

In addition, if $f\equiv 0$, $g(x)=\dfrac{1}{x}$ and $H(t,x)=e^{5+t}$, Assumption (4) takes the form

$$\frac{\|\varphi\|_p}{\Gamma(\alpha)(q(\alpha-1)+1)^{1/q}}\leq\frac{\mu}{2},\ \ \mu\leq e^5.$$

Remark 6.4. We also remark that, if we replace the function $x\to f(x)+g(x)$ by $x\to f(x)g(x)$ then it can be easily shown that the main results of our paper remain valid provided we replace Assumption (4) with the following one

$$2\left(c_2(\gamma)+\frac{f(\gamma)g(\mu)\,\|\varphi\|_p}{\Gamma(\alpha)(q(\alpha-1)+1)^{1/q}}\right)\leq 1. \tag{169}$$

Furthermore, if H independent of x and if $g(x)=[\psi(x)]^{-1}$, where $\psi:[0,\infty)\to[0,\infty)$ is continuous and nondecreasing, then Assumption (4) takes the form

$$\left(\frac{\|\varphi\|_p}{\Gamma(\alpha)(q(\alpha-1)+1)^{1/q}}\right)\leq\frac{\psi(\mu)}{2f(\gamma)},\ \ f(\gamma)\neq 0. \tag{170}$$

In account of the above remarks, we can see that the Assumption (4) is not too restrictive.

Now we give an example illustrating Theorem 6.11.

Example 6.5. Consider the quadratic integral equation of the fractional type

$$x(t)=\frac{1+t}{10}+\frac{x^2(t)e^{-t}}{30}+\frac{x(t)}{100\Gamma(0.5)}\int_0^t(t-s)^{-0.5}\left[\sqrt{x(s)}+\frac{1}{3\sqrt{x(s)}}\right]\frac{ds}{s^{1/6}},\ t\in[0,1]. \tag{171}$$

Observe that the above equation is a special case of Eq. (163) if we put $\alpha=1/2, p=3$ (Hence $q=1.5$) and

- $H(t,x)=\dfrac{1+t}{10}+e^{-t}\dfrac{x^2}{30}$,
- $\varphi(t)=\dfrac{1}{100t^{1/6}}$,
- $f(x)=\sqrt{x},\ g(x)=\dfrac{1}{3\sqrt{x}}$.

In what follows, we show that the functions involved in Eq. (171) satisfy the inequality (166) of Theorem 6.11. To do this, let $\mu=0.1$, then for every $x\geq\mu$ and every $t\in[0,1]$, we have $H(t,x)>0.1$, and $2H(t,0.1)<0.5$;

Now, let $\gamma = 0.5$, then $\gamma > \mu$ and for any $x, y \in [\mu, r],\ r > 0$ and every $t, s \in [0,1]$, we have

$$|H(t,x) - H(s,y)| \leq \frac{1}{10}|t-s| + \frac{1}{30}\left|e^{-t}x^2 - e^{-t}y^2\right| + \frac{1}{30}\left|e^{-t}y^2 - e^{-s}y^2\right|.$$

By the classical mean value theorem, it follows

$$\begin{aligned} |H(t,x) - H(s,y)| &\leq \frac{1}{10}|t-s| + \frac{2|x+y|}{30}|x-y| + \frac{r^2}{30}\left|e^{-t} - e^{-s}\right| \\ &\leq \frac{1}{10}|t-s| + \frac{2r}{30}|x-y| + \frac{r^2 e^{-\varsigma}}{30}|t-s|,\ \varsigma \in (0,1). \end{aligned}$$

Then it is clear that this choice of H satisfies the assumptions (1) with $\mu = 0.1$, $\gamma = 0.3$, $b(t) = t,\ t \in [0,1]$ and

$$c_1(r) = \frac{1}{10} + \frac{r^2}{30},\ c_2(r) = \frac{2r}{30}.$$

Also, we note that $c_2(\gamma) = c_2(0.5) = 0.03 < 0.5$ and $\|\varphi\|_3 \simeq 0.0125$. Therefore

$$2\left(c_2(\gamma) + 2\sqrt[3]{2}\frac{\sqrt{\gamma} + 1/(3\sqrt{\mu})}{\Gamma(1.5)}\|\varphi\|_3\right) \leq 0.83.$$

Thus, the hypotheses of Theorem 6.11 are satisfied, hence we conclude that the Equation. (171) has at least one solution $x \in C[0,1]$ such that $0.1 \leq x(t) \leq 0.3,\ t \in [0,1]$.

To encompass the full scope of this part, we investigate the problem of the existence of maximal and minimal solutions to the integral equation (163)

Maximal and minimal solutions

We recall with the following standard definition

Definition 6.1. Let m be a solution of the integral equation (163) in [0,1], then m is said to be a maximal solution of (163) if, for every solution x of (163) existing on [0,1], the inequality $x(t) \leq m(t),\ t \in [0,1]$, holds. A minimal solution may be define similarly by reversing the last inequality.

Theorem 6.12. *Suppose that the assumptions (1)-(6) are satisfied with $f(\cdot) = 0$ and with a constant function c_2. Then there exists either maximal or minimal solutions of the integral equation (163) in the space $C([0,1],(0,\infty))$.*

Proof. Consider the integral equation

$$x(t) = \frac{1}{n} + H(t,x(t)) + \ x(t)\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\varphi(s)g(x(s))\,ds,\ t \in [0,1],\ n \in \mathbb{N}. \tag{172}$$

In the view of our assumptions with $H_n(t,x) := \frac{1}{n} + H(t,x)$ it is clear that

$$H_n(t,x) \geq \mu,\ t \in [0,1] \text{ and } 2H_n(t,\mu) \leq \frac{2}{n} + \gamma \leq 2+\gamma := \gamma*$$

Since c_2 is constant function, then the assumptions of Theorem 6.11 hold, therefore the integral equation (172) has at least one solution $x_n \in C[0,1]$, such that the inequality $\mu \leq x_n(t) \leq \gamma^*,\ t \in [0,1]$ holds for every $n \in \mathbb{N}$.

Now, we claim that the family of functions $\{x_n\}$ is relatively compact on [0,1]. Indeed, the uniform boundedness of the family $\{x_n\}$ is owing to $\|x_n\| \leq \gamma^*,\ \forall n \in \mathbb{N}$. Moreover, as in the proof of Eq. (167), it can be easily shown that

$$|x_n(t_2) - x_n(t_1)| \leq b_{1,2} + K^*|x(t_2) - x(t_1)| + k|t_2 - t_1|^{\alpha - \frac{1}{p}}$$

which implies

$$|x_n(t_2) - x_n(t_1)|\,[1 - K^*] \leq b_{1,2} + k|t_2 - t_1|^{\alpha - \frac{1}{p}}.$$

That is

$$|x_n(t_2) - x_n(t_1)| \leq \frac{1}{1 - K^*}\left[b_{1,2} + k|t_2 - t_1|^{\alpha - \frac{1}{p}}\right].$$

This yields the equicontinuity of the family $\{x_n\}$ on [0,1], hence the relatively compactness on [0,1]. Therefore, we can extract a uniformly convergent subsequence $\{x_{n_i}\}$ of the sequence $\{x_n\}$ such that the limit $\lim_{n_i \to \infty} x_{n_i}(t)$ exists uniformly in $[0,1]$, we denote this limiting value by $\psi(t)$.

Obviously, the uniform continuity of g and $H(\cdot, x(\cdot))$ together with

$$x_{n_i}(t) = \frac{1}{n_i} + H(t, x_{n_i}(t)) +\ x_{n_i}(t)\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\varphi(s) g(x_{n_i}(s))\, ds,\ t \in [0,1],$$

yield ψ is a solution of integral equation (163). It remains to show that the solution ψ is either maximal or minimal solution to integral equation (163). To do this, we observe the problem (172) and we let $m,\ n \in \mathbb{N}$ with $m < n$. keeping in mind that $\varphi(\cdot)g(x_{n_i}(\cdot)) \in C[0,1]$, then we obtain, by the aid of Corollary (254) that

$$x_n(0) - x_m(0) = \frac{1}{n} - \frac{1}{m} + H(0, x_n(0)) - H(0, x_m(0)). \tag{173}$$

Therefore, we have $x_n(0) \neq x_m(0)$ for otherwise, we get $m = n$ which contradict our hypotheses that $m < n$. Thus, we have one of the following two possibilities

$$\text{either } x_n(0) > x_m(0) \text{ or } x_n(0) < x_m(0). \tag{174}$$

We start with the case $x_n(0) > x_m(0)$ and we will show, in this case, that the ψ is the minimal solution to the equation (163). We proceed in two steps as follows:

Firstly, we will show that if $x_n(0) > x_m(0)$ then

$$x_n(t) > x_m(t) \text{ for all } t \in [0,1]. \tag{175}$$

To prove the conclusion (175), we assume that it is false, then there exist a $t_1 \in [0,1]$ such that

$$x_m(t_1) = x_n(t_1) \text{ and } x_m(t) > x_n(t), \text{ for all } t \in [0,t_1).$$

Thus $H(t_1, x_m(t_1)) = H(t_1, x_n(t_1))$. So, since g is monotonic nonincreasing, it follows, using equation (172), that

$$\begin{aligned} x_m(t_1) &= \frac{1}{m} + H(t_1, x_m(t_1)) + x_m(t_1) \int_0^{t_1} \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} \varphi(s) g(x_m(s))\, ds \\ &> \frac{1}{n} + H(t_1, x_n(t_1)) + x_n(t_1) \int_0^{t_1} \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} \varphi(s) g(x_n(s))\, ds = x_n(t_1). \end{aligned}$$

Which contradict the fact that $x_m(t_1) = x_n(t_1)$. Hence (175) is true.

Secondly, we show that ψ is the minimal solution to the equation (163).

To achieve this goal, we let x be any solution of (163) existing on the interval [0,1]. Then $x_m(t) < x(t),\ t \in [0,1]$. Since the minimal solution is unique (see [114] and [142]), it is clear that $x_m(t)$ tends to $\psi(t)$ uniformly in $t \in [0,1]$ as $m \to \infty$. Which proves the existence of minimal solution to the integral equation (163).

A similar argument making up the possibility that $x_n(0) < x_m(0)$, implies the existence of maximal solution. This ends the proof. □

Corollary 6.1. *Suppose that the assumptions of Theorem 6.12 are satisfied. If H does not depend on x, then there exists a maximal solution to the integral equation (163) in the space $C([0,1],(0,\infty))$.*

Proof. By looking at the proof of Theorem 6.12, we observe that the existence of maximal solution to (163), follows directly by showing that the inequality $x_n(0) < x_m(0),\ m < n$ holds. To see this, we look at the equality (173). Since H independent of x, we have $H(0, x_n(0)) - H(0, x_m(0)) = 0$. This implies that the right-hand side of the equation (173) (and consequently the left-hand side) is negative. That is $x_n(0) - x_m(0) < 0,\ m < n$. Thus, $x_n(0) < x_m(0),\ m < n$. Now we are able to repeat the rest of the proof of theorem 6.12 to prove the existence of a maximal solution to the integral equation (163). □

7. Methods for solving fractional differential equations

In this section, we use the Laplace transform method and the successive approximations method to drive explicit solutions to the ordinary differential equations with fractional derivatives.

7.1. **Laplace transform method.** In this section, by the Laplace transform, we construct explicit solution to linear differential equations of the fractional type. We start by the following

Lemma 7.1. *The Laplace transform of Riemann-Liouville fractional integral operator of order $\alpha > 0$ can be obtained in the form of:*

$$£\left[I^{\alpha} f(t)\right] = \frac{F(s)}{s^{\alpha}}.$$

Proof. The Laplace transform of Riemann-Liouville fractional integral operator of order $\alpha > 0$ is

$$£\left[I^{\alpha} f(t)\right] = £\left[\frac{1}{\Gamma(\alpha)}\int_0^t (t-x)^{\alpha-1} f(x)dx\right] = \frac{1}{\Gamma(\alpha)} F(s)G(s),$$

where

$$G(s) = £\left[t^{\alpha-1}\right] = \frac{\Gamma(\alpha)}{s^{\alpha}}.$$

$\square$

Lemma 7.2. *The Laplace transform of Caputo fractional derivative for $m-1 < \alpha \leq m,\ m \in \mathbb{N}$, can be obtained in the form of:*

$$£\left(\frac{d^{\alpha} f(t)}{dt^{\alpha}}\right) = \frac{s^m F(s) - s^{m-1} f(0) - s^{m-2} f\prime(0) - \cdots - f^{(m-1)}(0)}{s^{m-\alpha}},\ t > 0.$$

Proof. The Laplace transform of Caputo fractional derivative of order $\alpha > 0$ is

$$£\left(\frac{d^{\alpha} f(t)}{dt^{\alpha}}\right) = £\left[I^{m-\alpha} f^{(m)}(t)\right] = \frac{£\left[f^{(m)}(t)\right]}{s^{m-\alpha}},$$

using equation (30).

$\square$

Lemma 7.3. *The Laplace transform of Riemann-Liouville fractional derivative for $m-1 < \alpha \leq m,\ m \in \mathbb{N}$, can be obtained in the form of:*

$$£\left(D^{\alpha} f(t)\right) = s^{\alpha} F(s) - \sum_{j=1}^{m} s^{j-1} D^{\alpha-j} f(0),\ t > 0.$$

Proof. The Laplace transform of Riemann-Liouville fractional derivative of order $\alpha > 0$ is

$$£\left\{D^\alpha f(t)\right\} = £\left\{D^m(I^{m-\alpha}f(t))\right\}$$

Using equation (30) and applying Lemma 3.24 result in

$$\begin{aligned} £\left\{D^\alpha f(t)\right\} &= s^m £\left\{I^{m-\alpha}f(t)\right\} - \sum_{k=0}^{m-1} s^{m-(k+1)}\left[D^k I^{m-\alpha}f(t)\right]_{t=0} \\ &= \frac{s^m F(s)}{s^{m-\alpha}} - \sum_{k=0}^{m-1} s^{m-(k+1)} D^{k-(m-\alpha)}f(0) \\ &= s^\alpha F(s) - \sum_{j=1}^{m} s^{j-1} D^{\alpha-j} f(0),\ j = m-k. \end{aligned}$$

□

<u>Comment</u>: The Laplace transform of the Caputo fractional derivative is a generalization of the Laplace transform of integer-order derivative. The same does not hold for the Riemann-Liouville case. This property is an important advantage of the Caputo operator over the Riemann-Liouville operator. Now,

we can transform fractional differential equations into algebraic equations and then by solving this algebraic equations, we can obtain the unknown Laplace function $F(s)$.

Firstly, we prove some facts which are useful for finding the function f from its Laplace transform.

Lemma 7.4. *For $\alpha, \beta > 0$, $a \in \mathbb{R}$ and $s^\alpha > |a|$ we have the following inverse Laplace transform formula*

$$£^{-1}\left[\frac{s^{\alpha-\beta}}{s^\alpha + a}\right] = t^{\beta-1}E_{\alpha,\beta}(-at^\alpha).$$

Proof. By using the series expansion of $(1+t)^{-n-1}$, we are able to rewrite the expression $s^{\alpha-\beta}/s^\alpha + a$ as

$$\frac{s^{\alpha-\beta}}{s^\alpha + a} = \frac{1}{s^\beta}\frac{1}{1+\frac{a}{s^\alpha}} = \frac{1}{s^\beta}\sum_{n=o}^{\infty}\left(\frac{-a}{s^\alpha}\right)^n = \sum_{n=o}^{\infty}\frac{(-a)^n}{s^{n\alpha+\beta}}.$$

The inverse Laplace transform of above function is

$$\sum_{n=o}^{\infty}\frac{(-a)^n t^{n\alpha+\beta-1}}{\Gamma(n\alpha+\beta)} = t^{\beta-1}\sum_{n=o}^{\infty}\frac{(-at^\alpha)^n}{\Gamma(n\alpha+\beta)} = t^{\beta-1}E_{\alpha,\beta}(-at^\alpha).$$

□

Lemma 7.5. *For $\alpha \geq \beta > 0, a \in \mathbb{R}$ and $s^{\alpha-\beta} > |a|$ we have*

$$£^{-1}\left[\frac{1}{(s^\alpha + as^\beta)^{n+1}}\right] = t^{\alpha(n+1)-1}\sum_{k=o}^{\infty}\frac{(-a)^k\left(\begin{array}{c} n+k \\ k \end{array}\right)}{\Gamma(k(\alpha-\beta)+(n+1)\alpha)}\, t^{k(\alpha-\beta)}.$$

Proof. Using the series expansion of $(1+t)^{-n-1}$ of the from

$$\frac{1}{(1+t)^{n+1}} = \sum_{k=o}^{\infty}\left(\begin{array}{c} n+k \\ k \end{array}\right)(-t)^k$$

we have:

$$\frac{1}{(s^\alpha + as^\beta)^{n+1}} = \frac{1}{(s^\alpha)^{n+1}}\frac{1}{\left(1+\frac{a}{s^{\alpha-\beta}}\right)^{n+1}} = \frac{1}{(s^\alpha)^{n+1}}\sum_{k=o}^{\infty}\left(\begin{array}{c} n+k \\ k \end{array}\right)\left(\frac{-a}{s^{\alpha-\beta}}\right)^k$$

Giving the inverse Laplace transform of above function can prove the Lemma. □

As we did at the previous Lemma, we obtain

Lemma 7.6. *For $\alpha \geq \beta$, $\alpha > \gamma$, $a \in \mathbb{R}$, $s^{\alpha-\beta} > |a|$ and $|s^\alpha + as^\beta| > |b|$ we have:*

$$£^{-1}\left[\frac{s^\gamma}{s^\alpha + as^\beta + b}\right] = t^{\alpha-\gamma-1}\sum_{n=o}^{\infty}\sum_{k=o}^{\infty}\frac{(-b)^n(-a)^k\left(\begin{array}{c} n+k \\ k \end{array}\right)t^{k(\alpha-\beta)+n\alpha}}{\Gamma(k(\alpha-\beta)+(n+1)\alpha-\gamma)}.$$

Proof. $s^\gamma/(s^\alpha + as^\beta + b)$ by using the series expansion can be rewritten as

$$\frac{s^\gamma}{s^\alpha + as^\beta + b} = \frac{s^\gamma}{s^\alpha + as^\beta}\frac{1}{1+\frac{b}{s^\alpha + as^\beta}} = \sum_{n=o}^{\infty}\frac{s^\gamma(-b)^n}{(s^\alpha + as^\beta)^{n+1}}.$$

Applying now Lemma 7.5 we completes the proof. □

Illustrative examples:

Now, we apply the method presented in this section to obtain an exact solution of some linear fractional differential equations.

Example 7.1. Consider the fractional order differential equation

$$D^{\frac{1}{2}}y(t) = \frac{\lambda y(t)}{t},\ t > 0,\ \lambda \in \mathbb{R}^+. \tag{176}$$

In our consideration, we rewrite (176) in the form

$$tD^{\frac{1}{2}}y(t) = \lambda y(t). \tag{177}$$

Applying the Laplace transform to (177), and using the relation $D\pounds[x(t)] = -\pounds(tx(t))$, with $x = D^{\frac{1}{2}}y$, we have

$$-\frac{d}{ds}\pounds[D^{\frac{1}{2}}y(t)] = \lambda\pounds[y(t)]$$

By Lemma 7.3 with $m = 1$, we arrive at

$$-\frac{d}{ds}\left(s^{\frac{1}{2}}\pounds y(t) - \zeta\right) = \lambda\pounds[y(t)],\ \zeta = I^{\frac{1}{2}}y(0)$$

This implies the ordinary differential equation of the first order

$$-s^{\frac{1}{2}}\frac{dF(s)}{ds} - \frac{1}{2}s^{-\frac{1}{2}}F(s) = \lambda F(s)$$

where $F := \pounds y$. Consequently

$$\frac{dF(s)}{ds} = -\left(\frac{1}{2s} + \frac{\lambda}{\sqrt{s}}\right)F(s).$$

It is not hard to see that the solution of this ordinary differential equation given by

$$F(s) = \pounds y(t) = \frac{c}{\sqrt{s}}e^{-2\lambda\sqrt{s}}$$

where c is an arbitrary real constant. Owing to the formula

$$\pounds^{-1}\left[\frac{e^{-\zeta\sqrt{s}}}{\sqrt{s}}\right] = \frac{e^{-\zeta^2/4t}}{\sqrt{\pi t}},\ \zeta > 0,$$

the differential equation (176) has its solution given by

$$y(t) = \frac{c}{\sqrt{t\pi}}e^{\frac{-\lambda^2}{t}}.$$

Example 7.2. Consider the following initial value problem in the case of the inhomogeneous Bagley-Torvik equation

$$\frac{d^2y}{dt^2} + \frac{d^{\frac{3}{2}}y}{dt^{\frac{3}{2}}} + y(t) = 1 + t \tag{178}$$

$$y(0) = y\prime(0) = 1$$

this equation by using Laplace transform is converted to

$$s^sF(s) - sy(0) - y\prime(0) + \frac{s^2F(s) - sy(0) - y\prime(0)}{s^{\frac{1}{2}}} + F(s) = \frac{1}{s} + \frac{1}{s^2}$$

$$s^2F(s) - s - 1 + \frac{s^2F(s) - s - 1}{s^{\frac{1}{2}}} + F(s) = \frac{1}{s} + \frac{1}{s^2}$$

$$F(s)(s^2 + s^{\frac{3}{2}} + 1) = (\frac{1}{s} + \frac{1}{s^2})(s^2 + s^{\frac{3}{2}} + 1)$$

$$F(s) = \frac{1}{s} + \frac{1}{s^2}$$

Using the inverse Laplace transform the exact solution of this problem $y(t) = 1 + t$ can be obtained.

Example 7.3. Our second example covers the inhomogeneous linear equation

$$\frac{d^\alpha y}{dt^\alpha} + y(t) = \frac{2t^{2-\alpha}}{\Gamma(3-\alpha)} - \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} + t^2 - t \tag{179}$$

$$y(0) = 0, \qquad 0 < \alpha \leq 1$$

Using the Laplace transform $F(s)$ is obtained as follows

$$\frac{sF(s) - y(0)}{s^{1-\alpha}} = \frac{2}{s^{3-\alpha}} - \frac{1}{s^{2-\alpha}} - F(s) + \frac{2}{s^3} - \frac{1}{s^2}$$

$$F(s)(s^\alpha + 1) = 2\frac{s^\alpha + 1}{s^3} - \frac{s^\alpha + 1}{s^2}$$

$$F(s) = \frac{2}{s^3} - \frac{1}{s^2}$$

then $y(t) = t^2 - t$ is obtained by using the inverse Laplace transform.

Example 7.4. Consider the following linear initial value problem

$$\frac{d^\alpha y}{dt^\alpha} + y(t) = 0 \tag{180}$$

$$y(0) = 1, \qquad y\prime(0) = 0$$

The second initial condition is for $\alpha > 1$ only.

In two cases of α, $\pounds\left[\frac{d^\alpha f(t)}{dt^\alpha}\right]$ is obtained as

1. For $\alpha < 1$ we have

$$\pounds\left[\frac{d^\alpha f(t)}{dt^\alpha}\right] = \frac{s^2 F(s) - s}{s^{2-\alpha}} = \frac{sF(s) - 1}{s^{1-\alpha}},$$

2. For $\alpha > 1$ we have

$$\pounds\left[\frac{d^\alpha f(t)}{dt^\alpha}\right] = \frac{sF(s) - 1}{s^{1-\alpha}}.$$

which are the same. Now the Laplace transform, $F(s)$ is obtained as

$$\frac{sF(s) - 1}{s^{\alpha-1}} + F(s) = 0,$$

$$F(s) = \frac{s^{\alpha-1}}{1 + s^\alpha}$$

Using the lemma 7.4, the exact solution of this problem can be obtained as:

$$y(t) = E_\alpha(-t^\alpha)$$

Example 7.5. Consider the following linear initial value problem

$$\frac{d^\alpha y(t)}{dt^\alpha} = y(t) + 1, \qquad 0 < \alpha \leq 1 \tag{181}$$

$$y(0) = 0$$

Using the Laplace transform $F(s)$ is obtained as follows

$$\frac{sF(s)}{s^{1-\alpha}} = F(s) + \frac{1}{s}$$

$$F(s) = \frac{s^{-1}}{s^{\alpha-1}}$$

Using the Lemma 7.4 the exact solution of this problem can be obtained as:

$$y(t) = t^\alpha E_{\alpha,\alpha+1}(t^\alpha)$$

Example 7.6. Consider the composite fractional oscillation equation

$$y''(t) - a\frac{d^\alpha y(t)}{dt^\alpha} - by(t) = 8, \qquad 1 < \alpha \leq 2$$

$$y(0) = y'(0) = 0$$

Using the Laplace transform, $F(s)$ is obtained as follows

$$s^2F(s) - a\frac{s^2F(s)}{s^{2-\alpha}} - bF(s) = \frac{8}{s},$$

$$F(s) = \frac{8s^{-1}}{s^2 - as^\alpha - b}$$

Using the lemma 7.6 the exact solution of this problem can be obtained as:

$$y(t) = 8t^2 \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{b^n a^k \begin{pmatrix} n+k \\ k \end{pmatrix} t^{(2-\alpha)k+2n}}{\Gamma\left((2-\alpha)k + 2(n+1) + 1\right)}$$

Last but not the least, we consider the following

Example 7.7. Consider the following system of fractional algebraic-differential equations

$$\frac{d^\alpha x(t)}{dt^\alpha} - ty'(t) + x(t) - (1+t)y(t) = 0, \qquad 0 < \alpha \leq 1$$

$$y(t) - \sin t = 0$$

subject to the initial conditions

$$x(0) = 1,\ y(0) = 0.$$

A bit calculations using the Laplace transform $F(s) = \pounds\left[y(t)\right]$ and $G(s) = \pounds\left[x(t)\right]$ implies

$$\frac{sG(s)-1}{s^{1-\alpha}} + F(s) = sF'(s) + G(s) - F(s) + F'(s) = 0,$$

$$F(s) = \frac{1}{s^2+1} \qquad , \qquad F'(s) = \frac{-2s}{(s^2+1)^2}$$

$$G(s)(s^{\alpha}+1) = \frac{2s(s+1)}{(1+s^2)^2} + \frac{1}{s^{1-\alpha}}$$

$$G(s) = \frac{2s}{s^{\alpha}+1}\frac{s+1}{(1+s^2)^2} + \frac{s^{\alpha-1}}{s^{\alpha}+1}$$

The exact solution for $\alpha = 1$ is $x(t) = t\sin t + e^{-t}$. Using the Lemma 7.4 and 7.5 the exact solution for $0 < \alpha \leq 1$ can be obtained as:

$$x(t) = 2t^{\alpha+1}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{n+k}(k+1)t^{2k}$$

$$\left(\frac{t^{n\alpha+1}}{\Gamma((n+1)\alpha+2k+3)} + \frac{t^{n\alpha}}{\Gamma((n+1)\alpha+2k+2)}\right) + E_{\alpha}(-t^{\alpha})$$

$$= 2t^{\alpha+1}\sum_{k=0}^{\infty}(-1)^{\left[\frac{k}{2}\right]}\left(\left[\frac{k}{2}\right]+1\right)t^{k}E_{\alpha,\alpha+k+2}(-t^{\alpha}) + E_{\alpha}(-t^{\alpha})$$

$$y(t) = \sin t,$$

where $[\cdot]$ denotes the integer part of the number.

In the following example, we apply Laplace transform method to drive explicit solutions to some ordinary differential equations with Reimann-Liouville fractional derivatives

Example 7.8. Let us consider the differential equation

$$D^{\alpha}y(t) - \lambda y(t) = 0,\ \alpha \in (m-1, m],\ m \in \mathbb{N},\ \lambda \in \mathbb{R}, \tag{182}$$

subject to the initial conditions

$$D^{\alpha-k}y(0) = b_k,\ k = 1, 2, \cdots, m.$$

Applying the Laplace transform $F(s) = [y(t)]$ to the equation (182), we obtain in the view of Lemma 7.3 that

$$s^{\alpha}F(s) - \sum_{j=1}^{m} s^{j-1}b_j - \lambda F(s) = 0,\ j = 1, 2, \cdots, m.$$

Then we obtain

$$F(s) = \frac{\sum_{j=1}^{m} s^{j-1}b_j}{s^{\alpha}-\lambda}.$$

Recalling, by Lemma 7.4 that

$$£^{-1}\left(\frac{s^{j-1}}{s^{\alpha}-\lambda}\right) = t^{\alpha-j}E_{\alpha,\alpha+1-j}(\lambda t^{\alpha}).$$

This yields

$$y(t) = £^{-1}\left[\sum_{j=1}^{m} b_j \frac{s^{j-1}}{s^{\alpha}-\lambda}\right] = \sum_{j=1}^{m} b_j t^{\alpha-j}E_{\alpha,\alpha+1-j}(\lambda t^{\alpha}).$$

In what follows, we shall solve some integral equation of fractional type. Evidently, we shall use the alternative technique of the Laplace transform, that makes easier treatment of such fractional integral equation, to obtain the solution of such integral equation.

Example 7.9. Let us consider the integral equation

$$y(t) + \frac{\lambda}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)\ ds = g(t),\ \alpha > 0,\ \lambda \in \mathbb{R}, \tag{183}$$

where g is a given function. We easily recognize that this equation can be expressed in terms of a fractional integral, i.e.

$$y(t) + \lambda I^{\alpha}y(t) = g(t).$$

Applying the Laplace transform $F(s) = [y(t)]$ and $G(s) = £\,[g(t)]$ to (183), implies

$$\left[1 + \frac{\lambda}{s^{\alpha}}\right]F(s) = G(s) \Longrightarrow F(s) = \frac{s^{\alpha}G(s)}{s^{\alpha}+\lambda}.$$

Then we get

$$\frac{F(s)}{s} = \frac{s^{\alpha-1}}{s^{\alpha}+\lambda}G(s).$$

Recalling that

$$£^{-1}\left(\frac{F(s)}{s}\right) = \int_0^t y(\tau)d\tau,\ \text{and}\ £^{-1}\left[\frac{s^{\alpha-1}}{s^{\alpha}+\lambda}\right] = E_{\alpha,1}(-\lambda t^{\alpha}),$$

then, by the formula

$$£\left(\int_0^t y(t-s)g(s)\,ds\right) = F(s)G(s),$$

we arrive at

$$\int_0^t y(\tau)d\tau = £^{-1}\left[\frac{s^{\alpha-1}}{s^{\alpha}+\lambda}G(s)\right] = \int_0^t E_{\alpha,1}(-\lambda\tau^{\alpha})g(t-\tau)d\tau.$$

Thus

$$y(t) = \frac{d}{dt}\int_0^t g(t-\tau)E_{\alpha,1}(-\lambda\tau^{\alpha})d\tau. \tag{184}$$

Remark 7.1. We note that, if the function g is differentiable, we can proceed in different way:
If we write

$$F(s) = \frac{s^{\alpha-1}}{s^\alpha+\lambda}[sG(s)-g(0)] - g(0)\frac{s^{\alpha-1}}{s^\alpha+\lambda}$$

Using the properties of the convolution of the Laplace transform we obtain

$$y(t) = \pounds^{-1}\left[\frac{s^{\alpha-1}}{s^\alpha+\lambda}\right] * \pounds^{-1}[sG(s)-g(0)] - g(0)\pounds^{-1}\left[\frac{s^{\alpha-1}}{s^\alpha+\lambda}\right]$$

In this case, we deduce that

$$y(t) = \int_0^t g'(t-\tau)E_{\alpha,1}(-\lambda\tau^\alpha)d\tau + g(0)E_{\alpha,1}(-\lambda t^\alpha).$$

Remark 7.2. According to the Equation (8.12), if $g \equiv 1$, then $y(t) = E_{\alpha,1}(-\lambda t^\alpha)$. Therefore, in the view of the integral equation (183), we obtain the following rule

$$\lambda I^\alpha E_{\alpha,1}(-\lambda t^\alpha) = 1 - E_{\alpha,1}(-\lambda t^\alpha).$$

In particular

$$I^\alpha E_{\alpha,1}(t^\alpha) = E_{\alpha,1}(t^\alpha) - 1.$$

7.2. **Successive approximations method.** In this section, we construct explicit solution to linear differential equations with the Caputo and the Riemann-Liouville fractional derivative.
The results are taken from [107] (Chapter 4).

First, we consider the cauchy type problem for fractional differential equation of order $\alpha \in (m-1, m),\ m \in \mathbb{N}$ with the initial conditions

$$\begin{cases} D^\alpha y(t) - \lambda y(t) = f(x),\ \lambda \in \mathbb{R},\ t \in [0,1] \\ D^{\alpha-j}y(0) = b_j,\ b_j\ \in \mathbb{R},\ j = 1, 2, \cdots, m. \end{cases} \tag{185}$$

In the view of Equation (64) and Lemma 3.24, it can be easily seen "formally" that the integral equation medelled off the problem (185) is given by

$$y(t) = \sum_{j=1}^m \frac{b_j\ t^{\alpha-j}}{\Gamma(\alpha-j+1)} + \lambda I^\alpha y(t) + I^\alpha f(t) \tag{186}$$

We apply the successive approximations method to solve the integral equation (186). According to this method, we set

$$y_0(t) = \sum_{j=1}^m \frac{b_j\ t^{\alpha-j}}{\Gamma(\alpha-j+1)},$$

$$y_n(t) = y_0(t) + \lambda I^\alpha y_{n-1}(t) + I^\alpha f(t),\ n \in \mathbb{N}. \tag{187}$$

Taking into account Lemma 3.2, we find for y_1 that

$$\begin{aligned}
y_1(t) &= y_0(t) + \lambda I^\alpha y_0(t) + I^\alpha f(t) \\
&= y_0(t) + \lambda \sum_{j=1}^{m} \frac{b_j I^\alpha t^{\alpha-j}}{\Gamma(\alpha-j+1)} + I^\alpha f(t) \\
&= y_0(t) + \lambda \sum_{j=1}^{m} \frac{b_j\ \Gamma(1+\alpha-j)\ t^{2\alpha-j}}{\Gamma(\alpha-j+1)\Gamma(1+\alpha+\alpha-j)} + I^\alpha f(t) \\
&= \sum_{j=1}^{m} \frac{b_j\ t^{\alpha-j}}{\Gamma(\alpha-j+1)} + \lambda \sum_{j=1}^{m} \frac{b_j\ t^{2\alpha-j}}{\Gamma(2\alpha-j+1)} + I^\alpha f(t) \\
&= \sum_{j=1}^{m} b_j \sum_{k=1}^{2} \frac{\lambda^{k-1}\ t^{\alpha k-j}}{\Gamma(\alpha k-j+1)} + I^\alpha f(t).
\end{aligned}$$

Similarly we find

$$\begin{aligned}
y_2(t) &= y_0(t) + \lambda I^\alpha y_1(t) + I^\alpha f(t) \\
&= y_0(t) + \lambda \left[\sum_{j=1}^{m} b_j \sum_{k=1}^{2} \frac{\lambda^{k-1}\ t^{\alpha k-j}}{\Gamma(\alpha k-j+1)} + I^{2\alpha} f(t) \right] + I^\alpha f(t) \\
&= \sum_{j=1}^{m} \frac{b_j\ t^{\alpha-j}}{\Gamma(\alpha-j+1)} + \lambda \left[\sum_{j=1}^{m} b_j \sum_{k=1}^{2} \frac{\lambda^{k-1}\ t^{\alpha k-j}}{\Gamma(\alpha k-j+1)} + I^{2\alpha} f(t) \right] + I^\alpha f(t) \\
&= \sum_{j=1}^{m} b_j \sum_{k=1}^{3} \frac{\lambda^{k-1}\ t^{\alpha k-j}}{\Gamma(\alpha k-j+1)} + \int_0^t \left(\sum_{k=1}^{2} \frac{\lambda^{k-1}}{\Gamma(\alpha k)} (t-s)^{\alpha k-1} \right) f(s)\, ds.
\end{aligned}$$

Also

$$y_3(t) = \sum_{j=1}^{m} b_j \sum_{k=1}^{4} \frac{\lambda^{k-1}\ t^{\alpha k-j}}{\Gamma(\alpha k-j+1)} + \int_0^t \left(\sum_{k=1}^{3} \frac{\lambda^{k-1}}{\Gamma(\alpha k)} (t-s)^{\alpha k-1} \right) f(s)\, ds.$$

Repeated application of this procedure gives the following relation for $y_n,\ n \in \mathbb{N}$:

$$y_n(t) = \sum_{j=1}^{m} b_j \sum_{k=1}^{n+1} \frac{\lambda^{k-1}\ t^{\alpha k-j}}{\Gamma(\alpha k-j+1)} + \int_0^t \left(\sum_{k=1}^{n} \frac{\lambda^{k-1}}{\Gamma(\alpha k)} (t-s)^{\alpha k-1} \right) f(s)\ \ ds.$$

Taking the limit as $n \to \infty$, we obtain the following solution to the integral equation (186):

$$\begin{aligned}
y(t) &= \sum_{j=1}^{m} b_j \sum_{k=1}^{\infty} \frac{\lambda^{k-1}\, t^{\alpha k - j}}{\Gamma(\alpha k - j + 1)} + \int_0^t \left(\sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{\Gamma(\alpha k)} (t-s)^{\alpha k - 1} \right) f(s)\, ds \\
&= \sum_{j=1}^{m} b_j \sum_{k=0}^{\infty} \frac{\lambda^{k}\, t^{\alpha k + \alpha - j}}{\Gamma(\alpha k + \alpha - j + 1)} + \int_0^t \left(\sum_{j=0}^{\infty} \frac{\lambda^{j}}{\Gamma(\alpha j + \alpha)} (t-s)^{\alpha j + \alpha - 1} \right) f(s)\, ds \\
&= \sum_{j=1}^{m} b_j \sum_{k=0}^{\infty} \frac{\lambda^{k} t^{\alpha j + \alpha - j}}{\Gamma(1 + \alpha k + \alpha - j)} + \int_0^t \left(\sum_{j=0}^{\infty} \frac{\lambda^{j} (t-s)^{\alpha j}}{\Gamma(\alpha j + \alpha)} (t-s)^{\alpha - 1} \right) f(s)\, ds \\
&= \sum_{j=1}^{m} b_j t^{\alpha - j}\ E_{\alpha, \alpha - j + 1}(\lambda t^{\alpha}) + \int_0^t (t-s)^{\alpha - 1} E_{\alpha,\alpha}(\lambda (t-s)^{\alpha}) f(s)\, ds. \qquad (188)
\end{aligned}$$

Where, $E_{\alpha,\beta}$ is the Mittag-Leffler function. This yields an explicit solution to

the integral equation (186), hence to the Cauchy problem (185). Now, we are in the position to state the following result

Theorem 7.1. *Let $\alpha > 0,\ \lambda \in [-1, 1]$ and $\rho \in [0, 1)$ such that $\rho \geq m - \alpha$. If $f \in C_\rho[0, b]$, then the cauchy type problem (185) has a unique solution given by (188).*

Proof. Define the operator $T : C_\rho[0, 1] \to C_\rho[0, 1]$ by

$$Ty(t) := \sum_{j=1}^{m} \frac{b_j\ t^{\alpha - j}}{\Gamma(\alpha - j + 1)} + \lambda I^{\alpha} y(t) + I^{\alpha} f(t) \qquad (189)$$

Since $C_{\rho-\alpha}[0, 1] \subset C_\rho[0, 1]$, we obtain in the view of Lemma 5.2 that the operator T is well-defined. We will apply the standard Banach fixed point theorem to show that T has a unique fixed point.

At first glance, Observe that $(C_\rho[0, 1], \|\cdot\|_{C_\rho})$ is a Banach space. The norm $\|\cdot\|_{C_\rho}$ is equivalent to the maximum norm, that is, $t^{\rho} \|y\| \leq \|y\|_{C_\rho} \leq \|y\|$. We now show that T is a contraction on $(C_\rho[0, 1], \|\cdot\|_{C_\rho})$. To see this let $y, z \in C_\rho[0, 1]$ and notice

$$Ty(t) - Tz(t) := \lambda I^{\alpha}[y(t) - z(t)]$$

Thus for $t \in [0,1]$

$$\begin{aligned}
t^{\rho}|Ty(t)-Tz(t)| &\leq \frac{|\lambda|t^{\rho}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}s^{-\rho}s^{\rho}|y(s)-z(s)|\,ds \\
&\leq \frac{|\lambda|t^{\rho}}{\Gamma(\alpha)}\left\|y-z\right\|_{\rho}\int_0^t (t-s)^{\alpha-1}s^{-\rho}\,ds \\
&= \frac{|\lambda|t^{\rho}\Gamma(1-\rho)}{\Gamma(1-\rho+\alpha)}\left\|y-z\right\|_{\rho} t^{\alpha-\rho}
\end{aligned}$$

and therefore

$$\|Ty-Tz\| \leq \frac{\Gamma(1-\rho)}{\Gamma(1-\rho+\alpha)}\left\|y-z\right\|_{\rho}.$$

Since $\frac{\Gamma(1-\rho)}{\Gamma(1-\rho+\alpha)} < 1$, the Banach contraction principle implies that there is a unique $y \in C_{\rho}[0,1]$. □

Example 7.10. The solution to the cauchy problem

$$\begin{cases} D^{0.5}y(t) - y(t) = f(t), \\ I^{0.5}y(0) = 1, \end{cases}$$

has the following form

$$y(t) = \frac{E_{0.5,0.5}(\sqrt{t})}{\sqrt{t}} + \int_0^t \frac{E_{0.5,0.5}(\sqrt{t-s})}{\sqrt{t-s}}f(s)\ \ ds$$

Next, we consider the Cauchy type problem for the following more general homogenous fractional differential equation than (185):

$$\begin{cases} D^{\alpha}y(t) - \lambda t^{\beta}y(t) = 0,\ \alpha \in (m-1,m),\ \beta > m,\ m \in \mathbb{N}\ \lambda \in \mathbb{R} \\ D^{\alpha-j}y(0) = b_j,\ b_j\ \in \mathbb{R},\ j = 1,2,\cdots,m. \end{cases} \tag{190}$$

In the view of Equation 64 and Lemma 3.24, it can be easily seen "formally" that the integral equation medelled off the problem (190) is given by

$$y(t) = \sum_{j=1}^{m} \frac{b_j\ t^{\alpha-j}}{\Gamma(\alpha-j+1)} + \lambda I^{\alpha}t^{\beta}y(t) \tag{191}$$

We again apply the successive approximations method to solve the integral equation (191). According to this method, we set

$$y_0(t) = \sum_{j=1}^{m} \frac{b_j\ t^{\alpha-j}}{\Gamma(\alpha-j+1)},$$

$$y_n(t) = y_0(t) + \lambda I^{\alpha}t^{\beta}y_{n-1}(t),\ n \in \mathbb{N}. \tag{192}$$

Using the same arguments as above, we find for y_1 that

$$y_1(t) = y_0(t) + \lambda I^{\alpha}t^{\beta}y_0(t) = y_0(t) + \lambda\sum_{j=1}^{m}\frac{b_j}{\Gamma(\alpha-j+1)}I^{\alpha}t^{\alpha+\beta-j},$$

By the condition $\beta > m$ and in the view of Lemma 3.2, all integrals $I^{\alpha} t^{\alpha+\beta-j}$, $(j = 1, 2, \cdots, m)$ exist. Thus

$$y_1(t) = y_0(t) + \lambda \sum_{j=1}^{m} \frac{b_j \; t^{2\alpha+\beta-j}}{\Gamma(\alpha - j + 1)\Gamma(2\alpha + \beta - j + 1)}$$

Similarly we find

$$y_2(t) = \sum_{j=1}^{m} \frac{b_j \; t^{\alpha-j}}{\Gamma(\alpha - j + 1)} \left[1 + \lambda c_1 t^{\alpha+\beta} + \lambda^2 c_2 t^{2\alpha+2\beta}\right].$$

where

$$c_1 = \frac{\Gamma(\alpha + \beta - j + 1)}{\Gamma(2\alpha + \beta - j + 1)}, \; c_2 = \frac{\Gamma(\alpha + \beta - j + 1)\Gamma(2\alpha + 2\beta - j + 1)}{\Gamma(3\alpha + 2\beta - j + 1)}$$

Continuing this process, we drive the following relation for $y_n, \; n \in \mathbb{N}$:

$$y_n(t) = \sum_{j=1}^{m} \frac{b_j \; t^{\alpha-j}}{\Gamma(\alpha - j + 1)} \left[1 + \sum_{k=1}^{n} \lambda c_k (\lambda t^{\alpha+\beta})^k\right],$$

where

$$c_k = \prod_{r=1}^{k} \frac{\Gamma[r(\alpha + \beta) - j + 1]}{\Gamma[r(\alpha + \beta) + \alpha - j + 1]}$$

Taking the limit as $n \to \infty$, we obtain the following solution to the integral equation (191):

$$\begin{aligned} y(t) &= \sum_{j=1}^{m} \frac{b_j \; t^{\alpha-j}}{\Gamma(\alpha - j + 1)} \left[1 + \sum_{k=1}^{\infty} c_k (\lambda t^{\alpha+\beta})^k\right] \\ &= \sum_{j=1}^{m} \frac{b_j \; t^{\alpha-j}}{\Gamma(\alpha - j + 1)} E_{\alpha,1+\beta/\alpha,1+(\beta-j)/\alpha}[\lambda t^{\alpha+\beta}]. \end{aligned} \tag{193}$$

Where, $E_{\alpha,\beta,\gamma}$ is the generalized Mittag-Leffler function.
This yields an explicit solution to the integral equation (191), hence to the Cauchy problem (190). Now we have the following

Theorem 7.2. *Let $\alpha > 0, \; \lambda \in \mathbb{R}$ and $\beta \geq 0$. Then the cauchy type problem (190) has a unique solution given by (193).*

The proof of this theorem beyond the scope of our notes, so we omit it.

Example 7.11. The solution to the cauchy problem

$$\begin{cases} D^{0.5} y(t) - t^{\beta} y(t) = 0, \\ I^{0.5} y(0) = 1, \end{cases}$$

has the following form

$$y(t) = \frac{1}{\sqrt{t\pi}} E_{0.5,1+2\beta,2\beta-1}(t^{\beta+0,5}).$$

Next, we consider the Cauchy type problem for linear differential equations with the Caputo fractional derivative:

$$\begin{cases} \dfrac{d^\alpha y(t)}{dt^\alpha} - \lambda t^\beta y(t) = f(t), \ \alpha \in (m-1, m), \ m \in \mathbb{N}, \ \lambda \in \mathbb{R} \\ y^{(j)}(0) = b_j, \ b_j \ \in \mathbb{R}, \ j = 0, 1, 2, \cdots, m-1. \end{cases} \tag{194}$$

In the view of Lemma 3.27, it can be easily seen "formally" that the integral equation which is medelled off the problem (194) is given by

$$y(t) = \sum_{j=1}^{m-1} \frac{b_j \ t^j}{j!} + \lambda I^\alpha y(t) + I^\alpha f(t) \tag{195}$$

To solve the integral equation (195), we apply the method of successive approximations by setting

$$y_0(t) = \sum_{j=1}^{m-1} \frac{b_j \ t^j}{j!},$$

and

$$y_n(t) = y_0(t) + \lambda I^\alpha y_{n-1}(t) + I^\alpha f(t), \ n \in \mathbb{N}. \tag{196}$$

Using the same arguments as above, we find for y_n that

$$y_n(t) = \sum_{j=1}^{m-1} b_j \sum_{k=0}^{n} \frac{\lambda^k \ t^{\alpha k + j}}{\Gamma(\alpha k + j + 1)} + \int_0^t \left[\sum_{k=1}^{n} \frac{\lambda^{k-1} (t-s)^{k\alpha - 1}}{\Gamma(\alpha k)} \right] f(s) \ ds.$$

Taking the limit as $n \to \infty$, we obtain the following solution to the integral equation (195):

$$y(t) = \sum_{j=1}^{m-1} b_j \ t^j \ E_{\alpha,j+1}[\lambda t^\alpha] + \int_0^t (t-s)^{\alpha-1} \ E_{\alpha,\alpha}[\lambda (t-s)^\alpha] \ f(s) \ ds. \tag{197}$$

Where, $E_{\alpha,\beta}$ is the Mittag-Leffler function.
This yields an explicit solution to the integral equation (195), hence to the Cauchy problem (194).

Theorem 7.3. *Let $\alpha \in (m-1, m)$, $m \in \mathbb{N}$, $\lambda \in \mathbb{R}$ and $\rho \in [0, 1)$ such that $\rho \leq \alpha$. If $f \in C_\gamma[0, b]$, then the cauchy type problem (194) has a unique solution given by (197).*

The proof of this theorem beyond the scope of our notes, so we omit it.

Example 7.12. The solution to the cauchy problem

$$\begin{cases} \dfrac{d^{0.5}y(t)}{dt^{0.5}} - y(t) = f(t) \\ y(0) = 1, \end{cases}$$

has the following form

$$y(t) = E_{0.5,1}(\sqrt{t}) + \int_0^t (t-s)^{-0.5} E_{0.5,0.5}(\sqrt{t-s}) f(s)\ ds.$$

Example 7.13. The solution to the cauchy problem

$$\begin{cases} \dfrac{d^{1.5}y(t)}{dt^{1.5}} - y(t) = f(t) \\ y(0) = a,\ y'(0) = b,\ a, b \in \mathbb{R}, \end{cases}$$

has the following form

$$y(t) = aE_{1.5,1}(\sqrt{t^3}) + btE_{1.5,2}(\sqrt{t^3}) + \int_0^t (t-s)^{0.5} E_{1.5,1.5}(\sqrt{(t-s)^3}) f(s)\ ds.$$

Finally, we consider the following differential equation

$$\begin{cases} \dfrac{d^{\alpha}y(t)}{dt^{\alpha}} - \lambda t^{\beta} y(t) = 0,\ \alpha \in (m-1, m),\ m \in \mathbb{N},\ \lambda \in \mathbb{R} \\ y^{(j)}(0) = b_j,\ b_j\ \in \mathbb{R},\ j = 0, 1, 2, \cdots, m-1,\ b > -\alpha. \end{cases} \tag{198}$$

In the view of Lemma 3.27, it can be easily seen "formally" that the integral equation which is medelled off the problem (198) is given by

$$y(t) = \sum_{j=1}^{m-1} \frac{b_j\ t^j}{j!} + \lambda I^{\alpha} t^{\beta} y(t) \tag{199}$$

To solve the integral equation (199), we apply the method of successive approximations by setting

$$y_0(t) = \sum_{j=1}^{m-1} \frac{b_j\ t^j}{j!},$$

Using the same arguments as above, we find for y_n that

$$y_n(t) = \sum_{j=1}^{m-1} b_j t^j \left[1 + \sum_{k=1}^{n}\ d_k (\lambda t^{\alpha+\beta})^k \right].$$

where

$$d_k = \prod_{r=1}^{k} \frac{\Gamma(r\alpha + r\beta + \beta + j - \alpha + 1)}{\Gamma(r\alpha + r\beta\alpha + 1)},\ (n \in \mathbb{N})$$

Taking the limit as $n \to \infty$, we obtain the following solution to the integral equation (199):

$$y(t) = \sum_{j=1}^{m-1} b_j \; t^j \; E_{\alpha,1+\beta/\alpha,(\beta+j)/\alpha}[\lambda t^{\alpha+\beta}]. \tag{200}$$

Where, $E_{\alpha,\beta,\gamma}$ is the generalized Mittag-Leffler function.
This yields an explicit solution to the integral equation (199), hence to the Cauchy problem (198).

Theorem 7.4. *Let* $\alpha \in (m-1, m),\ m \in \mathbb{N},\ \lambda \in \mathbb{R}$ *and* $\rho \in [0, 1)$ *such that* $\rho \leq \alpha$. *If* $\beta \geq 0$ *and* $f \in C_\rho[0, b]$, *then the cauchy type problem (198) has a unique solution given by (200).*

The proof of this theorem beyond the scope of our notes, so we omit it.

Example 7.14. The solution to the cauchy problem

$$\begin{cases} \dfrac{d^{0.5}y(t)}{dt^{0.5}} - t^{-1/4}y(t) = 0 \\ y(0) = 1, \end{cases}$$

has the following form

$$y(t) = E_{0.5,0.5,-0.5}(\sqrt[4]{t}).$$

8. Calculus of Vector-Valued Functions

8.1. Differentiation of Vector-valued functions.

As every body know, the Lebesgue integral has been generalized for vector-valued functions in two distinct ways. The first such generalized was developed by Bochner and the other type of generalization was developed by Pettis. In (1938) Pettis begins with a given topology on the Banach spaces and defines the integral relative to this topology.

In the following pages, based on the linear functional over a Banach spaces and on the definition of fractional integrals of real-valued functions, we define the fractional Pettis-integrals of vector-valued functions and the corresponding fractional derivatives. Also, we show that the well-known properties of fractional calculus over the domains of the Lebesgue integrable also hold in the Pettis space. To encompass the full scope of this section, we apply this abstract results to investigate the existence of pseudo-solutions to some initial and boundary value problems of fractional order.

For the sake of the readers convenience, here we present some notations and the main properties for Pettis and Bochner integrable and corresponding derivatives.

For further background and details pertaining to this section we refer the reader to (J. Diestel and J.J. Uhl Jr. [61] and J. Pettis [130]. All unexplained terminology can be found in the standard reference due to Hille and Phillips [99].

At the beginning, we summarize some results about the integration and differentiation of Banach space- valued functions of a single variable. In a rough sense, vector-valued integrals of integrable functions have similar properties, often with similar proofs, to scalar-valued Lebesgue integrals. Nevertheless, the existence of different topologies (such as the weak and strong topologies) in the range space of integrals that take values in an infinite-dimensional Banach space introduces significant new issues that do not arise in the scalar-valued case. Unless otherwise stated, E considered to be a Banach space with norm $\|\cdot\|$ and dual E^*. Also, E_w denotes the space E when endowed with its weak topology.

Before embarking on our study a firm foundation must be laid. Let us begin with the following

Definition 8.1. Given a field of sets $(\Omega, \mathcal{F})$ and a Banach space E a "finitely additive vector measure is a function $\mu : \mathcal{F} \to E$ such that for any two disjoint sets A and B in $\mathcal{F}$ one has

$$\mu(A \cup B) = \mu(A) + \mu(B).$$

A vector measure μ is called *countably additive* if for any sequence $(J_i)_{i=1}^{\infty}$ of disjoint sets in $\mathcal{F}$ such that their union is in $\mathcal{F}$ it holds that

$$\mu\left(\bigcup_{i=1}^{\infty} J_i\right) = \sum_{i=1}^{\infty} \mu(J_i)$$

with the series on the right-hand side convergent in the norm of the Banach space E. That is

$$\lim_{n\to\infty} \left\| \mu\left(\bigcup_{i=1}^{n} J_i\right) - \sum_{i=1}^{n} \mu(J_i) \right\| = 0.$$

It can be proved that an additive vector measure μ is countably additive if and only if for any sequence $(J_i)_{i=1}^{\infty}$ as above one has

$$\lim_{n\to\infty} \left\| \mu\left(\bigcup_{i=n}^{\infty} J_i\right) \right\| = 0, \qquad (*)$$

where $\|\cdot\|$ is the norm on E.

Countably additive vector measures defined on a sigma-algebras are more general than finite measures, and complex measures, which are countably additive functions taking values respectively on the real interval $[0, \infty)$, the set of real numbers, and the set of complex numbers.

Example 8.1. Let $\mathcal{F}$ be the family of all Lebesgue measurable sets contained in $[0, 1]$. For any such set A, define $\mu(A) := \chi_A$:

[1] μ viewed as a function from $\mathcal{F}$ to $L_\infty[0, 1]$, is a vector measure which is countably-additive. This statements follow quite easily from the criterion (*) stated above since

$$\left\| \mu\left(\bigcup_{i=n}^{\infty} A_i\right) \right\|_\infty = \operatorname*{ess\,sup}_{t\in[0,1]} \left| \chi_{\left(\bigcup_{i=n}^{\infty} A_i\right)}(t) \right| \to 0 \text{ as } n \to \infty$$

[2] μ, viewed as a function from $\mathcal{F}$ to $L_1[0, 1]$, is a countably-additive vector measure. This statements follow quite easily from the criterion (*) stated above since

$$\left\| \mu\left(\bigcup_{i=n}^{\infty} A_i\right) \right\|_{L_1} = \int_0^1 \left| \chi_{\left(\bigcup_{i=n}^{\infty} A_i\right)}(t) \right| dt = \int_{\bigcup_{i=n}^{\infty} A_i} dt \to 0 \text{ as } n \to \infty$$

Definition 8.2. A vector measure is a function $\Im : \mathcal{F} \to E$ is called

1. Absolutely continuous with respect to a finite measure $\mu : \mathcal{F} \to E$ if for each $\epsilon > 0$ there exists $\delta > 0$ such that $\|\Im(J)\| < \epsilon$ whenever $\mu(J) < \delta$, $J \in \mathcal{F}$,
2. Strongly absolutely continuous with respect to a finite measure μ if for each $\epsilon > 0$ there exists $\delta > 0$ such that $\sum_n \|\Im(J_n)\| < \epsilon$ for each sequence of disjoint sets $\{J_n\} \subset \mathcal{F}$ such that $\sum_n \mu(J_n) < \delta$.

Definition 8.3.
1. A sequence $\{x_n\}$ of elements from the normed space E is called a weak Cauchy sequence if $\{\varphi x_n\}$ is a Cauchy sequence for all $\varphi \in E^*$. That is, for every $\varphi \in E^*$, we have $|\varphi x_n - \varphi x_m| \to 0$ as $n, m \to 0$.
2. The normed space E is called weakly sequentially complete (shortly, weakly complete) if every weak Cauchy sequence converges in E.

Remark: According to the definition of the weakly complete spaces, the converges include all strongly convergent Cauchy sequences, weak convergent Cauchy sequences and even the weakly but not strongly convergent Cauchy sequences. Thus, every sequentially complete space is complete (Banach space in case of normed spaces). Furthermore, it can be shown that every reflexive Banach space is weakly complete (a nice exercise with the uniform boundedness principle). For example, L_1 is not reflexive, but it turns out to be weakly sequentially complete. Also the reflexive Banach space is separable if, and only if its dual is separable. Indeed, we have the implication

$$\text{Hilbert} \implies \text{reflexive} \implies \text{weakly complete} \implies \text{complete}.$$

Lemma 8.1. [184] *A subset of a reflexive Banach space is weakly compact if and only if it is closed in the weak topology and bounded in the norm topology.*

Lemma 8.2. [67] *A convex subset of a normed space E is closed if and only if it is weakly closed.*

Remark 8.1. An immediate consequence of Lemma 8.1 and Lemma 8.2 is the following observations:

1. When E is reflexive, it follows that all closed and bounded convex subsets of E are weakly compact. Therefore, the closed unit ball of a Banach space E is weakly compact if and only if E is reflexive.

   In a Hilbert space H, the weak compactness of the unit ball is very often used in the following way: Every bounded sequence in H has weakly convergent subsequences.
2. Any weakly convergent sequence in a Banach space has a sequence of convex combinations of its members that converges strongly to the same limit: Indeed, if $Q := \{x_n\}$ is weakly convergent sequence in a Banach

space E then, in the view of Lemma 8.2, it can be shown that the sequence $\{\zeta_n x_n\} \in \overline{conv(Q)}$, where $\sum_n \zeta_n = 1$ is strongly convergent. This is the well-known *Mazur's lemma* (Lemma 2.6))

Theorem 8.1. (Eberlein Šmulian) *Suppose that K is weakly closed subset of the Banach space E. Then the following are equivalent*

(1) *K is weakly compact,*
(2) *K is weakly sequentially compact, i.e., any sequence in K has a subsequence which converges weakly.*

Theorem 8.2. (Arzela-Ascoli) [49] *and* [121] *Let M be a weakly equicontinuous family of functions from $[a, b]$ into E, and let $\{x_n\}$ be a sequence in M such that for each $t \in [a, b]$, the set $\{x_n(t), n \geq 1\}$ is weakly relatively compact. then there exists a subsequence $\{x_{n_k}(t)\}$ which converges weakly uniformly on $[a, b]$ to a weakly continuous function.*

We also recall the following

Definition 8.4. 1. A subset F of a normed space E is said to be *dense* if every $x \in E$ is the limit of some sequence in F, that is, for every $x \in E$ there exists $x_1, x_2, \cdots \in F$ such that $\|x_n - x\| \to 0$ as $n \to \infty$, consequently $E \subset \overline{F}$. Also, E is called *separable* if it has a countable dense subset. It is well known that the rational numbers Q are countable and are dense in $\mathbb{R}$, so the real numbers are separable. A well-known examples of separable spaces are $C[a, b]$, $\ell_p, L_p[a, b]$, $p \in [1, \infty)$ and C_0. However, the spaces ℓ_∞ and $L_\infty[a, b])$ are non-separable.

2. The function $x : I \to E$ is said to be weakly continuous at $t_0 \in I$ if for every $\varphi \in E^*$, we have φx continuous at t_0,
3. Let $C[I, E]$ denotes the Banach space of strong continuous functions $x : I \to E$ endowed by the norm $\|x\|_0 = \sup_{t \in I} \|x(t)\|$. If Q is an equicontinuous subset of $C[I, E]$, it is shown that ([125], Lemma 1.9) a sequence $\{x_n\} \subset Q$ converges weakly to x in $C[I, E]$ (that is $\psi x_n \to \psi x$, for all $\psi \in C^*[I, E]$) if and only if $\{x_n(t)\}$ tends weakly to $x(t)$ for each $t \in I$ (that is $\varphi x_n(t) \to \varphi x(t)$, for all $\varphi \in E^*$).
4. Let $C[I, E_w]$ denote the Banach space of all weakly continuous functions $x : I \to E_w$ endowed with the topology of weak uniform convergence.
5. The function $x : I \to E$ is said to be weakly measurable if for every $\varphi \in E^*$, we have φx measurable,
6. Recall that a function $x : I \to E$ is `essentially separably valued` (or almost separably valued) if, off null set, it has separable rang. That is, there is a $J \subset I$ of measure zero such that $x(I/J)$ is separable, meaning that $x(I/J)$ contains a countable dense subset E,
7. The function $x : I \to E$ is said to be strongly measurable if it is a.e.the limit (in the norm topology in E) of a sequence of simple functions

(countable-valued functions). That is, x is strongly measurable if there exists a sequence of simple functions $\{\phi_n\}$ such that $\|x - \phi_n\| \to 0$ almost everywhere as $n \to \infty$.

Remark: An immediate consequence of strong measurability is that the norm function is measurable. That is, if $x : I \to E$ is strongly measurable, then $\|x\|$ is measurable. This follows directly from $|\,\|x\| - \|\phi_n\|\,| \leq \|x - \phi_n\| \to 0$ almost everywhere, so $\|\phi_n\| \to \|x\|$ almost everywhere. Since the pointwise limit of measurable functions is measurable, $\|x\|$ is measurable. It is evident that a strongly measurable functions are also weakly measurable and the converse is not (in general) true. The relationship between weak and strong measurability is given by Pettis theorem (1938):

Theorem 8.3. [99],[130] *A function $x : I \to E$ is strongly measurable if, and only if, x is weakly measurable and almost separably valued. In particular, if E is separable, the strong and weak measurability are equivalent. (Recall that a subset of separable spaces is separable)*

This fact, therefore reduces the verification of strong measurability of x of separably valued to the verification of measurability of the real-valued functions φx.

Example 8.2. Let $Q := \{t_1, t_2, \cdots\}$ denotes the "countable" set of rational numbers of $[0, 1]$. Let $e_k := \{0, 0, \cdots, 0, 1, 0 \cdots\}$ and $e_0 := \{0, 0, 0 \cdots\}$ Define the function $x : [0, 1] \to C_0$ by

$$x(t) := \begin{cases} e_k, & t = t_k \in Q, \\ e_0, & t \notin Q. \end{cases}$$

This function is well-defined and it is strongly measurable on $[0; 1]$, since it is a countable-valued function. Therefore, $\|x\|$ is measurable. Evidently, we have

$$\|x(\cdot)\| : t \to \|x(t)\| = \begin{cases} 1, & t \text{ rational}, \\ 0, & t \text{ irrrational}. \end{cases}$$

Consequently $\|x(\cdot)\| \in L_1[0, 1]$ and

$$\int_0^1 \|x(t)\|\, dt = 0. \tag{201}$$

Since C_0 is separable, we except (in the view of Theorem 8.3), that x is weakly measurable too. Indeed, if we let $\sum_{n=1}^{\infty} \lambda_n$ denotes the absolutely convergent series corresponding to $\varphi \in C_0^*$, then

$$\varphi x(t) = \begin{cases} \lambda_k, & t = t_k \in Q, \\ 0, & t \text{ irrational}. \end{cases}$$

Then $\varphi x(\cdot)$ is measurable for every $\varphi \in C_0^*$. Moreover

$$\int_0^1 |\varphi x(t)| dt = 0.$$

So $\varphi x(\cdot) \in L_1[0,1]$ for every $\varphi \in C_0^*$.

Now, we well examine the measurability of the mapping x of Example 8.2 is case of replacing the space C_0 by $\ell_p,\ p \in [1,\infty)$:

Example 8.3. Redefine the mapping x of Example 8.2 as $x : [0,1] \to \ell_p,\ p \in [1,\infty)$ by

$$x(t) := \begin{cases} e_k, & t = t_k \in Q, \\ e_0, & t \notin Q. \end{cases}$$

In this case, it is also easy to see that x is strongly and weakly measurable on $[0,1]$. Moreover

$$\int_0^1 \|x(t)\|\, dt = \int_0^1 |\varphi x(t)| dt = 0 \text{ for any } \varphi \in \ell_p^*. \tag{202}$$

Another example of strongly measurable function (*cf.* [127]), we have

Example 8.4. Define the function $x : [0,1] \to C_0$ by

$$x(t) := \left\{ n\chi_{(0,\frac{1}{n}]}(t) \right\} = (\chi_{(0,1]}(t)), 2\chi_{(0,\frac{1}{2}]}(t), \cdots). \tag{203}$$

Since $\chi_{(0,\frac{1}{n}]}(0) = 0$ for any n because $t = 0 \notin (0, \frac{1}{n}]$, then $x(0) = (0,0,0,\cdots)$. This function is well-defined for any $t \in [0,1]$, since for $t \in [0,1]$, we have $x = \{x_n\}$, where

$$x_n(t) := n\chi_{(0,\frac{1}{n}]}(t) = \begin{cases} n, & t \in I_n := (0, \frac{1}{n}], \\ 0, & t \in [0,1] - I_n \end{cases}$$

Clearly $\{x_n\}$ converges to 0 pointwise. The function x is also strongly (hence weakly) measurable on $[0,1]$, since it is a countable-valued function. Evidently, on each interval $(1/(n+1), 1/n]$, the function x has the same value, e.g. for every $t \in (1/4, 1/3]$, $x(t) = (1,2,3,0,0\cdots)$ and $x(t) = (1,0,0,\cdots)$, $\forall\, t \in (1/2, 1]$. We remark also that the function $y : [0,1] \to C_0$ defined by

$$y(t) := \left\{ \frac{1}{n}\chi_{[0,\frac{1}{n}]}(t) \right\} \tag{204}$$

is strongly measurable too.

Remark 8.2. For future purpose we gather together some properties of the functions x and y defined in Example 8.4

1. The strongly measurable functions x and y are simple functions. By the remark above Theorem 8.3, $\|x(\cdot)\|$ and $\|y(\cdot)\|$ are measurable on $[0,1]$

2. While y is norm bounded ($\|y(t)\| = \sup_n |y_n(t)| = 1,\ t \in [0,1])$), x is not norm bounded ($\|x(t)\| = \sup_n |x_n(t)| \to \infty,\ t \in (0,1])$),
3. $\|y(\cdot)\| \in L_1[0,1]$ and $\|x(\cdot)\| \notin L_1[0,1]$. Evidently
$$\int_0^1 \|x(t)\|\, dt \to \infty, \ \int_0^1 \|y(t)\|\, dt = \int_0^1 \sup_n |\frac{1}{n}\chi_{[0,\frac{1}{n}]}(t)| dt < \infty$$
4. $\varphi y(\cdot) \in L_1[0,1]$ and $\varphi x(\cdot) \in L_1[0,1]$ for every $\varphi \in C_0^*$. To see this, let $\{\lambda_n\} \in \ell_1$ be the corresponds to $\varphi \in C_0^*$, then $\varphi x = \sum_n \lambda_n x_n$. Since
$$\sum_n \int_0^1 |\lambda_n x_n(t)| dt = \sum_n |\lambda_n| \int_0^1 x_n(t) dt = \sum_n |\lambda_n| \int_0^{\frac{1}{n}} n dt = \sum_n |\lambda_n| < \infty,$$
by Beppo Levi Theorem (Theorem2.3), the sum
$$\sum \lambda_n x_n,$$
is integrable. Thus $\varphi x(\cdot) \in L_1[0,1]$, and similarly we can show that $\varphi y(\cdot) \in L_1[0,1]$.

It is well known that every Hilbert space has an orthonormal basis. In particular, if the Hilbert space is separable, then the orthonormal basis are countable.

Example 8.5. Suppose that H is a non-separable Hilbert space (those spaces whose orthonormal basis are uncountable) whose dimension is equal to the cardinality of $\mathbb{R}$. Let $\{e_t,\ t \in (0,1)\}$ be an orthonormal basis of H and define a function $x : (0,1) \to H$ by $x(t) := e_t$. Then x is weakly but not strongly measurable: To see this, we define
$$S_n := \{t \in (0,1) : |\varphi x(t)| = |\varphi e_t| \geq \frac{1}{n}\}.$$
We will prove that the set
$$S := \{t \in (0,1) : \varphi e_t \neq 0\} = \bigcup_n S_n$$
is countable. Let $t_1, t_2, \cdots t_k \in S_n$, and define $e \in H$ by
$$e := \sum_{j=1}^k c_j e_{t_j}, \text{ where } c_j = \frac{\overline{\varphi e_{t_j}}}{|\varphi e_{t_j}|\sqrt{k}}.$$
Compute
$$< e, e >= \sum_{j=1}^k c_j^2 < e_{t_j}, e_{t_j} >= \sum_{j=1}^k \frac{\left\|e_{t_j}\right\|^2}{k} = \sum_{j=1}^k \frac{1}{k} = 1.$$

Also, for any $\varphi \in H^*$ we have

$$\begin{aligned} \varphi e &= \sum_{j=1}^{k} c_j \varphi e_{t_j} = \sum_{j=1}^{k} \frac{\overline{\varphi e_{t_j}}}{|\varphi e_{t_j}|\sqrt{k}} \varphi e_{t_j} = \sum_{j=1}^{k} \frac{|\varphi e_{t_j}|^2}{|\varphi e_{t_j}|\sqrt{k}} \\ &= \sum_{j=1}^{k} \frac{|\varphi e_{t_j}|}{\sqrt{k}} \geq \sum_{j=1}^{k} \frac{1}{n\sqrt{k}} = \frac{\sqrt{k}}{n}. \end{aligned}$$

Now, we have

$$\|\varphi\| = \sup_{x\in H, x\neq 0} \frac{|\varphi(x)|}{\|x\|} \geq \frac{|\varphi(e)|}{\|e\|} \geq \frac{\sqrt{k}}{n},$$

and so we conclude $\|\varphi\| \geq \frac{\sqrt{k}}{n}$ or $k \leq n^2 \|\varphi\|^2$. Thus S_n is finite and so S is a countable union of finite sets, hence countable. Consequently $\varphi x(t) = 0$ a.e. (the null set is S). That is, φx is countable-valued function and so it is measurable, meaning that x is weakly measurable. On the other hand, for any $t, s \in (0,1)$ with $t \neq s$, we have

$$\|x(t) - x(s)\|_H = \|e_t - e_s\| = \sqrt{< e_t - e_s, e_t - e_s >} = \sqrt{2}.$$

Now, let $J \subset (0,1)$ be any null set. Then the set $(0,1)/J$ is uncountable, for otherwise it will be null set. So the set of balls $\left\{ B_{\frac{\sqrt{2}}{2}}(x(t)) : \ t \in (0,1)/J \right\} \subset H$ are uncountable and mutually disjoint. So if M is dense subset of H, then each of this balls must contains at least one element of M. Hence M can not be countable. Then $x[(0,1)/J]$ is non-separable for every null set $J \subset (0,1)$. Alternatively, neither H is separable nor x is almost separably valued, so x is not strongly measurable.

On the other hand, if we redefine $x : (0,1) \to H$ by $x(t) := \begin{cases} 0 \text{ if } t \notin J, \\ e_t \text{ if } t \in J. \end{cases}$ where J any countable subset of $(0,1)$. Then x is countable-valued function hence it is strongly and weakly measurable.

Furthermore, Let $J \subset (0,1)$ be the standard middle thirds Cantor set (that is, J is uncountable null set). Define the function $y : (0,1) \to H$ by

$$y(t) := \begin{cases} 0 \text{ if } t \notin J, \\ e_t \text{ if } t \in J. \end{cases}$$

It is clear that y is not a countable-valued function. We are able to proceed as before to show that y is weakly measurable. However, y is even strongly measurable: Obviously y is almost separably valued (since $y[(0,1)/J] = 0$), so it is strongly measurable.

Example 8.6. 1. Define $x : (0,1) \to L_\infty(0,1)$ by $x(t) := \chi_{(0,t)}$. This function is weakly measurable ($\varphi(x)$ is a function of bounded variation)(see [86]). Also, x is not almost separably valued. Indeed, the estimation

$$\|x(t) - x(s)\|_{L_\infty} = \sup_{\xi} |\chi_{(0,t)}(\xi) - \chi_{(0,s)}(\xi)| = 1 \text{ for } t \neq s,$$

showing that the range of x has no countable subset which dense in this range. Thus, neither $L_\infty(0,1)$ is separable nor x is almost separably valued, so x is not strongly measurable.

2. Define $y : (0,1) \to L_2(0,1)$ by $y(t) := \chi_{(0,t)}$, then y is strongly measurable. To see this, note that for every $\varphi \in L_2(0,1)$, which is the dual of $\in L_2(0,1)$, we have

$$\varphi y(t) = \int_0^1 \varphi(s)\chi_{(0,t)}(s)\, ds = \int_0^t \varphi(s)\, ds.$$

Thus, $\varphi y : (0,1) \to \mathbb{R}$ is absolutely continuous and therefore y is weakly measurable. Thus, since $L_2(0,1)$ is separable, then y is strongly measurable.

3. Define $z : (0,1) \to L_1(0,1)$ by $z(t) := \chi_{(0,t)}$, then z is strongly measurable. To see this, note that for every $\varphi \in L_\infty(0,1)$, which is the dual of $\in L_1(0,1)$, we have

$$\varphi z(t) = \int_0^1 \varphi(s)\chi_{(0,t)}(s)\, ds = \int_0^t \varphi(s)\, ds.$$

Thus, $\varphi z : (0,1) \to \mathbb{R}$ is absolutely continuous and therefore z is weakly measurable. Thus, since $L_1(0,1)$ is separable, then z is strongly measurable.

Now, we will show that the weak and strong separability are equivalent

Proposition 8.1. A subset M of a metric space is weakly separable (in the sense that there is a countable subset Q such that M is contained in the weak closure of Q) if, and only if, it is norm (strong) separable

Proof. The idea of the proof can be derived directly in the view of *Mazur's lemma* (Lemma 2.6) which shows that any weakly convergent sequence in a Banach space has a sequence of convex combinations of its members that converges strongly to the same limit. Indeed, we suppose that M is weakly separable. Then by *Mazur's lemma*, it follows that every element in $H := \overline{conv(Q)}$ is a strong limit of some sequence in Q. Since Q is countable, then the closed convex hull H of Q is separable in the norm topology. Because H is convex and closed, then it is weakly closed (*cf.* Lemma 8.2), it follows that H must contain M. Since subset of separable sets in metric spaces are separable, it follows that M is also norm separable. $\square$

It shown in ([99] page 73) that the weak continuity implies the strong measurability, see also Remark 2.8 in [43]. In fact, since the image of separable space under continuous function is separable, we have

Lemma 8.3. [99] The a.e. weakly continuous function $x : [0,1] \to E$ is strongly measurable. In particular, every $x \in C[I, E_\omega]$ is strongly measurable.

Proof. At the begging, we note that the result is automatically satisfied if E is separable since the weakly continuous function is weakly measurable.
Now, let Q be the null set where x is not weakly continuous. Let $J := [0,1]/Q$. For any $\varphi \in E^*$ the real function φx is continuous at each point in J. That is, the restriction $\varphi x|_J : J \to \mathbb{R}$ is continuous. Since J is separable (since a subsets of a separable metric spaces are separable) (In fact, the countable dense subset is the set of rationales in J), then the continuous image $\varphi x(J) \subset E$ separable. Thus, $x(J) = x([0,1]/Q)$ is weakly separable and so it is norm separable. Consequently $x([0,1])$ is essentially separably valued. Noting that the weak continuity yields the weak measurability, we directly obtain the result. $\square$

Example 8.7. Consider the strongly measurable functions y and z defined in Example 8.6. Here will show that the functions y and z are weakly continuous (hence strongly measurable by Lemma 8.3).

1. Consider the function $y : (0,1) \to L_2(0,1)$ given by $y(t) := \chi_{(0,t)}$. Clearly, we have for any $a, b \in (0,1)$, $a < b$

$$\begin{aligned} y(b)(s) - y(a)(s) &= \begin{cases} 1, & s \in (0,a], \\ 1, & s \in [a,b], \\ 0 & \text{otherwise.} \end{cases} - \begin{cases} 1, & s \in (0,a], \\ 0, & s \in (a,b], \\ 0 & \text{otherwise.} \end{cases} \\ &= \begin{cases} 0, & s \in (0,a], \\ 1, & s \in (a,b], \\ 0 & \text{otherwise.} \end{cases} = \chi_{[a,b]}(s). \end{aligned}$$

Thus, for any $t_1, t_2 \in (0,1)$ with $t_1 > t_2$ any $\varphi \in L_2^*$, there exists $\psi \in L_2$ such that

$$|\varphi(y(t_1)) - \varphi(y(t_2))| = |\varphi(y(t_1) - y(t_2))| = \int_0^1 \psi(\zeta)\chi_{[t_2,t_1]}(\zeta)d\zeta \le \|\psi\|_2 \sqrt{t_1 - t_2}.$$

Therefore, for any $\varphi \in L_2^*(0,1)$ the function $\varphi y : (0,1) \to \mathbb{R}$ is continuous and therefore y is strongly measurable (cf. Lemma 8.3). The function y is even strongly continuous. This is a direct consequence of the estimate

$$\|y(t_1) - y(t_2)\| = \left\|\chi_{[t_2,t_1)}\right\| = \left(\int_0^1 |\chi_{(t_2,t_1)}(s)|^2 \, ds\right)^{1/2} = \sqrt{t_1 - t_2}$$

2. Consider the function $z : (0,1) \to L_1(0,1)$ by $z(t) := \chi_{[0,t]}$. For any $t_1, t_2 \in (0,1)$ with $t_1 > t_2$ and any $\varphi \in L_1^*$, there exists $\psi \in L_\infty$ such that

$$|\varphi(z(t_1)) - \varphi(z(t_2))| = |\varphi(z(t_1) - z(t_2))| = \int_0^1 \psi(\zeta)\chi_{[t_2,t_1]}(\zeta)d\zeta \leq \|\psi\|_\infty |t_1 - t_2|.$$

Thus z is weakly continuous and consequently, by Lemma 8.3, strongly measurable. The function z is even strongly continuous. This is a direct consequence of the estimate

$$\|x(t_1) - x(t_2)\| = \int_0^1 \chi_{[s,t]}(\zeta)d\zeta = |t_1 - t_2|.$$

Definition 8.5. 1. A function $x : I \to E$ is said to be absolutely continuous on I (AC, for short) if for every $\epsilon > 0$ there exists a $\delta > 0$ such that $\left\|\sum_{k=1}^{m} x(b_k) - x(a_k)\right\| < \epsilon$ for every finite disjoint family $\{(a_k, b_k); 1 \leq k \leq m\}$ of subintervals of I such that $\sum_{k=1}^{m} (b_k - a_k) < \delta$. If the last inequality is replaced by $\sum_{k=1}^{m} \|x(b_k) - x(a_k)\| < \epsilon$, then we say that x is a strongly absolutely continuous sAC function.

2. A function $x : I \to E$ is said to be weakly absolutely continuous wAC on I if for every $\varphi \in E^*$ the real valued function $\varphi x(\cdot)$ is AC on I.

Each sAC function is anAC function, and each AC function is a wAC function. If E is a weakly sequentially complete space, then every wAC function is an AC function:

Lemma 8.4. [110] *A weakly absolutely continuous function on a compact real interval having values in a weakly sequentially complete space is absolutely continuous. That is, in a weakly sequentially complete spaces, the absolute and weakly absolute continuity are equivalent.*

In what follows, we show that the weak sequential completeness assumption imposed on the space is really essential and can not be omitted: **There exist weakly absolutely continuous functions, defined on compact real intervals and taking values in a Banach space, which are not absolutely continuous.** To show this, we construct the following example: In this example, we show that the hypothesis of weak sequential completeness in Lemma 8.4 cannot be omitted or extended to a separable spaces.

Example 8.8. Recall that the Banach space C_0 is a subspace of ℓ_∞ and it is separable, but not weak sequentially complete, with dual ℓ_1 (for example, the sequence $\{x_n\} \in C_0$ where $x_n := (1, 1/2, \cdots, 1/n, 0, 0, \cdots)$ is cauchy sequence

but it is not weakly convergent to $x := (1, 1/2, \cdots, 1/n, \cdots) \in C_0$).
Define $y : [0,1] \to C_0$ by

$$y(t) = \begin{cases} \circ = (0,0,0,\cdots), & \text{if } t = 0, \\ \{y_n(t)\}_{n=1}^{\infty} := \left\{\int_0^t x_n(s)\,ds\right\}_{n=1}^{\infty}, & \text{if } 0 < t \le 1. \end{cases} \tag{205}$$

where

$$x_n(t) = \begin{cases} n, & \text{if } 0 < t \le \frac{1}{2n}, \\ -n, & \text{if } \frac{1}{2n} \le t \le \frac{1}{n}, \\ 0, & \text{if } \frac{1}{n} \le t \le 1. \end{cases} = n(\chi_{[0,1/2n]}(t) - \chi_{[1/2n,1/n]}(t)) \tag{206}$$

Fix $t \in (0,1]$, then we have

$$\begin{aligned} y_n(t) &= \int_0^t x_n(s)\,ds = \int_0^t n(\chi_{[0,1/2n]}(s) - \chi_{[1/2n,1/n]}(s))\,ds \\ &= \begin{cases} nt, & \text{if } 0 < t \le \frac{1}{2n}, \\ \int_0^{1/2n} n\,ds - \int_{1/2n}^{t} n\,ds, & \text{if } \frac{1}{2n} \le t \le \frac{1}{n}, \\ \int_0^{1/2n} n\,ds - \int_{1/2n}^{1/n} n\,ds, & \text{if } \frac{1}{n} \le t \le 1. \end{cases} \\ &= \begin{cases} nt, & \text{if } 0 < t \le \frac{1}{2n}, \\ 1 - nt, & \text{if } \frac{1}{2n} \le t \le \frac{1}{n}, \\ 0, & \text{if } \frac{1}{n} \le t \le 1. \end{cases} \\ &= nt\chi_{[0,1/2n]}(t) + (1-nt)\chi_{[1/2n,1/n]}(t) \to 0 \text{ as } n \to \infty, t > 0. \end{aligned} \tag{207}$$

Therefore, for any $k = 1, 2, 3, \cdots$, we have

$$\left|y_n(\frac{1}{2k})\right| = \begin{cases} \frac{n}{2k}, & \text{if } \frac{n}{2k} \le \frac{1}{2}, \\ \left|1 - \frac{n}{2k}\right|, & \text{if } \frac{n}{2k} \in [\frac{1}{2}, 1], \\ 0, & \text{if } \frac{n}{2k} \ge 1. \end{cases}$$

Then

$$\left\|y(\frac{1}{2k}) - y(0)\right\| = \left\|y(\frac{1}{2k})\right\| = \sup_{n \ge 1}\left|y_n(\frac{1}{2k})\right| = \frac{1}{2}$$

So y is not continuous at zero. Certainly y is not absolutely continuous on $[0,1]$.

To show that y is weakly absolutely continuous on $[0,1]$, let $\varphi \in C_0^*$. Then there corresponds to φ an absolutely convergent series $\sum_{n=1}^{\infty} \lambda_n$ such that for all $y = \{y_n\}$ belonging to C_0,

$$\varphi y(t) = \sum_n \lambda_n y_n(t) = \sum_n \int_0^t \lambda_n x_n(s)\,ds.$$

Since

$$\sum_{n=1}^{\infty}\int_0^1 |\lambda_n x_n(s)|\,ds = \sum_{n=1}^{\infty}|\lambda_n|\left(\int_0^{1/2n} n\,ds + \int_{1/2n}^{1/n} |-n|\,ds\right) = \sum_{n=1}^{\infty}|\lambda_n| < \infty,$$

it follows by Beppo-Levi Theorem

$$\varphi y(t) = \sum_{n=1}^{\infty}\int_0^t \lambda_n x_n(s)\,ds = \int_0^t \left[\sum_{n=1}^{\infty}\lambda_n x_n(s)\right] ds, \tag{208}$$

and so the real valued function $t \to \varphi y(t)$ is absolutely continuous for every $\varphi \in C_0^*$. That is y is weakly absolutely continuous on $[0, 1]$.

In what follows, we need some standard notions related to the differentiations of vector valued functions. They summarized below for the reader's convenience

Definition 8.6.
A real number ω is called the approximate derivative of a real valued function $x : I \to \mathbb{R}$ at the point t, if there exists a measurable set $J \subset I$ that has t as a density point (see Definition 2.4) such that

$$\lim_{s\to t,\ s\in J}\frac{x(s)-x(t)}{s-t} = \omega.$$

We write $D_{ap}x$ to represent the real number ω.

Definition 8.7. Consider the vector-valued function $x : I \to E$:

1. Let φx be differentiable on I for every $\varphi \in E^*$. The function x is said to be weakly differentiable on I if there exists $y : I \to E$ such that for every $\varphi \in E^*$ we have

$$\frac{d\varphi x(t)}{dt} = \varphi y(t),\ \text{for every } t \in I.$$

That is, for every $t \in I$, there exists $y : I \to E$ such that the difference quotient $\frac{x(t+h)-x(t)}{h}$ converges weakly to $y(t)$ as $h \to 0$. In other words, for every $\varphi \in E^*$, we have

$$\lim_{h\to 0}\varphi\left(\frac{x(t+h)-x(t)}{h}\right) = \lim_{h\to 0}\frac{\varphi(x(t+h))-\varphi(x(t))}{h} = \varphi y(t).$$

The function y is called the weak derivative of the function x. It is obvious that if the function x is weakly differentiable on I, then the real function $\varphi(x)$ is differentiable on I. The converse holds in a weakly sequentially complete spaces: If E is weakly sequentially complete and $\varphi(x)$ is differentiable for every $\varphi \in E^*$, then every weakly Cauchy sequence (hence every weakly convergent sequence) converges to a unique element in E. In particular the sequence $\{\frac{x(t+h)-x(t)}{h}\}$ weakly converges in E as $h \to 0$. That is there exists a a unique function $y : I \to E$ such that

$$\lim_{h\to 0} \varphi\left(\frac{x(t+h)-x(t)}{h}\right) = \varphi y(t) \text{ for every } \varphi \in E^*.$$

So, x is weakly differentiable (see [172], Theorem 7.3.3]).

2. The function $x : I \to E$ is said to be strongly differentiable on I if there is a function $y : I \to E$ such that

$$\lim_{h\to 0} \left\| \frac{1}{h}[x(t+h)-x(t)] - y(t) \right\| = 0.$$

That is, the limit (in the norm topology in E)

$$\lim_{h\to 0} \frac{x(t+h)-x(t)}{h}$$

converge in E to $y(t)$. The function y is the strong derivative of the function x. Alternatively, since

$$\left| \left\| \frac{1}{h}[x(t+h)-x(t)] \right\| - \|y(t)\| \right| \leq \left\| \frac{1}{h}[x(t+h)-x(t)] - y(t) \right\|,$$

it follows that

$$\left\| \frac{x(t+h)-x(t)}{h} \right\| \to \|y(t)\| \text{ as } h \to 0.$$

3. Let φx be differentiable a.e. on I for every $\varphi \in E^*$ (the null set may varies with $\varphi \in E^*$). The function x is said to be pseudo (approximate pseudo) differentiable on I if there exists a function $y : I \to E$ such that for every $\varphi \in E^*$ there exists a null set $N(\varphi) \subset I$ such that

$$(\varphi x(t))' = \varphi y(t), \ (D_{ap}\varphi x(t) = \varphi y(t)) \text{ for every } t \in I/N(\varphi).$$

In this case, the function y is called the pseudo (approximate pseudo) derivative of the x.
If x is pseudo-differentiable on I and the null set invariant for every $\varphi \in E^*$, then x in this case is a.e. weakly differentiable on I.

Unfortunately, the pseudo-derivative of the pseudo-differentiable function is not unique and two pseudo-derivatives need not be a.e. equal [176] (see Example 9.1 in [130]). Indeed, the pseudo-derivatives of the same pseudo-differentiable function are weakly equivalent.

Caution!: It is very important to remark that, the weak absolute continuity of a function $x : I \to E$ yields φx is a.e. differentiable on I every $\varphi \in E^*$, but this can not guarantee the existence of a pseudo-derivative of x.

That is, the weak absolute continuity of x can not guarantee the existence of a function y such that $(\varphi x(t))' = \varphi y(t)$, for every $\varphi \in E^*$. In fact, even E is separable, and x is Lipschitz function, the pseudo-derivative of x need not exists [177].

Example 8.9. The weak absolute continuity of function y given by (205) yields the a.e. differentiability of φy. Thus for every $\varphi \in C_0^*$ there exists a null set $N(\varphi)$ such that the function $\varphi y(\cdot)$ is differentiable on $[0,1]/N(\varphi)$. It follows from (208) that

$$\frac{d\varphi y(t)}{dt} = \sum_{n=1}^{\infty} \lambda_n x_n(t) = \varphi x(t),\ t \in [0,1]/N(\varphi).$$

That is the function x is a pseudo (but not necessary a weak) derivative of y.

It is clear that, a strongly differentiable function x is also weakly (trivially pseudo) differentiable, but the converse is not (in general) true. Also, it is clear that, if the function $x : I \to E$ is a.e. weakly differentiable on I (where the null set independent on $\varphi \in E^*$) with weak derivative y_ω, then x is pseudo-differentiable on I with pseudo derivative y_p, say. Also $y_\omega = y_p$ a.e. on I. Since the null set in the definition of the pesudo derivative dependence on $\varphi \in E^*$, one can not expect that the a.e. weak derivative of the function x equivalent to its pesudo derivative. However, in [130, 176, 177], it was shown that, the weak (pesudo) derivative of an a.e. weakly (pesudo) differentiable function is strongly (weakly) measurable:

Lemma 8.5. (a) *If the function $x : I \to E$ is a.e. weakly differentiable on I, then the weak derivative of x is strongly measurable on I (see Theorem 1.1 in* [130]*),*

(b) *If the function $x : I \to E$ is pesudo differentiable on I, then the pesudo derivative of x is weakly measurable on I (see* [176]*)*

The following result is a slight generalization of Theorem 2.9 of Pettis [131]

Proposition 8.2. *Let $x : J \to E$, where $J \subset \mathbb{R}$*

1. *If x is strongly differentiable at t with derivative $y(t)$, then x is also weakly differentiable at t and the weak derivative of x at t is $y(t)$*
2. *If x is weakly differentiable on J to y, then x is strongly differentiable a.e. on J to y.*

Proof. For the first statement, we suppose that x is strongly differentiable at t with derivative $y(t) \in E$, that is the limit (in the norm topology in E)

$$\lim_{h\to 0} \frac{x(t+h) - x(t)}{h}$$

converge in E to $y(t)$. Thus, continuity of any $\varphi \in E^*$ yields

$$\lim_{h\to 0} \varphi\left(\frac{x(t+h)-x(t)}{h}\right) = \varphi\left(\lim_{h\to 0}\frac{x(t+h)-x(t)}{h}\right) = \varphi(y(t)).$$

Hence, $y(t)$ is also the weak derivative of x at t.

For the proof of the second statement, see Theorem 4 of Alexiewicz [5]. □

Caution!: It is clear that if x is not strongly differentiable for any t (or strongly differentiable at a finite numbers of points) then, in the view of the second part of Proposition 8.2, x is not weakly differentiable everywhere (that is, the weak derivative of x could exist for some, but not for all, t).

It is well-known that the absolutely continuous real-valued function is a.e. differentiable. This one of the few properties of the real-valued function that is not carry over in arbitrary Banach spaces: In fact, in the following example, we prove the existence of an absolutely continuous function but not strongly differentiable at any point (it is not even everywhere weakly differentiable).

Example 8.10. Define $x : (0,1) \to L_1(0,1)$ by $x(t) := t\chi_{[0,t]}$. For $t_1, t_2 \in (0,1)$, $t_1 > t_2$ we have

$$\begin{aligned}
\|x(t_1) - x(t_2)\| &= \int_0^{t_2} |t_1\chi_{[0,t_1]}(s) - t_2\chi_{[0,t_2])}(s)|\,ds \\
&+ \int_{t_2}^{t_1} |t_1\chi_{[0,t_1]}(s) - t_2\chi_{[0,t_2]}(s)|\,ds \\
&+ \int_{t_1}^{1} |t_1\chi_{[0,t_1]}(s) - t_2\chi_{[0,t_2]}(s)|\,ds \\
&= \int_0^{t_2} |t_1 - t_2|\,ds + \int_{t_2}^{t_1} |t_1|\,ds < 2|t_1 - t_2|,
\end{aligned}$$

Then x is Lipschitz continuous, and therefore absolutely continuous. Nevertheless, the strong derivative x' does not exist for any $t \in (0,1)$. Indeed, we have

$$\begin{aligned}
\left\|\frac{x(t+h)-x(t)}{h}\right\| &= \frac{1}{|h|}\int_0^1 |(t+h)\chi_{[0,t+h])}(s) - t\chi_{[0,t]}(s)|\,ds \\
&= \frac{1}{|h|}\left[\int_0^t |(t+h)\chi_{[0,t+h]}(s) - t\chi_{[0,t]}(s)|\,ds\right. \\
&+ \int_t^{t+h} |(t+h)\chi_{(0,t+h)}(s) - t\chi_{[0,t]}(s)|\,ds \\
&+ \left.\int_{t+h}^1 |(t+h)\chi_{[0,t+h]}(s) - t\chi_{[0,t]}(s)|\,ds\right] \\
&= \frac{1}{|h|}\left[\int_0^t |(t+h)-t|\,ds + \int_t^{t+h} |t+h|\,ds\right] \\
&= \frac{1}{|h|}\left[|h|t + |t+h|h\right] \to \begin{cases} 2t & : h>0 \\ 0 & : h<0 \end{cases}
\end{aligned}$$

Thus, the limit (in the norm topology in $L_1(0,1)$) as $h \to 0$ of the difference quotient

$$\frac{x(t+h)-x(t)}{h}$$

does not converge in $L_1(0,1)$ for all $t \in (0,1)$ (This limit would exist in case $t = 0 \notin (0,1)$). So by Proposition 8.2, x is not weakly differentiable everywhere on $(0,1)$.

Example 8.11. The function $x : [0,1] \to L_1[0,1]$ by $x(t) := \chi_{[0,t]}$ satisfies

$$\|x(t+h) - x(t)\| = \left\|\chi_{[t,t+h]}\right\| = \int_0^1 |\chi_{[t,t+h]}(s)|\,ds = |h|$$

and so x is absolutely continuous. The strong derivative of x does not exists on $[0,1]$. To see this, let $h > 0$ and $t \in [0,1]$

$$\left\|\frac{x(t+h)-x(t)}{h}\right\| = \frac{1}{h}\int_0^1 |\chi_{[t,t+h]}(s)|\,ds = \left[\frac{1}{h}\int_t^{t+h} ds\right] = 1.$$

Therefore, if the strong derivative y would exist at some t, then necessarily $\|y\| = 1$.

On the other hand, we would have

$$\left\|\frac{x(t+h)-x(t)}{h} - y(t)\right\| \to 0 \text{ as } h \to 0$$

where the term on the left-hand side is given by

$$\begin{aligned}
\int_0^1 \left| \frac{x(t+h)(s) - x(t)(s)}{h} - y(t)(s) \right| ds &= \int_0^1 \left| \frac{\chi_{[t,t+h]}(s)}{h} - y(t)(s) \right| ds \\
&\geq \int_0^t |y(t)(s)| \ ds + \int_{t+h}^1 |y(t)(s)| \ ds \\
&= \int_0^1 |y(t)(s)| \ ds - \int_t^{t+h} |y(t)(s)| \ ds \\
&\to \|y(t)\| \ \text{ as } h \to 0.
\end{aligned}$$

Combining the above relations, we find $\|y(t)\| = 0$, a contradiction. Again by Proposition 8.2, x is not weakly differentiable everywhere on $[0, 1]$.

8.2. **Vector-valued integrals.** The definition of the Lebesgue integral as a supremum of integrals of simple functions does not extend directly to vector- valued integrals because it uses the ordering properties of $\mathbb{R}$ in an essential way. A natural extension of the Lebesgue integral of real-valued function to a Banach-valued function is the so called Bachner, Dunford and Pettis integrals.

Definition 8.8. 1. The strongly measurable function $x : I \to E$ is said to be Bochner integrable on $J \subset I$ if there exists a sequence of simple function $\{x_n\}$ such that $x_n(t) \to x(t)$ pointwise a.e. in J, $\|x - x_n\|$ is Lebesgue integrable on J and

$$\lim_{n\to\infty} \int_J \|x(t) - x_n(t)\| \, ds = 0.$$

The Bochner integral of x on J defined as

$$(B)\int_J x\,ds := \lim_{n\to\infty} \int_J x_n\,ds,$$

where the limit is in the norm topology on E. If $x = \sum_{i=1}^n c_i \chi_{J_i}$ is a simple function, then

$$(B)\int_J x\,ds = \sum_{i=1}^n c_i \mu(J_i) \in E,$$

where $\mu(J_i)$ is the Lebesgue measure of J_i.

The function x is Bochner integrable on I if and only if it is Bochner integrable on every measurable subset $J \in I$.

2. The space of all $E-$valued Bochnar integrable functions in the interval I denoted by $L_1[I, E]$. This space equipped with the norm

$$\|x\| := \int_I \|x(s))\| \, ds$$

is a Banach space.

Let us state some simple facts about Bochner. Perhaps one of the main features of this integrals is that it generalize the Lebesgue integral.

Lemma 8.6. *If x is Bochner integrable, then*

$$\left\| \int_J x\,ds \right\| \le \int_J \|x\| \, ds \tag{209}$$

holds for all measurable set $J \subset I$.

Lemma 8.7. *If a strongly measurable y is Bochner integrable and if $\|x\| \leq \|y\|$, then x is Bochner integrable.*

Lemma 8.8. *If x is Bochner integrable and if $T : X \to Y$ be a bounded operator between the two Banach spaces X and Y, then $\int Tx\, ds = T \int x\, ds$.*

Theorem 8.4. *The strongly measurable function $x : I \to E$ is Bochner integrable if and only if $\int \|x\|\, ds < \infty$.*

The functions x defined in Example 8.2 and Example 8.3 are Bochner integrable on $[0, 1]$. Also in Example 8.4, the function y is Bochner integrable on $[0, 1]$ while the function x is not (see Remark 8.2).

Example 8.12. Consider the strongly measurable functions $x : [0, 1] \to C_0$ defined by

$$x(t) := \left\{ \sum_{n=1}^{\infty} n\chi_{(\frac{1}{n+1},\frac{1}{n}]}(t), 0, 0 \cdots \right\}.$$

Clearly

$$\int_0^1 \|x(t)\|_{C_0}\, dt = \int_0^1 \sum_{n=1}^{\infty} n\chi_{(\frac{1}{n+1},\frac{1}{n}]}(t)dt = \sum_{n=1}^{\infty} \frac{1}{n+1} = \infty,$$

which implies that $\|x(\cdot)\|_{C_0}$ is not Lebesgue integrable on $[0, 1]$. Hence, it is not Bochner integrable $[0, 1]$.

Definition 8.9. The weakly measurable function $x : I \to E$ is called Dunford (or Gelfand) integrable on I if there exists a bounded linear functional $\varphi_J^* \in E^{**}$ corresponding to each $J \subset I$ such that

$$\varphi_J^*(\varphi) := \int_J \varphi x(t)dt, \text{ for each } \varphi \in E^*.$$

That is, the weakly measurable function $x : I \to E$ is integrable on I in the sense of Dunford, if for every $J \subset I$, the mapping

$$E^* \to \mathbb{R},\ \varphi \longmapsto \int_J \varphi x(t)dt,$$

is bounded linear.

The functional φ_J^* is called the Dunford integral of x over J. We use the notation

$$(D) \int_J x(s)\, ds := \varphi_J^*.$$

Therefore, the set of integrals $\{(D) \int_J x(s)ds : J \subset I\ \}$ is a subset of E^{**}.

The proof of the following lemma is not the standard Baire Category argument as in ([89] page 78) but it is in the spirit of the closed graph theorem 2.6.

Lemma 8.9. *If φx is Lebesgue integrable on $[0,1]$ for each $\varphi \in E^*$, then x is Dunford integrable on $[0,1]$. That is, for each measurable $J \subset [0,1]$, there exists a vector $\varphi_J^* \in E^{**}$ such that*

$$\varphi_J^*(\varphi) = \int_J \varphi x(t)dt, \text{ for all } \varphi \in E^*.$$

Proof. Fix $J \subset [0,1]$ and define $\varphi_J^* : E^* \to \mathbb{R}$ by

$$\varphi_J^*(\varphi) := \int_J \varphi x(s)\, ds, \ \varphi \in E^*.$$

It is clear that φ_J^* is defined and linear on E^*. It remains to show that φ_J^* is bounded. To do this we define the linear operator $T : E^* \to L_1$ by $T(\varphi) := \varphi x(\cdot)\chi_J(\cdot)$. This operator is closed (The spaces E^* and L_1 are complete): Indeed, if we assume that $\varphi_n \to \varphi$ in E^* (that is, $\|\varphi_n - \varphi\|_{E^*} \to 0$ as $n \to \infty$) and $T(\varphi_n) \to g$ exists in L_1 (that is, $\int_J |T(\varphi_n)(t) - g(t)|dt \to 0$ as $n \to \infty$)) then, by Vitali's Theorem $\{T(\varphi_n)\}$ converges in measure to g. Using Egorov's theorem 2.7, we can find a subsequence, φ_{n_k} such that $T(\varphi_{n_k}) \to g$ for almost all t (that is $\varphi_{n_k}(x\chi_J) \to g$ a.e.). Since $\varphi_n(x\chi_J) \to \varphi(x\chi_J)$ everywhere, it follows that $T(\varphi) = \varphi(x\chi_J) = g$ a.e.. In other words $T\varphi = g$, so T is closed linear operator, hence it is bounded by the Closed Graph Theorem 2.6. It follow that

$$|\varphi_J^*(\varphi)| = \left|\int_J \varphi x(t)dt\right| = \left|\int_I \varphi(x\chi_J)(t)dt\right| \leq \int_I |\varphi(x\chi_J)(t)|\, dt = \|T\varphi\|_{L_1(I)} \leq \|T\|\, \|\varphi\|.$$

Hence φ_J^* identifies a bounded linear functional on E^* and, as such, defines a member in E^{**}. □

For the proof of Lemma 8.9 in the case when the function $x : I \to E$ having $\varphi x \in L_p$, for every $\varphi \in E^*$ with $p \in [1,\infty]$, see Theorem 3.2 in [130].

Example 8.13. Consider the strongly measurable function x defined in Example 8.4. It was showed in (Remark 8.2 part 4) by the aid of Lemma 8.9, that x is Dunford integrable on $[0,1]$. Then, the Dunford integral of x identifies an element $\varphi^* \in C_0^{**}$. However, let $\{\lambda_n\} \in \ell_1$ be the corresponds to $\varphi \in C_0^*$. Using the Beppo-Levi theorem, we get

$$\int_0^1 \varphi x(t)dt = \int_0^1 \sum_n \lambda_n x_n(t)dt = \sum_n \lambda_n \int_0^1 n\chi_{(0,1/n]}dt = \sum_n \lambda_n = \varphi(g(t)),$$

where $g(t) := \{1, 1, 1, \cdots\}$. Therefore, the Dunford integral of x is an element of C_0^{**} determined as a functional $\varphi \to \sum_n \lambda_n$. That is

$$(D)\int_0^1 x(t)dt : \varphi \in C_0^* \to \sum \lambda_n \in \mathbb{R}.$$

Example 8.14. Consider the strongly measurable functions $x : [0,1] \to C_0$ defined by

$$x(t) := \left\{ \sum_{n=1}^{\infty} n\chi_{(\frac{1}{n+1},\frac{1}{n}]}(t), 0, 0 \cdots \right\}.$$

Clearly

$$\int_0^1 |\varphi x(t)|dt = \int_0^1 \sum_n |\lambda_n x_n(t)|dt \leq \int_0^1 |\lambda_1| \sum_{n=1}^{\infty} n\chi_{(\frac{1}{n+1},\frac{1}{n}]}dt = |\lambda_1| \sum_{n=1}^{\infty} \frac{1}{n+1} = \infty,$$

That is φx is not Lebesgue integrable on $[0,1]$ and consequently, it is not Dunford integrable on $[0,1]$.

Recall that $E \subset E^{**}$ ($E = E^{**}$ when E is reflexive) (that is E embeddable in E^{**}), meaning that there is a canonical mapping from E to E^{**}" (that is, there is isometrical isomorphism between E and a closed subspace of E^{**}). Indeed, every element $\Im \in E$ identifies a bounded linear functional in $\varphi^* \in E^{**}$ as

$$\varphi^*_{\Im}(\varphi) := \varphi(\Im), \ \varphi \in E^* \qquad (**).$$

The map $\Im \longmapsto \varphi^*_{\Im}$ defined by (**) is called canonical map between E and a closed subspace of E^{**}.

Recall also, that a function $x : I \to E$ is Dunford integrable on I if $\varphi(x(\cdot)) \in L_1(I)$, for all $\varphi \in E^*$. In this case, consequence of Lemma 8.9, for every measurable $J \subset I$ there exists an element "denoted by $(D)\int_J x\,ds$" in E^{**} such that

$$\left[(D)\int_J x\,ds\right](\varphi) = \int_J \varphi x(t)dt, \text{ for all } \varphi \in E^*.$$

Now, the function $x : I \to E$ is said to be Pettis integrable on J if it is Dunford integrable on J and $(D)\int_J x$ belongs the canonical image of E in E^{**}. So, in the view of (**), there exists $\Im_J \in E$ such that

$$\varphi(\Im_J) = \int_J \varphi(x(s))\,ds, \text{ for all } \varphi \in E^*.$$

In this case, the element $\Im_J$ is the Pettis integrable of x over J and we write $(P)\int_J xds := \Im_J \in E$. We remark that, if $x : I \to E$ is Pettis integrable on

J, then

$$(P)\int_J x\,ds \in E, \text{ while } (D)\int_J x\,ds \in \text{ canonical image of } E.$$

However, x is pettis integrable on I if and only if it is Pettis integrable over every measurable subset $J \in I$. Thus $x : [a,b] \to E$ is Pettis integrable on $[a,b]$ if and only if

1. x is Dunford integrable on $[a,b]$,
2. The set of integrals $(D)\int_J x(s)ds$ belong to the canonical image of E for all measurable **subset** J of $[a,b]$.

Definition 8.10. Let $x : [a,b] \to E$ be a weakly measurable function such that

1. x is Dunford integrable on $[a,b]$,
2. The set of integrals $(D)\int_J x(s)ds$ belong to the canonical image of E for all **subinterval** J of $[a,b]$.

Then x is called Denjoy-Pettis integrable on $[a,b]$.

Obviously, if the $\sigma-$algebra of $\mu-$measurable subsets of I contains the subintervals, then every Pettis integrable function is Denjoy-Pettis integrable, the converse is not (in general) true: Evidently, in [85] (see also Example 8.25 below), the authors, construct an example of a strongly measurable function $x : [a,b] \to E$, where E contains isometric copy of C_0 (That is, there is a subspace of E isometric to C_0) such that

1. x is Dunford integrable on $[a,b]$,
2. The set of integrals $(D)\int_J x(s)ds$ belong to the canonical image of E for all **subinterval** J of $[a,b]$,
3. x is not Pettis integrable on $[a,b]$.

In the following lemma, we show that the Pettis and Denjoy-Pettis integrable coincide if E contains no isometric copy of C_0.

Lemma 8.10. *(* [89] *Theorem 23) Let E contains no isometric copy of C_0. If $x : I \to E$ is Dunford integrable function such that the set of integrals $\{(D)\int_J x(s)ds \in E\}$ for every subinterval $J \subset I$, then x is Pettis integrable on I. In other words, if E contains no isometric copy of C_0, and if the is Dunford integrable function $x : I \to E$ is Pettis integrable on every subinterval $J \subset I$, then x is Pettis integrable on I*

In the view of the above discussion, we introduce the following

Definition 8.11. 1. The weakly measurable function $x : I \to E$ is Pettis integrable on $J \subset I$ if there is an element $\Im_J \in E$ corresponding to each

$J \subset I$ such that

$$\varphi(\Im_J) = \int_J \varphi(x(s))\, ds, \text{ for all } \varphi \in E^*,$$

where the integral on the right is supposed to exist in the sense of Lebesgue.

The element $\Im_J$ is called the Pettis integral of x over J. We use the notation

$$(P) \int_J x(s)\, ds := \Im_J.$$

Thus, in order to prove that the weakly measurable function x is pettis integrable on J, it is necessary to show firstly that $\varphi(x(\cdot))$ is Lebesgue integrable on J for any $\varphi \in E^*$. *The function x is pettis integrable on I if and only if it is Pettis integrable on every measurable subset $J \in I$.*

2. The space of all $E-$valued Pettis integrable functions in the interval I denoted by $P[I,E]$. We define a norm on $P[I,E]$ by

$$\|x\| := \sup_{\varphi \in B(\varphi)} \left\{ \int_I |\varphi x(s)|\, ds \right\}, \text{ where } B(\varphi) \text{ is the closed unit ball of } E^*.$$

It has been shown by Thomas [187], that if E is infinite dimensional, then $P[I,E]$ is not complete.
3. For $1 \le p \le \infty$, we define the class $\mathcal{H}^p(E)$ to be the class of all functions $x : I \to E$ having $\varphi x \in L_p(I)$ for every $\varphi \in E^*$. If $p = \infty$, the added condition

$$l.u.b._{\|\varphi\|=1}(\operatorname*{ess\,sup}_{t \in I} |\varphi x(t)|) < \infty$$

must be satisfied by each $x \in \mathcal{H}^\infty(E)$. We also define the space $\mathcal{H}_0^p(E)$ to be the sub-class of $\mathcal{H}^p(E)$ composed of Pettis integrable functions. Obviously, all functions from the class $\mathcal{H}^p(E)$ are weakly measurable.

Let us remark, that the Pettis (Bochner) integrals of two weakly (strongly) measurable functions x, y coincide over every Lebesgue measurable set in I if and only if $x(\cdot) = y(\cdot)$ a.e. on I ([130], Theorem 5.2). Clearly, for any strong measurable function $x : I \to E$, we have the implication

$$x \text{ Bochner integrable } \Longrightarrow x \text{ Pettis integrable } \Longrightarrow x \text{ Dunford integrable.}$$

Indeed, if $x : I \to E$ is strongly measurable, by Lemma 8.8 we have $\varphi \int x\, ds = \int \varphi x\, ds$ for any $\varphi \in E^*$. That is x is Pettis integrable and

$$(P) \int x\, ds = (B) \int x\, ds.$$

In other words, every strongly measurable Bochner integrable function is also Pettis integrable and the integrals have the same value. When E is finite-dimension, the notation of Bochner integrability and Pettis integrability are coincide (see e.g [99]). When E is infinite-dimension, there are Pettis integrable functions, which are not Bochner integrable.

Example 8.15. *(A Dunford, Pettis but not Bochner integrable function)* Consider the function $x : [0,1] \to C_0$ defined by

$$x(t) := \{x_n(t)\},\ x_n := n\chi_{(\frac{1}{n+1},\frac{1}{n}]}.$$

Obviously, the sequence $\{n\chi_{(\frac{1}{n+1},\frac{1}{n}]}(t)\}$ has at most one non-zero term and hence is a member of C_0^*. So, x is well-defined. Moreover, x is countable-valued function, so it is strongly measurable. Consequently, it is weakly measurable, because C_0 is separable. Also, for any $\varphi \in C_0^*$, there is $\{\lambda_n\} \in \ell_1$ such that $\varphi x = \sum_n \lambda_n x_n$. Thus

$$\begin{aligned}\int_0^1 |\lambda_n x_n(t)|dt &= |\lambda_n| \int_0^1 |x_n(t)|dt = |\lambda_n| \int_{\frac{1}{n+1}}^{\frac{1}{n}} ndt \\ &= n|\lambda_n| \left[\frac{1}{n} - \frac{1}{n+1}\right] = \frac{|\lambda_n|}{n+1},\end{aligned}$$

whence

$$\sum_n \int_0^1 |\lambda_n x_n(t)|dt = \sum_n \frac{|\lambda_n|}{n+1} \le \sum_n |\lambda_n| < \infty.$$

Consequently, in the view of Beppo-Levi Theorem 2.3, it follows $\varphi x = \sum_n \lambda_n x_n \in L_1[0,1]$. That is, x is Dunford integrable on $[0,1]$. Now, we will prove that x is Pettis integrable on $[0,1]$. To do this, we recall that the function x is pettis integrable on $[0,1]$ if and only if it is Pettis integrable on every measurable subset $J \in [0,1]$. So we assume $J \subset [0,1]$ is measurable and note that

$$\int_J \varphi x(t)dt = \int_J \sum_n \lambda_n x_n(t)dt = \sum_n \lambda_n \int_J n\chi_{(\frac{1}{n+1},\frac{1}{n}]}(t)dt = \sum_n \lambda_n n\mu \left[J \cap (\frac{1}{n+1}, \frac{1}{n}]\right],$$

then

$$\int_J \varphi x(t)dt = \varphi(g(t)),\ \text{where } g(t) := \left\{n\mu \left[J \cap (\frac{1}{n+1}, \frac{1}{n}]\right]\right\}.$$

Obviously

$$n\mu \left[J \cap (\frac{1}{n+1}, \frac{1}{n}]\right] \le n\mu \left[(\frac{1}{n+1}, \frac{1}{n}]\right] \le \frac{1}{n+1} \to 0 \text{ as } n \to \infty.$$

Thus $g \in C_0$, it follows that x is pettis integrable on J and (consequently) on $[0,1]$. It is not hard to show that the Pettis integral of x $[0,1]$ equals $\left\{\dfrac{1}{n+1}\right\}$.
On the other hand, we note that $\|x(t)\|$ is not bounded, so it is not Lebesgue integrable on $[0,1]$ and thus x is not Bochner integrable on $[0,1]$.

In what follows, we construct an example of a Dunford integrable function $x:[0,1]\to C_0$ which is not Pettis integrable on $[0,1]$

Example 8.16. *(A Dunford but not Pettis integrable function)*
Consider the strongly measurable Dunford function x defined in Example 8.4. By Remark 8.2, it follows x is not Bochner integrable on $[0,1]$. Moreover, it was showed in Example 8.13 that

$$\int_0^1 \varphi x(t)dt = \varphi(\{1,1,1,\cdots\}),\ \varphi \in E^*,$$

Since $\{1,1,1,\cdots\} \notin C_0$, then x is not Pettis integrable on $[0,1]$. Thus there is no $g \in C_0$ such that $\int_{[0,1]} \varphi x = \varphi g$ for every $\varphi \in C_0^*$.

In the following lemma, we will show that if the $E-$valued Dunford integrable function x is not Pettis integrable, then it is not Pettis integrable even if we "enlarge" the space E. Consequently, if $x: I \to E$ is Dunford integrable function, by noting that $x(J) \subset x(I)$ for any $J \subset I$, it follows: If $x: I \to E$ is not Pettis integrable on J, then x is not Pettis integrable on I.

Lemma 8.11. [85] *Let Y be a Banach space having X as a sub-space (That is, X and Y endowed with the same topology). If $x: I \to X$ is Dunford (but not Pettis) integrable function on I, then $x: I \to Y$ is not Pettis integrable on I.*

Proof. The proof is an immediate consequence of the elementary fact: *If $x: I \to Y$ is Pettis integrable and $x(I)$ lies on $X \subset Y$, then $\int_J x$ lies in X for every measurable subset $J \subset I$.* To see this, we suppose that $x:[a,b]\to Y$ is Pettis integrable on $J \subset I$. Then there exists $x_J \in Y$ such that

$$\psi(x_J) = \int_J \psi(x(s))\,ds, \text{ holds for every } \psi \in Y^*.$$

In particular, if $\psi \in Y^*$ and ψ vanishes on X then $\psi(x_J) = \int_J \psi(x(s))\,ds = 0$ which implies that $\int_J x \in X$. □

Since x assumes only values in X, we are able to carry out the proof of Lemma 8.11 in different way making up the mean value theorem for Pettis integral (see Theorem 8.11).

Lemma 8.12. [85] *Let $(X, \|\cdot\|_X)$ be a subspace of the Banach space $(Y, \|\cdot\|_Y)$. If $x : I \to X$ is Pettis integrable function on I, then the function $x : I \to Y$ such that $x(I) \subset X$ is Pettis integrable function on I.*

Proof. Since $x : I \to X$ is Pettis integrable on I, then for every measurable subset $J \subset I$, we have $\int_J x \in X$. For any $\psi \in Y^*$, the restriction $\varphi := \psi|_X$ is linear and bounded (with respect to the norm $\|\cdot\|_X$ and thus also with respect to the norm $\|\cdot\|_Y$ *cf.* Hahn-Banach Theorem 2.9). So $\psi\left(\int_J x\right) = \varphi\left(\int_J x\right) = \int_J \varphi x$. Since x assume only values in X, it follows that $\psi x = \varphi x$ and so $\psi\left(\int_J x\right) = \int_J \psi x$ for every $\psi \in Y^*$. That is, $\int_J x$ is the Pettis integral of $x : I \to Y$ □

Example 8.17. If the Dunford integrable function $x : I \to C_0$ is Pettis integrable on I, then $x : I \to \ell_\infty$ is Pettis integrable on I too.

A fundamental property of Pettis integral contained in

Lemma 8.13. *(* [99] *Theorem 3.7.3) If x is Pettis integrable and if $T : X \to Y$ be a bounded operator between the two Banach spaces X and Y, then $\int Tx\, ds = T \int x\, ds$.*

Now, we will show that in reflexive Banach spaces the weakly measurable function is Pettis integrable if, and only if, it is Dunford integrable. Indeed we have

Lemma 8.14. *(*[99]*) If E is reflexive Banach space, the* ***weakly*** *measurable function $x : I \to E$ is Pettis integrable on I if, and only if $\varphi(x(\cdot)) \in L_1(I)$ for very $\varphi \in E^*$ (meaning that x is Dunford integrable).*

Proof. On the one hand, if $x : I \to E$ is Pettis integrable, then by the definition of Pettis integral, $\varphi(x(\cdot)) \in L_1(I)$ for very $\varphi \in E^*$. Consequently, by Lemma 8.9, x is Dunford integrable. One the other hand, if $\varphi(x(\cdot)) \in L_1(I)$ for very $\varphi \in E^*$, then x is Dunford integrable on I, so x defines a vector $\varphi^* \in E^{**}$. Since E is reflexive, then the duality between E and E^{**} defines $E-$valued integral of x of the corresponding real-valued integral, meaning that x is Pettis integrable on I. □

In what follows, we "relax" the reflexivity condition imposed on E into a weakly complete condition, as follows: If E is weakly complete, then the *strongly* measurable, Dunford integrable functions are Pettis integrable. A more bit, when $p > 1$, we will show that the class $\mathcal{H}_0^p(E)$ is "in fact" the sub-class of $\mathcal{H}^p(E)$, consists of the strongly measurable functions.

Proposition 8.3. *Let $p > 1$. If the* ***strongly*** *measurable $x : I \to E$ is in $\mathcal{H}^p(E)$ then, x is in $\mathcal{H}_0^p(E)$. If E is weakly complete, this is also true for $p = 1$.*

In the view of Example 8.16, the hypothesis of weak sequential completeness of the proposition 8.3 cannot be omitted or extended to a separable spaces (e.g. can not extended to C_0).
The space C_0 is exceptional: In fact, Proposition 8.3 with $p = 1$ remains valid for the strongly measurable functions if E contains no isometric copy of C_0. Indeed, we have the following results proved in [61], see also [85, 89].

Lemma 8.15. *Let E contain no isometric copy of C_0. The* ***strongly*** *measurable $x : I \to E$ is Dunford integrable I, if and only if x is Pettis integrable on I.*

Since the weak continuity implies a strong measurability (see Lemma 8.3), it follows in the view of Proposition 8.3 that

$$\begin{aligned} x : [0,1] \to E \text{ weakly continuous } &\Longrightarrow x \text{ strongly measurable and } x \in \mathcal{H}^\infty(E) \\ &\Longrightarrow x \in \mathcal{H}_0^\infty(E). \end{aligned}$$

An immediate consequence is

Lemma 8.16. *Every weakly continuous function $x : [0,1] \to E$ is Pettis integrable on* $[0,1]$.

In the view of Lemma 8.16, it follows that the weakly continuous functions x and z defined on the example 8.7 are Pettis integrable on $[0,1]$.

In summary, according to Lemma 8.14, Lemma 8.15, Lemma 8.16 and Proposition 8.3, it follows that

1. $\mathcal{H}_0^1(E) \supset \mathcal{H}_0^2(E) \supset \mathcal{H}_0^3(E) \supset \cdots \supset \mathcal{H}_0^\infty(E) \supset C[I, E_w]$,
2. If $p \in (1,\infty]$, we have $\{x \in \mathcal{H}^p(E) :\ x \text{ strongly measurable}\} \equiv \mathcal{H}_0^p(E)$,
3. If $p \in [1,\infty]$ and E is weakly complete or contains no isometric copy of C_0, we have
$$\{x \in \mathcal{H}^p(E) :\ x \text{ strongly measurable}\} \equiv \mathcal{H}_0^p(E),$$
4. If $p \in [1,\infty]$ and E is reflexive, we have $\mathcal{H}^p(E) \equiv \mathcal{H}_0^p(E)$ and $P[I,E] \equiv \mathcal{H}^1(E)$,
5. If $p \in [1,\infty]$ and $q \in [1,\infty)$, we have $\mathcal{H}^p(L_q(I)) \equiv \mathcal{H}_0^p(L_q(I))$ and $\mathcal{H}^p(\ell_q) = \mathcal{H}_0^p(\ell_q)$. This follows, by noting that the functions from the classes $\mathcal{H}^p(L_q(I))$ and $\mathcal{H}^p(\ell_q)$ are always weakly measurable, since $L_q(I)$ with $q \in [1,\infty)$ are weakly complete and separable, then the strong and the weak measurability coincides. The result now is a direct consequence of Proposition 8.3.

Example 8.18. The function y defined by (205) is weakly absolutely continuous on $[0,1]$, hence it is Pettis integrable on $[0,1]$. The function y is even

Bochner integrable: y is weakly measurable on $[0,1]$, hence strongly measurable since c_0 is separable, and y is bounded in norm.

Example 8.19. *(A Dunford, Pettis and Bochner integrable function)*
The weakly measurable function $y:(0,1)\to H$ defined in Example 8.5, is Dunford integrable: Indeed, the function y is weakly equivalent to the zero-function, so $\varphi y(\cdot)$ is Lebesgue integrable on $(0,1)$ for each $\varphi\in H^*$ and hence $y\in\mathcal{H}^1(H)$. Further

$$\int_0^1 \varphi y(t)dt = 0 = \varphi(\circ),\ \text{where } \circ \text{ the zero element in } H.$$

Because of the reflexivity of the Hilbert spaces H, it follows by Lemma 8.14 that y is Pettis integrable on $(0,1)$ and

$$(P)\int_0^1 x(t)dt = (P)\int_0^1 y(t)dt = \circ.$$

On the other hand, y is strongly measurable with

$$\|y(t)\| := \begin{cases} 0 \text{ if } t\notin J, \\ \sqrt{< e_t, e_t >} = 1 \text{ if } t\in J. \end{cases}$$

Thus $\|y(\cdot)\|\in L_1(0,1)$ and so y is Bochner integrable on $(0,1)$. Consequently, $(B)\int y = (P)\int y = \circ$. This is also evident by

$$\left\|\int_0^1 y(s)\,ds - 0\right\| \le \int_0^1 \|y(s)\|\,ds = 0.$$

Example 8.20. *(A Dunford, Pettis but not Bochner integrable function)*
As in the arguments in Example 8.19, the weakly measurable Dunford integrable function $x:(0,1)\to H$ defined in Example 8.5 is Pettis integrable on $(0,1)$ and

$$(P)\int_{[0,1]} x(t)dt = 0.$$

On the other hand, x is not strongly measurable. Consequently x is not Bochner integrable on $[0,1]$.

Example 8.21. *(A Dunford, Bochner and Pettis integrable function)*
The function $y:(0,1)\to L_2(0,1)$ in Example (8.6) is Bochner (trivially Pettis) integrable since it is strongly measurable and

$$\|y(t)\|_{L_2} = \left(\int_0^1 |\chi_{(0,t)}(s)|^2\,ds\right)^{\frac{1}{2}} = \left(\int_0^t ds\right)^{\frac{1}{2}} = \sqrt{t}\in L_1(0,1).$$

To compute its integral "as a Pettis integral", we let $\varphi \in L_2(0,1)$, which is the dual of $L_2(0,1)$. By the Riesz representation theorem there is $x \in L_2(0,1)$ such that

$$\varphi(y(t)) = \int_0^1 y(t)x(t)\ dt.$$

Hence

$$\begin{aligned}\int_0^1 \varphi(y(t))dt &= \int_0^1\int_0^1 \chi_{(0,t)}(s)x(s)\,dsdt\\ &= \int_0^1\int_0^t x(s)\,dsdt = \int_0^1\int_s^1 x(s)dt\,ds\\ &= \int_0^1 (1-s)x(s)\,ds = \varphi(w)\end{aligned}$$

with the function $w(s) = 1-s \in L_2(0,1)$. Since Bochner and Pettis integrals of y have the same value, then the function w must be the value of the integral. That is

$$\left[\int_0^1 y(t)dt\right](s) = 1-s.$$

Example 8.22. Consider the strongly measurable function $x : [0,1] \to C_0$ defined by

$$x(t) = \{x_n(t)\},\ x_n(t) = n^\gamma \chi_{(0,1/n](t)},\ \gamma \in (-\infty, 1).$$

Let $\{\lambda_n\} \in \ell_1$ denotes the corresponding to $\varphi \in C_0^*$. Therefore we have

$$\int_0^1 |\varphi x(t)|dt \le \sum_n |\lambda_n| \int_0^1 n^\gamma \chi_{(0,1/n]}dt = \sum_n \frac{|\lambda_n|}{n^{1-\gamma}} \le \sum_n |\lambda_n| < \infty,$$

Thus x is Dunford integrable on $[0,1]$ for any $\gamma < 1$. Now, we will prove that x is Pettis integrable on $[0,1]$ for any $\gamma < 1$. At the beginning, we recall that the function x is pettis integrable on $[0,1]$ if and only if it is Pettis integrable on every measurable subset $J \in [0,1]$. So we assume $J \subset [0,1]$ is measurable and note that

$$\int_J \varphi x(t)dt = \int_J \sum_n \lambda_n x_n(t)dt = \sum_n \lambda_n \int_J n^\gamma \chi_{(0,1/n]}dt = \sum_n \lambda_n n^\gamma \mu(J\cap(0,1/n]),$$

then

$$\int_J \varphi x(t)dt = \varphi(g(t))$$

where $g(t) := \{n^\gamma \mu(J \cap (0,1/n])\}$. Obviously, for $\gamma < 1$

$$n^\gamma \mu(J \cap (0,1/n]) \le n^\gamma \mu((0,1/n]) \le n^{\gamma-1} \to 0 \text{ as } n \to \infty.$$

Thus $g \in C_0$ for any $\gamma < 1$. That is, x is Pettis integrable on every measurable subset of $[0,1]$, consequently x is Pettis integrable on $[0,1]$ and

$$(P)\int_0^1 x(t)dt = g(t) \in C_0.$$

Example 8.23. *(A Dunford and Pettis integrable function)*
Defined the mapping x from the interval $(0,1)$ into the Hilbert space ℓ_2 as

$$x : t \longmapsto \left\{\frac{1}{1+t}, \frac{1}{2+t}, \frac{1}{3+t}, \cdots\right\},\ t \in (0,1).$$

We note that

$$\|x(t)\|_{\ell_2}^2 = \sum_n \left(\frac{1}{n+t}\right)^2 \le \sum_n \frac{1}{n^2} < \infty,\ t \in (0,1).$$

Thus the mapping x is well-defined. Moreover, x is Dunford, and consequently Pettis integrable on $(0,1)$ since ℓ_2 is reflexive. To see this, we let $\varphi \in (\ell_2)^* = \ell_2$. According to the Riesz representation theorem on Hilbert spaces there exists of a uniquely determined $a := \sum_n \lambda_n \in \ell_2$ such that $\varphi x =< x, a >$ for every $x \in \ell_2$. Hence

$$\varphi x(t) = \sum_n \frac{\lambda_n}{n+t}$$

and so

$$\int_0^1 |\varphi x(t)|dt = \int_0^1 |\sum_n \frac{\lambda_n}{n+t}|dt \le \sum_n \int_0^1 \frac{|\lambda_n|}{n+t}|dt = \sum_n |\lambda_n| \ln(1+1/n),$$

it follows, by Hölder inequality using $\ln(1+1/n) < 1/n$ that

$$\int_0^1 |\varphi x(t)|dt \le \sqrt{\sum_n |\lambda_n|^2}\sqrt{\sum_n \left(\frac{1}{n}\right)^2} < \infty.$$

Thus x is Dunford, hence Pettis, integrable on $(0,1)$ since ℓ_2 is reflexive. To calculate the pettis integral of x on $(0,1)$, we note that

$$\int_0^1 \varphi x(t)dt = \sum_n \lambda_n \ln(1+1/n) = \varphi(g(t)),\ g(t) := \{\ln(1+1/n)\}.$$

Since

$$\|g(t)\|_{\ell_2}^2 = \sum_n (\ln(1+1/n))^2 \le \sum_n \left(\frac{1}{n}\right)^2 < \infty,\ t \in (0,1).$$

we see that $g \in \ell_2$. Thus

$$(P)\int_0^1 xdt = \left\{\ln(1+\frac{1}{n})\right\}$$

We also note that

$$(P)\int_0^t x(s)\,ds = \left\{\ln(1+\frac{t}{n})\right\},\ t\in(0,1).$$

Further examples:

Example 8.24. *(A Dunford, not Pettis integrable function)*
Define $x:(0,1]\to C_0$ by

$$x(t) = \left\{2^n \chi_{(\frac{1}{2^n},\frac{1}{2^{n-1}}]}(t)\right\}.$$

If $\{\lambda_n\}\in\ell_1$ denotes the corresponding to $\varphi\in C_0^*$, then

$$\varphi x = \sum_n 2^n\lambda_n\chi_{(\frac{1}{2^n},\frac{1}{2^{n-1}}]},\ \text{and}\ \int_0^1 |\varphi x(t)|dt \le \sum_n |\lambda_n| < \infty,$$

Thus x is Dunford integrable on $(0,1]$. On the other hand, it is easily seen that for each measurable J

$$(D)\int_J x(t)dt = \left\{2^n\mu\left[J\cap\left(\frac{1}{2^n},\frac{1}{2^{n-1}}\right]\right]\right\}.$$

In particular,

$$(D)\int_{(0,1]} x(t)dt = (1,1,1,\cdots)\in\ell_\infty/C_0,$$

and so x is not Pettis integrable

Example 8.25. *(A Dunford, Denjoy-Pettis but not Pettis integrable function)*
For each $n\in\mathbb{N}$ let

$$I_n := \left(\frac{1}{n+1},\frac{n+1/2}{n(n+1)}\right),\ \text{and}\ J_n := \left(\frac{n+1/2}{n(n+1)},\frac{1}{n}\right)$$

and define the real-valued function $x_n:[0,1]\to\mathbb{R}$ by

$$x_n(t) := n(n+1)[\chi_{I_n}(t)-\chi_{J_n}(t)].$$

Then the sequence $\{x_n\}$ converges to 0 pointwise. Let the strongly measurable function $x:[0,1]\to C_0$ be defined by $x:=\{x_n\}$. The function x is Dunford integrable. Evidently, if we let $\{\lambda_n\}\in\ell_1$ to be the corresponds to $\varphi\in C_0^*$, then $\varphi x = \sum_n \lambda_n x_n$. Because of

$$\begin{aligned}\int_0^1 |x_n(t)|dt &= \int_0^1 n(n+1)|\chi_{I_n}(t)-\chi_{J_n}(t)|dt = \int_{I_n} n(n+1)dt + \int_{J_n} n(n+1)dt\\ &= n(n+1)\left[\frac{1}{n}-\frac{1}{n+1}\right] = 1,\end{aligned}$$

we deduce that

$$\sum_n \int_0^1 |\lambda_n x_n(t)| dt = \sum_n |\lambda_n| \int_0^1 |x_n(t)| dt = \sum_n |\lambda_n| < \infty,$$

by Beppo-Levi Theorem 2.3 applies to show that φx is Lebesgue (Hence x is Dunford) integrable on $[0,1]$. Now, we will show that, the strongly measurable function $x : [0,1] \to C_0$ is Denjoy-Pettis but it is not Pettis integrable on $[0,1]$. To see this, we notice that for any $J \subset [0,1]$ we have

$$\int_J \varphi x = \sum_n \int_J \lambda_n x_n = \sum_n \lambda_n \int_J x_n(t) dt = \varphi(g(t)), \text{ where } g := \left\{ \int_J x_n \right\}.$$

Now, we consider the following cases

1. For $J = [0,1]$, we have

$$g = \left\{ \int_0^1 x_n(t)\ dt \right\} = (0, 0, \cdots) \in C_0.$$

2. For any subinterval $J \in [0,1]$, we have $\left\{ \int_J x_n(t)\ dt \right\} \to 0$, that is

$$g = \left\{ \int_J x_n(t)\ dt \right\} \in C_0.$$

3. For the measurable subset $J = \cup_n I_n$, we have

$$g = \left\{ \int_J x_n(t)\ dt \right\} = \{n(n+1)[\mu(I_n \cap J) - \mu(J_n \cap J)]\} = (\frac{1}{2}, \frac{1}{2}, \cdots) \in \ell_\infty / C_0.$$

By the first and second cases, it follows that x is Denjoy-Pettis integrable on $[0,1]$ and $(DP) \int_0^1 x = (0, 0, \cdots)$. However, we recall that the necessary condition for the weakly measurable function x to be Pettis integrable on I, is that x is Pettis integrable on every measurable subset $J \in [0,1]$, therefore x is not be Pettis integrable on $[0,1]$.

Remark!: It is very important to remark here that, the function x is Example 8.25 is Dunford integrable on $[0,1]$ and the set of integrals $(D) \int_J x(s) ds$ belong to the canonical image of C_0 in ℓ_∞ (Note that the canonical image of C_0 in ℓ_∞ is C_0 itself).

Example 8.26. *(A pettis but not Bochner integrable function)*
Let r_k be a listing of the rational numbers in $[0,1)$ and for each pairs of positive integers n, j and k, define

$$I_{n,k} := \left(r_k + \frac{1}{1+n}, r_k + \frac{1}{n} \right), \text{ and } J_{i,k} := \left(r_k - \frac{1}{2^{j+k}}, r_k + \frac{1}{2^{j+k}} \right).$$

Let $A_j = \bigcup_k J_{j,k}$ and define $A := \bigcap_j A_j$. Clearly $\{r_k\} \in A$ and $\mu(A) = 0$ since $\mu(A) \leq \mu(A_j) < 2^{1-j}$ for each j.

For each k define $x_k : [0,1] \to \ell_2$ by

$$x_k(t) := \{2^{-2k}(n+1)\chi_{I_{n,k}}(t)\}.$$

The function x_k is well-defined. To see this, fix for a moment k and note that for any $t \notin A$, then $t \notin A_{j_0}$ for some j_0, it follows for $t \in [0,1] - A_{j_0}$ that

$$\chi_{I_{n,k}}(t) = \begin{cases} 0 & \text{if } I_{n,k} \subset J_{j_0,k}, \\ 1 & \text{otherwise.} \end{cases} = \begin{cases} 0 & \text{if } n > 2^{j_0+k}, \\ 1 & \text{if } n \leq 2^{j_0+k}. \end{cases}$$

On the other hand, for a fixed k, the intervals $\{I_{n,k}\}_n$ are disjoint, that is, if $t \in I_{n_0,k}$ for some positive integer n_0, then $t \notin I_{n,k}$ for $n \neq n_0$. So $x_k(t) = (0, 0, \cdots, n_0+1, 0, \cdots)$ for $t \in I_{n_0,k}$. Thus, for each k, we have $\|x_k(t)\| \leq 2^{-2k}[2^{j_0+k}+1] = 2^{j_0-k}+2^{-2k}$. Consequently x_k is ℓ_2-valued almost everywhere for all k. Moreover, we have

$$\sum_k \|x_k(t)\| \leq 2^{j_0} + 4/3 < \infty, \text{ for } t \in [0,1] - A_{j_0}.$$

Recall that in a Banach spaces the series converges if and only is it is absolutely summable, that is $\sum \|\cdot\| < \infty$ "see [124]". Hence the series $\sum_k x_k(t)$ of elements of the space ℓ_2 converges to an element belongs to ℓ_2. So, we are able to define a (countable-valued function, hence strongly measurable) function $g : [0,1] \to \ell_2$ by

$$g(t) := \begin{cases} \displaystyle\sum_k x_k(t) & \text{if } t \in [0,1] - A, \\ 0 & \text{if } t \in A. \end{cases}$$

Because of $\sum_k x_k(t) = \sum_k \{4^{-k}(n+1)\chi_{I_{n,k}}(t)\} = \{\sum_k 4^{-k}(n+1)\chi_{I_{n,k}}(t)\}$, the function $g(t) = \{g_n(t)\}$ where where $g_n : [0,1] \to \mathbb{R}$ given by

$$g_n(t) := \begin{cases} \displaystyle\sum_k 4^{-k}(n+1)\chi_{I_{n,k}}(t), & t \in [0,1] - A, \\ (0, 0, \cdots), & t \in A. \end{cases}$$

In order to show that g is Pettis integrable on $[0,1]$, we note that the space ℓ_2 is reflexive, so we need only to prove that φg is Lebesgue integrable on $[0,1]$ for each $\varphi \in \ell_2^* = \ell_2$. To do this, we note for any $t \in [0,1]$ that

$$\sum_{k=1} \int_0^1 4^{-k}(n+1)\chi_{I_{n,k}}(t)dt \leq \sum_{k=1} \frac{4^{-k}}{n} = \frac{1}{3n}.$$

So by Beppo-Levi Theorem, g_n is Lebesgue integrable on $[0,1]$ for each n. Now let $\{\alpha_n\} \in \ell_2$. Then $\varphi g = \sum_n \alpha_n g_n$. By Hölder inequality it follows that

$$\sum_n \int_0^1 |\alpha_n g_n| \le \sum_n \frac{\alpha_n}{3n} \le \sqrt{\sum_n |\alpha_n|^2} \sqrt{\sum_n \left(\frac{1}{3n}\right)^2} < \infty.$$

Again, Beppo-Levi Theorem yields the Lebesgue integrability of φg on $[0,1]$, hence g is Dunford integrable on $[0,1]$. Consequently, the reflexivity of ℓ_2 yields the Pettis integrability of g on $[0,1]$. Now, we will show that g is not Bochner integrable on $[0,1]$. To do this, we let $[a,b] \subset [0,1]$ and choose a rational number r_k and a positive integer N such that $(r_k, r_n + 1/N) \subset (a,b)$. We then have

$$\int_a^b \|g\| \ge \int_{r_k}^{r_k + \frac{1}{N}} \|g\| = 4^{-k} \sum_{n=N}^{\infty} \frac{1}{n} = \infty.$$

Therefore g is not Bochner integrable on $[a,b]$.

Example 8.27. *(A pettis but not Bochner integrable function)*
Let E be an infinite dimension Banach space. By Dvoretzky-Rogers [68] there exists a series $\sum x_n$ in E that unconditionally convergent (see Definition 2.1) but not absolutely convergent. Consider the partition $\{I_n\}_{n=1}$ of the interval $[0,1]$, where $I_n := (\frac{1}{n+1}, \frac{1}{n})$. Define a function $f : [0,1] \to E$ by

$$f(t) := \frac{x_n}{\mu(I_n)}, \text{ for } t \in I_n \text{ and } f(t) = \{0, 0, \cdots\} \text{ for all other values of } t.$$

In other words

$$f(t) := \sum_{n=1}^{\infty} \frac{x_n}{\mu(I_n)} \chi_{I_n}(t), \ t \in [0,1].$$

Obviously f is countable-valued function, hence strongly measurable but it is not Bochner integrable on $[0,1]$ since

$$\int_0^1 \|f(t)\| \, dt = \sum_n \int_{I_n} \|f(t)\| \, dt = \sum_n \int_{I_n} \frac{\|x_n\|}{\mu(I_n)} dt = \sum_n \|x_n\| = \infty.$$

To show that f is Pettis integrable on $[0,1]$, we let $\varphi \in E^*$. Then

$$\int_0^1 |\varphi f(t)| dt = \sum_n \int_{I_n} |\varphi f(t)| dt = \sum_n \int_{I_n} \frac{|\varphi x_n|}{\mu(I_n)} dt = \sum_n |\varphi x_n| < \infty,$$

since the series $\sum_n x_n$ is unconditionally convergent. Hence, the function f is Dunford integrable on $[0,1]$. Let $J \subset [0,1]$ be measurable subset of $[0,1]$, then for each $\varphi \in E^*$, we have

$$\begin{aligned}\int_J \varphi f(t)dt &= \sum_n \int_{J\cap I_n} \varphi f(t)dt = \sum_n \frac{\mu(J\cap I_n)}{\mu(I_n)}\varphi x_n \\ &= \varphi\left(\sum_n \frac{\mu(J\cap I_n)}{\mu(I_n)}x_n\right) = \varphi x_J,\end{aligned}$$

where x_J denotes the sum of the unconditionally convergent series $\sum_n \frac{\mu(J\cap I_n)}{\mu(I_n)}x_n$. This show that f is Pettis integrable on $[0,1]$. Also

$$\int_0^1 f = \sum_n x_n.$$

Alternatively, we have

Theorem 8.5. [128, Corollary 5.1] *Define the strongly measurable $f : I \to E$ by*

$$f := \sum_{n=1}^{\infty} x_n \chi_{A_n},$$

where $\{A_n\}$ is a sequence of pairwise disjoint, Lebesgue measurable subsets of I and $\{x_n\}$ a sequence in E. Then

1. *f is Bochner integrable on I if and only if, $\sum_{n=1}^{\infty} \|x_n\|\,\mu(A_n) < \infty$,*
2. *f is Pettis integrable on I if and only if, $\sum_{n=1}^{\infty} x_n\mu(A_n)$ is unconditionally convergent in E.*

Moreover, if either of the integrals exists, then it equals $\sum_{n=1}^{\infty} x_n\mu(A_n)$.

For any $x \in P[I,E]$ and any $\varphi \in E^*$, we know by the definition of Pettis integral that $\varphi x \in L_1[0,1]$. It follows that the integral $\int_J |\varphi(x(s))|\,ds,\ J \subset I$ is absolutely continuous for any $\varphi \in E^*$, that is

$$\int_J |\varphi x(s)|\,ds \to 0 \text{ whenever } \mu(J) \to 0.$$

A consequence, it can be easily seen that

Theorem 8.6. [130] *Let $x \in P[I,E]$ and define the set function $X(J) := (P)\int_J x(s)\,ds$, where J measurable sub-set of I. Then X is countably additive* *vector measure defined on I. Moreover, X is AC with respect to the measure μ defined on I (That is*

$$\left\|(P)\int_J x(s)\,ds\right\| < \epsilon, \text{ for } J \subset I,\ \mu(J) < \delta.)$$

Remark 8.3. The vector measure X defined in Theorem 8.6 is called the indefinite integral of x. Also, if we consider the measure space of the Lebesgue $\sigma-$algebra "where the sub-intervals are measurable", we have

$$X(t) := X([0,t]) = (P)\int_a^t x(s)\,ds.$$

Moreover, if $x \in P[I,E]$ and $J := \{(a_k,b_k)\}_{k=1}^m$ be a finite disjoint family of sub-intervals of $I := [a,b]$ such that $\mu(J) < \delta$ it follows by Theorem 8.6 that

$$\left\|\sum_k X(b_k) - X(a_k)\right\| = \left\|\sum_k (P)\int_{a_k}^{b_k} x(s)\,ds\right\| = \left\|(P)\int_J x(s)\,ds\right\| < \epsilon,$$

which is the classical definition of the absolute continuity on I (see Definition 8.5).

Example 8.28. Consider the function x defined in Example 8.4. This function is strongly (hence weakly) measurable on $[0,1]$, since it is a countable-valued function.

It was showed in Example 8.16 that x is Dunford, but it is not Pettis integrable on $[0,1]$. However, for $n \in \mathbb{N}$ we have

$$\int_0^{1/n} \varphi x = \int_0^{1/n} \sum_k \lambda_k k \chi_{(0,1/k]} = \frac{\lambda_1}{n} + \frac{2\lambda_2}{n} + \cdots + \frac{n\lambda_n}{n} + \lambda_{n+1} + \cdots ,$$

Thus, if x is Pettis integrable, we arrive at

$$(P)\int_0^{1/n} x dt = \{\frac{1}{n}, \frac{2}{n}, \cdots, \frac{n}{n}, 1, 1, \cdots\} \in \ell_\infty,$$

and

$$\left\|(P)\int_0^{1/n} x dt\right\|_\infty = 1.$$

Thus the indefinite Dunford integral is not AC. In the view of Theorem 8.6, the function x is not Pettis integrable on $[0,1]$.

Based on Theorem 8.6, it is not hard to prove the following

Proposition 8.4. *(*[130] *Corollary 2.51). If $x \in P[I,E]$, then for any bounded subset Ω of elements of E^*, the integrals*

$$\int_J |\varphi(x(s))|\,ds,\ \varphi \in \Omega$$

are weakly equi-absolutely continuous. Meaning that, for every $\epsilon >$, the exists $\delta = \delta(\epsilon)$ such that

$$\int_J |\varphi(x(s))|\, ds < \epsilon, \text{ whenever } \mu(J) < \delta,$$

holds for every $\varphi \in \Omega$. That is

$$\lim_{\mu(J)\to 0} \sup_{\varphi\in\Omega} \int_J |\varphi x(s)|\, ds = 0, \; x \in P[I, E].$$

Proof. Let $k > 0$ be a number such that $\|\varphi\| \leq k$ for every $\varphi \in \Omega$. In the view of the definition of Pettis integral and Theorem 8.6, we have

$$\left|\int_J \varphi x(s)\, ds\right| = \left|\varphi\left[(P)\int_J x(s)\, ds\right]\right| \leq \|\varphi\| \left\|(P)\int_J x(s)\, ds\right\| \leq k\frac{\epsilon}{k} = \epsilon,$$

for $\mu(J)$ less than a suitably chosen δ. Thus, if $J_\varphi^+ := J[\varphi x(t) \geq 0]$ and $J_\varphi^- := J[\varphi x(t) < 0]$, we have, by noting the graph of $|\varphi x|$

$$\int_J |\varphi x(s)|\; ds = \left|\int_{J_\varphi^+} \varphi x(s)\, ds\right| + \left|\int_{J_\varphi^-} \varphi x(s)\, ds\right| < 2\epsilon,$$

for $\mu(J) < \delta$. □

Example 8.29. In Example 8.16, we show that the strongly measurable function $x : [0, 1] \to C_0$ defined in Example 8.4 is not Pettis integrable on $[0, 1]$. This can be readily proved in different way making up Proposition 8.4. To do this, define for each set $J_n = [0, 1/n]$ the functionals $\varphi_n \in E^*$ (required by Proposition 8.4) to be the corresponding to the element $\{\lambda_n\} := (0, 0, \cdots, 0, 1, 0, 0, \cdots) \in \ell_1$ where the non-zero coordinate is in the $n-$th place. Then $\varphi_n(x(t)) = x_n(t)$ and thus $|\varphi_n(x(s))| = n\chi_{[0,1/n]}(s)$. Clearly, the family $\{\varphi_n\}$ runs through the unit ball of C_0^* since $|\varphi_n x| = |x_n| \leq \sup_n |x_n| = \|x\|$, that is $\|\varphi_n\| \leq 1\ \forall n$. Also we have

$$\int_{J_n} |\varphi_n(x(s))|\, ds = \int_{J_n} n\chi_{[0,1/n]}(s)\, ds = \int_0^{\frac{1}{n}} n\, ds = 1,$$

Therefore, by Proposition 8.4, x does not Pettis integrable on [0,1].

Now, recall that a strongly absolutely continuous function need not to be strongly differentiable anywhere (see Example 8.10). On the other hand, in what follows, we will show (and a bit more) that the indefinite Bochner (Pettis) integrable function is strongly (weakly) absolutely continuous and possesses a strong (pseudo) derivative almost everywhere. In fact, we have

Lemma 8.17. [125, 130]*(see also (* [99] *page 88)*
The indefinite integral of weakly continuous (Bochner integrable) (Pettis integrable) function is weakly (strongly) (weakly) absolutely continuous and it is

weakly (a.e. strongly) (pseudo) differentiable with respect to the right endpoint of the integration interval and its weak (strong) (pseudo) derivative equals the integrand at that point.

Proof. For the Bochner integral case, the proof is the same as for the real-valued case: If $x : [0, b] \to E$ is Bochner integrable, then the real-valued function $t \to \int_0^t \|x(s)\|)\, ds$ is absolutely continuous on $[0, b]$. Consequentially, by the formula (209), it follows that the integral $(B) \int_0^t x(s)\, ds$ is AC function (see Remark 8.3). Proceed as the same as the for the real-valued case, it can be shown that the integral $(B) \int_0^t x(s)\, ds$ is strongly differentiable with respect to t.

The proof for the Pettis integral case is similar.

□

Remark 8.4. We remark here that, J. Pettis closed his original paper (in 1939) by asking whether the Pettis integral of Pettis integrable function enjoys the stronger property of being a.e. weakly differentiable (that is, the null set independent on φ)). The answer, in general, is negative: However, if E is finite dimensional, then the indefinite integral of a Pettis integrable function is a.e. weakly differentiable. R.S. Phillips [134] (for $E = \ell_2$) and M.E. Munroe [81] (for $E = C[0, 1]$) each constructed an example of a Pettis integrable function whose indefinite Pettis integral is not a.e. weakly differentiable. In (1994) Kadets [101] prove that there exists a strongly measurable Pettis integrable function such that the indefinite Pettis integral is not weakly differentiable on a set of positive Lebesgue measure (That is, not weakly differentiable on a set of non-zero measure). In (1995) Dilworth and Girardi [64] showed that always there exists a strongly measurable Pettis integrable function such that the indefinite Pettis integral is nowhere weakly differentiable.

In the following lemma, we show that weakly absolutely continuous function is pseudo differentiable if, and only if, it is an indefinite pettis integral of a Pettis integrable function

Lemma 8.18. *(* [130] *p. 301) The weakly measurable function $x : [0, b] \to E$ is Pettis integrable on $[0, b]$, if and only if, there exists a weakly absolutely continuous $y : [0, b] \to E$ which has a pseudo derivative x on $[0, b]$. The Pettis integral of x over $[0, b]$ is y. In other words,*
The function $y : [0, b] \to E$ is an indefinite Pettis integrable, if and only if, y weakly absolutely continuous on $[0, b]$ and have a pseudo derivative on $[0, b]$. In this case, y is an indefinite Pettis integral of any of its pseudo derivatives.

Precisely, if we consider the measure space of the Lebesgue $\sigma-$algebra, we have

1. *If the weakly measurable function* $x : [0, b] \to E$ *is Pettis integrable on* $[0, b]$*, then the indefinite Pettis integral*

$$y(t) = \int_0^t x(s)\, ds,\ t \in [0, b],$$

is weakly absolutely continuous (even AC *if* E *is weakly sequentially complete space "cf. Lemma 8.4") on* $[0, b]$ *and* x *is a pseudo derivative of* y*,*

2. *If* y *is a weakly absolutely continuous on* $[0, b]$ *and it has pseudo derivative* x *on* $[0, b]$*, then* $x : [0, b] \to E$ *is Pettis integrable on* $[0, b]$ *and*

$$y(t) = y(0) + \int_0^t x(s)\, ds,\ \textit{for}\ \ t \in [0, b].$$

Proof. The sufficiency is a direct consequence of Lemma 8.17 (If E is weakly sequentially complete, then the claim that y is AC follows from Lemma 8.4). On the one hand, assume that y is weakly absolutely continuous and has a pseudo derivative x. By Lemma 8.5, the function x is weakly measurable on $[0, b]$. Since φy is absolutely continuous for every $\varphi \in E^*$, it follows

$$\varphi y(t) = \int_0^t (\varphi y(s))'\, ds = \int_0^t \varphi x(s)\, ds \text{ for all } \varphi \in E^*.$$

So x is Pettis integrable on $[0, b]$ to the value y. □

The next theorem extends Lemma 8.18 from the weakly sequentially complete spaces to the context of arbitrary Banach spaces.

Theorem 8.7. *(see (* [129]*, Theorem 5.1)*
The function $y : [0, b] \to E$ *is an indefinite pettis integrable, if and only if,* y *is weakly absolutely continuous on* $[0, b]$ *and have a pseudo derivative on* $[0, b]$*. In this case,* y *is an indefinite pettis integral of any of its pseudo derivatives.*

Theorem 8.7 asserts that every function that have a Pettis integrable pesudo derivative is weakly absolutely continuous, but it can happen that an weakly absolutely continuous function is not pesudo differentiable. However, a Banach space for which every weakly absolutely continuous has a Pettis integrable pesudo derivative is said to have a *Radon-Nikodym property* (See Definition 8.12).

We remark that, there is an interesting example due to Talagrand [186] of an indefinite Pettis integral whose range is not relatively compact. Moreover, Phillip [133] introduce an example of bounded weakly measurable function which is not be Pettis integrable (see also [87]). However, in Lindelof spaces "*cf.* [71] for the definition" (In particular, reflexive or separable spaces) every bounded weakly measurable functions is Pettis integrable (see [70] and [71]).

The following theorem shows that the nature of the convergence of $\varphi(x_n)$ to $\varphi(x)$ forms a large part of the distinction between the Bochner and Pettis integrals.

Theorem 8.8. [86] *Let $x : I \to E$ is bounded function. Then*

(a) *The function x is Pettis integrable if and only if there is a bounded sequence (x_n) of simple functions from I into E with $\lim_{n\to\infty} \varphi(x_n) = \varphi(x)$ a.e. for all $\varphi \in E^*$.*
(b) *The function x is Bochner integrable if and only if there is a null set A and a sequence (x_n) of simple functions from I into E with $\lim_{n\to\infty} \varphi(x_n(t)) = \varphi(x(t))$ for all $\varphi \in E^*$ and for all $t \notin A$.*

Caution!: By Theorem 8.8, it follows easily that the distinction between the Bochner and Pettis integrals is thus the distinction between stationary exceptional null sets and the mobile exceptional null sets. Indeed, the exceptional null set in which the convergence fails in the Pettis integral case may vary with φ.

Example 8.30. Let μ denote Lebesgue measure and define $x : [0, 1) \to L_\infty(\mu)$ by $x(t) = \chi_{[0,t)}$

This function is weakly measurable ($\varphi(x)$ is a function of bounded variation), and it is not Bochner integrable (It is not even strongly measurable *cf.* Example 8.6). To show that x is Pettis integrable, it is sufficient to show that x is the limit (in the sense introduced in Theorem 8.8) of a sequence of simple functions.

Let $\Im_n := \{[\frac{i-1}{2^n}, \frac{i}{2^n})\}_{i=1}^{2^n}$ be a partition of $[0, 1)$. For any point t belonging to $\Im_n$, define

$$x_n(t) = \chi_{[0,\frac{i}{2^n})}(s) \in L_\infty(\mu).$$

This defines a sequence $\{x_n\}$ of simple functions from $[0, 1)$ into $L_\infty(\mu)$. It is noted that (see e.g. [86]) any element $\varphi \in L^*_\infty(\mu)$ may be identified with a bounded additive measure ζ that vanishes on sets of $\mu-$measure 0. It is follows from this identification that

$$\varphi(x_n(t)) = \zeta([0, \frac{i}{2^n})) \text{ if } t \in [\frac{i-1}{2^n}, \frac{i}{2^n}),$$

and

$$\varphi(x(t)) = \zeta([0, t)) \text{ for all } \ t \in [0, 1).$$

Fix $t \in [0, 1)$ and let $A_{t,n} \in \Im_n$ be the element which contains t, hen we have

$$|\varphi(x(t)) - \varphi(x_n(t))| = |\zeta([0, t)) - \zeta([0, \frac{i}{2^n}))| \leq |\zeta(A_{t,n})| \leq |\zeta|(A_{t,n}).$$

It is follows from the boundedness of ζ that $\lim_{n\to\infty} |\zeta|(A_{t,n}) = 0$ for all (but countably) many t. We thus see that

$$\lim_{n\to\infty} \varphi(x_n) = \varphi(x) \ a.e. \ \text{ for all } \varphi \in L^*_\infty(\mu).$$

Thus, by Theorem 8.8, x is Pettis integrable on $[0, 1)$. To calculate the Pettis integral of x, we let $\rho \in L_1$ and let φ be the element in L^*_∞ corresponding to ρ. For any $t > 0$, we have

$$\begin{aligned}
\int_0^t \varphi\left(x(s)\right)\, ds &= \int_0^t \int_0^1 \rho(\theta)\chi_{[0,s]}(\theta) d\theta\, ds = \int_0^t \int_0^s \rho(\theta) d\theta\, ds \\
&= \int_0^t \int_\theta^1 \rho(\theta)\, ds d\theta = \int_0^t (1-\theta)\rho(\theta) d\theta \\
&= \int_0^1 (1-\theta)\rho(\theta)\chi_{[0,t]}(\theta) d\theta.
\end{aligned}$$

Consequently, we conclude that

$$\left[\int_0^t x(t) dt\right](\cdot) \ = (1-\cdot)\chi_{[0,t1]}(\cdot) \in L_\infty.$$

The Lebesgue dominated convergence theorem holds also for Pettis and Bochner integrals. The proof is the same as for the scalar-valued case, so we omit it.

Theorem 8.9. ***(Dominated convergence theorem for Bochner integral)***
Let $\{x_n\}$ be a sequence of Bochner integrable functions and $\|x - x_n\| \to 0$ almost everywhere. If there exists a Lebesgue integrable function g such that $\|x\| \leq g$ almost everywhere, then x is Bochner integrable and $\lim_{n\to\infty} \int_J x_n\, ds = \int_J x\, ds$ for all measurable set $J \subset I$.

Now, we recall the following results from the literature on Pettis integrals (see [86], [99] and [130])

Theorem 8.10. ***(Dominated convergence theorem for Pettis integral)***
Let $x : I \to E$. Suppose there is a sequence (x_n) of Pettis integrable functions from I into E such that $\lim\limits_{n\to\infty} \varphi(x_n) = \varphi(x)$ a.e. for $\varphi \in E^*$ (the null set

on which convergence fails may vary with φ). If there is a scalar function $\psi \in L_1(I)$ with $\|x_n(\cdot)\| < \psi(\cdot)$ a.e. for all n, then x is pettis integrable and

$$\int_J x_n(s)\,ds \to \int_J x(s)\,ds \text{ weakly for each measurable } J \subset I.$$

Theorem 8.11. ***(Mean value theorem for Pettis integral)***
If the function $x : I \to E$ is Pettis integrable on I, then

$$\int_J x(s)\ \ ds \in |J|\overline{conv}(x(J)),$$

where $J \subset I$, $|J|$ is the length of J and $\overline{conv}(x(J))$ is the closed convex hull of $x(J)$.

The following results probably are the deepest assertions proved by Pettis in [130] (Theorem 3.4 and Corollary 3.41).

Proposition 8.5. *In order that $x(\cdot)$ be in $\mathcal{H}_0^p(E)$, it is necessary and sufficient that $x(\cdot)u(\cdot)$ be Pettis integrable for every $u(\cdot) \in L_q(I)$.*

Proposition 8.6. *If $x : I \to E$ is Pettis integrable and u is a measurable and essentially bounded real-valued function, then $x(\cdot)u(\cdot)$ is Pettis integrable.*

Definition 8.12.
1. It is well-known (*cf.* Theorem 8.6) that, if the triple (I, Σ, μ) is a finite measure space, then indefinite Pettis integral of a function $x : I \to E$ defines a countably additive vector measure $\Im_x : \Sigma \to E$ (see also [70]). However, we will say that the Banach space E have the *Radon-Nikodym property* (*weak Radon-Nikodym property*) if and only if, for any probability measure space (I, Σ, μ) and any additive vector measure $\Im : \Sigma \to E$ with $\|\Im(J)\| \leq \mu(J)$ for all $J \in \Sigma$, it follows that $\Im$ is the indefinite integral of Bochner (Pettis) integrable function $x : I \to E$.
   It is known that the space ℓ_1 has the *Radon-Nikodym property*, but C_0 and the spaces $L_\infty(\Omega)$, $L_1(\Omega)$, for Ω an open bounded subset of $\mathbb{R}^n$ (see e.g Example 8.10), and $C(K)$, for K an infinite compact space, do not. Spaces with *Radon-Nikodym property* include separable, dual spaces (this is due to the Dunford-Pettis theorem) and reflexive spaces, which include, in particular, Hilbert spaces.
2. A Banach space E will be said to have the `property` (D), if every Lipschitz continuous function x from $J \subset \mathbb{R}$ to E is strongly differentiable a.e. on J. Example for the spaces with the property (D) are furnished by the uniform convex, the reflexive and locally weakly compact spaces (see Pettis [130] and [132] p. 262). As the Example 8.10 shows, the space $L_1(0, 1)$ does not has this property.

Proposition 8.7. [70, 71] *If the Banach space E have the Radon-Nikodym property, then very sAC (Hence every wAC, in case when E is weakly complete cf. Lemma 8.4) function has a Bochner (trivially Pettis) integrable weak derivative.*

We close this section by introducing the relation between the distributionaly and the weak derivative:

Let $L^1_{\text{Log}}[I, E]$ denote the space of strongly measurable functions $x : I \to E$ that are locally Bochner integrable on every compactly supported interval $J \subset I$. Also let $C_c^\infty(I)$ denote the space of smooth real-valued functions $\phi : I \to \mathbb{R}$ with compact support.

Definition 8.13. A function $x \in L^1_{\text{Loc}}[I, E]$ is distributionaly differentiable, with distributional derivative $x' = y \in L^1_{\text{Loc}}[I, E]$ if

$$(B)\int_I \phi' x \ \ ds = -(B)\int_I \phi y \ \ ds, \text{ for evey } \phi \in C_c^\infty(I).$$

Lemma 8.19. *For the Bochner integrable function $x \in L^1_{Loc}[I, E]$, the weak and the distributional derivatives are equivalent.*

Proof. On the one hand, if $x' = y$ then

$$\int_I \phi' x \ \ ds = -\int_I \phi y \ \ ds, \text{ for evey } \phi \in C_c^\infty(I),$$

where the integrals understood as a Bochner integrals. Acting by $\varphi \in E^*$ and using the continuity of the integral (Lemma 8.8) we get

$$\int_I \phi'(\varphi x) \ \ ds = -\int_I \phi(\varphi y) \ \ ds, \text{ for evey } \phi \in C_c^\infty(I),$$

Thus, the real function φx is distributionaly differentiable and $(\varphi x\ )' = \varphi y$ for all $\varphi \in E^*$. The converse is also easy to be proved. $\square$

9. Fractional Calculus of Vector-Valued Functions

Throughout this section, we outline some aspects of fractional calculus in Banach spaces. In what follows, we let μ denotes the Lebesgue measure and let $I := [0, b]$ of $\mathbb{R}$ endowed with the Lebesgue $\sigma-$algebra $\Delta(I)$ (that is, the sigma-algebra generated by the topology induced by the set $\tau \bigcup \Omega$, where τ is the topology induced by the Euclidean metric on $\mathbb{R}$ and Ω is the collection of null sets in $\mathbb{R}$).

Now we are in the position to define and study the fractional calculus in the Pettis and Bochner spaces. Firstly, we consider the particular case, where the integrals are in the Bochner sense. Secondly, based on the linear functional over a Banach space E and on the properties of the fractional calculus of real-valued functions, we define the fractional Pettis integrals of $E-$valued functions and the corresponding fractional derivatives. Also, we will show that the well-known properties of the fractional calculus over the domains of Lebesgue integrable functions also hold in the Pettis spaces. To encompass the full scope of this book, we apply our abstract results to investigate some existence result to some fractional order integral and differential equations in the Banach space $C[I, E]$.

Definition 9.1. Let $x : I \to E$. The fractional (arbitrary order) Pettis (Bochnar)-integral "shortly FPI (FBI)" of x of order $\alpha > 0$ is defined by

$$I^\alpha x(t) := \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} x(s)\, ds,\ t \in I. \tag{210}$$

In the above definition the sign " $\displaystyle\int$ " denotes the Pettis (Bochnar) integral. For the existence of FBI, we have

Theorem 9.1. *$x : I \to E$ be a strongly measurable function, such that $\|x(\cdot)\| \in L_1(I)$, then* FBI *of order $\alpha > 0$*

$$I^\alpha x(t) = (B) \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} x(s)\, ds,$$

exists for almost every $t \in I$.

In the above Theorem the sign " $\int$ " denotes the Bochner (trivially Pettis) integral.

Proof. Firstly, we recall that if x is strongly measurable and g is finite numerically-valued function, then $x(\cdot)g(\cdot)$ is strongly measurable (see [99]). So, if x be strongly measurable on I then for $s < t$, the function $s \mapsto (t-s)^{\alpha-1} x(s)$ is strongly measurable as well. Since $\|x(\cdot)\|$ is Lebesuge integrable, it follows by the properties of the fractional integral operators on the Lebesgue space that

$$\int_0^t \left\| \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} x(s) \right\| ds = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \|x(s)\| \, ds < \infty \ a.e. \ t \in I.$$

The result now follows immediately by Theorem 8.4.

$\square$

Remark 9.1. Theorem 9.1 remains valid for every $t \in I$ if x is norm bounded (say by a constant K) strongly measurable function, this follows from the inequality

$$\int_0^t \left\| \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} x(s) \right\| ds \leq \frac{K t^\alpha}{\Gamma(1+\alpha)} < \infty.$$

An immediate consequence of Theorem 9.1 and the Pettis measurability theorem (Theorem 8.3) is the following

Corollary 9.1. *Let $\alpha > 0$. Assume that the function $x : I \to E$ is weakly measurable and, off null set, the function x has separable rang such that $\|x(\cdot)\| \in L_1(I)$. Then* FBI *exists for almost every $t \in I$.*

Next, we introduce the following interesting properties of FPI, which is a basic tool for achieving our aims

Lemma 9.1. *Let $\alpha > 0$. If E is reflexive and $x : I \to E$ is Pettis integrable then the* FPI *of x exists for almost every $t \in I$ and $\varphi(I^\alpha x) = I^\alpha(\varphi x)$ holds for every $\varphi \in E^*$. If, in particular, $\alpha \geq 1$, the function $I^\alpha x : I \to E$ is weakly continuous (Hence $I^\alpha x \in \mathcal{H}_0^\infty(E)$).*

Proof. Since x is Pettis integrable on I, we have $\varphi x \in L_1(I)$ for every $\varphi \in E^*$, It follows by the Young inequality (*cf.* Proposition 2.1 with $p = q = r = 1$) that the function

$$s \mapsto \varphi\left(\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} x(s) \right) = \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \varphi(x(s)),$$

is Lebesgue integrable for almost every $t \in I$. Consequently, by the reflexivity of E, it follows in the view of Lemma 8.14, that the function $s \mapsto \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} x(s)$ is Pettis integrable on I for almost every t. By the definition of the Pettis integral there exists a function denoted $I^\alpha x(\cdot)$ from I into E that satisfies

$$\varphi\left(I^\alpha x(t)\right) = \int_0^t \varphi\left(\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} x(s) \right) ds = I^\alpha(\varphi x(t)), \ \varphi \in E^*,$$

for almost every $t \in I$. Also, if $\alpha \geq 1$, we have for any $\varphi \in E^*$

$$\varphi\left(I^\alpha x(t)\right) = I^\alpha \varphi x(t) = I I^{\alpha-1} \varphi x(t).$$

Since $\varphi x \in L_1(I)$ (and so is $I^{\alpha-1}\varphi x$) for every $\varphi \in E^*$, then $I^\alpha x$ is weakly continuous function from I to E. $\square$

A deep result is the following

Corollary 9.2. *Let E be reflexive. For any $\alpha > 0$ and arbitrary $p \geq 1$, the operator I^α takes $\mathcal{H}_0^p(E)$ into $\mathcal{H}_0^p(E)$ and is well-defined. In particular*

1. *For $\alpha \geq 1$, I^α maps $\mathcal{H}_0^p(E)$ into $\mathcal{H}_0^\infty(E)$ for any $p \geq 1$,*
2. *For $\alpha \in (0,1)$, I^α maps $\mathcal{H}_0^p(E)$ into $\mathcal{H}_0^{1/(1-\alpha)}(E)$ for any $p \geq 1$.*

Proof. At the beginning, we note in the view of Lemma 9.1 that, the FPI of any $x \in \mathcal{H}_0^p(E),\ p \geq 1$ exists for almost every $t \in I$ and $\varphi\left(I^\alpha x\right) = I^\alpha(\varphi x)$ holds for every $\varphi \in E^*$. Moreover, by Theorem 3.2 it follows that $\varphi\left(I^\alpha x\right) \in L_p$ for every $\varphi \in E^*$, meaning that $I^\alpha x \in \mathcal{H}^p(E)$. The reflexivity of E together with Lemma 8.14 result in $I^\alpha x \in \mathcal{H}_0^p(E)$. Now, we consider the following particular cases

1. For $\alpha \geq 1$, it follows by Lemma 9.1 that $I^\alpha x$ is weakly continuous and so $I^\alpha x \in \mathcal{H}_0^\infty(E)$.
2. Let $\alpha \in (0,1)$, $x \in \mathcal{H}_0^p(E) \subset \mathcal{H}_0^1(E)$ and note that, the function $g(t) := \frac{t^{\alpha-1}}{\Gamma(\alpha)},\ 0 < \alpha < 1$, belongs to the space $L_{1/(1-\alpha)}[I]$. Thus, by the Young inequality with $r = q = 1/(1-\alpha),\ p = 1$ it follows that the function
$$s \mapsto \varphi\left(\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}x(s)\right) = \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\varphi(x(s)),$$
is Lebesgue integrable for almost every $t \in I$ and $I^\alpha \varphi x = \varphi x * g \in L_{1/(1-\alpha)}[I]$. By the reflexivity of E, it follows in the view of Lemma 8.14, that the function $s \mapsto \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}x(s)$ is Pettis integrable on I for almost every t. Thus, the operator I^α makes sense. Further, I^α is well-defined. To see this, we define $y : I \to E$ by $y(t) := I^\alpha x(t),\ x \in \mathcal{H}_0^1(E)$. Then, for every $\varphi \in E^*$, we have
$$\varphi(y(t)) = \int_0^t \varphi\left(\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}x(s)\right) ds = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\varphi(x(s))\, ds = I^\alpha \varphi(x(t)).$$
It follows that $\varphi y(\cdot) = I^\alpha \varphi x(\cdot) \in L_{1/(1-\alpha)}[I]$ for every $\varphi \in E^*$. That is $y \in \mathcal{H}^{1/(1-\alpha)}(E)$. The reflexivity of E together with Lemma 8.14 result in $y \in \mathcal{H}_0^{1/(1-\alpha)}(E)$ which completes the proof.

$\square$

Example 9.1. Consider the Pettis integrable function function $y : (0,1) \to L_2(0,1)$ given in Example (8.21). By Lemma 9.1, the FPI of y exists for almost every $t \in I$. To calculate $I^\alpha y$, let $\rho \in L_2(0,1)$ be the element corresponding to $\varphi \in L_2(0,1)^*$. Then

$$\begin{aligned}
\int_0^t \varphi\left(\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\right) ds &= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\varphi(y(s))\,ds \\
&= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \int_0^1 \rho(\theta)y(s)(\theta)d\theta\,ds \\
&= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \int_0^1 \rho(\theta)\chi_{(0,s)}(\theta)d\theta\,ds \\
&= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \int_0^s \rho(\theta)d\theta\,ds \\
&= \int_0^t \int_0^s \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\rho(\theta)d\theta\,ds \\
&= \int_0^t \int_\theta^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\rho(\theta)\,dsd\theta = \int_0^t \frac{(t-\theta)^{\alpha}}{\Gamma(1+\alpha)}\rho(\theta)d\theta \\
&= \int_0^1 \frac{(t-\theta)^{\alpha}}{\Gamma(1+\alpha)}\rho(\theta)\chi_{[0,t]}(\theta)d\theta \\
&= \varphi\left(\frac{(t-s)^{\alpha}}{\Gamma(\alpha+1)}\chi_{(0,t)}(s)\right).
\end{aligned}$$

Whence

$$\begin{aligned}
\varphi\left(\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}x(s)\,ds\right) &= \int_0^t \varphi\left(\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}x(s)\right) ds \\
&= \varphi\left(\frac{(t-s)^{\alpha}}{\Gamma(\alpha+1)}\chi_{[0,t]}(s)\right).
\end{aligned}$$

Consequently, we conclude that

$$I^\alpha x(t)(\cdot) \;= \frac{(t-\cdot)^{\alpha}}{\Gamma(1+\alpha)}\chi_{(0,t)}(\cdot) \in L_2.$$

Lemma 9.2. *For any $\alpha \geq 1$ and arbitrary $p \geq 1$, the operator I^α takes $\mathcal{H}_0^p(E)$ into $\mathcal{H}_0^p(E)$ and is well-defined.*

Proof. Note first that, the function $g(t) := \frac{t^{\alpha-1}}{\Gamma(\alpha)}$, $\alpha \geq 1$ is bounded on I, that is $g \in L_\infty[I]$. Thus, for $x \in \mathcal{H}_0^p(E)$, it follows in the view of Proposition 8.6, that the operator I^α makes sense. Further, I^α is well-defined. To see this, we define $y : I \to E$ by $y(t) := I^\alpha x(t)$, $x \in \mathcal{H}_0^p(E)$. By Theorem 3.2 it follows that $\varphi y(\cdot) = I^\alpha \varphi x(\cdot) \in L_p$ for every $\varphi \in E^*$, meaning that $y \in \mathcal{H}^p(E)$. Since $x \in P[I, E]$ then x is Pettis integrable on $[0, b]$ for any $b \in I$. So the function $s \mapsto (b-s)^\alpha x(s)$ is Pettis integrable on $[0, b]$. That is, there exist an element

$x_{[0,b]} \in E$ such that every $\varphi \in E^*$, we have

$$\varphi x_{[0,b]} := \frac{1}{\Gamma(1+\alpha)} \int_0^b (b-s)^\alpha \varphi(x(s))\, ds.$$

Now, since φy is integrable, we have

$$\begin{aligned}\int_0^b \varphi y(t)dt &= \int_0^b I^\alpha \varphi x(t)dt = \frac{1}{\Gamma(\alpha)} \int_0^b \int_0^t (t-s)^{\alpha-1} \varphi x(s)\, dsdt \\ &= \frac{1}{\Gamma(\alpha)} \int_0^b \int_s^b (t-s)^{\alpha-1} \varphi x(s) dt\, ds \\ &= \int_0^b \frac{(b-s)^\alpha}{\Gamma(1+\alpha)} \varphi x(s)\, ds = \varphi(x_{[0,b]}),\end{aligned}$$

where

$$x_{[0,b]} := (P) \int_0^b \frac{(b-s)^\alpha}{\Gamma(1+\alpha)} x(s)\ \ ds.$$

A combination of these results yields $\varphi y \in L_p$ for every $\varphi \in E^*$ and there exists a $x_{[0,b]} \in E$ such that $\phi x_{[0,b]} = \int_0^b \varphi y(t)dt$ for every $b \in I$. Thus, by noticing that we consider the interval I endowed with the Lebesgue $\sigma-$algebra it follows that y is Pettis integrable on $[0,1]$ and consequently $y(\cdot) \in \mathcal{H}_0^p(E)$ and thus, we are finished. □

Lemma 9.3. *Let $\alpha > 0$ and $p > \max\{1, \frac{1}{\alpha}\}$. If $x \in \mathcal{H}_0^p(E)$, then the* FPI *of x exists for almost every $t \in I$ and $\varphi(I^\alpha x) = I^\alpha(\varphi x)$ holds for every $\varphi \in E^*$. Moreover the function $I^\alpha x : I \to E$ is weakly continuous on I.*

Proof. Fix $t \in I$ and note that, for $p > \max\{1, \frac{1}{\alpha}\}$, we have $q(\alpha-1) > -1$. Then, the real-valued function $y(\cdot) = (t-\cdot)^{\alpha-1}$, $t \in I$ belongs to the space $L_q([0,t])$. According to Proposition 8.5, the function $x(\cdot)y(\cdot)/\Gamma(\alpha)$ is Pettis integrable on $[0,t]$. Thus, the FPI

$$I^\alpha x(t) := (P) \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} x(s)\, ds,\ t \in I,$$

exists for every function $x \in \mathcal{H}_0^p(E)$ as a function from I to E. Moreover, we have

$$\varphi(I^\alpha x(t)) = \int_0^t \varphi\left(\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} x(s)\right) ds = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \varphi(x(s))\, ds, \quad (211)$$

holds for every $\varphi \in E^*$. So $\varphi(I^\alpha x(\cdot)) = I^\alpha \varphi x(\cdot)$ holds for every $\varphi \in E^*$.

Now the weak continuity of $I^\alpha x$ is a direct consequence of (211) and Theorem 3.2. (see also Young inequality with $r = \infty$).

□

An immediate consequence of Lemma 9.3, is the following

Corollary 9.3. *For any $\alpha > 0$ and $p > \max\{1, \frac{1}{\alpha}\}$, we have*

$$I^{\alpha} : \mathcal{H}_0^p(E) \to \mathcal{H}_0^{\infty}(E),$$

.

Remark 9.2. We remark here that for the interesting case (when $\alpha \in (0,1)$), Lemma 8.14, Lemma 8.15, Lemma 8.16 and Proposition 8.3 allows us to characterize the functions from $\mathcal{H}_0^p(E)$, $p > 1/\alpha$: In fact, Lemma 9.3 holds if at least one of the following cases hold

1. If x is a.e. weakly continuous (in this case we have $x \in \mathcal{H}^{\infty}(E)$),
2. If x is strongly measurable and having $\varphi x \in L^p(I),\ p > 1/\alpha$, for every $\varphi \in E^*$
3. If E is reflexive and $x \in \mathcal{H}^p(E),\ p > 1/\alpha$,
4. If $x \in \mathcal{H}^p(L_{p^*}(I)) \equiv \mathcal{H}_0^p(L_{p^*}(I))$ or $x \in \mathcal{H}^p(\ell_{p^*}) \equiv \mathcal{H}_0^p(\ell_{p^*})$ for $p \in \left(\frac{1}{\alpha}, \infty\right]$ and $p^* \in [1, \infty)$.

Example 9.2. Here, we consider the function $y : [0,1] \to C_0$ be defined by (205). This function is Pettis integrable on $[0,1]$. Indeed, by Example 8.8, $y \in \mathcal{H}_0^{\infty}(C_0)$. By Lemma 9.3 with $p = \infty$, it follows that the FPI of y exists. To calculate the FPI of y we put in mind that $C_0^* \equiv \ell_1$. Thus, for $\alpha \in (0,1)$ and any $\varphi \in C_0^*$, there exists a sequence of real numbers $\{\lambda_n\}_{n\geq 1}$ depends on φ such that the series $\sum_{n=1}^{\infty} \lambda_n$ is absolutely convergent. Since

$$\left| \sum_{n=1}^{\infty} \lambda_n y_n(t) \right| \leq \sum_{n=1}^{\infty} |\lambda_n| < \infty,$$

then, the series $\sum_{n=1}^{\infty} \lambda_n y_n(t)$ is absolutely convergent. Therefore, by the formula (207) and the generalized linearity of the fractional integrals (Lemma 3.8), it follows that

$$
\begin{aligned}
&\int_0^t \varphi\left(\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\right)\,ds = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\varphi y(s)\,ds\\
= \;& I^\alpha\left\{\sum_{n=1}^\infty \lambda_n y_n(t)\right\} = \sum_{n=1}^\infty \lambda_n I^\alpha y_n(t)\\
= \;& \sum_{n=1}^\infty \frac{\lambda_n}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}[ns\chi_{[0,1/2n]}(s) + (1-ns)\chi_{[1/2n,1/n]}(s)]\,ds\\
= \;& \sum_{n=1}^\infty \frac{\lambda_n}{\Gamma(\alpha)}\begin{cases} \int_0^t (t-s)^{\alpha-1}ns\,ds, & \text{if } 0\le t\le \frac{1}{2n},\\ \int_0^{1/2n}(t-s)^{\alpha-1}ns\,ds + \int_{1/2n}^t (t-s)^{\alpha-1}(1-ns)\,ds, & \text{if } \frac{1}{2n}\le t\le\frac{1}{n},\\ \int_0^{1/2n}(t-s)^{\alpha-1}ns\,ds + \int_{1/2n}^{1/n}(t-s)^{\alpha-1}(1-ns)\,ds, & \text{if } \frac{1}{n}\le t\le 1.\end{cases}\\
= \;& \sum_{n=1}^\infty \frac{\lambda_n}{\Gamma(2+\alpha)}\begin{cases} nt^{\alpha+1}, & \text{if } 0\le t\le\frac{1}{2n},\\ n(1/2n)^{\alpha+1} + \frac{\Gamma(2+\alpha)}{\Gamma(1+\alpha)}(t-1/2n)^\alpha + \\ \frac{n\Gamma(2+\alpha)}{\Gamma(\alpha)}\int_{1/2n}^t \left[(t-s)^\alpha - t(t-s)^{\alpha-1}\right]\,ds, & \text{if } \frac{1}{2n}\le t\le\frac{1}{n},\\ n(1/2n)^{\alpha+1} + \int_{1/2n}^{1/n}(t-s)^{\alpha-1}(1-ns)\,ds, & \text{if } \frac{1}{n}\le t\le 1.\end{cases}\\
= \;& \sum_{n=1}^\infty \frac{\lambda_n}{\Gamma(2+\alpha)}\begin{cases} nt^{\alpha+1}, & \text{if } 0\le t\le\frac{1}{2n},\\ n(1/2n)^{\alpha+1} + \frac{\Gamma(2+\alpha)}{\Gamma(1+\alpha)}(t-1/2n)^\alpha + \\ \frac{n\Gamma(2+\alpha)}{\Gamma(\alpha)}\left[\frac{(t-1/2n)^{\alpha+1}}{\alpha+1} - t\frac{(t-1/2n)^\alpha}{\alpha}\right], & \text{if } \frac{1}{2n}\le t\le\frac{1}{n},\\ n(1/2n)^{\alpha+1} + \int_{1/2n}^{1/n}(t-s)^{\alpha-1}(1-ns)\,ds, & \text{if } \frac{1}{n}\le t\le 1.\end{cases}\\
= \;& \sum_{n=1}^\infty \frac{\lambda_n}{\Gamma(2+\alpha)}\begin{cases} nt^{\alpha+1}, & \text{if } 0\le t\le\frac{1}{2n},\\ n(1/2n)^{\alpha+1} + (t-1/2n)^\alpha\left[1+\alpha/2 - nt\right], & \text{if } \frac{1}{2n}\le t\le\frac{1}{n},\\ n(1/2n)^{\alpha+1} + (1/2n)^\alpha\left[\alpha/2\right], & \text{if } \frac{1}{n}\le t\le 1.\end{cases}\\
= \;& \varphi z(t),
\end{aligned}
$$

where $z(t) = \{z_n(t)\} \in C_0$,

$$
z_n(t) = \frac{1}{\Gamma(2+\alpha)}\begin{cases} nt^{\alpha+1}, & \text{if } 0\le t\le\frac{1}{2n},\\ n(1/2n)^{\alpha+1} + (t-1/2n)^\alpha\left[1+\alpha/2-nt\right], & \text{if } \frac{1}{2n}\le t\le\frac{1}{n},\\ n(1/2n)^{\alpha+1} + (1/2n)^\alpha\left[\alpha/2\right], & \text{if } \frac{1}{n}\le t\le 1.\end{cases}.
$$

Thus

$$I^\alpha y(t) = z(t).$$

Example 9.3. Consider the weakly measurable Pettis integrable function $x : [0,1) \to L_\infty(\mu)$ defined by $x(t) = \chi_{[0,t)}(s)$(see Example 8.30). Recall that the real-valued function φx is of bounded variation on $[0,1)$ for any $\varphi \in L_\infty^*$. Since the functions of bounded variation are bounded, it follows that $\varphi x \in L_\infty$ for any $\varphi \in L_\infty^*$. Thus $x \in \mathcal{H}_0^\infty(L_\infty)$ and by Lemma 9.3 with $p = \infty$, it follows that the FPI of x exists. To compute the FPI of x, we let $\alpha > 0$ and consider $\rho \in L_1$ and let φ be the element in L_∞^* corresponding to ρ. For $t > 0$, we have

$$\begin{aligned}
\int_0^t \varphi\left(\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} x(s)\right) ds &= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \varphi(x(s))\, ds \\
&= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \int_0^1 \rho(\theta) x(s)(\theta) d\theta\, ds \\
&= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \int_0^1 \rho(\theta) \chi_{[0,s]}(\theta) d\theta\, ds \\
&= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \int_0^s \rho(\theta) d\theta\, ds \\
&= \int_0^t \int_0^s \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \rho(\theta) d\theta\, ds \\
&= \int_0^t \int_\theta^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \rho(\theta)\, ds d\theta = \int_0^t \frac{(t-\theta)^{\alpha}}{\Gamma(1+\alpha)} \rho(\theta) d\theta \\
&= \int_0^1 \frac{(t-\theta)^{\alpha}}{\Gamma(1+\alpha)} \rho(\theta) \chi_{[0,t]}(\theta) d\theta.
\end{aligned}$$

Thus

$$\begin{aligned}
\varphi\left(\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} x(s)\, ds\right) &= \int_0^t \varphi\left(\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} x(s)\right) ds \\
&= \varphi\left(\frac{(t-s)^{\alpha}}{\Gamma(\alpha+1)} \chi_{[0,t]}(s)\right).
\end{aligned}$$

Consequently, we conclude that

$$I^\alpha x(t)(\cdot) = \frac{(t-\cdot)^\alpha}{\Gamma(1+\alpha)} \chi_{[0,t]}(\cdot) \in L_\infty.$$

Lemma 9.4. *If $\alpha \geq 1$, $\theta \in L_q[0,1]$ and $y \in \mathcal{H}_0^p(E)$, then*

$$\int_0^1 \left[J^\alpha \theta(s)\right] y(s)\, ds = \int_0^1 \theta(s) \left[I^\alpha y(s)\right]\, ds. \tag{212}$$

Here, the operator J^α defined by

$$J^\alpha x(t) := \int_t^1 \frac{(s-t)^{\alpha-1}}{\Gamma(\alpha)} x(s)\, ds,\ t < 1. \tag{213}$$

Proof. Firstly, we note that, it is not so hard to verify that the operator J^α admits the same properties as I^α. Define the real-valued function h by

$$h(s) := \frac{1}{\Gamma(\alpha)} \int_s^1 (\tau - s)^{\alpha-1} \theta(\tau) d\tau = J^\alpha \theta(s).$$

Using the properties of fractional calculus in the Banach space $L_q[0,1]$, we deduce $h \in L_q[0,1]$. Now, for $y \in \mathcal{H}_0^p(E)$ we have, in the view of Lemma 9.2, that $I^\alpha y \in \mathcal{H}_0^p(E)$. Thanks to Proposition 8.5, the functions $t \to \theta(t) I^\alpha y(t)$ and $t \to h(t)y(t)$ are Pettis integrable on $[0,1]$. That is, the integrals in both sides of Equation (212) exist. So, there exists $\rho \in E$, such that

$$\rho = \int_0^1 \theta(s) I^\alpha y(s)\, ds.$$

By the definition of Pettis integral, we have

$$\begin{aligned}\varphi\rho &= \int_0^1 \theta(s)\varphi\left(I^\alpha y(s)\right)\, ds = \int_0^1 \theta(s) I^\alpha \varphi y(s)\, ds \\ &= \int_0^1 \theta(s) \int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} \varphi y(\tau) d\tau\, ds, \text{ for all } \varphi \in E^*.\end{aligned}$$

Interchanging the order of integrations results in

$$\begin{aligned}\varphi\rho &= \int_0^1 \left(\int_\tau^1 \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} \theta(s)\, ds\right) \varphi y(\tau) d\tau \\ &= \int_0^1 h(\tau)\varphi y(\tau) d\tau = \varphi\left(\int_0^1 h(s) y(s)\, ds\right), \text{ for all } \varphi \in E^*.\end{aligned}$$

Thus

$$\rho = \int_0^1 J^\alpha \theta(s) y(s)\, ds = \int_0^1 u(s) I^\alpha y(s)\, ds.$$

□

Arguing similarly as in the proof of Lemma 9.2, we are able to show that

Lemma 9.5. *Let $\alpha \geq 1$ and $a \in L_q(I)$. Assume that $x \in \mathcal{H}_0^p(E)$. Then,* FPI *of $a(\cdot)x(\cdot)$ exists as function from I to E. The result holds for all $\alpha > 0$, if E is reflexive. However,*

$$\varphi(I^\alpha a(t)x(t)) = I^\alpha a(t)\varphi(x(t)), \text{ for all } \varphi \in E^*$$

Proof. By Proposition 8.5, we have $a(\cdot)x(\cdot) \in \mathcal{H}_0^1(E)$. Arguing similarly as in the proof of Lemma 9.2, we complete the proof. When E is reflexive, the proof is a direct consequence of Lemma 9.1. □

Lemma 9.6. *Let $\alpha, \beta > 0$ and $p > \max\{1, 1/\alpha, 1/\beta\}$. Then*

1. $I^\alpha I^\beta x = I^\beta I^\alpha x = I^{\alpha+\beta} x$,

2. $\lim_{\alpha \to n} I^\alpha x = I^n x,\ n \in \mathbb{N}$,

hold for every $x \in \mathcal{H}_0^p(E)$. If E is reflexive, this is also true for every $p \geq 1$.

Proof. Firstly, we note that our assumption imposed on p yields $p > \max\{1, 1/(\alpha+\beta)\}$. Thus, by Lemma 9.3 and Corollary 9.3, it follows that the $I^\alpha x$, $I^\beta x$ and $I^{\alpha+\beta}x$ exist for every $x \in \mathcal{H}_0^p(E)$ as a weakly continuous functions from I to E. Indeed, $I^\alpha x$, $I^\beta x$ and $I^{\alpha+\beta}x$ belong to $x \in \mathcal{H}_0^\infty(E)$. Therefore, the operators $I^\alpha I^\beta$, $I^\beta I^\alpha$ and $I^{\alpha+\beta}$ makes sense. Consequently, for any $\varphi \in E^*$ we have

(1)

$$\varphi(I^\alpha I^\beta x(t)) = I^\alpha \varphi(I^\beta x(t)) = I^\alpha I^\beta \varphi(x(t)) = I^{\alpha+\beta}\varphi(x(t)) = \varphi(I^{\alpha+\beta}x(t)),$$

that is

$$\varphi(I^\alpha I^\beta x(t) - I^{\alpha+\beta}x(t)) = 0, \text{ for every } \varphi \in E^*.$$

Hence $I^\alpha I^\beta x(t) = I^{\alpha+\beta}x(t)$. Similarly, we are able to show that $I^\beta I^\alpha x(t) = I^{\alpha+\beta}x(t)$.

(2)

$$|\varphi(I^\alpha x(t)) - \varphi(I^n x(t))| = |I^\alpha \varphi(x(t)) - I^n \varphi(x(t))|,$$

since $\varphi x \in L_p$, we deduce (from the properties of the fractional calculus in the space $L_p([I])$) that

$$I^\alpha \varphi(x(t)) \to I^n \varphi(x(t)) \text{ uniformaly on } [0,1].$$

Hence $\varphi(I^\alpha x(t)) \to \varphi(I^n x(t))$ uniformly on the interval I, that is, $I^\alpha x(t) \to I^n x(t)$ weakly uniformly on the interval I and we are finished.

When E is reflexive, the result follows immediately as a direct consequence of Corollary 9.2. □

Proposition 9.1. *For any $\alpha > 0$, if the two functions $x, y \in \mathcal{H}_0^1(E)$ are weakly equivalent on the interval I such that $I^\alpha x$ and $I^\alpha y$ exist, then $I^\alpha x = I^\alpha y$ on I.*

Proof. Fix $t \in I$. Since x and y are weakly equivalent, then for every $\varphi \in E^*$ there exists a null set $N(\varphi)$ such that $(t-s)^{\alpha-1}\varphi x(s) = (t-s)^{\alpha-1}\varphi y(s)$ for every $s \in [0,t]/N(\varphi)$. It follows that

$$\begin{aligned}\varphi\left(\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} x(s)\,ds\right) &= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \varphi x(s)\,ds = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \varphi y(s)\,ds \\ &= \varphi\left(\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)\,ds\right),\end{aligned}$$

for every $\varphi \in E^*$ and every $t \in I$. Thus $I^\alpha x = I^\alpha y$ on I. □

Definition 9.2. Let $x : I \to E$. For the positive integer m such that $\alpha \in (m-1, m)$, $m \in \mathbb{N}_0 := \{0, 1, 2, \cdots$ we define the Riemann-Louville fractional-pesudo (-weak) derivative "shortly RFPD (RFWD)" of x of order α by

$$D^{\alpha}\ x(t) := D^m I^{m-\alpha} x(t). \tag{214}$$

Here D denote the pseudo-(weak-)differential operator. We use the notation $D_p^{m+\alpha}$ and $D_w^{m+\alpha}$ to characterize the Riemann-Louville fractional-pesudo derivatives and Riemann-Louville fractional-weak derivatives respectively.

In the following, we consider the most interesting case when $m = 1$. Indeed, we investigate the existence and the properties of the fractional differential operators in the space $\mathcal{H}_0^p(E)$. We start with the following existence result

Proposition 9.2. *Let $\alpha \in (0,1)$. If E is weakly sequentially complete, then the* RFWD *of a wAC function $x : I \to E$ exists for almost every $t \in I$.*

Proof. Firstly, we note that the weak absolute continuity of x yields $x \in \mathcal{H}_0^{\infty}(E)$. This guarantees, in the view of Lemma 9.3, the existence of $I^{1-\alpha}x$. Since φx is absolutely continuous on I for every $\varphi \in E^*$, we infer, as a consequence of Lemma 3.10 that the function $\varphi\left(I^{1-\alpha}x\right) = I^{1-\alpha}\varphi x$ is absolutely continuous on I. Since E is weakly sequentially complete and $\varphi\left(I^{1-\alpha}x\right)$ is a.e. differentiable for every $\varphi \in E^*$, we deduce, in the view of Theorem 7.3.3 in [172], that the function $I^{1-\alpha}x$ is a.e. weakly (trivially pesudo) differentiable on I. Thus, the RFWD $D_p^{\alpha}x$ exists a.e. on I. $\square$

Proposition 9.3. *Let $p > 1/(1-\alpha)$, $\alpha \in (0,1)$. The* RFPD *of a wAC function $x : I \to E$ having $D_p x \in \mathcal{H}_0^p(E)$ exists for almost every $t \in I$. If E is reflexive, this is also true for every $p \geq 1$.*

Proof. Since x is weakly absolutely continuous and having a pseudo derivative $D_p x \in \mathcal{H}_0^p(E)$, then by lemma Lemma 8.18 we obtain

$$x(t) = x(0) + \int_0^t D_p x(s)\, ds,\ t \in I.$$

Our assumption that $p > 1/(1-\alpha)$ together with Lemma 9.6 result in

$$I^{1-\alpha}x(t) = \frac{t^{1-\alpha}}{\Gamma(2-\alpha)}x(0) + I^{1-\alpha}I^1 D_p x(t) = \frac{t^{1-\alpha}x(0)}{\Gamma(2-\alpha)} + \int_0^1 I^{1-\alpha} D_p x(s)\, ds. \tag{215}$$

Since $t^{1-\alpha} = (1-\alpha)\displaystyle\int_0^t s^{-\alpha}\ ds$, then the first term of (215) is absolutely continuous. Moreover, by Lemma 8.17 we infer that the function $I^{1-\alpha}x$ is has a pesudo-derivative weakly equivalent to

$$\frac{t^{-\alpha}}{\Gamma(1-\alpha)}x(0) + I^{1-\alpha}D_p x(t).$$

Now, if E is reflexive, and $D_p x \in \mathcal{H}_0^p(E),\ p \geq 1$, then 215 holds which, of course, yields the existence of the fractional pesudo-derivative of x and we are finished.

$\square$

Remark 9.3. If x_1 and x_2 are two pesudo-derivatives of a wAC function x, it follows by Lemma 9.1 that $I^{1-\alpha}x_1(t) = I^{1-\alpha}x_2(t)$ on I and so $D_p^\alpha x$ does not depend on the choice of the pesudo-derivatives of x.

The following Lemma is folklore in case $E = \mathbb{R}$, but to see that it is also holds in the vector-valued case, we provide a proof

Lemma 9.7. *Let* $0 < \alpha \leq \beta < 1$ *and* $p > \max\{1, 1/\alpha, 1/(1-\beta)\}$. *For* $x \in C[I, E]$ *(* $x \in \mathcal{H}_0^p(E)$ *) we have* $D^0 x = x$, *and*

$$D^\beta I^\alpha x = \begin{cases} D^{\beta-\alpha}x, & 0 < \alpha \leq \beta, \\ I^{\alpha-\beta}x, & \alpha \geq \beta > 0. \end{cases} \tag{216}$$

If E is reflexive, this is also true for every $p \geq 1$.
In particular, when $\alpha = \beta$ (216) means that the operator $D^\alpha I^\alpha$ is defined on in $\mathcal{H}_0^p(E)$ and that D^α is the left-inverse of I^α. Here, D^α stands the fractional weak-(pesudo-) derivative of order α.

Proof. The first claim, i.e. $DI^1 x = x$, follows from Lemma 8.17. Then we have, in the view of $DI^1 x = x$ and Lemma 9.6 that

$$D^\beta I^\alpha x = DI^{1-\beta} I^\alpha x = \begin{cases} DI^{1-(\beta-\alpha)}x = D^{\beta-\alpha}x, & 0 < \alpha \leq \beta, \\ DI^1 I^{\alpha-\beta}x = I^{\alpha-\beta}x, & \alpha \geq \beta > 0. \end{cases}$$

Hence the result holds. $\square$

It is well known that, if E has a finite dimension, then the indefinite integral of a Pettis integrable function is a.e. weakly differentiable (see e.g. [64], [81] and [134]). However, the weakly absolutely continuous function is pseudo differentiable if, and only if, it is an indefinite pettis integral of a Pettis integrable function. In what follows, we construct an examples of weakly measurable functions that have no first order derivative and might have a RFWD fractional derivative of all orders less than one:

Example 9.4. Consider the weakly complete space $L_1[0,1]$. Define $x : [0,1] \to L_1[0,1]$ by

$$x(t) := \chi_{[0,t]} + A(t) \cdot \mathcal{W}_1(t),$$

Where $\mathcal{W}_1(t) := \sum_{n=0}^{\infty} b^{-n} \cos b^n t$ denotes the well-known Weierstrass function and $A(t)(\tau) := 1,\ t, \tau \in [0,1]$. It was shown in Example 8.11, that the Lipschitz continuous function $\chi_{[0,t]}$ is not weakly differentiable everywhere on

$(0, 1)$. Moreover, the Weierstrass function $\mathcal{W}_1(t))$ has continuous Riemann-Liouville fractional derivatives of any order less than one, but nowhere has the first order derivative (*cf.* [144]). By Proposition 9.2, it follows that that the function x has a.e. a RFWD of any order less than one.

Besides the fractional Riemann-Louville derivative (214) we define also the Caputo fractional-pesudo (-weak) derivative "shortly CFPD (CFWD)" :

Definition 9.3. Let $x : I \to E$. For the positive integer m such that $\alpha \in (m-1, m)$, we define the Caputo fractional-pesudo (-weak) derivative "shortly CFPD (CFWD)" of x of order α by

$$\frac{d^\alpha x(t)}{dt^\alpha} := I^{m-\alpha} D^m x(t), \tag{217}$$

where D denote the pseudo-(weak-)differential operator.

For the existence of CFWD, we introduce the following existence result

Proposition 9.4. *Let $\alpha \in (0, 1)$. The* CFWD *of a function $x : I \to E$ having a weak-derivative $D_w x \in C[I, E]$ exists almost everywhere on I.*
In particular, if E is reflexive, then the CFWD *of the wAC function $x : I \to E$ a.e. on I.*

Proof. The general case is a direct consequence of Lemma 9.3. Now if E is reflexive, then E has the *Radon-Nikodym property.* Thus, in the view of Proposition 8.7, the *wAC* function x has a Pettis integrable weak derivative (that is, $D_w x \in \mathcal{H}_0^1(E)$). Hence, in the view of Lemma 9.7 it follows that $I^{1-\alpha} D_w x$ exists for almost every $t \in I$. $\square$

Proposition 9.5. *Let $\alpha \in (0, 1)$. The* CFWPD *of a AC function $x : I \to E$ having a pesudo-derivative exists almost everywhere on I.*
In particular, if E is reflexive, then the result holds for the wAC function $x : I \to E$ a.e. on I.

Proof. Recall that (), the function $x : I \to E$ is an indefinite Pettis integrable of any of its pesudo-derivatives if, and only if, x is AC and has a pesudo-derivative on I. Thus we have $x(t) = \int_0^t D_w x(s) ds$. The general case is a direct consequence of Lemma 9.3. Now if E is reflexive, then E has the *Radon-Nikodym property.* Thus, in the view of Proposition 8.7, the *wAC* function x has a Pettis integrable weak derivative (that is, $D_w x \in \mathcal{H}_0^1(E)$). Hence, in the view of Lemma 9.7 it follows that $I^{1-\alpha} D_w x$ exists for almost every $t \in I$. $\square$

An immediate consequence of Lemma 9.1 and Lemma 9.3 is the following observation

Proposition 9.6. *Let $p > 1/(1-\alpha)$, $\alpha \in (0,1)$. The* CFPD *of function $x : I \to E$ having a pesudo-derivative $D_p x \in \mathcal{H}_0^p(E)$ exists almost everywhere on I. If E is reflexive, this is also true for every $p \geq 1$.*

The connection of Caputo fractional-pesudo (-weak) derivative with the Riemann-Liouville derivative is well-known and easy to see:

Proposition 9.7. *Let $p > 1/(1-\alpha)$, $\alpha \in (0.1)$. If $x : I \to E$ is weakly absolutely continuous and has a weak- (pesudo-)derivative $Dx \in C[I,E]$ ($Dx \in \mathcal{H}_0^p(E)$), then*

$$\frac{d^\alpha x}{dt^\alpha} = D^\alpha (x - x_0),$$

where x_0 denotes the constant function with value $x(0)$. If E is reflexive, this is also true for every $p \geq 1$.

Proof. We note first that under our assumption on x, we infer, in the view of Lemma 8.18, that $I^1 Dx = x - x_0$. Moreover, by Corollary 9.3, we have $I^{1-\alpha} Dx \in \mathcal{H}_0^\infty(E)$. Hence, in the view of Lemma 9.7 and Lemma 9.6, we have

$$\frac{d^\alpha x}{dt^\alpha} = D^\alpha I^\alpha I^{1-\alpha} D_p x = D^\alpha I^1 D_p x = D^\alpha (x - x_0).$$

□

The definition of the CFWD or CFPD of function $x : I \to E$ has a disadvantage that, it completely lose their meaning if x fails to be (almost everywhere) differentiable. For this reason, we are able to use the Proposition 9.7 to define the Caputo fractional derivative in general, that is, we put

$$\frac{d^\alpha x}{dt^\alpha}(t) := D^\alpha x(t) - \frac{t^\alpha x_0}{\Gamma(1+\alpha)}. \tag{218}$$

Proposition 9.7 implies that, for a weakly (pesudo) differentiable function with an Pettis integrable weak- (pesudo-)derivatives, the Definition 218 coincides with the usual definition of the CFWD (CFPD).

10. Existence results of some fractional differential and integral equations in abstract spaces

The main objects of this section are to consider the question of the existence of weak and pseudo solutions to some integral equations and and corresponding initial/boundary value problems of fractional order. Before embarking into details, we accept the following

Definition 10.1. Let $f : I \times E \to E$. Then $f(t,x)$ is said to be weakly-weakly continuous at (t_0, x_0) if given $\epsilon > 0$, $\varphi \in E^*$, there exists a $\delta > 0$ and a weakly open set U containing x_0 such

$$|\varphi(f(t,x) - f(t_0,x_0))| < \epsilon, \text{ whenever } |t - t_0| < \delta \text{ and } x \in U.$$

Let f be a weakly-weakly continuous function from $I \times E$ into E. Assume that $E_1 = \{x \in C[I,E] : \|x\|_0 < r\}$ $(r > 0)$, where $\|\cdot\|_0$ is the sup-norm. By B_r we will denote the set $\{x(t) \in E : x \in E_1, t \in I\}$.

Proposition 10.1. (1) *For each $t \in I$, $f(t,\cdot)$ is weakly continuous, hence weakly sequentially continuous (cf.* [14]*).*
(2) *f is norm bounded, i.e. there exists an M_r such that $\|f(t,x)\| \le M_r$ for all $(t,x) \in B_r$ (cf.* [184]*).*

We also have

Proposition 10.2. *For each weakly continuous $x : I \to E$, the map $f(\cdot, x(\cdot)) : I \to E$ is weakly continuous (Hence it is Pettis integrable on I).*

Proof. To prove this proposition, we equip E and $I \times E$ with weak topology and note that $t \longmapsto (t, x(t))$ is continuous as a mapping from I into $I \times E$, then $f(\cdot, x(\cdot))$ is a composition of this mapping with f and thus, for any $x \in C[I,E]$, $\varphi(f(\cdot, x(\cdot)))$, $\varphi \in E^*$ is continuous in I. □

For convenience of the readers, let us recall the following lemma describing particular sufficient conditions for Pettis-integrability of $f(\cdot, x(\cdot))$ ([48, Lemma 15]). It is obvious, that additional growth condition for f allows us to characterize the functions from $\mathcal{H}_0^p(E)$.

Lemma 10.1. *Assume that x is absolutely continuous and $f : I \times E \to E$. Thus $f(\cdot, x(\cdot))$ is Pettis-integrable if at least one of the following cases holds:*
(a) *f satisfies Carathéodory conditions i.e. $f(\cdot, x)$ is measurable, $f(t,\cdot)$ is continuous in E and there exists an integrable function $h : I \to \mathbb{R}$ such that $\|f(t,x)\| \le h(t)$ for all $x \in E$ and a.e. $t \in I$,*
(b) *$f(\cdot, x)$ is weakly measurable, $f(t,\cdot)$ is weakly–weakly continuous in E and E is a WCG-space (weakly compactly generated space),*

(c) *f is strongly measurable and there exists a Young function Γ such that $\lim_{x\to\infty}\Gamma(x)/x=\infty$ and φf belongs to the Orlicz space $L_\Gamma(I)$,*
(d) *f is strongly measurable and there exists $p>1$ such that $\varphi f\in L_p$ for each $\varphi\in E^*$ (here $f(\cdot,x(\cdot))\in\mathcal{H}_0^p(E)$). If E is weakly complete, this is also true for $p=1$ (see Proposition 8.3),*
(e) *$f(\cdot,x)$ is strongly measurable, $f(t,\cdot)$ is weakly sequentially continuous in E and f is bounded,*
(f) *$f(\cdot,x(\cdot))$ is Dunford strongly measurable, E contains no copy of c_0.*

For any bounded subset Λ of E we denote by $\beta(\Lambda)$ the De Blasi [7, 58] measure of weak noncompactness of Λ, i.e. the infimum of all $\epsilon>0$ for which there exists a weakly compact subset Ω of E such that $\Lambda\subset\epsilon B_1+\Omega$. We next state a proposition of Ambrosetti-type ([125], Theorem 2., see also [36]).

Proposition 10.3. *Let $X\subset C[I,E]$ be bounded and equicontinuous. Then the function $t\to\beta(X(t))$ is continuous on I and*

$$\beta(X)=\sup_{t\in I}\beta(X(t))=\beta(X(I)),$$

where $X(t)=\{x(t):x\in X\}$ and $X(I)=\bigcup_{t\in I}\{x(t):x\in X\}$.

We recall that β has the following properties:

1. If $A\subset B$, then $\beta(A)\leq\beta(B)$,
2. $\beta(A)=0$ if and only if A is relatively compact in E,
3. $\beta(A\cup B)=\max\{\beta(A),\beta(B)\}$,
4. $\beta(\bar{A}^\omega)=\beta(A)$, ($\bar{A}^\omega$ denotes the weak closure of A,)
5. $\beta(A+B)\leq\beta(A)+\beta(B)$
6. $\beta(\lambda A)=|\lambda|\beta(A)$,
7. $\beta(\mathrm{conv}(A))=\beta(A)$,
8. $\beta\left(\bigcup_{|\lambda|\leq h}\lambda A\right)=h\beta(A)$,
9. $\beta(A)\leq 2\,\mathrm{diam}(A)$.

The following results which are an immediate consequence of the Hahan Banach theorem

Proposition 10.4. *Let E be a Banach space with $x_0\neq 0$. Then there exists a $\varphi\in E^*$ with $\|\varphi\|=1$ such that $\|\varphi(x_0)\|=\|x_0\|$. Furthermore, if $\varphi(x_0)=0$, for each $\varphi\in E^*$, then $x_0=0$.*

Now, we will define weak and sequential weak continuity for mappings between Banach spaces. Let X, Y be a Banach spaces and $T:X\longrightarrow Y$. We use $\to$ and $\rightharpoonup$ to denote strong and weak convergence of sequences respectively.

Definition 10.2. (1) T is strongly continuous if and only if $x_n\to x$ implies $T(x_n)\to T(x)$,

(2) T is weakly continuous if and only if T is continuous with respect to the weak topologies on X and Y, (i.e. if $x_\alpha \rightharpoonup x$ weakly and (x_α) is a net in X, then $T(x_\alpha) \rightharpoonup T(x)$ see([83] and [138])),
(3) T is weakly sequentially continuous if and only if $x_n \rightharpoonup x$ implies $T(x_n) \rightharpoonup T(x)$

It is clear that (2) implies (3). If T is linear, then (1), (2) and (3) are equivalent. There exist many important examples of mappings that are weakly sequentially continuous but not weakly continuous. The relationship between strong, weak and weak sequential continuity of mapping is studied in details in [14].

Also, we state some fixed point theorems which will be used in the next section

Theorem 10.1. *(see* [139]*) Let E be a Banach space and let Q be nonempty, bounded, closed and convex subset of $C[I, E]$. Suppose, that $T : Q \to Q$ is weakly sequentially continuous and assume that $TQ(t)$ is relatively weakly compact in E for each $t \in [0, 1]$. Then the operator T has a fixed point in Q.*

Recall [184] that a subset of a reflexive Banach space is weakly compact if and only if it is closed in the weak topology and bounded in the norm topology. Thus putting in mind that $TQ(t)$ is bounded subset of E, then the condition $TQ(t)$ is weakly relatively compact is automatically satisfied. Accordingly, we immediately have

Theorem 10.2. *Let E be a reflexive Banach space with Q a nonempty, closed, convex and equicontinuous subset of $C[I, E]$. Assume that $T : Q \to Q$ is weakly sequentially continuous. Then T has a fixed point in Q.*

We recall the following fixed point theorem being an extension of results from [9]:

Theorem 10.3. *Let E be a Banach space with Q a nonempty, closed, convex and weakly compact subset of $C[I, E]$. Assume that $T : Q \to Q$ is weakly-weakly sequentially continuous. Then T has a fixed point in Q.*

Theorem 10.4. [139] *Let Q be a closed convex and equicontinuous subset of a metrizable locally convex vector space $C[I, E]$. Assume that $T : Q \to Q$ is wk-sequentially continuous (i.e. for any sequence (x_n) in Q with $x_n(t) \to x(t)$ in (E, w) for each $t \in [0, 1]$, then $Tx_n(t) \to Tx(t)$ in (E, w) for each $t \in [0, 1]$). If for some $y \in Q$ and the implication*

$$\overline{V} = \overline{conv}\,(\{y\} \cup T(V)) \Longrightarrow V \text{ is relatively weakly compact,} \tag{219}$$

holds for every subset $V \subset Q$, then T has a fixed point.

We remark here that the above theorems are extended and improved by Amar in (2011) [33].

While it is not always possible to show that a given mapping between Banach spaces is weakly continuous, quite often its weak sequential continuity and wk-sequentially continuous offers no problem. This follows from the fact that the Lebesgue's dominated convergence theorem is valid for sequences but not for nets, we recall the following counter example:

Example 10.1. If $\Im$ denotes the family of finite subsets of [0,1], ordered by inclusion, and if

$$\chi_\alpha(t) = \begin{cases} 1, & \text{if } t \in \alpha, \\ 0, & \text{if } t \notin \alpha. \end{cases}$$

Then $(\chi_\alpha) \to 1$ and

$$\int \chi_\alpha d\mu = 1 \cdot \mu(\alpha) + 0 = 0 \text{ for each } \alpha \in \Im.$$

10.1. **Existence results for initial value problem in Banach spaces.** We introduce the following set of example

Example F: (Multi-term fractional differential equation in reflexive Banach space)

In this example, we deal with the existence of weak solutions for the multi-term differential equation of the fractional type

$$\begin{cases} \left(D^{\alpha_n} - \sum_{i=1}^{n-1} a_i D^{\alpha_i}\right) x(t) = f(t, x(t)), \ t \in [0,1], \\ x(0) = 0. \end{cases} \tag{220}$$

Here, x takes values in a reflexive Banach space E endowed with the weak topology, $a_1, a_2, \cdots, a_{n-1}$, are constants, $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_n < 1$ and D^{α_j} denotes the weakly-(Riemann-Liouville) fractional differential operator of order α_j.

We assume that the vector-valued function $f : [0,1] \times E \to E$ is weakly-weakly continuous.

Keeping the Cauchy problem (231) in mind, we examine the following fractional order integral equation (modeled from problem (231))

$$x(t) = \sum_{i=1}^{n-1} a_i I^{\alpha_n - \alpha_i} + I^{\alpha_n} f(t, x(t)), \ n \in \mathbb{N}, \ t \in I := [0,1], \tag{221}$$

where I^α denotes the fractional Pettis-integral operator corresponding to D^α.

Definition 10.3. A function $x : I \to E$ is called weak (pseudo) solution of (231) if x has the weak (pseudo) derivatives of order $\alpha_i \in (0,1)$, $x(0) = 0$, $i =$

$1, \cdots, n$ and satisfies

$$\left(D^{\alpha_n} - \sum_{i=1}^{n-1} a_i D^{\alpha_i} \right) x(t) = f(t, x(t)),$$

for all (for almost every) $t \in [0, 1]$.

Definition 10.4. By a weak solution to (221), we mean a weakly continuous function x which satisfies the integral equation (221). This is equivalent to finding $x \in C[I, E_w]$ with

$$\varphi(x(t)) = \varphi \left(\sum_{i=1}^{n-1} a_i I^{\alpha_n - \alpha_i} + I^{\alpha_n} f(t, x(t)) \right), \ t \in I \text{ for all } \varphi \in E^*.$$

Thus, we are in a position to formulate and prove the following

Theorem 10.5. *Suppose $f : [0, 1] \times E \to E$ is weakly-weakly continuous. Assume that there are some $a_i \in \mathbb{R}, \ i = 1, \cdots, n-1, \ n \in \mathbb{N}$ such that there exists $r > 0$ satisfying*

$$\sum_{i=1}^{n-1} \frac{2|a_i|}{\Gamma(1 + \alpha_n - \alpha_i)} + \frac{M_r}{\Gamma(\alpha_n + 1)} < 1, \tag{222}$$

where M_r is the real number defined in Proposition 10.1. Then equation (221) has at least one weak solution $x \in C[I, E]$. In fact the solution we produce will be norm continuous.

Proof. Define the operator $T : C[I, E] \to C[I, E]$ by

$$Tx(t) = \sum_{i=1}^{n-1} a_i I^{\alpha_n - \alpha_i} + I^{\alpha_n} f(t, x(t)), \ n \in \mathbb{N}, \ t \in I, \tag{223}$$

Let us remark, in the view of Proposition 10.2, that for every weakly continuous function $x : I \to E$, $f(\cdot, x(\cdot))$ is weakly continuous. Consequentially, in the view of Lemma **??**, the operator T makes sense. Also, T is well-defined. To see this, let $t_1, t_2 \in [0, 1]$ with $t_2 > t_1$. Without loss of generality, assume $Tx(t_2) - Tx(t_1) \neq 0$. Then there exists (consequence of Proposition 10.4)

$\varphi \in E^*$ with $\|\varphi\| = 1$ and $\|Tx(t_2) - Tx(t_1)\| = \varphi(Tx(t_2) - Tx(t_1))$. Thus

$$\begin{aligned}
\|Tx(t_2) - Tx(t_1)\| &= |\varphi\left(Tx(t_2) - Tx(t_1)\right))| \\
&\leq \sum_{i=1}^{n-1} \frac{\|x\|\,|a_i|}{\Gamma(\alpha_n - \alpha_i)} \left[\int_0^{t_1} [(t_1 - s)^{\alpha_n - \alpha_i - 1} - (t_2 - s)^{\alpha_n - \alpha_i - 1}]\, ds\right. \\
&+ \left.\int_{t_1}^{t_2} (t_2 - s)^{\alpha_n - \alpha_i - 1}\, ds\right] \\
&+ \frac{M_r}{\Gamma(\alpha_n)} \left[\int_0^{t_1} [(t_1 - s)^{\alpha_n - \alpha_i - 1} - (t_2 - s)^{\alpha_n - \alpha_i - 1}]\, ds\right. \\
&+ \left.\int_{t_1}^{t_2} (t_2 - s)^{\alpha_n - \alpha_i - 1}\, ds\right] \\
&\leq \sum_{i=1}^{n-1} \frac{2\,\|x\|\,|a_i|(t_2 - t_1)^{\alpha_n - \alpha_i}}{\Gamma(\alpha_n - \alpha_i + 1)} + \frac{2M_r(t_2 - t_1)^{\alpha_n}}{\Gamma(\alpha_n + 1)}.
\end{aligned}$$

Therefore

$$\|Tx(t_2) - Tx(t_1)\| \leq \left[\sum_{i=1}^{n-1} \frac{2\,\|x\|\,|a_i|}{\Gamma(\alpha_n - \alpha_i + 1)} + \frac{2M_r}{\Gamma(\alpha_n + 1)}\right] (t_2 - t_1)^{\alpha_n - \alpha_{n-1}}. \tag{224}$$

This estimation shows that Tx is norm continuous i.e. T maps $C[I, E]$ into itself. Define the closed, convex, bounded, equicontinuous subset of $Q \subset C[I, E]$ by

$$\begin{aligned}
&Q := \{x \in C[I, E] : \|x\|_0 \leq r,\ \|x(t_2) - x(t_1)\| \leq \\
&\qquad \left[\sum_{i=1}^{n-1} \frac{2r|a_i|}{\Gamma(\alpha_n - \alpha_i + 1)} + \frac{2M_r}{\Gamma(\alpha_n + 1)}\right] (t_2 - t_1)^{\alpha_n - \alpha_{n-1}}\ t_1,\ t_2 \in [0, 1]\}.
\end{aligned}$$

We claim that $T : Q \to Q$ is weakly sequentially continuous. Once the claim is established, then Theorem 10.2 guarantees a fixed point of T, and hence (221) has a solution in $C[I, E]$. We begin by showing that $T : Q \to Q$. To see this, take $x \in Q,\ t \in [0, 1]$. Without loss of generality, assume $Tx(t) \neq 0$. Then there exists (consequence of Proposition 10.4) $\varphi \in E^*$ with $\|\varphi\| = 1$ and $\|Tx(t)\| = \varphi(Tx(t))$. Notice also since $\|x\|_0 \leq r$, then Proposition 10.1 guarantees the existence of a constant M_r with

$$\|f(t, x(t)\| \leq M_r \ \text{ for all } t \in [0, 1] \ \text{ and for all } x \in Q.$$

Thus, from inequality (224) we have

$$\|Tx(t_2) - Tx(t_1)\| \leq \left[\sum_{i=1}^{n-1} \frac{2r|a_i|}{\Gamma(\alpha_n - \alpha_i + 1)} + \frac{2M_r}{\Gamma(\alpha_n + 1)}\right] (t_2 - t_1)^{\alpha_n - \alpha_{n-1}}.$$

Secondly, we show that $\|Tx\|_0 = \sup_{t\in[0,1]} \|Tx(t)\| \leq r$ for any $x \in Q$. To see this, look at $I^{\alpha_n} f(t, x(t))$ for $t \in [0, 1]$. Without loss of generality, we may assume $I^{\alpha_n} f(t, x(t)) \neq 0$ for all $t \in [0, 1]$, that is $Tx(t) \neq 0$ for all $t \in [0, 1]$. Then there exists (consequence of Proposition 10.4) $\varphi \in E^*$ with $\|\varphi\| = 1$ and $\|Tx(t)\| = \varphi(Tx(t))$. Thus

$$\|Tx(t)\| \leq \left(\sum_{i=1}^{n-1} \frac{2|a_i|}{\Gamma(\alpha_n - \alpha_i + 1)}\right) r + \frac{M_r}{\Gamma(\alpha_n + 1)}.$$

In the view of our assumptions, we obtain $\|Tx\|_0 \leq r$. Thus $T : Q \to Q$. Finally, we will show that T is weakly sequentially continuous. To see this, let (x_n) be a sequence in Q and let $x_n(t) \to x(t)$ in E_w for each $t \in [0, 1]$. Recall [125], that a sequence (x_n) is weakly convergent in $C[I, E]$ if, and only if, it is weakly pointwise convergent in E. Fix $t \in I$. From the weak sequential continuity of $f(t, \cdot)$, the Lebesgue dominated convergence theorem for the Pettis-integral (Theorem 8.10) implies for each $\varphi \in E^*$ that $\varphi(Tx_n(t)) \to \varphi(Tx(t))$ a.e. on I, $Tx_n(t) \to Tx(t)$ in E_w. So $T : Q \to Q$ is weakly sequentially continuous. We have therefore shown that the operator T satisfies all requirements of Theorem 10.2. Consequently, the proof is now completed. □

Now, we restrict our attention to discuss the existence of weak solutions to the problem (231). To do this, we can make use of Theorem 10.5. Firstly, we recall the following

Lemma 10.2. *Suppose x be a solution to the problem (144) then*

$$\lim_{t\to 0} I^\beta x(t) = \lim_{t\to 0} I^\beta f(t, x(t)) = 0, \;\; \beta > 0.$$

Proof. Let $\varphi \in E^*$. The proof follows immediately from Proposition 10.4 and the inequalities

$$\left|\varphi(I^\beta x(t))\right| = \left|I^\beta \varphi(x(t))\right| \leq K_1 \frac{t^\beta}{\Gamma(1+\beta)},$$

$$\left|\varphi(I^\beta f(t, x(t)))\right| = \left|I^\beta \varphi(f(t, x(t)))\right| \leq K_2 \frac{t^\beta}{\Gamma(1+\beta)},$$

where $K_1 = \sup |\varphi(x(t))|$ and $K_2 = \sup |\varphi(f(t, x(t)))|$. □

Theorem 10.6. *If the assumptions of Theorem 10.5 are satisfied, the problem (231) has at least one weak solution $x(\cdot) \in C[I, E]$. In fact the solution we produce will be norm continuous.*

Proof. Let $x \in C[I, E]$ be a solution to the integral equation (221). In the view of Lemma 9.7, we arrive at the equation (231). Now, we show that the initial condition of the problem (231) also hold. Taking the limit as $t \to 0$ in both sides of equation (221). In the view of Lemma 10.2, we obtain the initial value, hence the result is proved. $\square$

In the remaining part of this section, we will consider the Cauchy problem

$$\begin{cases} \dfrac{dx}{dt} & = f(t, x(t)), \quad t \in I, \\ x(0) & = 0. \end{cases} \tag{225}$$

Definition 10.5. A function $x : I \to E$ is said to be a pseudo-solution of (225) if

(a) $x(\cdot)$ absolutely continuous,
(b) $x(0) = 0$,
(c) For each $\varphi \in E^*$ there exists a null set $N(\varphi)$ (i.e. N is depending on φ and $mes(N(\varphi)) = 0$), such that for each $t \notin N(\varphi)$

$$(\varphi x)'(t) = \varphi(f(t, x(t))) \tag{226}$$

where x' denote the pseudo-derivative (see Pettis [130]).

In other words by a pseudo-solution of (225) we will understand an absolutely continuous function such that $x(0) = 0$ and for each $\varphi \in E^*$, $x(\cdot)$ satisfies (226) a.e. on I. A strong (weak) solution of (225) is an absolutely (weakly) continuous function with strong (weak) derivative satisfying (225) a.e. on I. Each strong (or weak) solution is also a pseudo-solution. The converse is not (in general) true.

Corollary 10.1. *Under the assumptions of Theorem 10.5, the Cauchy problem (225) has a pseudo-solution. In fact this solution is a strong solution.*

Proof. Look at the problem (221). Put $a_1 = a_2 = \cdots = a_{n-1} = 0$ and $\alpha_n \to 1$. With the use of Lemma 9.6 and Theorem 10.5, it can be easily be seen that the problem

$$x(t) = \int_0^t f(s, x(s)) \ ds, \tag{227}$$

has a solution $x \in C[I, E]$. Observe the definition of Q. It follows that (as $\alpha_n \to 1$) every solution $x \in Q$ will be absolutely continuous. Since f is Pettis integrable, then the existence of a pseudo-solution of (227) is equivalent to the existence of the solution of (225) (cf. [130], sec. 8). Moreover, since E is reflexive, then E has the Radon-Nikodym property (see the comment below the Example 8.10) which implies that each pesudo-solution is a strong solution. $\square$

Remark 10.1. We can introduce a definition of solution to the equation (221) in the space $P(I, E)$ (understand as a solution a Pettis-integrable function), but in this case we can not use the classical fixed point theorems because a space $P(I, E)$ is incomplete.

Example G:

In this example we prove the existence of continuous solutions to the integral equation of the following fractional type

$$x(t) = g(t) + \lambda\, I^{\alpha}\, f(t, x(t)),\ t \in I,\ \alpha \in (0,1), \tag{228}$$

in the Banach space $C[I, E_w]$. Here, we assume that E is reflexive.
The problem (228) is special case of (221) and we proceed here in different way making up Tychonoff fixed point theorem (Theorem 10.2)

Let us state the following assumptions:

1. $g : I \to E$ is weakly continuous function,
2. $f : [0,1] \times B_r \to E$ is a weakly-weakly continuous.
3. There exists $r > 0$ such that $\frac{M_r|\lambda|}{\Gamma(1+\alpha)} < r$, where M_r is the real number defined in Proposition 10.1.

Now, we are in a position to state and prove the following theorem

Theorem 10.7. *Let the assumptions* $(1)-(3)$ *be satisfied. Then the integral equation (228) has at least one weak solution* $x(\cdot) \in C[I, E_w]$ *in fact the solution we produce will be norm continuous.*

Proof. Define the convex and norm closed subset

$$Q := \{x \in C[I, E_w] : x \text{ is norm continuous and } \ \|x\| < \|g\| + r.\}$$

First notice that $C[I, E_w]$ is locally convex topological space (see [83]). Since a convex subset of a normed space is closed if and only if it is weakly closed (Lemma 8.2), then Q is weakly closed.
Define the operator $T : C[I, E_w] \to C[I, E_w]$ by

$$Tx(t) = g(t) + \lambda\, I^{\alpha}\, f(t, x(t)),\ t \in I,\ \alpha \in (0,1). \tag{229}$$

According to Proposition 10.2, $f(\cdot, x(\cdot))$ is weakly continuous in I, so the operator T is makes sense. We claim that $T : Q \to Q$ is weakly continuous and $T(Q)$ is weakly relatively compact. Once the claim is established, then Theorem 10.2 with $F = C[I, E_w]$ guarantees a fixed point of T, and hence (228) has a solution in $C[I, E_w]$.
We start by showing that $T : Q \to Q$. To see this, take $x \in Q$ and look at $t \to I^{\alpha}\, f(t, x(t))$ for $t \in I$. Without loss of generality assume $I^{\alpha}\, f(t, x(t)) \neq 0$ for all $t \in I$. Then Proposition 10.4 implies that there exits a $\varphi \in E^*$ with

$\|\varphi\| = 1$ and $\varphi(I^\alpha\ f(t, x(t))) = \|I^\alpha\ f(t, x(t))\|$. Thus

$$\begin{aligned}\|Tx(t)\| &\le \|g(t)\| + \|\lambda I^\alpha f(t, x(t))\| \\ &= \|g(t)\| + \varphi(\lambda I^\alpha f(t, x(t)) \\ &\le \|g(t)\| + \lambda I^\alpha \varphi(f(t, x(t)) \\ &\le \|g(t)\| + |\lambda| M_r \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\ ds \\ &\le \|g(t)\| + \frac{|\lambda| M_r}{\Gamma(1+\alpha)} \\ &< \|g(t)\| + r.\end{aligned}$$

therefore

$$\|Tx(t)\| < \|g(t)\| + r. \tag{230}$$

It remain to show Tx is norm continuous for any $x \in Q$. To see this, let $t, s \in [0, 1]$ with $t > s$. Without loss of generality, assume $Tx(t) - Tx(s) \neq 0$. Then there exists (consequence of Proposition 10.4) $\varphi \in E^*$ with $\|\varphi\| = 1$ and $\|Tx(t) - Tx(s)\| = \varphi(Tx(t) - Tx(s))$. Thus

$$\begin{aligned}|\varphi(Tx(t) &- Tx(s))| \le |\varphi\left(g(t) - g(s)\right)| \\ &+ |\lambda| \left| \int_0^t \frac{(t-\theta)^{\alpha-1}}{\Gamma(\alpha)} \varphi(f(\theta, x(\theta)))\ d\theta - \int_0^s \frac{(s-\theta)^{\alpha-1}}{\Gamma(\alpha)} \varphi(f(\theta, x(\theta)))\ d\theta \right| \\ &\le \|g(t) - g(s)\| + |\lambda| \left| \int_0^t \left(\frac{(s-\theta)^{\alpha-1}}{\Gamma(\alpha)} - \frac{(t-\theta)^{\alpha-1}}{\Gamma(\alpha)} \right) \varphi(f(\theta, x(\theta)))\ d\theta \right| \\ &+ |\lambda| \left| \int_t^s \frac{(s-\theta)^{\alpha-1}}{\Gamma(\alpha)} \varphi(f(\theta, x(\theta)))\ d\theta \right| \\ &\le \|g(t) - g(s)\| + \frac{|\lambda| M_r}{\Gamma(1+\alpha)} \left\{ |t^\alpha - s^\alpha| + 2(t-s)^\alpha \right\},\end{aligned}$$

So TQ is norm continuous. Hence $T : Q \to Q$. Also $T : Q \to Q$ is weakly continuous. To see this, notice if $x_n \rightharpoonup x$ in Q (here (x_n) is a net in Q), i.e., x_n converges weakly uniformly to x on [0,1], then since f satisfies assumption (2) we have immediately Krasnosielski-type lemma ([185], Lemma 2) $f(s, x_n(s))$ converging weakly to $f(s, x(s))$ for $s \in [0, 1]$, hence

$$|\varphi(Tx_n(t) - Tx(t))| \le \frac{|\lambda|}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |\varphi(f(s, x_n(s)) - f(s, x(s))\ )|\ ds \le \epsilon.$$

So, T is weakly continuous.

Remark The usual idea at this stage is to apply the Lebesgue dominated convergence theorem as in **Example F** to deduce the weak continuity of TQ. However the Lebesgue dominated convergence theorem is certainly valid for sequences not for nets (see Example 10.1).

Next we show that $T(Q)$ is weakly relatively compact. To see this, we apply both the Arzelà-Ascoli and the Eberlein Šmulian theorems:
Choose sequence $x_n \in Q, n \geq 1$. Our aim is to show first that for each $t \in I$ the set $\{Tx_n(t) : n \geq 1\}$ is weakly relatively compact. This follows immediately from Theorem 8.1 since the inequality (230) shows that for each $t \in I$ the set $\{Tx_n(t) : n \geq 1\}$ is norm bounded.
Next we show that $T(Q)$ is weakly equicontinuous. Let $x \in Q$ be arbitrary, and let $t, s \in [0, 1]$ with $t > s$. Without loss of generality, assume $Tx(t) - Tx(s) \neq 0$. Then there exists (consequence of Proposition 10.4) $\varphi \in E^*$ with $\|\varphi\| = 1$ and $\|Tx(t) - Tx(s)\| = \varphi(Tx(t) - Tx(s))$. Thus

$$\begin{aligned}
|\varphi(Tx(t) \;\; &- \;\; Tx(s))| \leq |\varphi\left(g(t) - g(s)\right)| \\
&+ \;\; \left|\int_0^t \frac{(t-\theta)^{\alpha-1}}{\Gamma(\alpha)}\varphi(f(\theta, x(\theta)))\; d\theta - \int_0^s \frac{(s-\theta)^{\alpha-1}}{\Gamma(\alpha)}\varphi(f(\theta, x(\theta)))\; d\theta\right| \\
&\leq \;\; \|g(t) - g(s)\| + \left|\int_0^t \left(\frac{(s-\theta)^{\alpha-1}}{\Gamma(\alpha)} - \frac{(t-\theta)^{\alpha-1}}{\Gamma(\alpha)}\right)\varphi(f(\theta, x(\theta)))\; d\theta\right| \\
&+ \;\; \left|\int_t^s \frac{(s-\theta)^{\alpha-1}}{\Gamma(\alpha)}\varphi(f(\theta, x(\theta)))\; d\theta\right| \\
&\leq \;\; \|g(t) - g(s)\| + \frac{|\lambda| M_r}{\Gamma(1+\alpha)}\left\{|t^\alpha - s^\alpha| + 2(t-s)^\alpha\right\},
\end{aligned}$$

So $T(Q)$ is weakly equicontinuous. Theorem 8.2 guarantees that the weak closure of $T(Q)$ is weakly sequentially compact, and this together with Theorem 8.1 implies that the weak closure of $T(Q)$ is weakly compact, i.e. $T(Q)$ is weakly relatively compact. Theorem 10.2 now guarantees that the integral equation (228) has a at least one solution $x \in Q$. □

If we proceed as in the proof of Theorem 10.6, it follows that the differential equation of the fractional type

$$\left\{\begin{array}{l} D^\alpha x(t) = f(t, x(t)), \quad t \in [0, 1], \\ \;x(0) = 0. \end{array}\right. \tag{231}$$

has at least one weak solution $x \in C[I, E]$.

10.2. **Existence results for Boundary value problems in Banach spaces.**
We introduce the following set of examples

<u>Example H</u>: (On the fractional order $m-$point boundary value problem in reflexive Banach spaces)

In this example, sufficient conditions are given for the existence of pseudo solutions for the following nonlinear $m-$point boundary value problem of fractional type

$$\begin{cases} D^{\alpha}x(t)+q(t)f(t,x(t))=0, \; a.e. \text{ on } [0,1], \; \alpha\in(n-1,n], \; n\geq 2, \\ x(0)=x^{'}(0)=x^{''}(0)=\cdots=x^{(n-2)}(0)=0, \; x(1)=\sum_{i=1}^{m-2}\zeta_i x(\eta_i), \end{cases} \tag{232}$$

where, x takes values in a reflexive Banach space E. Here $0<\eta_1<\cdots<\eta_{m-2}<1$, $\zeta_i>0$ with $\displaystyle\sum_{i=1}^{m-2}\zeta_i\eta_i^{\alpha-1}<1$ and $x^{(k)}$ denotes the kth pesudo-derivative of x while D^{α} denotes the pseudo fractional differential operator of order α. We assume that $q(\cdot)$ is continuous real-valued function on [0,1], while f is vector-valued Pettis-integrable function. We remark that no compactness condition will be assumed on the nonlinearity of f, this will be due to the fact that a subset of a reflexive Banach space is weakly compact if and only if it is weakly closed and norm bounded.

To obtain the integral equation modeled off the problem (232), we keep the boundary value problem (232) in mind and we *formally* put (*cf.* [13], Lemma 2.3)

$$x(t)=-I^{\alpha}q(t)f(t,x(t))+ct^{\alpha-1}, \tag{233}$$

where I^{α} denotes the Pettis-fractional integral operator corresponding to the operator D^{α}. We solve equation (233) for c by $x(1)=\sum_{i=1}^{m-2}\zeta_i x(\eta_i)$, it follows that

$$\begin{aligned} c &= \int_0^1\frac{(1-s)^{\alpha-1}q(s)f(s,x(s))}{\Gamma(\alpha)}\,ds \\ &+ \sum_{i=1}^{m-2}\zeta_i\left(c\eta_i^{\alpha-1}-\int_0^{\eta_i}\frac{(\eta_i-s)^{\alpha-1}q(s)f(s,x(s))}{\Gamma(\alpha)}\,ds\right) \\ &= \int_0^1\frac{(1-s)^{\alpha-1}q(s)f(s,x(s))}{\Gamma(\alpha)}\,ds \\ &+ cA-\sum_{i=1}^{m-2}\zeta_i\int_0^{\eta_i}\frac{(\eta_i-s)^{\alpha-1}q(s)f(s,x(s))}{\Gamma(\alpha)}\,ds, \end{aligned}$$

where $A:=\sum_{i=1}^{m-2}\zeta_i\eta_i^{\alpha-1}$. Therefore

$$c=\int_0^1\frac{(1-s)^{\alpha-1}q(s)f(s,x(s))}{(1-A)\Gamma(\alpha)}\,ds-\sum_{i=1}^{m-2}\zeta_i\int_0^{\eta_i}\frac{(\eta_i-s)^{\alpha-1}q(s)f(s,x(s))}{(1-A)\Gamma(\alpha)}\,ds.$$

Substituting c into (233) implies

$$\begin{aligned}
x(t) &= -I^{\alpha}q(t)f(t,x(t)) + \int_0^1 \frac{t^{\alpha-1}(1-s)^{\alpha-1}q(s)f(s,x(s))}{(1-A)\Gamma(\alpha)}\,ds \\
&- \sum_{i=1}^{m-2}\zeta_i \int_0^{\eta_i} \frac{t^{\alpha-1}(\eta_i-s)^{\alpha-1}q(s)f(s,x(s))}{(1-A)\Gamma(\alpha)}\,ds \\
&= \int_0^t \frac{(t(1-s))^{\alpha-1}-(t-s)^{\alpha-1}}{\Gamma(\alpha)} q(s)f(s,x(s))\,ds \\
&+ \int_t^1 \frac{(t(1-s))^{\alpha-1}}{\Gamma(\alpha)} q(s)f(s,x(s))\ \,ds \\
&+ A\int_0^1 \frac{t^{\alpha-1}(1-s)^{\alpha-1}q(s)f(s,x(s))}{(1-A)\Gamma(\alpha)}\,ds \\
&- \sum_{i=1}^{m-2}\zeta_i \int_0^{\eta_i} \frac{t^{\alpha-1}(\eta_i-s)^{\alpha-1}q(s)f(s,x(s))}{(1-A)\Gamma(\alpha)}\ \,ds,
\end{aligned}$$

therefore,

$$x(t) = \int_0^1 G(t,s)q(s)f(s,x(s))\,ds,\ t\in[0,1], \tag{234}$$

where the Green function G given by $G(t,s) = G_1(t,s) + G_2(t,s)$, where

$$G_1(t,s) := \begin{cases} \dfrac{1}{\Gamma(\alpha)}\left[(t(1-s))^{\alpha-1} - (t-s)^{\alpha-1}\right],\ 0\le s\le t\le 1, \\ \dfrac{1}{\Gamma(\alpha)}\left[(t(1-s))^{\alpha-1}\right],\ 0\le t\le s\le 1, \end{cases} \tag{235}$$

$$G_2(t,s) := \frac{1}{(1-A)\Gamma(\alpha)}\left[A(t(1-s))^{\alpha-1} - \sum_{s\le\eta_i}\zeta_i(t(\eta_i-s))^{\alpha-1}\right]. \tag{236}$$

Lemma 10.3. *If* $A := \sum_{i=1}^{m-} \zeta_i\eta_i^{\alpha-1} < 1$, *there exists* $g\in L_\infty([0,1])$ *with* $g(s)\ge 0$ *a.e.* $t\in[0,1]$ *and* $\|g\|_\infty > 0$ *such that* $G_t(s) := G(t,s) \le g(s)$ *for each* $t\in[0,1]$, *a.e.* $s\in[0,1]$, *in addition the map* $t\to G_t$ *is continuous from* $[0,1]$ *to* $L_\infty[0,1]$.

Proof. Observe the expression of G. It is clear that, for $s\in(0,1)$, $G_1(t,s)$ is decreasing with respect to t for $s\le t$ and increasing with respect to t for $t\le s$. Consequently, with the use of the monotonicity of $G_1(t,s)$, we have

$$\max_{t\in[0,1]} G_1(t,s) = G_1(s,s) = \frac{1}{\Gamma(\alpha)}\left[s(1-s)\right]^{\alpha-1},\ s\in(0,1).$$

Define $g_1 \in C[0,1]$ by $g_1(s) := G_1(s,s) > 0,\ s \in (0,1)$. Furthermore, we have

$$G_2(t,s) \leq \frac{A}{(1-A)\Gamma(\alpha)}\left[(t(1-s))^{\alpha-1}\right].$$

Therefore

$$\max_{t\in[0,1]} G_2(t,s) = \frac{A}{(1-A)\Gamma(\alpha)}\left[(1-s)^{\alpha-1}\right],\ s \in (0,1).$$

Define $g_2 \in C[0,1]$ by

$$g_2(s) := \frac{A}{(1-A)\Gamma(\alpha)}\left[(1-s)^{\alpha-1}\right],\ s \in (0,1).$$

Therefore, $g := g_1 + g_2 \in L_\infty([0,1])$ and $g(s) \geq 0$ *a.e.* $t \in [0,1]$. Also $\|g\|_\infty > 0$ and $G_t(s) := G(t,s) \leq g(s)$ for each $t \in [0,1]$, *a.e.* $s \in [0,1]$, in addition the map $t \to G_t$ is continuous from $[0,1]$ to $L_\infty[0,1]$. Hence the result. □

To facilitate our discussion, let $f : [0,1] \times E \to E$ satisfy the following assumptions:

1. For each $t \in I = [0,1]$, $f(t,\cdot)$ is weakly sequentially continuous,
2. For each $x \in C[I,E]$, $f(\cdot,x(\cdot))$ is Pettis integrable on I,
3. For any $r > 0$ there exists an integrable function $M_r : I \to \mathbb{R}^+$ such that $\|f(t,x)\| \leq M_r(t)$ for all $t \in [0,1]$ and $\|x\| \leq r$.

Now, we are in the position to formulate and prove the following existence result

Theorem 10.8. *Let the Assumptions* $(1)-(3)$ *be satisfied. Then there exists* $\rho > 0$ *such that for any* $q \in C([0,1],\mathbb{R})$ *with* $\|q\|_0 < \rho$*, the integral equation (234) has at least one solution* $x \in C[I,E]$.

Proof. Let

$$\rho := \left(\sup_{r>0} \frac{r}{\displaystyle\int_0^1 g(s)M_r(s)\,ds}\right)$$

Fix $q \in C(I,\mathbb{R})$, $\|q\|_0 < \rho$ and choose $r_0 > 0$ such that

$$\|q\|_0 \int_0^1 g(s)M_{r_0}(s)\,ds \leq r_0. \tag{237}$$

Define the operator $T : C[I,E] \to C[I,E]$ by

$$Tx(t) := \int_0^1 G(t,s)q(s)f(s,x(s))\,ds,\ t \in [0,1].$$

First notice that, for $x \in C[I,E]$, $f(\cdot,x(\cdot)) \in P[I,E]$ (Assumption 3.). Since, $s \longmapsto q(s)G(t,s) \in L_\infty(I)$, then (thanks to Proposition 8.6), $q(\cdot)G(t,\cdot)f(\cdot,x(\cdot))$, $t \in$

$[0,1]$ is Pettis integrable and thus, the operator T makes sense. Also, T is well-defined. To see this, let $t_1, t_2 \in [0,1]$ with $t_2 > t_1$. Without loss of generality, assume $Tx(t_2) - Tx(t_1) \neq 0$. Then there exists (consequence of Proposition 10.4) $\varphi \in E^*$ with $\|\varphi\| = 1$ and $\|Tx(t_2) - Tx(t_1)\| = \varphi(Tx(t_1) - Tx(t_1))$. Thus

$$\begin{aligned}
\|Tx(t_2) - Tx(t_1)\| &= \varphi(Tx(t_2) - \lambda Tx(t_1)) \\
&\leq \int_0^1 |G(t_2,s) - G(t_1,s)||q(s)|M_r(s)\,ds \\
&\leq (\|q\|_0 \, \|G(t_2,s) - G(t_1,s)\|_\infty) \left(\int_0^1 |M_r(s)|\,ds\right)
\end{aligned}$$

This estimation shows that T maps $C[I,E]$ into itself. Define the convex, closed and equicontinuous subset $Q \in C[I,E]$ by

$$\begin{aligned}
Q &:= \{x \in C[I,E] : \|x\|_0 \leq r_0, \forall t_1,\ t_2 \in [0,1] \text{ we have } \|x(t_2) - x(t_1)\| \\
&\leq (\|q\|_0 \, \|G(t_2,s) - G(t_1,s)\|_\infty) \left(\int_0^1 |M_r(s)|\ ds\right)\}.
\end{aligned}$$

We claim that $T : Q \longrightarrow Q$ is weakly sequentially continuous. Once the claim is established, Theorem 10.2 guarantees the existence of a fixed point of T. Hence the integral equation (234) has a solution in $C[I,E]$. We begin by showing that $T : Q \to Q$. To see this, take $x \in Q,\ t \in [0,1]$. Without loss of generality, assume $Tx(t) \neq 0$. Then there exists (consequence of Proposition 10.4) $\varphi \in E^*$ with $\|\varphi\| = 1$ and $\|Tx(t)\| = \varphi(Tx(t))$. Thus

$$\begin{aligned}
\|Tx(t)\| &\leq \varphi\left(\int_0^1 G(t,s)q(s)f(s,x(s))\,ds\right) \\
&\leq \sup_{t\in[0,1]} \int_0^1 |G(t,s)q(s)||\varphi(f(s,x(s)))|\ ds \\
&\leq \left(\|q\|_0 \int_0^1 g(s)M_{r_0}(s)\ ds\right),
\end{aligned}$$

therefore $\|Tx\|_0 = \sup\limits_{t\in[0,1]} \|Tx(t)\| \leq r_0$. Hence $T : Q \to Q$.

To see that $T : Q \to Q$ is weakly sequentially continuous we let (x_n) be a sequence in Q and let $x_n(t) \to x(t)$ in (E,w) for each $t \in [0,1]$.

Recall [125], that a sequence (x_n) is weakly convergent in $C[I,E]$ if, and only if, it is weakly pointwise convergent in E. Fix $t \in I$. From the weak sequential continuity of $f(t,\cdot)$ (Assumption 1.), the Lebesgue dominated convergence theorem for the Pettis-integral (Theorem 8.10) implies for each $\varphi \in E^*$ that $\varphi(Tx_n(t)) \to \varphi(Tx(t))$ a.e. on I, $Tx_n(t) \to Tx(t)$ in E_w. We do it for each $t \in I$, so $T : Q \to Q$ is weakly sequentially continuous.

Applying now Theorem 10.2 we conclude that T has a fixed point in Q, which completes the proof. $\square$

In what follows, we looking for sufficient conditions to ensure the existence of pseudo solution to the boundary value problem (232) under the Pettis integrability as assumption. In order to obtain the existence of solutions of the problem (232), we can make use of Theorem 10.8. It is worth to recall the following:

Definition 10.6. A function $x : I \to E$ is called pseudo solution of (232) if $x \in C[I, E]$ has FPD of order $\alpha \in (n-1, n]$, $x(0) = x^{'}(0) = x^{''}(0) = \cdots = x^{(n-2)}(0) = 0$, $x(1) = \sum_{i=1}^{m-2} \zeta_i x(\eta_i)$ and satisfies

$$D^n \varphi(I^{n-\alpha} x(t)) = -q(t)\varphi(f(t, x(t))) \ a.e. \text{ on } [0, 1], \quad \text{for each } \varphi \in E^*.$$

Theorem 10.9. *Let $A = \sum_{i=1}^{m-2} \zeta_i \eta_i^{\alpha-1} < 1$ and assume that $q : [0, 1] \to \mathbb{R}$ be continuous. Suppose that $f : I \times E \to E$ satisfy the assumptions* (1) – (3)*. Then there exists $\rho > 0$ such that for any $q \in C([0, 1], \mathbb{R})$ with $\|q\|_0 < \rho$, the boundary value problem (232) has at least one solution $x \in C[I, E]$.*

Proof. By Theorem 10.8, the integral equation (234) has a solution $x \in C[I, E]$. Let us remark, by Proposition 8.6 the functions $q(\cdot)f(\cdot, x(\cdot))$ is Pettis integrable in [0,1]. In the view of Lemma **??**, the FPI of $q(\cdot)f(\cdot, x(\cdot))$ of order $\alpha > 1$ exists and

$$\varphi(I^\alpha q(t)f(t, x(t))) = I^\alpha \varphi(q(t)f(t, x(t)), \text{ for all } \varphi \in E^*.$$

Let x be a solution of equation (234) then

$$\begin{aligned} x(t) &= \int_0^1 [G_1(t, s) + G_2(t, s)]q(s)f(s, x(s))\,ds \\ &= -\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} q(s)f(s, x(s))\,ds + ct^{\alpha-1}, \end{aligned}$$

$$c = \int_0^1 \frac{(1-s)^{\alpha-1}q(s)f(s, x(s))}{(1-A)\Gamma(\alpha)}\,ds - \sum_{i=1}^{m-2} \zeta_i \int_0^{\eta_i} \frac{(\eta_i - s)^{\alpha-1}q(s)f(s, x(s))}{(1-A)\Gamma(\alpha)}\,ds.$$

Since $G_1(0, s) = G_2(0, s) = 0$, it can easily be seen that

$$x(0) = 0, \text{ and } x(1) = \sum_{i=1}^{m-2} \zeta_i x(\eta_i).$$

In fact we have

$$x(t) = \int_0^1 [G_1(0, s) + G_2(0, s)]q(s)f(s, x(s))\,ds = 0,$$

$$\begin{aligned}
x(1) &= \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} q(s)f(s,x(s)) \left[\frac{1}{1-A} - 1\right] ds \\
&- \sum_{i=1}^{m-2} \zeta_i \int_0^{\eta_i} \frac{(\eta_i - s)^{\alpha-1} q(s) f(s,x(s))}{(1-A)\Gamma(\alpha)} ds \\
&= \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} q(s)f(s,x(s)) \left[\frac{A}{1-A}\right] ds \\
&- \sum_{i=1}^{m-2} \zeta_i \int_0^{\eta_i} \frac{(\eta_i - s)^{\alpha-1} q(s) f(s,x(s))}{(1-A)\Gamma(\alpha)} ds,
\end{aligned}$$

and

$$\begin{aligned}
\zeta_i x(\eta_i) &= -\zeta_i \int_0^{\eta_i} \frac{(\eta_i - s)^{\alpha-1} q(s) f(s,x(s))}{\Gamma(\alpha)} ds \\
&+ \zeta_i \eta_i^{\alpha-1} \int_0^1 \frac{(1-s)^{\alpha-1}}{(1-A)\Gamma(\alpha)} q(s) f(s,x(s))\, ds \\
&- \zeta_i \eta_i^{\alpha-1} \left(\sum_{i=1}^{m-2} \zeta_i \int_0^{\eta_i} \frac{(\eta_i - s)^{\alpha-1} q(s) f(s,x(s))}{(1-A)\Gamma(\alpha)} ds \right).
\end{aligned}$$

From here we arrive at

$$\begin{aligned}
\sum_{i=1}^{m-2} \zeta_i x(\eta_i) &= -\sum_{i=1}^{m-2} \zeta_i \int_0^{\eta_i} \frac{(\eta_i - s)^{\alpha-1} q(s) f(s,x(s))}{\Gamma(\alpha)} ds \\
&+ A \int_0^1 \frac{(1-s)^{\alpha-1}}{(1-A)\Gamma(\alpha)} q(s) f(s,x(s))\, ds \\
&- A \left(\sum_{i=1}^{m-2} \zeta_i \int_0^{\eta_i} \frac{(\eta_i - s)^{\alpha-1} q(s) f(s,x(s))}{(1-A)\Gamma(\alpha)} ds \right) \\
&= A \int_0^1 \frac{(1-s)^{\alpha-1}}{(1-A)\Gamma(\alpha)} q(s) f(s,x(s))\, ds \\
&- \sum_{i=1}^{m-2} \zeta_i \int_0^{\eta_i} \frac{(\eta_i - s)^{\alpha-1} q(s) f(s,x(s))}{\Gamma(\alpha)} ds \left(\frac{A}{1-A} + 1 \right) \\
&= x(1).
\end{aligned}$$

Furthermore, we have

$$x(t) = -I^\alpha q(t) f(t,x(t)) + ct^{\alpha-1}, \tag{238}$$

Since $x, ct^{\alpha-1} \in C[I,E]$, then $t \to I^\alpha q(t) f(t,x(t)) \in C[I,E]$. In addition $\varphi(I^\alpha q(t)f(t,x(t))) = I^\alpha \varphi(q(t)f(t,x(t)))$ and $\varphi(I^{n-\alpha}x(t)) = I^{n-\alpha}\varphi(x(t))$, for all $\varphi \in E^*$.

From equation (265), we deduce that

$$\varphi x(t) = -\varphi(I^{\alpha}q(t)f(t,x(t)))+\varphi ct^{\alpha-1} = -I^{\alpha}\varphi(q(t)f(t,x(t)))+\varphi ct^{\alpha-1}. \quad (239)$$

Define the set J by $J := \{1,2,\cdots,n-2\}$ and let $j \in J$, $\alpha = (n-1)+\delta = 1+(n-2+\delta-j)+j$, $\delta > 0$. Thus we are able to rewrite equation (266) as

$$\varphi x(t) = -I^{1+j}I^{n-2+\delta-j}\varphi(q(t)f(t,x(t))) + \varphi ct^{n-2+\delta}$$

whence

$$D^{(j)}\varphi x(t) = -I^{1}I^{n-2+\delta-j}\varphi(q(t)f(t,x(t))) + \varphi c\frac{\Gamma(1+n-2+\delta)}{\Gamma(1+n-2+\delta-j)}t^{n-2+\delta-j}.$$

Then $x^{(j)}(0) = 0,\ j = 1,2,\cdots,n-2$.

Operating by $I^{n-\alpha}$ on both sides of the equation (266) and using the properties of fractional calculus in the space $L^1[0,1]$ result in

$$I^{n-\alpha}\varphi x(t) = -I^{n}\varphi(q(t)f(t,x(t))) + \varphi c\frac{\Gamma(\alpha)}{\Gamma(n)}t^{n-1}.$$

Therefore,

$$\varphi(I^{n-\alpha}x(t)) = -I^{n}\varphi(q(t)f(t,x(t))) + \varphi c\frac{\Gamma(\alpha)}{\Gamma(n)}t^{n-1}.$$

Thus

$$\frac{d^n}{dt^n}\varphi(I^{n-\alpha}x(t)) = -\varphi(q(t)f(t,x(t)))\ a.e. \text{ on } [0,1].$$

That is, x has the FPD of order $\alpha \in (n-1,n]$ and satisfies the assumptions of Definition 10.6. Therefore, x is a pseudo solution of the differential equation (232). This completes the proof. □

Example I: (On the fractional boundary value problem with integral boundary conditions in reflexive Banach spaces.)

in the present example, sufficient conditions are given for the existence of pseudo solutions for the following nonlinear $m-$point boundary value problem of fractional type

$$\begin{cases} D^{\alpha}x(t) + \lambda f(t,x(t)) = 0,\ a.e. \text{ on } [0,1],\ \alpha \in (n-1,n],\ n \geq 2, \lambda \in \mathbb{R}, \\ x(1) + \int_0^1 u(\tau)x(\tau)d\tau = l,\ x^{(k)}(0) = 0,\ k = 0,1,\cdots,n-2, \end{cases} \quad (240)$$

where, x takes values in a reflexive Banach space E and $l \in E$. Here $u \in L_q([0,1])$ for some $q \in [1,\infty]$ and $x^{(k)}$ denotes the kth pesudo-derivative of x while D^{α} denotes the pseudo fractional differential operator of order α. We assume that f is vector-valued Pettis-integrable function on $[0,1]$. We note, that no compactness condition will be assumed on the nonlinearity of f, this will be due to the fact that a subset of a reflexive Banach space is weakly compact if and only if it is weakly closed and norm bounded.

Now, to obtain the Hammerstein type integral equation modeled off the problem (240), we keep the boundary value problem (240) in mind and we *formally* put

$$x(t) = -I^{\alpha} q(t) f(t, x(t)) + ct^{\alpha-1}, \tag{241}$$

where I^{α} denotes the Pettis-fractional integral operator corresponding to the operator D^{α}.

To facilitate our discussion, let $q \in [1, \infty]$ be constant with the conjugate exponents p. Suppose $u \in L_q(I)$ be a nonnegative real-valued function and $f : [0,1] \times E \to E$ satisfy the following assumptions:

1. For each $t \in I = [0,1]$, $f(t, \cdot)$ is weakly sequentially continuous,
2. For each $x \in C[I, E]$, $f(\cdot, x(\cdot)) \in \mathcal{H}_0^p(E)$,
3. For any $r > 0$ there exists an L_p-integrable function $M_r : [0,1] \to \mathbb{R}^+$ such that $\|f(t,x)\| \leq M_r(t)$ for all $t \in [0,1]$ and $\|x\| \leq r$.

To solve the equation (241) for c by $x(1) + \int_0^1 u(\tau) x(\tau) d\tau = l$, it follows that

$$\begin{aligned} c &= l + \lambda \int_0^1 \frac{(1-s)^{\alpha-1} f(s, x(s))}{\Gamma(\alpha)} ds \\ &- \int_0^1 u(\tau) \left(c\tau^{\alpha-1} - \lambda \int_0^{\tau} \frac{(\tau - s)^{\alpha-1} f(s, x(s))}{\Gamma(\alpha)} ds \right) d\tau \\ &= l - c \int_0^1 u(\tau) \tau^{\alpha-1} d\tau + \lambda \int_0^1 u(\tau) \left(\int_0^{\tau} \frac{(\tau - s)^{\alpha-1} f(s, x(s))}{\Gamma(\alpha)} ds \right) d\tau, \end{aligned}$$

therefore

$$\begin{aligned} c(1+\gamma) &= l + \lambda \int_0^1 u(\tau) \left(\int_0^{\tau} \frac{(\tau - s)^{\alpha-1} f(s, x(s))}{\Gamma(\alpha)} \ ds \right) d\tau \\ &+ \ \lambda \int_0^1 \frac{(1-s)^{\alpha-1}) f(s, x(s)}{\Gamma(\alpha)} \ ds, \end{aligned}$$

where

$$\gamma = \int_0^1 u(\tau) \tau^{\alpha-1} d\tau.$$

Then (in account of Lemma 9.4) we have

$$\begin{aligned}
c &= \frac{1}{1+\gamma}\left[l+\lambda\int_0^1 \frac{(1-s)^{\alpha-1}f(s,x(s))}{\Gamma(\alpha)}\,ds\right.\\
&+ \left.\lambda\int_0^1\int_s^1 u(\tau)\frac{(\tau-s)^{\alpha-1}f(s,x(s))}{\Gamma(\alpha)}d\tau\,ds\right]\\
&= \frac{1}{1+\gamma}\left[l+\lambda\int_0^1 \frac{(1-s)^{\alpha-1}f(s,x(s))}{\Gamma(\alpha)}\;ds+\lambda\int_0^1 h(s)f(s,x(s))\;ds\right],
\end{aligned}$$

where $h(t)=J^\alpha u(t)$ (see formula 213).
Substituting c into (241), we obtain

$$\begin{aligned}
x(t) &= -\lambda I^\alpha f(t,x(t))+\frac{lt^{\alpha-1}}{1+\gamma}+\frac{\lambda t^{\alpha-1}}{1+\gamma}\int_0^1\left[\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}+h(s)\right]f(s,x(s))\,ds\\
&= \frac{lt^{\alpha-1}}{1+\gamma}+\lambda\int_0^t\left[\frac{(t(1-s))^{\alpha-1}}{(1+\gamma)\Gamma(\alpha)}-\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\right]f(s,x(s))\,ds\\
&+ \lambda\int_t^1\frac{(t(1-s))^{\alpha-1}}{(1+\gamma)\Gamma(\alpha)}f(s,x(s))\;ds+\frac{\lambda t^{\alpha-1}}{1+\gamma}\int_0^1 h(s)f(s,x(s)\;ds,
\end{aligned}$$

therefore,

$$x(t)=p(t)+\lambda\int_0^1 G(t,s)f(s,x(s))\;ds,\;t\in[0,1],\tag{242}$$

where $p(t)=\dfrac{lt^{\alpha-1}}{\Gamma(\alpha)}$ and the Green function G given by $G(t,s)=G_1(t,s)+G_2(t,s)$, where

$$G_1(t,s):=\begin{cases}\dfrac{1}{\Gamma(\alpha)}\left[\dfrac{(t(1-s))^{\alpha-1}}{1+\gamma}-(t-s)^{\alpha-1}\right],\;0\le s\le t\le 1,\\[2ex] \dfrac{1}{\Gamma(\alpha)}\left[\dfrac{(t(1-s))^{\alpha-1}}{1+\gamma}\right],\;0\le t\le s\le 1,\end{cases}\tag{243}$$

$$G_2(t,s):=\frac{t^{\alpha-1}}{1+\gamma}h(s),\;t,s\in[0,1].\tag{244}$$

The following results plays major rule in our analysis

Lemma 10.4. *If $u\in L_q([0,1])$ is nonnegative, there exists $g\in L_q([0,1])$ with $g(s)\ge 0$ a.e. $t\in[0,1]$ and $\|g\|_q>0$ such that $G_t(s):=G(t,s)\le g(s)$ for each $t\in[0,1]$, a.e. $s\in[0,1]$, in addition the map $t\to G_t$ is continuous from $[0,1]$ to $L_q[0,1]$.*

Proof. Observe the expression of G. It is clear that, for $s \in (0,1)$, $G_1(t,s)$ is decreasing with respect to t for $s \leq t$ and increasing with respect to t for $t \leq s$. Consequently, with the use of the monotonicity of $G_1(t,s)$, we have

$$\max_{t\in[0,1]} G_1(t,s) = G_1(s,s) = \frac{1}{\Gamma(\alpha)} \left[s(1-s)\right]^{\alpha-1}, \ s \in (0,1).$$

Define $g_1 \in C[0,1]$ by $g_1(s) := G_1(s,s) > 0, \ s \in (0,1)$. Furthermore, we have

$$\max_{t\in[0,1]} G_2(t,s) = \frac{h(s)}{(1+\gamma)}, \ s \in (0,1).$$

As in the proof of Lemma 9.4, we deduce the following implication

$$u \in L_q([0,1]) \Rightarrow h \in L_q([0,1])$$

Define $g_2 \in L_q([0,1])$ by

$$g_2(s) := \max_{t\in[0,1]} G_2(t,s), \ s \in (0,1).$$

Therefore, $g := g_1 + g_2 \in L_q([0,1]))$ and $g(s) \geq 0$ *a.e.* $t \in [0,1]$. Also, $\|g\|_q > 0$ and $G_t(s) := G(t,s) \leq g(s)$ for each $t \in [0,1]$, *a.e.* $s \in [0,1]$. In addition, It can be easily seen that the map $t \to G_t$ is continuous from $[0,1]$ to $L_q([0,1])$. Hence the result. □

Theorem 10.10. *Assume $u \in L_q(I)$ be a nonnegative real-valued function. Let the Assumptions (1)-(3) be satisfied. Then there exists $\rho > 0$ such that for any $\lambda \in \mathbb{R}$ with $|\lambda| < \rho$, the integral equation (242) has at least one solution $x \in C[I,E]$.*

Proof. Define the real number ρ by

$$\rho := \left(\sup_{r>0} \frac{r - \|p\|_0}{\displaystyle\int_0^1 g(s) M_r(s)\, ds} \right)$$

Fix $\lambda \in \mathbb{R}, \ |\lambda| < \rho$ and choose $r_0 > 0$ such that

$$\|p\|_0 + |\lambda| \int_0^1 g(s) M_{r_0}(s) \ ds \leq r_0. \tag{245}$$

Define the operator $T : C[I,E] \to C[I,E]$ by

$$Tx(t) := p(t) + \int_0^1 G(t,s) f(s,x(s)) \ ds, \ t \in [0,1].$$

First notice that, for $x \in C[I,E]$, $f(\cdot,x(\cdot)) \in \mathcal{H}_0^p(E)$ (Assumption 2.). By Lemma 10.4, $s \longmapsto G(t,s) \in L_q(I)$, for all $t \in [0,1]$. Then (thanks to Proposition 8.5), $G(t,\cdot)f(\cdot,x(\cdot)) \in p[I,E]$ for all $t \in [0,1]$ and thus, the operator T makes sense. Also, T is well-defined. To see this, let $t_1, t_2 \in [0,1]$ with $t_2 > t_1$. Without loss of generality, assume $Tx(t_2) - Tx(t_1) \neq 0$. Then there exists (consequence of Proposition 10.4) $\varphi \in E^*$ with $\|\varphi\| = 1$ and $\|Tx(t_2) - Tx(t_1)\| = \varphi(Tx(t_1) - Tx(t_1))$. Thus

$$\begin{aligned}
\|Tx(t_2) - Tx(t_1)\| &= \varphi(\lambda Tx(t_2) - \lambda Tx(t_1)) \leq \varphi(p(t_2) - p(t_1)) \\
&+ |\lambda| \int_0^1 |G(t_2, s) - G(t_1, s)| M_r(s)\ ds \\
&\leq \|p(t_2) - p(t_1)\| \\
&+ |\lambda| \left(\int_0^1 |G(t_2, s) - G(t_1, s)|^q \, ds \right)^{\frac{1}{q}} \left(\int_0^1 |M_r(s)|^p \ ds \right)^{\frac{1}{p}}
\end{aligned}$$

This estimation shows that T maps $C[I, E]$ into itself. Define the convex, closed and equicontinuous subset $Q \in C[I, E]$ by

$$\begin{aligned}
Q &:= \{x \in C[I, E] : \|x\|_0 \leq r_0, \forall t_1,\ t_2 \in [0,1] \text{ we have } \|x(t_2) - x(t_1)\| \\
&\leq \|p(t_2) - p(t_1)\| + |\lambda| \left(\int_0^1 |G(t_2, s) - G(t_1, s)|^q \, ds \right)^{\frac{1}{q}} \left(\int_0^1 |M_r(s)|^p \, ds \right)^{\frac{1}{p}} \}.
\end{aligned}$$

We claim that $T : Q \longrightarrow Q$ is weakly sequentially continuous. Once the claim is established, Theorem 10.2 guarantees the existence of a fixed point of T. Hence the integral equation (242) has a solution in $C[I, E]$. We begin by showing that $T : Q \to Q$. To see this, take $x \in Q,\ t \in [0,1]$. Without loss of generality, assume $Tx(t) \neq 0$. Then there exists (consequence of Proposition 10.4) $\varphi \in E^*$ with $\|\varphi\| = 1$ and $\|Tx(t)\| = \varphi(Tx(t))$. Thus

$$\begin{aligned}
\|Tx(t)\| &\leq \varphi(p(t)) + |\lambda| \varphi \left(\int_0^1 G(t, s) f(s, x(s))\ ds \right) \\
&\leq \|p(t)\| + |\lambda| \sup_{t \in [0,1]} \int_0^1 G(t, s) |\varphi(f(s, x(s)))|\ ds \\
&\leq \|p\|_0 + |\lambda| \left(\int_0^1 g(s) M_{r_0}(s) \, ds \right),
\end{aligned}$$

therefore $\|Tx\|_0 = \sup_{t \in [0,1]} \|Tx(t)\| \leq r_0$. Hence $T : Q \to Q$. Also $T : Q \to Q$ weakly sequentially continuous. To see this, let (x_n) be a sequence in Q and let $x_n(t) \to x(t)$ in (E, w) for each $t \in [0,1]$.

Recall [125], that a sequence (x_n) is weakly convergent in $C[I, E]$ if, and only if, it is weakly pointwise convergent in E. Fix $t \in I$. From the weak sequential continuity of $f(t, \cdot)$ (Assumption 1.), the Lebesgue dominated convergence theorem for the Pettis-integral (Theorem 8.10) implies for each $\varphi \in E^*$ that $\varphi(Tx_n(t)) \to \varphi(Tx(t))$ a.e. on I, $Tx_n(t) \to Tx(t)$ in E_w. We do it for each $t \in I$, so $T : Q \to Q$ is weakly sequentially continuous. Applying now Theorem 10.2 we conclude that T has a fixed point in Q, which completes the proof. $\square$

Now, we are in the position to formulate and prove the following main existence result

Theorem 10.11. *Let $u \in L_q(I)$ be a nonnegative real-valued function. Suppose that $f : I \times E \to E$ satisfy the Assumptions* $(1) - (3)$*. Then there exists $\rho > 0$ such that for any $\lambda \in \mathbb{R}$ with $|\lambda| < \rho$, the boundary value problem (240) has at least one solution $x \in C[I, E]$.*

Proof. Firstly, we remark that, the weakly measurable function $x(\cdot)$ from I to E is Pettis integrable on I if and only if $\varphi(x(\cdot))$ is Lebesgue integrable on I, for every $\varphi \in E^*$ (*cf.* [61]). So, if $x : I \to E$ is weakly continuous function, $x \in \mathcal{H}_0^\infty(E)$. According to Proposition 8.5, $x(\cdot)u(\cdot) \in P[I, E]$ for $u \in L_q$, for $q \in [1, \infty]$. Thus the integral boundary condition makes sense. In account of Theorem 10.10, it can be easily seen that, the integral equation (242) has a solution $x \in C[I, E]$. Let us remark, that in the view of Lemma **??**, the FPI of $f(\cdot, x(\cdot))$ of order $\alpha > 1$ exists and

$$\varphi(I^\alpha f(t, x(t))) = I^\alpha \varphi(f(t, x(t)), \text{ for all } \varphi \in E^*.$$

Let x be a solution of equation (242) then

$$\begin{aligned} x(t) &= \frac{t^{\alpha-1} l}{(1+\gamma)} + \lambda \int_0^1 [G_1(t,s) + G_2(t,s)] f(s, x(s))\ ds \\ &= -\lambda \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x(s))\, ds + c t^{\alpha-1}, \end{aligned}$$

$$c = \frac{1}{1+\gamma}\left[l + \lambda \int_0^1 \left(\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} + h(s) \right) f(s, x(s))\, ds \right].$$

In the view of $G_1(0, s) = G_2(0, s) = 0$ and Lemma 9.4, we obtain by a straightforward estimate that

$$x(0) = 0, \text{ and } x(1) + \int_0^1 u(\tau) x(\tau) d\tau = l.$$

Furthermore, we have

$$x(t) = -\lambda I^\alpha f(t, x(t)) + c t^{\alpha-1}, \tag{246}$$

Since $x, ct^{\alpha-1} \in C[I, E]$, then $t \to I^\alpha f(t, x(t)) \in C[I, E]$. In addition

$$\varphi(I^\alpha f(t, x(t))) = I^\alpha \varphi(f(t, x(t))) \text{ and } \varphi(I^{n-\alpha} x(t)) = I^{n-\alpha} \varphi(x(t)), \text{ for all } \varphi \in E^*.$$

From equation (246), we deduce that

$$\varphi x(t) = -\lambda \varphi(I^\alpha f(t, x(t))) + \varphi c t^{\alpha-1} = -I^\alpha \varphi(f(t, x(t))) + \varphi c t^{\alpha-1}. \tag{247}$$

Define the set J by $J := \{1, 2, \cdots, n-2\}$ and let $j \in J$, $\alpha = (n-1) + \delta = 1 + (n-2+\delta-j) + j,\ \delta > 0$. Then, we are able to rewrite equation (247) as

$$\varphi x(t) = -\lambda I^{1+j} I^{n-2+\delta-j} \varphi(f(t, x(t))) + \varphi c t^{n-2+\delta}$$

whence

$$D^{(j)} \varphi x(t) = -\lambda I^{1} I^{n-2+\delta-j} \varphi(f(t, x(t))) + \varphi c \frac{\Gamma(1+n-2+\delta)}{\Gamma(1+n-2+\delta-j)} t^{n-2+\delta-j}.$$

Then $x^{(j)}(0) = 0,\ j = 1, 2, \cdots, n-2$.

Operating by $I^{n-\alpha}$ on both sides of the equation (247) and using the properties of fractional calculus in the space $L_1[0,1]$ result in

$$I^{n-\alpha} \varphi x(t) = -\lambda I^{n} \varphi(f(t, x(t))) + \varphi c \frac{\Gamma(\alpha)}{\Gamma(n)} t^{n-1}.$$

Therefore,

$$\varphi(I^{n-\alpha} x(t)) = -\lambda I^{n} \varphi(f(t, x(t))) + \varphi c \frac{\Gamma(\alpha)}{\Gamma(n)} t^{n-1}.$$

Thus

$$\frac{d^n}{dt^n} \varphi(I^{n-\alpha} x(t)) = -\lambda \varphi(f(t, x(t)))\ a.e. \text{ on } [0,1].$$

That is, x has the fractional pseudo derivative of order $\alpha \in (n-1, n]$ and satisfies the assumptions of Definition 10.6. Therefore, x is a pseudo solution of the differential equation (240). This completes the proof. $\square$

Now, we consider an example to illustrate our result

Example 10.2. Let $u \in L_\infty([0,1])$. Define $E := \ell^2(I)$ to be Hilbert space of all real functions x on $[0,]$ vanishing off a countable set such that $\sum_{t\in[0,1]} |x(t)|^2 < \infty$, equipped with the inner product $< x, y >:= \sum_{t\in[0,1]} x(t)y(t)$.

For each $t \in I$ define $g(t) = e_t \in \ell^2(I)$ by

$$e_t(s) := \begin{cases} 1 \text{ if } s = t, \\ 0 \text{ if } s \neq t. \end{cases}$$

This function is Pettis, but not Bochner, integrable (it is not even strongly measurable, see also Example 8.20 with $H := \ell^2(I)$). And for any $\varphi \in \ell^2(I)$ we have,

$$\varphi e_t = \sum_{\tau\in[0,1]} \varphi(\tau) e_t(\tau) = \begin{cases} 0 \text{ if } \tau \neq t, \\ \varphi(\tau) \text{ if } \tau = t. \end{cases}$$

Since, $\left(\ell^2(I)\right)^* = \ell^2(I)$, the function φe_t is only countably nonzero for each $\varphi \in \left(\ell^2(I)\right)^*$. Hence $\varphi g \equiv 0$ $a.e.$ (with respect to the Lebesgue measure).

Define $f(t,x) : I \times E \to E$ by

$$f(t,x) := \begin{cases} \circ \text{ if } t < 1, \ x \in E, \\ e_1 \text{ if } t = 1, \ x \in E, \end{cases}$$

where $\circ(t) \equiv 0, \ t \in [0,1]$ and

$$e_1(s) = \begin{cases} 1 \text{ if } s = 1, \\ 0 \text{ if } s \neq 1. \end{cases}$$

Indeed, f is Pettis, but not Bochner, integrable. Therefore, one can not expect a strong solution, that one could only expect (in account of Theorem 10.11) a pseudo solution. Since f satisfies all requirements of Theorem 10.11 with $p = 1$, the problem

$$\begin{cases} D^\alpha x(t) + \lambda f(t, x(t)) = 0, \ a.e. \text{ on } [0,1], \ \alpha \in (n-1, n], \ n \geq 2, \lambda \in \mathbb{R}, \\ x(0) = x'(0) = x''(0) = \cdots = x^{(n-2)}(0) = 0, \ x(1) + \int_0^1 u(\tau)x(\tau)d\tau = l, \end{cases} \tag{248}$$

has a pseudo solution $x \in C[I, E]$ given by

$$x(t) = \frac{t^{\alpha-1}l}{(1+\gamma)} + \lambda \int_0^1 G(t,s)f(s,x(s))\,ds, \ t \in [0,1].$$

To calculate the above integral, let $\varphi \in \left(\ell^2(I)\right)^* = \ell^2(I)$ and note that (see [87])

$$\varphi(f(t,x(t))) = \sum_{\tau \in [0,1]} \varphi(\tau) f(\tau, x(\tau)) = \begin{cases} 0 \text{ if } t < 1, \\ \varphi(1) \text{ if } t = 1. \end{cases}$$

Therefore,

$$\varphi\left(\int_0^1 G(t,s)f(s,x(s))\,ds\right) = \int_0^1 G(t,s)\varphi\left(f(s,x(s))\right)\,ds = 0,$$

so,

$$\lambda \int_0^1 G(t,s)f(s,x(s))\,ds = \circ.$$

Consequently $x(t) = \dfrac{t^{\alpha-1}l}{(1+\gamma)}, \ t \in [0,1]$ is the pseudo solution of the problem (248). It is obvious that $\|x\| \leq r_0, \ r_0 = \frac{\|l\|}{(1+\gamma)}$ and $x^{(j)}(0) = 0, \ j = 0, 1, 2, \cdots, n-2$. Further, we have

$$x(1) + \int_0^1 u(\tau)x(\tau)d\tau = \frac{l}{(1+\gamma)} + l\int_0^1 u(\tau)\frac{\tau^{\alpha-1}}{(1+\gamma)}d\tau = \frac{l}{(1+\gamma)} + l\frac{\gamma}{(1+\gamma)} = l.$$

Moreover,

$$D^n I^{n-\alpha}\varphi x(t) = D^n I^{n-\alpha}\frac{t^{\alpha-1}}{(1+\gamma)}\varphi l = 0 \text{ for all } t \in [0,1] \text{ and } \lambda\varphi(f(t,x(t))) = 0 \text{ for } t \neq 1.$$

Then

$$D^\alpha \varphi x(t) = -\lambda \varphi(f(t, x(t)))\ a.e.\ [0,1].$$

Thus, x is a pseudo-(but not a weak-) solution to the problem (248)

Remark 10.2. If E is a WCG-space (weakly compactly generating space) (in particular, a reflexive Banach space) (see [66]), $f(\cdot, x)$ is strongly measurable, $f(t, \cdot)$ is weakly-weakly continuous and f is bounded, then for each absolutely strongly continuous function x, $f(\cdot, x(\cdot))$ is Pettis-integrable, so our assumption on f seems to be natural.

Example J: (On the fractional boundary value problem in a not necessary reflexive Banach spaces.) In this paper we establish the existence of a pseudo solution to the boundary value problem of fractional type

$$D^\alpha x(t) + \lambda f(t, x(t)) = 0,\ \alpha \in (1, 2],\ a.e. \text{ on } [0,1],\ x(0) = x(1) = 0, \tag{249}$$

where, x takes values in a Banach space E and D^α denotes the pseudo differential operator of fractional order.

As in above examples, keeping the boundary value problem (249) in mind, we begin with the following problem (modeled off the problem (249))

$$x(t) = \lambda \int_0^1 G(t,s) f(s, x(s))\, ds,\ a.e. \text{ on } [0,1], \tag{250}$$

where the Green function G given by

$$G(t,s) := \begin{cases} \dfrac{1}{\Gamma(\alpha)} \left[(t(1-s))^{\alpha-1} - (t-s)^{\alpha-1} \right],\ 0 \le s \le t \le 1, \\ \qquad \dfrac{1}{\Gamma(\alpha)} \left[(t(1-s))^{\alpha-1} \right],\ 0 \le t \le s \le 1. \end{cases}$$

Since the space of all Pettis integrable functions is not complete, we restrict our attention to the weakly continuous solution of the integral equation (250) (*cf.* [**?**]), hence the pseudo solution to the problem (249).

Now, we prove the existence of a weak solutions to the problem (250) in the space $C[I, E]$. By a weak solution to (250) we mean a function $x \in C[I, E_w]$ which satisfies the integral equation (250). This is equivalent to finding $x \in C[I, E_w]$ with

$$\varphi(x(t)) = \varphi\left(\lambda \int_0^1 G(t,s) f(t, x(s))\, ds \right),\ t \in [0,1] \text{ for all } \varphi \in E^*.$$

To facilitate our discussion, let $f : [0,1] \times E \to E$ satisfies the following assumptions:

1. For each $t \in I = [0,1]$, $f(t, \cdot)$ is weakly sequentially continuous,
2. For each $x \in C[I, E]$, $f(\cdot, x(\cdot))$ is Pettis integrable on [0,1],
3. For any $r > 0$ there exists an integrable function $M_r : [0,1] \to \mathbb{R}^+$ such that $\|f(t,x)\| \le M_r(t)$ for all $t \in [0,1]$ and $\|x\| \le r$.

following results plays major rule in our analysis.

Lemma 10.5. *Let f satisfies the Assumptions (1)-(3). Also, let for any $\epsilon > 0$ and for any subset X of B_r there exists a closed subset I_ϵ of the interval I such that $mess(I \setminus I_\epsilon) < \epsilon$ and*

$$\beta(f(T \times X)) \leq \sup_{t \in T} h(t)\beta(X), \tag{251}$$

for each closed subset T of I_ϵ, where $h \in L_1[I, \mathbb{R}^+]$. If V is an equicontinuous set of functions $x : I \to B_r$, then

$$\beta\left(\left\{\lambda \int_0^1 G(t,s)f(s,x(s))\,ds : x \in V\right\}\right) \leq |\lambda| \int_0^1 G(t,s)h(s)\beta(V(s))\,ds, \tag{252}$$

for any $t \in I$.

Proof. As V is equicontinuous, by Ambrosett's lemma (*cf.* [36], Lemma 3.)) the function $t \longmapsto v(t) := \beta(V(t))$ is continuous on I. Fix $t \in I$ and $\epsilon > 0$. Let $\delta = \delta(\epsilon)$ be a number such that

$$\sup_{t \in A} \int_A G(t,s)M_r(s)\,ds \leq \frac{\epsilon}{2} \text{ for } A \subset [0,1] \text{ such that } \mu(A) \leq \delta,$$

where μ denotes the Lebesgue measure in $\mathbb{R}$. By Luzin's theorem, there exists a closed subset K_ϵ of I with $\mu\,(I \setminus K_\epsilon) < \delta$ and such that h is continuous on K_ϵ. Put $J_\epsilon^{'} = I \setminus K_\epsilon$, So $J_\epsilon^{'}$ is open and $\mu\left(J_\epsilon^{'}\right) < \delta$. Hence h is continuous on $I \setminus J_\epsilon^{'}$ and

$$\sup_{t \in J_\epsilon^{'}} \int_{J_\epsilon^{'}} G(t,s)M_r(s)\,ds \leq \frac{\epsilon}{2}.$$

By (251), there exists an open subset $J_\epsilon^{''}$ of I such that $\mu\left(I \setminus J_\epsilon^{''}\right) < \delta$ and

$$\beta(f(T \times X)) \leq \sup_{t \in T} h(t)\beta(X), \text{ for every compact subset } T \text{ of } I \setminus J_\epsilon^{''}.$$

Let $J_\epsilon = J_\epsilon^{''} \cup J_\epsilon^{'}$ and $A := I \setminus J_\epsilon$. Then h is continuous on A, inequality (251) holds and

$$\sup_{t \in J_\epsilon} \int_{J_\epsilon} G(t,s)M_r(s)\,ds \leq \epsilon.$$

Let $W \subset V$, for any $x \in W$, we observe that

$$\left\|\lambda \int_{J_\epsilon} G(t,s)f(s,x(s))\,ds\right\| \leq |\lambda| \sup_{t \in J_\epsilon} \int_{J_\epsilon} G(t,s)M_r(s)\,ds \leq |\lambda|\epsilon.$$

Since $\beta(W) \leq$ diam (W) for bounded W, we have

$$\beta\left(\left\{\lambda \int_{J_\epsilon} G(t,s)f(s,x(s))\,ds : x \in W\right\}\right) \leq 2|\lambda|\epsilon.$$

Now, divide the interval I into n parts $0 = t_0 < t_1 < \cdots < t_n = 1$ in such a way that

$$|h(s_1)G(t,s_2)v(s_3) - h(r_1)G(t,r_2)v(r_3)| < \epsilon \text{ for } s_1, s_2, s_3, r_1, r_2, r_3 \in T_i := [t_{i-1}, t_i]\backslash J_\epsilon.$$

Let $G_i = \sup_{s\in T_i} |G(t,s)| = |G(t,s_i)|,\ h_i = \sup_{s\in T_i} |h(s)| = |h(\tau_i)|,\ s_i, \tau_i \in T_i,$
Set $V_i = \{x(s) : x \in V, s \in T_i\}$. In the view of Ambrosett's lemma there exists $s_i \in T_i$ such that $\beta(V_i) = v(s_i)$. Then

$$\begin{aligned}
&\beta\left(\int_0^1 G(t,s)f(s,V(s))\,ds\right)\\
&\leq \beta\left(\lambda\int_A G(t,s)f(s,V(s))\,ds + \lambda\int_{J_\epsilon} G(t,s)f(s,V(s))\right)\\
&\leq \beta\left(\lambda\int_A G(t,s)f(s,V(s))\,ds\right) + 2|\lambda|\epsilon.
\end{aligned}$$

By the Pettis-integral mean value theorem (Theorem 8.11) we obtain

$$\begin{aligned}
\lambda\int_A G(t,s)f(s,x(s))\,ds &= \sum_{i=1}^n \lambda\int_{T_i} G(t,s)f(s,x(s))\,ds\\
&\in |\lambda|\sum_{i=1}^n \mu(T_i)\overline{\text{conv}}\left\{|G(t,s)|f(s,x(s)) : x \in V, s \in T_i\right\}\\
&\subset |\lambda|\sum_{i=1}^n \mu(T_i)\overline{\text{conv}}\left\{\bigcup_{|\gamma|<G_i} \gamma f(s,x(s)) : x \in V, s \in T_i\right\}.
\end{aligned}$$

Furthermore, by the properties of the De Blasi measure of weak noncompactness β, we have

$$
\begin{aligned}
&\beta\left(\lambda\int_A G(t,s)f(s,V(s))\,ds\right)\\
&\quad\le\ \beta\left(\lambda\sum_{i=1}^{n}\mu(T_i)\overline{\mathrm{conv}}\left\{\bigcup_{|\gamma|<G_i}|\gamma|f(s,x(s)) : x\in V, s\in T_i\right\}\right)\\
&\quad\le\ |\lambda|\sum_{i=1}^{n}\mu(T_i)G_i\beta(f(T_i\times V_i))\\
&\quad\le\ |\lambda|\sum_{i=1}^{n}\mu(T_i)G_i\sup_{s\in T_i}h(s)\beta(V_i)\\
&\quad\le\ |\lambda|\sum_{i=1}^{n}\mu(T_i)|G(t,s_i)|h(\tau_i)v(q_i),
\end{aligned}
$$

where $s_i, \tau_i, q_i \in T_i$. Moreover as

$$|h(s)G(t,s)v(s) - h(\tau_i)G(t,s_i)v(q_i)| < \epsilon, \text{ for } s \in\in T_i,$$

we have

$$h(\tau_i)G(t,s_i)v(q_i)\mu(T_i) \le \int_{T_i} h(s)G(t,s)v(s)\,ds + \epsilon\mu(T_i).$$

Thus

$$\beta\left(\lambda\int_A G(t,s)f(s,V(s))\,ds\right) \le |\lambda|\int_A G(t,s)h(s)v(s)\,ds + |\lambda|\epsilon\sum_{i=1}^{n}\mu(T_i).$$

As ϵ is arbitrarily small, from this and (253) we deduce that

$$\beta\left(\lambda\int_0^1 G(t,s)f(s,V(s))\,ds\right) \le |\lambda|\int_0^1 G(t,s)h(s)v(s)\,ds.$$

□

Now, we are in the position to formulate and prove the following existence result

Theorem 10.12. *Let f satisfies the Assumptions (1)-(3). Assume that there exists $h \in L_1[I,\mathbb{R}^+]$ such that for any $\epsilon > 0$ and any bounded subset X of E there exists a closed subset I_ϵ of the interval I such that mess$(I\setminus I_\epsilon) < \epsilon$ and $\beta(f(T\times X)) \le \sup_{t\in T} h(t)\beta(X)$, for each closed subset T of I_ϵ. Then there exists $\rho > 0$ such that for any $\lambda \in \mathbb{R}$ with $|\lambda| < \rho$, the equation (250) has at least one solution $x \in C[I,E]$.*

Proof. Let $r(H) := \inf_n \sqrt[n]{\|H^n\|}$ be the spectral radius of the integral operator H defined by

$$Hx(t) := \int_0^1 G(t,s)h(s)u(s)\,ds,\ (u \in C[I,\mathbb{R}], t \in I),$$

and let

$$\rho = \min\left(\sup_{r>0} \frac{r}{\sup_{t\in I}\int_0^1 |G(t,s)|M_r(s)\,ds}, \frac{1}{r(H)}\right)$$

Fix $\lambda \in \mathbb{R}, |\lambda| < \rho$ and choose $r_0 > 0$ such that

$$|\lambda| \sup_{t\in I} \int_0^1 |G(t,s)|M_{r_0}(s)\,ds \le r_0. \tag{253}$$

Define the operator $T : C[I,E] \to C[I,E]$ by

$$Tx(t) := \lambda \int_0^1 G(t,s)f(s,x(s))\,ds,\ a.e. \text{ on } [0,1].$$

First notice that, for $x \in C[I,E]$, $f(\cdot,x(\cdot))$, is Pettis integrable (Assumption 1.). Since, $G(t,\cdot)$ is measurable and essentially bounded real-valued function, then (thanks to Proposition 8.6), $G(t,\cdot)f(\cdot,x(\cdot)),\ t \in [0,1]$ is Pettis integrable and thus, the operator T makes sense. Also, T well-defined. To see this, let $t_1, t_2 \in [0,1]$ with $t_2 > t_1$. Without loss of generality, assume $Tx(t_2) - Tx(t_1) \neq 0$. Then there exists (consequence of Proposition 10.4) $\varphi \in E^*$ with $\|\varphi\| = 1$ and $\|Tx(t_2) - Tx(t_1)\| = \varphi(Tx(t_1) - Tx(t_1))$. Thus

$$\begin{aligned}\|Tx(t_2) - Tx(t_1)\| &= \varphi(\lambda Tx(t_1) - \lambda Tx(t_1)) \\ \le\ & |\lambda| \int_0^1 |G(t_2,s) - G(t_1,s)|M_r(s)\,ds\end{aligned}$$

This estimation shows that T maps $C[I,E]$ into itself. Define the convex, closed and equicontinuous subset $Q \in C[I,E]$ by

$$\begin{aligned}Q := \{x \in C[I,E] : \|x\|_0 \le r_0, \forall t_1,\ t_2 \in [0,1] \text{ we have} \\ \|x(t_2) - x(t_1)\| \le |\lambda| \int_0^1 |G(t_2,s) - G(t_1,s)|M_{r_0}(s)\,ds\}.\end{aligned}$$

We claim that $T : Q \longrightarrow Q$ is *wk*-sequentially continuous and T satisfies the implication 219. Once the claim is established, Theorem 10.4 guarantees the existence of a fixed point of T. Hence (250) has a solution in $C[I,E]$. We begin by showing that $T : Q \to Q$. To see this, take $x \in Q,\ t \in [0,1]$. Without loss of generality, assume $Tx(t) \neq 0$. Then there exists (consequence of Proposition 10.4) $\varphi \in E^*$ with $\|\varphi\| = 1$ and $\|Tx(t)\| = \varphi(Tx(t))$. Thus

$$\begin{aligned}\|Tx(t)\| &\leq |\lambda|\varphi\left(\int_0^1 G(t,s)f(s,x(s))\,ds\right)\\ &\leq |\lambda| \sup_{t\in[0,1]}\int_0^1 G(t,s)|\varphi(f(s,x(s)))|\,ds \leq |\lambda|\left(\sup_{t\in[0,1]}\int_0^1 G(t,s)M_{r_0}(s)\,ds\right),\end{aligned}$$

therefore $\|Tx\|_0 = \sup_{t\in[0,1]}\|Tx(t)\| \leq r_0$. Hence $T : Q \to Q$. Also $T : Q \to Q$ wk-sequentially continuous. To see this, let (x_n) be a sequence in Q and let $x_n(t) \to x(t)$ in (E, w) for each $t \in [0,1]$. Fix $t \in I$. Then since f satisfies Assumption 1., we have $f(t, x_n(t))$ converging weakly uniformaly to $f(t, x(t))$, hence the Lebesgue dominated convergence theorem for Pettis integral (Theorem 8.10) implies $Tx_n(t)$ converging weakly uniformaly to $Tx(t)$ in E_w. We do it for each $t \in I$, so $T : Q \to Q$ wk-sequentially continuous. Finally, we will show that T satisfies the implication (219). Let V be a subset of Q such that

$$\overline{V} = \overline{\text{conv}}\,(T(V) \cup \{0\})\,. \tag{254}$$

Then V is equicontinuous. By (251) and (254) and Lemma 10.5, for any $t \in I$, we have

$$v(t) \leq \beta(\overline{\text{conv}}\,(T(V)(t) \cup \{0\})) \leq \beta((T(V)(t))) \leq |\lambda|\int_0^1 G(t,s)h(s)v(s)\,ds.$$

Since $|\lambda| r(H) < 1$, it is follows that $v(t) = 0$ for $t \in I$. By Theorem 2. in [125], V is weakly relatively compact in $C[I, E]$. Applying now Theorem 10.4 we conclude that T has a fixed point, which completes the proof. □

Definition 10.7. A function $x : I \to E$ is called pseudo solution of (249) if x has the pseudo derivative of order $\alpha \in (1, 2]$, $x(0) = x(1) = 0$ and satisfies

$$D^2\varphi(I^{2-\alpha}x(t)) = -\lambda\varphi(f(t, x(t)))\ a.e.\text{ on } [0,1], \quad \text{for each } \varphi \in E^*.$$

Now, we are in the position to formulate and prove the following main existence result

Theorem 10.13. *If the assumptions of Theorem 10.12 are satisfied, the boundary value problem (249) has at least one solution $x \in C[I, E]$.*

Proof. Let x be a solution of equation (250) then

$$\begin{aligned} x(t) &= \lambda \int_0^1 G(t,s) f(s,x(s))\,ds \\ &= \lambda \int_0^t \frac{(t(1-s))^{\alpha-1} - (t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,x(s))\,ds + \lambda \int_t^1 \frac{(t(1-s))^{\alpha-1}}{\Gamma(\alpha)} f(s,x(s))\,ds \\ &= -\lambda \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,x(s))\,ds + \lambda \int_0^1 \frac{(t(1-s))^{\alpha-1}}{\Gamma(\alpha)} f(s,x(s))\,ds, \end{aligned}$$

thus

$$x(t) = -\lambda I^\alpha f(t,x(t)) + ct^{\alpha-1}, \text{ where } c = \frac{\lambda}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} f(s,x(s))\,ds. \tag{255}$$

Since $x, ct^{\alpha-1} \in C[I,E]$, then $t \to I^\alpha f(t,x(t)) \in C[I,E]$. Operating by $I^{2-\alpha}$ on both sides of equation (255) and using Lemma **??** result in

$$I^{2-\alpha} x(t) = -\lambda I^2 f(t,x(t)) + c\frac{\Gamma(\alpha)}{\Gamma(2)} t.$$

Now, for every $\varphi \in E^*$ we have

$$\varphi(I^{2-\alpha} x(t)) = -\lambda I^2 \varphi(f(t,x(t))) + \varphi(c)\frac{\Gamma(\alpha)}{\Gamma(2)} t,$$

Since $f(\cdot,x(\cdot))$ is Pettis integrable, then $\varphi(f(\cdot,x(\cdot))) \in L_1[0,1]$, for every $\varphi \in E^*$. Thus

$$\frac{d^2}{dt^2}\varphi(I^{2-\alpha} x(t)) = -\lambda \varphi(f(t,x(t))) \ a.e. \text{ on } [0,1].$$

From here, we arrive at the differential equation (249). Now, we show that the boundary conditions of the problem (249) also holds. For this note that $G(0,s) = G(1,s) = 0$ which implies that $x(0) = x(1) = 0$, which completes the proof. □

Example K: (On solutions of fractional order boundary value problems with integral boundary conditions in Banach spaces) In this example, some sufficient conditions for the existence of solutions are presented for the following nonlinear $m-$point boundary value problem of fractional type

$$\begin{cases} D^\alpha u(t) + f(t,u(t),D^\beta u(t)) = 0, \ t \in [0,1], \ \alpha \in (1,2], \ \beta \in (0,1), \ \alpha > 1+\beta, \\ u(1) + \displaystyle\int_0^1 \Im(\tau) u(\tau) d\tau = l, \ u(0) = 0, \end{cases} \tag{256}$$

where x takes values in a Banach space E and $l \in E$. Here $\Im \in L_q[0,1]$ for some $q \in [1,\infty]$ and D^α denotes the pseudo fractional differential operator of order α. We will assume that f is Pettis-integrable function on $[0,1]$.

Definition 10.8. A function $u : I \to E$ is called pseudo-solution of the problem (256) if $u \in C[I, E_w]$ has fractional pseudo-derivative of order $\alpha \in (1, 2]$, $u(0) = 0$, $u(1) + \int_0^1 \Im(\tau)u(\tau)d\tau = l$ and satisfies

$$D^2\varphi(I^{2-\alpha}u(t)) + \varphi(f(t, u(t), D^\beta u(t))) = 0 \ a.e. \text{ on } [0, 1], \quad \text{for each } \varphi \in E^*.$$

The following auxiliary Lemma will be needed in our techniques.

Lemma 10.6. *If $v \in C[I, E_w]$ be a pseudo-solution to the problem*

$$\begin{cases} D^{\alpha-\beta}v(t) + f(t, I^\alpha v(t), v(t)) = 0, \ t \in [0, 1], \ \alpha \in (1, 2], \ \beta \in (0, 1], \ \alpha > 1 + \beta, \\ I^\beta v(1) + \displaystyle\int_0^1 \Im(\tau)I^\beta v(\tau)d\tau = l, \ v(0) = 0, \end{cases} \tag{257}$$

Then $u := I^\beta v$ is a pseudo-solution for the problem (256)

Proof. Let $v \in C[I, E_w]$ be a pseudo-solution to the problem (257) and $\varphi \in E^*$. As in the proof of Lemma 9.2 it follows that $I^\beta v$ exists and the real function φu is continuous for every $\varphi \in E^*$, moreover

$$\lim_{t \to 0^+} \varphi u(t) = \lim_{t \to 0^+} \left(I^\beta \varphi v\right)(t) = \lim_{t \to 0^+} \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \varphi v(s)\, ds = 0.$$

Thus $\varphi u(0) = 0$ for every $\varphi \in E^*$ i.e. $u(0) = 0$. Further

$$u(1) + \int_0^1 \Im(\tau)u(\tau)d\tau = l.$$

In the view of Lemma 9.7 we also have

$$D^\alpha u(t) = \left(D^\alpha I^\beta v\right)(t) = D^{\alpha-\beta}v(t).$$

$\square$

To obtain the Hammerstein type integral equation modeled off the problem (257), we keep the boundary value problem (257) in mind and we *formally* put

$$v(t) = -I^{\alpha-\beta}f(t, I^\beta v(t), v(t)) + ct^{\alpha-\beta-1}. \tag{258}$$

In the view of Lemma **??** and Lemma 9.6, we obtain

$$I^\beta v(t) = -I^\alpha f(t, I^\beta v(t), v(t)) + c\left[\frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)}\right] t^{\alpha-1}$$

To facilitate our discussion, let $q \in [1, \infty]$ be constant with the conjugate exponents p. Suppose $\Im \in L_q[0, 1]$ be a nonnegative real-valued function and $f : [0, 1] \times E \times E \to E$ satisfy the following assumptions:

1. For each $t \in I = [0, 1]$, $f(t, \cdot, \cdot) : [0, 1] \times E \times E \to E$ is weakly- weakly sequentially continuous,
2. For each $x, y \in C(I, E_w)$, $f(\cdot, x(\cdot), y(\cdot)) \in \mathcal{H}_0^p(E)$,

3. There exist a positive constant M such that for each $x \in E$, $y \in E$ and almost all $t \in I$
$$|f(t,x,y)| \leq M.$$
4. For any $r > 0$ and $\varphi \in E^*$ there exist a Pettis integrable function $\tilde{f} : I \to E$, function $\Psi \in L_p[I, \mathbb{R}^+]$ and nondecreasing continuous function $\Omega : [0,\infty) \to (0,\infty)$ such that $|\varphi(f(t,x,y))| \leq |\varphi(\tilde{f}(t))|\Omega(r) \leq \|\varphi\|\Psi(t)\Omega(r)$ for a.e. $t \in I$ and all $(x,y) \in B_r \times B_r$.

Now, we would like to pay our attention to solve the equation (258) for c by
$$I^\beta v(1) + \int_0^1 \Im(\tau) I_+^\beta v(\tau) d\tau = l,$$
it follows that
$$\begin{aligned}
c\left[\frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)}\right] &- \int_0^1 \frac{(1-s)^{\alpha-\beta-1} f(s, I^\beta v(s), v(s))}{\Gamma(\alpha-\beta)}\, ds = l \\
&- \int_0^1 \Im(\tau)\left(c\left[\frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)}\right]\tau^{\alpha-\beta-1}\right. \\
&- \left.\int_0^\tau \frac{(\tau-s)^{\alpha-\beta-1} f(s, I^\beta v(s), v(s))}{\Gamma(\alpha-\beta)}\, ds\right) d\tau \\
&= l - c\left[\frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)}\right]\int_0^1 \Im(\tau)\tau^{\alpha-\beta-1} d\tau \\
&+ \int_0^1 \Im(\tau)\left(\int_0^\tau \frac{(\tau-s)^{\alpha-\beta-1} f(s, I^\beta v(s), v(s))}{\Gamma(\alpha-\beta)}\, ds\right) d\tau,
\end{aligned}$$
therefore
$$\begin{aligned}
c\left[\frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)}\right](1+\gamma) &= l + \int_0^1 \Im(\tau)\left(\int_0^\tau \frac{(\tau-s)^{\alpha-\beta-1} f(s, I^\beta v(s), v(s))}{\Gamma(\alpha-\beta)}\, ds\right) d\tau \\
&+ \int_0^1 \frac{(1-s)^{\alpha-\beta-1} f(s, I^\beta v(s), v(s))}{\Gamma(\alpha-\beta)}\, ds,
\end{aligned}$$
where
$$\gamma = \int_0^1 \Im(\tau)\tau^{\alpha-\beta-1} d\tau.$$
Then (in account of Lemma 9.4), we have
$$\begin{aligned}
c &= \frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)(1+\gamma)}\left[l + \int_0^1 \frac{(1-s)^{\alpha-\beta-1} f(s, I^\beta v(s), v(s))}{\Gamma(\alpha-\beta)}\, ds\right. \\
&+ \left.\int_0^1 h(s) f(s, I^\beta v(s), v(s))\, ds\right].
\end{aligned}$$

Here $h = J^{\alpha-\beta}\Im$ (see formula 213). Substituting c into (258) implies

$$\begin{aligned}
v(t) &= -I^{\alpha-\beta} f(t, I^{\beta} v(t), v(t)) + \frac{l\Gamma(\alpha) t^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)(1+\gamma)} \\
&+ \frac{\Gamma(\alpha) t^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)(1+\gamma)} \int_0^1 \left[\frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} + h(s) \right] f(s, I^{\beta} v(s), v(s))\, ds \\
&= \frac{l\Gamma(\alpha) t^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)(1+\gamma)} \\
&+ \int_0^t \left[\Gamma(\alpha) \frac{(t(1-s))^{\alpha-\beta-1}}{(1+\gamma)(\Gamma(\alpha-\beta))^2} - \frac{(t-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} \right] f(s, I^{\beta} v(s), v(s))\, ds \\
&+ \int_t^1 \Gamma(\alpha) \frac{(t(1-s))^{\alpha-\beta-1}}{(1+\gamma)(\Gamma(\alpha-\beta))^2} f(s, I^{\beta} v(s), v(s))\, ds \\
&+ \frac{\Gamma(\alpha) t^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)(1+\gamma)} \int_0^1 h(s) f(s, I^{\beta} v(s), v(s))\, ds.
\end{aligned}$$

Thus

$$v(t) = p(t) + \int_0^1 G(t,s) f(s, I^{\beta} v(s), v(s))\, ds, \ t \in [0,1], \tag{259}$$

where $p(t) = \dfrac{l\Gamma(\alpha) t^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)(1+\gamma)}$ and the Green function G is given by $G(t,s) = G_1(t,s) + G_2(t,s)$ where

$$G_1(t,s) := \begin{cases} \dfrac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)} \left[\dfrac{(t(1-s))^{\alpha-\beta-1}}{(1+\gamma)\Gamma(\alpha-\beta)} - \dfrac{(t-s)^{\alpha-\beta-1}}{\Gamma(\alpha)} \right], \ 0 \le s \le t \le 1, \\ \dfrac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)} \left[\dfrac{(t(1-s))^{\alpha-\beta-1}}{(1+\gamma)\Gamma(\alpha-\beta)} \right], \ 0 \le t \le s \le 1, \end{cases} \tag{260}$$

$$G_2(t,s) := \frac{\Gamma(\alpha) t^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)(1+\gamma)} h(s), \ t,s \in [0,1]. \tag{261}$$

Because $\alpha - \beta > 1$, it can be easily seen that

Lemma 10.7. *The map $t \to G(t,\cdot)$ is continuous from [0,1] to $L_q[0,1]$.*

In the light of the Assumptions (1-3) imposed on f, we proceed to obtain a result which relies on the fixed point Theorem 10.3 to ensure the existence of weak solution to the integral equation (259). For the sake of convenience, we introduce the following

Definition 10.9. By a solution to (259) we mean a function $v \in C(I,E)$ which satisfies the integral equation (259). This is equivalent to finding $v \in$

$C(I,E)$ with

$$\varphi(v(t)) = \varphi\left(p(t) + \int_0^1 G(t,s) f(s, I^\beta v(s), v(s))\, ds\right),\ t \in I \text{ for all } \varphi \in E^*.$$

Now, we are in the position to state and prove the first existence result:

Theorem 10.14. *Assume $\alpha \in (1,2]$, $\beta \in (0,1)$ with $\alpha > 1+\beta$ and $\Im \in L_q([0,1])$ be a nonnegative real-valued function. If the Assumptions (1)-(3) hold along with*

$$\left(\sup_{t\in[0,1]} \int_0^1 |G(t,s)|\Psi(s)\, ds\right) \limsup_{r\to\infty} \frac{\Omega(r)}{r} < \Gamma(1+\beta). \tag{262}$$

Then the integral equation (259) has at least one solution $v \in C[I,E]$.

Proof. First of all, observe the expression of G and note that the following implications

$$\Im \in L_q[0,1] \Rightarrow h \in L_q[0,1] \Rightarrow G(t,\cdot) \in L_q[0,1],\ t \in [0,1],$$

hold. Consequently $G(t,\cdot)\Psi(\cdot) \in L_1[0,1]$ for any $t \in [0,1]$. Let

$$\rho = \left(\sup_{t\in[0,1]} \int_0^1 |G(t,s)|\Psi(s)\, ds\right) \limsup_{r\to\infty} \frac{\Omega(r)}{r}.$$

Consider the set S of real numbers $r \geq 0$ which satisfy the inequality

$$r\Gamma(1+\beta) \leq \|p\|_0 + \Omega(r)\left(\sup_{t\in[0,1]} \int_0^1 |G(t,s)|\Psi(s)\, ds\right).$$

Then S is bounded above, i.e. there exists a constant R_0 with

$$r \leq R_0 \text{ for all } r \in S. \tag{263}$$

To see this, suppose (263) is false. Then there exists a sequence $0 \neq r_n \in S$ with $r_n \to \infty$ as $n \to \infty$ and

$$\Gamma(1+\beta) \leq \frac{\|p\|_0}{r_n} + \frac{\Omega(r)}{r_n}\left(\sup_{t\in[0,1]} \int_0^1 |G(t,s)|\Psi(s)\, ds\right).$$

Since $\limsup(s_n + t_n) \leq \limsup s_n + \limsup t_n$ for any sequences $s_n \geq 0$, $t_n \geq 0$, we have $\rho \geq \Gamma(1+\beta)$. This contradicts (262). Then, for every $R > R_0$ the inequality

$$\|p\|_0 + \Omega(R)\left(\sup_{t\in[0,1]} \int_0^1 |G(t,s)|\Psi(s)\, ds\right) < \Gamma(1+\beta)R,$$

holds, for otherwise $R \in S$, and would contradict (263).

Now, define the operator $T : C[I,E] \to C[I,E]$ by

$$Tv(t) := p(t) + \int_0^1 G(t,s) f(s, I^\beta v(s), v(s))\, ds,\ t \in [0,1].$$

We remark that, for $v \in C[I,E]$ we have, by Lemma **??**, $I^\beta v$ is weakly continuous and consequently, $f(\cdot, I^\beta v(\cdot), v(\cdot)) \in \mathcal{H}_0^p(E)$ (Assumption 2.). Since $s \longmapsto G(t,s) \in L_q(I)$, for all $t \in [0,1]$, $G(t,\cdot)f(\cdot, I^\beta v(\cdot), v(\cdot))$ is Pettis integrable for all $t \in [0,1]$ (thanks to Proposition 10.4) and thus the operator T makes sense. Note that T is well-defined. To see this, let $t_1, t_2 \in [0,1]$ with $t_2 > t_1$. Since $\beta \in (0,1)$, we deduce that, if $\|v\| \leq \sigma_1$ and then $\|v\| < \sigma$ and $\left\|I^\beta v\right\| \leq \sigma$, where $\sigma = \frac{\sigma_1}{\Gamma(1+\beta)}$. Without loss of generality, assume $Tv(t_2) - Tv(t_1) \neq 0$. Then there exists (as a consequence of Proposition 10.4) $\varphi \in E^*$ with $\|\varphi\| = 1$ and $\|Tv(t_2) - Tv(t_1)\| = \varphi(Tv(t_1) - Tv(t_1))$.

Putting the Assumption (3.) in mind, one can write the following chain of inequalities

$$\begin{aligned}
&\|Tv(t_2) - Tv(t_1)\| = \varphi(Tv(t_2) - Tv(t_1)) \\
&\quad \leq \quad \varphi(p(t_2) - p(t_1)) + \Omega(\sigma) \int_0^1 |G(t_2,s) - G(t_1,s)| \Psi(s)\, ds \\
&\quad \leq \quad \|p(t_2) - p(t_1)\| + \Omega(\sigma) \left(\int_0^1 |G(t_2,s) - G(t_1,s)|^q\, ds \right)^{\frac{1}{q}} \left(\int_0^1 |\Psi(s)|^p\, ds \right)^{\frac{1}{p}} \\
&\quad \leq \quad \|p(t_2) - p(t_1)\| + \Omega(\sigma) \left\|G(t_2,\cdot) - G(t_1,\cdot)\right\|_q \|\Psi\|_p \\
&\quad \leq \quad \|p(t_2) - p(t_1)\| \\
&\quad\ + \quad \Omega(\sigma) \left(\left\|G_1(t_2,\cdot) - G_1(t_1,\cdot)\right\|_q + \left\|G_2(t_2,\cdot) - G_2(t_1,\cdot)\right\|_q \right) \|\Psi\|_p \\
&\quad \leq \quad \|p(t_2) - p(t_1)\| \\
&\quad\ + \quad \Omega(\sigma) \left(\left\|G_1(t_2,\cdot) - G_1(t_1,\cdot)\right\|_q + \frac{\Gamma(\alpha) \left\| (t_2^{\alpha-\beta-1} - t_1^{\alpha-\beta-1}) h(\cdot) \right\|_q}{\Gamma(\alpha-\beta)(1+\gamma)} \right) \|\Psi\|_p .
\end{aligned}$$

Then

$$\begin{aligned}
\|Tv(t_2) - Tv(t_1)\| \ \leq\ & \|p(t_2) - p(t_1)\| \\
+\ & \Omega(\sigma) \Bigg(\left\|G_1(t_2,\cdot) - G_1(t_1,\cdot)\right\|_q \\
+\ & \frac{\Gamma(\alpha) \left| t_2^{\alpha-\beta-1} - t_1^{\alpha-\beta-1} \right| \|h\|_q}{\Gamma(\alpha-\beta)(1+\gamma)} \Bigg) \|\Psi\|_p . \qquad (264)
\end{aligned}$$

Therefore we deduce, in the view of Lemma 10.7, that T maps $C[I,E]$ into itself.

Let $Q \in C[I, E]$ be the convex, closed and equicontinuous subset (required by Theorem 10.3). Define this set by

$$\begin{aligned} Q &:= \{v \in C[I, E] : \|v\|_0 \leq R_0, \forall t_1,\ t_2 \in [0,1] \text{ we have } \|v(t_2) - v(t_1)\| \\ &\leq \|p(t_2) - p(t_1)\| \\ &+ \Omega(\frac{R_0}{\Gamma(1+\beta)}) \left(\|G_1(t_2, \cdot) - G_1(t_1, \cdot)\|_q + \frac{\Gamma(\alpha) \left| t_2^{\alpha-\beta-1} - t_1^{\alpha-\beta-1} \right| \|h\|_q}{\Gamma(\alpha-\beta)(1+\gamma)} \right) \|\Psi\|_p \}. \end{aligned}$$

We claim that T restricted to the set Q maps this set into itself (i.e. $T : Q \longrightarrow Q$) and is weakly-weakly sequentially continuous. Once the claim is established, Theorem 10.3 guarantees the existence of a fixed point of T. Hence the integral equation (259) has a solution in $C[I, E]$.

We start by showing that $T : Q \to Q$. To see this, take $v \in Q$, $t \in [0, 1]$. Since $\beta \in (0, 1)$, we deduce $\|v\| \leq R_0 < \frac{R_0}{\Gamma(1+\beta)}$, $\left\|I^\beta v\right\| \leq \frac{R_0}{\Gamma(1+\beta)}$. The monotonicity of Ω and the inequality (264) imply that

$$\begin{aligned} \|Tv(t_2) - Tv(t_1)\| &\leq \|p(t_2) - p(t_1)\| \\ &+ \Omega(\frac{R_0}{\Gamma(1+\beta)}) \Big(\|G_1(t_2, \cdot) - G_1(t_1, \cdot)\|_q \\ &+ \frac{\Gamma(\alpha) \left| t_2^{\alpha-\beta-1} - t_1^{\alpha-\beta-1} \right| \|h\|_q}{\Gamma(\alpha-\beta)(1+\gamma)} \Big) \|\Psi\|_p . \end{aligned}$$

Now, without loss of generality, assume $Tv(t) \neq 0$. Then there exists (consequence of Proposition 10.4) $\varphi \in E^*$ with $\|\varphi\| = 1$ and $\|Tx(t)\| = \varphi(Tv(t))$. By the Assumption 3., we obtain

$$\begin{aligned} \|Tv(t)\| &\leq \varphi(p(t)) + \varphi\left(\int_0^1 G(t,s) f(s, I^\beta v(s), v(s))\, ds \right) \\ &\leq \|p(t)\| + \sup_{t\in[0,1]} \int_0^1 |G(t,s)| \cdot |\varphi(f(s, I^\beta v(s), v(s))|\, ds \\ &\leq \|p\|_0 + \Omega(\frac{R_0}{\Gamma(1+\beta)}) \left(\sup_{t\in[0,1]} \int_0^1 G(t,s)\Psi(s)\, ds \right) \\ &< \frac{R_0}{\Gamma(1+\beta)} \Gamma(1+\beta) = R_0, \end{aligned}$$

therefore $\|Tv\|_0 = \sup_{t\in[0,1]} \|Tv(t)\| \leq R_0$. Hence $T : Q \to Q$.

We need to prove now that $T : Q \to Q$ is weakly-weakly sequentially continuous. Let us recall, that the weak convergence in $Q \subset C(I, E)$ is exactly the weak pointwise convergence. Let (v_n) be a sequence in Q weakly convergent to v. Then $v_n(t) \to v(t)$ in E_w for each $t \in [0,1]$. It is obvious (by Lemma 8.2), that $v \in Q$.

Fix $t \in I$ and note, in the view of Lebesgue dominated convergence theorem for the Pettis integral, that $I^\beta v_n(s) \to I^\beta v(s)$ in E_w. Let us recall, that the topology on $C(I, E_w)$ on equicontinuous subsets coincide with the topology of weak pointwise convergence. Since f satisfies Assumption 1., we have $f(t, I^\beta v_n(t), v_n(t))$ converging weakly to $f(t, I^\beta v(t), v(t))$, hence again the Lebesgue dominated convergence theorem for Pettis integral yields $Tv_n(t)$ converging weakly to $Tv(t)$ in E. But Q is an equicontinuous subset of $C(I, E)$ and then $T : Q \to Q$ is weakly-weakly sequentially continuous. Applying now Theorem 10.3 we conclude that T has a fixed point in Q, which completes the proof. $\square$

In order to obtain the existence of solutions of the problem (256), we can make use of Theorem 10.14.

Theorem 10.15. *Let the Assumptions of Theorem 10.14 be satisfied. Then the boundary value problem (256) has at least one pseudo-solution* $u \in C(I, E_w)$.

Proof. Firstly, we remark that, for any $v \in C[I, E]$, we have (according to Proposition 8.5) that $v(\cdot)\Im(\cdot) \in P[I, E]$ for $\Im \in L_q$, for $q \in [1, \infty]$. Thus the integral boundary condition makes sense.

In account of Theorem 10.3 it can be easily seen that the integral equation (259) has a solution $v \in C[I, E]$. Let v be a weak solution of equation (259) then

$$\begin{aligned} v(t) &= \frac{t^{\alpha-\beta-1} l\Gamma(\alpha)}{\Gamma(\alpha-\beta)(1+\gamma)} + \int_0^1 [G_1(t,s) + G_2(t,s)] f(s, I^\beta v(s), v(s))\, ds \\ &= -\int_0^t \frac{(t-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} f(s, I^\beta v(s), v(s))\, ds + ct^{\alpha-\beta-1}, \end{aligned}$$

$$\begin{aligned} c &= \frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)(1+\gamma)} \left[l + \int_0^1 \frac{(1-s)^{\alpha-\beta-1} f(s, I^\beta v(s), v(s))}{\Gamma(\alpha-\beta)}\, ds \right. \\ &+ \left. \int_0^1 h(s) f(s, I^\beta v(s), v(s))\, ds \right]. \end{aligned}$$

By Lemma 9.4, using $G_1(0,s) = G_2(0,s) = 0$, a straightforward estimates show that

$$v(0) = 0, \text{ and } I^\beta v(1) + \int_0^1 \Im(\tau) I^\beta v(\tau) d\tau = l.$$

Furthermore, we have

$$v(t) = -I^{\alpha-\beta} f(t, I^\beta v(t), v(t)) + ct^{\alpha-\beta-1}. \tag{265}$$

Thus for any $\varphi \in E^*$ we have

$$\begin{aligned} \varphi v(t) &= -\varphi(I^{\alpha-\beta} f(t, I^{\beta}v(t), v(t))) + \varphi c t^{\alpha-\beta-1} \\ &= -I^{\alpha-\beta}\varphi(f(t, I^{\beta}v(t), v(t))) + \varphi c t^{\alpha-\beta-1}. \end{aligned} \tag{266}$$

Operating by $I^{2-(\alpha-\beta)}$ on both sides of the equation (266) and using the properties of fractional calculus in the space $L_1[0,1]$ result in

$$I^{2-(\alpha-\beta)}\varphi v(t) = -I^2\varphi(f(t, I^{\beta}v(t), v(t))) + \varphi c\frac{\Gamma(\alpha-\beta)}{\Gamma(2)}t.$$

Therefore

$$\varphi(I^{2-(\alpha-\beta)}v(t)) = -I^2\varphi(f(t, I^{\beta}v(t), v(t))) + \varphi c\frac{\Gamma(\alpha-\beta)}{\Gamma(2)}t.$$

Frequently

$$\frac{d^2}{dt^2}\varphi(I^{2-(\alpha-\beta)}v(t)) = -\varphi(f(t, I^{\beta}v(t), v(t))) \ a.e. \text{ on } [0,1].$$

That is, v has the fractional pseudo-derivative of order $\alpha-\beta \in (1,2)$ and satisfies

$$D^{\alpha-\beta}v(t)) = -f(t, I^{\beta}v(t), v(t)) \text{ on } [0,1].$$

Therefore v is a pseudo-solution to the problem (257). This together with Lemma 10.6 implies that the problem (256) has a pseudo-solution $u \in C(I, E_w)$ which completes the proof. □

Now, we consider an example to illustrate our result

Example 10.3. Let $\Im \in L^{\infty}([0,1])$ and assume $E := \ell^2(I)$ be the non-separable Hilbert space of countably nonzero functions on $I := [0,1]$ that are square-summable, under the ℓ^2-norm. For each $t \in I$ we define $g(t) := e_t \in \ell^2(I)$ by

$$e_t(s) := \begin{cases} 1 \text{ if } s = t, \\ 0 \text{ if } s \neq t. \end{cases}$$

This function is Pettis, but not Bochner, integrable (it is not even strongly measurable, see e.g. [87]) and for any $\varphi \in \ell^2(I)$ we have [87]

$$\varphi e_t = \sum_{\tau\in[0,1]} \varphi(\tau)e_t(\tau) = \begin{cases} 0 \text{ if } \tau \neq t, \\ \varphi(\tau) \text{ if } \tau = t. \end{cases}$$

Since $\left(\ell^2(I)\right)^* = \ell^2(I)$, the function φe_t is only countably nonzero for each $\varphi \in \left(\ell^2(I)\right)^*$. Hence $\varphi g = 0$ $a.e.$ (with respect to the Lebesgue measure).

Now, we investigate the existence of pseudo-solutions for the problem

$$\begin{cases} D^{\alpha}u(t)+\delta D^{\beta}u(t)+\mu u(t)=g(t),\ t\in[0,1],\ \delta,\ \mu\in\mathbb{R}, \\ u(1)+\displaystyle\int_0^1 \Im(\tau)u(\tau)d\tau=l,\ u(0)=0,\ \alpha\in(1,2],\ \beta\in(0,1),\ \alpha>1+\beta. \end{cases} \tag{267}$$

Let us define the function $f: I\times E\times E\to E$ by $f(t,x,y)=\delta x+\gamma y-g(t)$. Remark, that for any $x,y\in C[I,E]$, f is Pettis, but not Bochner, integrable and satisfies, by suitable choice of δ,γ, all requirements of Theorem 10.15 with $\psi\equiv 1,\ \Omega(r)=(\delta+\gamma)r$, $p=1$ and $q=\infty$.

Therefore one cannot expect the existence of weak or strong solutions to the problem (267), that one could only expect (in account of Theorem 10.15) the existence of pseudo-solutions. Since f satisfies the requirements of Theorem 10.15 with $p=1, q=\infty$, the problem (267) has a pseudo-solution $u\in C(I,E_w)$ given by $u=I^{\beta}v$, where v denotes the weak solution to the problem

$$v(t)=\frac{l\Gamma(\alpha)t^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)(1+\gamma)}+\int_0^1 G(t,s)[\mu v(s)+\delta I^{\beta}v(s)-g(s)]\,ds,\ t\in[0,1].$$

That is, v satisfies the problem

$$v(t)=\frac{l\Gamma(\alpha)t^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)(1+\gamma)}+\int_0^1 G(t,s)[\mu v(s)+\delta I^{\beta}v(s)]\,ds-\int_0^1 G(t,s)g(s)\,ds.$$

Since $\displaystyle\int_0^1 G(t,s)\varphi g(s)\,ds=0$, for each $\varphi\in\ell^2(I)$, then $\displaystyle\int_0^1 G(t,s)g(s)\,ds=\circ$. Hence

$$\begin{aligned} v(t) &= \frac{l\Gamma(\alpha)t^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)(1+\gamma)}+\int_0^1 G(t,s)[\mu v(s)+\delta I^{\beta}v(s)]\,ds \\ &= \frac{l\Gamma(\alpha)t^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)(1+\gamma)}-I^{\alpha-\beta}[\mu v(s)+\delta I^{\beta}v(s)]+ct^{\alpha-\beta-1}, \end{aligned}$$

where

$$\begin{aligned} c &:= \frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)(1+\gamma)}\left[l+\int_0^1\frac{(1-s)^{\alpha-\beta-1}[\mu v(s)+\delta I^{\beta}v(s)]}{\Gamma(\alpha-\beta)}\,ds\right. \\ &+ \left.\int_0^1 h(s)[\mu v(s)+\delta I^{\beta}v(s)]\,ds\right]. \end{aligned}$$

Therefore

$$v(t)+\mu I^{\alpha-\beta}v(t)+\delta I^{\alpha}v(t)=c^{*}t^{\alpha-\beta-1},$$

where

$$c^{*}=\left(c+\frac{l\Gamma(\alpha)}{\Gamma(\alpha-\beta)(1+\gamma)}\right)$$

Further

$$I^{\beta}v(t)+\mu I^{\alpha-\beta}I^{\beta}v(t)+\delta I^{\alpha+\beta}v(t)=c^{*}\frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)}t^{\alpha-1}.$$

Now if $u=I^{\beta}v$ we obtain

$$u(t)+\mu I^{\alpha-\beta}u(t)+\delta I^{\alpha}u(t)=c^{*}\frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)}t^{\alpha-1}.$$

Thus for every $\varphi\in\left(\ell^{2}(I)\right)^{*}$, we have

$$\varphi u(t)+\mu I^{\alpha-\beta}\varphi u(t)+\delta I^{\alpha}\varphi u(t)=\varphi c^{*}\frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)}t^{\alpha-1}.$$

Whence

$$I^{2-\alpha}\varphi u(t)+\mu I^{2-\beta}\varphi u(t)+\delta I^{2}\varphi u(t)=\varphi c^{*}\frac{\Gamma(\alpha-\beta)}{\Gamma(2)}t.$$

Moreover,

$$\frac{d^{2}}{dt^{2}}\varphi(I^{2-\alpha}u(t))+\frac{d^{2}}{dt^{2}}\varphi I^{2-\beta}u(t)+\delta\varphi u(t)=0.$$

Since

$$\begin{aligned}\frac{d^{2}}{dt^{2}}\varphi I^{2-\beta}u(t) &= \frac{d^{2}}{dt^{2}}I^{2-\beta}\varphi u(t)=\frac{d}{dt}\left(\frac{d}{dt}I^{1}I^{1-\beta}\varphi u(t)\right)\\ &= \frac{d}{dt}I^{1-\beta}\varphi u(t)=\frac{d}{dt}\varphi I^{1-\beta}u(t),\end{aligned}$$

we arrive at

$$D^{\alpha}\varphi u(t)+\mu D^{\beta}\varphi u(t)+\delta\varphi u(t)=0 \text{ for all } t\in[0,1] \text{ and all } \varphi\in E^{*},$$

meanwhile $\varphi g=0$ *a.e.* in $[0,1]$ and all $\varphi\in E^{*}$.

Combining all the estimates, we obtain

$$D^{\alpha}\varphi u(t)+\mu D^{\beta}\varphi u(t)+\delta u(t)=\varphi e_{t},\ a.e.\ [0,1],\ \forall\varphi\in E^{*}.$$

Therefore u is a pseudo- (but not a weak or strong) solution to the problem (267).

11. Applications of fractional calculus

As we mentation above, the basic mathematical ideas of fractional calculus (integral and differential operations of non integer order) were developed long ago by the mathematicians Leibniz (1695), Liouville (1834), Riemann (1892), and others and brought to the attention of the engineering world by Oliver Heaviside in the 1890s, it was not until 1974 that the first book on the topic was published by Oldham and Spanier. Recent monographs and symposia proceedings have highlighted the application of fractional calculus in e.g. physics, and continuum mechanics. In this section, we gather together some of such applications:

1. **First one:Tautochronous problem**
   It may be important to point out that the first application of fractional calculus was made by Abel (1802-1829) in the solution of an integral equation that arises in the formulation of the *tautochronous problem.* This problem deals with the determination of the shape of a frictionless plane curve through the origin in a vertical plane along which a particle of mass M can fall in a time that is independent of the starting position. If the sliding time is constant T, the acceleration due to gravity is g, (ξ, η) is the initial position and $s = f(y)$ is the equation of the sliding curve, then the **Abel integral equation** is
   $$\sqrt{2g}T = \int_0^\eta (\eta - y)^{\frac{-1}{2}} f'(y)dy. \tag{268}$$
   To see this, we apply the conservation energy law which says that the sum of the kinetic energy and the potential energy is a constant. In fact, we have the kinetic energy of the particle at any time t after being dropped is given by
   $$\frac{1}{2}M\left(\frac{ds}{dt}\right)^2,$$
   because the particle is dropped from rest, the kinetic energy of the particle at beginning vanishes. Also, the potential energy of the particle at any time t given by Mgy while the initial potential is $Mg\eta$. According to the conservation energy law, we have
   $$\frac{1}{2}M\left(\frac{ds}{dt}\right)^2 + Mgy = 0 + Mg\eta,$$
   from which we can find a formula
   $$\frac{ds}{dt} = -\sqrt{2g(\eta - y)},$$
   where we have chosen the negative sign for the square root, because as the bead falls, s decreases. Now, we have $s = f(y)$ that is, $ds = f'(y)dy$

which implies that

$$\int_{\eta}^{0} \frac{f'(y)dy}{\sqrt{(\eta - y)}} = -\int_{0}^{T} \sqrt{2g}dt,$$

whence, the integral equation (268) is obtained. It turns out that the integral equation (268) is equivalent to the fractional integral equation

$$T\sqrt{2g} = \Gamma(\frac{1}{2})I^{\frac{1}{2}}f'(\eta) = \sqrt{\pi}I^{\frac{1}{2}}f'(\eta). \tag{269}$$

Now, to find the function f, we take the Laplace transform of both sides of equation (268) which implies that

$$\frac{\sqrt{2g}}{s}T = \sqrt{\pi}\pounds(f'(\eta))\pounds\left(\eta^{-\frac{1}{2}}\right) = \pounds(f'(\eta))\sqrt{\frac{\pi}{s}}.$$

Thus

$$\pounds(f'(y)) = \sqrt{\frac{2g}{\pi s}}T,$$

which can in principle be inverted by the inverse Laplace transform to solve for f', we immediately see that

$$f'(\eta) = \sqrt{\frac{2g}{\pi}}\frac{\eta^{-\frac{1}{2}}}{\sqrt{\pi}}T = \frac{\sqrt{2g}}{\pi\sqrt{\eta}}T,$$

from which we obtain by integration that

$$s = f(y) = \frac{2\sqrt{gy}T}{\pi}.$$

This curve is called a *cycloid*. It is the curve traced by a point in the rim of a circle of rolling upside down without sliding.

Remark 11.1. We are able to proceed in different way making up Lemma (3.24) to obtain the function f. Evidently by applying the operator $D^{\frac{1}{2}}$ in both sides of equation (269) "modeled off the equation 268" we obtain

$$D^{\frac{1}{2}}\left(T\sqrt{2g}\right) = \Gamma(\frac{1}{2})f'(\eta).$$

By the result of example (3.30) we obtain, in the view of Lemma (3.24), that

$$\frac{T\sqrt{2g}\eta^{-\frac{1}{2}}}{\Gamma(\frac{1}{2})} = \Gamma(\frac{1}{2})f'(\eta)$$

Thus

$$f'(y) = \frac{\sqrt{2g}}{\pi\sqrt{y}}T.$$

which is the same result obtained by applying Laplace transform.

2. **Fluid Flow and the Design of a Weir Notch**
A weir notch is an opening in a dam (weir) that allows water to spill over the dam. Our problem here is to design the shape of the opening such that the rate of flow of water through the notch (say, in cubic feet per second) is a specified function of the height of the opening.

Starting from physical principles we derive the equation for determining the shape of the notch. It turns out to be an integral equation of fractional type. After formulating the problem, we shall, of course, solve it. Let us first provide some definitions and notions
(a) x-axis: direction of flow,
(b) y-axis: transverse direction along the face of the dam,
(c) z-axis: vertical direction of flow,
(d) h: height of notch,
(e) In our discussion, we assume that the shape of the notch is symmetrical about the z-axis measured from the lowest point of the notch where the origin of our coordinates system is located. So we are able to write the equation of the notch as $|y| = f(z)$, f is sufficiently smooth, unknown function.
(f) dA: a horizontal element of area of the notch. We may put $dA = 2|y|dz = 2f(z)dz$,
(g) dQ: the incremental rate of flow of water through the area dA. Then $dQ = V dA$, where V denotes the velocity of flow at height z,
(h) The two points $I \equiv (x_0, y_0, z_0)$ and $II \equiv (0, y_0, z_0)$ are supposed to indicate the same element of fluid as it moves from point I to the point II along the same "tube of flow".

We start by applying Bernoulli's formula from hydrodynamics:

$$\frac{P_I}{\rho} + gZ_0 + \frac{V_I^2}{2} = \frac{P_{II}}{\rho} + gZ_0 + \frac{V_{II}^2}{2},$$

where ρ is the density of water, g the acceleration of gravity, and P_I and V_I are the pressure and velocity at point I while P_{II} and V_{II} are the corresponding quantities at point II. We have

$$\begin{aligned} P_I &= \text{(atmospheric pressure)} \\ &+ \text{(the pressure exerted by a column of water of height } (h - z_0)). \end{aligned}$$

Since the point II is in the plane of the notch

$$P_{II} = \text{atmosheric pressure}$$

Thus the difference $P_I - P_{II}$ equals the pressure exerted by a column of water of height $h - z_0$. That is

$$P_I - P_{II} = \rho g(h - z_0).$$

On the other hand, if we assume that point I is far enough upstream, then V_I is negligible and we may write Bernoulli's formula as

$$P_I - P_{II} = \frac{\rho V_{II}^2}{2}$$

Therefore

$$V_{II} = \sqrt{g}(h - z_0)^{\frac{1}{2}}.$$

Thus, the total flow of water through the notch given by

$$Q = \int_0^h dQ = \int_0^h \sqrt{g}(h - z_0)^{\frac{1}{2}} dA = 2\Gamma(\frac{3}{2})\sqrt{g} I^{\frac{3}{2}} f(h) = \sqrt{g\pi} I^{\frac{3}{2}} f(h). \quad (270)$$

To solve (270) for f, we are able to apply Laplace transform or by different way making up Lemma (3.24) to obtain the function f. Evidently by applying the operator $D^{\frac{3}{2}}$ in both sides of equation (270) we obtain

$$f(h) = \frac{1}{\sqrt{g\pi}} D^{\frac{3}{2}} Q(h),$$

which is the desired solution. For example, suppose that $Q(z) = kz^\lambda$, where k is a constant and $\lambda > \frac{1}{2}$. Thus

$$f(z) = \frac{k\Gamma(1+\lambda)}{\sqrt{2g\pi}\Gamma(\lambda - \frac{1}{2})} z^{\lambda - \frac{3}{2}},$$

That is the equation of the notch is given by

$$|y| = \frac{k\Gamma(1+\lambda)}{\sqrt{2g\pi}\Gamma(\lambda - \frac{1}{2})} z^{\lambda - \frac{3}{2}}.$$

The reader is encouraged to compare the following special cases

(a) Suppose that $\lambda = 1$. Then $f(z) = \dfrac{k}{\pi\sqrt{2gz}}$. That is the shape of this notch is given by the equation

$$|y| = \frac{k}{\pi\sqrt{2gz}},$$

(b) Suppose that $\lambda = 3/2$. Then $f(z) = \dfrac{3k}{4\pi\sqrt{2g}}$. That is the shape of this notch is given by the equation

$$|y| = \frac{3k}{4\pi\sqrt{2g}},$$

(c) Suppose that $\lambda = 2$. Then $f(z) = \dfrac{2k}{\pi\sqrt{2g}}\sqrt{z}$. That is the shape of this notch is given by the equation

$$|y| = \frac{2k}{\pi\sqrt{2g}}\sqrt{z},$$

(d) Suppose that $\lambda = \frac{5}{2}$.Then $f(z) = \dfrac{15k}{8\sqrt{2g}}z$. That is the shape of this notch is given by the equation

$$|y| = \frac{15k}{8\sqrt{2g}}z,$$

(e) Suppose that $\lambda = \frac{7}{2}$. Then $f(z) = \dfrac{105k}{32\sqrt{2g\pi}}z^2$. That is the shape of this notch is given by the equation

$$|y| = \frac{105k}{32\sqrt{2g\pi}}z^2.$$

An immediate consequence is the following different shapes of the weir notch: When $\lambda = \frac{3}{2}$, the notch is a rectangle. When $\lambda \in (\frac{3}{2}, \frac{5}{2})$, the notch is a U-shape. In particular, when $\lambda = 2$, the notch is a parabolic. When $\lambda = \frac{5}{2}$, the notch is V-shaped. When $\lambda > \frac{5}{2}$, the notch is a cusp.

In what follows, we state without details, more applications of the fractional calculus

3. **Ultrasonic wave propagation in human cancellus bone**[174]
Fractional calculus is used to describe the viscous interactions between fluid and solid structure. Reflection and transmission scattering operators are derived for a slab of cancellus bone in the elastic frame using Blots theory. Experimental results are compared with theoretical predictions for slow and fast waves transmitted through human cancellous bone samples.

4. **Modeling of speech signals using fractional calculus** [10]
In [10], a novel approach for speech signal modeling using fractional calculus is presented. This approach is contrasted with the celebrated Linear Predictive Coding (LPC) approach which is based on integer order models. It is demonstrated via numerical simulations that by using a few integrals of fractional orders as basis functions, the speech signal can be modeled accurately.

5. **Modeling the Cardiac Tissue Electrode Interface Using Fractional Calculus**[119]
The tissue electrode interface is common to all forms of biopotential recording (e.g., ECG, EMG, EEG) and functional electrical stimulation (e.g., pacemaker, cochlear implant, deep brain stimulation). Conventional lumped element circuit models of electrodes can be extended by generalization of the order of differentiation through modification of the defining current-voltage relationships. Such fractional order models provide an improved description of observed bioelectrode behavior, but recent experimental studies of cardiac tissue suggest that additional mathematical tools may be needed to describe this complex system.

6. **Application of Fractional calculus to the sound Waves Propagation in Rigid Porous Materials**[82]
The observation that the asymptotic expressions of stiffness and damping in porous materials are proportional to fractional powers of frequency suggests the fact that time derivatives of fractional order might describe the behavior of sound waves in this kind of materials, including relaxation and frequency dependence.

7. **Using Fractional calculus for Lateral Longitudinal control of Autonomous Vehicles**[180]
Here it is presented the use of Fractional Order Controllers (FOC) applied to the path-tracking problem in an autonomous electric vehicle. A lateral dynamic model of a industrial vehicle has been taken into account to implement conventional and Fractional Order Controllers. Several control schemes with these controllers have been simulated and compared.

8. **Application of Fractional calculus in the theory of viscoelasticity**[175]
The advantage of the method of fractional derivatives in theory of viscoelasticity is that it affords possibilities for obtaining constitutive equations for elastic complex modulus of viscoelastic materials with only few experimentally determined parameters. Also the fractional derivative method has been used in studies of the complex modulo and impedances for various models of viscoelastic substances.

9. **Fractional differentiation for edge detection**[122]
In image processing, edge detection often makes use of integer-order differentiation operators, especially order 1 used by the gradient and order

2 by the Laplacian. This paper demonstrates how introducing an edge detector based on non-integer (Fractional) differentiation can improve the criterion of thin detection, or detection selectivity in the case of parabolic luminance transitions, and the criterion of immunity to noise, which can be interpreted in term of robustness to noise in general.

10. **Wave propagation in viscoelastic horns using a fractional calculus rheology model**[120]
The complex mechanical behavior of materials are characterized by fluid and solid models with fractional calculus differentials to relate stress and strain fields. Fractional derivatives have been shown to describe the viscoelastic stress from polymer chain theory for molecular solutions. Here the propagation of infinitesimal waves in one dimensional horns with a small cross-sectional area change along the longitudinal axis are examined. In particular, the linear, conical, exponential, and catenoidal shapes are studied. The wave amplitudes versus frequency are solved analytically and predicted with mathematical computation. Fractional rheology data from Bagley are incorporated in the simulations. Classical elastic and fluid "Webster equations" are recovered in the appropriate limits. Horns with real materials that employ fractional calculus representations can be modeled to examine design trade-offs for engineering or for scientific application.

11. **Application of Fractional Calculus to Fluid Mechanics**[112]
Application of fractional calculus to the solution of time-dependent, viscous-diffusion fluid mechanics problems are presented. Together with the Laplace transform method, the application of fractional calculus to the classical transient viscous-diffusion equation in a semi-infinite space is shown to yield explicit analytical (fractional) solutions for the shear-stress and fluid speed anywhere in the domain. Comparing the fractional results for boundary shear-stress and fluid speed to the existing analytical results for the first and second Stokes problems, the fractional methodology is validated and shown to be much simpler and more powerful then existing techniques.

12. Appendix: Generalizations of Fractional Calculus

1. Erdélyi-Kober's integral operator

In many literature, it was shown that (see e.g. [143], [193]) the integral equations where the integral kernel is expressed in terms of squares of the variables; that is, $(t^2 - s^2)^{\alpha-1}$ is related to the classical Abel's integral equation (and hence to Riemann-Liouville fractional integrals). Motivated by these results, the Riemann-Liouville integral has been generalized as

$$I_n^{\alpha} f(t) := \frac{n}{\Gamma(\alpha)} \int_0^t (t^n - s^n)^{\alpha-1} s^{n-1} f(s)\, ds,\ n \in \mathbb{N}$$

We can interpret this as a Stieljes integral where we are integrating with respect to the function s^n. But further, it is useful to consider power weights of these operators. Let us define

$$I_n^{\zeta,\alpha} f(t) := t^{-\zeta-\alpha} I_n^{\alpha} t^{\zeta} f(t),\ n \in \mathbb{N}$$

which are called Erdélyi-Kober operators after the two mathematicians who popularized their use. Although we have not specified the constraints on the orders, it is taken by convention that n is a positive integer and $\alpha, \zeta > 0$. Obviously the case when we have $\zeta = 0$ and $n = 1$ reduces to the case which we studied. The case when $n = 2$ has deep connections with the Hankel transform (McBride [123]).

2. Kiryakova's integral operator

Because of the popularity of the Erdélyi-Kober operators, there have been attempts to generalize even these operators by changing the integral kernel. The first attempt in this approach was by Saigo [147] who defined, using the hypergeometric function,

$$I^{\alpha,\beta,\gamma} f(t) := \frac{t^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} F_1(\alpha+\beta, -\gamma, \alpha, 1-\frac{s}{t}) f(s)\, ds,\ \alpha > 0,\ \beta, \gamma \geq 0.$$

For the properties of this operator see Saigo and Ryabogin [148] and the references therein.

This concept was pushed furthest by Kiryakova [108] who used the Meijer G-function to define a very flexible fractional calculus.

3. **Hadamard's integral operator**

The integral

$$J_{\mu}^{\alpha} f(t) := \frac{1}{\Gamma(\alpha)} \int_0^t \left(\frac{s}{t}\right)^{\mu} \log\left(\frac{t}{s}\right)^{\alpha-1} \frac{f(s)}{s}\, ds, \;\; \alpha > 0, \mu \geq 0.$$

was introduced by Hadamard [95]. Therefore, such an integral is often referred to as the Hadamard fractional integral [see Samko et al. [169] Section 18.3 and Section 23.1, notes to Section 18.3)]. The generalization of this definition introduced by Butzer et al. [38].

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