Scientific Computations 1

Numerical Solution of Ordinary Differential Equation

Ordinary differential equations(ODEs)

- Differential equation is an equation involving derivatives of a function.

- If a differential equation has a specific value, called initial condition, then the specific problem is called initial value problem (IVP).

- A first order initial value problem of ODE may be written in the form

$$y'(x) = f(x, y), \qquad y(x_0) = y_0$$

Numerical methods for solving ODEs

- **Numerical methods for ordinary differential equations**

are methods used to find numerical approximations to the solutions of ordinary differential equations (ODEs).

One-Step Methods	**Multi-Step Methods**
Estimates of the solution at a particular step are entirely based on information on the previous step	Estimates of the solution at a particular step are entirely based on information on more than one step

One-Step Methods

- Taylor Method
- Euler Method
- Runge-Kutta Method

Taylor's Series Method

Suppose y(x) is a solution of the differential equation

$$y'(x) = f(x, y), \qquad y(x_0) = y_0$$

From Taylor expansion

$$y(x_{n+1}) = y(x_n) + \frac{h}{1!}y'(x_n) + \frac{h^2}{2!}y''(x_n) + \frac{h^3}{3!}y'''(x_n) + \cdots$$

$$x_n = x_0 + nh \qquad\qquad n = 0,1,2,3, \dots \dots$$

Taylor's Series Method

❖ **Example**: using Taylor's method to find the solution of

$$y'(x) = x + y, \quad y(0) = 2, \quad then \quad find \quad y(0.1) \quad and \quad y(0.2).$$

- **Solution:** Differentiating, we get

$$y'(x) = x + y, \quad y''(x) = 1 + y'(x),$$
$$y'''(x) = y''(x), \quad y''''(x) = y'''(x)$$

Taylor's Series:

$$y(x_{n+1}) = y(x_n) + \frac{h}{1!}y'(x_n) + \frac{h^2}{2!}y''(x_n) + \frac{h^3}{3!}y'''(x_n) + \cdots$$

Taylor's Series Method

$(1) \quad find \quad y(0.1):$

Here $x_0 = 0,\ y_0 = 2,\ h = 0.1$

$$y(0.1) = y(0) + \frac{0.1}{1!}y'(0) + \frac{(0.1)^2}{2!}y''(0) + \frac{(0.1)^3}{3!}y'''(0) + \frac{(0.1)^4}{4!}y''''(0) \ldots$$

$$= 2 + 0.1(2) + \frac{(0.1)^2}{2!}(3) + \frac{(0.1)^3}{3!}(3) + \frac{(0.1)^4}{4!}(3) = 2.2$$

$(2) \quad find \quad y(0.2):$

$$x_1 = 0.1, \quad y_1 = 2.2, \quad h = 0.1$$

$$y(0.2) = y(0.1) + \frac{0.1}{1!}y'(0.1) + \frac{(0.1)^2}{2!}y''(0.1) + \frac{(0.1)^3}{3!}y'''(0.1) + \frac{(0.1)^4}{4!}y''''(0.1) \ldots$$

$$= 2.2 + 0.1(2.3) + \frac{(0.1)^2}{2!}(3.3) + \frac{(0.1)^3}{3!}(3.3) + \frac{(0.1)^4}{4!}(3.3) = 2.4470633$$

Taylor's Series Method

❖ **Exercise**: Employ Taylor's method to obtain approximate value of y at x = 0.2 for the differential equation

$$y'(x) = 2y + 3e^{x},\ \ y(0) = 0.$$

Compare the numerical solution obtained with the exact solution.

❖ **Exercise**: Solve by Taylor series method of third order the equation

$$y'(x) = x^2 y - 1,\ \ y(0) = 1.$$

at x = 0.1 and x=0.2

Euler method

- Euler method take three elements only From Taylor expansion.

$$y(x_{n+1}) = y(x_n) + \frac{h}{1!} y'(x_n) + O(h^2)$$

- Rewriting the above equation we have.

$$y_{n+1} = y_n + hy_n', \qquad y_n' = f(x_n, y_n)$$

- y_n is calculated as

$$y_1 = y_0 + hy_0' = y_0 + h\,f(x_0, y_0)$$
$$y_2 = y_1 + h\,f(x_1, y_1)$$
$$\vdots$$
$$y_n = y_{n-1} + h\,f(x_{n-1}, y_{n-1})$$

Euler method

❖ **Example**: use the Euler method to find the solution of

$$y'(x) = x + y, \quad y(0) = 1, \quad at \quad x = 0.1, \quad h = 0.02$$

- **Solution:**

$$y(0.02) = y_1 = y_0 + hy_0' = 1.02$$
$$y(0.04) = y_2 = y_1 + hy_1' = 1.0408$$
$$y(0.06) = y_3 = y_2 + hy_2' = 1.0624$$
$$y(0.08) = y_4 = y_3 + hy_3' = 1.0048$$
$$y(0.1) = y_5 = y_4 + hy_4' = 1.1081$$

Matlab code

```
function [ x,y ] = euler(h,fxy,a,b,x0,y0)
n=(b-a)/h;
x(1)=x0;
y(1)=y0;
for i=1:n
    y(i+1)=y(i)+h*fxy(x(i),y(i));
    x(i+1)=x(i)+h;
end
end
```

```
>> [ x,y ] = euler(0.02,@(x,y)(x+y),0,0.1,0,1)
```

Euler method

- **Exercise**: use the Euler method to find the solution of

$$y'(x) = \frac{y-x}{y+x}, \quad y(0)=1, \quad at \quad x=0.1, \quad h=0.02$$

- **Algorithm:**

?

Runge Kutta method

Second Order Runge-Kutta Method

$$y_{n+1} = y_n + \frac{1}{2}\left(k_1 + k_2\right)$$

$$\text{where} \quad k_1 = hf\,(x_n, y_n)$$

$$k_2 = hf\,(x_n + h, y_n + k_1)$$

Runge Kutta method

- **Example**: Using Runge Kutta of order two to find $y(0.1)$, $y(0.2)$ to solve $y'(x) = x^2 - y,\ y(0) = 1.$

- **Solution:** find $y(0.1)$

$$y'(x) = x^2 - y, \qquad x_0 = 0,\ \ y_0 = 1, \qquad \text{h} = 0.1$$

$$k_1 = (0.1)f(x_0, y_0) = -0.1,$$

$$k_2 = (0.1)f(x_0 + h, y_0 + k_1) = 0.098,$$

$$y_1 = y_0 + \frac{1}{2}(k_1 + k_2) = 0.9055,$$

Runge Kutta method

find $y(0.2)$

$$k_1 = (0.1)f(x_1, y_1) = -0.08955,$$

$$k_2 = (0.1)f(x_1 + h, y_1 + k_1) = -0.07759,$$

$$y_2 = y(0.2) = y_1 + \frac{1}{2}(k_1 + k_2) = 0.821975$$

Matlab Code

```
function [ x,y ] = RK2(h,fxy,a,b,x0,y0)
n=(b-a)/h;
x(1)=x0;
y(1)=y0;
for i=1:n
    k1=h*fxy(x(i),y(i));
    k2=h*fxy(x(i)+h,y(i)+k1);
    y(i+1)=y(i)+(0.5)*(k1+k2)
    x(i+1)=x(i)+h
end
end

%[ x,y ] = RK2(0.1,@(x,y)(x^2-y),0,0.2,0,1)
```

Runge Kutta method

Fourth Order Runge-Kutta Method

$$y_{n+1} = y_n + \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)$$

$$\text{where} \quad k_1 = hf\,(x_n, y_n)$$

$$k_2 = hf\,(x_n + \frac{h}{2}, y_n + \frac{k_1}{2})$$

$$k_3 = hf\,(x_n + \frac{h}{2}, y_n + \frac{k_2}{2})$$

$$k_4 = hf\,(x_n + h, y_n + k_3)$$

Runge Kutta method

❖ **Example**: Apply the Runge-Kutta fourth order method to find an approximate value of y when x= 0.2 given that $y'(x) = x + y,\ \ y(0) = 1$

❖ **Solution:** $x_0 = 0,\ \ y_0 = 1,\qquad \mathrm{h} = 0.2$

$$k_1 = (0.2)f(0,1) = 0.2,$$
$$k_2 = (0.2)f(0.1,1.1) = 0.2400,$$
$$k_3 = (0.2)f(0.1,1.12) = 0.2440,$$
$$k_4 = (0.2)f(0.2,1.244) = 0.2888,$$

$$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 1.2428,$$

Matlab Code

```
function [ x,y ] = RK4(h,fxy,a,b,x0,y0)
n=(b-a)/h;
x(1)=x0;
y(1)=y0;
for i=1:n
    k1=h*fxy(x(i),y(i));
    k2=h*fxy(x(i)+(h/2),y(i)+(k1/2));
    k3=h*fxy(x(i)+(h/2),y(i)+(k2/2));
    k4=h*fxy(x(i)+h,y(i)+k3);
    y(i+1)=y(i)+(1/6)*(k1+2*k2+2*k3+k4);
    x(i+1)=x(i)+h
end
end
```

```
>> [ x,y ] = RK4(0.2,@(x,y)(x+y),0,0.4,0,1)
```

Runge Kutta method

❖ **Exercise**: Using Runge Kutta of order four to find y(0.1), y(0.2), y(0.3) to solve $y'(x) = xy + y^2,\ y(0) = 1.$

❖ **Algorithm:**

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