Digital Signal Processing

Inverse Z Transform

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Z Transform

Z transform of a discrete time signal x(n) or we may call x(k) is :defined as

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

:For convenience, The z transform of a signal x(n) is denoted by

$X(Z) = \mathcal{Z}\,(X(n))$

The inverse procedure (obtaining x(k) from x(z) is called the inverse Z_Transform

$$\mathbf{X(n) = Z^{-1}(x(z))}$$

:Whereas the Relationship between X(n) and X(z) is indicated by

X(n) $\longleftrightarrow$ **X(Z)**

Z transform definition for causal Sequence

Defination 2.1:Given the causal sequence:

x(n)=0   n <0

x(n)={x0,x1,x2,x3,....} then its z transform is defined as:

$X(z)= x_0+x_1z^{-1}+x_2z^{-2}+x_3z^{-3}+....$

$$x(z) = \sum_{n=0}^{\infty} x_n z^{-n}$$

z^{-1} = time delay operator

Inversion of the z-Transform

1. **Long division:** gives as many terms of series as desired.
2. **Partial fraction expansion and table look-up:** similar to Laplace transform inversion.

Long Division

(*i*) Using long division, expand $F(z)$ as a series

$$F_i(z) = f_0 + f_1 z^{-1} + \ldots + f_i z^{-i}$$

$$= \sum_{k=0}^{i} f_k z^{-k}$$

(*ii*) Write the inverse transform as the sequence

$$\{f_0, f_1, \ldots, f_i, \ldots\}$$

Example

Inverse z-transform $F(z) = \dfrac{z+1}{z^2+0.2z+0.1}$

Solution:

(i) Long Division

$$\begin{array}{r|l} & z^{-1}+0.8z^{-2}-0.26z^{-3}+\ldots\ldots \\ \hline z^2+0.2z+0.1 & z+1 \\ & \underline{z+0.2+0.1z^{-1}} \\ & \quad 0.8-0.10z^{-1} \\ & \quad \underline{0.8+0.16z^{-1}+0.08z^{-2}} \\ & \quad -0.26z^{-1}-\ldots \end{array}$$

$$F_t(z) = 0 + z^{-1} + 0.8z^{-2} + (-0.26)z^{-3}$$

(ii) Inverse Transformation $\{f_k\} = \{0, 1, 0.8, -0.26, \ldots\}$

Partial Fraction Expansion

(i) Find the partial fraction expansion of $F(z)/z$.

(ii) Obtain the inverse transform $f(k)$ using the z-transform tables.

Three types of z-domain functions $F(z)$:

1. $\boldsymbol{F(z)}$ with simple (non-repeated) real poles.
2. $\boldsymbol{F(z)}$ with complex conjugate & real poles.
3. $\boldsymbol{F(z)}$ with repeated poles.

I: Simple Real Roots

Residue of a complex function $F(z)$ at a simple pole z_i

$$A_i = (z - z_i)F(z)\big]_{z \to z_i}$$

Residue = partial fraction coefficient of the i^{th} term of the expansion

$$F(z) = \sum_{i=1}^{n} \frac{A_i}{z - z_i}$$

Example

Obtain the inverse z-transform of the function

$$F(z) = \frac{z+1}{z^2 + 0.3z + 0.02}$$

Solution: Solve using two different methods.

(*i*) Partial Fraction Expansion (dividing by z)

$$\frac{F(z)}{z} = \frac{z+1}{z(z^2 + 0.3z + 0.02)}$$

$$= \frac{A}{z} + \frac{B}{z+0.1} + \frac{C}{z+0.2}$$

Example (cont.)

$$A = z\frac{F(z)}{z}\Bigg]_{z=0} = F(0) = \frac{1}{0.02} = 50$$

$$B = (z+0.1)\frac{F(z)}{z}\Bigg]_{z=-0.1} = \frac{1-0.1}{(-0.1)(0.1)} = -90$$

$$C = (z+0.2)\frac{F(z)}{z}\Bigg]_{z=-0.2} = \frac{1-0.2}{(-0.2)(-0.1)} = 40$$

Partial fraction expansion

$$F(z) = \frac{50z}{z} - \frac{90z}{z+0.1} + \frac{40z}{z+0.2}$$

Example (cont.)

(*ii*) Table Lookup

$$f(k)=\begin{cases}50\delta(k)-90(-0.1)^k+40(-0.2)^k, & k\geq 0\\ 0 & ,\quad k<0\end{cases}$$

Note

$f(0)=0$, so the time sequence can be rewritten as

$$f(k)=\begin{cases}-90(-0.1)^k+40(-0.2)^k, & k\geq 1\\ 0 & ,\quad k<1\end{cases}$$

Example (cont.)

(i) Partial Fraction Expansion (without dividing by *z*)

$$F(z) = \frac{z+1}{z^2 + 0.3z + 0.02}$$

$$= \frac{A}{z+0.1} + \frac{B}{z+0.2}$$

Partial fraction coefficients

$$A = (z+0.1)F(z)\big]_{z=-0.1} = \frac{1-0.1}{0.1} = 9$$

$$B = (z+0.2)F(z)\big]_{z=-0.2} = \frac{1-0.2}{-0.1} = -8$$

Example (cont.)

Partial Fraction Expansion

$$F(z)=\frac{9}{z+0.1}-\frac{8}{z+0.2}$$
$$=\frac{9z}{z+0.1}z^{-1}-\frac{8z}{z+0.2}z^{-1}$$

(*ii*) Table Lookup (use the delay theorem)

$$f(k)=\begin{cases}9(-0.1)^{k-1}-8(-0.2)^{k-1}, & k\geq 1\\ 0 & , \quad k<1\end{cases}$$

(Verify: same answer as before)

III: Repeated Roots

$$F(z) = \frac{N(z)}{(z-z_1)^r \prod_{j=r+1}^{n} z - z_j} = \sum_{i=1}^{r} \frac{A_{1i}}{(z-z_1)^{r+1-i}} + \sum_{j=r+1}^{n} \frac{A_j}{z-z_j}$$

$$A_{1,i} = \frac{1}{(i-1)!} \frac{d^{i-1}}{d\,z^{i-1}} (z-z_1)^r F(z) \Big]_{z \to z_1}, \quad i = 1, 2, \ldots, r$$

Example

Obtain the inverse **z**-transform of the function

$$F(z) = \frac{1}{z^2(z-0.5)}$$

Solution

(*i*) *Partial Fraction Expansion* (Dividing by *z*)

$$\frac{F(z)}{z} = \frac{1}{z^3(z-0.5)} = \frac{A_{11}}{z^3} + \frac{A_{12}}{z^2} + \frac{A_{13}}{z} + \frac{A_4}{z-0.5}$$

Partial Fraction Coefficients

$$A_{11} = z^3 \left.\frac{F(z)}{z}\right|_{z=0} = \left.\frac{1}{z-0.5}\right|_{z=0} = -2$$

$$A_{12} = \frac{1}{1!}\frac{d}{dz} z^3 \left.\frac{F(z)}{z}\right|_{z=0} = \left.\frac{d}{dz}\frac{1}{z-0.5}\right|_{z=0} = \left.\frac{-1}{(z-0.5)^2}\right|_{z=0} = -4$$

$$A_{13} = \frac{1}{2!}\frac{d^2}{dz^2} z^3 \left.\frac{F(z)}{z}\right|_{z=0}$$

$$= \left(\frac{1}{2}\right)\frac{d}{dz}\left.\frac{-1}{(z-0.5)^2}\right|_{z=0} = \left(\frac{1}{2}\right)\left.\frac{(-1)(-2)}{(z-0.5)^3}\right|_{z=0} = -8$$

$$A_4 = (z-0.5)\left.\frac{F(z)}{z}\right|_{z=0.5} = \left.\frac{1}{z^3}\right|_{z=0.5} = 8$$

Example (cont.)

Partial Fraction Expansion

$$F(z) = \frac{1}{z^2(z-0.5)} = \frac{8z}{z-0.5} - 2z^{-2} - 4z^{-1} - 8$$

(*ii*) *Table Lookup*

z-transform tables and Definition 2.1 yield

$$f(k) = \begin{cases} 8(0.5)^k - 2\delta(k-2) - 4\delta(k-1) - 8\delta(k), & k \ge 0 \\ 0 \qquad , & k < 0 \end{cases}$$

Example (cont.)

Evaluating $f(k)$ at $k = 0, 1, 2$, yields

$f(0) = 8 - 8 = 0$

$f(1) = 8(0.5) - 4 = 0$

$f(2) = 8(0.5)2 - 2 = 0$

$$f(k) = \begin{cases} (0.5)^{k-3}, & k \geq 3 \\ 0 \quad , & k < 3 \end{cases}$$

Using the delay theorem gives the same answer.

$$F(z) = \frac{z}{z-0.5} z^{-3}$$

Unit sample response using Z transform

Determine the system function and the unit sample response of the system described by the difference equation

$$y(n) = \tfrac{1}{2}y(n-1) + 2x(n)$$

Solution By computing the z-transform of the difference equation, we obtain

$$Y(z) = \tfrac{1}{2}z^{-1}Y(z) + 2X(z)$$

Hence the system function is

$$\frac{Y(z)}{X(z)} \equiv H(z) = \frac{2}{1 - \frac{1}{2}z^{-1}}$$

This system has a pole at $z = \frac{1}{2}$ and a zero at the origin. Using Table 3.3 we obtain the inverse transform

$$h(n) = 2(\tfrac{1}{2})^n u(n)$$

This is the unit sample response of the system.

Z-transform Solution of Difference Equations

Example 2.19: Solve the linear difference equation

$$x(k+2)-(3/2)x(k+1)+(1/2)x(k)=1(k)$$

with the initial conditions $x(0) = 1, x(1) = 5/2$

Solution

(i) Z-transform

$$\left[z^2X(z)-z^2x(0)-zx(1)\right]-(3/2)[zX(z)-zx(0)]+(1/2)X(z)$$
$$=z/(z-1)$$

(ii) Solve for X(z)

$$\left[z^2-(3/2)z+(1/2)\right]X(z)=z/(z-1)+z^2+(5/2-3/2)z$$

$$X(z)=\frac{z\left[1+(z+1)(z-1)\right]}{(z-1)(z-1)(z-0.5)}=\frac{z^3}{(z-1)^2(z-0.5)}$$

(*iii*) *Partial fraction expansion*

The partial fraction of $X(z)/z$ is

$$\frac{X(z)}{z} = \frac{z^2}{(z-1)^2(z-0.5)} = \frac{A_{11}}{(z-1)^2} + \frac{A_{12}}{z-1} + \frac{A_3}{z-0.5}$$

$$A_3 = (z-0.5)\frac{X(z)}{z}\bigg|_{z=0.5} = \frac{z^2}{(z-1)^2}\bigg|_{z=0.5} = \frac{(0.5)^2}{(0.5-1)^2} = 1$$

$$A_{11} = (z-1)^2\frac{X(z)}{z}\bigg|_{z=1} = \frac{z^2}{z-0.5}\bigg|_{z=1} = \frac{1}{1-0.5} = 2$$

Equating Coefficients

- Multiply by the denominator

$z^2 = A_{11}(z-0.5) + A_{12}(z-0.5)(z-1) + A_3(z-1)^2$

- Equate coefficient of z^2

$z^2 : 1 = A_{12} + A_3 = A_{12} + 1 \qquad \text{i.e. } A_{12} = 0$

$$X(z) = \frac{2z}{(z-1)^2} + \frac{z}{z-0.5}$$

(iv) *Inverse z-transformation:* z-transform tables

$$x(k) = 2k + (0.5)^k$$