Digital Signal Processing

Design of IIR filter (part1)

By: Prof. Howida A. Shedeed

Head of Scientific Computing Dep. Faculty of computers and information Sciences, Ain Shams University, Cairo, Egypt, dr_howida@cis.asu.edu.eg

IIR Filter

IIR filter is characterized by the following equation

$$y(n) = \sum_{k=0}^{\infty} h(k)x(n-k) \tag{5.1}$$

Which can be expressed in the recursive form:

$$y(n) = \sum_{k=0}^{\infty} h(k)x(n-k) = \sum_{k=0}^{N} a_k x(n-k) - \sum_{k=1}^{M} b_k y(n-k) \tag{7.1}$$

i.e.The current output y(n) is a function of the past output as well as the past and present input samples

$$\frac{Y(z)}{X(z)} = \quad H(z) = \frac{a_0 + a_1 z^{-1} + \ldots + a_N z^{-N}}{1 + b_1 z^{-1} + \ldots + b_M z^{-M}} = \frac{\sum_{k=0}^{N} a_k z^{-k}}{1 + \sum_{k=1}^{M} b_k z^{-k}} \tag{7.2}$$

An important part of the IIR filter design process is to find suitable values for the coefficients a_k and b_k such that some aspect of the filter characteristic, such as frequency response, behaves in a desired manner. Equations 7.1 and 7.2 are the characteristic equations for IIR filters.

IIR Filter

- Use IIR when the only important requirements are sharp cutoff filters and high throughput, as IIR filters, especially those using elliptic characteristics, will give fewer coefficients than FIR.

The Cost will be on the stability of the filter and the phase distortion (nonlinear phase characteristics).

Design stages for digital IIR filters

The design of IIR filters can be conveniently broken down into five main stages.

(1) Filter specification, at which stage the designer gives the function of the filter (for example, lowpass) and the desired performance.

(2) Approximation or coefficient calculation, where we select one of a number of methods and calculate the values of the coefficients, a_k and b_k, in the transfer function, $H(z)$, such that the specifications given in stage 1 are satisfied.

(3) Realization, which is simply converting the transfer function into a suitable filter structure. Typical structures for IIR filters are parallel and cascade of second and/or first-order filter sections.

(4) Analysis of errors that would arise from representing the filter coefficients and carrying out the arithmetic operations involved in filtering with only a finite number of bits.

(5) Implementation, which involves building the hardware and/or writing the software codes, and carrying out the actual filtering operation.

Step1: performance specifications

The frequency response specifications are often in the form of the tolerance scheme.

ε^2	passband ripple parameter
δ_p	passband deviation
δ_s	stopband deviation
f_{p1} and f_{p2}	passband edge frequencies
f_{s1} and f_{s2}	stopband edge frequencies

The passband attenuation in decibels is:

$$A_p = 10\log_{10}(1+\varepsilon^2) = -20\log_{10}(1-\delta_p)$$

And The Stopband attenuation in decibels is:

$$A_s = -20\log_{10}(\delta_s)$$

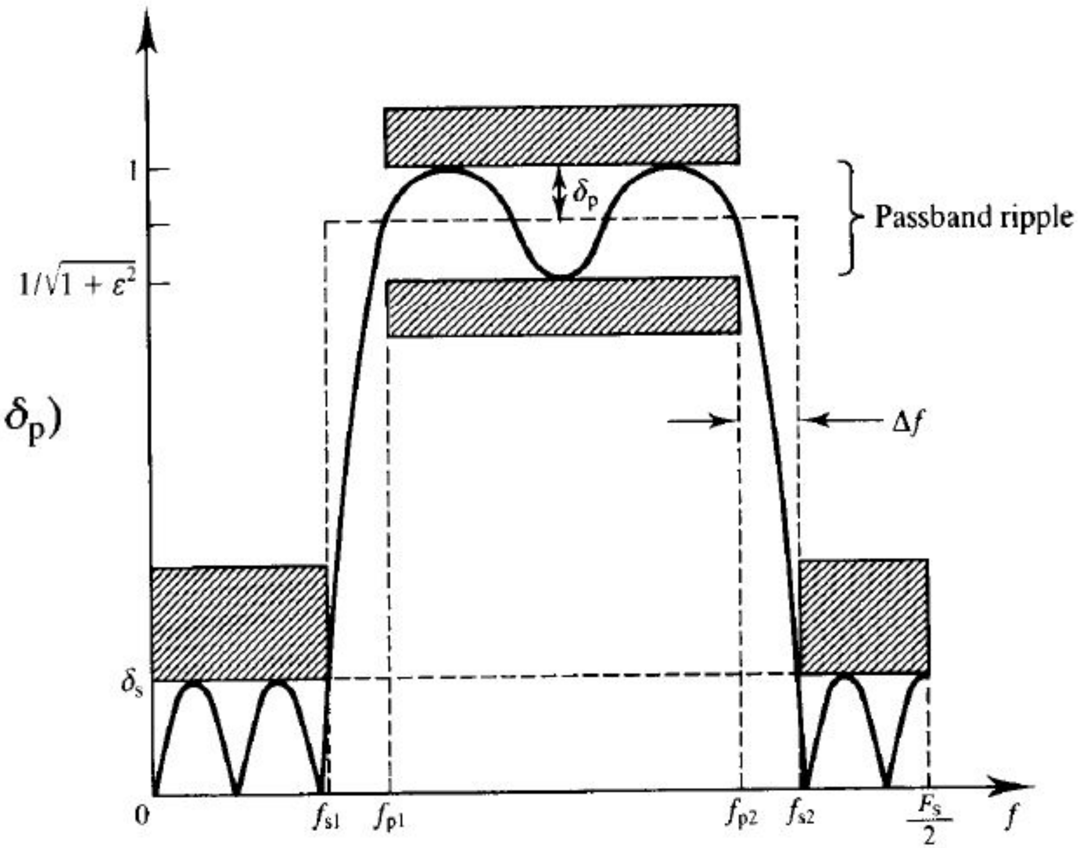

Figure 7.2 Tolerance scheme for an IIR bandpass filter.

Stage 2: Calculations of the filter coefficients

We begin design of digital filter in analog Domain and then convert it to the digital domain. H(s) converted to H(z)

An analog filter can be described by its system function.

$$H_a(s) = \frac{B(s)}{A(s)} = \frac{\sum_{k=0}^{M} \beta_k s^k}{\sum_{k=0}^{N} \alpha_k s^k} \tag{8.3.1}$$

$$H_a(s) = \int_{-\infty}^{\infty} h(t) e^{-st} dt$$

The two common approaches:

- Impulse invariant method
- Bilinear Z-Transform

Impulse Invariant method

Illustrating the impulse invariant method Digitize, using the impulse variant method, the simple analogue filter with the transfer function given by

$$H(s) = \frac{C}{s - p} \tag{7.6}$$

The impulse response, $h(t)$, is given by the inverse Laplace transform:

$$h(t) = L^{-1}[H(s)] = L^{-1}\left(\frac{C}{s - p}\right) = Ce^{pt}$$

where L^{-1} symbolizes the inverse Laplace transform. According to the impulse invariant method, the impulse response of the equivalent digital filter, $h(nT)$, is equal to $h(t)$ at the discrete times $t = nT$, $n = 0, 1, 2, \ldots$, that is

$$h(nT) = h(t)|_{t = nT} = Ce^{pnT}$$

The transfer function of $H(z)$ is obtained by z-transforming $h(nT)$:

$$H(z) = \sum_{n=0}^{\infty} h(nT)z^{-n} = \sum_{n=0}^{\infty} Ce^{pnT}z^{-n} \qquad \sum_{k=0}^{\infty} a^k = \frac{1}{1-a}, |a| < 1$$

$$= \frac{C}{1 - e^{pT}z^{-1}}$$

Impulse Invariant method

$$\frac{C}{s-p} \to \frac{C}{1-e^{pT}z^{-1}} \tag{7.7}$$

To apply the impulse invariant method to a high-order (for example, Mth-order) IIR filter with simple poles, the transfer function, $H(s)$, is first expanded using partial fractions as the sum of single-pole filters:

$$H(s) = \frac{C_1}{s-p_1} + \frac{C_2}{s-p_2} + \dots + \frac{C_M}{s-p_M}$$

$$= \sum_{K=1}^{M} \frac{C_K}{s-p_K} \tag{7.8}$$

where the p_K are the poles of $H(s)$. Each term on the right-hand side of Equation 7.8 has the same form as Equation 7.6 and so the transformation given in Equation 7.8 is applicable. Thus:

$$\sum_{K=1}^{M} \frac{C_K}{s-p_K} \to \sum_{K=1}^{M} \frac{C_K}{1-e^{p_K T}z^{-1}} \tag{7.9}$$

If we have two conjugate poles p1, p2 then C1, C2 are also complex conjugate

Example:

Applying the impulse invariant method to filter design It is required to design a digital filter to approximate the following normalized analogue transfer function:

$$H(s) = \frac{1}{s^2 + \sqrt{2}s + 1}$$

Using the impulse invariant method obtain the transfer function, $H(z)$, of the digital filter, assuming a 3 dB cutoff frequency of 150 Hz and a sampling frequency of 1.28 kHz.

Solution

Before applying the impulse invariant method, we need to frequency scale the normalized transfer function. This is achieved by replacing s by s/α, where $\alpha = 2\pi \times 150 = 942.4778$, to ensure that the resulting filter has the desired response. Thus

$$H'(s) = H(s)|_{s=s/\alpha} = \frac{\alpha^2}{s^2 + \sqrt{2}\,\alpha s + \alpha^2} = \frac{C_1}{s - p_1} + \frac{C_2}{s - p_2}$$

where

$$p_1 = \frac{-\sqrt{2}\alpha(1 - \mathrm{j})}{2} = -666.4324(1 - \mathrm{j}),\ p_2 = p_1^*$$

$$C_1 = \frac{\alpha}{\sqrt{2}}\mathrm{j} = -666.4324\mathrm{j};\ C_2 = C_1^*$$

Since the poles are complex conjugates, the transformation in Equation 7.11 is used to obtain the discrete-time transfer function, $H(z)$. For the problem, $C_r = 0$, $C_i = -666.4324$, $P_iT = 0.5207$, $P_rT = -0.5207$, $e^{P_rT} = 0.5941$, $\sin(P_iT) = 0.4974$, $\cos(p_iT) = 0.8675$, and $e^{P_rT} = 0.3530$. Substituting these values into Equation 7.11, we obtain $H(z)$:

$$H(z) = \frac{393.9264z^{-1}}{1 - 1.0308z^{-1} + 0.3530z^{-2}}$$

Example 8.3.3

Convert the analog filter with system function

$$H_a(s) = \frac{s + 0.1}{(s + 0.1)^2 + 9}$$

into a digital IIR filter by means of the impulse invariance method.

$$p_k = -0.1 \pm j3$$

$$H(s) = \frac{\frac{1}{2}}{s + 0.1 - j3} + \frac{\frac{1}{2}}{s + 0.1 + j3}$$

$$H(z) = \frac{\frac{1}{2}}{1 - e^{-0.1T} e^{j3T} z^{-1}} + \frac{\frac{1}{2}}{1 - e^{-0.1T} e^{-j3T} z^{-1}}$$

$$H(z) = \frac{1 - (e^{-0.1T} \cos 3T) z^{-1}}{1 - (2e^{-0.1T} \cos 3T) z^{-1} + e^{-0.2T} z^{-1}}$$

Bilinear z-Transform Method

The bilinear transform is a first-order approximation of the natural logarithm function that is an exact mapping of the *z*-plane to the *s*-plane. $\mathbf{Z=e^{sT}}$ $\mathbf{S = 1/T \ ln\ (Z)}$

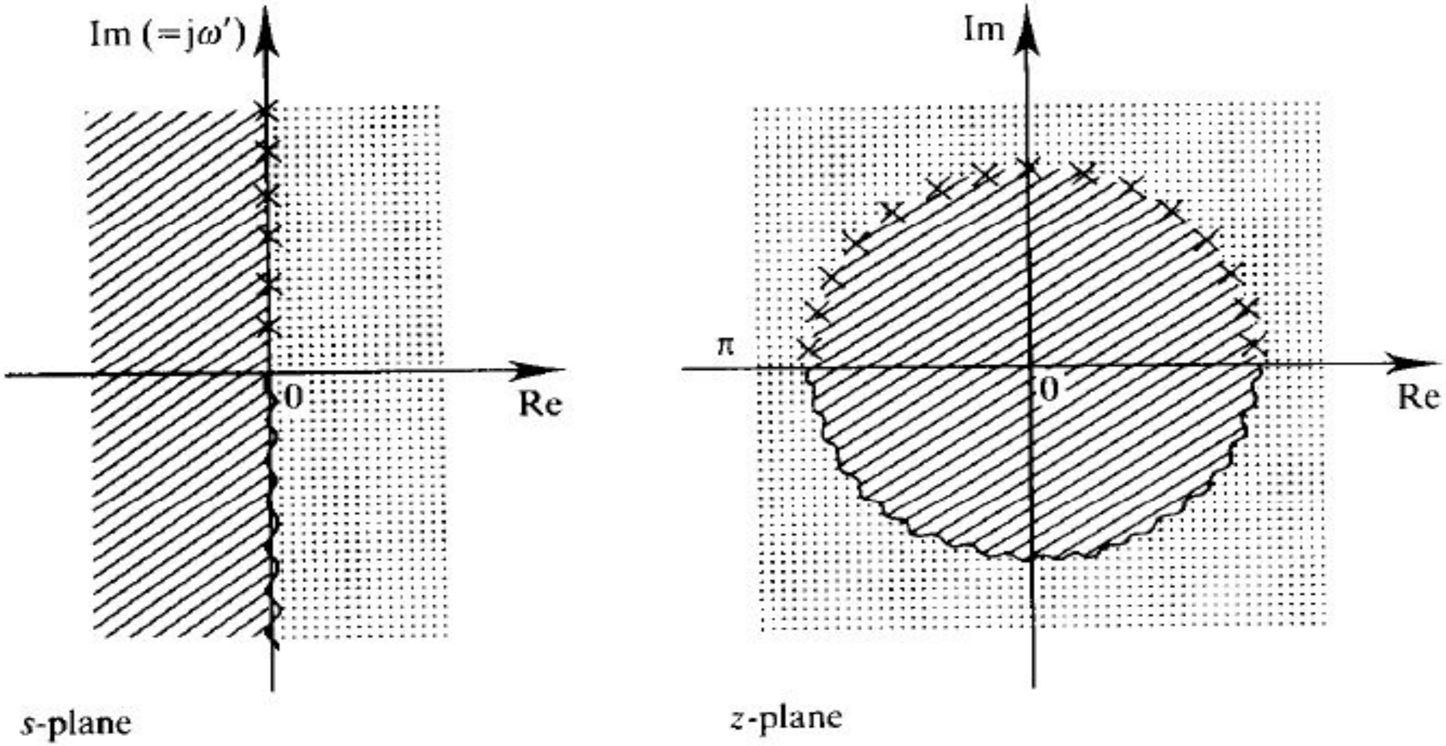

Figure 7.9 An illustration of the *s*-plane to *z*-plane mapping using the bilinear *z* transformation. Note that the positive jω′ axis in the *s*-plane (that is, $s = 0$ to $s = j\infty$) maps to the upper half of the unit circle, and the negative jω′ axis maps to the lower half.

Bilinear z-Transform Method

The inverse of this mapping (and its first-order bilinear approximation) is

$$
\begin{aligned}
s &= \frac{1}{T}\ln(z) \\
&= \frac{2}{T}\left[\frac{z-1}{z+1} + \frac{1}{3}\left(\frac{z-1}{z+1}\right)^3 + \frac{1}{5}\left(\frac{z-1}{z+1}\right)^5 + \frac{1}{7}\left(\frac{z-1}{z+1}\right)^7 + \cdots\right] \\
&\approx \frac{2}{T}\frac{z-1}{z+1} \\
&= \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}
\end{aligned}
$$

The bilinear transform essentially uses this first order approximation and substitutes into the continuous-time transfer function, $H_a(s)$

$$s \leftarrow \frac{2}{T}\frac{z-1}{z+1}.$$

That is

$$H_d(z) = H_a(s)\Big|_{s=\frac{2}{T}\frac{z-1}{z+1}} = H_a\left(\frac{2}{T}\frac{z-1}{z+1}\right).$$

Bilinear z-Transform Method

Converting $H(s)$ into an equivalent digital filter is to replace s as follows:

$$s = k\frac{z-1}{z+1}, \quad k = 1 \text{ or } \frac{2}{T}$$

$Z=e^{sT}$

substitution $z = e^{jwT}$ and $s = j\omega'$ Simplifying, we find that the analogue frequency ω' and the digital frequency ω are related as

$$jw' = K\frac{e^{jwT}-1}{e^{jwT}+1}$$

$$\omega' = k\tan\left(\frac{\omega T}{2}\right), \quad k = 1 \text{ or } \frac{2}{T}$$

Summary of the procedure for calculating digital filter coefficients by the BZT method

(1) Use the digital filter specifications to determine a suitable normalized transfer function, $H(s)$.

(2) Determine the cutoff frequency (or passband edge frequency) of the digital filter and call this ω_p.

(3) Obtain an equivalent analogue filter cutoff frequency (ω'_p) using the relation (prewarped)

$$\omega'_p = k \tan\left(\frac{\omega_p T}{2}\right), \quad k = 1 \text{ or } \frac{T}{2}$$

(4) Denormalize the analogue filter by frequency scaling $H(s)$. This is achieved by replacing s with s/ω'_p.

(5) Apply the bilinear transformation to obtain the desired digital filter transfer function $H(z)$ by replacing s by $(z-1)/(z+1)$.

BZT Example 1:

An illustration of the BZT method Determine, using the BZT method, the transfer function and difference equation for the digital equivalent of the resistance–capacitance (RC) filter. Assume a sampling frequency of 150 Hz and a cutoff frequency of 30 Hz.

Solution

The normalized transfer function for the RC filter is

$$H(s) = \frac{1}{s+1}$$

The critical frequency for the digital filter is $\omega_p = 2\pi \times 30$ rad. The analogue frequency, after prewarping, is $\omega_p' = \tan(\omega_p T/2)$. With $T = 1/150$ Hz, $\omega_p' = \tan(\pi/5) = 0.7265$. The denormalized analogue filter transfer function is obtained from $H(s)$ as

$$H'(s) = H(s) \quad S{=}s/0.7265 = \frac{1}{s/0.7265 + 1} = \frac{0.7265}{s + 0.7265}$$

$$H(z) = H'(s)|_{s=(z-1)/(z+1)} = \frac{0.7265(1+z)}{(1+0.7265)z + 0.7265 - 1}$$

$$= \frac{0.4208(1+z^{-1})}{1 - 0.1584z^{-1}}$$

The difference equation is

$$y(n) = 0.1584y(n-1) + 0.4208[x(n) + x(n-1)]$$

Thanks for good listening

Best wishes

Prof. Howida A. Shedeed

dr_howida@cis.asu.edu.eg