# Digital Signal Processing

## Z Transform

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# Z Transform

Z transform of a discrete time signal x(n) is defined as:

$$X(z) \equiv \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

**For convenience, The z transform of a signal x(n) is denoted by:**

$$X(Z) = \mathcal{Z}\,(X(n))$$

**The inverse procedure (obtaining x(n) from x(z) is called the inverse Z_Transform**

$$\mathbf{X(n) = Z^{-1}(x(z))}$$

**Whereas the Relationship between X(n) and X(z) is indicated by:**

$$\mathbf{X(n)} \overset{\mathcal{Z}}{\longleftrightarrow} \mathbf{X(Z)}$$

# Z transform definition for causal Sequence

Definition: Given the causal sequence:

$x(n)=0 \quad n<0$

$x(n)=\{x0,x1,x2,x3,....\}$ then its z transform is defined as:

$X(z)= x_0+x_1z^{-1}+x_2z^{-2}+x_3z^{-3}+....$

$$x(z) = \sum_{n=0}^{\infty} x_n z^{-n} \longleftarrow X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$z^{-1}$ = time delay operator

# Example

Obtain the Z transform of the sequence

X(n)= {1,3,2,0,4,0,0,0}

By definition: $$x(z) = \sum_{n=0}^{\infty} x_n z^{-n}$$

X(Z) $= 1+3z^{-1}+2z^{-2}+4z^{-4}$

# Why Z transform

The $z$-transform plays the same role in the analysis of discrete-time signals and LTI systems as the Laplace transform does in the analysis of continuous-time signals and LTI systems. For example, we shall see that in the $z$-domain (complex $z$-plane) the convolution of two time-domain signals is equivalent to multiplication of their corresponding $z$-transforms. This property greatly simplifies the analysis of the response of an LTI system to various signals. In addition, the $z$-transform provides us with a means of characterizing an LTI system, and its response to various signals, by its pole–zero locations.

# Relation between Z transform and Convolution

**Convolution of two sequences.** If

$$x_1(n) \stackrel{z}{\longleftrightarrow} X_1(z)$$

$$x_2(n) \stackrel{z}{\longleftrightarrow} X_2(z)$$

then

$$x(n) = x_1(n) * x_2(n) \stackrel{z}{\longleftrightarrow} X(z) = X_1(z)X_2(z)$$

The ROC of $X(z)$ is, at least, the intersection of that for $X_1(z)$ and $X_2(z)$.

**1.** Compute the $z$-transforms of the signals to be convolved.

$$X_1(z) = Z\{x_1(n)\}$$

(time domain $\longrightarrow$ $z$-domain)

$$X_2(z) = Z\{x_2(n)\}$$

**2.** Multiply the two $z$-transforms.

$$X(z) = X_1(z)X_2(z) \qquad (z\text{-domain})$$

**3.** Find the inverse $z$-transform of $X(z)$.

$$x(n) = Z^{-1}\{X(z)\} \qquad (z\text{-domain} \longrightarrow \text{time domain})$$

This procedure is, in many cases, computationally easier than the direct evaluation of the convolution summation.

# System Function for LTI system

Y(n)= x(n)*h(n)

In z domain: Y(z)= x(z)h(z)

$$Y(n) = z^{-1}Y(z)$$

If we know $h(n)$ and $x(n)$, we can determine their corresponding $z$-transform: $H(z)$ and $X(z)$, multiply them to obtain $Y(z)$, and therefore determine $y(n)$ by evaluating the inverse $z$-transform of $Y(z)$. Alternatively, if we know $x(n)$ and we observe the output $y(n)$ of the system, we can determine the unit sample response by first solving for $H(z)$ from the relation

$$H(z) = \frac{Y(z)}{X(z)} \tag{3.3.5}$$

and then evaluating the inverse $z$-transform of $H(z)$.

Since

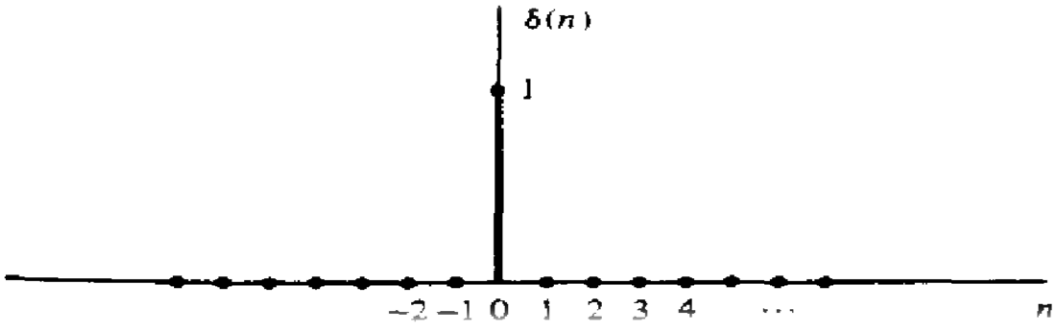
$$H(z) = \sum_{n=-\infty}^{\infty} h(n)z^{-n} \tag{3.3.6}$$

# Unit Sample (Impulse)

X(n)= $$\delta(n) \equiv \begin{cases} 1, & \text{for } n = 0 \\ 0, & \text{for } n \neq 0 \end{cases}$$

By definition: $$X(z) \equiv \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

X(Z)= 1



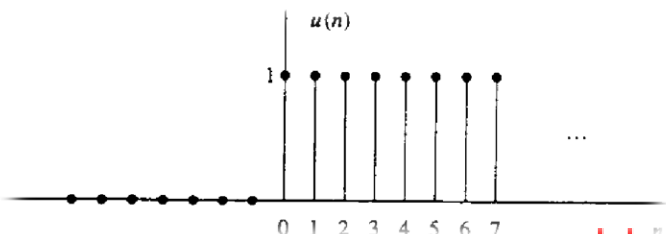
# Region of Convergence Roc

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

Since the $z$-transform is an infinite power series, it exists only for those values of z for which this series converges. The *region of convergence* (ROC) of $X(z)$ is the set of all values of $z$ for which $X(z)$ attains a finite value. Thus any time we cite

Determine x(z) and ROC for the following unit step x(n):
x(n) = u(n) = {1,1,1,1,…}   = 1 n>=0   usually referred as x(n)= 1(n)

$$u(n) = \begin{cases} 1, & \text{for } n \geq 0 \\ 0, & \text{for } n < 0 \end{cases}$$




$U(z)= 1+z^{-1}+z^{-2}+z^{-3}+\ldots$

$$U(z)= \frac{1}{1-Z^{-1}}$$

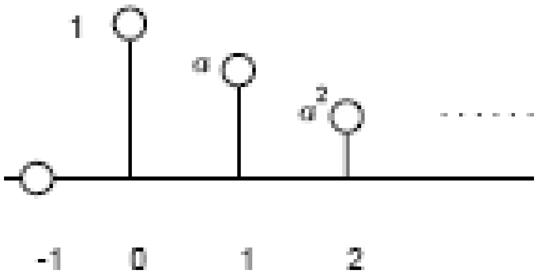
ROC: $|z^{-1}| <1 \rightarrow \frac{1}{|z|} <1 \rightarrow |z| >1$

Identities Used Repeatedly

$$\sum_{k=0}^{n} a^k = \frac{1-a^{n+1}}{1-a}, a \neq 1$$

$$\sum_{k=0}^{\infty} a^k = \frac{1}{1-a}, |a| < 1$$

# Sampled Exponential




$X(n) = a^n u(n)$

$$X(z) = 1 + a\,z^{-1} + a^2 z^{-2} + a^3 z^{-3} + \ldots + a^k z^{-k} + \ldots$$

$$= \sum_{n=0}^{\infty} (az^{-1})^n$$

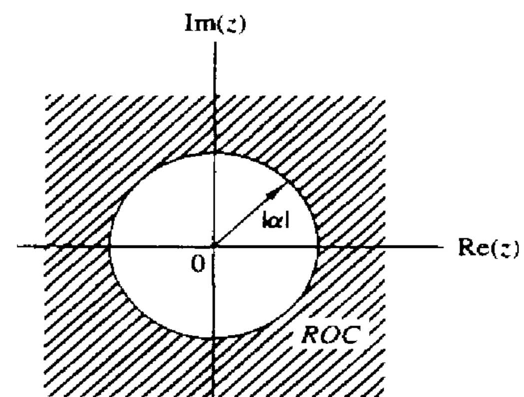
$$= \frac{1}{1 - a\,Z^{-1}}$$

$$= \frac{z}{z - a}$$

**Identities Used Repeatedly**

$$\sum_{k=0}^{n} a^k = \frac{1 - a^{n+1}}{1 - a}, a \neq 1$$

$$\sum_{k=0}^{\infty} a^k = \frac{1}{1 - a}, |a| < 1$$




**ROC in z domain:** $|a\,Z^{-1}| < 1$

**:** $\left|\frac{a}{z}\right| < 1 \qquad |Z| > a$

# ROC example

1- x1(n)= {1,2,5,7,0,1}

2- x2(n)= {1,2,5,7,0,1}

3- x3(n)= {0,0,1,2,5,7,0,1}

4- x4(n) =δ (n) = {1,0,0,0,..}

5- x5(n)= δ(n-2)= {0,0,1,0,0,0,..}

6- x6(n)= δ(n+2)= {1,0,0,0,..}

$$X(z) \equiv \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

Sol: 1- $X_1(z) = 1 + 2z^{-1} + 5z^{-2} + 7z^{-3} + z^{-5}$, ROC: entire $z$-plane except $z = 0$

2- $X_2(z) = z^2 + 2z + 5 + 7z^{-1} + z^{-3}$, ROC: entire $z$-plane except $z = 0$ and $z = \infty$

3- $X_3(z) = z^{-2} + 2z^{-3} + 5z^{-4} + 7z^{-5} + z^{-7}$, ROC: entire $z$-plane except $z = 0$

4- x4(z)= 1 ROC: entire $z$-plane

5- $x5(z)=0+0+z^{-2} = 1/z^2$ ROC: entire $z$-plane except $z = 0$

6- $x6(z)= z^2$ ROC: entire $z$-plane except $z = \infty$

# Z – transform Properties

**Linearity.** If

$$x_1(n) \overset{z}{\longleftrightarrow} X_1(z)$$

and

$$x_2(n) \overset{z}{\longleftrightarrow} X_2(z)$$

then

$$x(n) = a_1x_1(n) + a_2x_2(n) \overset{z}{\longleftrightarrow} X(z) = a_1X_1(z) + a_2X_2(z)$$

# Example

Find the z transform of the following signal x(n)

$X(n) = 3\ 2^n\ u(n) - 4\ 3^n\ u(n)$

$X1(n) = 2^n\ u(n)$ $\quad X1(z) = \frac{1}{1-2\ Z^{-1}} = \frac{z}{z-2}$ $\quad$ ROC: $|z| > 2$

$X2(n) = 3^k\ u(n)$ $\quad X2(z) = \frac{1}{1-3\ Z^{-1}} = \frac{z}{z-3}$ $\quad$ ROC: $|z| > 3$

$x(z) = 3x1(z) - 4\ x2(z)$

$$X(z) = \frac{3}{1-2z^{-1}} - \frac{4}{1-3z^{-1}} \qquad \text{ROC: } |z| > 3$$

# Time Delay

**If**

$$x(n) \xleftrightarrow{z} X(z)$$

$$x(n-k) \xleftrightarrow{z} z^{-k} X(z)$$

Example:

X(n)= 4    n=2,3,4,....

$X(z) = \mathcal{Z}\{4x\ u(n-2)\} = 4\ z^{-2}\ \mathcal{Z}(u(n)) = 4\ z^{-2}\ \dfrac{1}{1-Z^{-1}}$

$$= z^{-2}\frac{4z}{z-1} = \frac{4}{z(z-1)}$$

# Time Advance

$\mathcal{Z}$(x(n+1))=ZX(z)-ZX(0)

$\mathcal{Z}$(x(n+k))=$z^k$ X(z)-$z^k$ X(0)- $z^{k-1}$ X(1)-.....zX(k-1)

Example: use the time advance property to find the z transform of the causal sequence:

X(n)= {4,8,16,..}

X(n)= $2^{n+2}$ = g(n+2) where g(n) = $2^n$ n=0,1,2,...

X(z) $= z^2 G(z) - z^2 g(0) - z g(1) = z^2 \dfrac{z}{z-2} - z^2 - 2z = \dfrac{4z}{z-2}$

Another Sol: x(n) = 4 {1,2,4,...} = 4 . $2^n$

$$X(z) = 4\frac{z}{z-2}$$

# Differentiation in z domain

$$\mathcal{Z}(n^m\, x(n)) = \left(-z\frac{d}{dz}\right)^m X(z)$$

$$x(n) \stackrel{z}{\longleftrightarrow} X(z) \qquad nx(n) \stackrel{z}{\longleftrightarrow} -z\frac{dX(z)}{dz}$$

Example: find the z transform of the sampled ramp sequence

$x(n) = n\,u(n) \qquad n=0,1,2,..$

let $x_1(n) = u(n)$

$$\mathcal{Z}(n\, x_1(n)) = (-z\frac{d}{dz})\, x_1(z)$$

$$X(z) = \left(-z\frac{d}{dz}\right)\left(\frac{z}{z-1}\right) = (-z)\frac{(z-1)-z}{(z-1)^2} = \frac{z}{(z-1)^2}$$

# Differentiation in z domain

$\mathcal{Z}(n^m\, x(n)) = \left(-z\frac{d}{dz}\right)^m X(z)$

**Example 3.2.7**

Determine the $z$-transform of the signal

$$x(n) = na^n u(n)$$

$$x_1(n) = a^n u(n) \overset{z}{\longleftrightarrow} X_1(z) = \frac{1}{1 - az^{-1}} \qquad \text{ROC: } |z| > |a|$$

$$na^n u(n) \overset{z}{\longleftrightarrow} X(z) = -z\frac{dX_1(z)}{dz} = \frac{az^{-1}}{(1 - az^{-1})^2} \qquad \text{ROC: } |z| > |a| \tag{3.2.15}$$

If we set $a = 1$ in (3.2.15), we find the $z$-transform of the unit ramp signal

$$nu(n) \overset{z}{\longleftrightarrow} \frac{z^{-1}}{(1 - z^{-1})^2} \qquad \text{ROC: } |z| > 1 \tag{3.2.16}$$

# Multiplication by Exponential

$$\mathcal{Z}(a^n x(n)) = X(a^{-1}z)$$

$$u(n) = \begin{cases} 1, & \text{for } n \geq 0 \\ 0, & \text{for } n < 0 \end{cases}$$

$X(n) = a^n u(n)$

$U(z) = 1 + z^{-1} + z^{-2} + z^{-3} + \ldots$

$$X(z) = \frac{1}{1-(a^{-1}\,z)^{-1}}$$

$$U(z) = \frac{1}{1-Z^{-1}}$$

$$= \frac{1}{1-a\,Z^{-1}}$$

$$= \frac{z}{z-a}$$

# Z transform of trigonometric functions

Determine the Z transform of the signals:

1)x(n)= cos($w_0$n )u(n)

Using: cos($w_0$n) u(n) = ½ [ $e^{jw0n}$ + $e^{-jw0n}$) u(n)

$$X(z) = \frac{1}{2}\frac{1}{1-e^{j\omega_0}z^{-1}} + \frac{1}{2}\frac{1}{1-e^{-j\omega_0}z^{-1}}$$

ROC: $|z| > 1$

If we set $\alpha = e^{\pm j\omega_0}$ ($|\alpha| = |e^{\pm j\omega_0}| = 1$)

$$(\cos\omega_0 n)u(n) \overset{z}{\longleftrightarrow} \frac{1-z^{-1}\cos\omega_0}{1-2z^{-1}\cos\omega_0+z^{-2}}$$

ROC: $|z| > 1$

2) x(n)= sin($w_0$n) u(n)

Using: Sin($w_0$n) u(n) = ½j [ $e^{jw0n}$ – $e^{-jw0n}$) u(n)

$$X(z) = \frac{1}{2j}\left(\frac{1}{1-e^{j\omega_0}z^{-1}} - \frac{1}{1-e^{-j\omega_0}z^{-1}}\right)$$

ROC: $|z| > 1$

$$(\sin\omega_0 n)u(n) \overset{z}{\longleftrightarrow} \frac{z^{-1}\sin\omega_0}{1-2z^{-1}\cos\omega_0+z^{-2}}$$

ROC: $|z| > 1$

# Discrete Convolution Using Z- transform

**Convolution of two sequences.** If

$$x_1(n) \overset{z}{\longleftrightarrow} X_1(z)$$

$$x_2(n) \overset{z}{\longleftrightarrow} X_2(z)$$

then

$$x(n) = x_1(n) * x_2(n) \overset{z}{\longleftrightarrow} X(z) = X_1(z)X_2(z)$$

The ROC of $X(z)$ is, at least, the intersection of that for $X_1(z)$ and $X_2(z)$.

1. Compute the $z$-transforms of the signals to be convolved.

$$X_1(z) = Z\{x_1(n)\}$$

(time domain $\longrightarrow$ $z$-domain)

$$X_2(z) = Z\{x_2(n)\}$$

2. Multiply the two $z$-transforms.

$$X(z) = X_1(z)X_2(z) \qquad (z\text{-domain})$$

3. Find the inverse $z$-transform of $X(z)$.

$$x(n) = Z^{-1}\{X(z)\} \qquad (z\text{-domain} \longrightarrow \text{time domain})$$

This procedure is, in many cases, computationally easier than the direct evaluation of the convolution summation.

# Discrete Convolution Example

**Example 3.2.9**

Compute the convolution $x(n)$ of the signals

$$x_1(n) = \{1, -2, 1\}$$

$$x_2(n) = \begin{cases} 1, & 0 \le n \le 5 \\ 0, & \text{elsewhere} \end{cases}$$

**Solution** From (3.1.1), we have

$$X_1(z) = 1 - 2z^{-1} + z^{-2}$$

$$X_2(z) = 1 + z^{-1} + z^{-2} + z^{-3} + z^{-4} + z^{-5}$$

According to (3.2.17), we carry out the multiplication of $X_1(z)$ and $X_2(z)$. Thus

$$X(z) = X_1(z)X_2(z) = 1 - z^{-1} - z^{-6} + z^{-7}$$

Hence

$$x(n) = \{\underset{\uparrow}{1}, -1, 0, 0, 0, 0, -1, 1\}$$

# Correlation using z Transform

**Correlation of two sequences.** If

$$x_1(n) \overset{z}{\longleftrightarrow} X_1(z)$$

$$x_2(n) \overset{z}{\longleftrightarrow} X_2(z)$$

then

$$r_{x_1x_2}(l) = \sum_{n=-\infty}^{\infty} x_1(n)x_2(n-l) \overset{z}{\longleftrightarrow} R_{x_1x_2}(z) = X_1(z)X_2(z^{-1})$$

# Rational Z transform

An important family of z transform are those for which x(Z) is a rational function.

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}} \qquad (3.3.8)$$

Therefore, a linear time-invariant system described by a constant-coefficient difference equation has a rational system function.

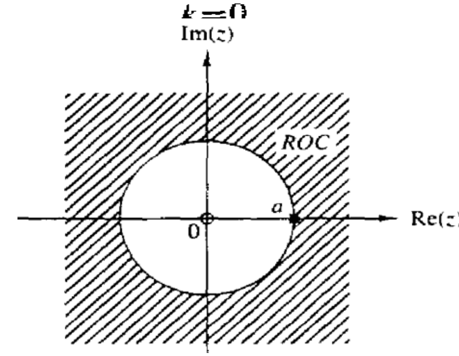
**Poles and zeros:**

The *zeros* of a $z$-transform $X(z)$ are the values of $z$ for which $X(z) = 0$. The *poles* of a $z$-transform are the values of $z$ for which $X(z) = \infty$. If $X(z)$ is a rational function, then

$$X(z) = \frac{N(z)}{D(z)} = \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{a_0 + a_1 z^{-1} + \cdots + a_N z^{-N}} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

Example:

$X(n) = a^n u(n) \longleftrightarrow X(z) = \frac{z}{z-a}$ ROC: $|z| > a$




Zeros: one zero at z1=0

Poles : and one pole p1=a

# Poles location and Time domain Behavior for Causal signals

$$x(n) = a^n u(n) \overset{z}{\longleftrightarrow} X(z) = \frac{1}{1 - az^{-1}} \qquad \text{ROC: } |z| > |a|$$

having one zero at $z_1 = 0$ and one pole at $p_1 = a$ on the real axis. Figure 3.11

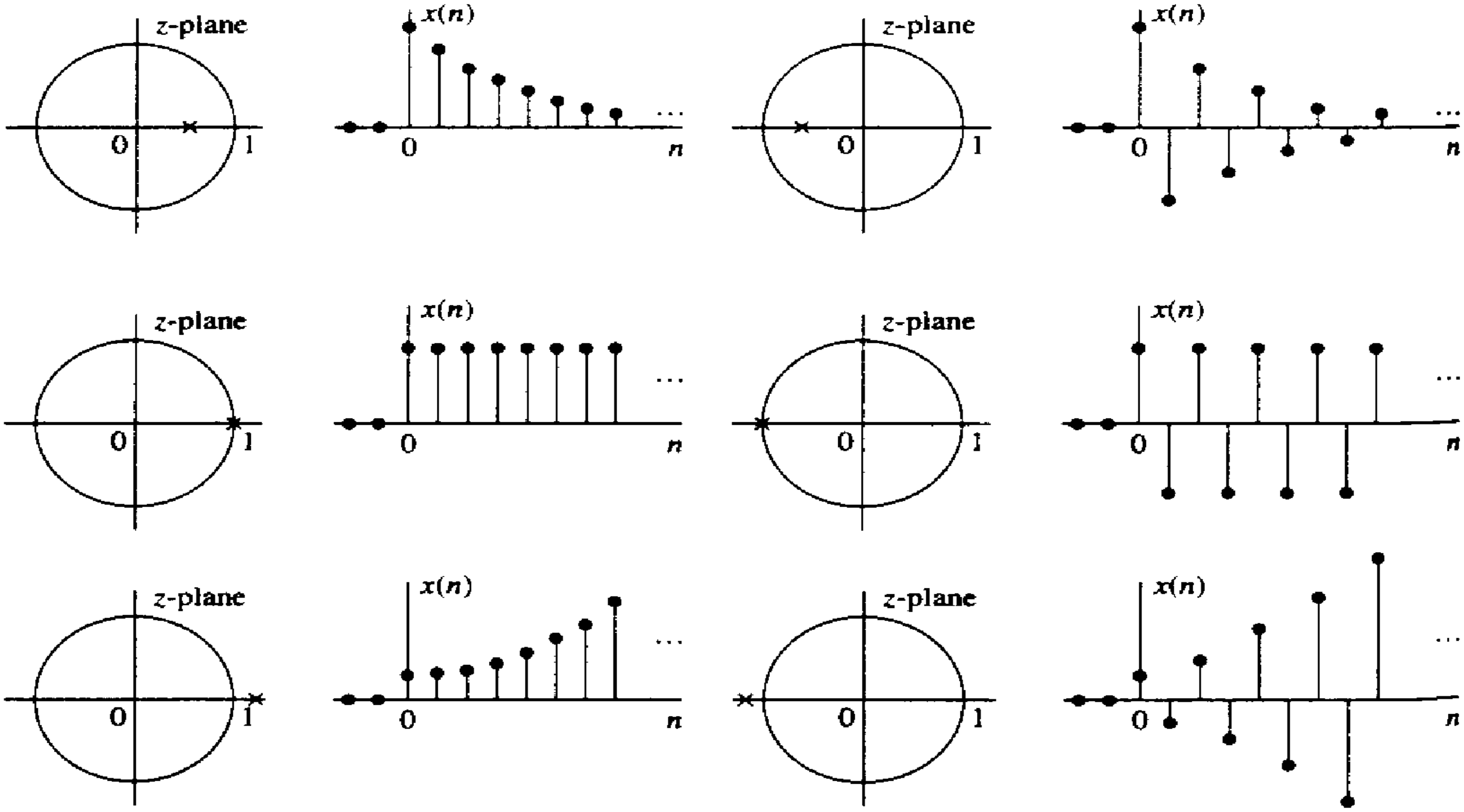



**Figure 3.11** Time-domain behavior of a single-real pole causal signal as a function of the location of the pole with respect to the unit circle.

# Thanks