Digital Signal Processing

Design of Digital Filters

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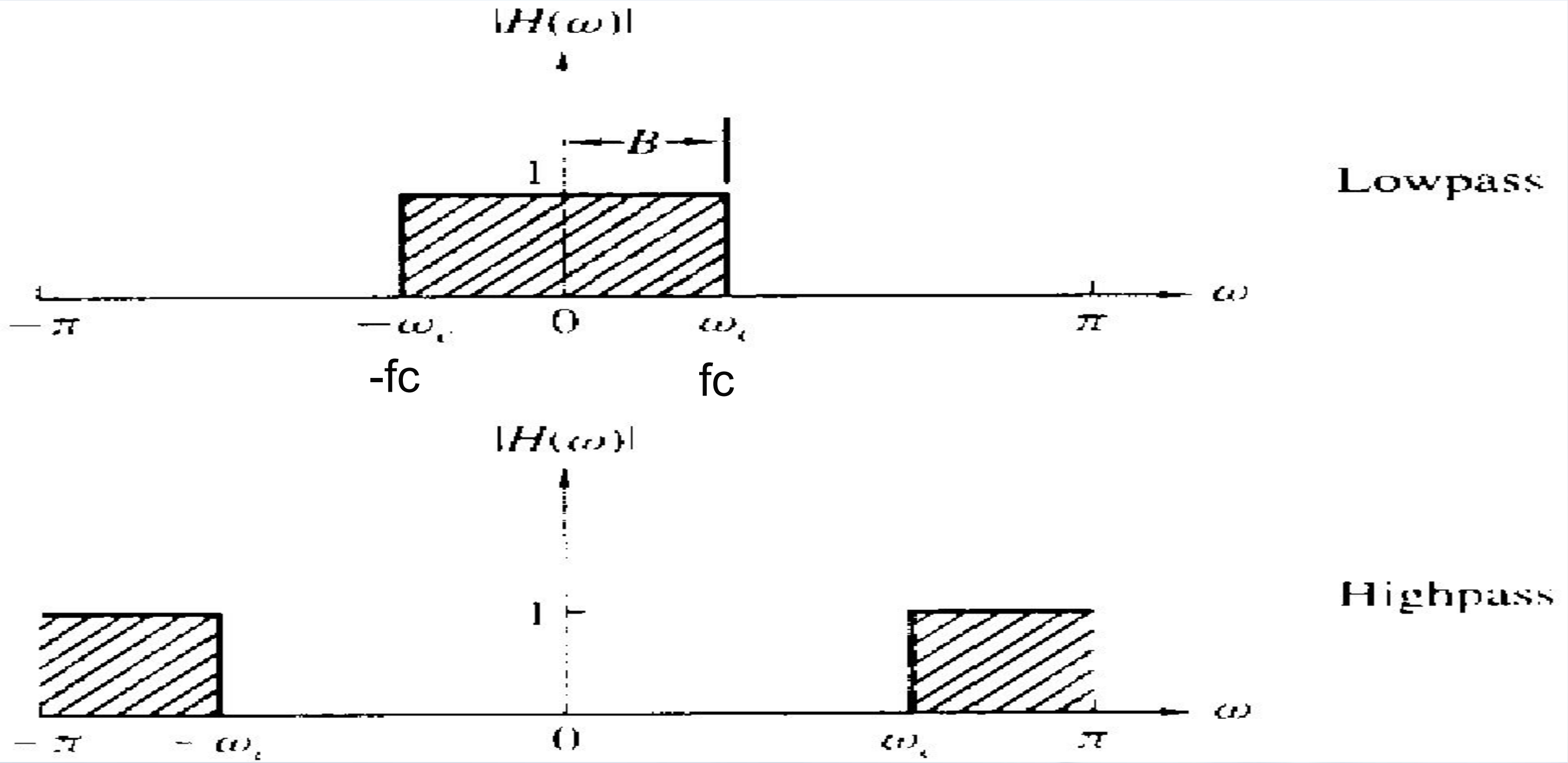
Filter

A filter is essentially a system or network that selectively changes the wave-shape, amplitude–frequency and/or phase–frequency characteristics of a signal in a desired manner. Common filtering objectives are to improve the quality of a signal (for example, to remove or reduce noise), to extract information from signals or to separate two or more signals previously combined to make, for example, efficient use of an available communication channel.

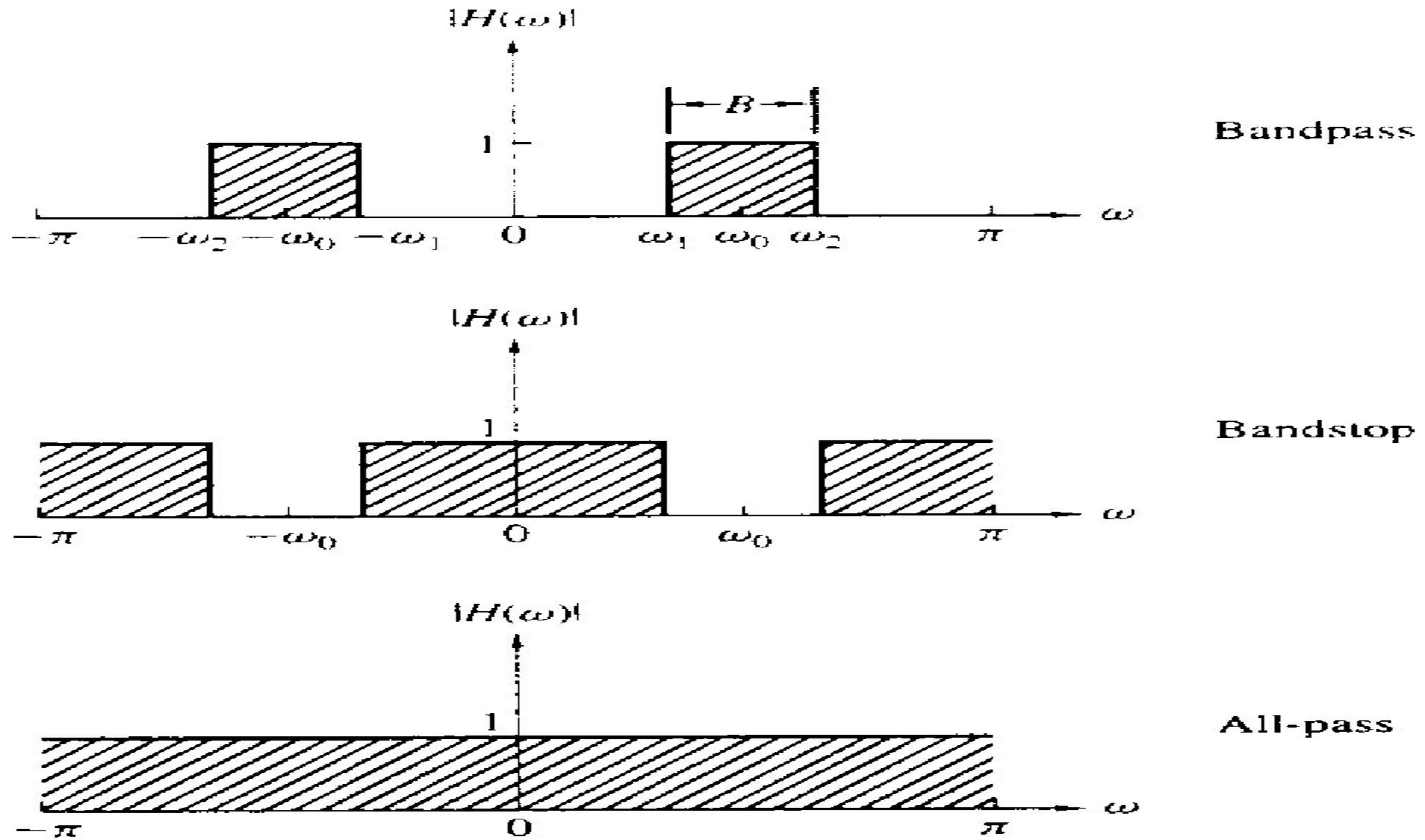
Type of filters: frequency characteristics

Ideal Filter Characteristics

Filters are usually classified according to their frequency-domain characteristics as lowpass, highpass, bandpass, and bandstop or band-elimination filters. The ideal magnitude response characteristics of these types of filters are illustrated



Ideal filters



Types of digital filters: FIR and IIR filters

Digital filters are broadly divided into two classes, namely infinite impulse response (IIR) and finite impulse response (FIR) filters. Either type of filter, in its basic form, can be represented by its impulse response sequence, $h(k)$ $(k = 0, 1, \ldots)$, The input and output signals to the filter are related by the convolution sum, which is given in Equations 5.1 for the IIR and in 5.2 for the FIR filter.

$$y(n) = \sum_{k=0}^{\infty} h(k)x(n-k) \tag{5.1}$$

$$H(z) = \sum_{k=0}^{N-1} h(k)z^{-k}$$

$$y(n) = \sum_{k=0}^{N-1} h(k)x(n-k) \tag{5.2}$$

IIR Filter

$$y(n) = \sum_{k=0}^{\infty} h(k)x(n-k) \tag{5.1}$$

$$y(n) = \sum_{k=0}^{N-1} h(k)x(n-k) \tag{5.2}$$

It is evident from these equations that, for IIR filters, the impulse response is of infinite duration whereas for FIR it is of finite duration, since $h(k)$ for the FIR has only N values. In practice, it is not feasible to compute the output of the IIR filter using Equation 5.1 because the length of its impulse response is too long (infinite in theory). Instead, the IIR filtering equation is expressed in a recursive form:

$$y(n) = \sum_{k=0}^{\infty} h(k)x(n-k) = \sum_{k=0}^{N} a_k x(n-k) - \sum_{k=1}^{M} b_k y(n-k) \tag{5.3}$$

i.e.The current output y(n) is a function of the past output as well as the past and present input samples

$$H(z) = \sum_{k=0}^{N} a_k z^{-k} \Big/ \left(1 + \sum_{k=1}^{M} b_k z^{-k}\right)$$

Choosing between FIR and IIR

The most Important Relative advantages of the two filters can be summarized as:

1) FIR can have exactly linear phase response (no phase distortion), whoever the response of the IIR filters are nonlinear especially at the band edges.

2) FIR filters realized nonrecursively (always stable). Stability of IIR can not always be guaranteed.

3) FIR requires more coefficient for sharp cutoff filters than IIR (more processing time and storage required for FIR).

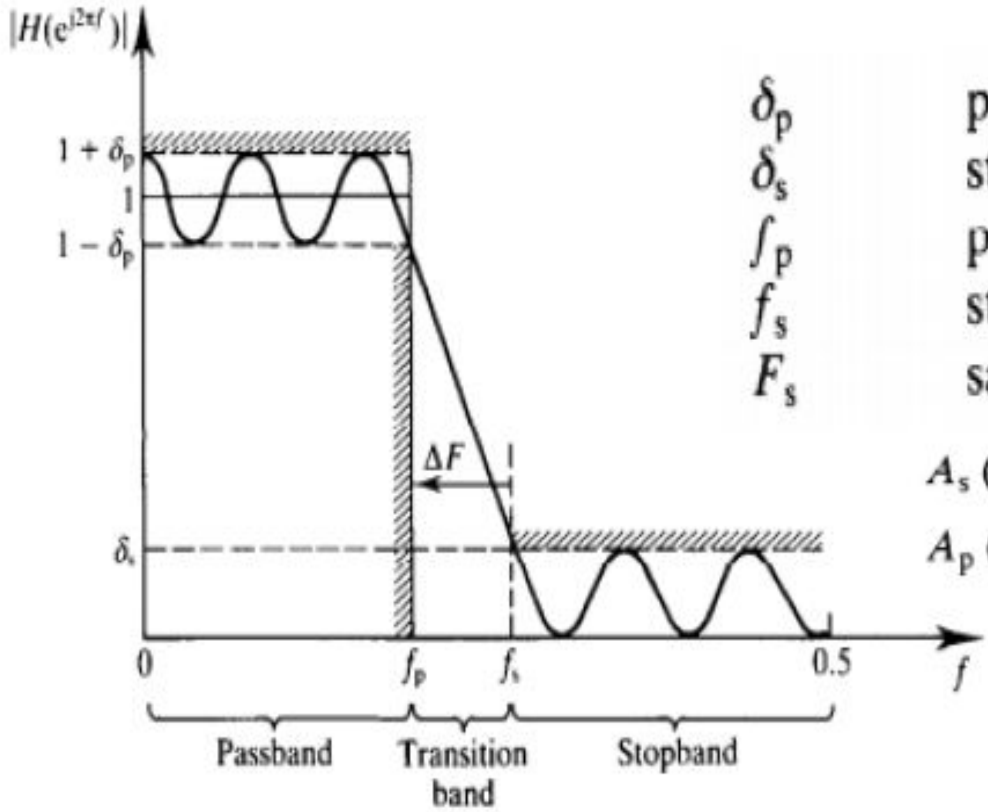
4) Analogue filters can be readily transformed into equivalent IIR digital filters meeting similar specifications. This is not possible with FIR filters.

5) The effect of using limited number of bits to implement (to achieve law quantization errors) are much less severe in FIR than in IIR.

When to use:

- Use IIR when the only important requirements are sharp cutoff filters and high throughput, as IIR filters, especially those using elliptic characteristics, will give fewer coefficients than FIR.
- Use FIR if the number of filter coefficients is not too large and, in particular, if little or no phase distortion is desired. One might also add that newer DSP processors have architectures that are tailored to FIR filtering, and indeed some are designed specifically for FIRs

Filter Design: practical filter specifications



δ_p	peak passband deviation (or ripples)
δ_s	stopband deviation
f_p	passband edge frequency
f_s	stopband edge frequency
F_s	sampling frequency

A_s (stopband attenuation) $= -20\log_{10}\delta_s$

A_p (passband ripple) $= 20\log_{10}(1+\delta_p)$

FIR Filter

Another characteristic of an ideal filter is a linear phase response.

- Suppose x(n) with frequency components w1<w<w2 passed through filter with freq. resp.

$$H(\omega) = \begin{cases} Ce^{-j\omega n_0}, & \omega_1 < \omega < \omega_2 \\ 0, & \text{otherwise} \end{cases} \tag{4.5.1}$$

where C and n_0 are constants. The signal at the output of the filter has a spectrum

$$\begin{aligned} Y(\omega) &= X(\omega)H(\omega) \\ &= CX(\omega)e^{-j\omega n_0} \qquad \omega_1 < \omega < \omega_2 \end{aligned} \tag{4.5.2}$$

$$y(n) = Cx(n - n_0)$$

Consequently, the filter output is simply a delayed and amplitude-scaled version of the input signal. A pure delay is usually tolerable and is not considered a distortion of the signal. Neither is amplitude scaling. Therefore, ideal filters have a linear phase characteristic within their passband, that is,

$$\Theta(\omega) = -\omega n_0 \tag{4.5.4}$$

$$\tau_g(\omega) = -\frac{d\Theta(\omega)}{d\omega}$$

$\tau_g(\omega)$ is usually called the *envelope delay* or the *group delay* of the filter. We interpret $\tau_g(\omega)$ as the time delay that a signal component of frequency ω undergoes as it passes from the input to the output of the system. Note that when $\Theta(\omega)$ is linear as in (4.5.4), $\tau_g(\omega) = n_0 =$ constant. In this case all frequency components of the input signal undergo the same time delay.

In conclusion, ideal filters have a constant magnitude characteristic and a linear phase characteristic within their passband.

Design of FIR filter
Symmetric and antisymmetric FIR

An FIR filter of length M with input x(n) and output y(n) is described by the difference equation

$$y(n) = b_0x(n) + b_1x(n-1) + \cdots + b_{M-1}x(n-M+1)$$
$$= \sum_{k=0}^{M-1} b_k x(n-k)$$

And by convolution we have:

$$y(n) = \sum_{k=0}^{M-1} h(k)x(n-k)$$

The filter can also characterized by:

$$H(z) = \sum_{k=0}^{M-1} h(k)z^{-k}$$

Linear phase FIR

$H(w) = hr(w)\, e^{j\theta(w)}$

ɵ(w) is a phase function related to the phase

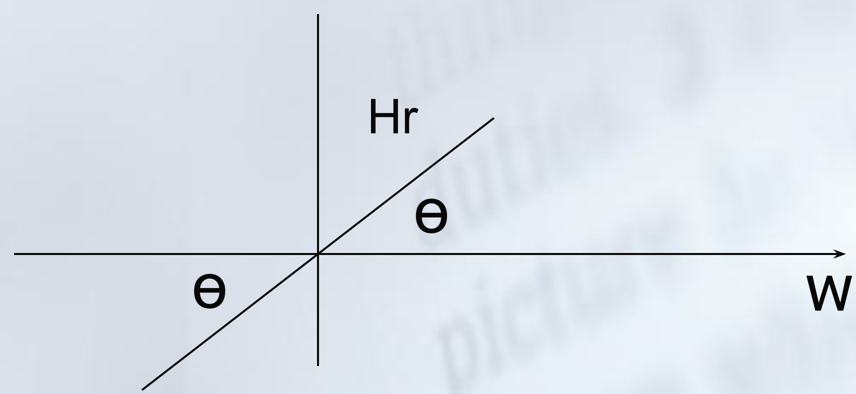
$\Theta(\omega)$ of h(w) by $\Theta(\omega)$ = ɵ(w)   if hr(w) > 0

= ɵ(w) +Π   if hr(w) < 0

For linear phase suppose that $\theta(w) = \beta - \alpha w$

Where α is the filter's group delay, β is a constant phase shift.

Then



Linear phase

$$H_r(w)e^{j(\beta-\alpha w)} = \sum_{n=0}^{n=N-1} h(n)e^{-jwn}$$

$$then\ H_r(w)\cos(\beta-\alpha w) = \sum_{n=0}^{n=N-1} h(n)\cos wn$$

$$H_r(w)\sin(\beta-\alpha w) = -\sum_{n=0}^{n=N-1} h(n)\sin wn$$

Multiply the first by $\sin(\beta-\alpha w)$ and the second by $-\cos(\beta-\alpha w)$

$$\sum_{n=0}^{N-1} h(n)[\sin(wn)\cos(\beta-\alpha w)+\cos(wn)\sin(\beta-\alpha w)] = 0$$

$$then \sum_{n=0}^{N-1} h(n)\sin(\beta+(n-\alpha)w) = 0$$

$$yields\ \alpha = N-1/2,\ \beta = 0,\ h(n) = h(N-1-n)$$

$$OR\ \alpha = N-1/2,\ \beta = \pi/2,\ h(n) = -h(N-1-n)$$

Linear phase FIR

Thus for linear phase: $h(n) = \pm h(M-1-n) \qquad n = 0, 1, \ldots, M-1$

When $h(n) = h(M-1-n)$, $H(\omega)$ can be expressed as

$$H(\omega) = H_r(\omega)e^{-j\omega(M-1)/2}$$

where $H_r(\omega)$ is a real function of ω and can be expressed as

$$H_r(\omega) = h\left(\frac{M-1}{2}\right) + 2\sum_{n=0}^{(M-3)/2} h(n)\cos\omega\left(\frac{M-1}{2} - n\right) \qquad M \text{ odd}$$

$$H_r(\omega) = 2\sum_{n=0}^{(M/2)-1} h(n)\cos\omega\left(\frac{M-1}{2} - n\right) \qquad M \text{ even}$$

The phase characteristic of the filter for both M odd and M even is

$$\Theta(\omega) = \begin{cases} -\omega\left(\dfrac{M-1}{2}\right), & \text{if } H_r(\omega) > 0 \\ -\omega\left(\dfrac{M-1}{2}\right) + \pi, & \text{if } H_r(\omega) < 0 \end{cases}$$

Linear phase antisymmetric FIR

When

$$h(n) = -h(M-1-n)$$

the unit sample response is *antisymmetric*. For M odd, the center point of the antisymmetric $h(n)$ is $n = (M-1)/2$. Consequently,

$$h\left(\frac{M-1}{2}\right) = 0$$

an antisymmetric unit sample response can be expressed as

$$H(\omega) = H_r(\omega)e^{j[-\omega(M-1)/2+\pi/2]}$$

where

$$H_r(\omega) = 2\sum_{n=0}^{(M-3)/2} h(n)\sin\omega\left(\frac{M-1}{2}-n\right) \qquad M \text{ odd}$$

$$H_r(\omega) = 2\sum_{n=0}^{(M/2)-1} h(n)\sin\omega\left(\frac{M-1}{2}-n\right) \qquad M \text{ even}$$

The phase characteristic of the filter for both M odd and M even is

$$\Theta(\omega) = \begin{cases} \dfrac{\pi}{2} - \omega\left(\dfrac{M-1}{2}\right), & \text{if } H_r(\omega) > 0 \\ \dfrac{3\pi}{2} - \omega\left(\dfrac{M-1}{2}\right), & \text{if } H_r(\omega) < 0 \end{cases}$$

Thus for symmetric h(n):

no of filter coef. We need=(N+1)/2 for N odd or N/2 when N is even.

If h(n) is negative (Antisymmetric).

we need (N-1)/2 coef. when N is odd, N/2 coef. When N is even.

The choice of symmetry or Neg-symmetry unit sample response depend on application.

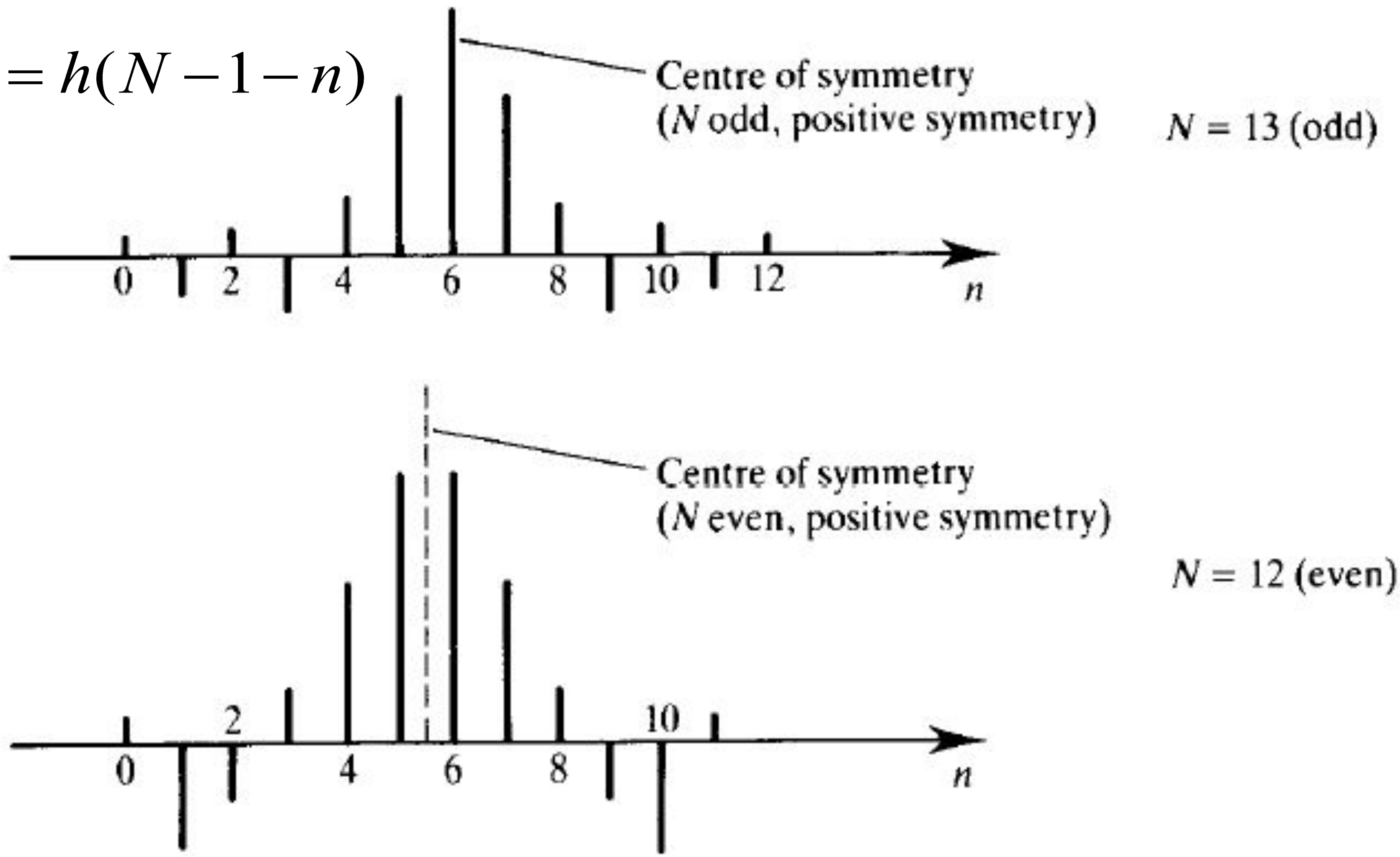
Table 6.1 A summary of the key points about the four types of linear phase FIR filters.

Impulse response symmetry	*Number of coefficients N*	*Frequency response* $H(\omega)$	*Type of linear phase*
Positive symmetry, $h(n) = h(N-1-n)$	Odd	$e^{-j\omega(N-1)/2} \sum_{n=0}^{(N-1)/2} a(n)\cos(\omega n)$	1
	Even	$e^{-j\omega(N-1)/2} \sum_{n=1}^{N/2} b(n)\cos[w(n-\frac{1}{2})]$	2
Negative symmetry, $h(n) = -h(N-1-n)$	Odd	$e^{-j[\omega(N-1)/2-\pi/2]} \sum_{n=1}^{(N-1)/2} a(n)\sin(\omega n)$	3
	Even	$e^{-j[\omega(N-1)/2-\pi/2]} \sum_{n=1}^{N/2} b(n)\sin[\omega(n-\frac{1}{2})]$	4

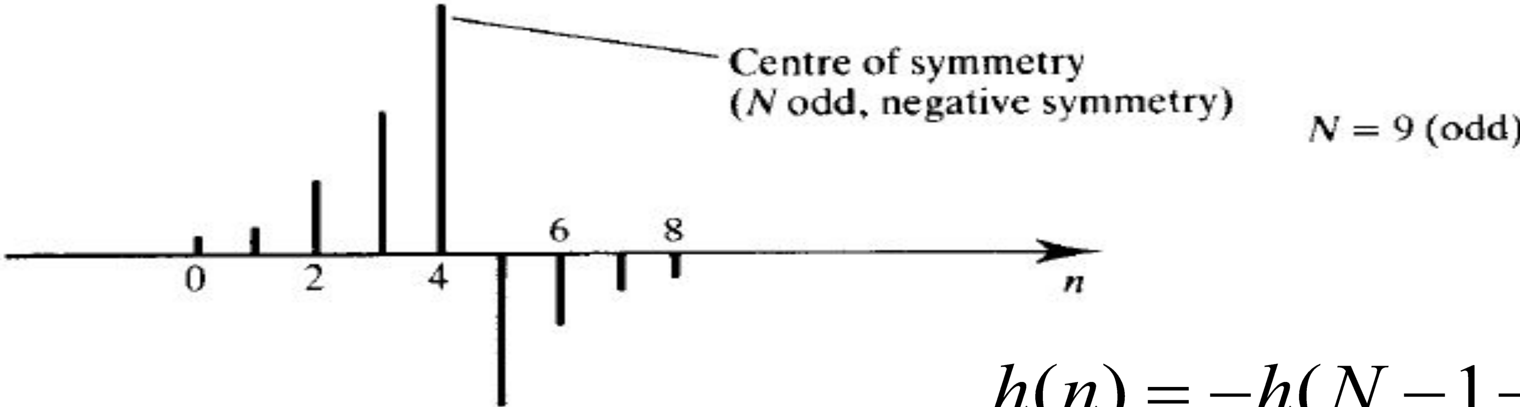
- Type 1 is the most versatile of the all
- Type 2 is not suitable to high pass
- Type 3 is not suitable to law &high pass
- Type 4 is not suitable to law pass

Impulse response positive Symmetry

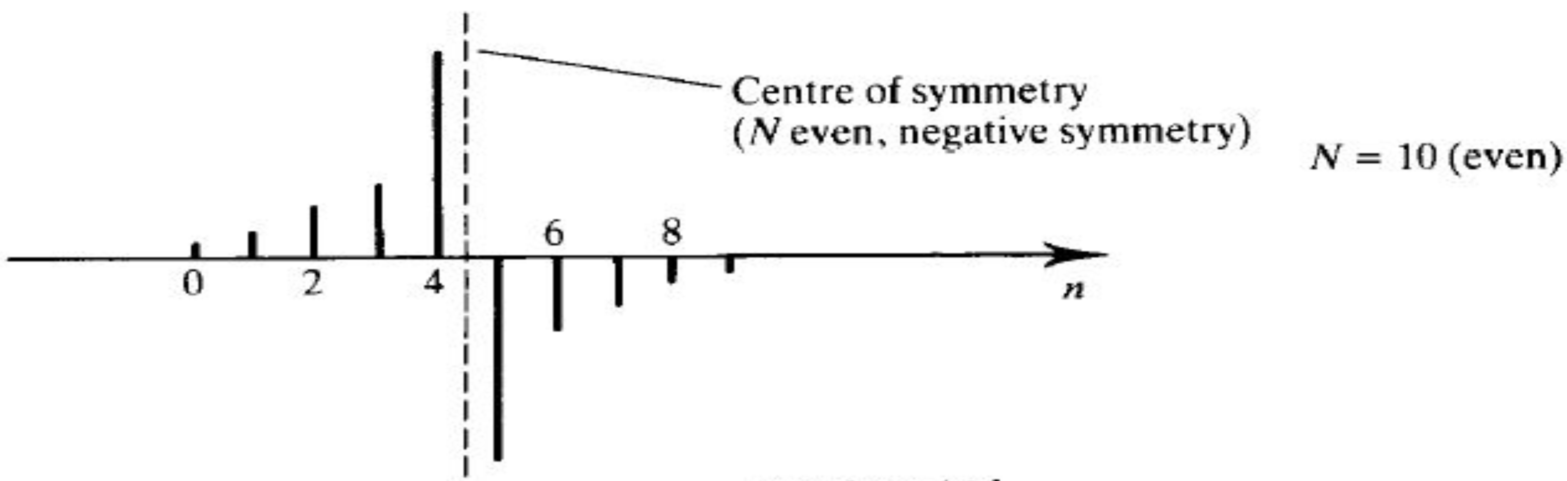
$$h(n) = h(N-1-n)$$



Impulse response negative symmetry



$$h(n) = -h(N-1-n)$$



responses

Figure 6.1 A comparison of the impulse responses of the four types of linear phase filters.

Conclusion

In summary the problem of FIR filter is to determine the N coef. H(n), n=0,1,2,..,N-1 from the specification of the desired frequency response Hd(w).

Filter Design Steps

(1) *Filter specification*. This may include stating the type of filter, for example lowpass filter, the desired amplitude and/or phase responses and the tolerances (if any) we are prepared to accept, the sampling frequency, and the wordlength of the input data.

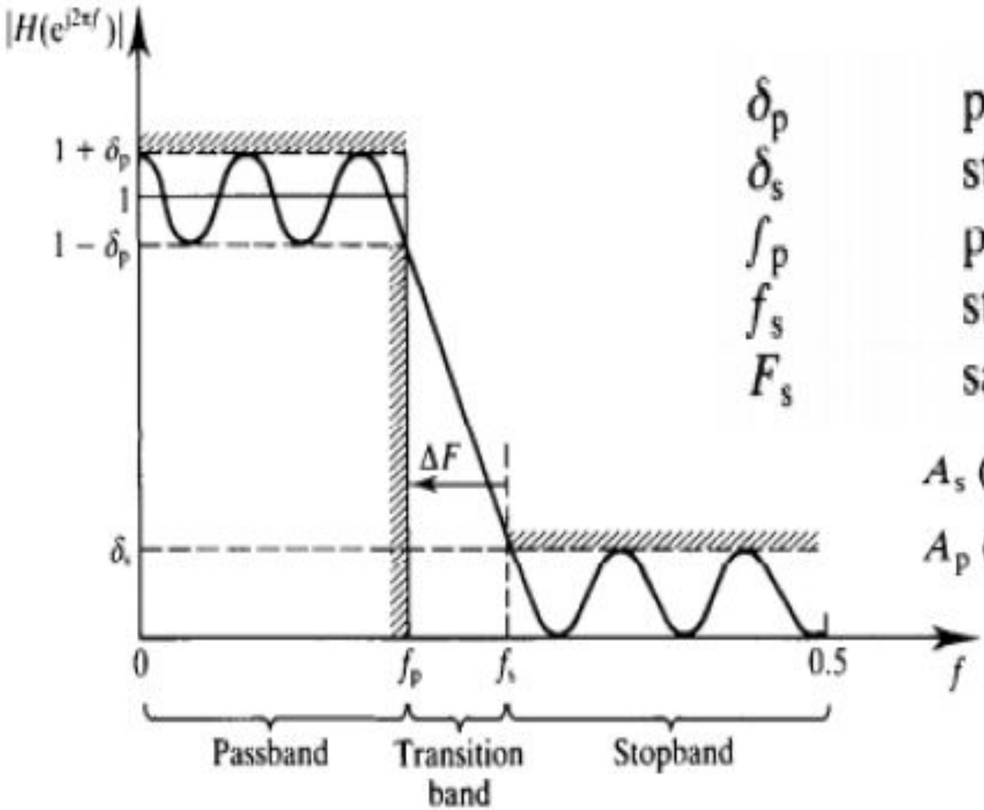
(2) *Coefficient calculation*. At this step, we determine the coefficients of a transfer function, $H(z)$, which will satisfy the specifications given in (1). Our choice of coefficient calculation method will be influenced by several factors, the most important of which are the critical requirements in step (1).

(3) *Realization*. This involves converting the transfer function obtained in (2) into a suitable filter network or structure.

(4) *Analysis of finite wordlength effects*. Here, we analyse the effect of quantizing the filter coefficients and the input data as well as the effect of carrying out the filtering operation using fixed wordlengths on the filter performance.

(5) *Implementation*. This involves producing the software code and/or hardware and performing the actual filtering.

Filter Design:
1-practical filter specifications



δ_p peak passband deviation (or ripples)
δ_s stopband deviation
f_p passband edge frequency
f_s stopband edge frequency
F_s sampling frequency

A_s (stopband attenuation) $= -20\log_{10}\delta_s$

A_p (passband ripple) $= 20\log_{10}(1+\delta_p)$

Design of FIR filter
2- coefficients calculations window Method

$$H_d(\omega) = \sum_{n=0}^{\infty} h_d(n) e^{-j\omega n}$$

where

$$h_d(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(\omega) e^{j\omega n} d\omega$$

hd(n) obtained is infinite in duration and must be truncated at some point n=N-1, to yields an FIR filter of length N.

Truncation is equivalent to multiplication of hd(n) by a rectangular window w(n) in a method called the window method

$$w(n) = \begin{cases} 1, & n = 0, 1, \ldots, N-1 \\ 0, & \text{otherwise} \end{cases}$$

Thus the unit sample response of the FIR filter becomes

$$h(n) = h_d(n)w(n)$$

$$= \begin{cases} h_d(n), & n = 0, 1, \ldots, N-1 \\ 0, & \text{otherwise} \end{cases}$$

We know that Convolution in time domain yields multiplication in frequency domain and vice versa.

$$\mathbf{H(w)=H_d(W)* W(w)}$$

2- coefficients calculations

- For example for law pass filter

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(\omega) e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} 1 \cdot e^{j\omega n}\, d\omega$$

$$= \begin{cases} \dfrac{2 f_c \sin(n\omega_c)}{n\omega_c} & \text{for } n \neq 0 \\ 2 f_c & \text{for } n = 0 \end{cases}$$

- High pass $\int_{-\pi}^{wc} h(w)$ $\int_{wc}^{\pi} H(w)$ wc=2 πfc
- Band pass w1 to w2 (f1 to f2) –w1 to –w2
- Band stop opposite to the band pass

Table 6.2 Summary of ideal impulse responses for standard frequency selective filters.

Filter type	*Ideal impulse response, $h_D(n)$*	
	$h_D(n),\ n \neq 0$	$h_D(0)$
Lowpass	$2f_c \dfrac{\sin(n\omega_c)}{n\omega_c}$	$2f_c$
Highpass	$-2f_c \dfrac{\sin(n\omega_c)}{n\omega_c}$	$1 - 2f_c$
Bandpass	$2f_2 \dfrac{\sin(n\omega_2)}{n\omega_2} - 2f_1 \dfrac{\sin(n\omega_1)}{n\omega_1}$	$2(f_2 - f_1)$
Bandstop	$2f_1 \dfrac{\sin(n\omega_1)}{n\omega_1} - 2f_2 \dfrac{\sin(n\omega_2)}{n\omega_2}$	$1 - 2(f_2 - f_1)$

f_c, f_1 and f_2 are the passband or stopband edge frequencies; N is the length of filter.

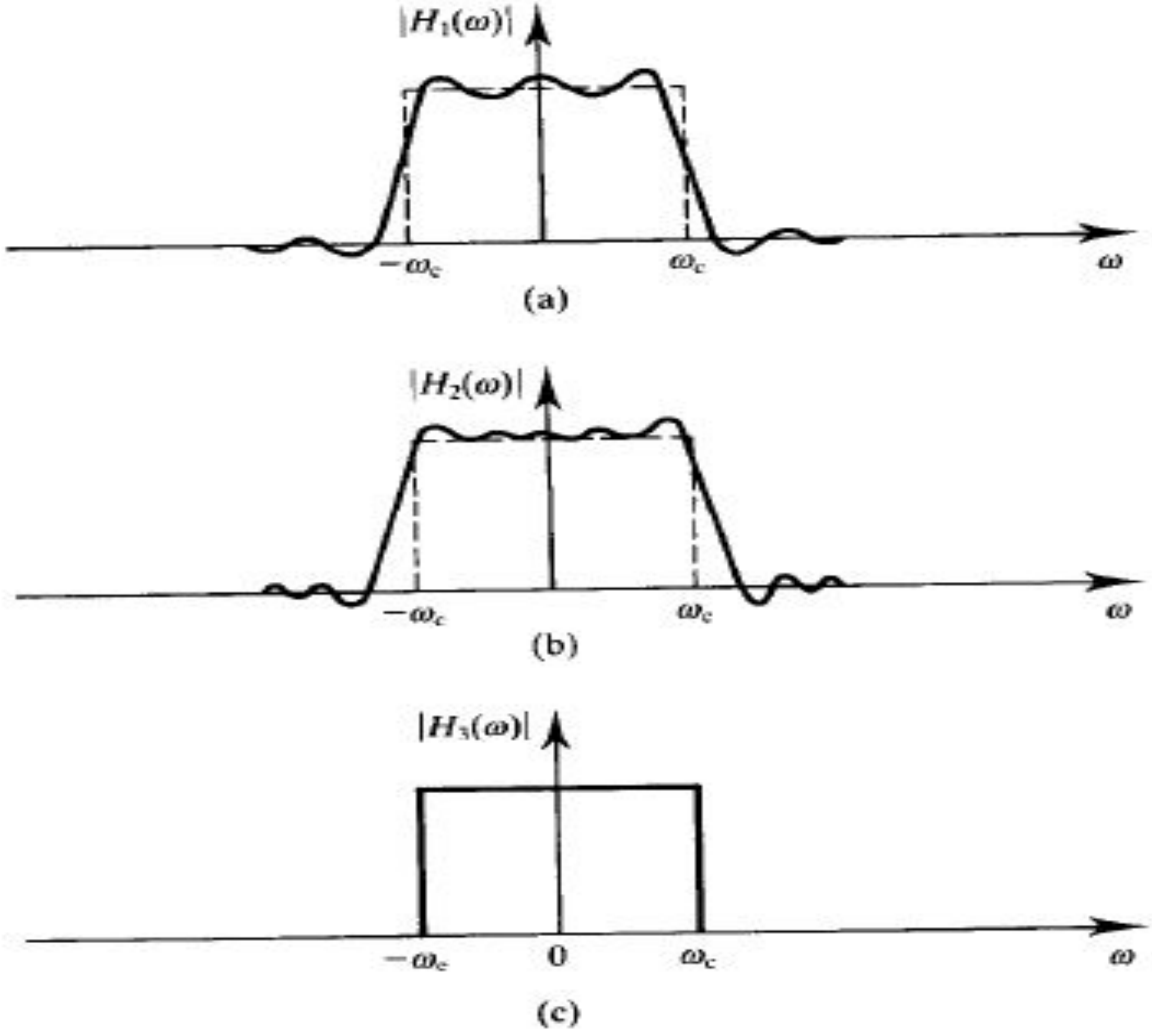


Figure 6.5 Effects on the frequency response of truncating the ideal impulse resp
to (a) 13 coefficients, (b) 25 coefficients and (c) an infinite number of coefficients.

Window method

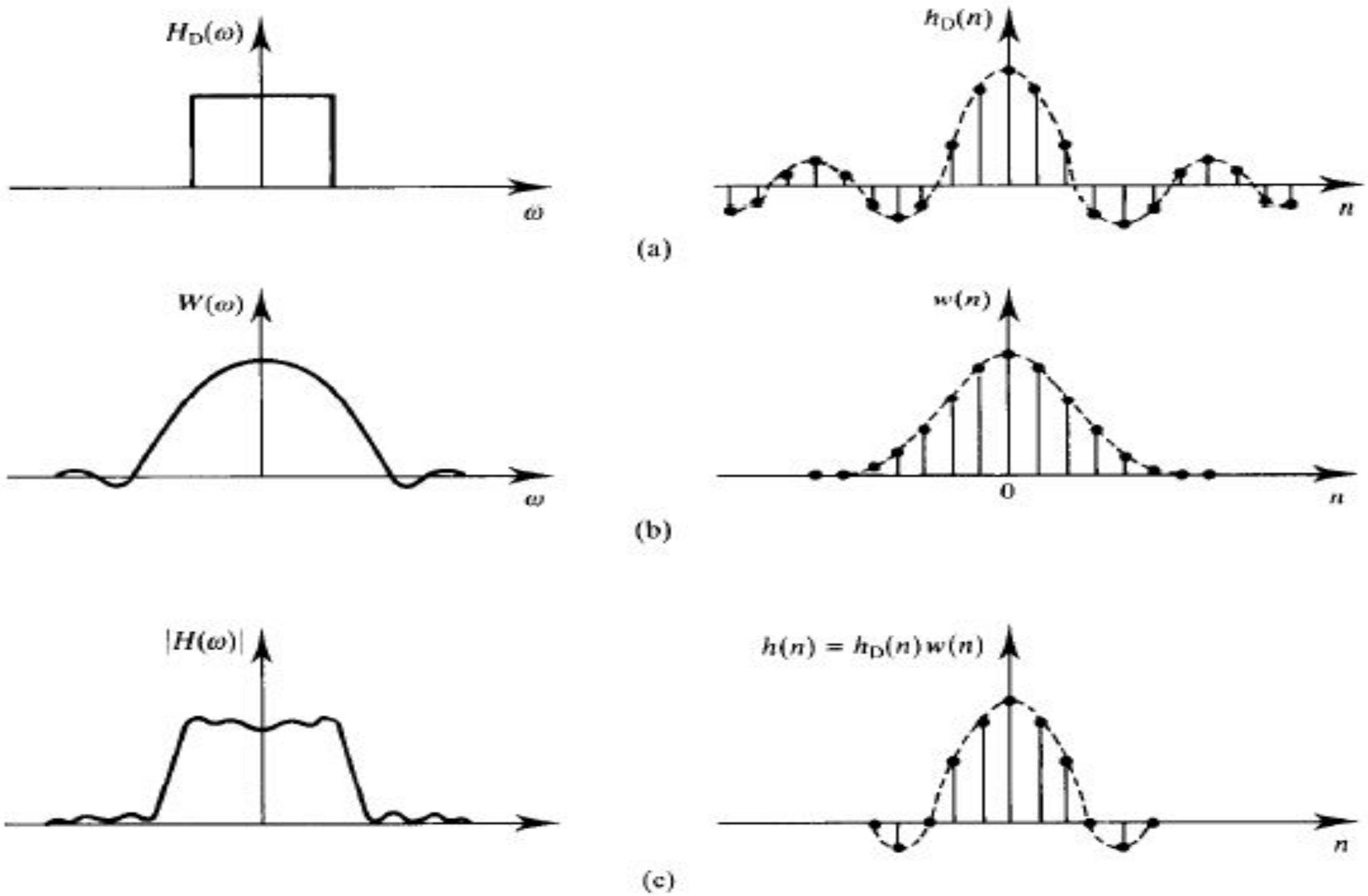
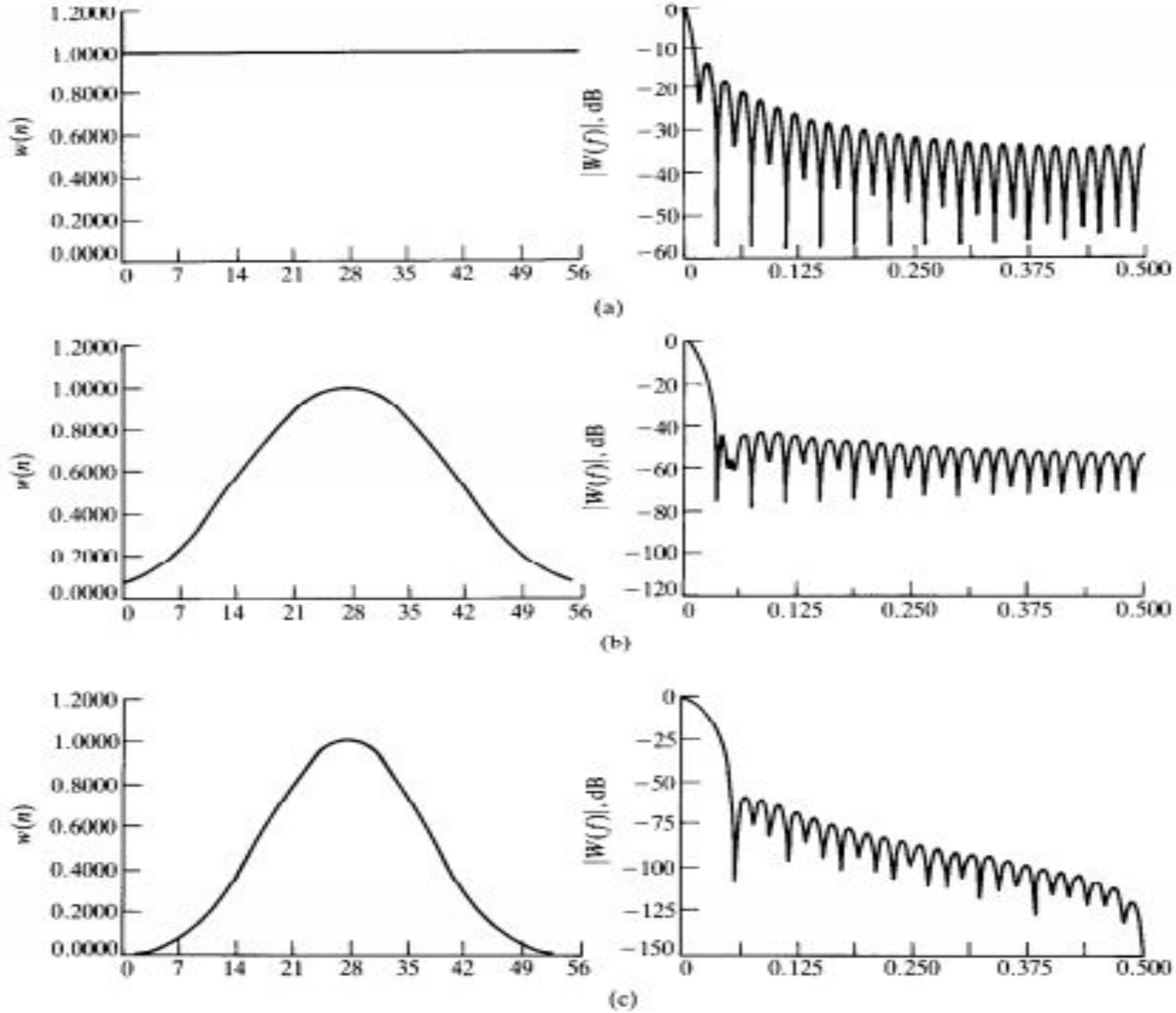


Figure 6.6 An illustration of how the filter coefficients, $h(n)$, are determined by the window method.

Common window functions



Comparison of the time and frequency domain characteristics of common window functions: (a) rectangular; (b) Hamming; (c) Blackman.

Specification table for every window type

- Selection of suitable window function will be according to the desired specifications for the filter: table 6.3

Name of window function	Δf Transition width (Hz) (normalized)	Passband ripple (dB)	Main lobe relative to side lobe (dB)	Stopband attenuation (dB) (maximum)	Window function $w(n)$, $\lvert n\rvert \le (N-1)/2$
Rectangular	$0.9/N$	0.7416	13	21	1
Hanning	$3.1/N$	0.0546	31	44	$0.5 + 0.5\cos\left(\frac{2\pi n}{N}\right)$
Hamming	$3.3/N$	0.0194	41	53	$0.54 + 0.46\cos\left(\frac{2\pi n}{N}\right)$
Blackman	$5.5/N$	0.0017	57	74	$0.42 + 0.5\cos\left(\frac{2\pi n}{N-1}\right) + 0.08\cos\left(\frac{4\pi n}{N-1}\right)$

Summary of the window method of calculating FIR filter coefficients

- *Step 1.* Specify the 'ideal' or desired frequency response of filter, $H_D(\omega)$.
- *Step 2.* Obtain the impulse response, $h_D(n)$, of the desired filter by evaluating the inverse Fourier transform (. For the standard frequency selective filters the expressions for $h_D(n)$ are summarized in Table 6.2.
- *Step 3.* Select a window function that satisfies the passband or attenuation specifications and then determine the number of filter coefficients using the appropriate relationship between the filter length and the transition width, Δf (expressed as a fraction of the sampling frequency).
- *Step 4.* Obtain values of $w(n)$ for the chosen window function and the values of the actual FIR coefficients, $h(n)$, by multiplying $h_D(n)$ by $w(n)$:

$$h(n) = h_D(n)w(n) \qquad (6.12)$$

Example: Design a law pass filter with the following Spec.

passband edge frequency	1.5 kHz
transition width	0.5 kHz
stopband attenuation	> 50 dB
sampling frequency	8 kHz

Solution

From Table 6.2, we select $h_D(n)$ for lowpass filter which is given by

$$h_D(n) = 2f_c \frac{\sin(n\omega_c)}{n\omega_c} \qquad n = 0$$

$$h_D(n) = 2f_c \qquad n \neq 0$$

Table 6.3 indicates that the Hamming, Blackman or Kaiser window will satisfy the stopband attenuation requirements. We will use the Hamming window for simplicity. Now $\Delta f = 0.5/8 = 0.0625$. From $N = 3.3/\Delta f = 3.3/0.0625 = 52.8$, let $N = 53$. The filter coefficients are obtained from

$$h_D(n)w(n) \qquad -26 \leq n \leq 26$$

Example cont.

$$h_D(n) = \frac{2f_c \sin(n\omega_c)}{n\omega_c} \qquad n = 0$$

$$h_D(n) = 2f_c \qquad n \neq 0$$

$$w(n) = 0.54 + 0.46\cos(2\pi n/53) \qquad -26 \leqslant n \leqslant 26$$

Because of the smearing effect of the window on the filter response, the cutoff frequency of the resulting filter will be different from that given in the specifications. To account for this, we will use an f_c that is centred on the transition

band:

$$f_c' = f_c + \Delta f/2 = (1.5 + 0.25)\ \text{kHz} = 1.75\ \text{kHz} \equiv 1.75/8 = 0.21875$$

Noting that $h(n)$ is symmetrical; we need only compute values for $h(0)$, $h(1)$, ..., $h(26)$ and then use the symmetry property to obtain the other coefficients.

Example Cont.

$$n = 0: \quad h_D(n) = 2f_c = 2 \times 0.21875 = 0.4375$$

$$w(0) = 0.54 + 0.46\cos(0) = 1$$

$$h(0) = h_D(0)w(0) = 0.4375$$

$$n = 1: \quad h_D(1) = \frac{2 \times 0.21875}{2\pi \times 0.21875}\sin(2\pi \times 0.21875)$$

$$= \frac{\sin(360 \times 0.21865)}{\pi} = 0.31219$$

$$w(1) = 0.54 + 0.46\cos(2\pi/53) = 0.54 + 0.46\cos(360/53) = 0.98713$$

$$h(1) = h(-1) = h_D(1)w(1) = 0.31119$$

Example Cont.

$$n = 26: \quad h_D(26) = \frac{2 \times 0.21875}{26 \times 2\pi \times 0.15} \frac{\sin(26 \times 2\pi \times 0.21875)}{2\pi}$$

$$= 0.01131$$

$$w(26) = 0.54 + 0.46\cos(2\pi \times 26/53)$$

$$= 0.54 + 0.46\cos(720^\circ/53) = 0.08081$$

$$h(26) = h(-26) = h_D(26)w(26) = 0.000913$$

We note that the indices of the filter coefficients run from -26 to 26. To make the filter causal (necessary for implementation) we add 26 to each index so that the indices start at zero. The filter coefficients, with indices adjusted, are listed in Table 6.4. The spectrum of the filter (not shown) indicates that the specifications were satisfied.

Example Cont.

Table 6.4 FIR coefficients for Example 6.3 ($N = 53$, Hamming window, $f_c = 1750$ Hz).

h[0] =	−9.1399895e−04	= h[52]
h[1] =	2.1673690e−04	= h[51]
h[2] =	1.3270280e−03	= h[50]
h[3] =	3.2138355e−04	= h[49]
h[4] =	−1.9238177e−03	= h[48]
h[5] =	−1.4683633e−03	= h[47]
h[6] =	2.3627318e−03	= h[46]
h[7] =	3.4846558e−03	= h[45]
h[8] =	−1.9925839e−03	= h[44]
h[9] =	−6.2837232e−03	= h[43]
h[10] =	4.5320247e−09	= h[42]
h[11] =	9.2669460e−03	= h[41]
h[12] =	4.3430586e−03	= h[40]
h[13] =	−1.1271299e−02	= h[39]
h[14] =	−1.1402453e−02	= h[38]
h[15] =	1.0630714e−02	= h[37]
h[16] =	2.0964392e−02	= h[36]
h[17] =	−5.2583216e−03	= h[35]
h[18] =	−3.2156086e−02	= h[34]
h[19] =	−7.5449714e−03	= h[33]
h[20] =	4.3546153e−02	= h[32]
h[21] =	3.2593190e−02	= h[31]
h[22] =	−5.3413653e−02	= h[30]
h[23] =	−8.5682029e−02	= h[29]
h[24] =	6.0122145e−02	= h[28]
h[25] =	3.1118568e−01	= h[27]
h[26] =	4.3750000e−01	= h[26]

Thanks for good listening