Digital Signal Processing

Discrete Fourier Transform DFT

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Fourier series

Fourier series

Any periodic waveform, $f(t)$, can be represented as the sum of an infinite number of sinusoidal and cosinusoidal terms, together with a constant term, this representation being the Fourier series given by

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega t) + \sum_{n=1}^{\infty} b_n \sin(n\omega t) \quad \textbf{(2.1)}$$

where t is an independent variable which often represents time but could, for example, represent distance or any other quantity, $f(t)$ is often a varying voltage versus time waveform, but could be any other waveform, $\omega = 2\pi/T_{\mathrm{p}}$ is known as the first harmonic, or fundamental, angular frequency, related to the fundamental frequency, f, by $\omega = 2\pi f$, T_{p} is the repetition period of the waveform,

Fourier transform

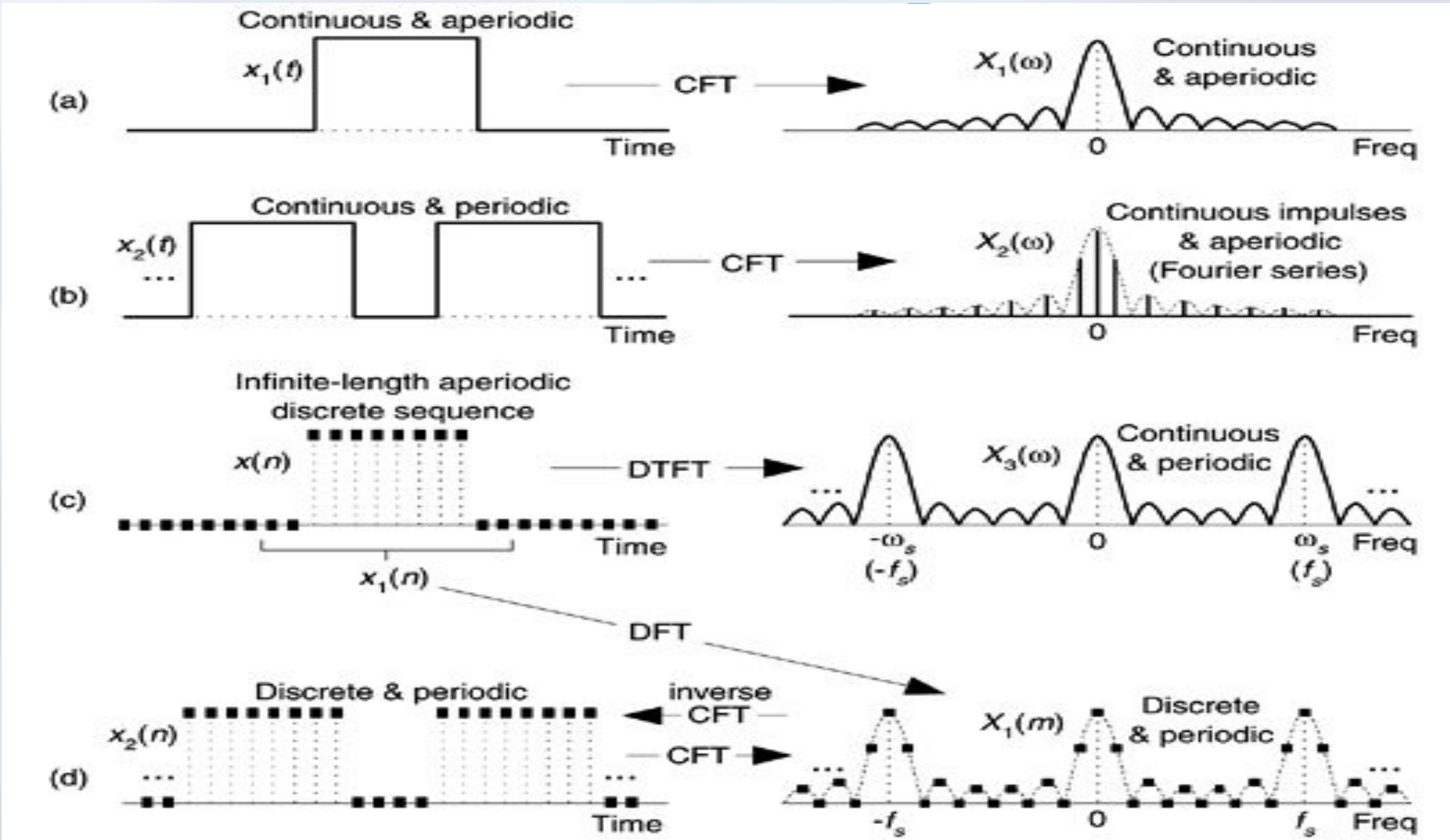
$$x(w) = \int_{-\infty}^{\infty} x(t)\, e^{-jwt}\, dt$$

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} \qquad 5.1.1$$

Discrete Fourier transform DFT x(k)

The DFT is a mathematical procedure used to determine the harmonic, or frequency, content of a discrete signal sequence.

Figure 1

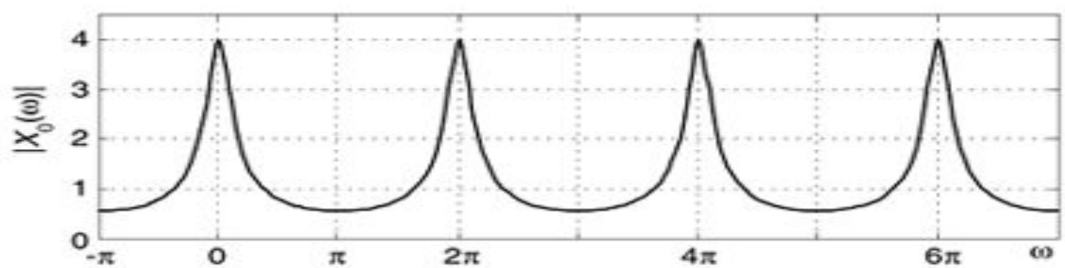


$x_o(n) = (0.75)^n$ for $n \geq 0$. Its DTFT would be

$$X_o(\omega) = \sum_{n=0}^{\infty} 0.75^n e^{-j\omega n} = \sum_{n=0}^{\infty} (0.75e^{-j\omega})^n = \frac{1}{1-0.75e^{-j\omega}} = \frac{e^{j\omega}}{e^{j\omega}-0.75}.$$

X0(w) is periodic with period 2π

Figure 3-47. DTFT magnitude $|X_o(w)|$.



$$X(\omega + 2\pi k) = \sum_{n=-\infty}^{\infty} x(n)e^{-j(\omega+2\pi k)n} = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}e^{-j2\pi kn}$$

$$= \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n} = X(\omega)$$

$X_3(w)$ in Figure 1(c) also has a 2π periodicity represented by $w_s = 2\pi f_s$ where the cyclic frequency f_s is the reciprocal of the time period between the x(n) samples.

DFT

We recall that aperiodic finite-energy signals have continuous spectra. Let us consider such an aperiodic discrete-time signal $x(n)$ with Fourier transform

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n} \tag{5.1.1}$$

Suppose that we sample $X(\omega)$ periodically in frequency at a spacing of $\delta\omega$ radians between successive samples. Since $X(\omega)$ is periodic with period 2π, only samples in the fundamental frequency range are necessary. For convenience, we take N equidistant samples in the interval $0 \leq \omega < 2\pi$ with spacing $\delta\omega = 2\pi/N$, as shown

or sampled frequencies df where (w=2Πf)

DFT

- Consider the selection of N the number of samples in frequency domain

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} \qquad (5.1.1)$$

If we evaluate (5.1.1) at $\omega = 2\pi k/N$, we obtain

$$X\left(\frac{2\pi}{N}k\right) = \sum_{n=-\infty}^{\infty} x(n) e^{-j2\pi kn/N} \qquad k = 0, 1, \ldots, N-1$$

N=4 we evaluate at w= $2\pi/4$ * K k=0,1,2,3
W=0 , $\pi/2$, π, 3 $\pi/2$

DFT

The summation in (5.1.2) can be subdivided into an infinite number of summations, where each sum contains N terms. Thus

$$
\begin{aligned}
X\left(\frac{2\pi}{N}k\right) = {} & \cdots + \sum_{n=-N}^{-1} x(n)e^{-j2\pi kn/N} + \sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N} \\
& + \sum_{n=N}^{2N-1} x(n)e^{-j2\pi kn/N} + \cdots \\
= {} & \sum_{l=-\infty}^{\infty} \sum_{n=lN}^{lN+N-1} x(n)e^{-j2\pi kn/N}
\end{aligned}
$$

$$
X\left(\frac{2\pi}{N}k\right) = \sum_{n=0}^{N-1}\left[\sum_{l=-\infty}^{\infty} x(n-lN)\right]e^{-j2\pi kn/N}
$$

for $k = 0, 1, 2, \ldots, N-1$.

The signal

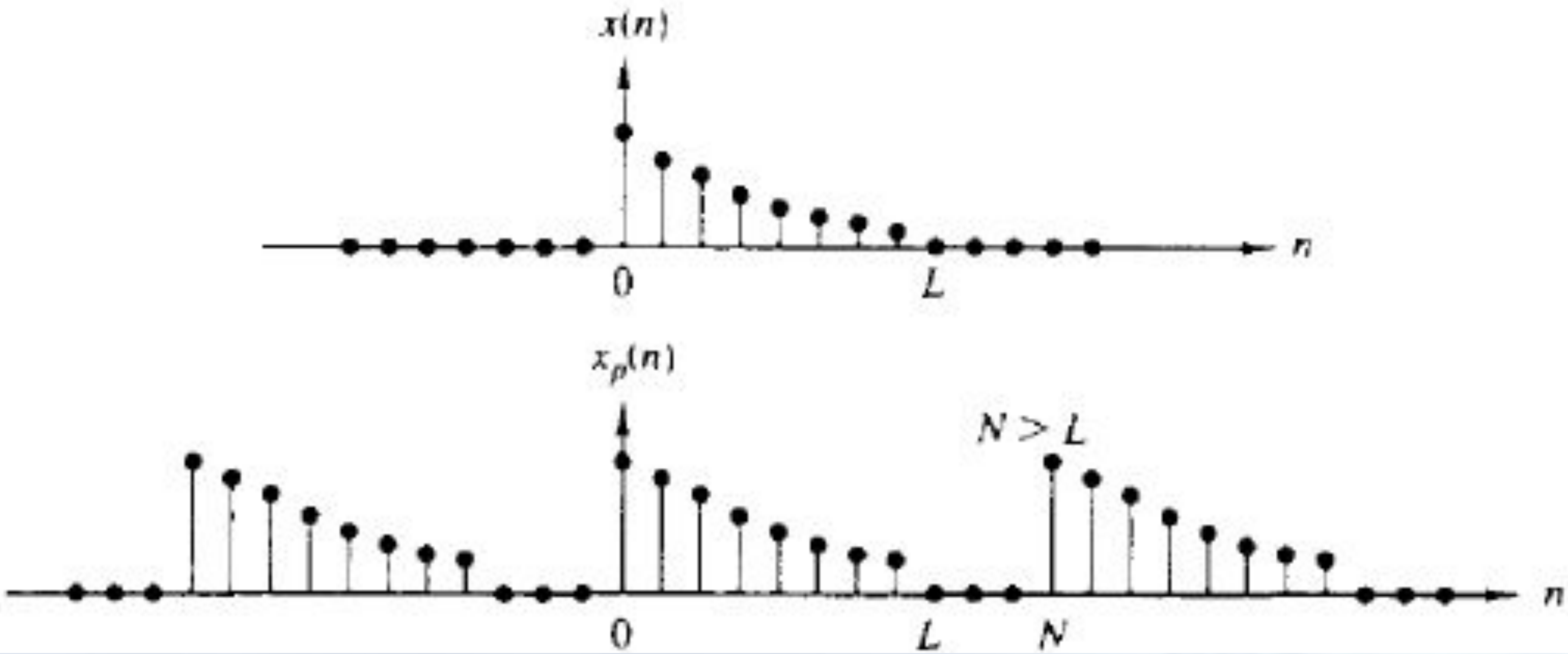
$$
x_p(n) = \sum_{l=-\infty}^{\infty} x(n-lN)
$$

obtained by the periodic repetition of $x(n)$ every N samples, is clearly periodic with fundamental period N. Consequently, it can be expanded in a Fourier

DFT

when $N \geq L$.

$$x(n) = x_p(n) \qquad 0 \leq n \leq N-1$$



DFT

In summary, a finite-duration sequence $x(n)$ of length L [i.e., $x(n) = 0$ for $n < 0$ and $n \geq L$] has a Fourier transform

$$X(\omega) = \sum_{n=0}^{L-1} x(n) e^{-j\omega n} \qquad 0 \leq \omega \leq 2\pi \tag{5.1.17}$$

where the upper and lower indices in the summation reflect the fact that $x(n) = 0$ outside the range $0 \leq n \leq L - 1$. When we sample $X(\omega)$ at equally spaced frequencies $\omega_k = 2\pi k/N$, $k = 0, 1, 2, \ldots, N - 1$, where $N \geq L$, the resultant samples are

$$\begin{aligned} X(k) &\equiv X\left(\frac{2\pi k}{N}\right) = \sum_{n=0}^{L-1} x(n) e^{-j2\pi kn/N} \\ X(k) &= \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N} \qquad k = 0, 1, 2, \ldots, N - 1 \end{aligned} \tag{5.1.18}$$

where for convenience, the upper index in the sum has been increased from $L - 1$ to $N - 1$ since $x(n) = 0$ for $n \geq L$.

Understanding DFT

- From Euler's relationship $e^{-j\varnothing} = \cos(\varnothing) - j\sin(\varnothing)$,

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N} \qquad k = 0, 1, 2, \ldots, N-1$$

Rectangular form

$$x(k) = \sum_{n=0}^{N-1} x(n)\left(\cos\frac{2\pi Kn}{N} - j\sin\frac{2\pi Kn}{N}\right)$$

x(k): the k^{th} DFT output component, i.e., X(0), X(1), X(2), X(3), etc.,

k =the index of the DFT output in the frequency domain,

k =0, 1, 2, 3, . . ., N–1,

n = the time-domain index of the input samples, n = 0, 1, 2, 3, . . ., N–1,

N = the number of samples of the input sequence and the number of frequency points in the DFT output.

Understanding DFT

- For example, when N = 16, n and k both go from 0 to 15

$$x(k) = \sum_{n=0}^{15} x(n)\left(\cos\frac{2\pi Kn}{16} - j\sin\frac{2\pi Kn}{16}\right)$$

- if we are sampling a continuous signal at a rate of 500 samples/s and, then, perform a 16-point DFT on the sampled data, the fundamental frequency of the sinusoids is $f_s/N = 500/16$ or 31.25 Hz. The other X(k) analysis frequencies are integral multiples of the fundamental frequency, i.e., we examine at frequencies : Kf_s/N

X(0)=1st frequency term, with analysis frequency =0*31.25 = 0 Hz,

X(1)=2nd frequency term, with analysis frequency = 1 · 31.25 = 31.25 Hz,

. . .

X(15)=16th frequency term, with analysis frequency = 15 · 31.25 = 468.75 Hz.

DFT example

Let's say we want to sample and perform an 8-point DFT on a continuous input signal x (t) containing components at 1 kHz and 2 kHz, expressed as:

$$x_{in}(t) = \sin(2\pi \cdot 1000 \cdot t) + 0.5\sin(2\pi \cdot 2000 \cdot t + 3\pi/4)$$

$$x(n) = x_{in}(nt_s) = \sin(2\pi \cdot 1000 \cdot nt_s) + 0.5\sin(2\pi \cdot 2000 \cdot nt_s + 3\pi/4)$$

If we choose to sample $x_{in}(t)$ at f_s = 8000 samples/s our DFT results will indicate what signal amplitude exists in x(n) at the analysis frequencies of Kf_s/N, or 0 kHz, 1 kHz, 2 kHz, . . ., 7 kHz. With f_s = 8000 samples/s, our eight x(n) samples are:

$$x(0) = 0.3535, \quad x(1) = 0.3535,$$
$$x(2) = 0.6464, \quad x(3) = 1.0607,$$
$$x(4) = 0.3535, \quad x(5) = -1.0607,$$
$$x(6) = -1.3535, \quad x(7) = -0.3535$$

Figure 3-2. DFT Example 1: (a) the input signal; (b) the input signal and the m = 1 sinusoids; (c) the input signal and the m = 2 sinusoids; (d) the input signal and the m = 3 sinusoids.

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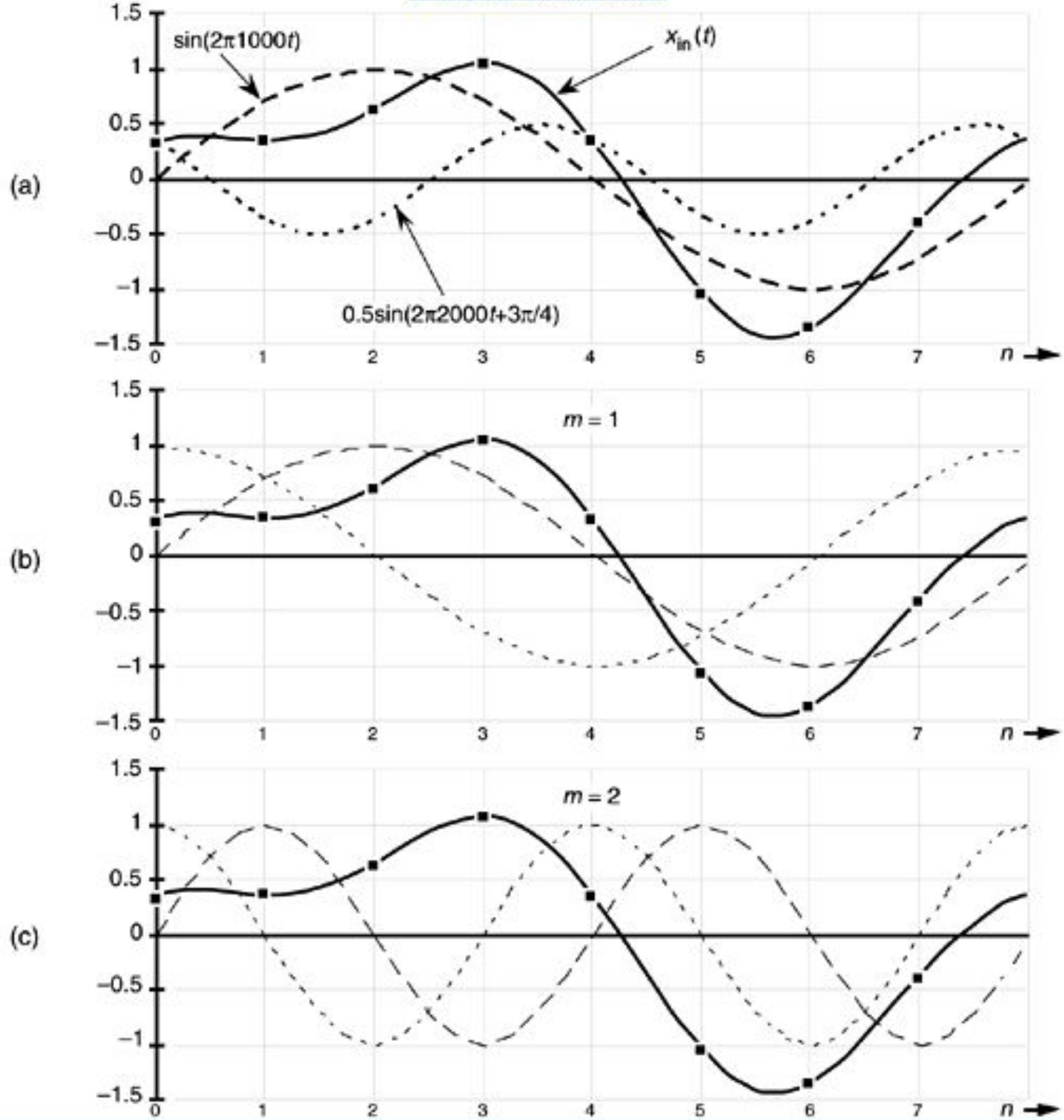

- X(1)=0.0 – j4.0 = 4 , -90°.
- X(2)= 1.414 + j1.414 = 2, 45°.
- x(3)=x(4)=….x(7)=0.0 – j0.0 = 0, 0°.

x(n) contains a signal component at a frequency of 1 kHz. and X(1)'s phase angle relative to a 1-kHz cosine is $X_ø(1)$ = –90°.

And also the input x(n) contains a signal at a frequency of 2 kHz whose relative amplitude is 2, and whose phase angle relative to a 2-kHz cosine is 45°.

DFT magnitude

When a real input signal contains a sinewave component of peak amplitude A_o with an integral number of cycles over N input samples, the output magnitude of the DFT for that particular sinewave is M_r where

Mr=A0 N/2

If the DFT input is a complex sinusoid of magnitude A_o (i.e., $A_o e^{j2(pi)ft}$) with an integral number of cycles over N samples, the output magnitude of the DFT is M_c where

Mc=Ao N

Example 5.1.2

A finite-duration sequence of length L is given as

$$x(n) = \begin{cases} 1, & 0 \le n \le L-1 \\ 0, & \text{otherwise} \end{cases}$$

Determine the N-point DFT of this sequence for $N \ge L$.

Solution The Fourier transform of this sequence is

$$X(\omega) = \sum_{n=0}^{L-1} x(n) e^{-j\omega n}$$

$$= \sum_{n=0}^{L-1} e^{-j\omega n} = \frac{1-e^{-j\omega L}}{1-e^{-j\omega}} = \frac{\sin(\omega L/2)}{\sin(\omega/2)} e^{-j\omega(L-1)/2}$$

The magnitude and phase of $X(\omega)$ are illustrated in Fig. 5.5 for $L = 10$. The N-point DFT of $x(n)$ is simply $X(\omega)$ evaluated at the set of N equally spaced frequencies $\omega_k = 2\pi k/N$, $k = 0, 1, \ldots, N-1$. Hence

$$X(k) = \frac{1-e^{-j2\pi kL/N}}{1-e^{-j2\pi k/N}} \qquad k = 0, 1, \ldots, N-1$$

$$= \frac{\sin(\pi kL/N)}{\sin(\pi k/N)} e^{-j\pi k(L-1)/N}$$

DFT

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N} \qquad k = 0, 1, 2, \ldots, N-1$$

IDFT

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j2\pi kn/N} \qquad n = 0, 1, 2, \ldots, N-1$$

5.17* Determine the eight-point DFT of the signal

$$x(n) = \{1, 1, 1, 1, 1, 1, 0, 0\}$$

$$\begin{aligned} X(k) &= \sum_{n=0}^{7} x(n) e^{-j\frac{2\pi}{8}kn} \\ &= \{6, -0.7071 - j1.7071, 1 - j, 0.7071 + j0.2929, 0, 0.7071 - j0.2929, 1 + j, \\ &\quad -0.7071 + j1.7071\} \end{aligned}$$

DFT properties

Periodicity. If $x(n)$ and X(k) are an N-point DFT pair, then

$$x(n+N) = x(n) \quad \text{for all } n$$

$$X(k+N) = X(k) \quad \text{for all } k$$

Linearity. If

$$x_1(n) \overset{\text{DFT}}{\underset{N}{\longleftrightarrow}} X_1(k)$$

and

$$x_2(n) \overset{\text{DFT}}{\underset{N}{\longleftrightarrow}} X_2(k)$$

then for any real-valued or complex-valued constants a_1 and a_2,

$$a_1x_1(n) + a_2x_2(n) \overset{\text{DFT}}{\underset{N}{\longleftrightarrow}} a_1X_1(k) + a_2X_2(k) \tag{5.2.6}$$

DFT symmetry

We can state that when the DFT input sequence is real, $X(k)$ is the complex conjugate of $X(N–k)$, or

$X(k)= x^*(N-k)$

we need only compute the first $N/2+1$ values of $X(k)$

where $0\leq k \leq(N/2)$;

EXercice

5.1 The first five points of the eight-point DFT of a real-valued sequence are $\{0.25, 0.125 - j0.3018, 0, 0.125 - j0.0518, 0\}$. Determine the remaining three points.

5.1

Since $x(n)$ is real, the real part of the DFT is even, imaginary part odd. Thus, the remaining points are $\{0.125 + j0.0518, 0, 0.125 + j0.3018\}$

X(k)= x*(N-k)
X(5)= x*(8-5)= x*(3) = 0.125+j0.0518
X(6)= x*(8-6)= x*(2) = 0
X(7)= x*(8-7)= x*(1) = 0.125+j0.3018

DFT shifting

If we decide to sample x(n) starting at n equals some integer L, as opposed to n = 0, the DFT of those time-shifted sample values is $X_{shifted}(k)$ where:

$$X_{shifted}(k)= e^{j2\Pi Lk/N}\ x(k).$$

5.23 Compute the N-point DFTs of the signals

(a) $x(n) = \delta(n)$

(b) $x(n) = \delta(n - n_0) \quad 0 < n_0 < N$

(c) $x(n) = a^n \quad 0 \leq n \leq N - 1$

(d) $$x(n) = \begin{cases} 1, & 0 \leq n \leq N/2 - 1 (N \text{ even}) \\ 0, & N/2 \leq n \leq N - 1 \end{cases}$$

(e) $x(n) = e^{j(2\pi/N)k_0} \quad 0 \leq n \leq N - 1$

(f) $x(n) = \cos \frac{2\pi}{N} k_0 n \quad 0 \leq n \leq N - 1$

(g) $x(n) = \sin \frac{2\pi}{N} k_0 n \quad 0 \leq n \leq N - 1$

(h) $$x(n) = \begin{cases} 1, & n \text{ even} \\ 0, & n \text{ odd} \quad 0 \leq n \leq N - 1 \end{cases}$$

(a) $X(k) = \sum_{n=0}^{N-1} \delta(n) e^{-j\frac{2\pi}{N}kn} = 1, \qquad 0 \le k \le N-1$

(b)

$$\begin{aligned} X(k) &= \sum_{n=0}^{N-1} \delta(n-n_0) e^{-j\frac{2\pi}{N}kn} \\ &= e^{-j\frac{2\pi}{N}kn_0}, \qquad 0 \le k \le N-1 \end{aligned}$$

(c)

$$\begin{aligned} X(k) &= \sum_{n=0}^{N-1} a^n e^{-j\frac{2\pi}{N}kn} \\ &= \sum_{n=0}^{N-1} (ae^{-j\frac{2\pi}{N}k})^n \\ &= \frac{1-a^N}{1-ae^{-j\frac{2\pi}{N}k}} \end{aligned}$$

$$\sum_{k=0}^{n} a^k = \frac{1-a^{n+1}}{1-a}, a \neq 1$$

$$e^{-j2\pi K} = 1$$

(d)

$$
\begin{aligned}
X(k) &= \sum_{n=0}^{\frac{N}{2}-1} e^{-j\frac{2\pi}{N}kn} \\
&= \frac{1-e^{-j\frac{2\pi}{N}\frac{N}{2}k}}{1-e^{-j\frac{2\pi}{N}k}} \\
&= \frac{1-(-1)^k}{1-e^{-j\frac{2\pi}{N}k}}
\end{aligned}
$$

(e)

$$
\begin{aligned}
X(k) &= \sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}nk_0} e^{-j\frac{2\pi}{N}kn} \\
&= \sum_{n=0}^{N-1} e^{-j\frac{2\pi}{N}(k-k_0)n} \\
&= N\delta(k-k_0)
\end{aligned}
$$

DFT of complex signal

$$x_c(n) = e^{j2\pi nk/N}$$

where k is the number of complete cycles occurring in the N samples

DFT of a complex sinusoid: →

$$X_c(m) = e^{j[\pi(k-m)-\pi(k-m)/N]} \cdot \frac{\sin[\pi(k-m)]}{\sin[\pi(k-m)/N]}$$

DFT for cos input $x_r(n) = \cos(2\pi nk/N)$,

where k is the integral number of complete cycles occurring in the N samples. Remembering Euler's relationship $\cos(ø) = (e^{jø} + e^{-jø})/2$, we can show the desired DFT as $X_r(m)$

$$X_r(m) = \sum_{n=0}^{N-1} x_r(n) e^{-j2\pi nm/N} = \sum_{n=0}^{N-1} \cos(2\pi nk/N) \cdot e^{-j2\pi nm/N}$$

$$= \sum_{n=0}^{N-1} (e^{j2\pi nk/N} + e^{-j2\pi nk/N})/2 \cdot e^{-j2\pi nm/N}$$

$$= \frac{1}{2}\sum_{n=0}^{N-1} e^{j2\pi n(k-m)/N} + \frac{1}{2}\sum_{n=0}^{N-1} e^{-j2\pi n(k+m)/N}.$$