Digital Signal Processing

Convolution Continued

Remember: The convolution sum

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$

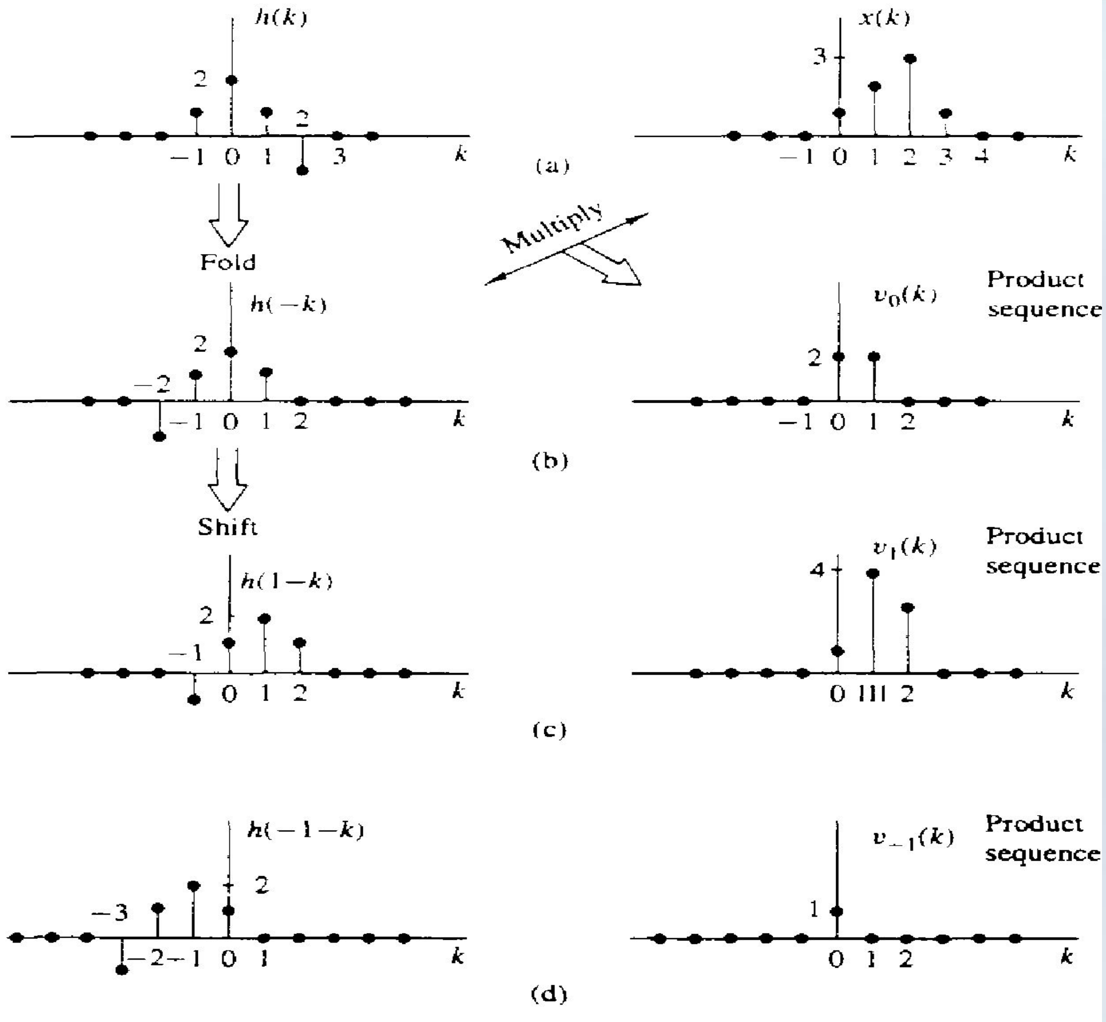


Figure 2.23 Graphical computation of convolution.

$$y(0) = \sum_{k=-\infty}^{\infty} v_0(k) = 4$$

$$y(1) = \sum_{k=-\infty}^{\infty} v_1(k) = 8$$

$$y(n) = \{\ldots, 0, 0, 1, \underset{\uparrow}{4}, 8, 8, 3, -2, -1, 0, 0, \ldots\}$$

The convolution sum: $y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$

$$v_n(k) = x(k)h(n-k)$$

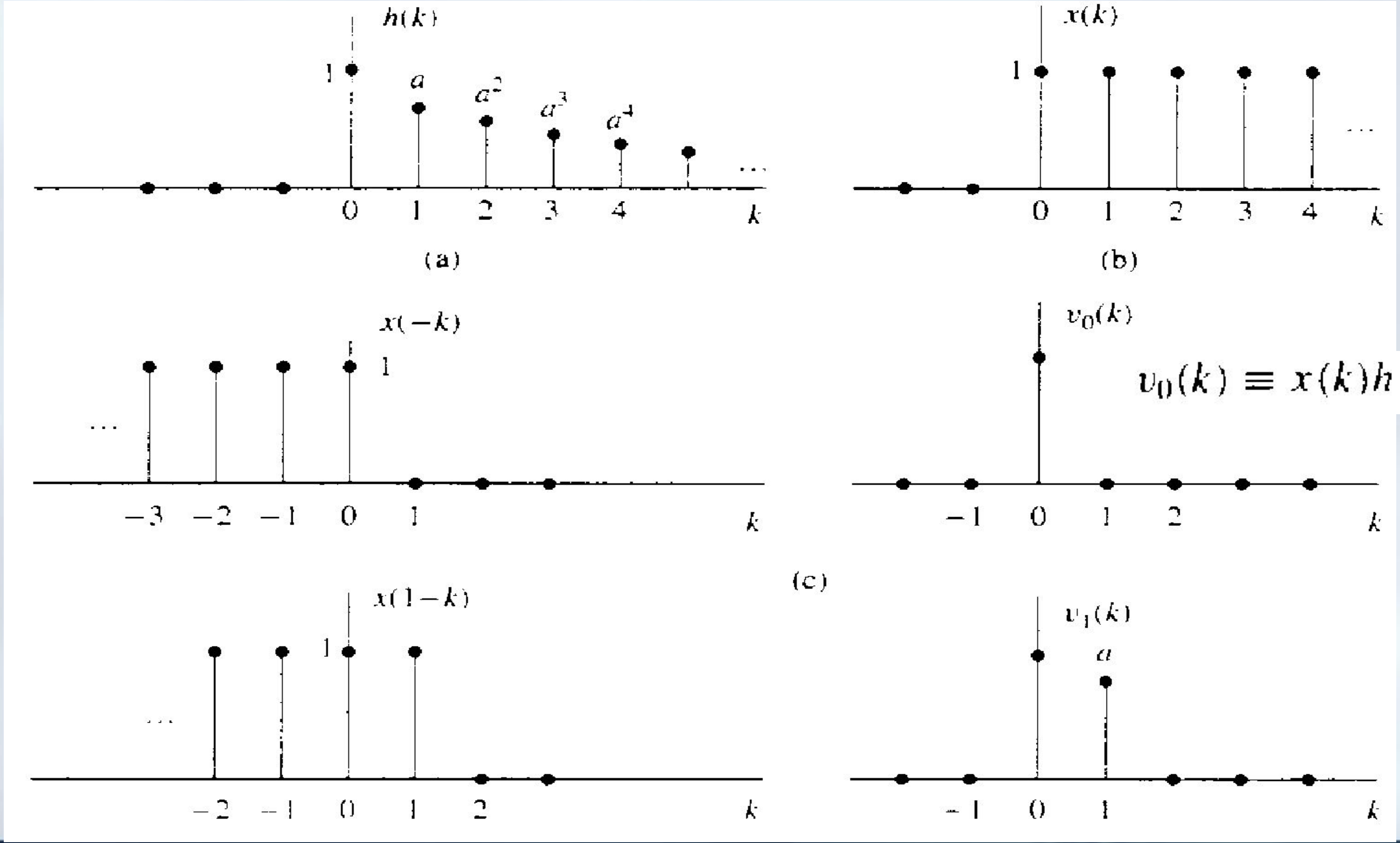
$$w_n(k) = x(n-k)h(k)$$

$$y(n) = \sum_{k=-\infty}^{\infty} v_n(k) = \sum_{k=-\infty}^{\infty} w_n(n-k)$$

Example: determine the output y(n) of a relaxed LTI system with impulse $h(n) = a^n u(n), |a| < 1$

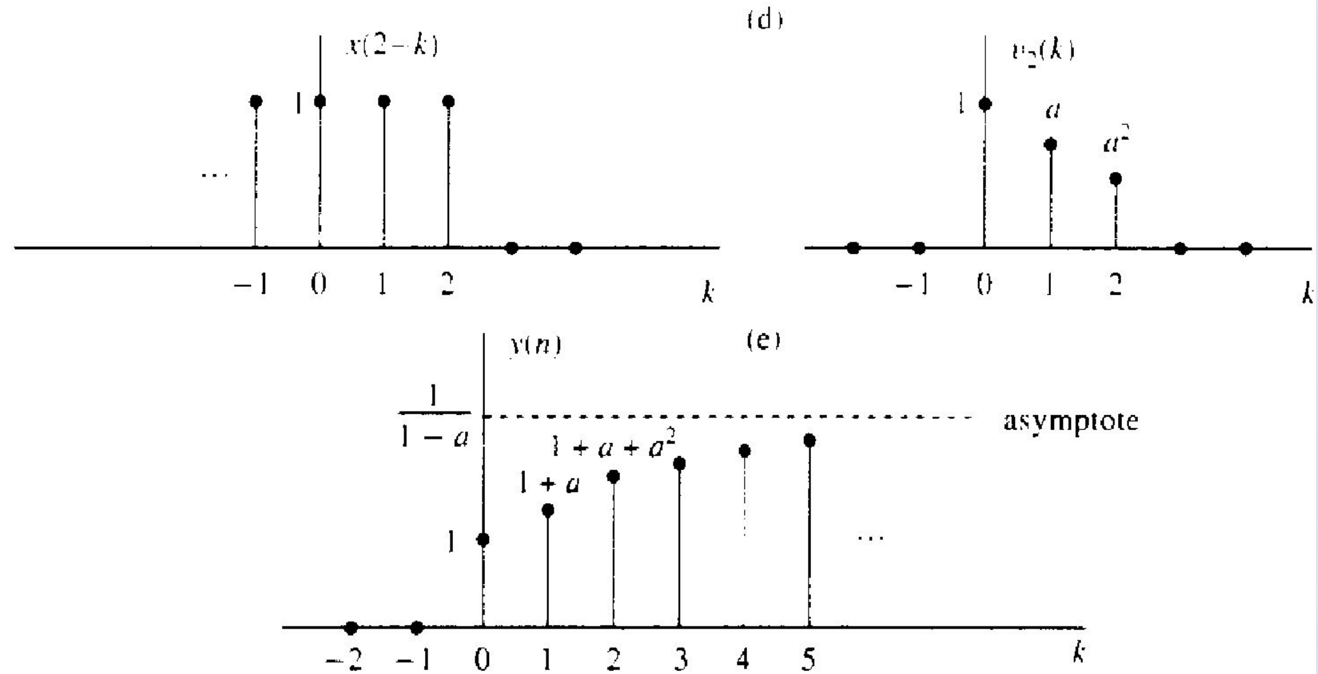
When the input x(n) =u(n).

The convolution sum: $y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$



$$v_0(k) \equiv x(k)h(-k)$$

The convolution sum:



Clearly, for $n > 0$, the output is

$$y(n) = 1 + a + a^2 + \cdots + a^n$$

$$= \frac{1 - a^{n+1}}{1 - a}$$

$$y(n) = 0 \qquad n < 0$$

$$y(\infty) = \lim_{n \to \infty} y(n) = \frac{1}{1 - a}$$

Properties of the convolution

$$y(n) = x(n) * h(n) \equiv \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$

$$y(n) = h(n) * x(n) \equiv \sum_{k=-\infty}^{\infty} h(k)x(n-k)$$

Commutative law

$$x(n) * h(n) = h(n) * x(n)$$

Associative law

$$[x(n) * h_1(n)] * h_2(n) = x(n) * [h_1(n) * h_2(n)]$$

Distributive law

$$x(n) * [h_1(n) + h_2(n)] = x(n) * h_1(n) + x(n) * h_2(n)$$

Causal LTI system

$$y(n_0) = \sum_{k=-\infty}^{\infty} h(k)x(n_0 - k)$$

$$\begin{aligned} y(n_0) &= \sum_{k=0}^{\infty} h(k)x(n_0 - k) + \sum_{k=-\infty}^{-1} h(k)x(n_0 - k) \\ &= [h(0)x(n_0) + h(1)x(n_0 - 1) + h(2)x(n_0 - 2) + \cdots] \\ &\quad + [h(-1)x(n_0 + 1) + h(-2)x(n_0 + 2) + \cdots] \end{aligned}$$

For causal system we have: $h(n) = 0 \qquad n < 0$ $\qquad y(n_0) = \sum_{k=0}^{\infty} h(k)x(n_0 - k)$

And if the input x(n) is also causal (: x(n)=0 n<0 →x(-1)=0, x(-2)=0,….)

Then : we have

$$\begin{aligned} y(n) &= \sum_{k=0}^{n} h(k)x(n-k) \\ &= \sum_{k=0}^{n} x(k)h(n-k) \end{aligned}$$

Example: determine the unit step response for LTI system with impulse response

$$h(n) = a^n u(n) \qquad |a| < 1$$

Sol: $x(n) = 1 \text{ for } n \geq 0$ the input is causal and the system is also causal, then using the previous formula 2.3.41

$$y(n) = \sum_{k=0}^{n} a^k$$

$$= \frac{1 - a^{n+1}}{1 - a}$$

Systems with Finite-Duration and Infinite-Duration Impulse Response (FIR and IIR)

- For causal FIR system

$$h(n) = 0 \qquad n < 0 \text{ and } n \geq M$$

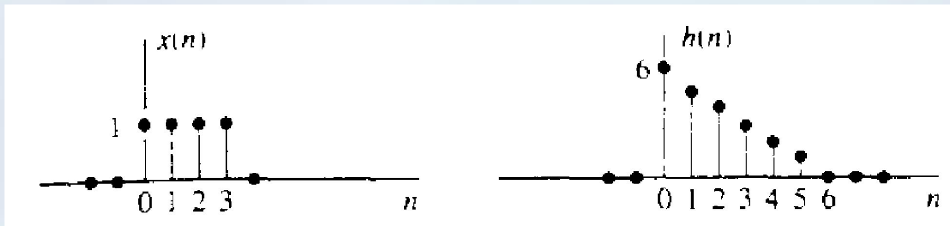
$$y(n) = \sum_{k=0}^{M-1} h(k)x(n-k)$$

- But for causal IIR system

$$y(n) = \sum_{k=0}^{\infty} h(k)x(n-k)$$

problems

- Compute the convolution for the following pairs



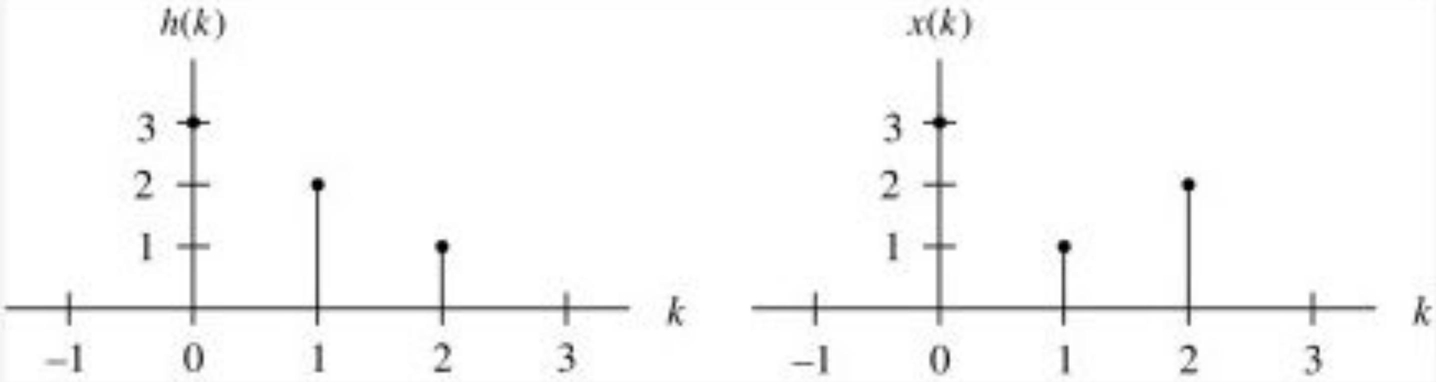
(a)

(a)

$$
\begin{aligned}
x(n) &= \left\{\underset{\uparrow}{1}, 1, 1, 1\right\} \\
h(n) &= \left\{\underset{\uparrow}{6}, 5, 4, 3, 2, 1\right\} \\
y(n) &= \sum_{k=0}^{n} x(k)h(n-k) \\
y(0) &= x(0)h(0) = 6, \\
y(1) &= x(0)h(1) + x(1)h(0) = 11 \\
y(2) &= x(0)h(2) + x(1)h(1) + x(2)h(0) = 15 \\
y(3) &= x(0)h(3) + x(1)h(2) + x(2)h(1) + x(3)h(0) = 18 \\
y(4) &= x(0)h(4) + x(1)h(3) + x(2)h(2) + x(3)h(1) + x(4)h(0) = 14 \\
y(5) &= x(0)h(5) + x(1)h(4) + x(2)h(3) + x(3)h(2) + x(4)h(1) + x(5)h(0) = 10 \\
y(6) &= x(1)h(5) + x(2)h(4) + x(3)h(2) = 6 \\
y(7) &= x(2)h(5) + x(3)h(4) = 3 \\
y(8) &= x(3)h(5) = 1 \\
y(n) &= 0, n \geq 9 \\
y(n) &= \left\{\underset{\uparrow}{6}, 11, 15, 18, 14, 10, 6, 3, 1\right\}
\end{aligned}
$$

Table method

$Y(n)= \{9,9,11,5,2,0,0….\}$



k:	-2	-1	0	1	2	3	4	5	
$x(k)$:			3	1	2				
$h(-k)$:	1	2	3						$y(0) = 3 \times 3 = 9$
$h(1-k)$		1	2	3					$y(1) = 3 \times 2 + 1 \times 3 = 9$
$h(2-k)$			1	2	3				$y(2) = 3 \times 1 + 1 \times 2 + 2 \times 3 = 11$
$h(3-k)$				1	2	3			$y(3) = 1 \times 1 + 2 \times 2 = 5$
$h(4-k)$					1	2	3		$y(4) = 2 \times 1 = 2$
$h(5-k)$						1	2	3	$y(5) = 0$ (no overlap)

problems

2.18 Determine and sketch the convolution $y(n)$ of the signals

$$x(n) = \begin{cases} \frac{1}{3}n, & 0 \le n \le 6 \\ 0, & \text{elsewhere} \end{cases}$$

$$h(n) = \begin{cases} 1, & -2 \le n \le 2 \\ 0, & \text{elsewhere} \end{cases}$$

(a) Graphically
(b) Analytically

(a)

$$\begin{aligned} x(n) &= \left\{ \underset{\uparrow}{0}, \frac{1}{3}, \frac{2}{3}, 1, \frac{4}{3}, \frac{5}{3}, 2 \right\} \\ h(n) &= \left\{ 1, 1, \underset{\uparrow}{1}, 1, 1 \right\} \\ y(n) &= x(n) * h(n) \\ &= \left\{ \frac{1}{3}, \underset{\uparrow}{1}, 2, \frac{10}{3}, 5, \frac{20}{3}, 6, 5, \frac{11}{3}, 2 \right\} \end{aligned}$$

2.19 Compute the convolution $y(n)$ of the signals

$$x(n) = \begin{cases} \alpha^n, & -3 \le n \le 5 \\ 0, & \text{elsewhere} \end{cases}$$

$$h(n) = \begin{cases} 1, & 0 \le n \le 4 \\ 0, & \text{elsewhere} \end{cases}$$

$$\begin{aligned} y(n) &= \sum_{k=0}^{4} h(k)x(n-k), \\ x(n) &= \left\{\alpha^{-3}, \alpha^{-2}, \alpha^{-1}, \underset{\uparrow}{1}, \alpha, \ldots, \alpha^{5}\right\} \\ h(n) &= \left\{\underset{\uparrow}{1}, 1, 1, 1, 1\right\} \end{aligned}$$

$$\begin{aligned} y(n) &= \sum_{k=0}^{4} x(n-k), -3 \le n \le 9 \\ &= 0, \text{ otherwise.} \end{aligned}$$