Digital Signal Processing

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ANALYSIS OF DISCRETE TIME LINEAR TIME INVARIANT SYSTEMS

Analysis of discrete time linear time invariant systems

- These systems are characterized in the time domain by their response to a simple unit sequence
- Any arbitrary input signal can be decomposed and represented as a weighted sum of unit sample sequences

Techniques for Analysis of linear time systems-1

- **Direct Solution:** the input output relationship is represented as

$$y(n) = F\left[y(n-1), y(n-2), \ldots, y(n-N), x(n), x(n-1), \ldots, x(n-M)\right]$$

  where F[.] represent function of the quantities in the brackets

- <u>Can also presented as:</u>

$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k)$$

Techniques for Analysis of linear time systems-2

Decomposition-1:

- The input signal is resolved into sum of elementary signals
- Then the response of the system to the elementary signals are added to obtain total response to the system to the given input signal

$$x(n) = \sum_{k} c_k x_k(n)$$

Resolution of discrete time signals into impulses

- Suppose we have an arbitrary signal x(n) that we wish to resolve into a sum of unit sample sequence.
- Let the elementary signal $\delta(n-k)$ Which is zero everywhere except at n=k its value is unity.
- Multiplication of x(n) by $x_k(n) = \delta(n-k)$ results another sequence that is zero everywhere except at n=k, where its value is x(k).
- That is

$$x(n) = \sum_{k=-\infty}^{\infty} x(k)\delta(n-k)$$

Resolution of discrete time signals into impulses

Example: consider the special case of a finite duration sequence given as:

$$x(n) = \{2, \underset{\uparrow}{4}, 0, 3\}$$

Resolve the sequence x(n) into a sum of weighted impulse sequences.

Solution: since the sequence x(n) is nonzero for the time instances n=-1,0,2. we need three impulses at delays k=-1,0,2. we find that:

$$x(n) = 2\delta(n+1) + 4\delta(n) + 3\delta(n-2)$$

Response of LTI systems to Arbitrary inputs: the convolution sum

- We denote the response y(n,k) of any relaxed linear system to the input unit sample sequence at n=k by the special symbol h(n,k), $-\infty < k < \infty$, that is:

$$y(n,k) \equiv h(n,k) = \mathcal{T}[\delta(n-k)]$$

Response of LTI systems to Arbitrary inputs: the convolution sum

- Suppose we have an arbitrary input signal, expressed as:

$$x(n) = \sum_{k=-\infty}^{\infty} x(k)\delta(n-k)$$

- Then

$$y(n) = \mathcal{T}[x(n)] = \mathcal{T}\left[\sum_{k=-\infty}^{\infty} x(k)\delta(n-k)\right]$$

$$= \sum_{k=-\infty}^{\infty} x(k)\mathcal{T}[\delta(n-k)]$$

$$= \sum_{k=-\infty}^{\infty} x(k)h(n,k) \qquad (2.3.14)$$

- And if $h(n) \equiv \mathcal{T}[\delta(n)]$ then, by TI property : $h(n-k) = \mathcal{T}[\delta(n-k)]$

and (2.3.14) reduced to $y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$ (2.3.17) **Convolution Sum**

The convolution sum:

- The process of computing the convolution between x(k) and h(k) involves the following 4 steps:
- 1- Folding : fold h(k) about k=0 to obtain h(-k) .
- 2- shifting, shift h(-k) by n0 to the right (left) if n0 is positive(negative) to obtain h(no-k).
- 3- multiplication. Multiply x(k) by h(n0-k) to obtain the product sequence vn0(k) = x(k)h(n0-k).
- 4- Summation. Sum all the values of the product sequence vn0(k) to obtain the value of the output at time n=n0.
- Step 2-4 repeated for all possible time shifts $-\infty < n < \infty$.

The convolution sum:

Example: the impulse response of LTI system is

$$h(n) = \{1, \underset{\uparrow}{2}, 1, -1\}$$

Determine the response of the system to the input signal

$$x(n) = \{\underset{\uparrow}{1}, 2, 3, 1\}$$

Sol: we use graphs of the sequences to aid us in the computation.

The convolution sum:

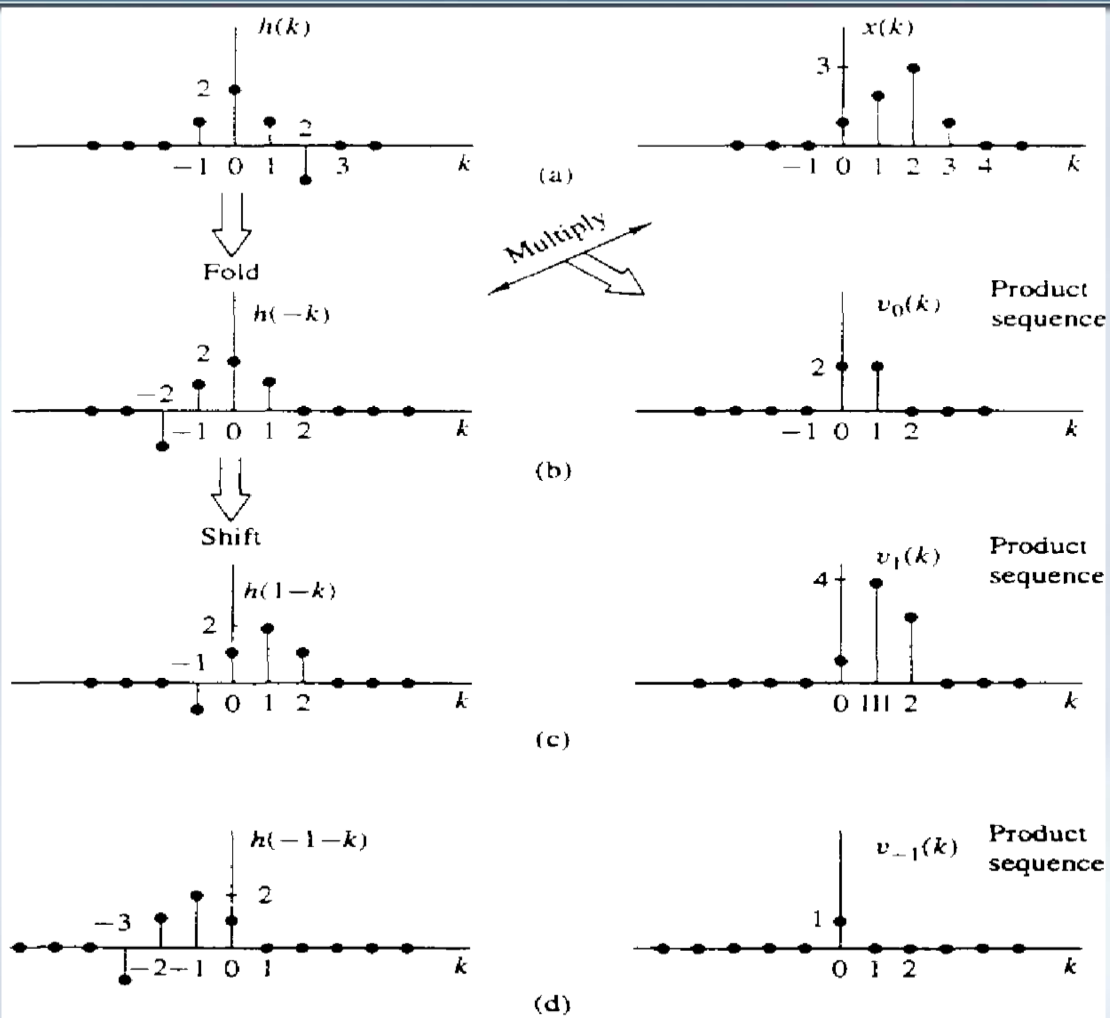


Figure 2.23 Graphical computation of convolution.

The convolution sum:

$$y(0) = \sum_{k=-\infty}^{\infty} x(k)h(-k)$$

$$v_0(k) \equiv x(k)h(-k)$$

$$y(0) = \sum_{k=-\infty}^{\infty} v_0(k) = 4$$

$$v_1(k) = x(k)h(1-k)$$

$$y(1) = \sum_{k=-\infty}^{\infty} v_1(k) = 8$$

In a similer mannaer we obtain y(2) by shifting h(-k) two units to the right, forming the product sequence v2(k), and then summing all the terms in the product obtaining y(2)= 8. repeat for n=3,4. for n> 5 y(n)=0.

Repeat for n=-1 (shift h(-k) to the left one unit) . Notice : y(n)=0 for n≤ 2.
Then we have:

$$y(n) = \{\ldots, 0, 0, 1, \underset{\uparrow}{4}, 8, 8, 3, -2, -1, 0, 0, \ldots\}$$

The convolution sum:

$$v_n(k) = x(k)h(n-k)$$

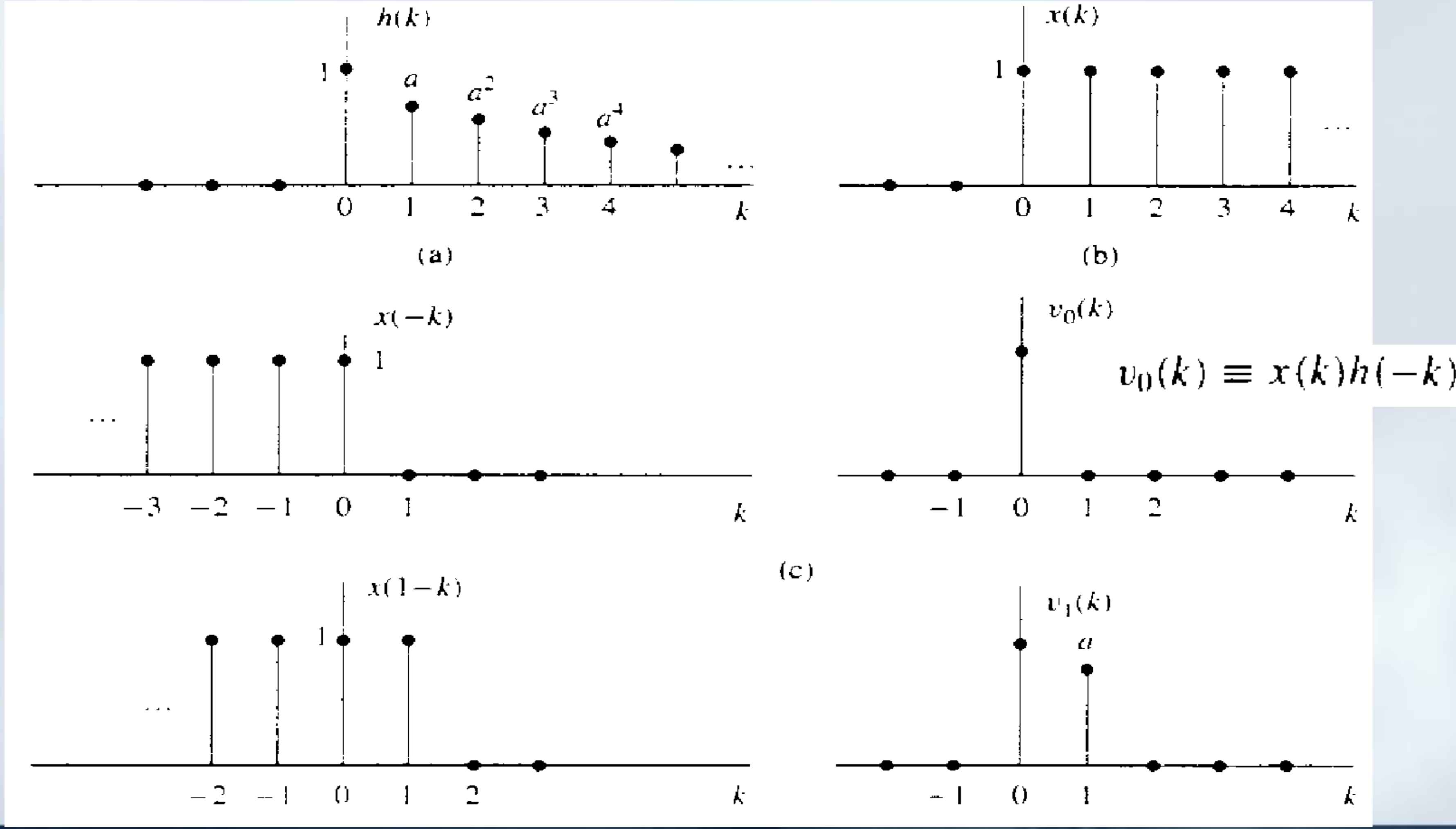
$$w_n(k) = x(n-k)h(k)$$

$$y(n) = \sum_{k=-\infty}^{\infty} v_n(k) = \sum_{k=-\infty}^{\infty} w_n(n-k)$$

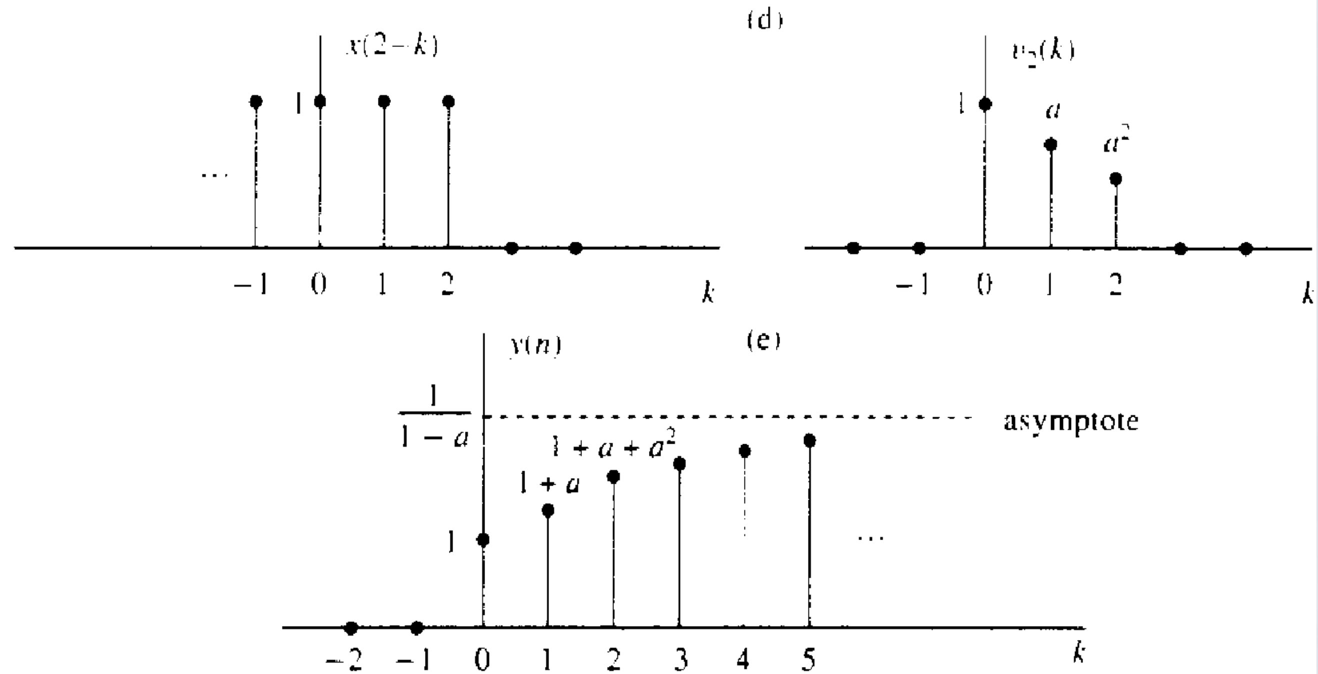
Example: determine the output y(n) of a relaxed LTI system with impulse $h(n) = a^n u(n), |a| < 1$

When the input x(n) =u(n).

The convolution sum: $y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$



The convolution sum:



Clearly, for $n > 0$, the output is

$$y(n) = 1 + a + a^2 + \cdots + a^n$$
$$= \frac{1 - a^{n+1}}{1 - a}$$

$y(n) = 0 \qquad n < 0$

$$y(\infty) = \lim_{n \to \infty} y(n) = \frac{1}{1-a}$$