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% TJAA - Example Article - Example.tex

%

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%

% Version: 2015-06-15 - 1.1

% Version: 2019-08-01 - 2.0

% Version: 2021-04-01 - 2.1

% Version: 2022-01-01 - 2.2

% - added Example-Turkce file using \TRlang

% which explains multi language abstract

% - updated explanations in manuscript meta-data notes

% - added sections: Astro. Macros, Astro. Journals

% - added description for new columntypes

% - added section for auth-year commands

% Version: 2022-01-22 - 2.3

% - added Disclosure section

% Version: 2024-09-01 - 2.4

% - updated author & address usage with new commands

% - updated abstract description

% - updated keyword description

% - added descr. for new astronomical macros

%---------------------------------------------------------------------

\documentclass[usenatbib]{tjaa}

%---------------------------------------------------------------------

% Türkçe karakterler ile yazmak ve Türkçe heceleme icin aşağıdaki

% paketin etkin olması zorunludur.

\usepackage[utf8]{inputenc}

%%%%% AUTHORS - PLACE YOUR OWN PACKAGES HERE %%%%%

\usepackage{lipsum}

\usepackage{amsmath}

\usepackage{graphicx}

\usepackage{caption}

\usepackage{listings}

\usepackage{xcolor}

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\definecolor{codegray}{rgb}{0.5,0.5,0.5}

\definecolor{codepurple}{rgb}{0.58,0,0.82}

\definecolor{backcolour}{rgb}{0.95,0.95,0.92}

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numberstyle=\tiny\color{codegray},

stringstyle=\color{codepurple},

basicstyle=\ttfamily\footnotesize,

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keepspaces=true,

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showspaces=false,

showstringspaces=false,

showtabs=false,

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\lstset{style=mystyle}

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%%%% PLEASE DONT DELETE LINES CONTAINING "%%%%TJAA-OZEL" %%%%

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%%%%TJAA-OZEL:BASLIK%

% NOTES (title):

% 1) If you need to put a \newline in the title

% then you HAVE TO specify the short title

% DONT USE [] (short title) if your title not overlapping short author

\title[Stability of a Body in Figure-Eight Orbits]{Analyzing the Stability of the Zero-Angular-Momentum Figure-Eight solution with a Fourth Body Introduced: Effects of Different Masses, Velocities, and Positions }%%%%TJAA-OZEL:TITLE%

% NOTES (authors):

% 1) For single author don't number single affiliation addreess.

% 2) If you need two or more lines of authors, use \newauthor (see below usage)

% 3) ORCID is required for ALL authors

% 4) Short author should be either A.U. Thor et.al -or- A. U. Thor

% 5) Depending on the language \others will be either "et al." or "v.ark."

\author[F. Author \others]{%

Ahmed Abd El-Hamid,

Mohamed Abdelghani,

Ziad Abu Shanab

\newauthor

\\

% NOTES (List of institutions):

% 1) Don't put \\ at the last institution

STEM High School for Boys - 6th of October\\

STEM October Physics Club}%%%%TJAA-OZEL:AUTHOR%

% These dates and numbers will be filled out by the publisher

\date{}

%

\renewcommand{\pubyear}{0000}

\renewcommand{\volume}{0}

\renewcommand{\issue}{0}

% NOTES (language):

% 1) Default language of TJAA is ENGLISH.

% 2a) If your manuscript is in ENGLISH then leave both commands as commented

%\ENlang

% 2b) If your manuscript is in TURKISH then uncomment only below command

%\TRlang

\begin{document}

% Don't change these 3 lines

\label{firstpage}

\pagerange{\pageref{firstpage}--\pageref{lastpage}}

\maketitle{M00-0000}

\begin{abstract}%%%%TJAA-OZEL:ABS%

% NOTES (abstract):

% >>>>>IF LANGUAGE IS IN ENGLISH<<<<< | >>>>>IF LANGUAGE IS IN TURKISH<<<<< |

% \begin{abstract}%%%%TJAA-OZEL:ABS% | \begin{abstract}%%%%TJAA-OZEL:ABS%

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% 0) Please follow above example for abstract

% 1) If your language is in ENGLISH

% (a) put your abstract below and

% (b) delete (2) option text below

The figure-eight solution of the three-body problem is one of the few known stable, zero-angular-momentum configurations. However, introducing a fourth body to this system presents new challenges to the stability of the configuration. This study investigates the effects of varying the mass, velocity, and position of the fourth body on the dynamics, energy, and angular momentum of the figure-eight solution. The analysis was conducted under the condition of equal masses for the three primary bodies ($m=1$), while restricting the mass of the fourth body to the range ($0.001$ to $0.01$). The results demonstrate that the system exhibits a wide range of parameter values that maintain gravitational binding, but only a small subset of these configurations exhibit long-term stability. Notably, a configuration with a mass of 0.001, a velocity of 0.1414213, and a position of 0.17 retained its gravitational binding and stability for over 1000 orbital periods. These results show how sensitive the system is to changes and give a clearer picture of the conditions needed to keep the orbits stable over time.

% 2) If your manuscript is in TURKISH (otherwise ignore below usage)

% a) Write TURKISH abstract above

% b) "uncomment" the command '\ozet' below

% c) Put ENGLISH abstract below the command

%\ozet

\end{abstract}

% NOTES (Keywords):

% 1) Select minimum one and maximum six entries from the approred list

% 2) Don't make up new ones.

% 3) THEY HAVE TO BE ALL IN ENGLISH

%

\begin{keywords}

% W A R N I N G : \*\*\*\*\* N O T U R K I S H K E Y W O R D S \*\*\*\*\*

Figure-Eight -- Bokeh$^{[1]}$ -- Matplotlib$^{[2]}$ -- Gravitational Binding -- Zero-Angular-Momentum

% W A R N I N G : \*\*\*\*\* N O T U R K I S H K E Y W O R D S \*\*\*\*\*

\end{keywords}

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\section{Introduction}

The three-body problem examines the motion of three masses under mutual gravitational attraction. Unlike the two-body case, which has a simple and predictable solution, adding a third body makes the system highly complex and generally chaotic. Periodic solutions to the three-body problem, like the Euler and Lagrange configurations, show order under specific conditions but are often unstable over long periods [3].

Among these solutions, the figure-eight trajectory, discovered numerically by Moore in 1993 and later studied by Montgomery, Šuvakov, and Dmitrašinović [4], stands out. In this solution, three bodies of equal mass follow a symmetric, figure-eight path while maintaining zero angular momentum. This symmetry, as shown in Figure 1, ensures a unique balance of forces, allowing the system to remain stable over long periods in ideal conditions. The figure-eight solution is rare because most gravitational systems naturally generate non-zero angular momentum due to mass or velocity asymmetries [5].

Introducing a fourth body into the figure-eight system disrupts its delicate equilibrium. The additional gravitational force introduces new degrees of freedom, leading to perturbations that alter the system's energy and angular momentum. Scientifically, this increases the likelihood of chaotic motion, including collisions, escapes, or deviations from the figure-eight trajectory [6]. Physically, even a small fourth body exerts a cumulative effect that destabilizes the system over time.

This study investigates the effects of the fourth body by analyzing its mass, velocity, and position. The mass of the fourth body is restricted to the range to reflect a small perturbing influence, while velocities and positions are varied between 0.1 and 1.0 in steps of 0.01. The results demonstrate that while many configurations maintain gravitational binding, only specific combinations of these parameters allow the system to remain stable over extended periods. For instance, a fourth body with a mass of 0.001, velocity 0.1414213, and initial position of 0.17 retained stability for over 1000 orbital periods.

From these parameter ranges, we are able to understand how the figure-eight solution responds to external disturbances. This work contributes to the understanding of the forces at play in zero-angular-momentum systems and improves the knowledge of their stability under other gravitational effects.

\section{Methodology}

\subsection{Simulation of the Figure-Eight solution}

The simulation of the figure-eight solution was implemented to establish a baseline configuration for the three-body problem under gravitational interaction. This solution is characterized by three equal-mass bodies following a periodic, symmetric trajectory under Newtonian gravity. The motion of each body is governed by Newton's second law and the gravitational force between the bodies. For three bodies $m\_1$, $m\_2$, $m\_3$, their accelerations can be written as:

$$\vec{a}\_i = G \sum\_{j\neq i } m\_j\frac{\vec{r}\_j -\vec{r}\_i}{|\vec{r}\_j - \vec{r}\_i|^3}$$

To determine the trajectory of the bodies with such acceleration, a system of second-order degree differential equations should be solved, as acceleration is the second derivative of position. The differential equation for each body $i$ is written as follows:

$$\frac{\text{d}^2\vec{r}\_i}{\text{d}t^2} = G \sum\_{j\neq i}m\_j\frac{\vec{r}\_j -\vec{r}\_i}{|\vec{r}\_j - \vec{r}\_i|^3}$$

To solve these second-order equations numerically, they were converted into a system of first-order differential equations by introducing the velocities $\vec{v}\_i$ of each body. This transformation is expressed as:

$$\frac{\text{d} \vec{r}\_i}{\text{d}t} = \vec{v}\_i ,\hspace{0.1cm} \frac{\text{d}\vec{v}\_i}{\text{d}t} = G \sum\_{j\neq i}m\_j\frac{\vec{r}\_j -\vec{r}\_i}{|\vec{r}\_j - \vec{r}\_i|^3}$$

Afterwards, this system was implemented in Python using the following function:

\lstinputlisting[language=Octave]{methodologies differential equations.py}

To begin the simulation, the positions and velocities of the three bodies were first initialized such that the velocities follow the following rules:

$$\vec{v}\_{1} = (p\_1, p\_2), \hspace{0.1cm} \vec{v}\_{2} = (p\_1, p\_2), \hspace{0.1cm} \vec{v}\_{3} = (-2p\_1, -2p\_2)$$

The certain values were then set as follows:

\begin{center}

$\vec{v}\_1 = (0.466203685,0.43236573)$

\end{center}

\begin{center}

$ \vec{v}\_2 = (0.466203685,0.43236573)$

\end{center}

\begin{center}

$\vec{v}\_3 = (−0.93240737,−0.86473146) $

\end{center}

and

\begin{center}

$\vec{r}\_1 = (−0.97000436,0.24308753)$

\end{center}

\begin{center}

$\vec{r}\_2 = (0.97000436,-0.24308753)$

\end{center}

\begin{center}

$\vec{r}\_3 = (0,0) $

\end{center}

The system of differential equations was next integrated numerically using Runge-Kutta 4(5) [7], and he system was then configured with a time span corresponding to 10 orbital periods ($T \approx 6.32591398$) and 10,000 time steps.

\lstinputlisting[language=Octave]{Time span figure eight.py}

The outputs $\vec{r}\_1(t), \vec{r}\_2(t), \vec{r}\_3(t)$ was finally plotted as an HTML plot using Bokeh, as shown in Figure 1:

\begin{center}

\includegraphics[scale=0.2]{Figure Eight Normal Figure Introduction.jpg}

\captionof{figure}{A simulation for the Figure-Eight solution}

\end{center}

\subsection{Stability-Distance range determeination}

To define the stability-distance range at which the fourth body remains stable, the net gravitationtal force needs to be calculated. The net gravitational force acting on a body within a multi-body system is obtained from the vector sum of all forces exerted by other bodies. The sum can be written as:

\[

\vec{F}\_4 = G \sum\_{j \neq 4} m\_j \frac{\vec{r}\_j - \vec{r}\_4}{|\vec{r}\_j - \vec{r}\_4|^3}.

\]

This expression was written with the following function:

\lstinputlisting[language=Octave]{netforce.py}

For a set of test position of body 4, the net force was calculated and marked on a graph, shown in Figure 2. The graph showed a steep decline at very small distnaces, a slight stablization at intermediate distances, and a near-to-zero values at larger ones.

\begin{center}

\includegraphics[scale=0.32]{Net Force copy.jpg}

\captionof{figure}{Net gravitational force on the fourth body vs Distance}

\end{center}

At $x = 0.5$, the turning point, that force started to be more stablized after some turbulence. On the other hand, $x= 2.0$ is the distance at which the force started to be much smaller. Beyond this point, the fourth body is too distant to be influenced by the forces, that the motion of the body dominates, making it escape the system. Hence, having $x=0.5$ as the turning point, $x=2.0$ as the threshold point, the stability-distance range should be as follows:

$$0.5\leq x \leq 2.0$$

\subsection{Determining the ideal initial conditions for the fourth body}

To determine the conditions under which the fourth body stays stable within the previously calculated range, the initial mass, velocity, and position were varied. The stability was evaluated based on the distance of the fourth body from two of the three primary bodies over a certain time span. A time span was said to be stable of the distance satisfy the following condition:

$$0.5 \leq r\_{41}, r\_{42} \leq 2.0$$

The stability percentage was then calculated as the number of stable time steps to the total number of steps:

$$ \text{Stability Percentage } = \frac{\text{Number of Stable Time Steps}}{\text{Total Number of Time Steps}} \times 100$$

To do this, we used a python function that computes the distance and evaluates the stability percentage for some trajectories:

\lstinputlisting[language=Octave]{stabilitypercentage.py}

After setting the stability conditions, two ways of analysis were taken, Mass vs Velocity and Mass vs Position.

\subsubsection{Mass vs Velocity Heat Map}

We first analyzed the effect of varying the mass and velocity of the fourth body while keeping its initial position fixed. The mass of the fourth body was varied between 0.001 and 0.01 with an increment of 0.00025. Its velocity magnitude was similarly varied between 0.1 and 1 with a step size of 0.025.

Having a total number of 1369 combinations, the system of equations for each one was solved using the Runge-Kutta 4(5) method, with a tolerance of $rtol = 10^{-8}$ and $atol = 10^{-10}$. The following code shows the entire process, including the implementation of the derivatives of position and velocity:

\lstinputlisting[language=Octave]{massvelocityheatmap.py}

The output of the code was stored in a 2D array, heat map, where each pixel represents the stability of a certain combination.

\subsubsection{Mass vs Position Heat Map}

Similarly, we examined the effect of varying the mass and initial position of the fourth body. The initial velocity was fixed, and the position were varied from -0.5 to 0.5 with an increments of 0.025. The mass was varied within the same range and increments as in the previous case.

The same procedures as in the previous case were made and are summarized in the following code:

\lstinputlisting[language=Octave]{masspositionheatmap.py}

\subsection{Energy \& Angular Momentum calculation}

To finally verify the stability of the system for the final conditions, the total energy and angular momentum should be calculated throughout the simulation. These quantities are conserved in isolated systems and calculating them ensures that the fourth body remains stable over time.

\subsubsection{Energy Conservation}

The total energy of any system is defined as the sum of its kinetic energy and gravitational potential energy. The kinetic energy of the system is given by:

$$K = \frac{1}{2} \sum\_{i=1}^4m\_i |\vec{v}\_i|^2,$$

where $m\_i$ and $v\_i$ are the mass and velocity of body i, respectively. The gravitational potential energy is given by:

$$U = -G \sum\_{i<j } \frac{m\_im\_j}{|\vec{r}\_i - \vec{r}\_j|},$$

where $|\vec{r}\_i - \vec{r}\_j|$ is the distance between bodies $i$ and $j$. The total energy is therefore:

$$E\_{total} = K+U$$

This was implemented using the following function:

\lstinputlisting[language=Octave]{kinetic.py}

\subsubsection{Angular momentum}

The angular momentum has to also be calculated to further confirm the stability of the motion. The total angular momentum is given by:

$$\vec{L} = \sum\_{i=1}^4 m\_i(\vec{r}\_i \times \vec{v}\_i),$$

where $\vec{r}\_i$ and $\vec{v}\_i$ are the position and velocity vectors of body $i$, respectively. Angular momentum was implemented using the following function:

\lstinputlisting[language=Octave]{angular.py}

The results of these functions were stored and plotted as functions of time to be observed. If both quantities remains approximately constant, it indicates that the simulation was stable and the integration was performed accurately.

% - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - -

\section{Results and Discussion}

\subsection{Initial conditions}

Figures 3 and 4, respectively, present the results of analyzing the stability of the system under varying initial conditions, specifically mass-velocity and mass-position combinations. The color gradient in both heat maps reflects the percentage of stability, with purple indicating instability and bright yellow representing the most stable state.

Figure 3 showed that the most stable conditions lie at lower values of mass and velocity, specifically within a mass range 0.001 to 0.004 and a velocity range of 0.1 to 0.5. As the initial velocity increases beyond this range, the stability decreases significantly.

\begin{center}

\includegraphics[scale=0.34]{output.png}

\captionof{figure}{Mass vs Velocity heatmap showing stability}

\end{center}

These results further show that increasing the mass of the fourth body has a similar destabilizing effect as velocity. Higher masses introduce greater gravitational forces, disturbing the balance of the system. Similarly, higher velocities alter the balance of the system.

On the other hand, Figure 4 shows the relationship between the mass of the body and its initial position. Unlike the previous case, the figure indicates that stability is highly sensitive to the initial position of the fourth body rather than its mass. The most stable regions are concentrated within a narrower range of approximately 0.002 to 0.2 units. Across this range, a wide range of masses, up to 0.01, exhibit high the same thing, high stability. This suggests that as long as the fourth body starts within a critical positional range, its mass has a minimal impact on the system's stability.

\begin{center}

\includegraphics[scale=0.34]{mass-position 2.png}

\captionof{figure}{Mass vs Position heatmap showing stability}

\end{center}

\subsection{Selection of the most optimal conditions}

In this section, the most stable regions from both the Mass vs Velocity and Mass vs Position heatmaps were analyzed. The goal was to identify specific conditions for the mass, velocity, and position of the fourth body that exhibit high stability over extended periods. The method involved selecting the ideal points from both graphs and identifying overlaps between them to determine the ideal conditions for these three parameters. Doing this, five points were selected to be simulated over different time periods.

\subsubsection{First point}

For each of the following points, a figure will show the tarjectory of the body over four different intervals: 1 period, 3 periods, 5 periods, and 10 periods. The first point has the following initial conditions:

\begin{center}

$m = 0.003421$ \\

$r= 0.2121320$ \\

$v=0.5$

\end{center}

Figure 5 shows that the trajectory of the body remains stable for 1 and 3 periods, closely following the figure-eight path. By the 5th period, the it starts to drift, and by the 10th period, it significantly deviates, indicating instability.

\begin{center}

\includegraphics[scale=0.34]{2 for the paper.jpg}

\captionof{figure}{Trajectory of the first point over 1, 3, 5, and 10 intervals}

\end{center}

\subsubsection{Second point}

The second point has an initial conditions of:

$$m = 0.003421, \hspace{0.1cm} r = 0.2121320, \hspace{0.1cm} v = 0.3605551$$

As shown in Figure 6, the trajectory remains stable for 1 and 3 periods, with the it slightly deviating from the path. By the 5th period, the deviation becomes more noticeable, and by the 10th period, it escaped completely from the stable configuration.

\begin{center}

\includegraphics[scale=0.34]{3 for the paper.jpg}

\captionof{figure}{Trajectory of the second point over 1, 3, 5, and 10 intervals}

\end{center}

\subsubsection{Third point}

Having the following initial conditions,

$$m = 0.002, \hspace{0.1cm} r= 0.2, \hspace{0.1cm} v = 0.2,$$

the system displays clear stability for the first 1, 3, and 5 periods. However, by the 10th period, the deviations of the body became more noticable, though the shape of the figure-eight is still visible.

\begin{center}

\includegraphics[scale=0.34]{4 for the paper.jpg}

\captionof{figure}{Trajectory of the third point over 1, 3, 5, and 10 intervals}

\end{center}

\subsubsection{Fourth body}

With these conditions:

$$m=0.001, \hspace{0.1cm} r=0.17, \hspace{0.1cm} v= 0.1414213, $$

this configuration showed the most stability across all four intervals.

\begin{center}

\includegraphics[scale=0.34]{Ideal conditions for paper.jpg}

\captionof{figure}{Trajectory of the fourth point over 1, 3, 5, and 10 intervals}

\end{center}

While the third and fourth points both show similar stability at first, they behave differently over time. In the third point, the fourth body slowly drifts away and eventually escapes the system by the 21st period. In the fourth point, however, the fourth body remains very stable, staying close to the figure-eight path even after 1000 periods, as shown in Figure 9. This shows that the fourth point has much better long-term stability.

Hence, with a mass of $m= 0.001$, a position of $r=(-0.008, -0.015)$, and a velocity of $v=(0.1, 0.1)$, this point was identified as having the most ideal conditions for introducing a stable fourth body into the figure-eight solution. Unlike other configurations, this set of values allowed the fourth body to remain consistently bound to the system without noticeable deviations over extended periods of time, demonstrating exceptional long-term stability.

\begin{center}

\includegraphics[scale=0.2]{1000.png}

\captionof{figure}{The trajectory of the obtained solution over 1000 periods}

\end{center}

\subsection{Energy \& Angular Momentum}

\subsubsection{Energy}

To further examine and verify the solution obtained, the energy of the system was calculated and plotted. Figure 10 represents the change in the total energy and the changes that occur in the kinetic energy and the potential energy of the system.

\begin{center}

\includegraphics[scale=0.4]{total energies PE and Ke.png}

\captionof{figure}{Tracing of the total, potential, and kinetic energy of the system over time}

\end{center}

As shown in the figure, the total energy of the system stays constant due to the absence of any external work that done on the system.

Additionally, the kinetic and potential energy of the system behave with a descending relationship with each other. As the potential energy increases with a certain value, the kinetic energy decreases with the same value and the opposite. This figure further proved the stability of the proposed solution.

\subsubsection{Angular Momentum}

Figure 11 shows how the angular momentum of the system changes with time.

\begin{center}

\includegraphics[scale=0.4]{Angular Momentum.png}

\captionof{figure}{The angular momentum of the system over time}

\end{center}

The angular momentum turned out to be nearly constant as there is no external torque acting on the system. Despite that, there was still a little bit of distribution due to the imbalance of the forces acting on the system.

% - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - -

\section{Conclusion}

The simulations demonstrated that a solution with the initial conditions of a mass $m = 0.001$, a position $r=(-0.008,-0.015)$, and a velocity $v=(0.1,0.1)$ achieved remarkable stability within the figure-eight solution. This configuration allowed the fourth body to remain stable for more than 1000 periods while maintaining a constant total energy of approximately -0.92

and an angular momentum consistently ranging between 2.90 and 3.10.

These results highlight that, under precise initial conditions, the figure-eight solution can accommodate an additional body without disrupting the dynamics of the system. These findings demonstrate the possibility of a solution to the four-body problem and open pathways for exploring similar stable solutions in celestial mechanics.

\section{References}

[1] Bokeh Development Team (2018). Bokeh: Python library for interactive visualization. URL: http://www.bokeh.pydata.org. \newline

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\section{Appendices}

\subsection{Mass-Velocity Heat Map}

\lstinputlisting[language=Octave]{Mass Velocity Heatmap.py}

\subsection{Mass-Position Heat Map}

\lstinputlisting[language=Octave]{Mass-Position Heatmap.py}

\subsection{Trajectory Plot}

\lstinputlisting[language=Octave]{Trajectory Plot.py}

\subsection{Energy Graph}

\lstinputlisting[language=Octave]{Energy Graph.py}

\subsection{Angular Momentum Graph}

\lstinputlisting[language=Octave]{Angular Momentum Graph.py}

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