

Generating Sets and Basis

In a vector space V , we are particularly interested in sets of vectors A that possess the property that any vector $v \in V$ can be obtained by a linear combination of vectors in A . These vectors are special vectors, and in the following, we will characterize them.

Definition 1 (Generating Set/Span). Consider a vector space V and set of vectors $\mathcal{A} = \{\mathbf{x}_1, \dots, \mathbf{x}_k\} \subseteq \mathcal{V}$. If every vector $v \in V$ can be expressed as a linear combination of $\mathbf{x}_1, \dots, \mathbf{x}_k$, \mathcal{A} is called a *generating set* or *span*, which spans the vector space V . In this case, we write $V = [\mathcal{A}]$ or $V = [\mathbf{x}_1, \dots, \mathbf{x}_k]$.

generating set
span

Generating sets are sets of vectors that span vector (sub)spaces, i.e., every vector can be represented as a linear combination of the vectors in the generating set. Now, we will be more specific and characterize the smallest generating set that spans a vector (sub)space.

Definition 2 (Basis). Consider a real vector space V and $\mathcal{A} \subseteq \mathcal{V}$.

- A generating set \mathcal{A} of V is called *minimal* if there exists no smaller set $\tilde{\mathcal{A}} \subseteq \mathcal{A} \subseteq \mathcal{V}$ that spans V . minimal
- Every linearly independent generating set of V is minimal and is called basis of V . basis

Let V be a real vector space and $\mathcal{B} \subseteq \mathcal{V}, \mathcal{B} \neq \emptyset$. Then, the following statements are equivalent:

- \mathcal{B} is a basis of V
- \mathcal{B} is a minimal generating set
- \mathcal{B} is a maximal linearly independent subset of vectors in V .
- Every vector $\mathbf{x} \in V$ is a linear combination of vectors from \mathcal{B} , and every linear combination is unique, i.e., with

$$\mathbf{x} = \sum_{i=1}^k \lambda_i \mathbf{b}_i = \sum_{i=1}^k \psi_i \mathbf{b}_i \quad (1)$$

and $\lambda_i, \psi_i \in \mathbb{R}, \mathbf{b}_i \in \mathcal{B}$ it follows that $\lambda_i = \psi_i, i = 1, \dots, k$.

Example:

- In \mathbb{R}^3 , the *canonical/standard basis* is

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}. \quad (2)$$

canonical/standard
basis

- Different bases in \mathbb{R}^3 are

$$\mathcal{B}_1 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}, \quad \mathcal{B}_2 = \left\{ \begin{bmatrix} 0.5 \\ 0.8 \\ -0.4 \end{bmatrix}, \begin{bmatrix} 1.8 \\ 0.3 \\ 0.3 \end{bmatrix}, \begin{bmatrix} -2.2 \\ -1.3 \\ 3.5 \end{bmatrix} \right\} \quad (3)$$

- The set

$$\mathcal{A} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ -4 \end{bmatrix} \right\} \quad (4)$$

is linearly independent, but not a generating set (and no basis): For instance, the vector $[1, 0, 0, 0]^\top$ cannot be obtained by a linear combination of elements in \mathcal{A} .

Remark. Every vector space V possesses a basis \mathcal{B} . The examples above show that there can be many bases of a vector space V , i.e., there is no unique basis. However, all bases possess the same number of elements, the *basis vectors*.

basis vectors
dimension

We only consider finite-dimensional vector spaces V . In this case, the *dimension* of V is the number of basis vectors, and we write $\dim(V)$. If $U \subseteq V$ is a subspace of V then $\dim(U) \leq \dim(V)$ and $\dim(U) = \dim(V)$ if and only if $U = V$. Intuitively, the dimension of a vector space can be thought of as the number of independent directions in this vector space.