

## Formula Sheet

### Vector operations

$$\mathbf{r} + \mathbf{s} = \mathbf{s} + \mathbf{r}$$

$$2\mathbf{r} = \mathbf{r} + \mathbf{r}$$

$$\|\mathbf{r}\|^2 = \sum_i r_i^2$$

- dot or inner product:

$$\mathbf{r} \cdot \mathbf{s} = \sum_i r_i s_i$$

commutative  $\mathbf{r} \cdot \mathbf{s} = \mathbf{s} \cdot \mathbf{r}$

distributive  $\mathbf{r} \cdot (\mathbf{s} + \mathbf{t}) = \mathbf{r} \cdot \mathbf{s} + \mathbf{r} \cdot \mathbf{t}$

associative  $\mathbf{r} \cdot (a\mathbf{s}) = a(\mathbf{r} \cdot \mathbf{s})$

$$\mathbf{r} \cdot \mathbf{r} = \|\mathbf{r}\|^2$$

$$\mathbf{r} \cdot \mathbf{s} = \|\mathbf{r}\| \|\mathbf{s}\| \cos \theta$$

- scalar and vector projection:

scalar projection:  $\frac{\mathbf{r} \cdot \mathbf{s}}{\|\mathbf{r}\|}$

vector projection:  $\frac{\mathbf{r} \cdot \mathbf{s}}{\mathbf{r} \cdot \mathbf{r}} \mathbf{r}$

### Basis

A basis is a set of  $n$  vectors that:

(i) are not linear combinations of each other

(ii) span the space

The space is then  $n$ -dimensional.

### Matrices

$$\mathbf{A}\mathbf{r} = \mathbf{r}'$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e \\ f \end{pmatrix} = \begin{pmatrix} ae + bf \\ ce + df \end{pmatrix}$$

$$\mathbf{A}(n\mathbf{r}) = n(\mathbf{A}\mathbf{r}) = n\mathbf{r}'$$

$$\mathbf{A}(\mathbf{r} + \mathbf{s}) = \mathbf{A}\mathbf{r} + \mathbf{A}\mathbf{s}$$

Identity:  $\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

clockwise rotation by  $\theta$ :  $\begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}$

determinant of 2x2 matrix:  $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$

inverse of 2x2 matrix:  $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

- summation convention for multiplying matrices  $a$  and  $b$ :

$$ab_{ik} = \sum_j a_{ij} b_{jk}$$

### Change of basis

Change from an original basis to a new, primed basis. The columns of the transformation matrix  $B$  are the new basis vectors in the original coordinate system. So

$$B\mathbf{r}' = \mathbf{r}$$

where  $\mathbf{r}'$  is the vector in the  $B$ -basis, and  $\mathbf{r}$  is the vector in the original basis. Or;

$$\mathbf{r}' = B^{-1}\mathbf{r}$$

If a matrix  $A$  is *orthonormal* (all the columns are of unit size and orthogonal to each other) then:

$$A^T = A^{-1}$$

### Gram-Schmidt process for constructing an orthonormal basis

Start with  $n$  linearly independent basis vectors  $\mathbf{v} = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ . Then

$$\mathbf{e}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|}$$

$$\mathbf{u}_2 = \mathbf{v}_2 - (\mathbf{v}_2 \cdot \mathbf{e}_1)\mathbf{e}_1 \quad \text{so} \quad \mathbf{e}_2 = \frac{\mathbf{u}_2}{\|\mathbf{u}_2\|}$$

... and so on for  $\mathbf{u}_3$  being the remnant part of  $\mathbf{v}_3$  not composed of the preceding  $\mathbf{e}$ -vectors, etc. ...

### Transformation in a Plane or other object

First transform into the basis referred to the reflection plane, or whichever;  $E^{-1}$ .

Then do the reflection or other transformation, in the plane of the object  $T_E$ .

Then transform back into the original basis  $E$ .

So our transformed vector  $\mathbf{r}' = ET_E E^{-1}\mathbf{r}$ .

### Eigenstuff

To investigate the characteristics of the  $n$  by  $n$  matrix  $\mathbf{A}$ , you are looking for solutions the the equation,

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$$

where  $\lambda$  is a scalar eigenvalue. Eigenvalues will satisfy the following condition

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = 0$$

where  $\mathbf{I}$  is an  $n$  by  $n$  dimensional identity matrix

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To find the dominant eigenvector of link matrix  $\mathbf{L}$ , the Power Method can be iteratively applied, starting from a uniform initial vector  $\mathbf{r}$ .

$$\mathbf{r}^{i+1} = \mathbf{L}\mathbf{r}^i$$

A damping factor,  $d$ , can be implement to stabilize this method as follows.

$$\mathbf{r}^{i+1} = d\mathbf{L}\mathbf{r}^i + \frac{1-d}{n}$$